COMPLEMENTED LATTICES

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CHAPTER I

INTRODUCTION

Until about thirty years ago the area of algebra now called lattice theory was virtually unknown. Since that time it has emerged as an important area of mathematical study, and rapid progress has marked its development.

Although it is a comparatively new subject area, the beginnings of lattice theory can be traced back more than 100 years. While doing research in mathematical logic in 1842, Boole developed an important class of lattices which were later named Boolean Algebras. In 1890 Schroder introduced the lattice concept as it is understood today; and in 1897 Dedekind defined, according to present-day terminology, modular and distributive lattices [14].[1] The development of lattice theory proper began in the 1930's with Garrett Birkhoff being the dominate figure in the field.

The significance of lattice theory in the field of mathematics is due not only to the important theorems it has yielded but also to its unifying nature. Its concepts and techniques have found fundamental applications in many areas of mathematics.

In an article for the American Mathematical Society, Garrett Birkhoff, an outstanding mathematician in lattice theory and head of

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[1]Numbers in brackets refer to references in the bibliography.

the Department of Mathematics at Harvard University, wrote that

> ... some familiarity with . . . lattice theory is an essential preliminary to the full understanding of logic, set theory, probability, functional analysis, projective geometry, the decomposition theorems of abstract algebra, and many other branches of mathematics [2].

Another mathematician who stresses the unifying nature of lattice theory is Lillian Lieber. In her book on the subject, she states that lattices help the mathematician to see connections between various branches of mathematics, thus revealing their common structure even though they seem so different [10].

In spite of the fact that lattice theory is an important branch of algebra and has manifold applications in mathematics, it has not reached the prominence it deserves in college curricula. Currently, very few colleges and universities offer a course in lattice theory, and when one is offered it is generally at the graduate level. Although an occasional text in abstract algebra will include some work in the field, elementary textbooks in lattice theory are few.

This paper has been prepared to be the basis for an undergraduate course in lattice theory. It is self-contained to the extent that the only prerequisites are a basic knowledge of abstract algebra and set theory. In fact, this material has already been used in a senior seminar course at Harding College, Searcy, Arkansas. The paper attempts to meet two important objectives. First, it will help show the basic relationships that exist between different areas of mathematics, and will illuminate some important properties of a variety of mathematical systems. The second basic objective of the paper is to expose the undergraduate student to the frontier in a specific area of mathematics.

Because lattice theory is relatively new and is based on very simple postulates, undergraduates can utilize it to engage in mathematical research and work on unsolved problems.

CHAPTER II

BASIC DEFINITIONS AND EXAMPLES

In trying to relate some of the many areas of mathematics, it could be asked, "What do set theory, group theory, theory of numbers, projective geometry, theory of probability, and mathematical logic have in common?" One answer to this question would be to observe that the notion of "part of" or "contained in" is basic to each of these subjects. Consider for example how important the study of subgroups is in group theory. In this context, A is a "part of" B would mean that A is a subgroup of B. Similar comments could be made in each of these other areas. The notion of a "lattice" arises in the attempt to obtain an abstract system that includes such systems as the subgroups of a group, subspaces of a vector space, etc. as special cases.

Partially Ordered Sets

How this relation of "part of" is used in defining a "lattice" will soon be made clear, but first some preliminary definitions are needed.

Definition 1.1. A partially ordered set is a system consisting of a set S and a binary relation $\subset$ ("part of" or "contained in") satisfying the following postulates: For all elements a, b, c, in S,

i) $a \subset a$,

ii) If $a \subset b$ and $b \subset a$, then $a = b$, and

iii) If $a \subset b$ and $b \subset c$, then $a \subset c$.

Definition 1.2. If $a, b \in S$, then $a \supset b$ (read $\supset$ as "contains") if, and only if, $b \subset a$.

If $a, b \in S$ and $a \subset b$ or $b \subset a$ then a and b are said to be comparable; however, it is not necessarily true that each two elements of S are comparable. If $a, b \in S$ implies that either $a \subset b$ or $b \subset a$, then S is called linearly ordered or sometimes the word chain is used.

When thinking intuitively of partially ordered sets, the next two examples will prove very useful.

Example 1.1. For an arbitrary set M, let **P** (M) be the set of all subsets of M. The set **P** (M) is partially ordered with respect to the relation of set inclusion.

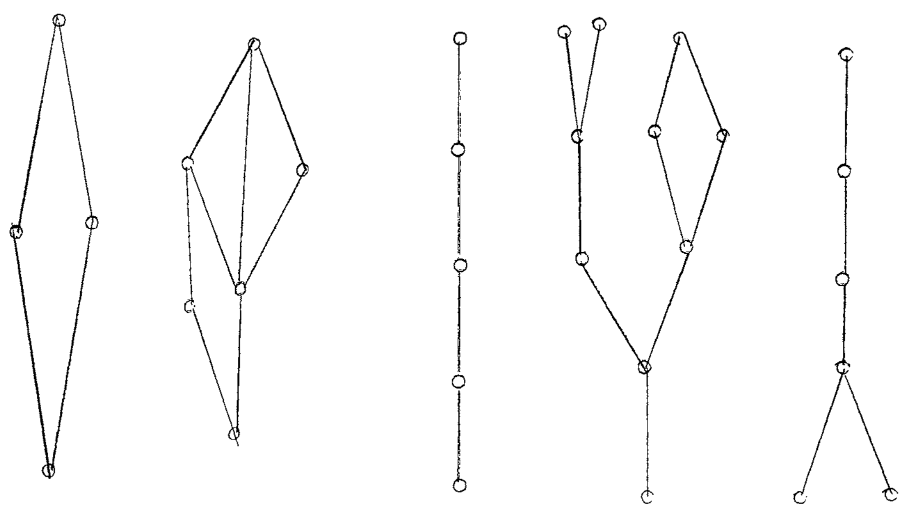
Example 1.2. The real numbers ordered by the relation "$\leq$". Notice that in Example 1.2 all the elements are comparable, thus this example represents a chain.

Diagrams

The ordering of finite sets can be very clearly illustrated with the use of diagrams. Before describing the method some preliminary concepts need to be introduced.

If for a pair of element a, b of a partially ordered set, $a \subset b$ and there is no element x such that $a \subset x \subset b$, then it is said that a is covered by b or b covers a. The notation for "a covered by b" is $a < b$. Notice that in Example 1.1, B covers A if, and only if, $B = A \cup \{a\}$ where $a \in B \setminus A$.

In a finite partially ordered set the relation $\subset$ can be expressed in terms of the relation of covering, for if $a \subset b$ then a chain $a = a_1 \subset a_2 \subset \ldots \subset a_n = b$ can be found in which each a_{i+1} covers a_i [14]. This is called a <u>principle</u> <u>chain</u> from a to b and is denoted by $a = a_1 < a_2 < \ldots < a_n = b$. Conversely, the existence of a principle chain implies that $a \subset b$. Thus, a diagram can be obtained by representing the elements of S by small circles (or dots) and placing the circle for a_2 above that for a_1 and connecting by a line if a_2 covers a_1. Therefore, $a \subset b$ if, and only if, there is a descending broken line connecting b to a. Some examples of such diagrams are given below:



(a) (b) (c) (d) (e)

Figure 1.1

It is evident that the notion of a diagram of a partially ordered set gives us another way to construct examples of such sets.

By appropriate conventions, some infinite partially ordered sets can be illustrated by diagrams of a similar nature but of generalized meaning. Thus, the natural ordering of the positive integers will be represented by the diagram in Figure 1.2.

Figure 1.2

One more concept must be considered before the definition of a lattice can be given.

Definition 1.3. An element p of a partially ordered set S is said to be an upper bound for the subset A of S if $a \subset p$ for every $a \in A$. The element p is a least upper bound (l.u.b.) if p is an upper bound and $p \subset v$ for any upper bound v of A. Similar definitions are made for lower bound and greatest lower bound (g.l.b.).

In Figure 1.1, the partially ordered set represented by (d) does not have an upper bound, and (e) does not have a lower bound.

Lattices

A particular type of partially ordered set will now be considered. This partially ordered set was first called a "Dual gruppe" by Dedekind in 1900 and was given the name "lattice" by Garrett Birkhoff in 1933 [5].

Definition 1.4. A lattice is a partially ordered set in which any two elements have a least upper bound, p, and a greatest lower bound, q.

It follows immediately from Definition 1.3 that if a least upper bound or greatest lower bound exists then it is unique.

There are many different notations used for the least upper bound, p, and the greatest lower bound, q, of two elements a and b. Some of these notations are:

$$p = a + b = \text{l.u.b.}\ (a,b) = a \cup b = \sup\ (a,b)$$

$$q = a \cdot b = \text{g.l.b.}\ (a,b) = a \cap b = \inf\ (a,b).$$

The notations used in this paper will be $a \cdot b$ (lattice product or the meet of a and b) for the greatest lower bound and $a + b$ (lattice sum or the join of a and b) for the least upper bound.

Thus, a lattice is a set S and two binary relations, = (binary relation of equivalence) and $\subset$ (binary relation of "part of" or ordering relation) such that

i) S is partially ordered with respect to $\subset$, and

ii) S contains, with each 2 of its elements, a g.l.b. and a l.u.b.

The notation for a lattice will be $L \doteq [S, =, \subset]$.

Notice that Examples 1.1 and 1.2 form lattices and that in Figure 1.1, (d) and (e) do not represent lattices.

Now consider the following examples of lattices. These will not only be a help in understanding more fully the concept of a lattice but will also begin to show how many of the areas of mathematics can be related through a study of lattices. The justification that each of these examples forms a lattice is relatively simple.

Example 1.3. Let S denote a ring of sets;[1] = will be set equality and $\subset$ will be set inclusion.

i) $a \cdot b$ is set intersection.

ii) $a + b$ is set union.

---

[1] A family of sets F is a ring of sets provided F contains with each two of its elements their union and intersection.

Example 1.4. Let S be the set of all convex subsets of E_2.[2] Equality is set equality and $\subset$ is set inclusion.

i) $a \cdot b$ is the set intersection since the intersection of two convex subsets of E_2 is convex [15].

ii) $a + b$ is the convex hull of a and b, where the convex hull of a set S is defined to be the intersection of all convex sets that contain a and b.

Example 1.5. Let S be the set of all subgroups of a given group G. Equality is set equality and $a \subset b$ means that a is a subgroup of b.

i) $a \cdot b$ is the intersection of a and b.

ii) $a + b$ is the smallest subgroup of G that contains both a and b, that is, $a + b$ is the intersection of all subgroups of G that contain both a and b.

Example 1.6. S is the set of all positive integers. Equality means identity and $a \subset b$ means that a is a divisor of b.

i) $a \cdot b$ is the greatest common divisor of a and b.

ii) $a + b$ is the least common multiple of a and b.

Example 1.7. Let S be the set of all linear subspaces of a projective 3-space, that is, S is formed by the null set, points, lines, etc. Equality is set equality and $a \subset b$ means that a is on b and the dimension of a is less than or equal to the dimension of b.

i) $a \cdot b = a \cap b$.

ii) $a + b$ is the smallest linear subspace containing both a and b.

---

[2] C is a convex subset of E_2 if C contains with each two of its elements the line segment joining them.

Example 1.8. S consists of all propositions; $a = b$ means that the occurrence of either implies the occurrence of the other, that is, $a \Rightarrow b$, and $b \Rightarrow a$; $a \subset b$ means that the occurrence of a implies the occurrence of b.

i) $a \cdot b$ is the conjunction of a and b ($a \wedge b$).

ii) $a + b$ is the inclusive disjunction of a and b ($a \vee b$).

Example 1.9. Let S be the set of all real functions f defined on the interval $[0,1]$ such that $0 \leq f(x) \leq 1$ for all x. If $f, g \in S$, then $f = g$ means that $f(x) = g(x)$ for all $x \in [0,1]$; $f \subset g$ means that $f(x) \leq g(x)$ for all $x \in [0,1]$.

i) $f \cdot g : x \rightarrow (f \cdot g)\, x = \text{minimum}\, [f(x), g(x)]$

ii) $f + g : x \rightarrow (f + g)\, x = \text{maximum}\, [f(x), g(x)]$

Example 1.10. Let S be the set of all subspaces of a vector space. Equality is set equality and $a \subset b$ means that a is a subspace of b.

i) $a \cdot b$ is the intersection of a and b.

ii) $a + b$ is the linear hull of a and b, that is, the intersection of all subspaces of S that contain both a and b.

These examples are very important since they will be used throughout this paper to illustrate certain types of lattices.

Some diagrams of finite lattices will not be considered with their corresponding addition and multiplication tables (see Figure 1.3).

An Alternate Definition

If $L = [S, =, \subset]$ is a lattice, then the following three properties are satisfied.

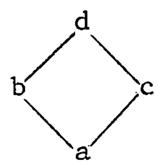
1°. S is partially ordered with respect to $\subset$, that is,

(a)

.	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

+	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

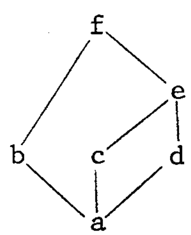
(b)



.	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

+	a	b	c	d
a	a	b	c	d
b	b	b	d	d
c	c	d	c	d
d	d	d	d	d

(c)



.	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	a	a	a	b
c	a	a	c	a	c	c
d	a	a	a	d	d	d
e	a	a	c	d	e	e
f	a	b	c	d	e	f

+	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	b	f	f	f	f
c	c	f	c	e	e	f
d	d	f	e	d	e	f
e	e	f	e	e	e	f
f	f	f	f	f	f	f

Figure 1.3

i) $a \in S \Rightarrow a \subset a$;

ii) $a,b \in S$; $a \subset b$ and $b \subset a \Rightarrow a = b$; and

iii) $a,b,c \in S$; $a \subset b$ and $b \subset c \Rightarrow a \subset c$.

2^o. $a,b \in S$ implies the existence of $a \cdot b \in S$ such that $a \cdot b \subset a$, $a \cdot b \subset b$, and if $x \in S$ where $x \subset a$ and $x \subset b$, then $x \subset a \cdot b$.

3^o. $a,b \in S$ implies the existence of $a + b \in S$ such that $a \subset a + b$, $b \subset a + b$ and if $y \in S$ where $a \subset y$ and $b \subset y$, then $a + b \subset y$.

If in 1^o, 2^o, 3^o the transformations

$$\begin{pmatrix} = \subset & a \cdot b & a + b \\ = \supset & a + b & a \cdot b \end{pmatrix}$$

are made, then the new statements will be called the "duals" of the corresponding old statements. Notice that 1^o is a self dual. By this it is meant that if the substitution is carried out, nothing is changed. The dual of 2^o is 3^o and the dual of 3^o is 2^o. Therefore, the three conditions 1^o, 2^o, 3^o remain valid under the transformations

$$\begin{pmatrix} = \subset & a \cdot b & a + b \\ = \supset & a + b & a \cdot b \end{pmatrix}$$

i.e., the principle of duality holds in a free lattice (a lattice with no special properties placed on it).

Now it is possible to approach a lattice as a purely algebraic structure. Consider the system $[S,=,\cdot,+]$[3] satisfying the following conditions:

---

[3]S is a set and "$\cdot$" and "+" are binary operations defined on S.

1'. $a, b, c \in S$ implies that:

(a) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(b) $a \cdot b = b \cdot a$

(c) $a \cdot a = a$.

2'. $a, b, c \in S$ implies that:

(a) $(a + b) + c = a + (b + c)$

(b) $a + b = b + a$

(c) $a + a = a$.

3'. $a, b \in S$; $a \cdot b = a$ iff $a + b = b$.

Theorem 1.1. Each lattice L is a system $[S, =, \cdot, +]$ where the meet of a and b is $a \cdot b$ and the join of a and b is $a + b$.

Proof: Conditions 1', 2', and 3' must be proved to hold given that 1^{o}, 2^{o}, and 3^{o} hold. To prove 1' part (a), notice that

(1) $(a \cdot b) \cdot c \subset a \cdot b \subset a$ follows directly from the definition of lattice product. Similarly,

(2) $(a \cdot b) \cdot c \subset a \cdot b \subset b$ and $(a \cdot b) \cdot c \subset c$, so

(3) $(a \cdot b) \cdot c \subset b \cdot c$.

Now on combining (1) and (3),

$$(a \cdot b) \cdot c \subset a \cdot (b \cdot c).$$

Using a similar argument,

$$a \cdot (b \cdot c) \subset b \cdot c \subset c \text{ and } a \cdot (b \cdot c) \subset a \cdot b.$$

This implies that $a \cdot (b \cdot c) \subset (a \cdot b) \cdot c$. Therefore, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Now to prove part (b) of 1', notice that $a \cdot b \subset b$ and $a \cdot b \subset a$, so $a \cdot b \subset b \cdot a$. Similarly, it follows that $b \cdot a \subset a \cdot b$. Therefore, $a \cdot b = b \cdot a$.

For part (c) of 1', $a \subset a$ and $a \subset a \Rightarrow a \subset a \cdot a$. This combined with the fact that $a \cdot a \subset a$ proves that $a = a \cdot a$.

The principle of duality gives 2' directly so all that remains is to prove 3'. Suppose that $a \cdot b = a$. This combined with $a \cdot b \subset b$ implies that $a \subset b$; therefore, $a + b = b$. The converse is proved similarly.

The next theorem is the converse of Theorem 1.1.

Theorem 1.2. $[S, =, \cdot, +]$ is a lattice when for each $a, b \in S$, $a \subset b$ is defined to mean that $a \cdot b = a$. Moreover, $a \cdot b$ is the meet of a and b, and $a + b$ is the join of a and b.

Proof: For 1^o part (i), merely observe that $a \cdot a = a \Rightarrow a \subset a$.

To prove 1^o part (ii), notice that by definition $a \subset b$ iff $a \cdot b = a$ and $b \subset a$ iff $b \cdot a = b$. These together with 1' (b) imply that $a = b$.

Part (iii) also follows directly, for $a \subset b \Rightarrow a \cdot b = a$ and $b \subset c \Rightarrow b \cdot c = b$, thus $a \cdot c = (a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot (b) = a$. Hence, $a \subset c$.

To prove 2^o, it must be shown that $a \cdot b$ is the g.l.b. (a,b). First, observe that $a \cdot b \subset a$ since $(a \cdot b) \cdot a = a \cdot (b \cdot a) = a \cdot (a \cdot b) = (a \cdot a) \cdot b = a \cdot b$. Also, $a \cdot b \subset b$ since $(a \cdot b) \cdot b = a \cdot (b \cdot b) = a \cdot b$. Now, suppose that $c \subset a$ and $c \subset b$. This means that $c \cdot a = c$ and $c \cdot b = c$, which implies that $c \cdot (a \cdot b) = (c \cdot a) \cdot b = c$. Thus, by definition, $c \subset a \cdot b$ and hence $a \cdot b$ is the greatest lower bound of a and b.

3^o follows from 2^o and the principle of duality.

Lattices With First and Last Elements

Attention will now be focused on an extremely important type of lattice.

<u>Definition</u> <u>1.5</u>. If $a \subset x$ for all x of the lattice, then a is called the <u>first</u> <u>element</u> of the lattice, and if $a \supset y$ for all y then a is called the <u>last</u> <u>element</u>.

<u>Theorem</u> <u>1.3</u>. Every finite lattice has a first element and a last element.

<u>Proof</u>: Rather than prove this theorem the way it is stated, it is desirable to prove the somewhat more general result that each finite subset of a lattice has a g.l.b. (meet) and a l.u.b. (join). This result is valid for each set of two elements of the lattice. Make the inductive assumption that the result holds for all subsets of k elements. Let $a_1, a_2, \ldots, a_k, a_{k+1} \in L$ and denote the g.l.b. $(a_1, a_2, \ldots, a_k)$ by $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ and the g.l.b. $(a_1 \cdot a_2 \cdot \ldots \cdot a_k, a_{k+1})$ by $a_1 \cdot a_2 \cdot \ldots \cdot a_k \cdot a_{k+1}$. To be shown: $a_1 \cdot a_2 \cdot \ldots \cdot a_{k+1} =$ g.l.b. $(a_1, a_2, \ldots, a_{k+1})$.

First note that:

$a_1 \cdot a_2 \cdot \ldots \cdot a_{k+1} \subset a_1 \cdot a_2 \cdot \ldots \cdot a_k \subset a_1, a_2, \ldots, a_k$ and $a_1 \cdot a_2 \cdot \ldots \cdot a_{k+1} \subset a_{k+1}$. This shows that $(a_1 \cdot \ldots \cdot a_k) \cdot a_{k+1}$ is a lower bound of the set. Now let Z be any other lower bound, i.e., $Z \in L$ and $Z \subset a_1, a_2, \ldots, a_{k+1}$ then $Z \subset a_1, a_2, \ldots, a_k$ and $Z \subset a_{k+1}$.

Thus, $Z \subset a_1 \cdot a_2 \cdot \ldots \cdot a_k$ and $Z \subset a_{k+1}$, which implies that $Z \subset a_1 \cdot a_2 \cdot \ldots \cdot a_{k+1} =$ g.l.b. $(a_1 \cdot a_2 \cdot \ldots \cdot a_k, a_{k+1})$.

Thus, each finite subset of a lattice has a g.l.b. and a l.u.b.

and hence it follows that every finite lattice has a first element and a last element.

Now, consider the concepts of first and last elements in regard to some of the examples considered previously.

Example 1.1. (All subsets of the given set). The set M is the last element and the null set is the first element.

Example 1.2. (Real numbers). There is no first element or last element.

Example 1.3. (Ring of sets). Neither a first nor a last element necessarily present. Consider, for example, all the sets which properly contain a set C_1 and are properly contained in a set C_2.

Example 1.8. (Algebra of propositions). A logically false statement (f) is the first element and a logically true statement (t) is the last element.

Example 1.9. The first element is the function f such that $f(x) = 0$ for $x \in [0,1]$ and the last element is the function g such that $g(x) = 1$ for $x \in [0,1]$.

The first element of a lattice is denoted by 0, and the last element by 1. Notice that $0 \subset a$ is equivalent to $0 \cdot a = 0$ and $0 + a = a$. Also, $a \subset 1$ is equivalent to $a \cdot 1 = a$ and $a + 1 = 1$ for all a. Thus, the principle of duality is valid in a lattice with both a first element and a last element, i.e., duality is preserved under the transformations:

$$\begin{pmatrix} = & \subset & a \cdot b & a + b & 0 & 1 \\ = & \supset & a + b & a \cdot b & 1 & 0 \end{pmatrix}$$

In the next two definitions, L is a lattice with first and last elements, and $a \subset b$.

<u>Definition 1.5</u>. <u>A complementary element of the first kind</u> (denoted by b - a) is any element x = b - a such that a · x = 0 and a + x = b.

<u>Definition 1.6</u>. <u>A complementary element of the second kind</u> (denoted by a / b) is any y = a / b such that b + y = 1 and y · b = a.

Now consider some of the examples with respect to these elements.

In Example 1.1, if $A \subset B$ then $B \setminus A$ is a complementary element of the first kind and $U \setminus (B \setminus A)$ is a complementary element of the second kind (See Figure 1.4).

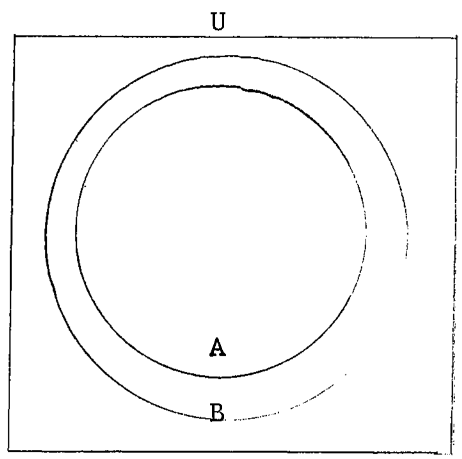


Figure 1.4.

To see that complementary elements of the first kind do not necessarily exist in the lattice of convex sets (Example 1.4), consider the case where a and b are intervals in E_1 with $a \subset b$. If x is any complementary element of the first kind, then a · x = 0 implies that x is disjoint from a. This combined with a + x = b shows that x = $b \setminus a$. However, $b \setminus a$ is not a convex set (see Figure 1.5).

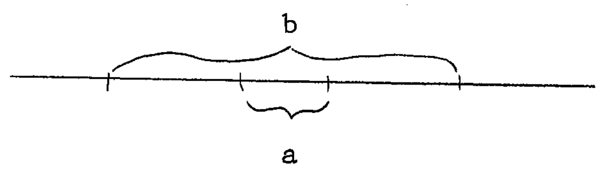


Figure 1.5

The same type argument can be used in considering complementary elements of the second kind.

In Example 1.7, if b is a line and a is a point on b, the complementary elements of the first kind are those points of b that are different from a. Complementary elements of the second kind would be planes which pass through a but do not contain b (see Figure 1.6).

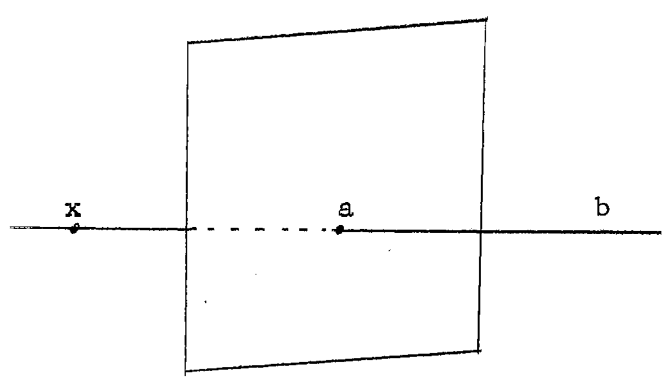


Figure 1.6

In Example 1.9 comparable elements must be of a very special type if they are to have either first or second complementary elements. To prove this, let $f \subset g$ and assume that there exists a point $a \in [0,1]$ such that $f(a) \neq 0$ and $f(a) \neq 1$. The existence of a complementary element of the first kind, k, means that $f \cdot k$ is the zero function and this implies that $k(a) = 0$. Thus, $[f + k](a) = f(a)$ but

$[f + k](a) = g(a)$, since k is a complementary element of the first kind. Thus, f(a) must be equal to g(a) and it is clear that in Example 1.9 most comparable elements would not have complementary elements of the first kind. A similar discussion could be given concerning complementary elements of the second kind.

To illustrate types of functions that would not contain these special elements see Figure 1.7.

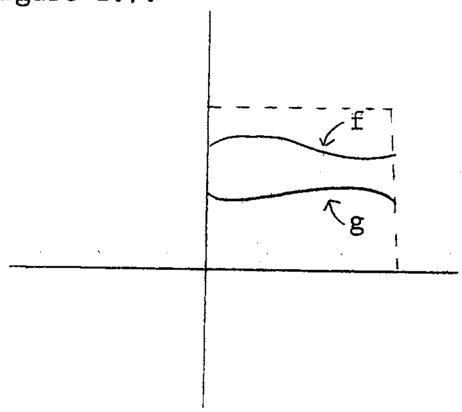


Figure 1.7

There is a related concept that should be discussed at this point.

Definition 1.7. If $c \in L$, any element c' of L such that $c \cdot c' = 0$ and $c + c' = 1$ is called a complement of c. A lattice is said to be complemented if it has first and last elements and if every element has a complement.

The next two theorems show the relationship between complementary elements of the first and second kind and the complement of an element.

Theorem 1.4. For each $a,b \in L$ for which $a \subset b$, each complementary element of the first kind is the complement of each complementary element of the second kind.

Proof: Let $x = b - a$ and $y = a / b$, i.e., $a \cdot x = 0$, $a + x = b$ and $y \cdot b = a$, $y + b = 1$.

Since $x = (a + x) \cdot x$,

$$x \cdot y = y \cdot x = y \cdot [(a + x) \cdot x] = y \cdot [b \cdot x] = (y \cdot b) \cdot x = a \cdot x = 0.$$

Similarly, $y = y + y \cdot b$[4] implies that

$$x + y = y + x = (y + y \cdot b) + x = (y + a) + x = y + (a + x) = y + b = 1.$$

Therefore, x is the complement of y and conversely since their lattice product is 0 and their sum is 1.

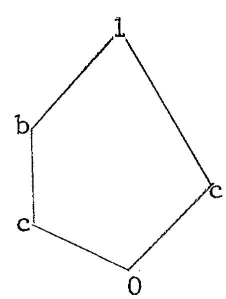
The next theorem follows directly from the definitions and is very important in showing the relationship between these concepts.

Theorem 1.5. If each pair of comparable elements of a lattice have complementary elements of the first kind (or second kind) then the lattice is complemented.

Proof: For any element $c \in L$, $c \subset 1$. Thus by hypothesis, there exists an $x \in L$ such that $x \cdot c = 0$ and $x + c = 1$.

A similar proof can be given under the assumption that every pair have second complementary elements

The following example shows that the converse of this theorem is not true. Consider the lattice:



[4]For convenience, $y + y \cdot b$ will be used in place of $y + (y \cdot b)$.

This lattice is complemented for the complement of 0 is 1, a is c, b is c, c is a or b, and 1 is 0. However, $a \subset b$, but there is no element x such that $a \cdot x = 0$ and $a + x = b$. The only possibilities for x are 0, c, and 1; however, $a + 0 \neq b$, $a + 1 \neq b$, and $a + c \neq b$.

In most of the basic examples it is quite easy to check whether or not the lattice is complemented. For example, in the lattice of propositions (Example 1.8), the complement of each statement p is its negative $\sim p$.

In projective 3-space, Example 1.7, the complement of a point is a plane not containing it. The complement of a plane is a point not lying in the plane, and the complement of a line is a skew line.

The lattice of real functions in Example 1.9 is not a complemented lattice since, for example, $f(x) = 1/2$ for $x \in [0,1]$ would not have a complement. This is easy to see, for $(f \cdot g)\, x = 0$ means that $g(x)$ is the zero function and, thus, $f(x) + g(x) = f(x) \neq 1$.

Properties of Lattices in General

Theorem 1.6. If $a \subset c$ and $b \subset d$ then $a \cdot b \subset c \cdot d$ and $a + b \subset c + d$.

Proof: $a \cdot b \subset a \subset c \Rightarrow a \cdot b \subset c$. Similarly, $a \cdot b \subset d$. Thus, by definition, $a \cdot b \subset c \cdot d$. The remaining part of the theorem follows from the principle of duality.

Corollary 1.6: If $a \subset b$, then for every $c \in L$, $a \cdot c \subset b \cdot c$ and $a + c \subset b + c$.

Proof: Follows directly from Theorem 1.6 and fact that $c \subset c$.

Theorem 1.7. If $a \subset c \subset b$ and $a = b$, then $a = c = b$.

Proof: Since $a = b$, $b \subset c \subset a \subset b$ and, thus, $a = c = b$.

Theorem 1.8. $a \cdot b = a + b$ if, and only if, $a = b$.

Proof:

i) If $a = b$, then $a \cdot b = a \cdot a = a$ and $a + b = a + a = a$. Thus, $a \cdot b = a + b$.

ii) $a \cdot b = a + b$ together with $a \cdot b \subset a \subset a + b$ implies that $a \cdot b = a = a + b$. Using a similar argument, it follows that $a \cdot b = b = a + b$ and, thus, $a = b$.

Theorem 1.9. If $a,b,c \in L$, then $a \cdot c + b \cdot c \subset (a + b) \cdot c \subset c \subset a \cdot b + c \subset (a + c) \cdot (b + c)$.

Proof:

i) Show that $a \cdot c + b \cdot c \subset (a + b) \cdot c$.

$a \cdot c \subset c$ and $b \cdot c \subset c$ implies that $a \cdot c + b \cdot c \subset c$. Also, $a \cdot c \subset a \subset a + b$ and $b \cdot c \subset b \subset a + b$ proves that $a \cdot c + b \cdot c \subset a + b$. Therefore, $(a \cdot c + b \cdot c) \cdot (a \cdot c + b \cdot c) = a \cdot c + b \cdot c \subset (a + b) \cdot c$.

ii) Show that $a \cdot b + c \subset (a + c) \cdot (b + c)$.

$a \cdot b \subset a$ and $c \subset c \Rightarrow a \cdot b + c \subset a + c$. Also, $a \cdot b \subset b$ and $c \subset c \Rightarrow a \cdot b + c \subset b + c$. Hence, $a \cdot b + c \subset (a + c) \cdot (b + c)$.

Modular and Distributive Lattices

The next special class of lattices to be considered was first studied by Dedekind [14], and the definition is due to him.

Definition 1.8. A lattice will be called a modular lattice if it satisfies the following condition:

If $a \subset c$ then $(a + b) \cdot c = a + b \cdot c$ for every b. Another way to state this same property is that $(a + b) \cdot c = a \cdot c + b \cdot c$ holds if $a \subset c$.

Notice that if L is any lattice, then for each $a,b,c, \in L$, $a \cdot c + b \cdot c \subset (a + b) \cdot c$. This means that if $a \subset c$, then $a + b \cdot c \subset (a + b) \cdot c$.

The following results follow immediately from the definition:

i) The Dedekind condition is a self-dual, thus duality holds in a Dedekind lattice.

ii) If a lattice is distributive, that is, $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$, then the lattice is modular.

Another rather interesting consequence of the modular condition is that if $a,b,c \in L$, with $a \subset c$, and L is modular lattice, then these elements are distributive. To prove this, the following three equalities must be proved to hold:

(1) $(a + b) \cdot c = a \cdot c + b \cdot c$

(2) $(a + c) \cdot b = a \cdot b + c \cdot b$

(3) $(b + c) \cdot a = b \cdot a + c \cdot a$

Equation (1) follows directly from modularity. In equation (2), $a \subset c \Rightarrow a + c = c$ and $a \cdot b \subset c \cdot b$. Thus, $(a + c) \cdot b = c \cdot b$ and $a \cdot b + c \cdot b = c \cdot b$. In equation (3), $a \subset c \Rightarrow a \subset b + c$ and $c \cdot a = a$. Thus, $(b + c) \cdot a = a$ and $b \cdot a + c \cdot a = b \cdot a + a = a$. Notice that this does not prove that the lattice, L, is distributive; it only proves that these particular elements satisfy the distributive property. Thus, modularity is certainly a kind of "weak" distributive law.

There is another rather useful way of characterizing modular lattices.

Theorem 1.10. A lattice L is modular if and only if it satisfies the following condition. If $a \subset b$ and there exists an element c such that $a \cdot c = b \cdot c$ and $a + c = b + c$, then $a = b$.

Proof:

i) Suppose L is modular, and $a, b, c \in L$ such that $a \subset b$, $a \cdot c = b \cdot c$, and $a + c = b + c$. It must be shown that $a = b$.

The following equalities all follow directly from the hypothesis:

(1) $a + c \cdot b = a + c \cdot a = a$

(2) $(a + c) \cdot b = (b + c) \cdot b = b$

(3) $(a + c) \cdot b = a + c \cdot b$

Therefore, $b = a$.

ii) Suppose now that L saisfies this new condition, i.e., if $a \subset b$ and there exists an element c such that $a \cdot c = b \cdot c$ and $a + c = b + c$, then $a = b$.

Let $p = a + b \cdot c$ and $q = (a + b) \cdot c$ where $a \subset c$. Since $a + b \cdot c \subset (a + b) \cdot c$ when $a \subset c$, then $p \subset q$ (see Theorem 1.9). It remains now to prove that $q \subset p$. $p \subset q \Rightarrow b \cdot p \subset b \cdot q$ and $b + p \subset b + q$. But $b \cdot q = b \cdot (a + b) \cdot c = b \cdot c$, and $b \cdot c \subset b \cdot p$ since $b \cdot c \subset a + b \cdot c = p$ and $b \cdot c \subset b$. Thus, $b \cdot q \subset b \cdot p$ and hence $b \cdot q = b \cdot p$. Also, $b + q = b + (a + b) \cdot c \subset b + a \subset b + b \cdot c + a = b + p$. Therefore, $b + q \subset b + p$, and hence, $b + q = b + p$. Thus, by hypothesis, $p = q$.

This theorem gives an extremely useful way of characterizing modular lattices in terms of their <u>sublattices</u>, where a subset M of a lattice

L is called a <u>sublattice</u> if it is closed relative to the operation + and ·. This characterization was first given by Dedekind [4] and is stated in the next corollary.

<u>Corollary 1.10</u>. (Dedekind's modularity criterion). A lattice is modular iff it does not contain the following sublattice:

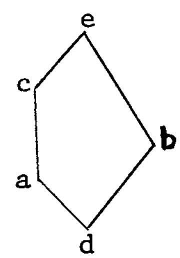
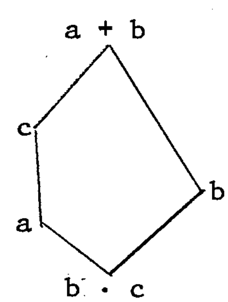


Figure 1.8

<u>Proof</u>:

i) If this sublattice is present in L, then L is not modular since $a \subset c$ but $(a + b) \cdot c \neq a + b \cdot c$.

ii) Assume that L is not modular, then by Theorem 1.10, there exist elements $a, b, c \in L$ such that $a \subset c$, $a + b = a + c$, and $a \cdot b = a \cdot c$ but $a \neq c$. Thus, L contains the sublattice:



A similar type result was obtained by Garret Birkhoff [4] for distributive lattices. He proved the following theorem:

<u>Theorem 1.11</u>. A lattice is distributive if and only if it does not contain either of the following sublattices shown in Figure 1.9.

Figure 1.9

A simple, though tedious, proof of this theorem is found in [14].

Now, consider some of the examples with respect to the properties of modularity and distributivity.

The convex sets (Example 1.4) do not form a modular lattice. Consider the following example. Let a,b, and c be non-overlapping unit circles whose centers are colinear (see Figure 1.10). If d = a + b, then $a \subset d$, $a \cdot c = d \cdot c$, and $a + c = d + c$; however, $a \neq d$. Thus, by Theorem 1.10, the lattice is not modular.

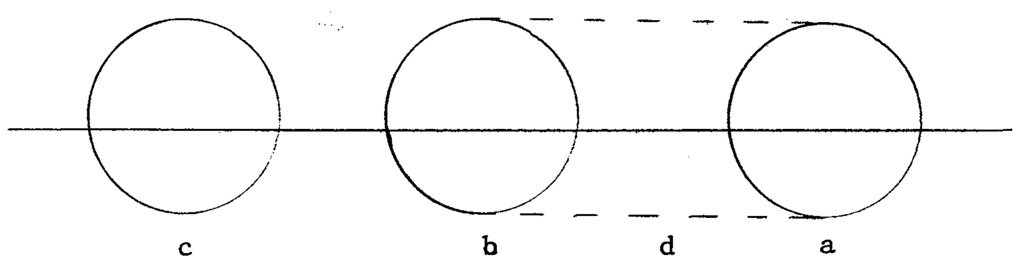
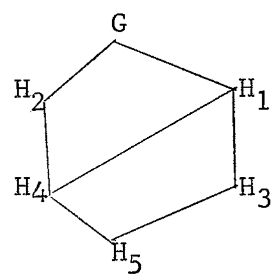


Figure 1.10

In the lattice formed by the subgroups of a group (Example 1.5), every possible combination can occur, depending on the particular group that is being considered. This is clearly seen in the following examples.

i) In the group of integers modulo 12, where the operation is addition, the subgroups form a distributive lattice. This can be proved by considering all the subgroups of this group and drawing the lattice diagram. The subgroups are: $G = \{0,1,2,3,4,5,6,7,8,9,10,11\}$, $H_1 = \{0,2,4,6,8,10\}$, $H_2 = \{0,3,6,9\}$, $H_3 = \{0,4,8\}$, $H_4 = \{0,6\}$, and $H_5 = \{0\}$. The diagram is:



Thus, the lattice is modular and distributive since it contains neither of the sublattices of Figure 1.9.

ii) In the permutation group S_3, the lattice of subgroups forms a modular, non-distributive lattice. The elements in this group are:

1 $\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} = (i)$

2 $\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} = (b\ c)$

3 $\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} = (a\ b)$

4 $\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} = (a\ b\ c)$

5 $\begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} = (a\ c\ b)$

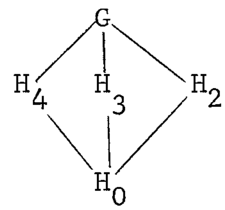
6 $\begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} = (a\ c)$

The multiplication table for this group is:

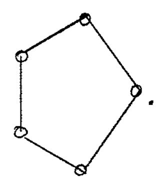
•	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	5	1	6	2	4
4	4	6	2	5	1	3
5	5	3	6	1	4	2
6	6	4	5	2	3	1

The subgroups are $H_0 = \{1\}$, $H_1 = \{1, 2\}$, $H_2 = \{1, 3\}$, $H_3 = \{1, 6\}$, $H_4 = \{1, 2, 3\}$, and $G = \{1, 2, 3, 4, 5, 6\}$.

From the lattice diagram shown in Figure 1.11, it follows that the lattice contains the sublattice



but not a sublattice of the form



Thus, the lattice is modular but not distributive.

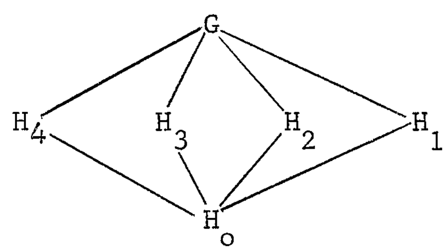


Figure 1.11

iii) The alternating group A_4 is an example of a lattice of subgroups that is not even modular. This group has elements:

1 $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix} = i$

2 $\begin{pmatrix} a & b & c & d \\ a & c & d & b \end{pmatrix} = (b\ c\ d)$

3 $\begin{pmatrix} a & b & c & d \\ a & d & b & c \end{pmatrix} = (b\ d\ c)$

4 $\begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix} = (a\ b)\ (c\ d)$

5 $\begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix} = (a\ b\ c)$

6 $\begin{pmatrix} a & b & c & d \\ b & d & c & a \end{pmatrix} = (a\ b\ d)$

7 $\begin{pmatrix} a & b & c & d \\ c & a & b & d \end{pmatrix} = (a\ c\ b)$

8 $\begin{pmatrix} a & b & c & d \\ c & b & d & a \end{pmatrix} = (a\ c\ d)$

9 $\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix} = (a\ c)\ (b\ d)$

10 $\begin{pmatrix} a & b & c & d \\ d & a & c & b \end{pmatrix} = (a\ d\ b)$

11 $\begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix} = (a\ d\ c)$

12 $\begin{pmatrix} a & b & c & d \\ d & c & b & a \end{pmatrix} = (a\ d)\ (b\ c)$

The subgroups of this group are:

$G_0 = \{1\}$

$G_1 = \{1, 4\}$

$G_2 = \{1, 9\}$

$G_3 = \{1, 12\}$

$G_4 = \{1, 2, 3\}$

$G_5 = \{1, 5, 7\}$

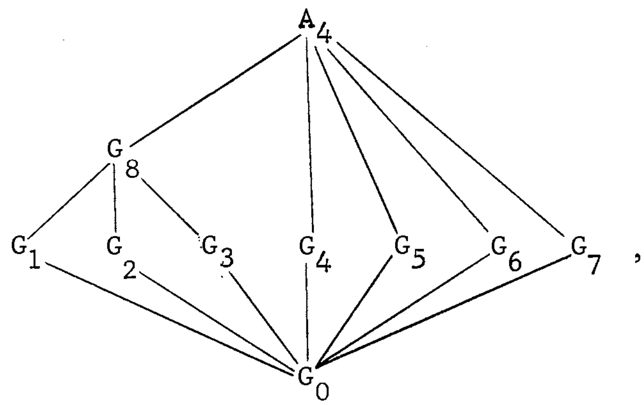
$G_6 = \{1, 6, 10\}$

$G_7 = \{1, 8, 11\}$

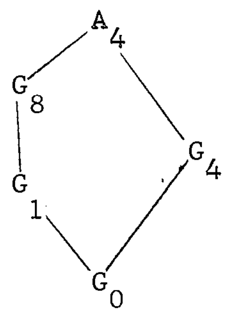
$G_8 = \{1, 4, 9, 12\}$

$G_9 = A_4$.

The lattice diagram is



and the existence of the sublattice



proves that the lattice is not modular.

In the set of real functions discussed in Example 1.9, the lattice formed is distributive. Let $x_0 \in [0, 1]$, and assume that $f(x_0) \geq g(x_0) \geq h(x_0)$. Then,

$$[(f + g) \cdot h]\, x_0 = \min\ [(f + g)\, x_0,\ h(x_0)]$$
$$= \min\ [\ \max\ [f(x_0),\ g(x_0)],\ \ h(x_0)]$$
$$= \min\ [\ f(x_0),\ h(x_0)]\ = h(x_0),$$

and $(f \cdot h)\, x_0 + (g \cdot h) x_0 =$

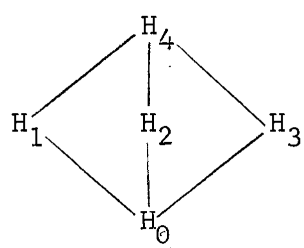
$$\max\ [(f \cdot h)x_0,\ \ (g \cdot h)x_0]\ = \max\ [h(x_0),\ h(x_0)] = h(x_0).$$

Thus $[(f + g) \cdot h]\ (x_0) = (f \cdot h)x_0 + (g \cdot h)x_0$ for this particular ordering. The same would be true for all possible orderings and hence $(f + g) \cdot h = (f \cdot h) + (g \cdot h)$.

The lattice of subspaces of a vector space (Example 1.10) is not distributive for consider Z_2^2 (Z_2). This is a vector space with elements (0,0), (1,0), (0,1), and (1,1). The subspaces are:

$$H_0 = \{(0,0)\}$$
$$H_1 = \{(0,0),\ (1,0)\}$$
$$H_2 = \{(0,0),\ (0,1)\}$$
$$H_3 = \{(0,0),\ (1,1)\}$$
$$H_4 = \{(0,0),\ (0,1),\ (1,0),\ (1,1)\}.$$

The lattice diagram is



,

and the lattice is modular but not distributive. In general, the lattice of subspaces is a modular lattice. To prove this, let a,b,c be subspaces with $a \subset c$. It must be proved that $(a + b) \cdot c = (a \cdot c) + (b \cdot c) = a + (b \cdot c)$. Since $a \cdot c + b \cdot c$ is always contained in $(a + b) \cdot c$, this reduces to showing that $(a + b) \cdot c \subset a \cdot c + b \cdot c$. Now, let $z \in (a + b) \cdot c$, then $z = y_3 \in c$ and $z = y_1 + y_2$, where y_1 and y_2 are in a and b respectively. Hence, $y_3 = y_1 + y_2$, which implies that $y_2 = y_3 - y_1 \in c + a = c$. Thus, $y_2 \in b \cdot c$ and $z = y_3 = y_1 + y_2 \in a + b \cdot c$. This shows that $(a + b) \cdot c \subset a + (b \cdot c)$; therefore, $(a + b) \cdot c = a + b \cdot c$ [9].

CHAPTER III

RELATIONSHIPS BETWEEN DIFFERENT TYPES OF LATTICES

In Chapter II some basic types of lattices were discussed and several examples given that illustrated these different types. As an aid in seeing how the many kinds of lattices are related, it is extremely illuminating to construct Venn diagrams where each circle represents a different type of lattice. This chapter will be concerned with the construction of such Venn diagrams.

Venn Diagrams of Lattices

First, consider a Venn diagram to illustrate the relationships between the following types of lattices--modular, distributive, complemented, and lattices with 0 and 1 elements. The Venn diagram is simplified considerably by the fact that all distributive lattices are modular; this means that the circle representing the distributive lattices is placed inside the circle for modular ones. A similar relationship holds between complemented lattices and lattices which have 0 and 1 elements. Figure 3.1 is the desired Venn Diagram.

To show that the Venn diagram is drawn correctly, it is necessary to find a lattice which will satisfy the specific properties of each region of Figure 3.1. In most instances the easiest way to find these lattices is by drawing lattice diagrams. The diagrams actually define lattices, with the meet and join of each pair of elements easily

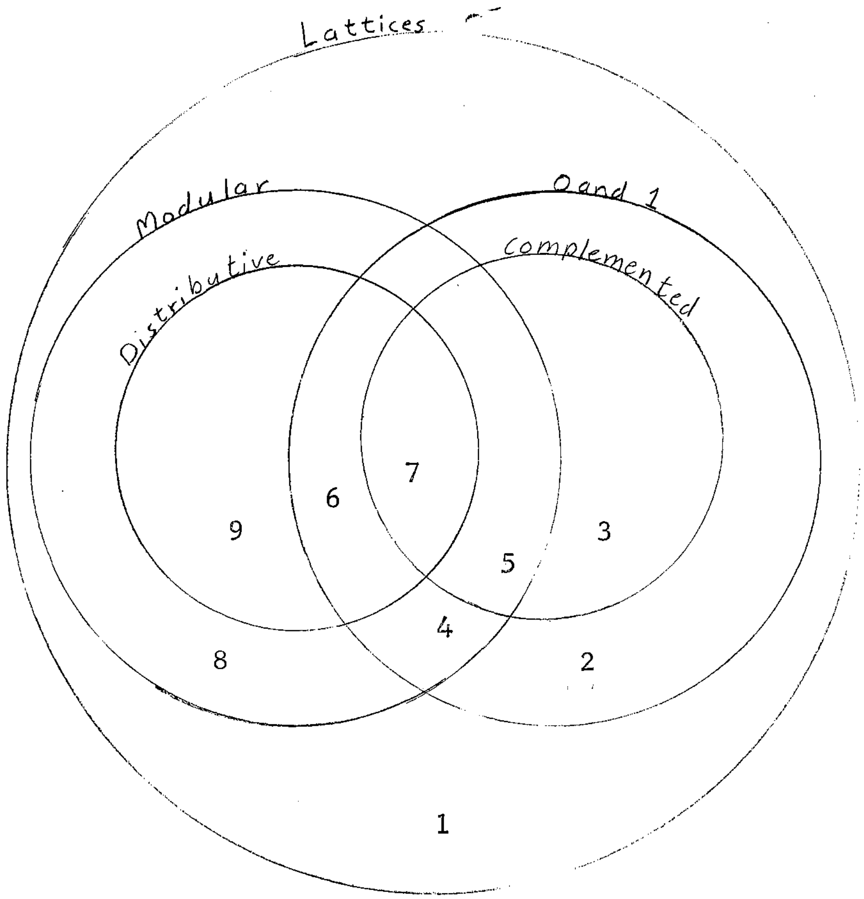
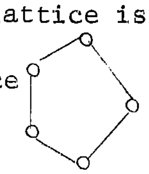


Figure 3.1

obtained from the diagram. To illustrate the technique used in constructing these diagrams, consider region (1) of Figure 3.1. A lattice in this region is non-modular and does not have both first and last elements. Since the desired lattice is not modular, it must, by Corollary 1.10, contain the sublattice  ; however, this lattice by itself contains 0 and 1 elements. Thus the infinite chain  (see page 7) is combined with it to form the desired lattice. It should be noted that this is certainly not the only lattice that satisfies these properties.

Examples of lattices in the different regions of Figure 3.1 are shown in Figure 3.2 with the number of the lattice diagram corresponding to the number of the region.

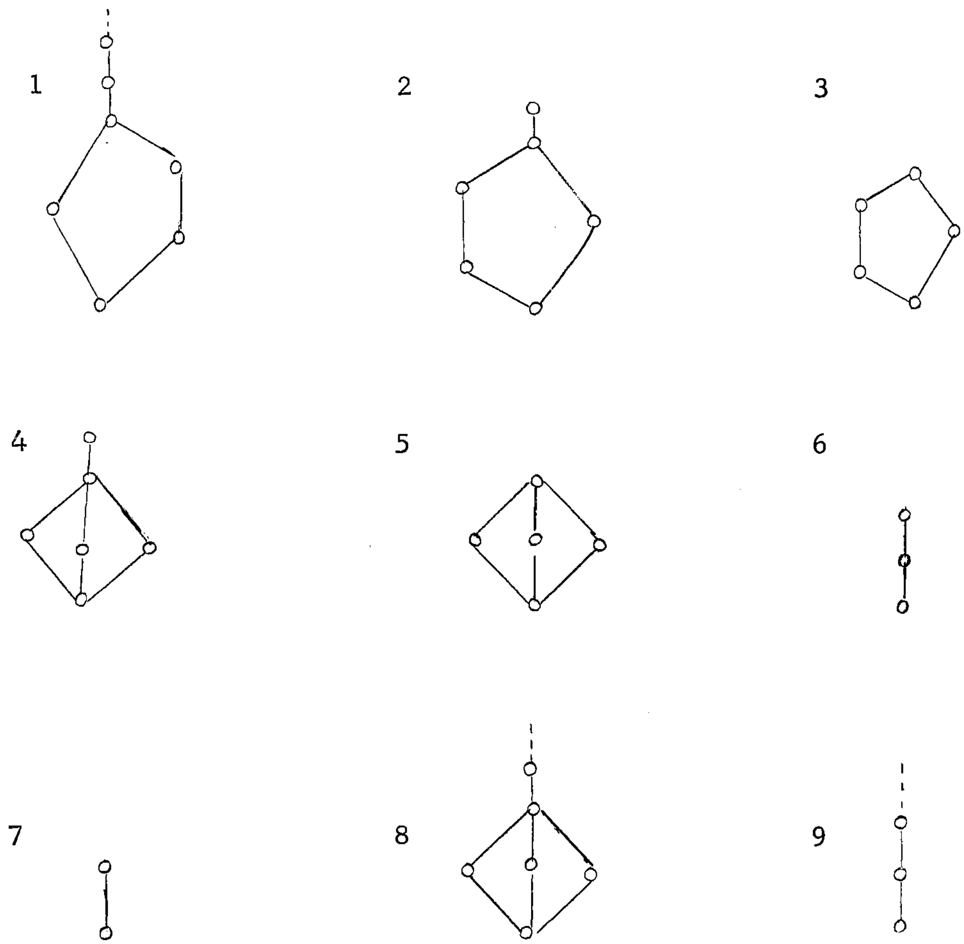


Figure 3.2

Next, consider how the Venn diagram is changed when the additional property of being "linearly ordered" is considered. As in the case of the first Venn diagram, this one is considerably simplified by relationships that exist among the lattices. For example, it follows immediately from Theorem 1.11 that every linearly ordered lattice is distributive.

The Venn diagram with the additional property included is given in

Figure 3.3. Perhaps it should be mentioned that the Venn diagram is drawn in its most general form; this means that every possible region is included. It is certainly conceivable that there might be some region that is empty, that is, no lattice satisfies all the conditions of that region. If this is found to be the case, then the Venn diagram will be redrawn with that region omitted.

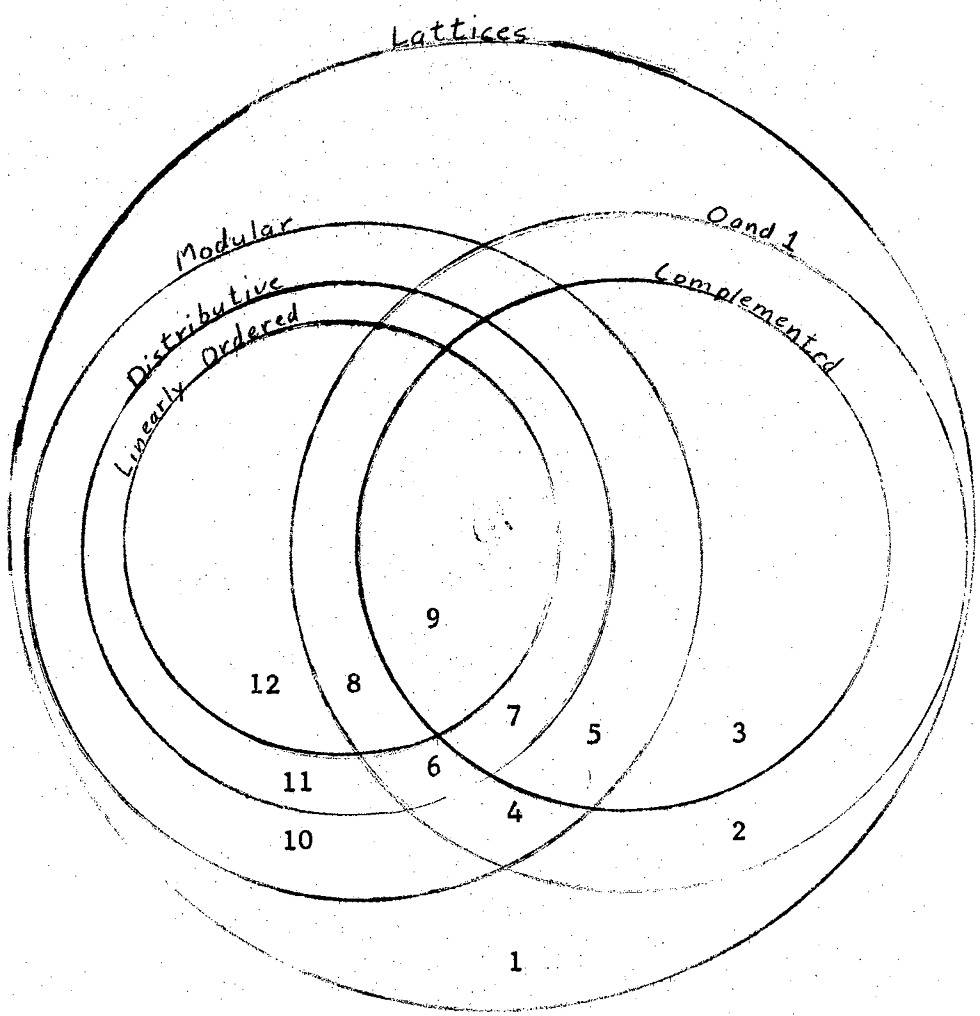


Figure 3.3

Again it is necessary to exhibit lattices that satisfy the particular properties of each of the regions in Figure 3.3; these lattices are given in Figure 3.4.

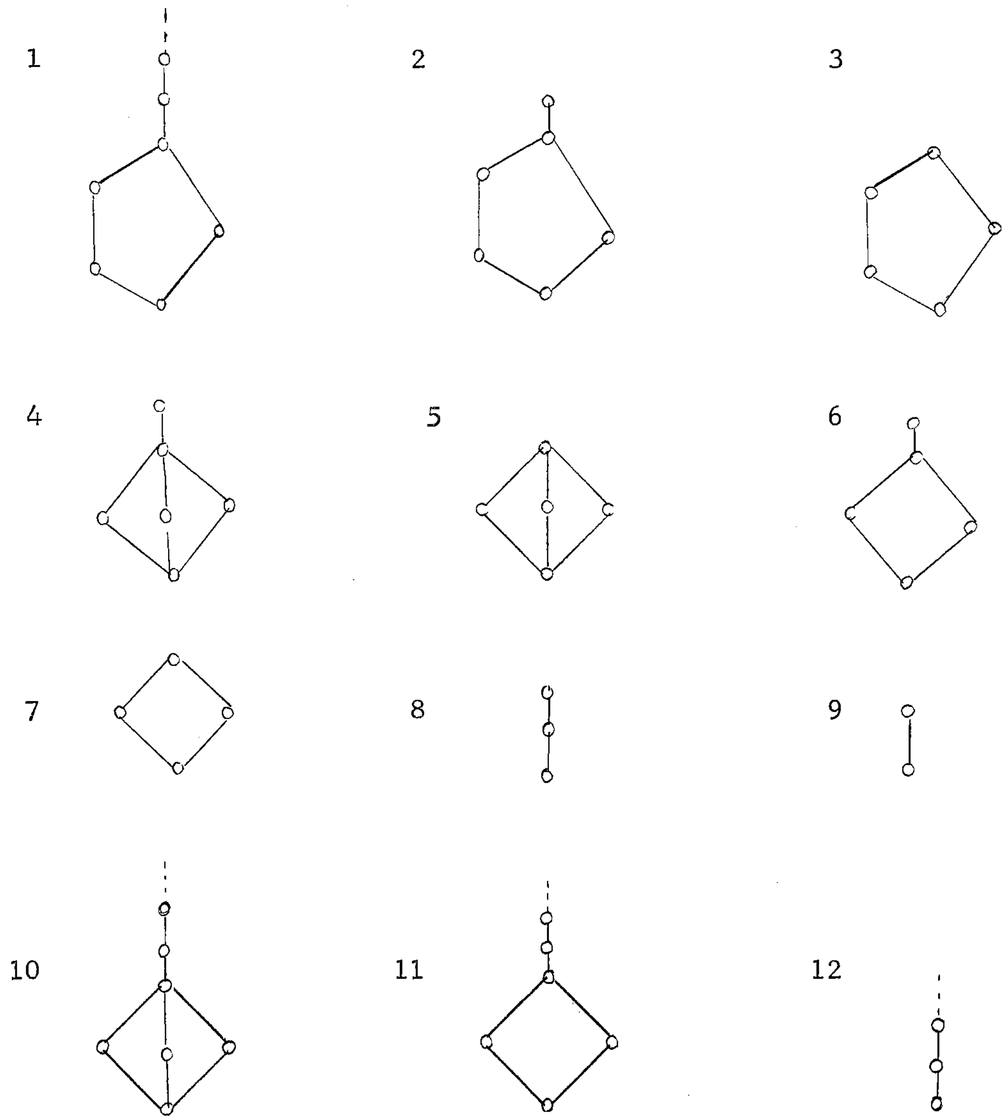


Figure 3.4.

For most of the regions there are many other lattices which would work just as well as the one used. In this respect, region (9) is unique since the only linearly ordered lattice that is complemented is the one given in Figure 3.4. This is easily seen, for if b is an element of the chain, where $b \neq 0$ and $b \neq 1$, then $0 \subset b \subset 1$. Thus, $b \cdot x = 0$ implies that $x = 0$; therefore, $b + x = b \neq 1$.

Complete Lattices

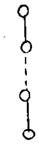
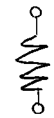
There are certainly many more properties of lattices that could be considered and the corresponding Venn diagram drawn; however, only one more property will be considered in this chapter. This is the property of "completeness."

Definition 3.1. A lattice is said to be complete if every (finite or infinite) subset $A = \{a_\alpha\}$ has a l.u.b. $\cup a_\alpha$ and a g.l.b. $\cap a_\alpha$ in the lattice.

In the proof of Theorem 1.3 it was shown that every finite subset of a lattice has a g.l.b. (meet) and a l.u.b. (join). This implies that finite lattices are complete. Thus, when trying to show a particular lattice incomplete, only infinite lattices and infinite subsets of these lattices need to be considered.

To illustrate a lattice that is not complete, consider the rational numbers between 0 and 2. This certainly forms a lattice, L, under the normal ordering--to be specific, it forms a lattice which is linearly ordered and has first and last elements. Now consider the set

$A = \{a_\alpha \in L \mid a_\alpha^2 < 2\}$. $A \subset L$ but $\cup\{a_\alpha\} = \sqrt{2} \notin L$. Thus L is not complete.

Notice that in this lattice there is no element which covers 0; in fact, there is no element that covers any other element. Thus, the lattice cannot be represented by the diagram  . For convenience in representing certain types of lattices that are not complete, this particular lattice will be represented by the diagram  .

In drawing the Venn diagram when this new property is considered, the diagram is again simplified by a basic relationship, namely, that every complete lattice must have first and last elements. This follows directly from the definition and the fact that every set is a subset of itself.

The most general Venn diagram representing these types of lattices is given in Figure 3.5.

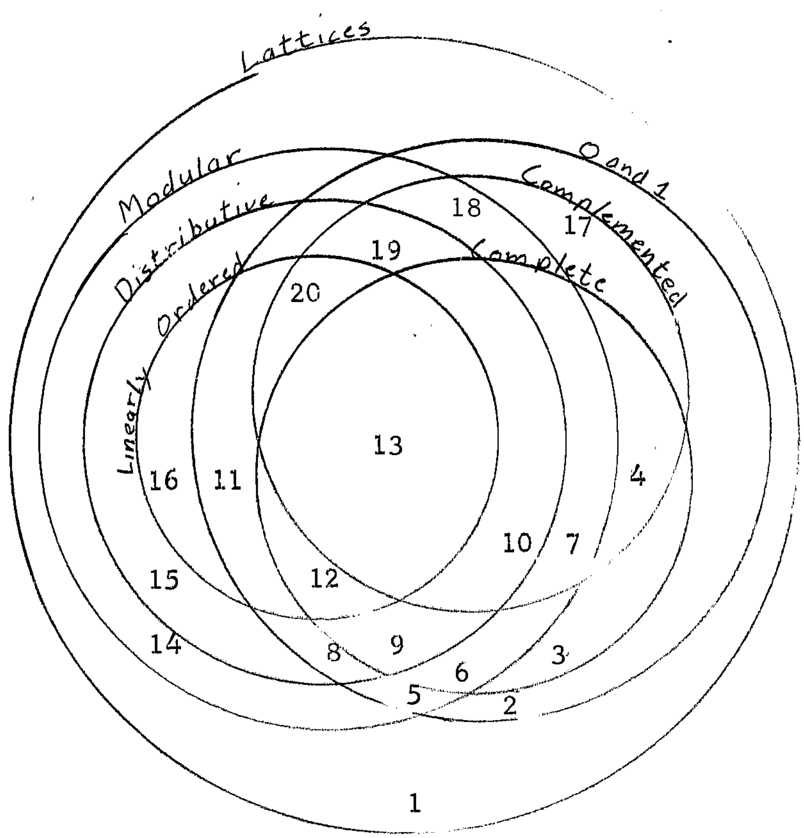


Figure 3.5

The lattices which correspond to regions (1) through (17) of Figure 3.5 are given below.

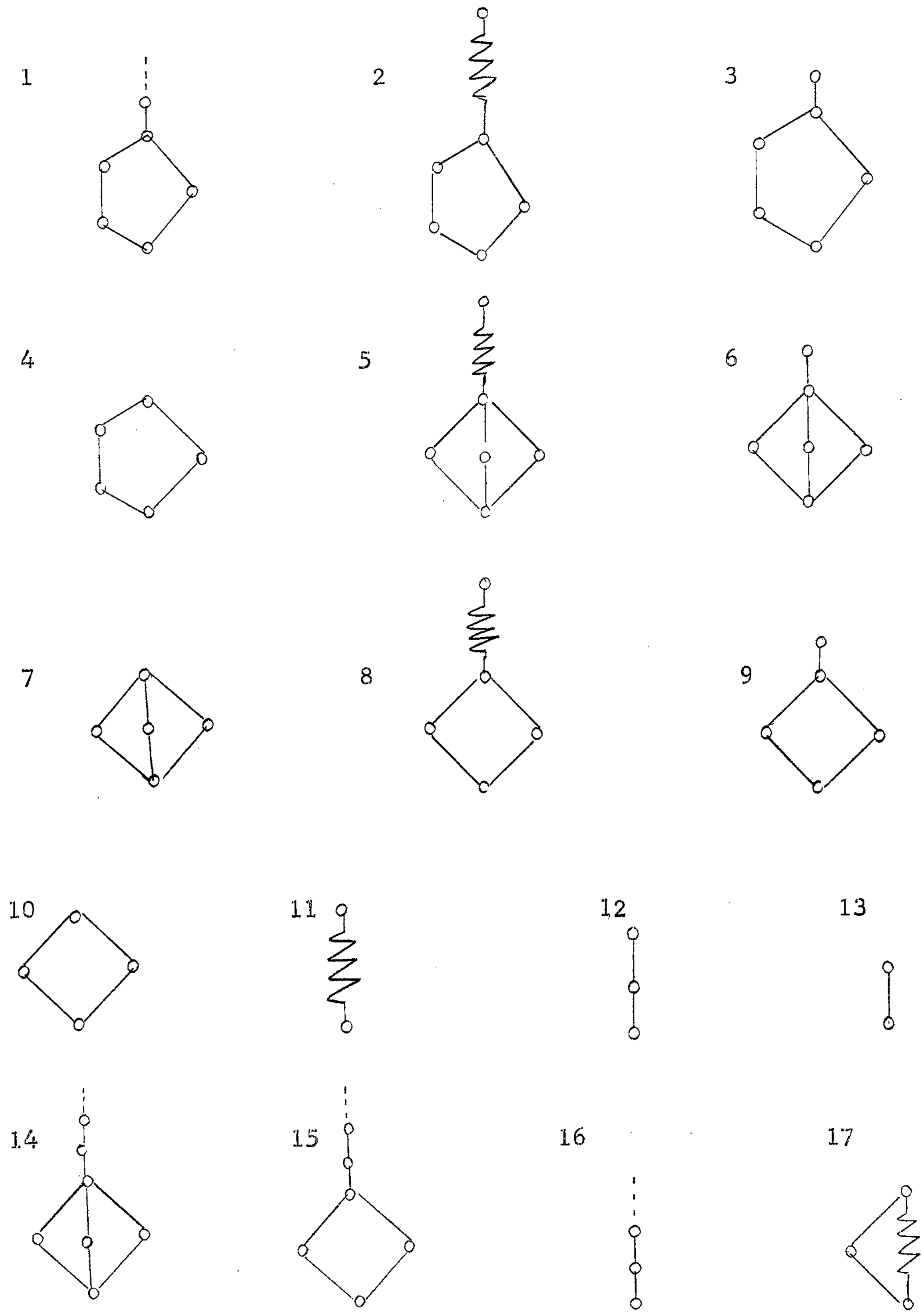


Figure 3.6

Regions (18) and (19) involve finding modular lattices that are complemented but not complete. It is extremely difficult to construct meaningful diagrams that would represent these lattices; therefore, these lattices will be obtained without the use of diagrams.

First, consider region (19). A lattice satisfying the particular properties of this region must be (i) distributive, (ii) complemented, (iii) not linearly ordered, and (iv) not complete.

To construct a lattice satisfying these properties, let B be a collection of subsets of the integers such that $S \in B$ iff S is finite or S' is finite. It will now be proved that B is a lattice that satisfies the desired conditions.

It must be established that B is a lattice. B is partially ordered by set inclusion so all that needs to be proved is that $S_1, S_2 \in B \Rightarrow S_1 \cup S_2 \in B$ and $S_1 \cap S_2 \in B$. There are three cases to be considered.

Case I: S_1, S_2 are both finite. It follows that $S_1 \cap S_2$ is finite and $S_1 \cup S_2$ is finite. Thus, $S_1 \cap S_2 \in B$ and $S_1 \cup S_2 \in B$.

Case II: S_1, S_2 are both infinite. This means that S_1' and S_2' are both finite. Thus, $(S_1 \cap S_2)' = S_1' \cup S_2'$ and $(S_1 \cup S_2)' = S_1' \cap S_2'$ are both finite sets of integers. Therefore, $S_1 \cup S_2$ and $S_1 \cap S_2$ are elements of B.

Case III: One of the sets, say S_1, is infinite and the other set, S_2, is finite. $S_1 \cap S_2$ is finite, and $(S_1 \cup S_2)' = S_1' \cap S_2'$ is finite. Thus, $S_1 \cap S_2$, $S_1 \cup S_2 \in B$.

Hence, B is a lattice with $S_1 \cdot S_2 = S_1 \cap S_2$ and $S_1 + S_2 = S_1 \cup S_2$.

This lattice is distributive, since sets are always distributive, and it follows immediately from the definition of B that the lattice is complemented.

All that remains is to show that the lattice is not complete. Consider the subset **P** of B defined as follows:

$$\mathbf{P} = \{\{2\}, \{4\}, \{6\}, \dots\}.$$

Then $\cup\mathbf{P}$ is $\{2\} \cup \{4\} \cup \dots \cup \{2n\} \cup \dots = \{2, 4, 6, 8, \dots\}$. However, $\{2, 4, 6, \dots, 2n, \dots\} \notin B$ since it is an infinite set and its complement is also infinite.

For region (18), the lattice must be modular, complemented, not distributive, and not complete. To obtain a lattice satisfying these conditions, let F be the set of all real functions defined on **Z** that are zero except at a finite number of points, where **Z** is the set of positive integers. That F forms a vector space over R, the set of real numbers, follows easily and the details will be omitted. Now define **L** to be the collection of subspaces of F that have finite dimension or finite co-dimension. It is now asserted that L is a lattice satisfying the desired conditions.

Before the proof of this assertion is begun, a basis needs to be obtained for the real vector space F. First define $f_n \in F$ as follows:

$$f_n(x) = \begin{cases} 1 \text{ if } x = n \\ 0 \text{ elsewhere.} \end{cases}$$

Then, $\mathbf{S} = \{f_1, f_2, \dots, f_n, \dots\}$ forms a basis for F. To prove this it must be shown that (1) **S** spans F, and (2) **S** is a linearly independent set of vectors.

<u>Proof of (1)</u>: If $g \in F$ then there exists a set of integers $B = \{i_1, i_2, \ldots, i_p\}$ such that $g(x) \neq 0$ if $x \in B$ and $g(x) = 0$ if $x \notin B$. Let $g(i_t) = a_t$ if $i_t \in B$. Then, $g = a_1 f_{i_1} + a_2 f_{i_2} + a_3 f_{i_3} + \ldots + a_p f_{i_p} = \sum_{t=1}^{p} a_t f_{i_t}$ and thus $S = \{f_1, f_2, \ldots, f_n, \ldots\}$ spans F.

<u>Proof of (2)</u>: Let $\{f_{i_1}, f_{i_2}, \ldots, f_{i_n}\}$ be any subset of S and suppose that $\alpha_1 f_{i_1} + \alpha_2 f_{i_2} + \alpha_3 f_{i_3} + \ldots + \alpha_n f_{i_n} = 0$.[1] Then $(\alpha_1 f_{i_1} + \alpha_2 f_{i_2} + \alpha_3 f_{i_3} + \ldots + \alpha_n f_{i_n})\, i_1 = 0\,(i_1) = 0$ implies that $\alpha_1 f_{i_1}(i_1) = 0$ and, hence, $\alpha_1 = 0$. Similarly, $\alpha_2 = \alpha_3 = \ldots = \alpha_n = 0$.

Now, it will be proved that L is a lattice. L is partially ordered by set inclusion so it remains to prove that $S_1, S_2 \in L$ implies that $S_1 \cdot S_2 = S_1 \cap S_2 \in L$ and $S_1 + S_2 = [S_1 \cup S_2] \in L$. Consider the following cases:

<u>Case I</u>: S_1, S_2 both of finite dimension. This implies that $S_1 \cdot S_2$ and $S_1 + S_2$ are both of finite dimension and thus $S_1 \cdot S_2, S_1 + S_2 \in L$.

<u>Case II</u>: S_1 of finite dimension and S_2 of infinite dimension. It follows directly that $S_1 \cap S_2$ is of finite dimension. S_2 being of infinite dimension implies that S_2' is of finite dimension; therefore, dim $(S_1 + S_2)'$ is finite

[1]In this context 0 means the zero function.

since $S_2 \subset S_1 + S_2 \Rightarrow \dim (S_1 + S_2)' \leq \dim S_2'$. Thus, $S_1 \cdot S_2$ and $S_1 + S_2$ are both elements of L.

Case III: S_1, S_2 both of infinite dimension. $S_1 \subset S_1 + S_2 \Rightarrow (S_1 + S_2)' \subset S_1'$ and thus $\dim (S_1 + S_2)' \leq \dim S_1'$, which is finite. Thus, $S_1 + S_2 \in L$. S_1' of finite dimension means that only a finite number of elements of B, the basis for F given on page 41, can be in S_1'. Hence, there exists an integer n_1 such that $f_m \in S_1$ if $m > n_1$, where $f_m \in B$. Similarly, there exists an integer n_2 such that $f_k \in S_2$ if $k > n_2$. Therefore, for $n = \max (n_1, n_2)$, $f_r \in (S_1 \cap S_2)$ if $r > n$. From this it follows that dim $(S_1 \cap S_2)' \leq r$.

Hence, L is a lattice with $A \cdot B = A \cap B$ and $A + B = [A \cup B]$.

This lattice is modular since the subspaces of a vector space are always modular; however, L is not distributive, for if $S_1 = [f_1]$,[2] $S_2 = [f_2]$, and $S_3 = [f_1 + f_2]$ then $S_1 \cdot (S_2 + S_3) = S_1$ but $S_1 \cdot S_2 + S_1 \cdot S_3 = 0$. Thus, L is not a distributive lattice.

All that now remains is to prove that L is not complete. Consider the subspace $S_2 = [f_2]$, $S_4 = [f_4]$, ... , $S_{2n} = [f_{2n}]$, ... Each of these is of dimension 1 and, thus, each is in L, but $S = \{ S_2, S_4, \ldots, S_{2n}, \ldots \}$ does not have a least upper bound in L since the l.u.b. $S = [f_2] + [f_4] + [f_6] + \ldots + [f_{2n}] + \ldots$ is of infinite dimension

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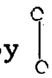
[2] $[f_1]$ means the space generated by f_1.

and its complement is also of infinite dimension.

A lattice satisfying the conditions of region (20) must be (i) linearly ordered, (ii) complemented, and (iii) not complete. It is impossible for any lattice to satisfy these three conditions since the only linearly ordered lattice that is complemented is represented by  . This is a finite lattice and is thus complete. Therefore, Figure 3.5 needs to be redrawn with region (20) excluded. (Figure 3.7).

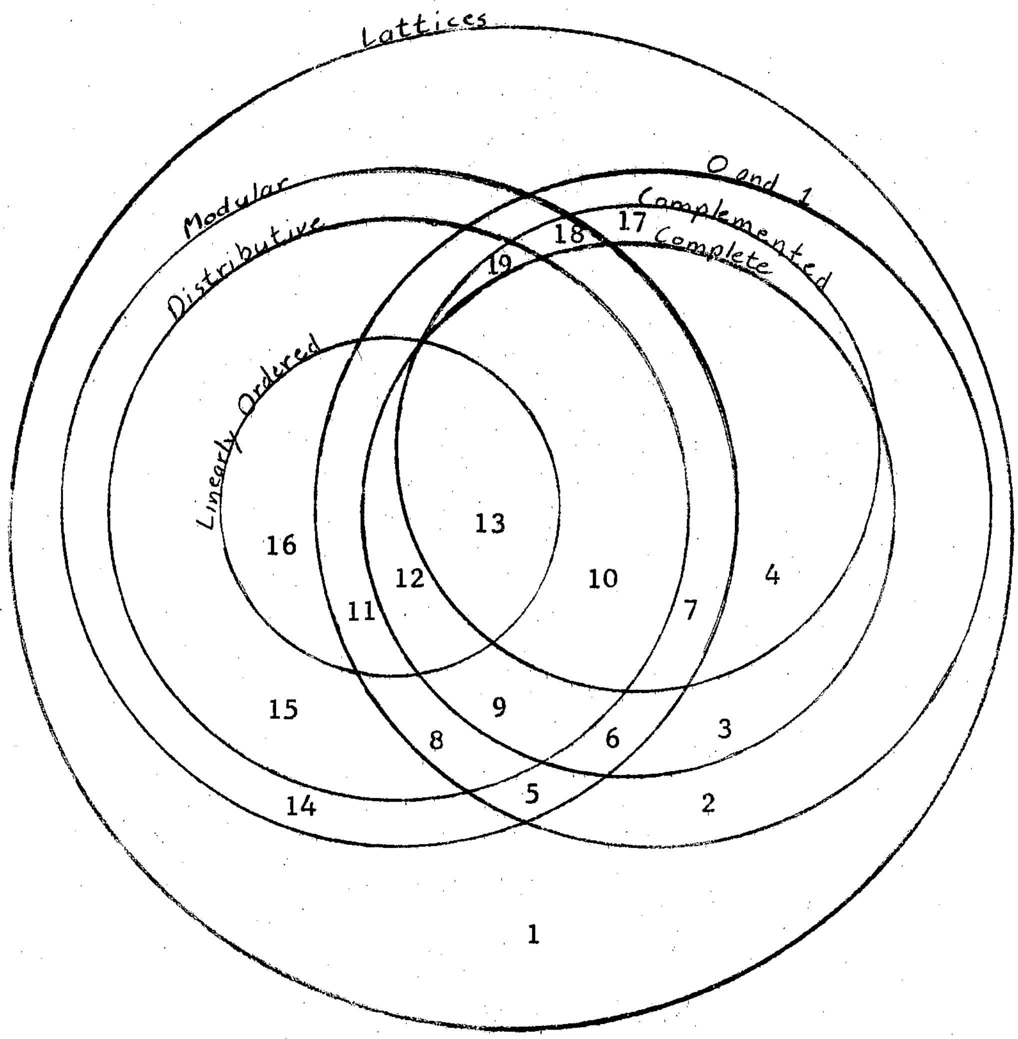


Figure 3.7

CHAPTER IV

COMPLEMENTED LATTICES

In this chapter attention is focused on various theorems concerning complemented lattices. Most of the results considered are from recent publications in mathematics journals, and the relationships between the various theorems is discussed in detail.

First, J. Von Neumann's Theorem [4] relating complemented and relatively complemented lattices is proved along with a recent converse to it that is due to G. Szasz [12]. The remainder of the chapter primarily deals with recent applications of this converse.

Relatively Complemented Lattices

One additional property of lattices is to be considered in this section - - that of being <u>relatively</u> <u>complemented</u>.

<u>Definition 4.1.</u> Let L be an arbitrary lattice and [a, b] be some interval[1] of L with u an element of [a, b]. If some element x of L satisfies the equations

$$u \cdot x = a \text{ and } u + x = b,$$

then x is included in the sublattice [a, b]. On this basis, an element

---

[1] If a and b are elements of a lattice L and $a \subset b$, then the set of all elements $x \in L$ such that $a \subset x \subset b$ is called the <u>interval</u> bounded by a and b and is denoted by [a, b].

x of L for which these two equations hold is called a <u>relative complement</u> of u in [a, b]. The adjective "relative" indicates that the complement is considered relative to a sublattice. It is also said that x is a relative complement of u with respect to the pair of elements a, b. A lattice is said to be <u>relatively complemented</u> if for any triplet of its elements a, b, u ($a \subset u \subset b$), there can be found at least one complement of u in [a, b], in other words, if every interval of L is a complemented lattice.

The similarity of the definitions of complemented and relatively complemented lattices would seem to imply that certain basic relationships exist between lattices satisfying each of these properties. Possibly the most basic theorem relating these two types of lattices is due to J. Von Neumann [14, p. 115].

<u>Theorem 4.1</u>. (<u>Neumann's Theorem</u>). Any complemented modular lattice is relatively complemented.

<u>Proof</u>. Let $a \subset r \subset b$ be given, and let t be any complement of r. Then $[(a + t) \cdot b] \cdot r = (a + t) \cdot (b \cdot r) = (a + t) \cdot r = a + (t \cdot r) = a + 0 = a$, and $[(a + t) \cdot b] + r = [a + (t \cdot b)] + r = r + [a + (t \cdot b)] = (r + a) + (t \cdot b) = r + (t \cdot b) = (r + t) \cdot b = 1 \cdot b = b$.

Thus, if L is modular, then for any complement t of r, the element

$$(1) \quad s = (a + t) \cdot b = a + (t \cdot b)$$

is a relative complement of r in [a, b].

In 1957, G. Szasz [12] published the following converse to Neumann's Theorem.

Theorem 4.2. Let L be any relatively complemented lattice with greatest and least elements, and let a, b, r be any elements of L such that $a \subset r \subset b$ holds. Furthermore, let s be any relative complement of r in [a, b]. Then there exists at least one complement t of r which satisfies (1).

Rather than prove this theorem the way it is stated, Szasz chose to prove the somewhat more general result.

Theorem 4.3. Let L, a, b, r, s be as in Theorem 4.2 and let t ($\in$ L) be any solution of the equation system

(2) i) $r \cdot t = 0$,

ii) $r + t = 1$,

iii) $(a + t) \cdot b = s$, and

iv) $a + (t \cdot b) = s$.

Then there exists a relative complement y of a in [0, s] and a relative complement z of b in [s, 1] such that t is a relative complement of s in [y, z].

Conversely, if y is any relative complement of a in [0, s] and z is any relative complement of b in [s, 1], then any relative complement t of s in [y, z] satisfies the equation system (2). (See Figure 4.1).

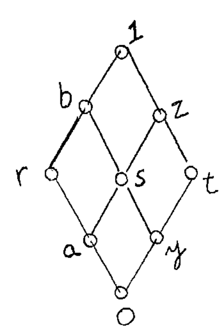


Figure 4.1

<u>Proof of Theorem 4.3</u>. To prove the first part, consider any solution t of (2) and define two elements y, z by $y = s \cdot t$ and $z = s + t$. By the definition of these elements, t is a relative complement of s in [y, z]. Also, by the last two equations of (2), it follows that

$$y = s \cdot t = [(a + t) \cdot b] \cdot t = (a + t) \cdot (b \cdot t) = b \cdot t, \text{ and}$$

$$z = s + t = [a + (t \cdot b)] + t = t + [a + (t \cdot b)] = (t + a) + (t \cdot b) = t + a.$$

It will now be shown that y is a relative complement of a in [0, s] and that z is a relative complement of b in [s, 1]. First,

$$a \cdot y = a \cdot (b \cdot t) = (a \cdot b) \cdot t = a \cdot t \subset r \cdot t = 0, \text{ and}$$

$a + y = a + (b \cdot t) = s$ by the last equation of (2).

Similarly,

$b \cdot z = b \cdot (t + a) = (a + t) \cdot b = s$ by the third equation of (2), and $b + z = b + (t + a) = b + (a + t) = (b + a) + t = b + t \supset r + t = 1$. Thus, the first part of the theorem is proved.

Conversely, let y be any relative complement of a in [0, s], z any relative complement of b in [s, 1], and t any relative complement of s in [y, z]. It must be proved that t satisfies the equation system (2). Thus, we have given:

(3) $a \subset x \subset b$

(4) $y \cdot a = 0,\ y + a = s$

(5) $z \cdot b = s,\ z + b = 1$

(6) $t \cdot s = y,\ t + s = z.$

Using these equations, the different parts of (2) are obtained as follows:

(i) $r \cdot t = (r \cdot b) \cdot (z \cdot t)$

$= r \cdot (b \cdot z) \cdot t$

$= r \cdot s \cdot t$

$= (r \cdot s) \cdot (s \cdot t)$

$= a \cdot y$

$= 0$

(ii) $r + t = (r + a) + (y + t)$

$= r + (a + y) + t$

$= r + s + t$

$= (r + s) + (s + t)$

$= b + z$

$= 1$

(iii) $(a + t) \cdot b = [a + (y + t)] \cdot b$

$= [(a + y) + t] \cdot b$

$= (s + t) \cdot b$

$= z \cdot b$

$= s$

(iv) $a + (t \cdot b) = a + [(t \cdot z) \cdot b]$

$= a + [t \cdot (z \cdot b)]$

$= a + [t \cdot s]$

$= a + y$

$= s$

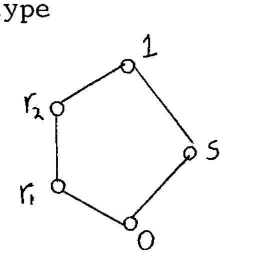
Thus, the proof of Theorem 4.3 is complete, and Theorem 4.2 is an immediate consequence of the second part of this theorem.

As an important consequence of this result, notice that if L, a, b are as in Theorem 4.3 and r_1, r_2 are two distinct elements of [a, b] that have a common relative complement s in [a, b], then it follows immediately that r_1 and r_2 have at least one common complement t.

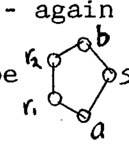
A Modularity Condition

The theorem of Dedekind (Corollary 1.10) is certainly a remarkable one, and its usefulness was demonstrated in Chapter III. Using Theorem 4.3 and the remark following it, Szasz [13] obtained a similar type modularity condition for relatively complemented lattices with greatest and least elements. This result is given in the following theorem.

<u>Theorem 4.4</u>. Let L be any relatively complemented lattice with greatest and least elements. Then L is modular if and only if it contains no sublattice of the type



.

<u>Proof</u>. By Corollary 1.10, the condition is necessary. To show that it is also sufficient, let L be any relatively complemented lattice with greatest and least elements. If L is non-modular, then -- again by the theorem of Dedekind -- it contains a sublattice of the type  .
Thus, there exists elements a, b, r_1, r_2, s in L such that r_1 and r_2 have a common relative complement s in [a, b]. Hence by the remark

following Theorem 4.3, r_1 and r_2 have a common complement t. It follows that the set of elements 0, r_1, r_2, t, 1 forms the desired type of sublattice.

Similar conditions have been obtained for complemented lattices of finite dimension by R. P. Dilworth [7], and for complemented, atomic lattices by J. E. McLaughlin [11].

Lattices With Unique Complements

Some interesting properties of uniquely complemented lattices will now be considered. In the first theorem, Szasz [13] gives a simple proof of a known theorem [4, p. 171].

Theorem 4.6. Any modular lattice with unique complements is a Boolean Algebra.

Proof. From Theorem 1.11, all that needs to be proved is that any lattice satisfying the given conditions has unique relative complements. To do this, let a, b, r be any elements of a complemented modular lattice L such that $a \subset r \subset b$. By Neumann's Theorem, L is relatively complemented; therefore, by Theorem 4.2 it follows that to each relative complement s of r there exists at least one complement t of r such that $s = a + (t \cdot b)$. Hence, if L also has the property of being a lattice with unique complements, then r has a unique complement t and, consequently, a unique relative complement s in [a, b]. Thus, the theorem is proved.

The next theorem follows immediately from Theorems 4.4 and 4.6.

Theorem 4.7. A lattice with unique complements is relatively complemented if and only if it is distributive.

Since any distributive, complemented lattice is relatively complemented, the "if" part is immediate. To prove the "only if" part, let L be any lattice with unique complements. If L is non-distributive, then by Theorem 4.6, it is non-modular. But from Theorem 4.4, it follows that non-modular lattices with unique complements are not relatively complemented. Thus, the proof is complete.

The last result included in this chapter was published by R. Bumcrot [6] in 1964. It provides an extension of DeMorgans laws to complemented modular lattices.

Theorem 4.8. If L is a modular lattice with 0 and 1, if a,b in L have unique complements a', b' respectively, and if a + b and a · b have complements, then a' + b' is the complement of a · b and a' · b' is the complement of a + b.

Proof. It will first be proved that $x = (a + b)' + [(a \cdot b)' \cdot b]$ is a complement of a and $y = (a + b)' + [(a \cdot b)' \cdot a]$ is a complement of b, where $(a + b)'$ and $(a \cdot b)'$ are any complements of a + b and a · b, respectively. The steps in which the modularity condition is used will be denoted by =*.

$$
\begin{aligned}
a + x &= \{a + (a \cdot b)\} + \{(a + b)' + [(a \cdot b)' \cdot b]\} \\
&= \{a + (a + b)'\} + \{(a \cdot b) + [(a \cdot b)' \cdot b]\} \\
&=^* \{a + (a + b)'\} + \{[(a \cdot b) + (a \cdot b)'] \cdot b\} \quad (\text{Note: } a \cdot b \subset b) \\
&= a + (a + b)' + b \\
&= (a + b) + (a + b)' \\
&= 1,
\end{aligned}
$$

and

$$
\begin{aligned}
a \cdot x &= \{a \cdot (a + b)\} \cdot \{(a + b)' + [(a \cdot b)' \cdot b]\} \\
&= a \cdot \langle (a + b) \cdot \{(a + b)' + [(a \cdot b)' \cdot b]\}\rangle \\
&\overset{*}{=} a \cdot \{(a + b)(a + b)' + [(a \cdot b)' \cdot b]\} \quad \text{(Note: } (a \cdot b)' \cdot b \subset b \subset a + b\text{)} \\
&= a \cdot \{0 + [(a \cdot b)' \cdot b]\} \\
&= (a \cdot b) \cdot (a \cdot b)' \\
&= 0.
\end{aligned}
$$

Thus, x is a complement of a. Similarly, $y = (a + b)' + [(a \cdot b)' \cdot a]$ is a complement of b. Also, by dualizing each step of the proofs above, it can be shown that $\bar{x} = (a \cdot b)' [(a + b)' + b]$ and $\bar{y} = (a \cdot b)' \cdot [(a + b)' + a]$ are also complements of a and b, respectively. Thus, since a' and b' are unique, $a' = x = \bar{x}$ and $b' = y = \bar{y}$. Now,

$$
\begin{aligned}
&(a' + b') + (a \cdot b) \\
&\quad = (x + y) + (a \cdot b) \\
&\quad = \langle \{(a + b)' + [(a \cdot b)' \cdot b]\} + \{(a + b)' + [(a \cdot b)' \cdot a]\}\rangle \\
&\qquad\qquad + (a \cdot b) \\
&\quad = (a + b)' + [(a \cdot b)' \cdot b] + (a + b)' + [(a \cdot b)' \cdot a] \\
&\qquad\qquad + (a \cdot b) \\
&\quad = (a + b)' + \{(a \cdot b) + [(a \cdot b)' \cdot b]\} + \{(a \cdot b) \\
&\qquad\qquad + [(a \cdot b)' \cdot a]\} \\
&\quad \overset{*}{=} (a + b)' + \{[(a \cdot b) + (a \cdot b)'] \cdot b\} + \{[(a \cdot b) \\
&\qquad\qquad + (a \cdot b)'] \cdot a\} \\
&\quad = (a + b)' + (1 \cdot b) + (1 \cdot a) \\
&\quad = (a + b)' + (a + b) \\
&\quad = 1,
\end{aligned}
$$

and

$$(a' + b') \cdot (a \cdot b)$$

$$= (\bar{x} + \bar{y}) \cdot (a \cdot b)$$

$$= \Big\langle \{(a \cdot b)' \cdot [(a + b)' + b]\} + \{(a \cdot b)' \cdot [(a + b)' + a]\} \Big\rangle \cdot (a \cdot b)$$

$$=^{*} (a \cdot b)' \cdot \Big\langle (a + b)' + b + \{(a \cdot b)' \cdot [(a + b)' + a]\} \Big\rangle \cdot (a \cdot b)$$

$$= (a \cdot b)' \cdot (a \cdot b) \cdot \Big\langle (a + b)' + b + \{(a \cdot b)' \cdot [(a + b)' + a]\} \Big\rangle$$

$$= 0.$$

Therefore, $(a' + b')$ is a complement of $a \cdot b$.

Also,

$$(a' \cdot b') + (a + b) = (x \cdot y) + (a + b) =$$

$$= \Big\langle \{(a + b)' + [(a \cdot b)' \cdot b]\} \cdot \{(a + b)' + [(a \cdot b)' \cdot a]\} \Big\rangle + (a + b)$$

$$=^{*} (a + b)' + \Big\langle [(a \cdot b)' \cdot b] \cdot \{(a + b)' + [(a \cdot b)' \cdot a]\} \Big\rangle + (a + b)$$

(Note: $(a + b)' \subset (a + b)' + [(a \cdot b)' \cdot a]$)

$$= \{(a + b)' + (a + b)\} + \Big\langle [(a \cdot b)' \cdot b] \cdot \{(a + b)' + [(a \cdot b)' \cdot a]\} \Big\rangle$$

$$= 1 + \Big\langle [(a \cdot b)' \cdot b] \cdot \{(a + b)' + [(a \cdot b)' \cdot a]\} \Big\rangle$$

$$= 1,$$

and

$$(a' \cdot b') \cdot (a + b) = (\bar{x} \cdot \bar{y}) \cdot (a + b) =$$

$$= \left\langle \left\{(a \cdot b)' \cdot [(a + b)' + b]\right\} \cdot \left\{(a \cdot b)' \cdot [(a + b)' + a]\right\}\right\rangle \cdot (a + b)$$

$$= (a \cdot b)' \cdot \left\{(a + b) \cdot [(a + b)' + b]\right\} \cdot \left\{(a + b) \cdot [(a + b)' + a]\right\}$$

$$\overset{*}{=} (a \cdot b)' \cdot \left\{[(a + b) \cdot (a + b)'] + b\right\} \cdot \left\{[(a + b) \cdot (a + b)'] + a\right\}$$

(Note: $b \subset a + b$ and $a \subset a + b$)

$$= (a \cdot b)' \cdot (0 + b) \cdot (0 + a)$$

$$= (a \cdot b)' \cdot (a \cdot b)$$

$$= 0.$$

Thus, $a' \cdot b'$ is a complement of $a + b$ and the proof is complete.

CHAPTER V

SUMMARY AND EDUCATIONAL IMPLICATIONS

This thesis presents a discussion of many basic types of lattices with particular emphasis given to complemented lattices. It attempts to do so in such a manner that the material can be understood by undergraduate mathematics students who have a basic knowledge of abstract algebra and set theory.

Summary

Chapter I serves as an introduction and includes a brief history of the development of lattice theory and its current significance in the field of mathematics. The two basic objectives of the paper are stated in the first chapter. These are

(1) to show the basic relationships that exist between different areas of mathematics and to illuminate some important properties of a variety of mathematical systems; and

(2) to expose the undergraduate to the frontier in a specific area of mathematics.

Chapter II discusses the basic structure of a lattice, and the concept of a lattice is then related to other areas of mathematics -- logic, group theory, convex sets, etc. -- by considering basic examples of lattices in the various areas. These examples are then used to illustrate certain special properties of lattices. To be specific, when the property of modularity is defined the basic examples are analyzed with respect to this particular property. The examples discussed are those that seem most relevant and interesting, and no attempt is made to discuss all the examples with respect to each property.

Chapter III illustrates in detail the relationships among the various lattices discussed in Chapter II. Venn diagrams are constructed with each circle representing a particular type lattice. To prove the diagram is accurately drawn a lattice is found that satisfies the particular properties of each region in the Venn diagram. The following types of lattices are considered: modular, distributive, complemented, complete, linearly ordered, and those with first and last elements.

Using Chapters II and III as background material, Chapter IV focuses on the specific area of complemented lattices. Most of the material discussed is from recent publications in mathematical journals, and the theorems and developments in the area of complemented lattices are related in detail.

Educational Implications

There seems to be a need in most mathematics curricula for a course that in some way shows underlying relationships among various areas of mathematics. This thesis uses a recently developed study -- lattice theory -- as a unifying course.

Not only does this paper help to show the relationship among different areas of mathematics, it also exposes the student to the frontier in the specific area of complemented lattices. This exposure will give the student an opportunity to reach the level of original work and will be a valuable asset in helping the more capable student prepare himself for future work on the graduate level.

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APPENDIX

INDEX

VITA

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Doctor of Education

Thesis: COMPLEMENTED LATTICES

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Biographical:

Personal Data: Born in Pocahontas, Arkansas, February 1, 1934 the son of Lester and Daisy Perrin; married in 1957 to Elizabeth Lansdon; father of three children.

Education: Attended public schools in De Kalb, Illinois and Pocahontas, Arkansas; graduated from Pocahontas High School, Pocahontas, Arkansas, in 1951; received Bachelor of Science degree from Harding College, Searcy, Arkansas in 1955 with a major in mathematics; received Master of Science degree from Kansas State University, Manhattan, Kansas in 1957 with major in mathematics; attended National Science Foundation Summer Institutes at the University of Kansas, Lawrence, Kansas, the summers of 1959 and 1960 and at Oklahoma State University, Stillwater, Oklahoma, 1962; also attended University of Missouri the summer of 1961; completed requirements for the Doctor of Education degree at Oklahoma State University, Stillwater, Oklahoma, in July of 1967.

Professional Experience: Was a graduate assistant in mathematics at Kansas State University, Manhattan, Kansas, while working on Master of Science degree, 1955-1957; was assistant professor of mathematics at Harding College, 1957-1964 and associate professor 1966-67; became chairman of the department of mathematics in 1961; was a graduate assistant in mathematics at Oklahoma State University while working on doctorate, 1964-1966.

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