SINGULAR EXTENSIONS AND COHOMOLOGY

OF LIE ALGEBRAS

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INTRODUCTION

It is well known that the second cohomology of modules over an algebra can be interpreted as extensions of modules and that singular extensions of a Lie algebra L can be interpreted as the second cohomologies of the enveloping algebra L^e when L is free or R is a field [1]. However the details of such interpretation over an arbitrary commutative ring R with unity have not yet been fully investigated, although Dixmier [2] and Shukla [7] have related singular extensions to a second cohomology assuming additional conditions on L.

We investigate the interrelations among extensions of Lie algebras over R and extensions of modules over Lie algebras. We also consider closely relations among these extensions and second dimensional cohomologies of Lie algebras over R.

In Chapter I we show that the classical bijection between equivalence classes of singular extensions of R-free Lie algebras L and those of their enveloping algebras L^e is in general replaced by a natural injection. The classical bijection between such classes of extensions of R-projective augmented algebras and classes of module extensions of their augmentation ideal holds true in general.

In Chapter II we consider first that the second cohomology derived from the classical bar construction for an augmented algebra A is in one-to-one correspondence with the "R-split" classes of singular extensions of A. The L^e-complex V(L) derived from the exterior algebra of L yields a cohomology $H^*(V(L),M)$. In general we inject the "R-split" classes of singular extensions of L into $H^2(V(L),M)$. If $H_2(V(L)) = 0$, then this correspondence is a bijection. The second cohomology with respect to an A-projective resolution is in one-to-one correspondence with all classes of singular extensions of A. Each class of R-split Lie algebra extensions is canonically a class of singular extensions of L^e, provided that $H_2(V(L)) = 0$. Shukla has put a second cohomology of L into one-to-one correspondence with the classes of singular extensions of L, when 2 is invertible in R. Therefore we have found the interrelations existing among four different cohomologies and several extensions. These interrelations are explicitly shown by a simple example in Chapter III.

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CHAPTER I

EXTENSIONS

1. Preliminaries

In this paper, R is a commutative ring with unity. $L \otimes L'$ denotes the tensor product over R.

<u>Definition</u>. An R-module L is a <u>Lie algebra</u> in case there is (1) a monomorphism of R-modules $j: L \longrightarrow A$ for some R-algebra A and (2) a morphism of R-modules $[\ ,\ ]: L \otimes L \longrightarrow L$ such that

$$j([x,x']) = j(x)j(x') - j(x')j(x).$$

This definition follows [6], 5.1.

<u>Proposition I.1</u>. (i) $[x,x] = 0$, (ii) $[x,x'] = -[x',x]$, and (iii) $[x_1,[x_2,x_3]] + [x_2,[x_3,x_1]] + [x_3,[x_1,x_2]] = 0$ (Jacobi's identity).

Since $j([x,x]) = (j(x))^2 - (j(x))^2 = 0$, (i) follows because j is injective. Likewise (ii) follows from $j([x,x']+[x',x]) = 0$, or from (i) by writing $[x+x',x'+x] = 0$. A similar computation implies (iii).

The <u>associated Lie algebra</u> A_L of an R-algebra A is defined to be the R-module A with 'Lie product' $[a,a'] = aa' - a'a$. If L and L' are Lie algebras we define a <u>morphism of Lie algebras</u> $f: L' \longrightarrow L$ to be a morphism of R-modules such that $f([x,x']) = [f(x),f(x')]$.

We have defined an R-module L furnished with a bilinear bracket operation [,] to be a Lie algebra in case there exists an algebra A and a monomorphism $j:L \longrightarrow A_L$ which respects the bracket operation.

The <u>tensor algebra</u> T(L) of an R-module L is as an R-module the direct sum $\sum_{n=0}^{\infty} T_n$ with $T_0 = R$, $T_1 = L$, $T_2 = L\otimes L$, and in general T_n the tensor product of n copies of L. With the natural multiplication induced by $T_n \otimes T_m \longrightarrow T_{n+m}$, T(L) becomes an R-algebra. The <u>universal enveloping algebra</u> L^e of a Lie algebra L is the quotient algebra T(L)/I, where I is the ideal in T(L) generated by elements of the form

$$x\otimes y - y\otimes x - [x,y], \text{ where } x,y\in L.$$

<u>Proposition I.2.</u> The composition $i_L : L = T_1 \subset T(L) \longrightarrow L^e$ has this 'universal property': if $j:L \longrightarrow A_L$ is any morphism of Lie algebras then there is a unique morphism of algebras $\tilde{j}:L^e \longrightarrow A$ such that $\tilde{j}i_L = j$.

As is shown in [6], 5.4, this follows from the corresponding 'universal property' of the tensor algebra.

<u>Proposition I.3.</u> The natural map $i_L : L \longrightarrow (L^e)_L$ is a monomorphism of Lie algebras.

Given an algebra A and a monomorphism of Lie algebras $j:L \longrightarrow A_L$, by the universal property of i_L there is a function $\tilde{j}:L^e \longrightarrow A$ such that $\tilde{j}i_L = j$. Since j is an injection, so is i_L. Finally $i_L([x,y]) = i_L(x)i_L(y) - i_L(y)i_L(x)$ by definition of the quotient algebra L^e.

We shall frequently identify L with $i_L(L) \subset L^e$.

Proposition I.4. L^e is an augmented algebra.

We have an identity injection $\eta_T : R \longrightarrow R = T_0 \subset T(L)$ and a natural morphism of algebras $\epsilon_T : T(L) \longrightarrow R$. Since $\epsilon_T \eta_T = 1_R$, it follows that $T(L)$ is as an R-module the direct sum $\operatorname{Im}(\eta_T) + \operatorname{Ker}(\epsilon_T) = R + \sum_{n=1}^{\infty} T_n$. Denote the quotient map $T(L) \longrightarrow L^e$ by p_T. Since $I = \operatorname{Ker}(p_T)$ is a subset of $\operatorname{Ker}(\epsilon_T)$, it follows that ϵ_T induces a morphism of algebras $\epsilon : L^e \longrightarrow R$ with $\epsilon p_T = \epsilon_T$. Defining $\eta = p_T \eta_T$, we see that

$$\epsilon \eta = \epsilon(p_T \eta_T) = \epsilon_T \eta_T = 1_R.$$

Thus we can express L^e as a direct sum of R-modules $\operatorname{Im}(\eta) + \operatorname{Ker}(\epsilon)$.

Proposition I.5. Let C be an ideal in the Lie algebra L and let D be an ideal in L^e. Then (i) $i_L(C)D$ is a two sided ideal in L^e. Consequently (ii) the two sided ideal generated in L^e by $i_L(C)$ is $i_L(C)L^e = L^e i_L(C)$.

We recall that any ideal C in L is necessarily two sided because $[y,c] = [-c,y]$ for any $c \in C$ and $y \in L$. To show (i) we prove (1) $(i_L(C)D)L^e \subset i_L(C)D$ and (2) $L^e(i_L(C)D \subset i_L(C)D$. For any $a \in L^e$ we have $(i_L(c)d)a = i_L(c)(da) \in i_L(C)D$. Thus assertion (1) is clear. To show (2) we consider in L^e that $i_L(y)i_L(c) = i_L(c)i_L(y) - i_L([c,y])$, for any $y \in L$. Thus $i_L(y)(i_L(c)d) = i_L(c)(i_L(y)d) - i_L([c,y])d \in i_L(C)D$. Since any element in L^e is either in R or is a finite sum of products of the form $i_L(y_1)\ldots i_L(y_n)$, (2) follows by induction on n. To prove (ii) denote by Y the two sided ideal $L^e i_L(C)L^e$ generated in L^e by $i_L(C)$. By (2) $Y \subset i_L(C)L^e$. Since L^e has a unit element $i_L(C)L^e \subset Y$ also. Therefore $Y = i_L(C)L^e$, as desired. Similarly, $Y = L^e i_L(C)$.

<u>Corollary</u>. The augmentation ideal of L^e is $i_L(L)L^e = L^e i_L(L)$.

Let Q denote the augmentation ideal, $\mathrm{Ker}(\epsilon)$. Since p_T is a surjection and $\epsilon p_T = \epsilon_T$, we have $Q = p_T(\mathrm{Ker}\,\epsilon_T)$. Clearly $\mathrm{Ker}\,\epsilon_T$ is the two sided ideal in $T(L)$ generated by $T_1 = L$. Thus $Q = p_T(T(L)T_1T(L))$ $= L^e i_L(L)L^e$. It follows that $Q = i_L(L)L^e = L^e i_L(L)$ by (ii) of Proposition I.5.

<u>Proposition I.6</u>. If $f:G \longrightarrow L$ is a morphism of Lie algebras, then there is a unique morphism of algebras $f^e:G^e \longrightarrow L^e$ such that $i_L f = f^e i_G$. If f is surjective, f^e is also surjective and $\mathrm{Ker}(f^e)$ $= i_G(\mathrm{Ker}(f))G^e$.

Since $i_L f:G \longrightarrow L^e$ is a morphism of Lie algebras, the universal property of i_G gives a unique morphism of algebras $f^e:G^e \longrightarrow L^e$ such that $f^e i_G = i_L f$. We obtain a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\;f\;} & L \\ {\scriptstyle i_G}\downarrow & & \downarrow{\scriptstyle i_L} \\ G^e & \xrightarrow{\;f^e\;} & L^e \end{array} .$$

If f is surjective it is clear that necessarily f^e is surjective. Since $\mathrm{Ker}(f)$ is an ideal in G, $G/\mathrm{Ker}(f)$ is a Lie algebra. Identifying $G/\mathrm{Ker}(f)$ with L, we can apply [1], p. 269, Proposition 1.3 to conclude that $\mathrm{Ker}(f^e)$ is the ideal generated in G^e by $i_G(\mathrm{Ker}(f))$. By the corollary $\mathrm{Ker}(f^e) = i_G(\mathrm{Ker}(f))G^e$.

We are now going to compare two definitions. Define a <u>singular extension</u> of a Lie algebra L to be an epimorphism $f:G \longrightarrow L$ of Lie algebras satisfying $[w,w'] = 0$ for $w,w' \in \mathrm{Ker}(f)$.

<u>Definition</u>. F is a <u>singular extension of L by M</u> in case (1) F is an exact sequence $0 \longrightarrow M \xrightarrow{i} G \xrightarrow{f} L \longrightarrow 0$ of R-modules, (2) f is a morphism of Lie algebras, and (3) $i:M \longrightarrow \text{Ker}(f)$ is a morphism of left L-modules, with an L-module structure of Ker(f) defined by $xw = [y,w]$ where $f(y) = x \in L$.

If C is any ideal in a Lie algebra G, define a left G operation on C by $yw = [y,w] \in C$ for any $y \in G$, $w \in C$. The condition that C be a left G-module is $([y,y'])w = y(y'w) - y'(yw)$, which is Jacobi's identity in G. In particular if f is a singular extension, the ideal $C = \text{Ker}(f)$ can be given the structure of a left L-module by defining $xw = [y,w]$, where $f(y) = x \in L$. For if $f(y') - f(y) = 0$ then $[y-y',w] = 0$. Thus a singular extension of L by Ker(f) is given by

$$0 \longrightarrow \text{Ker}(f) \xrightarrow{i} G \xrightarrow{f} L \longrightarrow 0$$

where $i:\text{Ker}(f) \longrightarrow G$ the identity injection.

On the other hand, suppose that F is a singular extension of L by M. Let f be the epimorphism in the exact sequence F. Choose any $w,w' \in \text{Ker}(f)$. Then $[w,w'] = f(w)w' = 0$, and hence f is a singular extension.

<u>Proposition I.7</u>. The following condition is equivalent to part (3) of the above definition. M is a left L-module and $i(xm) = [y,i(m)]$ where $f(y) = x \in L$.

For assume the condition. Given $w \in \text{Ker}(f)$ we have a unique $m \in M$ with $i(m) = w$. We are given $i(xm) = [y,i(m)]$ where $f(y) = x \in L$. As above we can well-define an L-module structure on Ker(f) by $xw = [y,w]$ where $f(y) = x \in L$. Then $i(xm) = [y,i(m)] = [y,w] = xw = xi(m)$. This shows that $i:M \longrightarrow \text{Ker}(f)$ is a morphism of L-modules. Thus the condition of the proposition implies condition (3).

Conversely, suppose (3) holds. We are given that $i(xm) = xi(m)$ for $m \in M$. Writing $w = i(m) \in \mathrm{Ker}(f)$, we are also given that $xw = [y,w]$ where $f(y) = x \in L$. Thus $i(xm) = xi(m) = xw = [y,w] = [y,i(m)]$. We conclude that (3) implies the condition of the proposition and the equivalence is proved.

We define two singular extensions F and F* of L by M to be <u>equivalent</u> in case there is a morphism of Lie algebras $k:G \longrightarrow G^*$ such that the diagram

$$\begin{array}{lccccccccc} F: & 0 & \longrightarrow & M & \xrightarrow{i} & G & \xrightarrow{f} & L & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle k} & & \| & & \\ F^*: & 0 & \longrightarrow & M & \xrightarrow{i^*} & G^* & \xrightarrow{f^*} & L & \longrightarrow & 0 \end{array}$$

commutes. By the five-lemma, such a k is necessarily a bijection. Hence the definition does give an equivalence relation. We shall abbreviate the equivalence by $k:F \sim F^*$. We denote the set of equivalence classes by $E_{\mathcal{L}}(L,M)$. A singular extension of L by M is defined to be <u>R-split</u> in case there is a morphism of R-modules $u:L \longrightarrow G$ such that $fu = 1_L$. We denote by $E^s_{\mathcal{L}}(L,M)$ the subset of R-split classes of $E_{\mathcal{L}}(L,M)$.

We shall consider any left L-module canonically as a left L^e-module, and conversely.

We now turn our attention to an augmented algebra A with augmentation $\varepsilon:A \longrightarrow R$. We shall always consider any left A-module as an A-bimodule with right operation of A defined by the augmentation. We again compare two definitions. Define a <u>singular extension</u> of an augmented algebra A to be an epimorphism $f:B \longrightarrow A$ of algebras satisfying $(\mathrm{Ker}\ f)(\mathrm{Ker}\ \varepsilon f) = 0$.

Definition. E is a singular extension of A by M in case (1) E is an exact sequence $0 \longrightarrow M \xrightarrow{i} B \xrightarrow{f} A \longrightarrow 0$ of R-modules, (2) f is a morphism of algebras, and (3) $i:M \longrightarrow \mathrm{Ker}(f)$ is a morphism of A-bimodules, with an A-bimodule structure of Ker(f) defined by aw = bw and $wa = wb = w\epsilon(a)$ where $f(b) = a \in A$.

Let F be a singular extension. Since Ker(f) is an ideal in B, Ker(f) is a B-bimodule. We can well-define an A-bimodule structure on Ker(f) by aw = bw and wa = wb, where f(b) = a. For if f(b') = f(b), both (b-b')w and w(b-b') belong to $(\mathrm{Ker}(f))^2 = 0$. We must verify that $wa = w\epsilon(a)$. If f(b) = a, then $f(b-\epsilon f(b)) = a - \epsilon(a)$. Thus $w(a-\epsilon(a)) = w(b-\epsilon f(b)) \in (\mathrm{Ker}\ f)(\mathrm{Ker}\ \epsilon f) = 0$, as required. Write $i:\mathrm{Ker}(f) \longrightarrow B$ for the identity injection. Thus a singular extension of B by Ker(f) is given by $0 \longrightarrow \mathrm{Ker}(f) \xrightarrow{i} B \xrightarrow{f} A \longrightarrow 0$.

Conversely, suppose that E is a singular extension of A by M. Let f be the corresponding epimorphism. Select any $w \in \mathrm{Ker}(f)$ and $b \in \mathrm{Ker}(\epsilon f)$. Then $wb = wf(b) = w\epsilon f(b) = 0$, and hence f is a singular extension.

Proposition I.8. The following conditions are equivalent to part (3) of the definition of a singular extension of A by M. If f(b) = a, then bi(m) = i(am) and $i(m)b = i(m\epsilon(a))$.

The proof is similar to that of proposition I.7.

We define two extensions E and E* to be equivalent in case there is a morphism of algebras $k:B \longrightarrow B^*$ such that the diagram

$$\begin{array}{ccccccccc} E: 0 & \longrightarrow & M & \xrightarrow{i} & B & \xrightarrow{f} & A & \longrightarrow & 0 \\ & & \| & & \downarrow k & & \| & & \\ E^*: 0 & \longrightarrow & M & \xrightarrow{i^*} & B^* & \xrightarrow{f^*} & A & \longrightarrow & 0 \end{array}$$

commutes. Again, such a k is necessarily a bijection. We abbreviate

$k:E\sim E^*$, and we denote by $E_{\mathcal{A}}(A,M)$ the set of classes of singular extensions of an augmented algebra A by M. A singular algebra extension E is defined to be R-split in case there is a morphism of R-modules $u:A \longrightarrow B$ such that $fu = 1_A$. We denote by $E^s{}_{\mathcal{A}}(A,M)$ the subset of R-split classes of $E_{\mathcal{A}}(A,M)$.

We shall denote by ${}_A\mathfrak{M}$ the category of all left A-modules; $\mathfrak{M}$ will denote the category of all left R-modules.

Definition. F is an extension of Q by M in case F is an exact sequence $0 \longrightarrow M \xrightarrow{i} X \xrightarrow{f} Q \longrightarrow 0$ in the category ${}_A\mathfrak{M}$.

We define two extensions F and F* to be equivalent in case there is a morphism of A-modules $k:X \longrightarrow X^*$ such that the diagram

$$\begin{array}{lcccccccc} F: & 0 & \longrightarrow & M & \xrightarrow{i} & X & \xrightarrow{f} & Q & \longrightarrow 0 \\ & & & \| & & \downarrow k & & \| & \\ F^*: & 0 & \longrightarrow & M & \xrightarrow{i^*} & X^* & \xrightarrow{f^*} & Q & \longrightarrow 0 \end{array}$$

commutes. As before, k is a bijection. We write $k:F\sim F^*$ and denote by $E_{{}_A\mathfrak{M}}(Q,M)$ the set of classes of extensions of an A-module Q by M. A module extension F is defined to be R-split in case there is a morphism of R-modules $u:Q \longrightarrow X$ such that $fu = 1_Q$. We denote by $E^s{}_{{}_A\mathfrak{M}}(Q,M)$ the subset of R-split classes of $E_{{}_A\mathfrak{M}}(Q,M)$.

§ 2. The Injection $\alpha:E_{\mathcal{A}}(L^e,M) \longrightarrow E_{\mathcal{L}}(L,M)$ of Singular Extension Classes of L^e by M Into Singular Extension Classes of a Lie Algebra L by a Left L-Module M

We first define $\alpha:E_{\mathcal{A}}(L^e,M) \longrightarrow E_{\mathcal{L}}(L,M)$. Given $[E]\in E_{\mathcal{A}}(L^e,M)$ write

$E: 0 \longrightarrow M \xrightarrow{i} B \xrightarrow{f} L^e \longrightarrow 0$. Define $G = f^{-1}(L) = \{y \in B;\ f(y) \in L \subset L^e\}$. Define $f' = f|_G : G \longrightarrow L$. Since $i(M) = f^{-1}(0) \subset f^{-1}(L) = G$, we can define $i': M \longrightarrow G$ by $i'(m) = i(m)$.

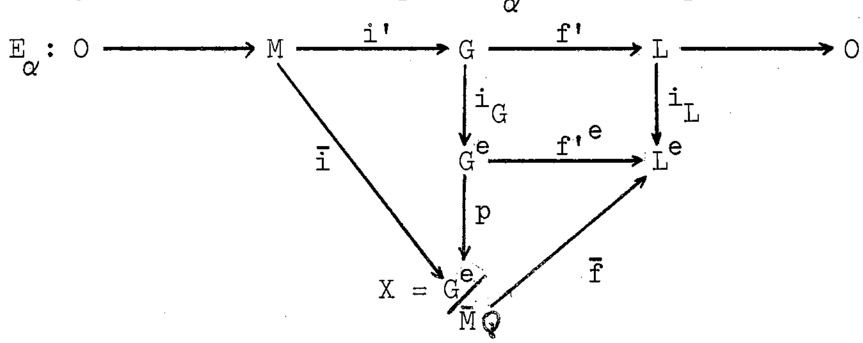
<u>Lemma</u>. $E_\alpha: 0 \longrightarrow M \xrightarrow{i'} G \xrightarrow{f'} L \longrightarrow 0$ is a singular extension of L by M.

Evidently exactness of E implies exactness of E_α. We first compute $f(yy'-y'y) = f(y)f(y') - f(y')f(y) = xx' - x'x = [x,x'] \in L$, where $f(y) = x \in L$ and $f(y') = x' \in L$. This shows that if $y, y' \in G$ then $yy' - y'y \in G$. It follows that G is closed with respect to $[y,y'] = yy' - y'y$. The natural injection $j: G = f^{-1}(L) \subset B_L$ satisfies $j([y,y']) = [j(y), j(y')]$, so by definition G is a Lie algebra. Also by the above computation $f' = f|_G$ is a morphism of Lie algebras. Finally if $f'(y) = x \in L$, the condition of Proposition I.7 implies that $yi'(m) - i'(m)y = i'(xm) - 0$.

To show we can well-define α by $\alpha([E]) = [E_\alpha]$, we suppose given $k: E \sim E^*$. Then evidently $k|_G : G \longrightarrow G^*$ is a morphism of Lie algebras and in fact $k|_G : E_\alpha \sim E^*_\alpha$.

<u>Theorem I.1</u>. $\alpha: E_{\mathcal{Q}}(L^e, M) \longrightarrow E_{\mathcal{L}}(L,M)$ is an injection.

<u>Proof</u>. We shall define $\omega: \mathrm{Im}(\alpha) \longrightarrow E_{\mathcal{Q}}(L^e, M)$ and prove that $\omega\alpha$ is the identity function. We are given E_α as the top row in the diagram



Since $i_L f':G \longrightarrow (L^e)_L$ is a morphism of Lie algebras, by Proposition I.6 there is a unique morphism of algebras $f'e$ such that $f'^e i_G = i_L f'$. Define $\bar{M} = i_G i'(M)$, $\mathcal{Q}$ the augmentation ideal of G^e, X the quotient R-module $G^e/\bar{M}\mathcal{Q}$, and $p:G^e \longrightarrow X$ the natural morphism of R-modules. Since $(f'^e i_G)i' = i_L(f'i') = 0$, we see that $\bar{M} \subset \mathrm{Ker}(f'^e)$. Since f'^e is a morphism of algebras, $\bar{M}\mathcal{Q} \subset \mathrm{Ker}(f'^e)$. Thus f'^e induces a morphism of R-modules $\bar{f}:X \longrightarrow L^e$ with $\bar{f}p = f'^e$. Defining $\bar{i} = p i_G i'$, it follows from the commutativity of the diagram that $\bar{f}\bar{i} = 0$. Define E* to be the sequence $0 \longrightarrow M \xrightarrow{\bar{i}} X \xrightarrow{\bar{f}} L^e \longrightarrow 0$. Since f'^e is a surjection and $\bar{f}p = f'^e$, $\bar{f}$ is also a surjection.

By part (ii) of Proposition I.5 the ideal generated by $i_G(i'(M)) = \bar{M}$ in G^e is $\bar{M}G^e$. Since f' is an epimorphism and $i'(M) = \mathrm{Ker}(f')$, it follows from Proposition I.6 that $\bar{M}G^e = \mathrm{Ker}(f'^e)$. Since $\bar{M} = \bar{M}R$ and $\bar{M}R + \bar{M}\mathcal{Q} = \bar{M}G^e$, we obtain $\mathrm{Im}(\bar{i}) = p(\bar{M}R) = (\bar{M}R + \bar{M}\mathcal{Q})/\bar{M}\mathcal{Q} = \bar{M}G^e/\bar{M}\mathcal{Q}$ $= \mathrm{Ker}(f'^e)/\bar{M}\mathcal{Q} = \mathrm{Ker}(\bar{f})$. Now $\mathcal{Q}$ is a two sided ideal in G^e, being the kernel of the augmentation, a morphism of algebras. Thus by part (i) of Proposition I.5, $\bar{M}\mathcal{Q}$ is a two sided ideal in G^e; consequently the quotient X is an algebra. Necessarily $p:G^e \longrightarrow X$ is a morphism of algebras. Since $\bar{f}p = f'^e$ we conclude that $\bar{f}:X \longrightarrow L^e$ is a morphism of algebras.

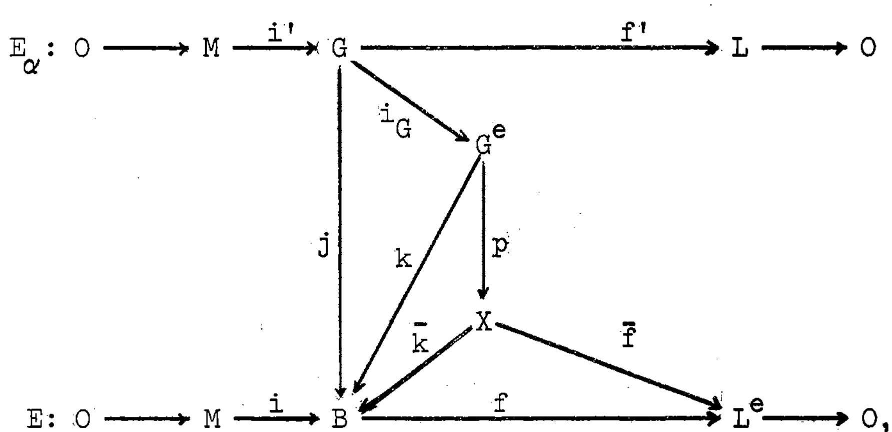
To complete the argument that E* is a singular extension of L^e by M we need to show that $\bar{i}$ is an injection and to verify the condition of Proposition I.8. We suppose given $\bar{f}(b) = a$ and write $b = p(z)$.

To show $b\bar{i}(m) = \bar{i}(am)$, we use induction on the degree of a representative of z in T(G). If $z = i_G(y)$ then $a = \bar{f}(b) = i_L f'(y)$. Denoting $x = f'(y) \in L$, we see that $am = xm$ by definition of the induced module structure. Since $[E_\alpha] \in E_{\mathcal{L}}(L,M)$, we have $i'(xm) = [y, i'(m)]$ and $i_G i'(xm)$ $= i_G(y)i_G(i'(m)) - i_G(i'(m))i_G(y)$. Since $i_G(G) \subset \mathcal{Q}$ and $p(i_G i'(M)\mathcal{Q}) = 0$

we conclude that $\bar{i}(am) = \bar{i}(xm) = p(i_G i'(xm)) = b\bar{i}(m) - 0$, as desired. Now suppose that $z = i_G(y)z'$. By the induction hypothesis, if $w = \bar{f}(p(z')) \in L^e$ then $p(z')\bar{i}(m) = \bar{i}(wm)$. It follows that $b\bar{i}(m) = pi_G(y)\bar{i}(wm) = \bar{i}(a'wm)$, where $a' = \bar{f}pi_G(y)$. Since $\bar{f}(b) = \bar{f}(pi_G(y))\bar{f}(p(z')) = a'w$, the induction is completed.

It remains to show $\bar{i}(m)b = i(m\epsilon(a))$. For the case $z = i_G(y)$ we have $\bar{i}(m)b = p(i_G i'(m)i_G(y)) = 0$ because $\bar{M}i_G(G) \subset \bar{M}\mathcal{Q}$. As before $a = i_L(f'(y))$. But $\epsilon i_L = 0$ implies $\bar{i}(m\epsilon(a)) = 0$ also. The induction step follows as before.

At this point in the construction of ω we have used only the assumption that E_α defined a class in $E_{\mathcal{X}}(L,M)$. To prove that $\bar{i}$ is injective we do use the assumption that $[E_\alpha] \in \mathrm{Im}\,\alpha$. In this case we are given an algebra B such that the inclusion map $j:G \longrightarrow B_L$ is a morphism of Lie algebras. By the universal property of $i_G:G \longrightarrow G^e$ there is a morphism of algebras $k:G^e \longrightarrow B$ such that $ki_G = j$. In the diagram



we are given that E is a singular algebra extension of L^e by M and that $\alpha([E]) = [E_\alpha]$.

<u>Lemma</u>. $\bar{M}\mathcal{Q} \subset \mathrm{Ker}(k)$.

If we show that $k(\bar{\bar{M}}i_G(G)) = 0$ then the result follows by induction. In B, for any $y \in G \subset B$, $i(m)y = i(m\epsilon(f(y))) = 0$ because $fj(G) = i_L f'(G) = i_L(L) \subset Ker(\epsilon)$. That is, $i(m)y = 0 \in G \subset B$. Thus we can write $0 = j(0) = j(i(m)y) = j(i'(m))j(y) = ki_G(i'(m))ki_G(y) = k(i_G(i'(m))i_G(y))$. It follows that $\bar{\bar{M}}i_G(G) \subset Ker(k)$, as desired.

By the lemma, k induces a morphism of R-modules $\bar{k}:X \longrightarrow B$ such that $\bar{k}p = k$. Thus $i = ji' = (ki_G)i' = \bar{k}\bar{\bar{i}}$. Since i is injective we can conclude that $\bar{\bar{i}}$ is injective.

Starting with a singular algebra extension E of L^e by M, we have completed the construction of a class [E*] in $E_{\mathcal{Q}}(L^e,M)$.

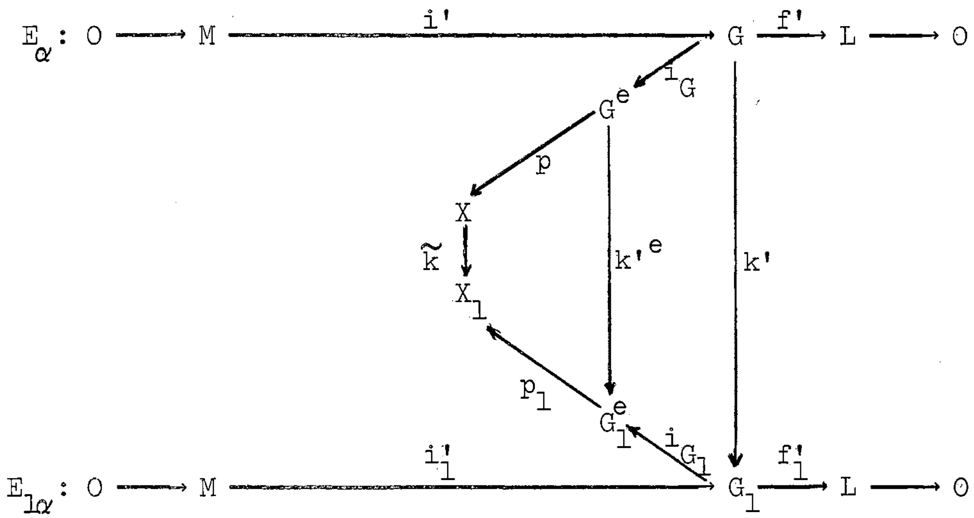
<u>Lemma</u>. $\bar{\bar{k}}:E^*\sim E$.

We just observed that $i = \bar{\bar{k}}\bar{\bar{i}}$. Since p and k are morphisms of algebras and $\bar{k}p = k$, clearly $\bar{\bar{k}}$ is a morphism of algebras. Finally we must show that $f\bar{\bar{k}} = \bar{\bar{f}}$. We observe that $f'^e i_G = i_L f' = fj = f(ki_G)$. Since $i_G(G)$ generates $\mathcal{Q}$ and f, k, and f'^e are morphisms of algebras, it follows that $f'^e = fk$. Therefore $\bar{f}p = f'^e = fk = f(\bar{k}p)$. Since p is an epimorphism, we obtain $\bar{\bar{f}} = f\bar{\bar{k}}$. This completes the proof of the lemma.

If we can well-define ω on $Im\alpha$ by $\omega([E_\alpha]) = [E^*]$ then by the lemma $\omega\alpha([E]) = \omega([E_\alpha]) = [E^*] = [E]$. This will complete the proof of Theorem I.1.

ω is well defined if given $k':E_\alpha \sim E_{1\alpha}$ we can construct $\tilde{k}:E^*\sim E_1{}^*$.

In the diagram



we are given that $k':G \longrightarrow G_1$ is a morphism of Lie algebras. By Proposition I.6 there is a morphism of algebras $k'^e:G^e \longrightarrow G_1{}^e$ such that $k'^e i_G = i_{G_1} k'$. Since k' is an isomorphism, necessarily also k'^e is an isomorphism. By construction $\operatorname{Ker}(p) = \bar{M}Q$. Denoting by Q_1 the augmentation ideal of G_1, likewise $\operatorname{Ker}(p_1) = i_{G_1} i_1'(M)Q_1$. Since $k'^e(Q) = Q_1$ and $(k'^e i_G)i' = (i_{G_1} k')i' = i_{G_1} i_1'$, we infer that $k'^e(\operatorname{Ker} p) = \operatorname{Ker}(p_1)$. We obtain an isomorphism of algebras $\widetilde{k}:X \longrightarrow X_1$. It can be shown that $\widetilde{k}$ commutes as required. This completes the proof that ω is well-defined and establishes theorem I.1.

§ 3. The Bijection $\beta:E_{\mathcal{A}}(A,M) \longrightarrow E_A\mathfrak{M}(Q,M)$ Onto Module Extension Classes by M of the Augmentation Ideal Q Of an Augmented Algebra A

Suppose that $E:0 \longrightarrow M \xrightarrow{i} B \xrightarrow{f} A \longrightarrow 0$ is a singular extension of A by M. As before, let Q denote the augmentation ideal of A. We define a sequence $E_\beta:0 \longrightarrow M \xrightarrow{i'} X \xrightarrow{f'} Q \longrightarrow 0$ as follows. Let

$X = f^{-1}(Q) = \{b \in B; f(b) \in Q\}$ and let $f' = f|_X : X \longrightarrow Q$. Since $i(M) = f^{-1}(0) \subset f^{-1}(Q)$, we can define $i' : M \longrightarrow X$ by $i'(m) = i(m)$.

Lemma. $[E_\beta] \in E_A\mathfrak{M}(Q,M)$.

From the construction, E_β is an exact sequence of R-modules. X can be considered a left A-module if we define $ax = bx$ where $f(b) = a$. To see that this multiplication is well-defined, suppose $f(b') = f(b)$. Then there is an $m \in M$ such that $b - b' = i(m)$. Since $f(X) = Q = \mathrm{Ker}(\epsilon)$, we conclude that $bx - b'x = i(m)x = i(m\epsilon(f(x))) = 0$. We next show that f' and i' are morphisms of A-modules. Given any $a \in A$ fix $b \in B$ such that $f(b) = a$. Then at once $af'(x) = f(b)f(x) = f(bx) = f(ax)$. Likewise $i'(am) = i(am) = bi(m) = ai(m)$. This completes the proof of the lemma.

We show that we can well-define $\beta : E\ (A,M) \longrightarrow E_A\mathfrak{M}(Q,M)$ by $\beta([E]) = [E_\beta]$. From a given $k : E \sim E^*$ we want to define $k_\beta : E_\beta \sim E^*_\beta$. Write $E^* : 0 \longrightarrow M \xrightarrow{i^*} X^* \xrightarrow{f^*} Q \longrightarrow 0$. For any $x \in X$ we have $f^*(k(x)) = f(x) \in Q$. This can be written as $k(x) \in f^{*-1}(Q) = X^*$, which implies that $k|_X(X) \subset X^*$. Then $k_\beta = k|_X : X \longrightarrow X^*$ gives the desired equivalence.

Theorem I.2. β is a bijection.

Proof. We shall define $J : E_A\mathfrak{M}(Q,M) \longrightarrow E_{\mathcal{A}}(A,M)$. Then we shall prove that $J\beta$ and βJ are identity maps.

Let $F : 0 \longrightarrow M \xrightarrow{i} X \xrightarrow{f} Q \longrightarrow 0$ be an extension of Q by M. We construct a sequence $F_J : 0 \longrightarrow M \xrightarrow{\bar{i}} \bar{B} \xrightarrow{\bar{f}} A \longrightarrow 0$. As an R-module, we define $\bar{B}$ to be the direct sum $X + R$. We define a product in $\bar{B}$ by $(x,r)(y,s) = (ry + sx + f(x)y,\ rs)$. If e_R is the identity element of R then $(0,e_R)$ is a two sided identity element for $\bar{\bar{B}}$. Since clearly the multiplication distributes over addition, we verify the associative property to conclude that $\bar{\bar{B}}$ is an algebra. We compute $((x,r)(y,s))(z,t)$ $= ((rs)z + t(ry + sx + f(x)y + f(ry + sx + f(x)y)z,(rs)t)$ and $(x,r)((y,s)(z,t)) = (r(sz+ty+f(y)z) + st(x) + f(x)(sz+ty+f(y)z),r(st))$. Since f is a morphism of A-modules and $f(x) \in Q \subset A$, necessarily $f(f(x)y)z = f(x)f(y)z$. It follows that the multiplication in $\bar{\bar{B}}$ is associative. We define $\bar{\bar{f}}(x,r) = f(x) + r \in Q + R = A$ and define $\bar{\bar{i}}(m)$ $= (i(m),0) \in \bar{\bar{B}}$. Then evidently F_J is an exact sequence of R-modules. Clearly $\bar{\bar{f}}$ preserves the identity element. Furthermore, $\bar{\bar{f}}((x,r)(y,s))$ $= f(ry+sx+f(x)y) + rs = rf(y) + sf(x) + f(x)f(y) + rs = \bar{\bar{f}}(x,r)\bar{\bar{f}}(y,s)$.

If we verify the conditions of Proposition I.8, then we can conclude that F_J is a singular extension of A by M. If $\bar{\bar{f}}(b) = a$ then necessarily $b = (x,r)$ with $f(x) + r = a$. It follows from the definition of multiplication in $\bar{\bar{B}}$ that $b\bar{\bar{i}}(m) = (x,r)(i(m),0) = (ri(m)+0+f(x)i(m),0)$ $= (ai(m),0) = (i(am),0) = \bar{\bar{i}}(am)$. Likewise, $\bar{\bar{i}}(m)b = (i(m),0)(x,r)$ $= (0+ri(m)+f(i(m))x,0) = (i(mr),0) = (i(m\epsilon(a)),0) = \bar{\bar{i}}(m\epsilon(a))$.

To show that we can well-define J by $J([F]) = [F_J]$, we suppose given $k:F\sim F^*$ and construct $\bar{\bar{k}}:\bar{\bar{B}} \longrightarrow \bar{\bar{B}}^*$. Given $k:X \longrightarrow X^*$ we define $\bar{\bar{k}}(x,r) = (k(x),r)$. Then $\bar{\bar{k}}((x,r)(y,s)) = (rk(y)+sk(x)+f(x)k(y),rs)$ $= \bar{\bar{k}}(x,r)\bar{\bar{k}}(y,s)$ because $f(x) = f^*(k(x))$. Also $\bar{\bar{k}}$ preserves the identity element. We have shown that $\bar{\bar{k}}$ is a morphism of algebras. We verify that $\bar{\bar{f}}^*\bar{\bar{k}} = \bar{\bar{f}}$ and $\bar{\bar{k}}\bar{\bar{i}} = \bar{\bar{i}}^*$. To see the first condition we compute $\bar{\bar{f}}^*\bar{\bar{k}}(x,r) = f^*(k(x)) + r = f(x) + r = \bar{\bar{f}}(x,r)$. Likewise $\bar{\bar{k}}\bar{\bar{i}}(m) = (ki(m),0)$ $= (i^*(m),0) = \bar{\bar{i}}^*(m)$. We have shown that $\bar{\bar{k}}:F_J\sim F_J^*$, and consequently that J is well-defined.

We next show that $J\beta$ is the identity map on $E_{\mathcal{A}}(A,M)$. We suppose that E is any singular extension of A by M. Since E_β was defined by restriction, the diagram

$$\begin{array}{lcccccccc} E: & 0 & \longrightarrow & M & \xrightarrow{\ i\ } & B & \xrightarrow{\ f\ } & A & \longrightarrow & 0 \\ & & & \| & & \uparrow k & & \| & & \\ (E_\beta)_J: & 0 & \longrightarrow & M & \xrightarrow{\ \bar{i}\ } & \bar{B} & \xrightarrow{\ \bar{f}\ } & A & \longrightarrow & 0 \end{array}$$

suffices to recall the construction of $(E_\beta)_J$. If we define $k:\bar{B} \longrightarrow B$ by $k(x,r) = x + r \in X + R = B$ then evidently the diagram commutes. Since by definition of the A-module structure of X, $f(x)y = xy$ we have $k((x,r)(y,s)) = k(ry+sx+f(x)y,rs) = k(x,r)k(y,s)$. Clearly k preserves the identity element, hence k is a morphism of algebras. We have shown that $k:(E_\beta)_J \sim E$. It follows that $J\beta([E]) = J([E_\beta]) = [(E_\beta)_J] = [E]$ and we conclude $J\beta$ is the identity map.

Finally we show that βJ is the identity map on $E_A\mathfrak{M}(Q,M)$. The construction is indicated in the diagram

$$\begin{array}{lcccccccc} F: & 0 & \longrightarrow & M & \xrightarrow{\ i\ } & X & \xrightarrow{\ f\ } & Q & \longrightarrow & 0 \\ & & & \| & & \downarrow j & & & & \\ F_J: & 0 & \longrightarrow & M & \xrightarrow{\ \bar{i}\ } & \bar{B} & \xrightarrow{\ \bar{f}\ } & A & \longrightarrow & 0 \\ & & & \| & & \downarrow p & & & & \\ (F_J)_\beta: & 0 & \longrightarrow & M & \xrightarrow{\ i'\ } & \bar{f}^{-1}(Q) & \xrightarrow{\ f'\ } & Q & \longrightarrow & 0, \end{array}$$

in which F is a given extension of Q by M. We observe that $\bar{f}^{-1}(Q)$ $= \{(x,r);f(x)+r \in Q\} = \{(x,0);x \in X\}$. We define $j(x) = (x,0)$, $p(x,r)$ $= (x,0)$, and $k = pj$. If $a \in A$, then $a = f(x) + r$ for some $x \in X$ and $r \in R$; that is, $\bar{f}(x,r) = a$. We compute $ak(y) = (x,r)(y,0) = (ry+0+f(x)y,0)$ $= (ay,0) = k(ay)$, to see that k is a morphism of A-modules. Since commutativity is evident, we conclude that $k:F \sim (F_J)_\beta$. Thus

$\beta J([F]) = \beta([F_J]) = [(F_J)_\beta] = [F]$ and we have shown that βJ is also the identity map. This completes the proof of theorem I.2.

§ 4. The Injection $\Delta: E_{L^e\mathfrak{M}}(Q,M) \longrightarrow E_{\mathcal{L}}(L,M)$ and the Restrictions of α, β, and Δ to Classes of R-Split Extensions

Let Q be the augmentation ideal of L^e. Let $F: 0 \longrightarrow M \xrightarrow{i} X \xrightarrow{f} Q \longrightarrow 0$ be an extension of Q by M. Identifying L with $i_L(L) \subset Q \subset L^e$, define $G = f^{-1}(L) \subset X$. For $y, y' \in G$ define $[y,y']_G = f(y)y' - f(y')y \in X$. Define $f': G \longrightarrow L$ by $f'(x) = f(x)$. As before, since $i(M) = f^{-1}(0) \subset f^{-1}(L) = G$, we can define $i': M \longrightarrow G$ by $i'(m) = i(m)$. Define F_Δ to be the sequence of R-modules $0 \longrightarrow M \xrightarrow{i'} G \xrightarrow{f'} L \longrightarrow 0$.

<u>Proposition I.9</u>. F_Δ is a singular extension of L by M equivalent to $(F_J)_\alpha$.

We refer to the diagram

$$\begin{array}{rccccccccc}
F: & 0 & \longrightarrow & M & \xrightarrow{\quad i \quad} & X & \xrightarrow{\quad f \quad} & Q & \longrightarrow & 0 \\
 & & & \| & & \downarrow j & & & & \\
F_J: & 0 & \longrightarrow & M & \xrightarrow{\quad \bar{i} \quad} & \bar{B} & \xrightarrow{\quad \bar{f} \quad} & L^e & \longrightarrow & 0 \\
 & & & \| & & & & & & \\
(F_J)_\alpha: & 0 & \longrightarrow & M & \xrightarrow{\quad \bar{i}' \quad} & \bar{G} & \xrightarrow{\quad \bar{f}' \quad} & L & \longrightarrow & 0.
\end{array}$$

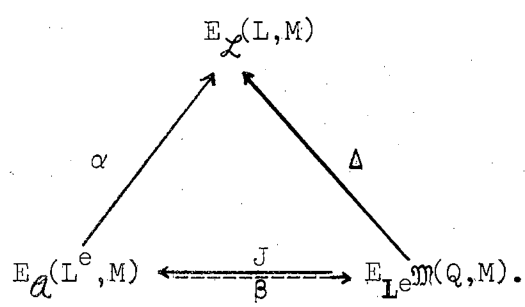
The sequence F_J in the middle row is the singular extension of L^e by M defined in §3 with $\bar{B} = X + R$. The sequence $(F_J)_\alpha$ in the bottom row is the singular extension of L by M defined in §2 with $\bar{G} = \bar{f}^{-1}(L)$. As before, we define the natural injection $j:X \longrightarrow \bar{B}$ by $j(y) = (y,0)$. Since $\bar{f}j = f$, as an R-module $\bar{G} = \bar{f}^{-1}(L) = j(f^{-1}(L)) = j(G)$. The Lie product in $\bar{G}$ was defined for any $\bar{y},\bar{y}' \in \bar{G}$ by $[\bar{y},\bar{y}'] = \bar{y}\bar{y}' - \bar{y}'\bar{y}$. Writing $\bar{y} = j(y)$ and $\bar{y}' = j(y')$, we use the definition of multiplication in $\bar{B}$ to compute $[j(y),j(y')] = (y,0)(y',0) - (y',0)(y,0) = (0+0+f(y)y',0) - (0+0+f(y')y,0) = j(f(y)y'-f(y')y) = j([y,y']_G)$. This result implies not only that G with Lie product $[\ ,\ ]_G$ is a Lie algebra, but also that $j|_G:G \longrightarrow \bar{G}$ is a morphism of Lie algebras. Since $\bar{f}'j = f'$ and $ji' = \bar{i}'$, necessarily as asserted F_Δ is a singular extension of L by M. Moreover, $j|_G:F \sim (F_J)_\alpha$ and the proposition is proved.

We define $\Delta:E_{L^e\mathfrak{M}}(Q,M) \longrightarrow E_{\mathcal{L}}(Q,M)$ by $\Delta([F]) = [F_\Delta]$. Since $F_\Delta \sim (F_J)_\alpha$ and the functions J and α are well-defined, so is Δ.

<u>Corollary</u>. $\Delta = \alpha J$ and consequently Δ is an injection.

α is an injection by theorem I.1, and $J = \beta^{-1}$ is a bijection by theorem I.2.

The commutative diagram



exhibits these maps.

<u>Lemma</u>. We can define $\alpha_s:E^s_{\mathcal{A}}(L^e,M) \longrightarrow E^s_{\mathcal{L}}(L,M)$ to be the restriction of α.

We suppose that $E: 0 \longrightarrow M \xrightarrow{i} B \underset{u}{\overset{f}{\rightleftarrows}} L^e \longrightarrow 0$ is an R-split singular extension of L^e by M, where $fu = 1_{L^e}$. We suppose

$$F: 0 \longrightarrow M \xrightarrow{i'} G \xrightarrow{f'} L \longrightarrow 0$$

is the representative we constructed of $\alpha([E])$. Let u' be the restriction of u to $L \subset L^e$. Then in fact $\mathrm{Im}(u') \subset G = f^{-1}(L)$ so we can consider $[F] \in E^s_{\mathcal{L}}(LM)$.

It is clear that from the maps $E_{\mathcal{Q}}(L^e,M) \underset{J}{\overset{\beta}{\rightleftarrows}} E_{L^e\mathfrak{M}}(Q,M) \xrightarrow{\Delta} E_{\mathcal{L}}(L,M)$ we can also obtain by restriction to equivalence classes of R-split extensions the maps $E^s_{\mathcal{Q}}(L^e,M) \underset{J_s}{\overset{\beta_s}{\rightleftarrows}} E^s_{L^e\mathfrak{M}}(Q,M) \xrightarrow{\Delta_s} E^s_{\mathcal{L}}(L,M)$.

<u>Proposition I.10.</u> $\Delta_s = \alpha_s J_s$ and $J_s = \beta_s^{-1}$.

As in the proof of the lemma, this is clear from the definitions.

CHAPTER II

COHOMOLOGY AND EXTENSIONS

§1. Definition of $H(V(L),M)$ and of the Relative Cohomologies $\mathrm{Ext}_{\mathcal{E}_0}$ and $\mathrm{Ext}_{\mathcal{E}_1}$

Definition. The exterior algebra $E(L)$ of a Lie algebra L is the quotient algebra $T(L)/\mathcal{J}$, where $\mathcal{J}$ is the ideal in the tensor algebra $T(L)$ generated by elements of the form $x\otimes x$ for $x \in L$.

We write $p:T(L) \longrightarrow E(L)$ for the quotient map. We denote $p(T_n)$ by $\wedge^n L$ or by $L\wedge \dots \wedge L$. In particular we identify $p(T_0) = R$ and $p(T_1)=L$. We denote $p(x_1\otimes \dots \otimes x_n)$ by $x_1\wedge \dots \wedge x_n \in \wedge^n L$ for $x_i \in L$, $i \geq 2$.

Proposition II.1. $x\wedge y = - y\wedge x$ for $x,y \in L$.

This follows from $p((x+y)\otimes(y+x)) = 0$.

Consider $V_n(L) = L^e\otimes\wedge^n L$ as a left L^e-module by defining $a'(a\otimes w) = a'a\otimes w$. We identify $V_0(L) = L^e\otimes R$ with L^e. Let $d_n:V_n(L) \longrightarrow V_{n-1}(L)$ be the morphism of L^e-modules defined on the generators of $V_n(L)$ by

$$d_n(a\otimes x_1\wedge \dots \wedge x_n) = \sum_{i=1}^{n} (-1)^{i+1} ax_i\otimes x_1\wedge \dots \wedge \check{x}_i\wedge \dots \wedge x_n$$
$$+ \sum_{1\leq i<j\leq n} (-1)^{i+j} a\otimes[x_i,x_j]\wedge x_1\wedge \dots \wedge \check{x}_i\wedge \dots \wedge \check{x}_j\wedge \dots \wedge x_n, \quad \text{for } n \geq 2.$$

For $n = 1$, omitting the second summation, we define $d_1:L^e\otimes L \longrightarrow L^e$.

on generators by $d_1(a\otimes x) = ax \in Q \subset L^e$, where Q is the augmentation ideal.

<u>Definition</u>. We define $V(L)$ to be the L^e-complex

$$\cdots \longrightarrow V_n(L) \xrightarrow{d_n} \cdots \xrightarrow{d_1} L^e \longrightarrow 0.$$

We observe that if we interpret $V(L)$ as $L^e\otimes E(L)$, we can define an R-algebra structure for $V(L)$. We denote an element x of L by $\bar{x}$ when we consider $L = p(T_1)$ as a subset of $E(L)$. For $y \in L$, considered as a subset of L^e, define a multiplication in $V(L)$ by $y\bar{x} = y\otimes\bar{x}$ and $\bar{x}y = y\bar{x} + \overline{[x,y]}$.

We further define a derivation $d{:}V(L) \longrightarrow V(L)$. For $\bar{x} \in L \subset E(L)$, let $d(\bar{x}) = x$ and for $y \in L \subset L^e$ let $d(y) = 0$. Extending d as a derivation to the algebra $V(L)$, it can be shown that the restriction of d to $V_n(L)$ is d_n as defined above.

Returning to the definition of $V(L)$ as an L^e-complex, denote as usual its n^{th} homology $\mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1})$ by $H_n(V(L))$.

<u>Proposition II.2</u>. $H_1(V(L)) = 0$.

Consider the diagram

$$\begin{array}{ccccccc} & \longrightarrow & T(L)\otimes L & \xrightarrow{t} & T(L) & & \\ & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle q} & & \\ \cdots & \xrightarrow{d_2} & L^e\otimes L & \xrightarrow{d_1} & L^e & \longrightarrow & 0, \end{array}$$

in which q is the quotient map and $s = q\otimes 1_L$. Given any $v = \Sigma a_i\otimes x_i \in T(L)\otimes L$, we define $t(v) = \Sigma a_i x_i \in T(L)$. Since the formation of the tensor product defines the multiplicative operation in $T(L)$, it follows that t is a monomorphism of R-modules. Since clearly $d_1 s = qt$, we obtain $d_1^{-1}(0) = s\,t^{-1}q^{-1}(0)$. We recall that $q^{-1}(0)$ is the ideal I generated by elements of the form $x\otimes y - y\otimes x - [x,y]$ where $x,y \in L \subset T(L)$. Therefore, given any $\bar{v} \in \mathrm{Ker}(d_1)$ we have $\bar{v} = s(v)$ where $v \in t^{-1}(I)$. Write $t(v) = \Sigma w_i$ with $w_i = u_i(x_i y_i - y_i x_i - [x_i, y_i])v_i \in I$. Write $v_i = r_i + \bar{v}_i \in R + Q = T(L)$, where Q is the augmentation ideal of T(L). Since $s\,t^{-1}(\bar{v}_i) \in L^e \otimes L$, we can conclude that $d_2\{r_i q(u_i)\otimes x_i \wedge y_i\} = s\,t^{-1}(w_i)$. We have shown that $\bar{v} = \Sigma s\,t^{-1}(w_i) \in \mathrm{Im}(d_2)$ and we conclude that $H_1(V(L)) = 0$.

If $M \in {}_{L^e}\mathfrak{M}$, let $\hom_{L^e\mathfrak{M}}(V(L),M)$ be the complex

$$\cdots \longleftarrow \hom_{L^e\mathfrak{M}}(V_{n+1}(L),M) \xleftarrow{\delta^n} \hom_{L^e\mathfrak{M}}(V_n(L),M) \xleftarrow{\delta^{n-1}} \cdots$$

with $\delta^n(g_n) = g_n d_{n+1}$ for a morphism $g_n : V_n(L) \longrightarrow M$ of left L^e-modules.

<u>Definition</u>. $H^n(V(L),M)$ is the n^{th} cohomology of $\hom_{L^e\mathfrak{M}}(V(L),M)$, namely $\mathrm{Ker}\,\delta^n/\mathrm{Im}\,\delta^{n-1}$.

We now recall definitions and certain properties of a relative cohomology theory which we shall need in this paper. Consider any $N \in {}_A\mathfrak{M}$, the category of left A-modules for an augmented algebra A. Let $\mathcal{E}$ be a projective class of sequences in ${}_A\mathfrak{M}$. We know from [3], page 6, Proposition 3.1, that there is a complex $P^*: \cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_o \longrightarrow 0$ and a morphism $\epsilon : P_o \longrightarrow N$ such that each P_i is an $\mathcal{E}$-projective module, each sequence $P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1}$ is in $\mathcal{E}$, and $P_1 \xrightarrow{d_1} P_o \xrightarrow{\epsilon} N$ is in $\mathcal{E}$. Let $\hom_{A\mathfrak{M}}(P^*,M)$ be the complex

$$\cdots \longleftarrow \hom_{A\mathfrak{M}}(P_{n+1},M) \xleftarrow{\delta^n} \hom_{A\mathfrak{M}}(P_n,M) \xleftarrow{\delta^{n-1}} \cdots$$

with $\delta^n(g_n) = g_n d_{n+1}$ for $g_n : P_n \longrightarrow M \in {}_A\mathfrak{M}$.

<u>Definition</u>. $\mathrm{Ext}^n_{\mathcal{E}}(N,M)$ is the n^{th} cohomology of $\hom_{A\mathfrak{M}}(P^*,M)$, namely $\mathrm{Ker}(\delta^n)/\mathrm{Im}(\delta^{n-1})$.

It follows from the general theory ([3], p. 7) that (up to isomorphism) this definition is independent of the $\mathcal{E}$-projective resolution P^* chosen for N.

We shall consider two particular classes of sequences, $\mathcal{E}_0$ and $\mathcal{E}_1$, in ${}_A\mathfrak{M}$.

<u>Definition</u>. $\mathcal{E}_0$ is the class of all R-split exact sequences in ${}_A\mathfrak{M}$.

<u>Definition</u>. $\mathcal{E}_1$ is the class of all exact sequences in ${}_A\mathfrak{M}$.

We recall that $\mathcal{E}_0$ and $\mathcal{E}_1$ are projective classes in ${}_A\mathfrak{M}$. We shall apply the notion of $\mathrm{Ext}^2_{\mathcal{E}}(N,M)$ with $N = R$, the underlying ring considered an A-bimodule by "pull-back" along the augmentation $\epsilon : A \longrightarrow R$. We shall use the "adjoint isomorphisms" $\hom_{{}_A\mathfrak{M}}(A\otimes C, M) \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \hom_{\mathfrak{M}}(C,M)$ defined by $\psi(g)(c) = g(e\otimes c)$ and $\varphi(g')(e\otimes c) = g'(c)$, where e is the identity element of A.

Denote the n-fold tensor product of $A/\eta(R)$ with itself by Q^{*n}. Let $B^*_n = A\otimes Q^{*n}\otimes R$, with a left A-module structure given by the algebra multiplication in the left component A, as above. Denote a generator $a\otimes\bar{x}_1\otimes \dots \otimes\bar{x}_n\otimes r \in B^*_n$ by $a(x_1,\dots,x_n)r$. Define a morphism of A-modules $d^*_n : B^*_n \longrightarrow B^*_{n-1}$ on generators by $d^*_n(a(x_1,\dots,x_n)r) = ax_1(x_2,\dots,x_n)r + \sum_{i=1}^{n-1} (-1)^i a(x_1,\dots,x_ix_{i+1},\dots,x_n)r + (-1)^n a(x_1,\dots,x_{n-1})x_nr$. Let B^* be the A-complex $\dots \longrightarrow B^*_n \xrightarrow{d^*_n} \dots \xrightarrow{d^*_1} B^*_0 = A\otimes R \longrightarrow 0$. With $\epsilon^* : A\otimes R \longrightarrow R$ induced by ϵ, B^* has been shown [9] to be a canonical $\mathcal{E}_0$-resolution of R.

For computation we replace B^* by a simpler complex of A-modules, $B(A,R)$: $\dots \longrightarrow B_n \xrightarrow{d_n} \dots \xrightarrow{d_1} A \longrightarrow 0$, defined as follows. Let Q^n be the n-fold tensor product of Q with itself and let $B_n = A\otimes Q^n$.

Define $d_n : B_n \longrightarrow B_{n-1}$ by $d_n(a(x_1,\ldots,x_n)) = ax_1(x_2,\ldots,x_n)$ $+ \sum_{i=1}^{n-1} (-1)^i a(x_1,\ldots,x_i x_{i+1},\ldots,x_n)$ on generators, with $a(x_1,\ldots,x_n)$ an abbreviation for $a \otimes x_1 \otimes \ldots \otimes x_n$. $B(A,R)$ with augmentation $\epsilon : A \longrightarrow R$ is often called the "bar resolution" of R.

Proposition II.4. With the augmentation $\epsilon : A \longrightarrow R$, $B(A,R)$ is an $\mathcal{E}_0$-resolution of R.

As in [9], we can consider $B(A,R)$ derived from the canonical resolution B^*.

We mention that the $\mathcal{E}_1$ -projective modules are the classical projective modules in ${}_A\mathfrak{M}$. For any R-module B, let F_B denote the free R-module with base B. To construct inductively a complex

$$X_R : \ldots \longrightarrow X_n \xrightarrow{d_n} \ldots \xrightarrow{d_1} X_0 \longrightarrow 0,$$

if given $X_n \xrightarrow{d_n} X_{n-1}$, let K_n be any set of generators for $Ker(d_n)$ as an R-module. (In particular we can fix $K_n = Ker(d_n)$.) Let $\bar{X}_{n+1} = F_{K_n} \in \mathfrak{M}$ and define $\bar{d}_{n+1} : \bar{X}_{n+1} \longrightarrow X_n$ by $d_{n+1}(e_k) = k$ for any $k \in K_n$, extending to the free R-module by R-linearity. Let $X_{n+1} = A \otimes \bar{X}_{n+1}$ and define $d_{n+1} = \varphi \bar{d}_{n+1}$. Setting $X_0 = A$, we complete the inductive definition by (temporarily for this induction) denoting the augmentation $\epsilon : A \longrightarrow R$ by d_0 and R by X_{-1}.

Proposition II.5. With $\epsilon : A \longrightarrow R$, X_R is an $\tilde{\mathcal{E}}_1$ -projective resolution of R.

By construction each triple is an exact sequence in ${}_A\mathfrak{M}$. $X_0 = A$ is a free A-module, and hence projective. Since, for $n \geq 1$, $\bar{X}_n$ is a free R-module it follows that $X_n = A \otimes \bar{X}_n$ is a projective A-module.

§2. The Bijection $\Psi: \mathrm{Ext}^2_{\mathcal{E}_0}(R,M) \longrightarrow E^s_{\mathcal{A}}(A,M)$ of the Second $\mathcal{E}_0$ Cohomology of $R \in {}_A\mathfrak{M}$ Onto the R-Split Classes in $E_{\mathcal{A}}(A,M)$

We considered in seminar [9] the diagram

$$\begin{array}{ccccccc}
\cdots \longleftarrow & \hom_{{}_A\mathfrak{M}}(B_3,M) & \xleftarrow{\delta^2} & \hom_{{}_A\mathfrak{M}}(B_2,M) & \longleftarrow \cdots \\
& \psi \big\downarrow & & \big\uparrow \varphi & \\
\cdots \longleftarrow & \hom_{\mathfrak{M}}(Q^3,M) & \xleftarrow{\bar{\delta}^2} & \hom_{\mathfrak{M}}(Q^2,M) & \longleftarrow \cdots
\end{array}$$

in which $\hom_{{}_A\mathfrak{M}}(B(A,R),M)$ is related to a complex of R-modules in the bottom row by the "adjoint isomorphisms" of §1. In fact we define $\bar{\delta}^n$ to be $\psi\delta^n\varphi$.

<u>Proposition II.6</u>. $g \in \hom_{{}_A\mathfrak{M}}(B_2,M)$ is a 2-cocycle if and only if $x_1\psi g(x_2 \otimes x_3) + \psi g(x_1 \otimes x_2x_3) = \psi g(x_1x_2 \otimes x_3)$ for any $x_1,x_2,x_3 \in Q$.

Suppose $\delta^2(g) = 0$. In particular, we use the definition of d_3 to compute $0 = \delta^2(g)(e \otimes x_1 \otimes x_2 \otimes x_3) = g(d_3(e \otimes x_1 \otimes x_2 \otimes x_3)) = x_1 g(e \otimes x_2 \otimes x_3)$ $- g(e \otimes x_1x_2 \otimes x_3) + g(e \otimes x_1 \otimes x_2x_3) = x_1 g(x_2 \otimes x_3) - g(x_1x_2 \otimes x_3) + g(x_1 \otimes x_2x_3)$, as required. Conversely if the condition holds, the computation shows that $0 = \delta^2(g)(e \otimes x_1 \otimes x_2 \otimes x_3)$. From the definition of left A-module structure for B_3 we conclude that $\delta^2(g)$ is the zero function $B_3 \longrightarrow M$, as asserted.

Since A is the direct sum $R + Q$, we can define $g' \in \hom_{\mathfrak{M}}(A \otimes A, M)$ as follows. Let e denote the identity element of A. For any $x,x' \in Q$ define $g'(x \otimes x') = \psi g(x \otimes x')$ and let $g'(e \otimes e) = g'(e \otimes x) = g'(x \otimes e) = 0$.

Let B_g as an R-module be the direct sum $M + A$. Let E_g denote the sequence $0 \longrightarrow M \xrightarrow{i} B_g \xrightarrow{f} A \longrightarrow 0$ where $i(m) = (m,0) \in B_g$ and $f(m,a) = a \in A$. Clearly E_g is exact. Define a multiplication in B_g by

$$(m,a)(m',a') = (am' + m\epsilon(a') + g'(a\otimes a'),aa').$$

This multiplication distributes over addition, and $(0,e)$ is a two sided identity. Also $f(0,e) = e$ and $f((m,a)(m',a')) = aa' = f(m,a)f(m',a')$. We next verify the conditions of Proposition I.8. If $f(b) = a$ necessarily $b = (m,a)$. Then $bi(m') = (am'+0,0) = i(am')$ and $i(m')b = (0+m'\epsilon(a),0) = i(m'\epsilon(a))$. Thus E_g is a singular extension of A by M if the product in B is associative.

To consider associativity, let $b_i = (m_i,a_i) \in B_g$. After computing $b_1(b_2b_3)$ and $(b_1b_2)b_3$, we see the two are equal in case

$$a_1g'(a_2\otimes a_3) + g'(a_1\otimes a_2a_3) = g'(a_1\otimes a_2)\epsilon(a_3) + g'(a_1a_2\otimes a_3).$$

Writing a_i as $r_i + x_i \in R + Q$, this condition is equivalent to

$$x_1\psi g(x_2\otimes x_3) + \psi g(x_1\otimes x_2x_3) = \psi g(x_1x_2\otimes x_3).$$

By Proposition II.6, if g is a 2-cocycle then the product in B_g is associative. Therefore, E_g determines a class in $E_{\mathcal{A}}(A,M)$. Let $u:A \longrightarrow B_g$ be defined by $u(a) = (0,a)$. Since $fu = 1_A$ we see that E_g is R-split.

<u>Definition</u>. $\Psi:\mathrm{Ext}^2_{\mathcal{E}_0}(R,M) \longrightarrow E^s_{\mathcal{A}}(A,M)$ is defined by $\Psi([g]) = [E_g]$, with E_g constructed from the cocycle g as above.

We show Ψ is well-defined. Given $g - g^* = \delta^1(h)$, define $h' \in \hom_{\mathfrak{M}}(A,M)$ by $h'(x) = \psi h(x)$ for $x \in Q$ and $h'(e) = 0$. Define $k:B_g \longrightarrow B_{g^*}$ by $k(m,a) = (m+h'(a),a)$. We wish to show that $k:E_g \sim E_{g^*}$. Evidently $k(0,e) = (0,e)$. Writing $b = (m,a)$ and $b' = (m',a')$, we consider $k(bb')$ and $k(b)k(b')$. By definition,

$k(bb') = (am'+m\epsilon(a')+g'(a\otimes a') + h'(aa'),aa')$, while $k(b)k(b')$ $= (a(m'+h'(a')) + (m+h'(a))\epsilon(a') + g^{*\prime}(a\otimes a'),aa')$. We consider three cases. Suppose first that $a = x \in Q$ and $a' = x' \in Q$. Since $g - g^* = \delta^1 h$, we obtain $(\psi g-\psi g^*)(x\otimes x') = x\psi h(x') - \psi h(xx')$. Since $\epsilon(x') = 0$, we can conclude that $k(bb') = k(b)k(b')$. For the second case, suppose that $a = r \in R$ and $a' \in A$. Then $(g'-g^{*\prime})(r\otimes a') = 0 = rh'(a') - h'(ra')$. Since $h'(r)\epsilon(a') = 0$, again we can conclude that $k(bb') = k(b)k(b')$. Finally, suppose that $a = x \in Q$ and $a' = r' \in R$. Again $(g'-g^{*\prime})(x\otimes r') = 0$. But now $xh'(r') - h'(xr') + h'(x)\epsilon(r') = 0 - h'(xr) + h'(x)r = 0$. Therefore in this third case, we also conclude that $k(bb') = k(b)k(b')$. Since clearly k commutes as desired ($ki=i^*$ and $f^*k=f$), we have shown that $k:E_g \sim E_{g^*}$. Thus Ψ is well-defined.

<u>Theorem II.1.</u> $\Psi:\mathrm{Ext}^2_{\mathcal{E}_0}(R,M) \longrightarrow E^S_{\mathcal{A}}(A,M)$ is a bijection.

<u>Proof</u>. We define $\zeta:E^S_{\mathcal{A}}(A,M) \longrightarrow \mathrm{Ext}^2_{\mathcal{E}_0}(R,M)$ and show that $\zeta\Psi$ and $\Psi\zeta$ are identity maps.

Let $E:0 \longrightarrow M \xrightarrow{i} B \underset{u}{\overset{f}{\rightleftarrows}} A \longrightarrow 0$ be an R-split singular extension of A by M. Since $u(a)u(a') - u(aa') \in \mathrm{Ker}(f)$, we can define a morphism of R-modules $g:Q\otimes Q \longrightarrow M$ by $g(x\otimes x') = i^{-1}(u(x)u(x') - u(xx'))$. We compute $u(x_1)\{u(x_2)u(x_3)\} = u(x_1)\{ig(x_2\otimes x_3) + u(x_2x_3)\} = i(x_1g(x_2\otimes x_3)) + ig(x_1\otimes x_2x_3) + u(x_1x_2x_3)$ because $u(x_1)i(m) = i(x_1m)$. Likewise $\{u(x_1)u(x_2)\}u(x_3) = \{ig(x_1\otimes x_2) + u(x_1x_2)\}u(x_3) = 0 + ig(x_1x_2\otimes x_3) + u(x_1x_2x_3)$ because $i(m)u(x_3) = i(m\epsilon(x_3)) = 0$. Since the products in the algebras are associative and i is a monomorphism, we deduce that $g(x_1x_2\otimes x_3) = x_1g(x_2\otimes x_3) + g(x_1\otimes x_2x_3)$. By Proposition II.6, $\varphi(g)$ is a 2-cocycle. To recall the construction, we write $g = g^u_E$, and we define

$\zeta([E]) = [\varphi(g_E^u)] \in \mathrm{Ext}^2_{\mathcal{E}_0}(R,M)$.

We show ζ is well-defined. Suppose $k:E^* \sim E$, where E^* is the sequence $0 \longrightarrow M \xrightarrow{i^*} B^* \underset{u^*}{\overset{f^*}{\rightleftarrows}} A \longrightarrow 0$. We need to show that $g_{E^*}^{u^*} \sim g_E^u$. We are given that $f(ku^*) = f^*u^* = 1_A$. We conclude that $g_E^{ku^*} = g_{E^*}^{u^*}$ because $i = ki^*$ implies that $i^{-1}\{ku^*(x)ku^*(x')-ku^*(xx')\} = i^{*-1}\{u^*(x)u^*(x')-u^*(xx')\} \in M$. Writing $u' = ku^*$ for simplicity, it will therefore suffice to show that $g_E^{u'} \sim g_E^u$. Since $(u-u')(x) \in \mathrm{Ker}(f) = \mathrm{Im}(i)$, write $h(x)$ for the unique element in M such that $ih(x) = (u-u')(x)$. Clearly h may be considered in $\mathrm{hom}_{\mathfrak{M}}(Q\otimes Q,M)$. We compute, for $x,x' \in Q$,

$i(g_E^u-g_E^{u'})(x\otimes x') = \{u(x)u(x')-u(xx')\} - \{u'(x)u'(x')-u'(xx')\}$

$= u(x)\{u(x')-u'(x')\} - \{u(x)-u'(x)\}u'(x') - ih(xx')$

$= i(xh(x')) + i(h(x)\varepsilon(x')) - ih(xx')$. Since $\varepsilon(x') = 0$ and $\overline{\delta}^1(h)(x\otimes x')$ $= h(d_2(e\otimes x\otimes x')) = xh(x') - h(xx')$, we conclude that $g_E^u - g_E^{u'} = \overline{\delta}^1(h)$. Thus ζ is well-defined. We shall henceforth write g^u instead of g_E^u.

We show $\zeta\Psi$ is the identity map on $\mathrm{Ext}^2_{\mathcal{E}_0}(R,M)$. We defined $\Psi([g]) = [E_g]$, where $E_g:0 \longrightarrow M \xrightarrow{i} B_g \underset{u}{\overset{f}{\rightleftarrows}} A \longrightarrow 0$. Then we defined $\zeta([E_g]) = [\varphi g^u]$ with $ig^u(x\otimes x') = u(x)u(x') - u(xx')$ for any $x,x' \in Q$. Since $u(x)u(x') - u(xx') = (0+0+\psi g(x\otimes x'),xx') - (0,xx') = i\psi g(x\otimes x')$, we can conclude that $g^u = \psi g$ or $g = \varphi g^u$. All the more, $\zeta\Psi$ is the identity map.

Finally consider $\Psi\zeta$ defined on $E^s_{\mathcal{Q}}(A,M)$. Given $E:0 \longrightarrow M \xrightarrow{i} B \underset{u}{\overset{f}{\rightleftarrows}} A \longrightarrow 0$, we defined $\zeta([E]) = [\varphi g^u]$ with $ig^u(x\otimes x')$ $= u(x)u(x') - u(xx')$. Then $\Psi([\varphi g^u]) = [F]$ where

$$F:0 \longrightarrow M \xrightarrow{i^u} B^u \xrightarrow{f^u} A \longrightarrow 0.$$

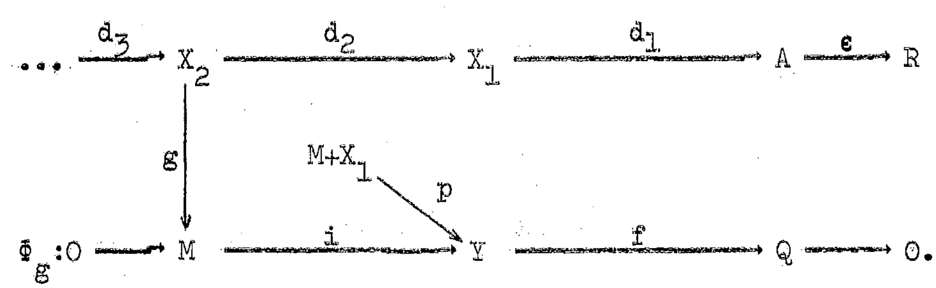
The product in B^u is given by $(m,a)(m',a') = (am'+m\varepsilon(a') + (g^u)'(a\otimes a'), aa')$ with $(g^u)'\big|_{Q\otimes Q} = g^u$, otherwise zero.

Since ζ was shown above to be independent of the choice of "right inverse" for f, we assume u preserves the identity element. To see that this is possible, suppose $u(e_A) = b_o \neq e_B$. Define $m_o = i^{-1}(b_o - e_B)$, and let $u^*(a) = u(a) - i(am_o)$. Then $fu^*(a) = a - 0$ and $u^*(e_A) = u(e_A) - (b_o - e_B) = e_B$. That is, u^* is a right inverse for f which preserves the identity element.

Define $k: B^u \longrightarrow B$ by $k(m,a) = i(m) + u(a)$. Then $k(0,e_A) = u(e_A) = e_B$. The required commutativity ($i = ki^u$ and $fk = f^u$) is evident. Finally we compare $k(bb')$ and $k(b)k(b')$. For $b = (m,a)$ and $b' = (m',a')$, we have $k(b)k(b') = i(m)i(m') + u(a)i(m') + i(m)u(a') + u(a)u(a')$ and $k(bb') = k(am' + m\epsilon(a') + (g^u)'(a \otimes a'), aa') = i(am') + i(m\epsilon(a')) + i(g^u)'(a \otimes a') + u(aa')$. Thus k is a morphism of algebras in case $i(g^u)'(a \otimes a') + u(aa') = u(a)u(a')$. If a and a' belong to Q, $(g^u)' = g^u$, and the equality holds. If either a or a' is in R, $(g^u)'(a \otimes a') = 0$ and $u(aa') = u(a)u(a')$ because $u(e_A) = e_B$. This completes the demonstration that $k: F \sim E$, and we conclude that $\Psi\mathfrak{F}$ is the identity function on $E^s\mathcal{Q}(A,M)$. Thus theorem II.1 is proved and $\mathfrak{F} = \Psi^{-1}$.

§3. The Bijection $\Phi: \mathrm{Ext}^2_{\mathcal{E}_1}(R,M) \longrightarrow E_A\mathfrak{M}(Q,M)$

We want to define $\Phi: \mathrm{Ext}^2_{\mathcal{E}_1}(R,M) \longrightarrow E_A\mathfrak{M}(Q,M)$. For a given cocycle g we construct a sequence Φ_g as in the diagram



Define $I = \{(g(w), -d_2(w));\ w \in X_2\}$ and let Y be the quotient of the direct sum of A-modules $M + X_1$ by its left A-submodule I. Let $p:M + X_1 \longrightarrow Y$ be the quotient map. Define $i:M \longrightarrow Y$ by $i(m) = p(m,0)$. We want to define $f(p(m,v)) = d_1(v)$ where $v \in X_1$. To see that this is possible, suppose $p(m,v) = p(m',v')$. Then for some $w \in X_2$ we have $d_2(w) = v - v'$, and thus $0 = d_1(d_2(w)) = d_1(v) - d_1(v')$. Clearly i and f are morphisms of A-modules. Since $\mathrm{Im}(d_1) = Q$, f is surjective. Evidently $\mathrm{Im}(i) \subset \mathrm{Ker}(f)$. On the other hand if $y = p(m,v) \in \mathrm{Ker}(f)$, then $0 = f(y) = d_1(v)$. Since $\mathrm{Ker}\, d_1 = \mathrm{Im}\, d_2$, $v = d_2(w)$ for some $w \in X_2$. It follows that $i(m+g(w)) = p(m,0) + p(g(w),0) = p(m,d_2(w)) = y$ and consequently $\mathrm{Im}(i) \supset \mathrm{Ker}(f)$. To conclude that Φ_g is an exact sequence in ${}_A\mathfrak{M}$, it remains only to show that i is injective. If $i(m) = 0 \in Y$ then $(m,0) = (g(w),-d_2(w))$ for some $w \in X_2$. This implies that $m = g(w)$ and $d_2(w) = 0$. Thus $w = d_3(x)$ for some $x \in X_3$. Since g is a 2-cocycle, $0 = \delta^2(g) = gd_3$ and $m = g(d_3(x)) = 0$. We have shown that i is injective and therefore that $[\Phi_g] \in E_{A\mathfrak{M}}(Q,M)$.

<u>Definition</u>. $\Phi([g]) = [\Phi_g]$.

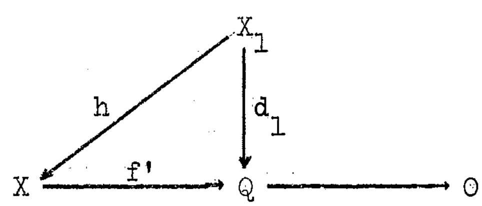
We must show that if $g - g^* = \delta^1(h)$ then $\Phi_g \sim \Phi_{g^*}$. Let Y^* be the quotient of $M + X_1$ by $I = \{(g^*(w),-d_2(w)); w \in X_2\}$ and write $p^*:M + X_1 \longrightarrow Y^*$. If $p(m,v) = p(m',v')$ then $g(w) = m - m'$ and $d_2(w) = v' - v$ for some $w \in X_2$. From these conditions we obtain $(m+h(v)) - (m'+h(v')) = g(w) - h(d_2(w)) = g^*(w)$. We have shown that $k:Y \longrightarrow Y^*$ can be well-defined by $k(p(m,v)) = p^*(m+h(v),v)$. Evidently k is a morphism of A-modules. It follows that $k:\Phi_g \sim \Phi_{g^*}$, as required.

<u>Theorem II.2</u>. $\Phi:\mathrm{Ext}^2_{\mathcal{E}_1}(R,M) \longrightarrow E_{A\mathfrak{M}}(Q,M)$ is a bijection.

<u>Proof</u>. We show that Φ is injective and surjective.

We suppose that $\Phi([g]) = \Phi([g^*])$. Then we are given $k:\Phi_g \sim \Phi_{g^*}$, that is $k:Y \longrightarrow Y^*$. Let $j:X_1 \longrightarrow M + X_1$ be the natural injection. Since $f^*(kp-p^*)j = (fp-f^*p^*)j = d_1 - d_1 = 0$, we can define $h:X_1 \longrightarrow M$ by $h = i^{*-1}(kp-p^*)j$. We observe that $i^*(g-g^*) = kig - i^*g^* = kpjd_2 - p^*jd_2 = (kp-p^*)jd_2 = i^*hd_2 = i^*\delta^1(h)$. Since i^* is a monomorphism, we conclude that $g - g^* = \delta^1(h)$ and $[g] = [g^*]$. We have proved that Φ is an injection.

Select any $[E] \in E_{{}_A\mathfrak{M}}(Q,M)$. Then E is an exact sequence in ${}_A\mathfrak{M}$, say $0 \longrightarrow M \xrightarrow{i'} X \xrightarrow{f'} Q \longrightarrow 0$. Consider $d_1:X_1 \longrightarrow Q \subset A$. Since X_1 is a projective module in ${}_A\mathfrak{M}$ and f' is an epimorphism, there is some $h \in \hom_{{}_A\mathfrak{M}}(X_1,X)$ such that the diagram



commutes. Since $f'(hd_2) = d_1d_2 = 0$, we can define $g \in \hom_{{}_A\mathfrak{M}}(X_2,M)$ by $g = i'^{-1}hd_2$. We want to construct $k:\Phi_g \sim E$. For any $p(m,v) \in Y = p(M+X_1)$ define $k(p(m,v)) = i'(m) + h(v)$. To show this is possible suppose $p(m,v) = p(m',v')$. Then $m - m' = g(w)$ and $v' - v = d_2(w)$ for some $w \in X_2$. This implies that $i'(m) + h(v) = i'(m'+g(w)) + h(v'-d_2(w)) = i'(m') + h(v')$ because $i'g = hd_2$. We have defined a morphism of A-modules $k:Y \longrightarrow X$. Clearly $ki = i'$. If $y = p(m,v) \in Y$, we evaluate $f'k(y) = f'(i'(m)+h(v)) = 0 + d_1(v) = f(y)$. Thus also $f'k = f$, which completes the proof that $k:\Phi_g \sim E$. We have demonstrated that $\Phi([g]) = [E]$. This completes the proof of the theorem.

§ 4. The Injection $\mu : E^s_{\mathscr{L}}(L,M) \longrightarrow H^2(V(L),M)$

To define μ, we shall define a cocycle $\mu_F = \tilde{g}$ from a given $[F]$ in $E^s_{\mathscr{L}}(L,M)$. Then we let $\mu[F]$ be the cohomology class determined by μ_F. Writing F as the sequence $0 \longrightarrow M \xrightarrow{i} G \underset{u}{\overset{f}{\rightleftarrows}} L \longrightarrow 0$, define $g \in \hom_{\mathfrak{M}}(L\wedge L,M)$ by $ig(x\wedge x') = [u(x),u(x')] - u([x,x'])$. Define $\tilde{g} \in \hom_{L^e\mathfrak{M}}(V_2(L),M)$ by $\tilde{g}(a\otimes w) = ag(w)$ where $a \in L^e$ and $w \in L\wedge L$.

<u>Lemma</u>. $\tilde{g}$ is a 2-cocycle in $\hom_{L^e\mathfrak{M}}(V(L),M)$.

For a generator $e\otimes z \in V_3(L)$, with $z = x_1\wedge x_2\wedge x_3 \in \wedge^3 L$ and e the identity of L^e, we have $\delta^2(\tilde{g})(e\otimes z) = \tilde{g}(d_3(e\otimes z)) = x_1 g(x_2\wedge x_3) - x_2 g(x_1\wedge x_3) + x_3 g(x_1\wedge x_2) - g([x_1,x_2]\wedge x_3) + g([x_1,x_3]\wedge x_2) - g([x_2,x_3]\wedge x_1)$. Abbreviate $[u(x_i),[u(x_j),u(x_k)]]$ by u_{ijk} and write x_{ijk} for $[[x_i,x_j],x_k]$. Since $i(xm) = [u(x),i(m)]$ for $x \in L$, we obtain $i(\delta^2(\tilde{g}))(e\otimes z)$

$$
\begin{aligned}
= &\left\{u_{123} - [u(x_1),u[x_2,x_3]]\right\} - \left\{u_{213} - [u(x_2),u[x_1,x_3]]\right\} \\
&+ \left\{u_{312} - [u(x_3),u[x_1,x_2]]\right\} - \left\{[u[x_1,x_2],u(x_3)] - u(x_{123})\right\} \\
&+ \left\{[u[x_1,x_3],u(x_2)] - u(x_{132})\right\} - \left\{[u[x_2,x_3],u(x_1)] - (x_{231})\right\}
\end{aligned}
$$

$= (u_{123}+u_{231}+u_{312}) + g(x_{123}+x_{231}+x_{312}) = 0$, by Jacobi's identity. We conclude that $\delta^2(\tilde{g}) = 0$, as asserted.

To show μ is well-defined, suppose given $k : F \sim F^*$ such that the solid arrows in the diagram

$$
\begin{array}{ccccccccc}
F:0 & \longrightarrow & M & \xrightarrow{i} & G & \underset{u}{\overset{f}{\rightleftarrows}} & L & \longrightarrow & 0 \\
 & & \| & & \downarrow k & & \| & & \\
F^*:0 & \longrightarrow & M & \xrightarrow{i^*} & G^* & \underset{u^*}{\overset{f^*}{\rightleftarrows}} & L & \longrightarrow & 0
\end{array}
$$

commute and $k[y_1,y_2] = [ky_1,ky_2]$. Define $h \in \hom_{\mathfrak{M}}(L,M)$ by $i^*h = ku - u^*$ and define $\widetilde{h} \in \hom_{L^e\mathfrak{M}}(V_1(L),M)$ by $\widetilde{h}(a\otimes x) = ah(x)$. We observe that

$i^*(\mu_F - \mu_{F^*})(e\otimes x_1 \wedge x_2)$

$= \{[ku(x_1),ku(x_2)] - ku[x_1,x_2]\} - \{[u^*(x_1),u^*(x_2)] - u^*[x_1,x_2]\}$

$= [ku(x_1),ku(x_2)-u^*(x_2)] - [u^*(x_1)-ku(x_1),u^*(x_2)] - (ku-u^*)[x_1,x_2]$

$= [ku(x_1),i^*h(x_2)] - [u^*(x_2),i^*h(x_1)] - i^*h[x,y]$

$= i^*\{x_1h(x_2)-x_2h(x_1)-h[x,y]\}$, because $[ku(x),i^*(m)] = i^*(m) = [u^*(x),i^*(m)]$.

We conclude that $\mu_F - \mu_{F^*} = \delta^1(\widetilde{h})$, and μ is well-defined.

<u>Theorem II.3.</u> $\mu: E^s{}_{\mathcal{L}}(L,M) \longrightarrow H^2(V(L),M)$ is an injection.

We define $\nu: \mathrm{Im}(\mu) \longrightarrow E^s{}_{\mathcal{L}}(L,M)$ and show that $\nu\mu$ is the identity map. Given $[g] \in H^2(V(L),M)$, define an R-module G^* to be the direct sum $M + L$. Define a bracket operation in G^* by $[(m,x),(m',x')]$ $= (xm'-x'm+g(e\otimes x\wedge x'),[x,x'])$. Define ν_g to be the sequence of R-modules $0 \longrightarrow M \xrightarrow{i^*} G^* \underset{u^*}{\overset{f^*}{\rightleftarrows}} L \longrightarrow 0$ where $i^*(m) = (m,0)$, $f^*(m,x) = x$ and $u^*(x) = (0,x)$. Clearly ν_g is R-split exact. Suppose $f(y) = x \in L$ for $y \in G^*$. Since necessarily $y = (m,x)$, it follows that $[y,i^*(m')]$ $= (xm'-0,0) = i(xm')$. Also with $y' = (m',x')$ we see that $f^*([y,y'])$ $= [x,x'] = [f^*(y),f^*(y')]$.

To be able to define $\nu([g])$ to be the class of the extension ν_g, we must yet show that G is a Lie algebra and that such a definition is independent of the choice of g. We have not yet used the condition that $[g]$ is in the image of μ. Now assuming that $g = \mu_F$, as defined above, we obtain the diagram

$$\begin{array}{ccccccccc}
F:0 & \longrightarrow & M & \xrightarrow{i} & G & \underset{u}{\overset{f}{\rightleftarrows}} & L & \longrightarrow & 0 \\
 & & \| & & \uparrow k & & \| & & \\
\nu_g:0 & \longrightarrow & M & \xrightarrow{i^*} & G^* & \xrightarrow{f^*} & L & \longrightarrow & 0.
\end{array}$$

Define $k:G^* \longrightarrow G$ by $k(m,x) = i(m) + u(x)$. Clearly the diagram commutes, hence by the five-lemma k is an isomorphism of R-modules. Given $y_i = (m_i,x_i) \in G^*$, we observe that $[k(y_2),k(y_1)] = [i(m_1),i(m_2)] + [u(x_1),i(m_2)] - [u(x_2),i(m_1)] + [u(x_1),u(x_2)] = 0 + i(x_1m_2) - i(x_2m_1) + i\mu_F(e\otimes x_1\wedge x_2) + u[x,y] = k[y_1,y_2]$. Since G is a Lie algebra, $i_G:G \longrightarrow (G^e)_L$ is a monomorphism of Lie algebras by Proposition I.3. It follows that $i_Gk:G^* \longrightarrow (G^e)_L$ is also a monomorphism of Lie algebras. This shows that G^* is a Lie algebra, and moreover that $k:\nu_g \sim F$.

To show ν is well-defined, we suppose $g - g^* = \delta^1(h)$ and construct $k:\nu_g \sim \nu_{g^*}$. Define $h' \in \hom_{\mathfrak{M}}(L,M)$ by $h'(x) = h(e\otimes x)$. We are given that $(g-g^*)(e\otimes x_1\wedge x_2) = x_1h'(x_2) - x_2h'(x_1) - h'([x_1,x_2])$. We define $k:G \longrightarrow G^*$ by $k(m,x) = (m+h'(x),x)$. The required commutativity ($f^*k = f$ and $ki = i^*$) is obvious. Writing $y_i = (m_i,x_i) \in G$, we compute

$k([y_1,y_2]) = k(x_1m_2-x_2m_1+g(e\otimes x_1\wedge x_2),[x_1,x_2])$

$= (x_1(m_2+h'(x_2)) - x_2(m_1+h'(x_1)) + g^*(e\otimes x_1\wedge x_2),[x_1,x_2]) = [k(y_1),k(y_2)]$.

We conclude that $k:\nu_g \sim \nu_{g^*}$. While proving that G was a Lie algebra, we demonstrated that given $[F] \in E^s_{\mathcal{L}}(L,M)$ it follows that $k:\nu_{(\mu_F)} \sim F$. Thus $\nu\mu([F]) = [\nu_{(\mu_F)}] = [F]$, and the theorem is proved.

§5. The Injection $\Theta: H^2(V(L),M) \longrightarrow E_{L^e\mathfrak{M}}(Q,M)$, With Assumption $H_2(V(L)) = 0$

Given a cohomology class $[g]$ in $H^2(V(L),M)$, we shall define a sequence Θ_g. We first consider the direct sum $M + V_1(L)$ in ${}_{L^e}\mathfrak{M}$. Let I be the left L^e-submodule $\{(g(w),-d_2(w)); w \in V_2(L)\}$. Let Y be the quotient L^e-module $(M+V_1(L))/I$ and let $p: M + V_1(L) \longrightarrow Y$ be the quotient map. The construction is indicated in the diagram

$$\begin{array}{ccccccc}
\cdots \xrightarrow{d_3} & V_2(L) & \xrightarrow{d_2} & V_1(L) & \xrightarrow{d_1} & Q \subset L^e \\
 & \Big\downarrow g & M+V_1(L) & \searrow p & & \Vert \\
 & M & \xrightarrow{i} & Y & \xrightarrow{f} & Q
\end{array}.$$

Define $i: M \longrightarrow Y$ by $i(m) = p(m,0)$. We want to define $f: Y \longrightarrow Q$ by $fp(m,v) = d_1(v)$. If $(m,v) - (m',v') \in I$ then $v - v' = d_2(w)$ for some $w \in Y_2(L)$. Thus $0 = d_1(d_2(w)) = d_1(v) - d_1(v')$ and f can be well-defined. Clearly $\mathrm{Im}(i) \subset \mathrm{Ker}(f)$ and i and f are morphisms of L^e-modules. To show that f is surjective, choose any $z \in Q$. By the corollary to Proposition 1.5, $z = \Sigma a_i x_i$ for some $a_i \in L^e$ and $x_i \in L$. Denote $v = \Sigma a_i \otimes x_i \in V_1(L)$. It follows that $fp(0,v) = z$, and f is surjective.

Let Θ_g be the sequence $0 \longrightarrow M \xrightarrow{i} Y \xrightarrow{f} Q \longrightarrow 0$. To conclude the demonstration that Θ_g is exact we need to show that $\mathrm{Im}(i) = \mathrm{Ker}(f)$ and that i is an injection. To prove the inclusion, select any $y = p(m,v)$ in the kernel of f. Since $0 = f(y) = d_1(v)$, we can write $v = d_2(w)$ for some $w \in V_2(L)$ by Proposition II.2. Therefore $i(m+g(w)) = p(m,0) + p(0,d_2(w)) = y$ and $\mathrm{Ker}(f) \subset \mathrm{Im}(i)$, as desired. Now suppose $i(m) = p(0,0) \in Y$; we shall show that $m = 0$. We are given that $m = g(w)$

and $0 = d_2(w)$ for some $w \in V_2(L)$. With the assumption that $H_2(V(L)) = 0$, we can write $w = d_3(z)$ for some $z \in V_3(L)$. Since g is a 2-cocycle, $m = g(d_3(z)) = \delta^2(g)(z) = 0$. We have demonstrated that i is injective and consequently that Θ_g is exact.

We show that we can well-define Θ by $\Theta([g]) = [\Theta_g]$. Suppose $g - g^* = \delta^1(h)$, for some $h \in \hom_{{}_L\mathfrak{e}\mathfrak{M}}(V_1(L),M)$. Let Θ_g be the sequence $0 \longrightarrow M \xrightarrow{i^*} Y^* \xrightarrow{f^*} Q \longrightarrow 0$ constructed from g^*. We want to define $k:Y \longrightarrow Y^*$ by $kp(m,v) = p^*(m+h(v),v)$. To see this is possible, suppose $(m,v) - (m',v') \in I$. Then for some $w \in V_2(L), m - m' = g(w)$ and $v - v' = - d_2(w)$. This implies that $(m+h(v),v) - (m'+h(v'),v')$ $= (g(w) - h(d_2(w)),v-v') = (g^*(w),-d_2(w)) \in I^*$. Clearly k, defined in this manner, is a morphism in ${}_L\mathfrak{e}\mathfrak{M}$ and commutes as desired ($ki=i^*$ and $f^*k=i$). Since this shows that $k:\Theta_g \sim \Theta_{g^*}$, we can conclude that Θ is well-defined.

<u>Lemma</u>. $\mathrm{Im}(\Delta\Theta) \subset E^s{}_{\mathcal{L}}(L,M)$.

Given a cohomology class $[g]$ in $H^2(V(L),M)$, let Θ_g denote the top sequence in the diagram

$$\begin{array}{llllllllll} \Theta_g: 0 & \longrightarrow & M & \xrightarrow{\quad i \quad} & Y & \xrightarrow{\quad f \quad} & Q & \longrightarrow & 0 \\ & & \| & & & & & & \\ F_\Delta: 0 & \longrightarrow & M & \xrightarrow{\quad i' \quad} & G' = f^{-1}(L) & \xrightarrow{\quad f' \quad} & L & \longrightarrow & 0. \end{array}$$

Let F_Δ, the bottom row of the diagram, be the representative of $\Delta([\Theta_g])$ which we constructed from Θ_g by restriction. Define a morphism of R-modules $u':L \longrightarrow G'$ by $u'(x) = p(0,e\otimes x)$. Clearly $f'u'$ is the identity function on L, which proves the lemma.

The lemma motivates consideration of commutativity of the diagram

$$\begin{array}{ccc} E^s_{\mathcal{L}}(L,M) & \xrightarrow{\quad\mu\quad} & H^2(V(L),M) \\ & \nwarrow^{\Delta} & \downarrow \Theta \\ & & \operatorname{Im}(\Theta) \subset E_{L^e\mathfrak{M}}(Q,M). \end{array}$$

Given any cohomology class $[g]$ in $H^2(V(L),M)$, we constructed in §4 an R-split exact sequence ν_g. With the assumption $H_2(V(L)) = 0$ we are going to show that we can define $\bar{\nu}: H^2(V(L),M) \longrightarrow E^s_{\mathcal{L}}(L,M)$ by $\bar{\nu}([g]) = \nu_g$. It will suffice to show that $\nu_g \sim F_\Delta$ where F_Δ is constructed from g as in the lemma. The constructions are exhibited in the diagram

$$\begin{array}{lcccccccc} F_\Delta: 0 & \longrightarrow & M & \xrightarrow{i'} & G'=f^{-1}(L) & \xrightarrow{f'} & L & \longrightarrow & 0 \\ & & \| & & \uparrow k & & \| & & \\ \nu_g: 0 & \longrightarrow & M & \xrightarrow{i^*} & G^* & \xrightarrow{f^*} & L & \longrightarrow & 0. \end{array}$$

We shall define an isomorphism $k: G^* \longrightarrow G'$ which respects the bracket operation. Since $i_{G'}: G' \longrightarrow G'^e$ was shown in Chapter I, §4, to be a monomorphism of Lie algebras, it will then follow that $i_{G'}k: G^* \longrightarrow G'^e$ is also a monomorphism of R-modules which preserves the bracket operation. This will show that G^* is a Lie algebra.

The formula $k(m,x) = p(m,e\otimes x)$ defines a morphism of R-modules $k: G \longrightarrow p(M+(e\otimes L)) \subset G'$. Notice that $f'k(m,x) = d_1(e\otimes x) = x = f^*(m,x)$. Since clearly $ki^* = i'$, k is an isomorphism of R-modules by the five-lemma. We recall that the Lie product in G' is defined by $[y_1,y_2] = f'(y_1)y_2 - f'(y_2)y_1$. Denoting $y_i = (m_i,x_i) \in G^*$, we calculate

$[k(y_1),k(y_2)] = f'(k(y_1))k(y_2) - f'k(y_2)k(y_1)$

$= x_1p(m_2,e\otimes x_2) - x_2p(m_1,e\otimes x_1) = p(x_1m_2-x_2m_1 \ , \ x_1\otimes x_2-x_2\otimes x_1)$

$= p(x_1m_2-x_2m_1+g(e\otimes x_1\wedge x_2),e\otimes[x_1,x_2]) = k[y_1,y_2]$. We have shown that k respects the bracket operation and consequently we can conclude that G* is a Lie algebra. Also k gives an equivalence of ν_g with F_Δ. We have explicitly defined $\bar{\nu}:H^2(V(L),M) \longrightarrow E^s_{\mathcal{L}}(L,M)$ such that $\bar{\nu} = \Delta\Theta$ and $\bar{\nu}\big|_{Im(\mu)} = \nu$.

<u>Theorem II.4.</u> $\mu:E^s_{\mathcal{L}}(L,M) \longrightarrow H^2((L),M)$ is a bijection.

<u>Proof.</u> The argument of Theorem II.3 can be used to show that $\bar{\nu}\mu$ is the identity map on $E^s_{\mathcal{L}}(L,M)$. We shall prove that $\mu\bar{\nu}$ is the identity map on $H^2(V(L),M)$. Given a cocycle g we have defined an R-split singular extension of L by M which we denoted by $F:0 \longrightarrow M \longrightarrow G^* \rightleftarrows L \longrightarrow 0$. Since we defined $\mu^*(x) = (0,x) \in G^*$, we obtain $i^*\mu_F(e\otimes x\wedge x')$ $= [u^*(x),u^*(x')] - u^*[x,x'] = (0-0+g(e\otimes x\wedge x'),[x,x']) - u^*[x,x']$ $= i^*g(e\otimes x\wedge x')$. This computation shows that the cocycle μ_F coincides with g. All the more, $\mu\bar{\nu}$ is the identity map on $E^s_{\mathcal{L}}(L,M)$, because $\mu\bar{\nu}([g]) = \mu[F] = [\mu_F] = [g]$.

<u>Corollary.</u> $\Theta:H^2(V(L),M) \longrightarrow E_{L\,e\mathfrak{M}}(Q,M)$ is an injection.

It was shown in the proof of the theorem that $\mu\Delta\Theta$ is the identity map on $H^2(V(L),M)$.

<u>Theorem II.6.</u> $E^s{}_{L\,e\mathfrak{M}}(Q,M) \subset Im(\Theta)$ and $\Theta\mu\Delta\big|_{Im\Theta}$ is the identity function on $Im(\Theta) \subset E_{L\,e\mathfrak{M}}(Q,M)$.

<u>Proof</u>. We suppose given an R-split extension F of Q by M. Let F be the top row in the diagram

$$\begin{array}{ccccccccc}
F:0 & \longrightarrow & M & \xrightarrow{\ i\ } & X & \underset{u}{\overset{f}{\rightleftarrows}} & Q & \longrightarrow & 0 \\
 & & \| & & & & & & \\
F_\Delta:0 & \longrightarrow & M & \xrightarrow{\ i'\ } & G=f^{-1}(L) & \underset{u'}{\overset{f'}{\rightleftarrows}} & L & \longrightarrow & 0 \\
 & & \| & & & & & & \\
E:0 & \longrightarrow & M & \xrightarrow{\ i^*\ } & Y & \xrightarrow{\ f^*\ } & Q & \longrightarrow & 0.
\end{array}$$

We constructed by restriction the R-split singular extension F_Δ of L by M. We defined $\mu([F_\Delta])$ to be $[g]$, where $ig(e\otimes x_1\wedge x_2)$ $= [u'(x_1),u'(x_2)] - u'[x_1,x_2]$. Then we defined $\Theta([g])$ to be the class of the bottom row E of the diagram, where $Y = (M+V_1(L))/\{(g(w),-d_2(w))\}$. We are going to show that E and F are equivalent. Define $k':M + V_1(L) \longrightarrow X$ by $k'(m,a\otimes x) = i(m) + au'(x)$ for $a \in L^e$ and extend by R-linearity. This is possible because u' is a morphism of R-modules. For $w = e\otimes x_1\wedge x_2 \in V_2(L)$, we compute

$$k'(0,d_2(w)) = k'(0,x_1\otimes x_2-x_2\otimes x_1-e\otimes[x_1,x_2])$$

$$= 0+x_1u'(x_2)-x_2u'(x_1) - u'[x_1,x_2].$$

We also compute $k'(g(w),0) = ig(w) + 0 = [u'(x_1),u'(x_2)] - u'[x_1,x_2]$. But in G we defined $[y_1,y_2] = f'(y_1)y_2 - f'(y_2)y_1$. We conclude that $k'(0,d_2(w)) = k'(g(w),0)$. Consequently k' annihilates the L^e submodule $\{(g(w),-d_2(w));\ w \in V_2(L)\}$. Therefore there is a map $k:Y \longrightarrow X$ such that $kp = k'$, where $p:M + V_1(L) \longrightarrow Y$ is the quotient map. We want to show that $k:E \sim F$. Obviously $ki^* = i$ and $fk = f^*$. We verify that k is a morphism of L^e-modules. Writing $y = p(m,a\otimes x)$, for any $a' \in L^e$

we have $k(a'y) = kp(a'm,a'a\otimes x) = i(a'm) + a'au(x) = a'(i(m) + au(x))$ $= a'k(y)$. We conclude that $k:E \sim F$, as asserted. Therefore $\Theta([g])$ $= [E] = [F]$ and $E^S{}_{L}\mathfrak{e}\mathfrak{M}(Q,M) \subset \mathrm{Im}\Theta$.

To prove the second assertion, notice in the above argument we used only the existence of $u' = u\big|_L : L \longrightarrow G$ satisfying $f'u' = 1_L$. The argument did not require that $[F]$ be in $E^S{}_{L}\mathfrak{e}\mathfrak{M}(Q,M)$, but only that F_Δ represent a class in $E^S\mathcal{L}(L,M)$. By the lemma, the image of $\Delta\big|_{\mathrm{Im}\Theta}$ is a subset of $E^S\mathcal{L}(L,M)$. Therefore by the above argument, if F represents any class in $\mathrm{Im}(\Theta)$, then $\Theta\mu\Delta([F]) = \Theta\mu(F_\Delta) = \Theta([g]) = [F]$ and the theorem is proved.

CHAPTER III

EXAMPLES

In this chapter we consider the ring Z of integers as our underlying ring R. In this case, $\mathfrak{M}$ is the category of all abelian groups, and any commutative ring with unity is a Z-algebra. Let Z_2 denote the additive group of integers modulo two. Let L be the direct sum $Z_2 + Z_2$ of two copies of Z_2 with generators x and y, respectively. Define a bilinear mapping of L x L into L by $[x,y] = 0$. Let Q be the ideal generated by x and y in the polynomial ring $Z_2[x,y]$ in x and y with coefficients in Z_2. Let L^e denote the direct sum Z + Q. Since xy = yx in $Z_2[x,y]$, it follows that $i_L : L \longrightarrow (L^e)_L$ is a group monomorphism preserving the bracket operation. Therefore we can conclude that L is a Lie algebra. Clearly L^e may be considered as the enveloping algebra of L. Let M, as an abelian group, be Z_2 with generator m. Define an L-module structure on M by xm = 0 = ym.

<u>Proposition III.1.</u> $\mathrm{Ext}^2_{\mathcal{E}_0}(R,M) = 0$.

In the $\mathcal{E}_0$-cohomology we can consider $g \in \hom_{\mathfrak{M}}(Q\otimes Q,M)$ as a 2-cocycle in case g satisfies $0 = u_1g(u_2\otimes u_3) - g(u_1u_2\otimes u_3) + g(u_1\otimes u_2u_3)$ for $u_i \in Q$. Thus g is a cocycle if and only if $g(u_1u_2\otimes u_3) = g(u_1\otimes u_2u_3)$ for $u_i \in Q$. We can write any element u of Q in the form x^iy^j where i and j are non-negative and $1 \leq i + j$. We define $h \in \hom_{\mathfrak{M}}(Q,M)$ as follows. Define $h(x) = 0$ and $h(y) = 0$. If $2 \leq i$ define $h(x^i) = g(x^{i-1}\otimes x)$, and if $2 \leq j$

define $h(y^j) = g(y^{j-1}\otimes y)$. If $1\leq i$ and $1\leq j$ define $h(x^iy^j) = g(x^{i-1}y^j\otimes x)$.

<u>Lemma</u>. If $1\leq m + r$ and $1\leq n + s$, then $g(x^my^r\otimes x^ny^s) = h(x^{m+n}y^{r+s})$.

Suppose first that $1\leq n$. Then $h(x^{m+n}y^{r+s}) = g(x^{m+n-1}y^{r+s}\otimes x)$ $= g(x^my^r\otimes x^ny^s)$, as required. We consider next the case when $n = 0$ and $m = 0$. Then $h(x^{m+n}y^{r+s}) = h(y^{r+s}) = g(y^{r+s-1}\otimes y) = g(y^r\otimes y^s) = g(x^my^r\otimes x^ny^s)$. The case $1\leq m$ follows like the first case, and the lemma is proved.

Let $u_1 = x^my^r$ and $u_2 = x^ny^s$. Using the lemma, we compute $g(u_1\otimes u_2)$ $= g(x^my^r\otimes x^ny^s) = h(x^{m+n}y^{r+s}) = h(u_1u_2) = u_1h(u_2)-h(u_1u_2)$. This demonstrates that g is the coboundary of h. Since g was an arbitrary cocycle, this completes the proof of the proposition.

<u>Proposition III.2</u>. $H^2(V(L),M) = Z_2$.

If $h \in \hom_{\mathfrak{M}}(L,M)$ then the coboundary of h evaluated at the generator $x\wedge y$ of $L\wedge L$ is $xh(y)-yh(x)-h[x,y] = 0 + 0 + h(0) = 0$. Any g in hom $(L\wedge L,M)$ may be considered as a cocycle because $L\wedge L\wedge L = 0$. In particular, let g be defined by mapping $x\wedge y$ to m. The proposition follows because this g is clearly the only possible nonzero cocycle.

<u>Proposition III.3</u>. For $L = Z_2 + Z_2$, as above, $H_2(V(L)) = 0$.

We consider an arbitrary element w in $L^e\otimes L\wedge L$. We recall that L consists of the four elements $0,x,y$, and $x+y$. Since $x\wedge x = 0$, $y\wedge y = 0$, and $y\wedge x = -x\wedge y$, we can write w as $a\otimes x\wedge y$ for some $a \in L^e$. If w is in the kernel of $d_2: L^e\otimes L\wedge L \longrightarrow L^e\otimes L$, then $0 = d_2(w) = ax\otimes y - ay\otimes x - a\otimes 0$. We have obtained $ay\otimes x = ax\otimes y$ in $L^e\otimes L$. But $L^e\otimes L$ decomposes into the direct sum $L^e\otimes x$ and $L^e\otimes y$. Consequently, $a = 0$ or a has a factor of 2. In either case, $w = 0$ and the proposition follows.

We now construct for computation the portion up to $n = 3$ of an $\mathcal{E}_1$-projective resolution for Z as an L^e-module. For $n \geq 3$ we define the resolution canonically. We shall denote the resolution by

$$P^*: \cdots \longrightarrow P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_o \longrightarrow 0.$$

We let $P_o = L^e$ and we let ϵ be the augmentation of L^e which maps the direct summand Q to zero. We define P_1 to be the direct sum of two copies of $L^e \otimes Z$. Denote the identity elements of these copies of Z by r_1 and r_2 respectively. With e the identity element of L, define $d_1(e\otimes r_1) = x$ and $d_1(e\otimes r_2) = y$. We recall that $d_1(a\otimes r_i) = ad_1(e\otimes r_i)$ for $a \in L^e$. If $u \in Q$ then u is a sum of products $a_{ij}x^i y^j$, where $i+j \geq 1$. We can consider $a_{ij} \in Z$, and we recall that such a product is read modulo two. Then

$$d_1\{(\textstyle\sum_{j=0} a_{ij}x^{i-1})\otimes r_1 + (\sum_{j>0} a_{ij}x^i y^{j-1})\otimes r_2\} = u,$$

hence $\text{Im}(d_1) = \text{Ker } \epsilon$.

Let P_2 be the direct sum of three copies of $L^e \otimes Z$ with identity elements s_1, s_2, and s_3 for the copies of Z. Define $d_2 : P_2 \longrightarrow P_1$ by $d_2(e\otimes s_1) = 2\otimes r_1$, $d_2(e\otimes s_2) = 2\otimes r_2$, $d_2(e\otimes s_3) = y\otimes r_1 - x\otimes r_2$. We see that $d_1 d_2(e\otimes s_1 + e\otimes s_2) = d_1(2\otimes r_1 + 2\otimes r_2) = 2x+2y = 0$. Also, $d_1 d_2(1\otimes s_3) = yx-xy = 0$. To show that conversely $\text{Ker } d_1 \subset \text{Im } d_2$, we decompose P_1 into the direct sum $Z\otimes Z$, $Q\otimes Z$, $Q\otimes Z$, and $Z\otimes Z$ with generators $e\otimes r_1, x^i y^j \otimes r_1$, $x^i y^j \otimes r_2$, and $e\otimes r_2$, respectively. We observe that $d_1(n\otimes r_1) = nx \in L \subset Q$ and $d_1(m\otimes r_2) = my \in L \subset Q$. Decompose Q into the direct sum $L + Q'$, where an element of Q' is of the form $x^i y^j$ with $2 \leq i + j$. We observe that the image of d_1 restricted to $(Q\otimes Z) + (Q\otimes Z)$ lies in Q'. Now an arbitrary element in P_1 is of the form $w = n\otimes r_1 + u\otimes r_1 + v\otimes r_2 + m\otimes r_2$, where u and v belong to Q. Consequently if $d_1(w) = 0 \in P_1$ then

$d_1(n\otimes r_1) = nx = 0$, $d_1(m\otimes r_2) = my = 0$, and $d_1\{u\otimes r_1 + v\otimes r_2\} = ux + vy = 0$. It follows by unique factorization in the polynomial ring that $u = u'y$ and $v = v'x$. We obtain in the polynomial ring $u'yx = -v'xy$. This implies that $u' = -v'$. Define $a = u'(y\otimes r_1 - x\otimes r_2) = u\otimes r_1 + v\otimes r_2$. Since n and m are even, $d_2\{(n/2)\otimes s_1 + u'\otimes s_3 + (m/2)\otimes s_2\} = n\otimes r_1 + u'(y\otimes r_1 - x\otimes r_2) + m\otimes r_2 = w$. We have demonstrated that $\mathrm{Im}(d_2) = \mathrm{Ker}(d_1)$.

Let P_3 be the direct sum of five copies of $L^e\otimes Z$. Denote the identity elements of the copies of Z by t_1, t_2, t_3, t_4, and t_5, respectively. An arbitrary element in P_3 is of the form $w = \sum_{i=1}^{5} (a_i\otimes t_i)$ where $a_i \in L^e$. Define $d_3(a\otimes t_1) = ax\otimes s_1$, $d_3(a\otimes t_2) = ay\otimes s_1$, $d_3(a\otimes t_3) = ax\otimes s_2$, $d_3(a\otimes t_4) = ay\otimes s_2$, and $d_3(a\otimes t_5) = 2a\otimes s_3$. Then $d_2 d_3(w)$

$$= d_2\{(a_1x + a_2y)\otimes s_1 + (a_3x + a_4y)\otimes s_2 + 2a_5\otimes s_3\}$$
$$= 2(a_1x + a_2y)\otimes r_1 + 2(a_3x + a_4y)\otimes r_2 + 2a_5(y\otimes r_1 - x\otimes r_2) = 0 + 0 + 0 = 0.$$

To show that $\mathrm{Ker}(d_2) \subset \mathrm{Im}(d_3)$ we decompose P_2 into six direct summands as follows. The decomposition consists of three pairs $(Z\otimes Z + Q\otimes Z)$ with the identity element in the two right hand components of Z denoted by s_1 in the first pair, s_2 in the second pair, and s_3 in the third. We have $d_2(n\otimes s_1) = 2n\otimes r_1 \in Z\otimes Z$, $d_2(m\otimes s_2) = 2m\otimes r_2 \in Z\otimes Z$, $d_2(u\otimes s_1) = 2u\otimes r_1 = 0$ and $d_2(v\otimes s_2) = 2v\otimes r_2 = 0$ for any $u, v \in Q$. Moreover, $d_2(w\otimes s_3) = w(y\otimes r_1 - x\otimes r_2) \in Q'$ for any $w \in Q$. Finally, $d_2(p\otimes s_3) = p(y\otimes r_1 - x\otimes r_2) \in I\otimes Z + I\otimes Z$, a direct sum. If z is an arbitrary element of P_2, we can write $z = \{n\otimes s_1 + u\otimes s_1\} + \{m\otimes s_2 + v\otimes s_2\} + \{p\otimes s_3 + w\otimes s_3\}$. We have indicated the manner in which direct summands in P_2 map into direct summands in P_1. It follows that if $d_2(z) = 0$ then $2n\otimes r_1 = 0$, $2m\otimes r_2 = 0$, $w(y\otimes r_1 - x\otimes r_2) = 0$, and $p(y\otimes r_1 - x\otimes r_2) = 0$. From the first two conditions, necessarily $n = 0$ and $m = 0$. From the third condition, w must be zero because $wy\otimes r_1$ and $wx\otimes r_2$ lie in different direct summands.

Since $py\otimes r_1$ and $px\otimes r_2$ lie in different direct summands, from the last condition p must be even. Consequently, if z is in the kernel of d_2, then $z = u\otimes s_1 + v\otimes s_2 + 2p'\otimes s_3$. Since u and v are in Q we can write $u = a_1x + a_2y$ and $v = a_3x + a_4y$ for some $a_i \in L^e$. Then $d_3\{\sum_{i=1}^{4} a_i\otimes t_i + p'\otimes t_5\}$ $= \{a_1x\otimes s_1 + a_2y\otimes s_1\} + \{a_3x\otimes s_2 + a_4y\otimes s_2\} + 2p'\otimes s_3 = z$. We have proved that $\mathrm{Im}(d_3) = \mathrm{Ker}(d_2)$.

Since L^e is L^e-projective, each $L^e\otimes Z$ is also L^e-projective. Consequently each of P_1, P_2, and P_3 is L^e-projective, and P^* is an $\mathcal{E}_1$-projective resolution of the L^e-module Z.

<u>Proposition III.4.</u> $\mathrm{Ext}^2_{\mathcal{E}_1}(Z,M) = Z_2 + Z_2 + Z_2$.

Consider any $f \in \hom_{L^e\mathfrak{M}}(P_1,M)$. Observe that $fd_2(e\otimes s_1) = 2f(e\otimes r_1) = 0$, $fd_2(e\otimes s_2) = 2f(e\otimes r_2) = 0$ and $fd_2(e\otimes s_3) = yf(e\otimes r_1) - xf(e\otimes r_2) = 0$. Thus zero is the only coboundary. Consider an arbitrary $g \in \hom_{L^e\mathfrak{M}}(P_2,M)$. Since Q operates trivially on $M = Z_2$, we obtain $gd_3(e\otimes t_1) = g(x\otimes s_1)$ $= xg(e\otimes s_1) = 0$, $gd_3(e\otimes t_2) = yg(e\otimes s_1) = 0$, $gd_3(e\otimes t_3) = xg(e\otimes s_2)$, $gd_3(e\otimes t_4) = yg(e\otimes s_2)$, and $gd_3(e\otimes t_5) = 2g(e\otimes s_3) = 0$. Therefore any morphism of L^e-modules $g:P_2 \longrightarrow M$ is a 2-cocycle. With $\delta_{ij} = 1$ when $i=j$, otherwise zero, define $g_i(e\otimes s_j) = \delta_{ij}m \in M$. Let $h_i(e\otimes s_i) = 0$, otherwise m. We have defined cocycles h_i which satisfy $h_i(e\otimes s_j) = (\delta_{ij}+1)m$. With the usual addition of functions, $h_3 = g_1 + g_2, h_2 = g_1 + g_3$, and $h_1 = g_2 + g_3$. Finally define k by $k(e\otimes t_i) = m$ for all i. We mention that $k = g_1+g_2+g_3$. Explicitly, this set of 2-cocycles $\{0, g_1, g_2, g_3, h_1, h_2, h_3, k\}$ has the additive structure of the direct sum $Z_2 + Z_2 + Z_2$.

<u>Corollary.</u> $E_{\mathcal{L}}(L,M)$ contains at least eight elements.

$\Phi:\mathrm{Ext}^2_{\mathcal{E}_1}(R,M) \longrightarrow E_{L^e\mathfrak{M}}(Q,M)$ is a bijection, and $\Delta:E_{L^e\mathfrak{M}}(Q,M) \longrightarrow E_{\mathcal{L}}(L,M)$

is an injection.

<u>Proposition III.5</u>. There are exactly eight elements in $E_{\mathcal{L}}(L,M)$.

With Ext(,) the classical extension functor on $\mathfrak{M}$, we recall that $\mathrm{Ext}(Z_2+Z_2,Z_2) = \mathrm{Ext}(Z_2,Z_2) + \mathrm{Ext}(Z_2,Z_2) = Z_2 + Z_2$. Hence, as an abelian group, we know L has exactly four classes of extensions by M. For $0 \leq j \leq 3$, we shall explicitly define an exact sequence of abelian groups $0 \longrightarrow M \xrightarrow{i_j} G_j \xrightarrow{f_j} L \longrightarrow 0$, which we denote by E_j. First, let G_o denote the direct sum $Z_2 + Z_2 + Z_2$ with generators a, b, and c, respectively, for the cyclic groups of order two. Define $f_o(a) = x, f_o(b) = y, f_o(c) = 0$, and $i_o(m) = c$. Let $G_1 = G_2 = G_3$ be the direct sum $Z_4 + Z_2$ with generator a for Z_4 and b for Z_2. Let each of i_1, i_2, and i_3 map m to 2a. Define $f_1(a) = x, f_1(b) = y, f_2(a) = y, f_2(b) = x, f_3(a) = x$, and $f_3(b) = x + y$.

We may consider each G_j as a trivial Lie algebra; that is, let the Lie product of any two elements be zero. Clearly each f_j is a morphism of Lie algebras because the Lie product in L is also trivial. The condition of Proposition I.7 is obviously satisfied because both the module operation on M and Lie products in G_j are zero. We therefore can consider each E_j as a singular extension of L by M.

<u>Lemma</u>. The classes $[E_j]$ and $[E_{j'}]$ in $E_{\mathcal{L}}(L,M)$ are distinct unless $j = j'$.

Obviously E_o is not equivalent to E_j for $1 \leq j$ because G_o is not isomorphic as an abelian group with G_j if $1 \leq j$. We consider E_1, E_2, and E_3. First, suppose that there is an equivalence $k: E_1 \sim E_2$. Then $x = f_1(a) = f_2(k(a))$. But a is of order 4 and $f_2^{-1}(x) = \{b, b+2a\}$ consists of elements of order 2. Therefore the classes $[E_1]$ and $[E_2]$ are distinct.

Second, suppose that there is an equivalence $k:E_1 \sim E_3$. Then $y = f_1(b) = f_3(k(b))$. But b is of order 2 and $f_3^{-1}(y) = \{b-a, b+a\}$ consists of elements of order 4. Therefore $[E_1]$ and $[E_3]$ are distinct. Finally, if we assume that $k:E_2 \sim E_3$, then $x = f_2(b) = f_3(k(b))$. But b is of order 2 and $f_3^{-1}(x) = \{a, 3a\}$ consists of elements of order 4. This completes the proof of the lemma.

We are now going to define non-zero Lie products in G_j. We shall let F_j denote the corresponding singular extension of L by M. In G_o define $[a,b] = c$ and $[b,a] = -c = c$. Otherwise let the Lie product be zero. Since $f(c) = 0$, f respects this bracket operation. Moreover this is the only possible non-zero bracket operation such that the condition of Proposition I.7 holds. For example, if the condition holds then necessarily $[a,c] = [a, i_o(m)] = xm = 0$. Clearly the class determined by F_o is distinct from all the E_j.

In G_1, G_2, and G_3 define $[a,b] = 2a$ and $[b,a] = -2a = 2a$, otherwise zero. Up to equivalence, this is again the only definition which can yield singular extensions of L by M. Again since each F_j has a non-zero Lie product, F_j cannot be equivalent to E_j. Since as abelian groups there is no map satisfying the commutatively condition between F_j and $F_{j'}$, necessarily $[F_j]$ and $[F_{j'}]$ are distinct unless $j = j'$. We have exhibited representatives E_j and F_j for $0 \leq j \leq 3$ of the eight distinct classes in $E_{\mathcal{L}}(L,M)$.

For clarity, we shall prove that the natural map i_{G_1} of G_1 into its enveloping algebra is an injection; the argument that this property holds for the other G_j is similar. Suppose that in the tensor algebra $T(G)$ we have $y = \Sigma c_i(a \otimes b - b \otimes a - 2a)d_i$ for some $y \in G$. Notice that any element in the kernel of the quotient map $T(G) \longrightarrow G^e$ can be written in this form. Decompose this summation as

$$y = mn(a\otimes b - b\otimes a - 2a) + \Sigma c'_j(a\otimes b - b\otimes a - 2a)d'_j \quad \ldots \quad (*).$$

We have collected first all terms with both c_i and d_i in Z. Thus at least one of c'_j or d'_j has degree greater than zero. Equate the terms in equation (*) of degree one to conclude that $y = mn(-2a)$. Equate the terms of degree two to obtain

$$0 = mn(a\otimes b - b\otimes a) + \Sigma c'_j(-2a)d'_j \quad \ldots \quad (**).$$

In equation (**), exactly one of the c'_j, d'_j has degree one, the other zero. We observe that $G\otimes G$ is a direct sum with generators $a\otimes a$, $a\otimes b$, $b\otimes a$, and $b\otimes b$. Suppose that y is non-zero. Then $0 \neq mn(-2a)$ implies that mn must be odd. But if $d'_j = b$ then $c'_j(-2a)(d_j) = c'_j(-2a\otimes b)$ $= c'_j(a\otimes -2b) = 0$. Consequently if y is non-zero, we can deduce from equation (**) the contradiction $0 = mn(a\otimes b) + 0 = a\otimes b$. We conclude that if $i_{G_1}(y) = 0 \in G_1^e$, we must have $y = 0$.

SUMMARY AND CONCLUSIONS

For an arbitrary commutative ring R with unity, we construct a bijection of singular extension classes $E_{\mathcal{A}}(A,M)$ of an augmented R-algebra A by an A-module M with extension classes $E_{A}\mathfrak{M}(Q,M)$ of the augmentation ideal Q by M. We give an injection of $E_{\mathcal{A}}(L^e,M)$ into the singular extension classes $E_{\mathcal{L}}(L,M)$ of L by M. Considering R as an A-bimodule, we show that $\mathrm{Ext}^2_{\mathcal{E}_0}(R,M)$ is in one-to-one correspondence with R-split extension classes of $E^s_{A}\mathfrak{M}(Q,M)$. We construct a bijection $\mathrm{Ext}^2_{\mathcal{E}_1}(R,M)$ with $E_{A}\mathfrak{M}(Q,M)$. We show that in general $\mu: E^s_{\mathcal{L}}(L,M) \longrightarrow H^2(V(L),M)$ is an injection. If $H_2(V(L)) = 0$, then μ is a bijection and we can define an injection of $H^2(V(L),M)$ into $E_{A}\mathfrak{M}(Q,M)$. In the diagram

$$\mathrm{Ext}^2_{\mathcal{E}_0}(R,M) \xrightarrow[\Psi]{=} E^s_{\mathcal{A}}(L^e,M) \xrightarrow[\beta]{=} E^s_{L^e}\mathfrak{M}(Q,M) \xrightarrow[\Delta_s]{\subset} E^s_{\mathcal{L}}(L,M)$$

$$\xrightarrow[\mu]{=} H^2(V(L),M) \xrightarrow[\Theta]{\subset} E_{L^e}\mathfrak{M}(Q,M) \xrightarrow[\Delta]{\subset} E_{\mathcal{L}}(L,M) \;,$$

we write " = " above a map to symbolize a bijection, and we write " $\subset$ " to symbolize an injection. We show by example that the $\mathcal{E}_0$,V(L), and $\mathcal{E}_1$ cohomologies are distinct.

Recent developments in homological algebra show strong evidence that $H^*(V(L),M)$ and the cohomology of Dixmier and Shukla could be included within the general framework of relative cohomology theory. It is expected that this problem will be settled by a most recent result of my adviser and my colleagues concerning triple cohomology in relative homological algebra.

BIBLIOGRAPHY

1. Cartan, Henri, and Samuel Eilenberg. Homological Algebra. Princeton: Princeton University Press, 1956.

2. Dixmier, J. "Homologie des Anneaux de Lie." Ann. Sci. École Norm. Sup. Vol. 74. (1957) 25-83.

3. Eilenberg, Samuel, and John C. Moore. "Foundations of Relative Homological Algebra." Mem. Amer. Math. Soc., No. 55. (1965).

4. MacLane, Saunders. "Categorical Algebra." Bul. of the Amer. Math. Soc. Vol. 71. (1965) 40-106.

5. MacLane, Saunders. Homology. Berlin: Springer-Verlag, 1963.

6. Milnor, John W., and John C. Moore. "On the Structure of Hopf Algebras." Ann. Math. Vol. 81. (1965) 211-264.

7. Shukla, U. "A Cohomology for Lie Algebras." Jour. Math. Soc. Japan. Vol. 18. (1966) 275-289.

8. Shukla, U. "Cohomologie des Algèbres Associative." Ann. Sci. École Norm. Sup. t. 78. (1961) 163-209.

9. Uehara, Hiroshi. Seminar Notes, Fall 1966.

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