A STUDY OF ORBITS AND TRANSITIVE CONTINUOUS FUNCTIONS

By

W. DAVID MOON

Bachelor of Science
Arkansas State Teachers College
Conway, Arkansas
1958

Master of Arts
University of Tennessee
Knoxville, Tennessee
1962

Submitted to the faculty of
the Graduate College of
the Oklahoma State University
in partial fulfillment of the require-
ments for the degree of
DOCTOR OF PHILOSOPHY
July, 1967

A STUDY OF ORBITS AND TRANSITIVE

CONTINUOUS FUNCTIONS

Thesis Approved:

R K Deal
Thesis Adviser

Hiroshi Uehara

L. Wayne Johnson

Herbert A. Pohl

N. N. Durham
Dean of the Graduate College

PREFACE

This paper will be concerned with results related to the orbit of a point and the orbit of a component under a continuous transformation. Chapter I is an introductory chapter where the definitions of orbit and related terms are given. The fundamental properties of transitive continuous functions are developed in Chapter II. In a major portion of the paper, it is assumed that the topological spaces are compact, connected, metric spaces. In the chapter, some necessary and sufficient conditions are established for a function f to be transitive at a point x in the topological space. Chapter III is devoted primarily to the construction of transitive continuous functions. Examples of transitive continuous functions on the closed unit interval and on the open unit interval are given.

In Chapter IV a theorem of G. E. Schweigert concerning the limit set of a closed component orbit under a periodic function in a compact metric space will be generalized. The same result is obtained in a compact Hausdorff space with a continuous, componentwise periodic, monotone transformation. The summary of all results is given in Chapter V.

Numbers in brackets refer to the bibliography at the end of the paper. For example, (3, 720) refers to bibliography reference number 3, page 720.

It is a pleasure to acknowledge my gratitude to Professor R. B. Deal for the experience of working with a man who is an outstanding mathematician; to the members of my advisory committee; to Professor O. H. Hamilton who gave me my start in research; Dr. A. Glen Haddock who verified many proofs and examples; to the National Science Foundation for a research grant in the Summer of 1964; and for a Science Faculty Fellowship in 1965-66; and, most of all, to my family; Virginia, Debra and Pat.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

This paper will be devoted to the consideration of certain results in connection with the orbit of a point and the orbit of a component under a continuous transformation. In particular, the theory of transitive continuous functions is developed in Chapter II. In a major portion of the paper it is assumed that the spaces are compact, connected, metric spaces.

The concept of transitive functions is not new. However, in the past it seems that the notion of transitive functions has been reserved for homeomorphisms. The concept of transitive functions is related to what was known as the Ergodic Hypothesis which was introduced about 1879 by two physicists, Ludwig Bolzman and James Maxwell. This hypothesis states that any system will return to its original state after a sufficiently long time. This theory was not very acceptable; however, the Quasi-Ergodic Hypothesis introduced by P. and T. Ehrenpst is probably true for all systems of interest. It states that after sufficiently long time any system will return arbitrarily close to its original state.

Quasi-Ergodic theory aroused considerable interest in measure preserving transformations. Oxtobey generalized the notion to homeomorphisms (4). Schweigert defined a component orbit for a pointwise periodic transformation, and has shown that the limit set of a convergent sequence of closed component orbits is a closed component orbit (5). This result will be generalized in Chapter III.

Some of the definitions and notation that are needed in this paper are the following:

Notation: f will denote a continuous mapping on a topological space X to X. Let

$F = \{g : g = f^n$ for some non-negative integer $n\}$ where by definition $f^o(x) = x$.

Definition 1.1: Let $A \subset F$ and $x \in X$. The A-orbit of x will be

$$\{y : y = g(x),\ g \in A\}$$

and will be denoted by xA.

Definition 1.2: The A-orbit closure of x shall mean the closure of xA and is denoted by $\overline{xA}$.

Definition 1.3: A point $x \in X$ is called a transitive point of X with respect to F (or just transitive) if $\overline{xF} = X$.

Definition 1.4: A subset Y of X is called A-invariant under $A \subset F$ if $Y(A-e) = \bigcup_{g \in (A-e)} g(Y) = Y$, where e is the identity mapping.

Definition 1.5: If there is a point x in X such that x is transitive with respect to F, then we will say that (X,F) is transitive.

Definition 1.6: A mapping f of a topological space X to a topological space Y is interior if and only if the image of each open set is open.

CHAPTER II

PROPERTIES OF TRANSITIVE CONTINUOUS FUNCTIONS

Some examples of functions that have transitive points will be constructed in Chapter III. Verification of that construction will depend upon the theory developed in Chapter II. It will especially use the characterizations of a transitive point with respect to F given in Theorem 2.3 and the Corollary to Theorem 2.3. A characterization of an F-invariant, nonempty, open subset is given in Theorem 2.4 and Theorem 2.5 provided F is generated by an interior transformation, and the relation of an F-invariant open subset which is dense in the space to an F-invariant proper closed subset of the space is given in Theorem 2.8.

Throughout the chapter, examples are given, whenever it seems appropriate, to show that certain conditions of theorems cannot be relaxed and still obtain the same result.

The following theorem gives a characterization of a transitive point in an arbitrary topological space.

<u>Theorem 2.1:</u> Let X be an arbitrary topological space. A point x in X is a transitive point with respect to F if and only if, given any non-empty open subset V of X, there is a $g \in F$ such that $g(x) \in V$.

<u>Proof:</u> This follows immediately from the trivially

equivalent fact that a set A is everywhere dense if and only if every open set contains a point of A.

The next theorem characterizes the set of transitive points in a connected T_1-space.

Theorem 2.2: Let X be a connected T_1 space. If there is a point x that is transitive with respect to F, then there is a dense subset Y of X such that if $y \in Y$, then y is transitive with respect to F. In particular, the set of iterates of a transitive point is an everywhere dense set of transitive points.

Proof: Let $A = \bigcup_{n=0}^{\infty} f^n(x)$ where x is transitive. Since A is dense, then $\overline{A} = X$. Suppose the set of transitive points is not dense. Then some point of A is not transitive. Let k be the smallest positive integer such that $f^k(x)$ is not transitive. Now $k \geq 1$ since $x = f^o(x)$ is transitive. Let $y = f^{k-1}(x)$, then y is a transitive point and f(y) is not. Let $B = \{y\}$ and $C = \bigcup_{n=1}^{\infty} f^n(y)$. Then $\overline{C} \neq X$, in particular since $\overline{\overline{C}} = \overline{C}$, $y \notin \overline{C}$. Since X is a connected T_1-space and B consists of a single point y, $\overline{B} = B \neq X$. However, $\overline{B \cup C} = \overline{\bigcup_{n=0}^{\infty} f^n(y)} = X$ and $\overline{B \cup C} = \overline{B} \cup \overline{C}$ in any topological space. By above $B \cap \overline{C} = \emptyset$ and $B = \overline{B}$, so $\overline{B} \cap \overline{C} = \emptyset$. Hence $X = \overline{B} \cup \overline{C}$, two closed disjoint sets contradicting the connectedness of X.

Remark: From this point on, X will denote a compact, connected metric space unless it is specified otherwise.

A characterization of transitive points in a compact, connected metric space will now be given. This theorem will be used extensively in Chapter III to prove that the examples given there are actually transitive.

Theorem 2.3: The pair (X,F) is transitive if and only if, given two arbitrary non-empty open subsets U and V of X, then $UF \cap V \neq \emptyset$.

Proof: Assume (X,F) is transitive. Let U and V be two arbitrary open subsets of X. Since U is open and $\overline{xF} = X$, there is a point y in xF such that $y \in U$. Thus, by Theorem 2.2, $\overline{yF} = X$ which means that $yF \cap V \neq \emptyset$. Therefore, there is an element g in F such that g(y) is in V. Thus, $g(U) \cap V \neq \emptyset$ which means that $UF \cap V \neq \emptyset$.

Suppose that for two arbitrary open subsets U and V of X we have $UF \cap V \neq \emptyset$. This means that there is an element g in F such that $g(U) \cap V \neq \emptyset$. Since X is a compact metric space, we choose a covering of X as follows: For each n let V_{ni}, $i = 1, 2, \cdots, k_n$, be a finite covering of X such that the diameter of each V_{ni} is less than 1/n. Let U_{10} be a spherical neighborhood in X with diameter less than one. By assumption $U_{10}F \cap V_{11} \neq \emptyset$. Thus, there is an element g_{11} of F such that $g_{11}(U_{10}) \cap V_{11} \neq \emptyset$. Let U_{11} be a spherical neighborhood of X such that $\overline{U_{11}} \subset U_{10}$ and $g_{11}(U_{11}) \subset V_{11}$. In general, define a spherical neighborhood U_{1k} such that $\overline{U_{1k}} \subset U_{1(k-1)}$, $k = 2, 3, \cdots, k_n$ and $g_{1k}(U_{1k}) \subset V_{1k}$.

Define $U_{20} = \bigcap_{k=0}^{k_n} U_{1k} = U_{1k_n}$ and continue inductively. then there is a point p in $\bigcap_{n=1}^{\infty} \overline{U_{n0}}$ and it follows that $pF \cap V \neq \emptyset$ for an arbitrary open subset V of X. Hence $\overline{pF} = X$.

Corollary: The pair (X,F) is transitive if and only if for any open subset U of X, it is true that $\overline{UF} = X$.

Proof: This follows immediately from the theorem.

The following example shows that Theorem 2.3 is not true in an arbitrary topological space.

Example 2.1: Let X be the unit interval with the following topology. A subset U of X is open if and only if the complement of U is countable. Let F be the identity map, e. Now if U and V are nonempty open subsets of X, we have $e(U) \cap V = U \cap V \neq \emptyset$. The orbit of a point x is just the point x, therefore $\overline{xF} \neq X$ and there are no transitive points.

Theorem 2.4 and Theorem 2.5 will serve to decide when (X, F) is transitive provided F is generated by an interior transformation f. It is noted that the condition that F be generated by an interior transformation is not required in Theorem 2.4. We follow Theorem 2.5 with an example which shows that the condition that f be an interior transformation cannot be relaxed to the condition that the generating function f be a monotone transformation.

Theorem 2.4: If (X, F) is transitive, then every F-invariant, non-empty, open subset of X is everywhere dense in X.

Proof: Suppose that (X, F) is transitive. Let U be an arbitrary open subset of X such that U is F-invariant. It will now be shown that U is everywhere dense in X.

Suppose U is not everywhere dense in X, then there exists a non-empty open subset V of X such that $U \cap V = \emptyset$.

Since (X, F) is transitive, then by Theorem 2.2, there is a point y in U such that $\overline{yF} = X$. Now V is a non-empty open subset of X thus there is an element $g \neq e$ in F such that $g(y) \in V$. This means that $g(U) \cap V \neq \emptyset$, but U is F-invariant so $g(U) \subset U$ for every g in F. Therefore $U \cap V \neq \emptyset$ contrary to assumption and U is everywhere dense in X.

<u>Theorem 2.5:</u> If F is generated by an interior transformation f, and every F-invariant open subset of X is everywhere dense in X, then (X, F) is transitive.

<u>Proof:</u> Assume that (X, F) is not transitive. Then there exist two open subsets U and V of X such that $UF \cap V = \emptyset$ by Theorem 2.3. Let f be the interior transformation that generates F. Now

$$UF = U \cup f(U) \cup \cdots \cup f^n(U) \cup \cdots$$

is F-invariant and UF is open because it is the union of open sets since f in an interior transformation. It is seen that U = UF is not everywhere dense in X because $UF \cap V = \emptyset$, and this contradicts the fact that every F-invariant open subset of X is everywhere dense in X.

<u>Example 2.2:</u> Let $X = [0, 1]$ with the usual topology. Define $f(x) = \frac{1}{2}$ for all $x \in X$. Since there are no F-invariant open subsets of X, the hypothesis of the theorem is satisfied vacuously, but obviously (X,F) is not transitive.

The following three theorems deal with F-invariant subsets when (X, F) is transitive. In particular, Theorem 2.8 will deal with the relationship between F-invariant open subsets and F-invariant closed subsets.

<u>Theorem 2.6:</u> If (X, F) is transitive then every F-invariant, proper, closed, subset of X is nowhere dense in X.

<u>Proof:</u> Suppose (X, F) is transitive and H is a proper, closed, F-invariant subset of X which is not nowhere dense. Then, since H is closed, there is an open subset U of X such that $U \subset \overline{H} = H$ which means that every open subset V of U contains points of H.

By Theorem 2.3 we have, for any two non-empty open subsets V_1 and V_2 of X, $V_1 F \cap V_2 \neq \emptyset$. Choose a covering of X as follows: For each n let the set V_{ni}, $i = 1, 2, \cdots, k_n$, be a finite covering of X such that the diameter of each V_{ni} is less than I/n. Let U_{10} be a spherical neighborhood in H with diameter less than one and by Theorem 2.3, $U_{10} F \cap V_{11} \neq \emptyset$ thus there is an element g_{11} in F such that $g_{11}(U_{10}) \cap V \neq \emptyset$. Define U_{11} so that $\overline{U_{11}} \subset U_{10}$ and $g_{11}(U_{11}) \subset V_{11}$. In general, define U_{1k}, $k = 2, 3, \cdots, k_n$, such that $\overline{U_{1k}} \subset U_{1(k-1)}$, and there is an element g_{1k} in F such that $g_{1k}(U_{1k}) \subset V_{1k}$. Let S_1 be a spherical neighborhood of diameter less than one such that $\overline{S}_1 \subset \bigcap_{k=1}^{k_1} U_{1k}$ and let $D_1 = \overline{S_1} \cap H$. Then D_1 is nonempty. Continue inductively in the same manner as in the proof of Theorem 2.3 to obtain two sequences $\{D_n\}$ and $\{S_n\}$. Now H is closed hence there is a point p in $\bigcap_{n=1}^{\infty} D_n = \bigcap_{n=1}^{\infty} S_n$. Furthermore p is transitive with respect to F by construction. Since p belongs to H, then H must be everywhere dense in X because H is F-invariant. But H is closed and, therefore, $H = \overline{H}$ which means that

H = X. This contradicts the assumption that H was a proper subset of X.

Theorem 2.7: For any set X and F generated by any function $f: X \to X$, a subset Y of X is F-invariant if and only if $f(Y) = Y$, where f is the function which generates F.

Proof: Let $f(Y) = H \neq Y$. Since Y is F-invariant and $f(Y) = H$, then certainly $Y = \bigcup_{g \in (F-e)} g(Y) \supsetneq H$. Thus there is a point y in Y such that y is not in H. Now $f^n(Y) \subset H$ for all $n \geq 1$ and thus $Y = \bigcup_{g \in (F-e)} g(y) \subset H \subsetneq Y$. Hence $f(Y) = Y$.

If $f(Y) = Y$, then for all g in F-e, $g(Y) = Y$. Hence, $\bigcup_{g \in (F-e)} g(Y) = Y$ and Y is F-invariant.

Theorem 2.8: Let F be generated by an interior transformation f. Every F-invariant open subset of X is everywhere dense in X if and only if every F-invariant proper closed subset of X is nowhere dense in X.

Proof: Suppose that every F-invariant open subset of X is everywhere dense in X and suppose there exists a proper closed subset H of X such that H is not nowhere dense in X. Since H is closed, and H is not nowhere dense, then H contains a nonempty open subset U. Let U_0 be the interior of H. Suppose that H is F-invariant, then since $U_0 \subset H$ we have $U_0F \subset H$. Now by assumption F is generated by an interior transformation, thus U_0F is open. But U_0 is the maximal open subset of F and $U_0F \subset H$ therefore $U_0F \subset U_0$. Hence, U_0 is an F-invariant open set and thus by assumption is everywhere dense in X. Thus $\overline{U_0} \subset H = \overline{H} = X$, and this contradicts the fact that H was a proper subset of X.

Suppose that every F-invariant proper closed subset of X is nowhere dense in X. Suppose further that there exists an open subset U of X which is F-invariant and not everywhere dense in X. By Theorem 2.7, f(U) = U and since f is continuous $f(\overline{U}) \subset \overline{U}$. Also $f(\overline{U}) \supset U$ and, therefore, contains $\overline{U}$ since $f(\overline{U})$ is closed.

Therefore, $\overline{U}$ is nowhere dense in X since it is F-invariant. This gives a contradiction, and thus U is everywhere dense in X.

It is noticed that if the restriction that the generating transformation of F be interior is removed, then the following result remains true; every F-invariant proper closed subset of X is nowhere dense in X implies that every F-invariant open subset of X is everywhere dense in X.

The following results concern the nature of F-invariant subsets of a topological space X when (X, F) is transitive.

<u>Theorem 2.9:</u> Let (X, F) be transitive, then

i) If B and C are F-invariant, open subsets of X such that $B \cap C = \emptyset$, then either $B = \emptyset$ or $C = \emptyset$.

ii) If B and C are F-invariant, closed subsets of X such that $B \cup C = X$, then B = X or C = X.

iii) If B and C are F-invariant subsets of X such that $B \cap C = \emptyset$ and $B \cup C = X$, then

a) The interior of B is empty or the interior of C is empty, and

b) $\overline{B} = X$ or $\overline{C} = X$

iv) If B is an F-invariant subset of X, then the interior of B is empty or $\overline{B} = X$.

Proof: i) Suppose B and C are F-invariant open subsets of X such that $B \cap C = \emptyset$. If both B and C are non-empty, then by Theorem 2.4, each of B and C is everywhere dense in X. But each of B and C is open, hence, $B \cap C \neq \emptyset$. This gives a contradiction, therefore, either $B = \emptyset$ or $C = \emptyset$.

ii) Suppose B and C are F-invariant closed subsets of X such that $B \cup C = X$. Suppose B and C are both proper subsets of X, then by Theorem 2.5, both B and C are nowhere dense in X. Therefore, $B \cup C$ is nowhere dense in X since it is the union of two nowhere dense sets. This contradicts $B \cup C = X$ and means that either $B = X$ or $C = X$.

iii) a) Suppose that both B and C contain interior points. Since B is an F-invariant subset of X, we have $B = BF$. But B contains an open set and, therefore, contains a transitive point since (X, F) is transitive, thus BF intersects every non-empty open set because this set is everywhere dense in X. Now since $B = BF$, we have $B \cap C \neq \emptyset$.

b) Since (X, F) is transitive and $B \cup C = X$, then there is a point x in $B \cup C$ such that $\overline{xF} = X$. Suppose $x \in B$, then $xF \subset B$ because B is F-invariant. Thus $\overline{B} = X$.

Similarly, if $x \in C$ we have $\overline{C} = X$ because C is also F-invariant.

iv) Suppose B is an F-invariant subset of X and B contains an interior point, then B contains a transitive point and thus $\overline{B} = \overline{BF} = X$.

Theorem 2.10: The set of transitive points is F-invariant.

Proof: Let Y be the set of transitive points. It is easily seen that if $y \in Y$, then each point of yF is in Y. Let f be the generating function for F, then $f(Y) \subset Y$. Suppose there is a point $z \in [Y-f(Y)]$. Now Y is non-empty, and f is onto. Since f(X) is everywhere dense in X and X is a compact metric space $f(X) = X$. Therefore, there is a point $x \in X$ such that $f(x) = z$. This means that $xF \supset zF$ because z is a transitive point, thus $x \in Y$ and $f(x) = z \in f(Y)$. But by assumption $z \in [Y-f(Y)]$ and this gives a contradiction. Thus, $f(Y) = Y$, and Y is F-invariant.

Given a transitive continuous function f from a topological space S onto S and a homeomorphism g from S onto a topological space X, it will now be shown how one can construct a transitive continuous function from X onto X.

Theorem 2.11: Let f be a transitive continuous function from a topological space S onto S and let g be a homeomorphism from S onto a topological space X. Then gfg^{-1} is a transitive continuous function from X onto X.

Proof: Clearly gfg^{-1} is continuous as the composite of three continuous functions.

Since f is transitive on S, there is a point $s \in S$ such that $\overline{\bigcup_{n=0}^{\infty} f^n(s)} = S$. Now g is a homeomorphism, so there is exactly one point $x_1 \in X$ such that $g(s) = x_1$. This means that $g^{-1}(x_1) = s$. It is also seen that

$$(gfg^{-1})^n(x_1) = gf^n g^{-1}(x_1).$$

Choose an arbitrary point $x' \in X$. It will now be shown that $x' \in \overline{\bigcup_{n=0}^{\infty} gf^n g^{-1}(x_1)}$: Now $g^{-1}(x') \in S$, and s is a transitive point with respect to f so $g^{-1}(x') \in \overline{\bigcup_{n=0}^{\infty} f^n(s)}$.

Case 1: If there is an integer $m \geq 0$ such that $f^m(s) = g^{-1}(x')$ then $gf^m g^{-1}(x_1) = x'$. Hence, $x' \in \bigcup_{n=0}^{\infty} gf^n g^{-1}(x_1)$.

Case 2: Suppose there is no integer m such that $f^m(s) = g^{-1}(x')$, then, since $\bigcup_{n=0}^{\infty} f^n(s)$ is dense in S, $g^{-1}(x')$ is a limit point of $\bigcup_{n=0}^{\infty} f^n(s)$. Therefore, there is a subsequence $\{f^{n_i}(s)\}$ of $\{f^n(s)\}_{n=0}^{\infty}$ that converges to $g^{-1}(x')$. Now $s = g^{-1}(x_1)$ so $\{f^{n_i}[g^{-1}(x_1)]\}$ converges to $g^{-1}(x')$ and since g is continuous $\{gf^{n_i}g^{-1}(x_1)\}$ converges to $g(g^{-1}(x')) = x'$. Thus, $x' \in \overline{\bigcup_{n=0}^{\infty} gf^n g^{-1}(x_1)}$ and, since x' is an arbitrary point of X, gfg^{-1} is transitive.

CHAPTER III

SPECIAL CASES AND EXAMPLES

It seems reasonable and desirable at this point to investigate the existence of transitive continuous functions in particular cases. The existence of transitive continuous functions will now be established by means of examples, and some theorems will be proved for particular topological spaces.

In Example 3.1, a transitive continuous function from the closed unit interval into itself is established.

Example 3.1: The following function f is a continuous mapping of the closed unit interval I onto itself such that there exists a transitive point $x \in I$:

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2-2x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

In general

$$f^n(x) = \left.\begin{cases} 2^n x - K, & K/2^n \leq x \leq (K+1)/2^n, \ K \text{ even} \\ (K+1) - 2^n x, & K/2^n \leq x \leq (K+1)/2^n, \ K \text{ odd} \end{cases}\right\} 0 \leq K < 2^n,$$

K an integer.

Notice that f^n maps the interval $[K/2^n, (K+1)/2^n]$ onto I.

Proof: Let $\{g_n\}_{n=1}^{\infty}$ be a countable base, consisting of open intervals, for the topology for the interval I (for example, intervals with rational centers and rational length). Then given any open set U of I there exists a base element g_i such that $g_i \subset U$. A sequence of closed nested intervals will be defined inductively. Let $[a_1, b_1] = \overline{g_1}$. Choose an integer n_1 such that $4(1/2^{n_1}) \leq b_1 - a_1$, then $f^{n_1}[a_1, b_1] = I$, and since f^{n_1} is a continuous function, there exists a closed subinterval $[a_2, b_2] \subset [a_1, b_1]$ such that $f^{n_1}[a_2, b_2] \subset g_1$. In general, given $[a_k, b_k]$, choose an integer n_k such that $4(1/2^{n_k}) \leq b_k - a_k$, then $f^{n_k}[a_k, b_k] = I$, and since f^{n_k} is continuous, there exists a subinterval $[a_{k+1}, b_{k+1}]$ of $[a_k, b_k]$ such that $f^{n_k}[a_{k+1}, b_{k+1}] \subset g_k$.

The sequence $\{[a_k, b_k]\}$ satisfies the following:

1) $[a_{k+1}, b_{k+1}] \subset [a_k, b_k]$;

2) Given a base element g_n there exists an integer m such that $f^m[a_{n+1}, b_{n+1}] \subset g_n$.

Let $x = \bigcap_{n=1}^{\infty} [a_n, b_n]$, then

$$f^{n_1}(x) \in g_1,\ f^{n_2}(x) \in g_2,\ \cdot\ \cdot\ \cdot,\ f^{n_k}(x) \in g_k,\ \cdot\ \cdot\ \cdot.$$

Therefore, $\bigcup_{m=0}^{\infty} f^m(x)$ forms an everywhere dense set in I by Theorem 2.1. (Figure 1).

An interesting theorem concerning continuous functions on the unit interval is now given.

Theorem 3.1: Let f be a continuous function such that $f(I) = I$ (I is the unit interval). Then $f^2 = f(f)$ allows at least two fixed points in I.

Proof: If f allows two fixed points, then f^2 allows two fixed points, and the proof is complete. Suppose f allows only one fixed point. Since f is onto, this fixed

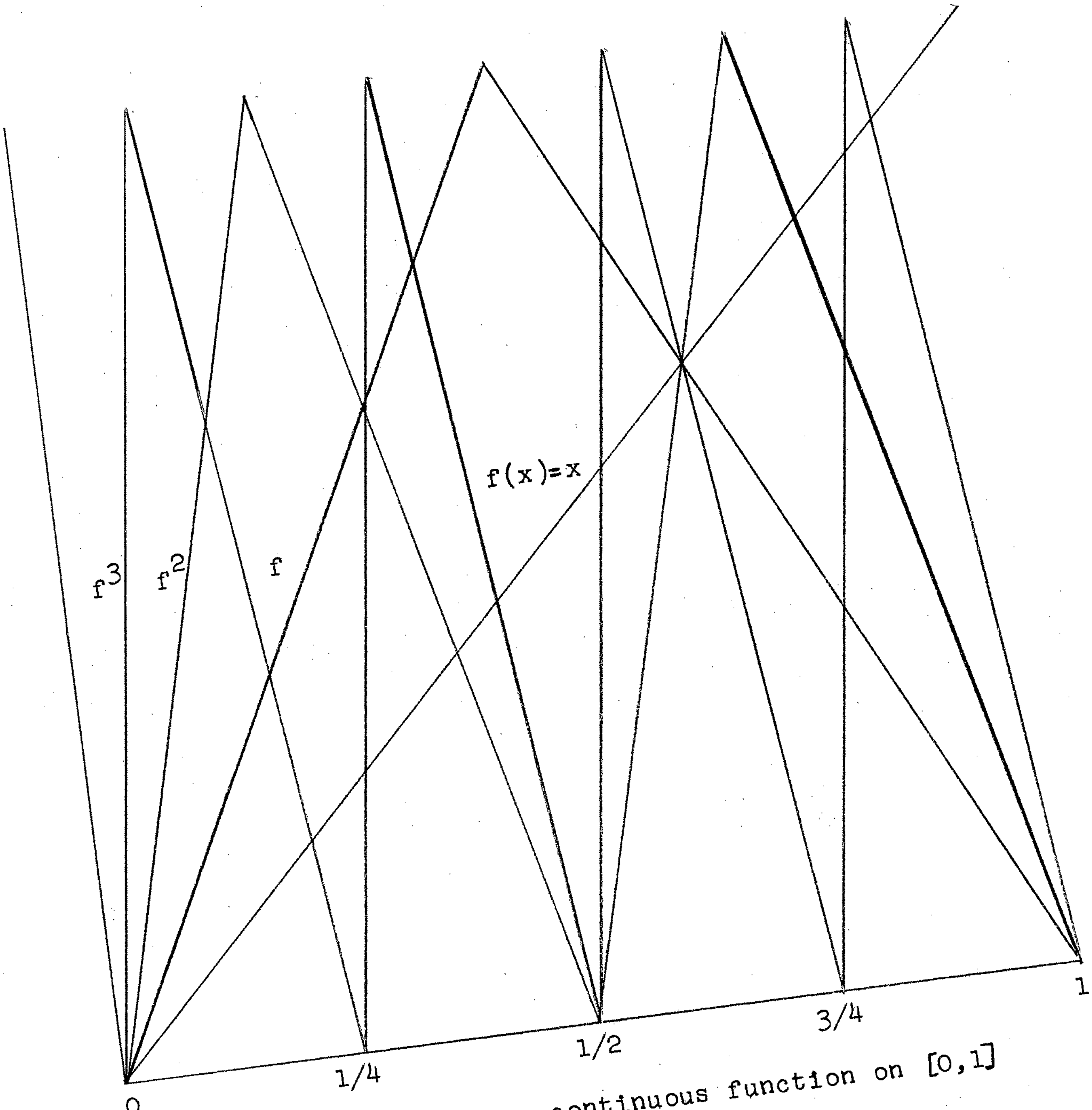

Figure 1. Transitive continuous function on [0,1]

point must be an interior point of the interval. Hence, $f(0) > 0$ and $f(1) < 1$. Suppose p is the fixed point. It follows that there exists c, $0 \leq c < p$, such that $f(c) = 1$. Hence, $f(c, p) \supset [p, 1]$. Similarly, there exists a point d, $p < d \leq 1$, such that $f(d) = 0$. Let $e \in (c, p)$ such that $f(e) = d$. Then $f^2(e) = 0$. Now either $f^2(0) = 0$ or there exists x such that $0 < x \leq e$ and $f^2(x) = x$.

Example 3.2: The following function f is a continuous mapping of the closed unit square I x I onto itself such that there exists a transitive point $(x, y) \in I \times I$.

$$f(x,y) = \begin{cases} (2x, 2y), & 0 \leq x \leq 1/2,\ 0 \leq y \leq 1/2 \\ (2x, 2-2y), & 0 \leq x \leq 1/2,\ 1/2 \leq y \leq 1 \\ (2-2x, 2y), & 1/2 \leq x \leq 1,\ 0 \leq y \leq 1/2 \\ (2-2x, 2-2y), & 1/2 \leq x \leq 1,\ 1/2 \leq y \leq 1 \end{cases}$$

In general,

$$f^n(x,y) = \begin{cases} (2^n x-k,\ 2^n y-j), & k/2^n \leq x \leq (k+1)/2^n, \\ & \quad j/2^n \leq y \leq (j+1)/2^n,\ k, j \text{ even} \\ (2^n x-k,\ (j+1)-2^n y), & k/2^n \leq x \leq (k+1)/2^n, \\ & \quad j/2^n \leq y \leq (j+1)/2^n,\ k \text{ even},\ j \text{ odd} \\ ((k+1)-2^n x,\ 2^n y-j), & k/2^n \leq x \leq (k+1)/2^n, \\ & \quad j/2^n \leq y \leq (j+1)/2^n,\ k \text{ odd},\ j \text{ even} \\ ((k+1)-2^n x,\ (j+1)-2^n y), & k/2^n \leq x \leq (k+1)/2^n, \\ & \quad j/2^n \leq y \leq (j+1)/2^n,\ k, j \text{ odd} \end{cases}$$

$$0 \leq k < 2^n,\ 0 \leq j < 2^n,\ k, j \text{ integers}.$$

Notice that f^n maps the rectangle $k/2^n \leq x \leq (k+1)/2^n$, $j/2^n \leq y \leq (j+1)/2^n$ onto $I \times I$.

Proof: i) f(x,y) is continuous.

It is first shown that $f(1/2, 1/2) = (1, 1)$ in each representation for f. $f(1/2, 1/2) = (2(1/2), 2(1/2)) = (1, 1)$, $f(1/2, 1/2) = (2(1/2), 2-2(1/2)) = (1, 1)$, $f(1/2, 1/2) = (2-2(1/2), 2(1/2)) = (1, 1)$, and $f(1/2, 1/2) = (2-2(1/2), 2-2(1/2)) = (1, 1)$.

It will now be shown that for each open neighborhood U of $f(x_0, y_0)$ of radius ϵ, there exists an open neighborhood V of (x_0, y_0) of radius δ_ϵ such that if $(x,y) \in V$, then $f(x, y) \in U$.

Choose an arbitrary $\epsilon > 0$ and suppose $0 \leq x_0 < 1/2$, $0 \leq y_0 < 1/2$. $\rho(f(x,y), f(x_0, y_0)) < \epsilon$ implies $\sqrt{(2x-2x_0)^2 + (2y-2y_0)^2} < \epsilon$ implies $4[(x-x_0)^2 + (y-y_0)^2] < \epsilon^2$ which implies $(x-x_0)^2 + (y-y_0)^2 < \epsilon^2/4$ so let $\delta_\epsilon = \epsilon^2/4$ and choose ϵ small enough so that $0 \leq x < 1/2$, $0 \leq y < 1/2$. Hence, it is seen that $\rho(f(x,y), f(x_0, y_0)) < \epsilon$ whenever $\rho((x,y), (x_0, y_0)) < \delta_\epsilon$ in this case.

Now suppose $0 \leq x_0 < 1/2$, $1/2 < y_0 \leq 1$, then $\rho(f(x,y), f(x_0, y_0)) = \sqrt{(2x-2x_0)^2 + (2-2y-2+2y_0)^2}$ implies $4[(x-x_0)^2 + (y-y_0)^2] < \epsilon^2$, so let $\delta_\epsilon = \epsilon^2/4$.

The other two cases are done similarly.

ii) f is onto.

Let $(a,b) \in I \times I$.

Case 1: Suppose $0 \leq a \leq 1/2$, $0 \leq b \leq 1/2$. Let $x = a/2$, and $y = b/2$, then $f(x,y) = (2(a/2), 2(b/2)) = (a, b)$.

Case 2: Suppose $0 \leq a \leq 1/2$, $1/2 \leq b \leq 1$. Let $x = a/2$, and $y = (2-b)/2$, then

$f(x,y) = (2(a/2), 2-2((2-b)/2)) = (a, b)$.

Case 3: Suppose $1/2 \leq a \leq 1$, $0 \leq b \leq 1/2$. Let $x = (2-a)/2$, and $y = b/2$, then

$$f(x,y) = (2-2((2-a)/2),\ 2(b/2)) = (a,\ b).$$

Case 4: Suppose $1/2 \leq a \leq 1$, $1/2 \leq b \leq 1$. Let $x = (2-a)/2$ and $y = (2-b)/2$, then

$$f(x,y) = (2-2((2-a)/2),\ 2-2((2-b)/2)) = (a,\ b).$$

Hence, f is onto.

iii) There exists a transitive point $(x,y) \in I \times I$.

Let $\{G_n\}_{n=1}^{\infty}$ be a countable base for the topology for $I \times I$ consisting of open rectangles (for example, rectangles with rational centers, rational length and rational width). Then given an open set U of I, there exists a base element G_i such that $G_i \subset U$. A sequence of closed nested rectangles will be defined inductively. Let $A_1 = \overline{G_1}$. Choose an integer n_1 such that $4(1/2^{n_1}) \leq \min$ (length A_1, width A_1), then $f^{n_1}(A_1) = I$ and since f^{n_1} is a continuous function, there exists a closed subrectangle $A_2 \subset A_1$ such that $f^{n_1}(A_2) \subset G_1$. In general, given $A_k \subset \overline{G_1}$, choose an integer n_k such that $4(1/2^{n_k}) \leq \min$ (length A_k, width A_k), then $f^{n_k}(A_k) = I \times I$ and since f^{n_k} is continuous, there exists a subrectangle A_{k+1} of A_k such that $f^{n_k}(A_{k+1}) \subset G_k$.

The sequence $\{A_k\}$ satisfies the following conditions:

1) $A_{k+1} \subset A_k$.

2) Given a base element G_n there exists an integer m such that $f^m(A_{n+1}) \subset G_n$.

Let $(x,y) = \bigcap_{n=1}^{\infty} A_n$, then

$$f^{n_1}(x,y) \in G_1,\ f^{n_2}(x,y) \in G_2,\ \cdots,\ f^{n_k}(x,y) \in G_k,\ \ldots.$$

Therefore, $\overset{\infty}{\underset{m=0}{\cup}} f^m(x,y)$ forms an everywhere dense set in I by Theorem 2.1.

It should be evident to the imaginative reader that one could construct an example of a transitive continuous function from I^n to I^n.

It seems reasonable to believe that if $f:X \rightarrow X$ is a transitive continuous function and $g:X \rightarrow X$ is a homeomorphism, then $g(f)$ is a transitive continuous function. The following theorem will show that this is not necessarily true.

Theorem 3.2: Let $f:A \rightarrow A$ be a transitive continuous function and let $g:A \rightarrow A$ be a homeomorphism, then $g(f)$ is not necessarily a transitive function.

Proof: Let

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2 \\ 2-2x, & 1/2 \leq x \leq 1 \end{cases}$$

then $f:I \rightarrow I$ is transitive continuous by a previous example.

Let $g(x) = x^2$, then g is a homeomorphism from I onto I.

$$g(f(x)) = \begin{cases} 4x^2, & 0 \leq x \leq 1/2 \\ 4x^2-8x+4, & 1/2 \leq x \leq 1 \end{cases}$$

If $0 \leq x \leq 1/4$, then $0 \leq g(f(x)) \leq 1/4$, hence $0 \leq (gf)^n \leq 1/4$. Let $F = \{h:h(x) = (gf)^n(x)$ for some non-negative integer $n\}$.

By Theorem 2.3, we have x is transitive with respect to F if and only if given two non-empty open sets U and V of X, then $UF \cap V \neq \emptyset$.

Let U be the open interval (1/16, 1/8) and let V be the open interval (5/8, 3/4) then $UF \subset [0, 1/4]$ and hence,

$UF \cap V = \emptyset$. Therefore, g(f) is not transitive.

An example of a transitive continuous function from the open unit interval onto itself will now be given.

<u>Example 3.3:</u> Example of a transitive continuous function from the open unit interval onto the open unit interval.

$$R_1 = L_1 = \overline{(5/12,\ 1/6)\ (7/12,\ 5/6)}$$

that is, $$y-1/6 = 4(x-5/12)$$

or, $$y = 4x-3/2 \quad 5/12 \leq x \leq 7/12$$

$$L_2 = \overline{(1/4,\ 1/8)\ \ (1/3,\ 1-1/6)}$$

In general, if $k \geq 2$

$$L_k = \overline{(1/2k,\ 1/4k)\ \ (1/(2k-1),\ 1-1/2(2k-1))}$$

or $$y = ((8k^2-8k+1)/2)x + (2-2k),\ 1/2k \leq x \leq 1/(2k-1).$$

Notice that the absolute value of the slope is greater than or equal to 4.

$$L'_1 = \overline{(1/3,\ 5/6)\ (5/12,\ 1/6)}$$

or, $$y = -8x + 19/6,\ 1/3 \leq x \leq 5/12$$

$$L'_2 = \overline{(1/5,\ 1-1/10)\ (1/4,\ 1/8)}$$

and in general if $k \geq 2$

$$L'_k = \overline{(1/(2k+1),\ 1-1/2(2k+1))\ (1/2k,\ 1/4k)}$$

or, $$y = ((-8k^2+1)/2)x + 2k,\ 1/(2k+1) \leq x \leq 1/2k.$$

For $k \geq 2$ we have

$$R_k = \overline{(1-1/(2k-1),\ 1/2(2k-1))\ (1-1/2k,\ 1-1/2(2k))}$$ or,

$$y = (8k^2-8k+1)x/2+(16k^3-24k^2+14k-2)/4k,\ 1-1/(2k-1) \leq x \leq 1-1/2k$$

$$R'_1 = \overline{(7/12,\ 5/6)\ (2/3,\ 1/6)}$$

or, $$y = -8x + 33/6,\ 7/12 \leq x \leq 2/3.$$

In general, if $k \geq 2$ we have

$R'_k = \overline{(1-1/2k,\ 1-1/2(2k))\ (1-1/(2k+1),\ 1/2(2k+1))}$

or, $y = (-8k^2+4k+1)x/2+(16k^3-8k^2-2k+1)/(4k+2)$

for $1-1/2k \leq x \leq 1-1/(2k+1)$.

Now define f to be the function whose graph is

$\bigcup_{k=1}^{\infty} L_k \cup (\bigcup_{k=1}^{\infty} L'_k) \cup (\bigcup_{k=1}^{\infty} R_k) \cup (\bigcup_{k=1}^{\infty} R'_k)$.

Choose an arbitrary open set $U \subset (0,1)$ then U contains an open interval (a, b).

It will now be shown that if (a, b) is any interval containing an interval of the form (1/(2k+1), 1/2k), (1/2k, 1/(2k-1)), (1-1/(2k-1), 1-1/2k), or (1-1/2k, 1-1/2k+1) and x is any point in (0, 1) then there exists an integer n such that $x \in f^n(a, b)$.

Case 1: Suppose (a,b) contains an interval of the form (1/2k, 1/(2k-1)).

Let L_k^0 denote the domain of L_k and let L_k^I denote the range of L_k and similarly let $L_k'^0$ and $L_k'^I$, R_k^0 and R_k^I, $R_k'^0$, and $R_k'^I$ denote the range and domain of L_k', R_k, and R_k' respectively. Then we have

$L_k^0 = [1/2k,\ 1/(2k-1)]$, $L_k^I = [1/4k,\ 1-1/2(2k-1)]$

$L_k'^0 = [1/(2k+1),\ 1/2k]$, $L_k'^I = [1/4k,\ 1-1/2(2k+1)]$

$R_k^0 = [1-1/(2k-1),\ 1-1/2k]$, $R_k^I = [1/2(2k-1),\ 1-1/2(2k)]$

$R_k'^0 = [1-1/2k, 1-1/(2k+1)]$, $R_k'^I = [1/2(2k+1),\ 1-1/2(2k)]$.

Since (a, b) contains an interval of the form (1/2k, 1/(2k-1)) by assumption then $(a, b) \supset L_k^0$ thus $f(a, b) \supset f(L_k^0) = L_k^I = [1/4k,\ 1-1/2(2k-1)] = [1/2(2k),\ 1-1/2(2k-1)] \supset [1/2(2k), 1/(2(2k)-1)] = L_{2k}^0$. Therefore, $f(a,b) \supset L_{2k}^0$ which by the same argument as used above contains $L^0_{4k} = L^0_{2^2k}$.

Hence, in general

$$f^n(a, b) \supset f(L^0_{2^{n-1}k}) = [1/2^{n+1}k, 1-1/(2^{n+1}k-2)] .$$

Now let $x \in (0,1)$ then there exists an integer j such that $1/2^j < \min[d(x,0), d(x,1)]$ which means that $1/2^j < x < 1-1/2^j$.

Now $1/2^{j+1}k = (1/2^k)(1/2^j) < 1/2^j$ and $1/2^j > 1/(2^{j+1}k-2)$ because $2^{j+1}k-2 = 2(2^jk-1) > 2^j$, therefore $-1/(2^{j+1}k-2) > -1/2^j$ which means that $1-1/(2^{j+1}k-2) > 1-1/2^j$. Therefore, $1/2^{j+1}k < x < 1-1/(2^{j+1}k-2)$. Therefore, in this case $x \in f^j(a,b)$ for some integer j. Thus, $\bigcup_{n=0}^{\infty} f^n(a,b) \supset I$.

Case 2: Now suppose (a, b) contains an interval of the form $[1/(2k+1), 1/2k]$. Then (a, b) contains L'^0_k, therefore, $f(a,b) \supset f(L'^0_k) = L'^I_k = [1/4k, 1-1/2(2k+1)] \supset$ $[1/2(2k+1) \quad +1), 1/2(2k)] = L'^0_{2k}$.

Therefore, $f(a, b) \supset L'^0_{2k}$ which means that $f^2(a, b) \supset f(L'^0_{2k})$ and by the same argument as used above, we have $f^2(a, b) \supset L'^0_{4k} = L'^0_{2^2k}$. In general,

$$f^n(a, b) \supset f(L'^0_{2^{n-1}k}) = [1/2^nk, 1-1/2^nk-2] .$$

Now let $x \in (0,1)$, then there exists an integer j such that $1/2^{j-1} < \min[d(x, 0), d(x, 1)]$ therefore, $1/2^{j-1} < x < 1-1/2^{j-1}$ and by the same argument as used in Case 1, we have $1/2^jk < x < 1-1/2^jk-2$ and in this case $x \in f^j(a, b)$ for some integer j. Thus, $\bigcup_{n=0}^{\infty} f^n(a,b) \supset I$.

Case 3: Now suppose (a, b) contains an interval of the form $[1-1/(2k-1), 1-1/2k]$. Then (a, b) contains R^0_k thus, $f(a, b) \supset f(R^0_k) = R^I_K = [1/2(2k-1), 1-1/2(2k)] \supset$ $[1-1/(2(2k)-1), 1-1/2(2k)] = R^0_{2k}$.

Therefore, $f(a, b) \supset R^0_{2k}$ which means that $f^2(a, b) \supset f(R^0_{2k})$ and by the same argument as above $f^2(a, b) \supset R^0_{4k} = R^0_{2}2k$. In general, $f^n(a,b) \supset f(R^0_{2^{n-1}k}) = [1/(2^n k-2), 1-1/2^n k]$.

Let $x \in (0, 1)$, then there exists an integer j such that $1/2^j < \min[(d(x,0), d(x,1))]$ thus, $1/2^{j-1} < x < 1-1/2^{j-1}$.

By the argument used in Case 1, we have $1/2^{j-1} > 1/(2^j k-2)$ and $1-1/2^j k > 1-1/2^{j-1}$ hence, $1/2^j k-2 < x < 1-1/2^{j-1}$ and $x \in f^j(a, b)$ for some integer j. Thus, $\bigcup_{n=0}^{\infty} f^n(a, b) \supset I$.

Case 4: Suppose (a, b) contains an interval of the form $[1-1/2k, 1-1/(2k+1)]$, then (a, b) contains $R_k^{'0}$. Thus, $f(a, b) \supset f(R_k^{'0}) = R_k^{'I} = [1/2(2k+1), 1-1/2(2k)] \supset$

$$[1-1/2(2k-2), 1-1/2(2k)] = R^{'0}_{2k-1}.$$

Thus $f^2(a, b) \supset f(R^{'0}_{2k-1})$ thus by the same argument $f^2(a, b) \supset R^{'0}_{4k-3} = R^{'0}_{2^2(k-1)+1}$. In general $f^n(a, b) \supset f(R^{'0}_{2^{n-1}(k-1)+1}) =$ $[1/(2^{n+1}(k-1)+1), 1-1(2^n(k-1)+1)]$.

Let $x \in (0, 1)$ then there exists an integer j such that $1/2^j < \min[d(x, 0), d(x, 1)]$ which means that $1/2^j < x < 1-1/2^j$. Now $1/(2^{j+1}(k-1)+1) < 1/2^j$ and $1-1/(2^j(k-1)+1 > 1-1/2^j$, hence, $1/(2^{j+1}(k-1)+1 < x < 1-1/(2^j(k-1)+1$. Therefore, $x \in f^j(a, b)$ for some integer j and, therefore, $\bigcup_{n=0}^{\infty} f^n(a, b) = I$.

Now suppose $(a, b) \supset (1/3, 5/12)$ then $f(a, b) \supset (1/6, 5/6)$ which contains L^0_2 hence, $\bigcup_{n=0}^{\infty} f^n(a, b) \supset (0, 1)$.

Similarly, if $(a, b) \supset (5/12, 7/12)$ or $(2/3, 3/4)$ then f(a, b) contains L_2^0 hence, $\bigcup_{n=0}^{\infty} f^n(a, b) \supset (0, 1)$.

Now suppose (a, b) does not contain any of the intervals, and it will be shown that there is an integer k such that $f^k(a, b)$ contains one of the intervals and hence, $\bigcup_{n=0}^{\infty} f^{k+n}(a, b) = \bigcup_{k=0}^{\infty} f^k(a, b) \supset (0, 1)$.

It is observed that the absolute value of the minimum slope of f is 4 which occurs on the interval $5/12 \leq x \leq 7/12$. Hence, given any subinterval (c, d) of L_1, L_1', R_1, R_1', L_k^0, $L_k'^0$, R_k^0 or $R_k'^0$, then the arc length of f(c, d) is greater than or equal to 4d(c, d).

Now f is a continuous function, therefore, for any subinterval $(a, b) \subset (0, 1)$, we have f(a, b) is a continuous curve. Let I_k^0 represent any of L_1, L_1', R_1, R_1', L_k^0, $L_k'^0$, R_k^0, or $R_k'^0$, then if (a, b) does not contain an I_k^0, it must intersect at most two of them. Let (c_1, d_1) be the maximum intersection of f(a, b) with an I_k^0. Then arc length $f^n(c_1, d_1) \geq 4^n d(c_1, d_1)$. Then certainly there is an integer N such that the inverse image of $f^N(c_1, d_1)$ contains an I_k^0, then $\bigcup_{j=0}^{\infty} f^{N+j}(c_1, d_1) \supset (0, 1)$ by the argument above, which means that $\bigcup_{n=0}^{\infty} f^n(c_1, d_1) \supset (0, 1)$. Thus, f is transitive. (Figure 2).

It is well known that the open unit interval is homeomorphic to E^1, thus if one applies Theorem 2.11 to Example 3.3, it is seen that there exists a transitive continuous function from E^1 onto E^1.

The next theorem will show that there does not exist a transitive homeomorphism from the open unit interval onto

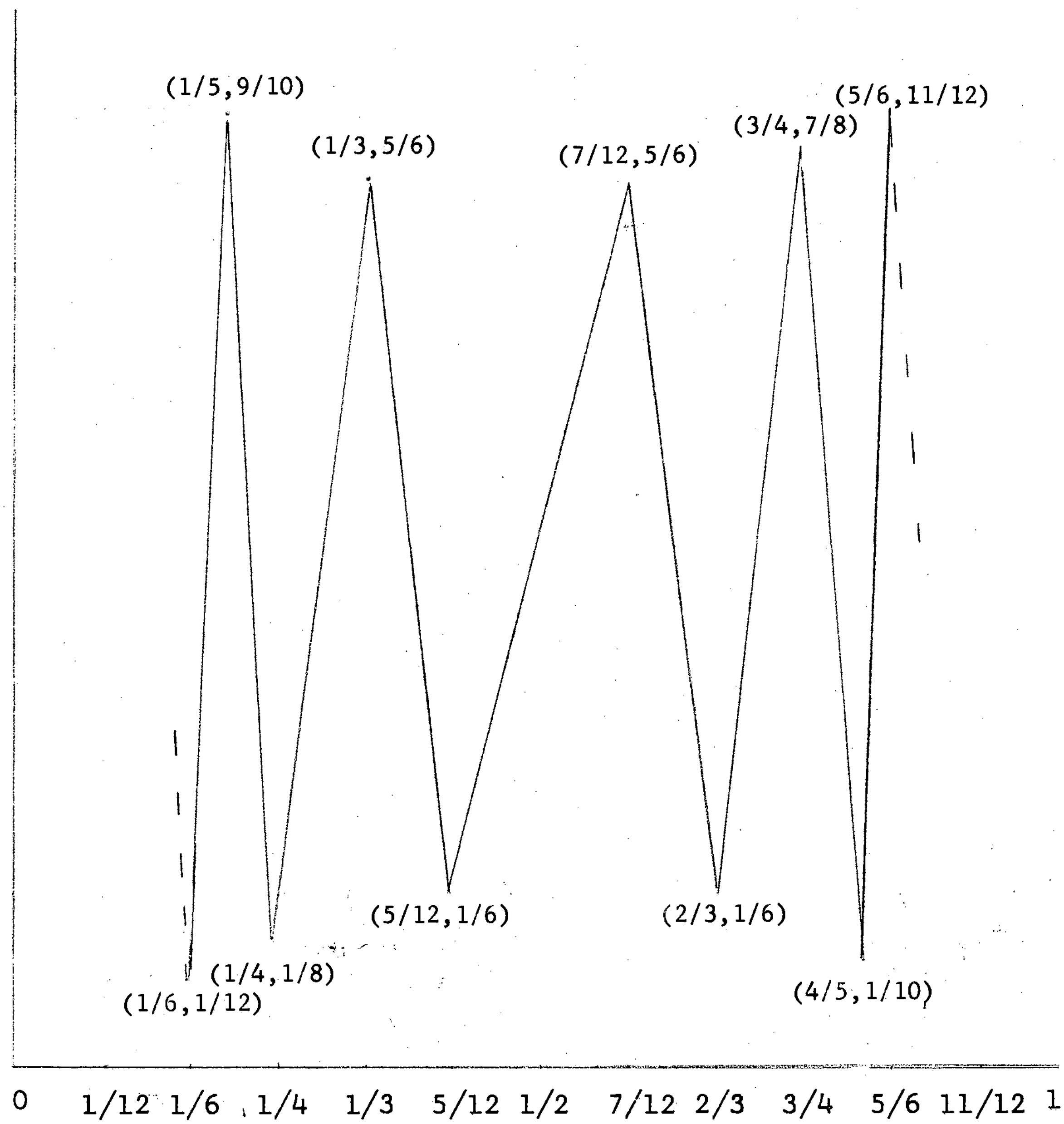

Figure 2. Transitive continuous function on (0,1)

itself. It follows from Theorem 2.11 that there does not exist a transitive homeomorphism from the reals onto the reals.

Theorem 3.3: There is no transitive homeomorphism from the open unit interval (0,1) onto (0,1).

Proof: Suppose $f:(0,1) \to (0,1)$ is a homeomorphism, then f is monotone and either $\lim_{x \to 0} f(x) = 0$ and $\lim_{x \to 1} f(x) = 1$, or $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to 1} f(x) = 0$.

Suppose f is monotone increasing. We consider the following three cases:

Case 1: $f(x) > x$ for every $x \in (0, 1)$.

Choose a point $a \in (0, 1)$, then $f(x) > a$ for every $x > a$ since f is monotone. This means that $f^n(x) > a$ for each integer $n = 1, 2, \cdots$. Therefore, $f(a, 1) \subset (a, 1)$ which means that $f^n(a, 1) \subset (a, 1)$ for an arbitrary $a \in (0, 1)$. But then f is not transitive by Theorem 2.2 because the set of transitive points of a transitive function must be dense in (0, 1).

Case 2: Suppose there is a point $a \in (0, 1)$ such that $f(a) = a$. Then since f is monotone increasing, we have $f(x) > a$ for all $x > a$ and f is not transitive by case 1.

Case 3: $f(x) < x$ for each $x \in (0, 1)$.

Choose an arbitrary point $a \in (0, 1)$ then $f(0, a) \subset (0,a)$ and thus $f^n(0, a) \subset (0, a)$ and f is not transitive by Theorem 2.2.

Now suppose f is monotone decreasing, then $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to 1} f(x) = 0$, so there exists a point x_1 such that

$f(x) > x$ if $x < x_1$ and $f(x) < x$ if $x > x_1$. Let $a = x_1 + (1-x_1)/2$, then $f(a) < a$ so let $f(a) = b$. Now if $f(b) < a$, then $f^2(b) < f(a) = b$ and $f^3(b) < f^2(a) = f(b)$. Continuing in this manner, we see that $b \leq f^n(a) < a$ so there is no integer k such that $f^k(a) > a$ thus $F(b, a) \cap (a, 1) = \emptyset$, and f is not transitive by Theorem 2.2.

Now suppose $f(b) > a$, then $f^2(b) < f(a) = b$ and $f^3(b) > f^2(a) = f(b) > a$. Hence, if n is odd $f^n(a) < b$ and if n is even, then $f^n(a) > a$. Thus, $f^n(a, 1) \leq b$ if n is odd and $f^n(a, 1) > a$ if n is even, thus $f^n(a, 1) \cap (b, a) = \emptyset$ and f is not transitive by Theorem 2.2.

A. S. Besicovitch has proved that the following homeomorphism is a transitive function from the plane into the plane (1, 63). Define

$$T(re^{i\phi}) = re^{i\phi}e^{f(\phi)+\delta}$$

where f is periodic of period 2π such that $\int_0^{2\pi} f(\phi)d\phi = 0$ and δ is an irrational multiple of π.

The previous example shows a transitive continuous function on $E^1 \times E^1$. One is tempted to conjecture that if transitive functions exist on X and Y, they do on X x Y. However, the next example will show that it is possible to have transitive continuous functions on spaces X, Y and no transitive function on X x Y. It is noticed, however, that X is not a connected topological space.

<u>Example 3.4:</u> Let $X = \{1/n\}_{n=1}^{\infty} \cup \{0\}$, with the relative topology of the reals. Let $f: X \to X$ be defined by $f(1/n) = 1/(n+1)$, $f(0) = 0$, then f is a transitive

continuous function. Moreover, it will be shown that any transitive function on X is topologically equivalent to f.

Let g be any transitive continuous function defined on X. Suppose $x \neq y$, x, $y \in X$ and $g(x) = g(y)$.

Case 1: $y = 0$, x not a limit point of X-x.

Let x_o be a transitive point then there is an integer n such that $g^n(x_o) = x$ because x is not a limit point of X-x, hence, not a limit point of $\{g^n(x_o)\}$ but $x \in \overline{g^n(x_o)}$.

Since g is continuous, for any sequence $\{x_n\}$ converging to zero, we have $\{g(x_n)\}$ converges to g(0). Now the only limit point in the space is zero, therefore, $g(0) = 0$. Hence, for the point x_o chosen above, we have

$$g^{n+1}(x_o) = g(g^n(x_o)) = g(x) = g(y) = g(0) = 0.$$

Therefore, the orbit of x_o under g is finite, namely x_o, $g(x_o)$, $g^2(x_o)$, $\cdot \cdot \cdot$, $g^n(x_o) = x_o$. Thus g is not transitive in this case.

Case 2: Suppose $x \neq 0$ and $y \neq 0$.

In this case $\overline{X-x} = X-x$ and $\overline{X-y} = X-y$. Assume x_o is a transitive point, then as in case 1 we have $g^n(x_o) = x$ for some n and $g^m(x_o) = y$ for some m. Choose the smallest of the integers m, n. Now $g(g(x_o)) = g^{n+1}(x_o) = g(x) = g(y) = g^{m+1}(x_o)$. If $m = n$, we have $x = y$, a contradiction. There is no loss in generality in assuming $n > m$, then $n = m+k$ for some integer k, therefore, $g^{m+1}(x_o) = g^{n+1}(x_o)$ implies $g^{m+1}(x_o) = g^{m+k+1}(x_o)$, and thus g is periodic of period k at the point $g^{m+1}(x_o)$, and, therefore, the orbit of x_o under g is $\{x_o, g(x_o), g^2(x_o), \cdot \cdot \cdot, g^{m+1}(x_o), \cdot \cdot \cdot, g^{m+k}(x_o)\}$ which is finite. Hence, g is not transitive in case 2.

Hence, if $x \neq y$, we have $g(x) \neq g(y)$ which implies that g is one-to-one. Now g is continuous and one-to-one and X is compact, therefore, g is a homeomorphism.

It will now be shown that there is only one transitive point relative to g.

Suppose $x_o \neq y_o$ and $\overline{x_oF} = \overline{y_oF} = X$. Then $x_o \neq 0$ and $y_o \neq 0$ since $g(0) = 0$.

Thus, there is an integer n such that $g^n(x_o) = y_o$, and an integer m such that $g^m(y_o) = x_o$. Therefore, $g^{mn}(y_o) = g^n(g^m(y_o)) = g^n(x_o) = y_o$, and thus g is periodic at y_o of period mn.

Define $h:X \to X$ as follows: $h(x_o) = 1$, $h(g^n(x_o)) = 1/(n+1)$ and $h(0) = h(g(0)) = 0$, then h is a homeomorphism.

Now,

$$hgh^{-1}(1/n) = hg(g^{n-1}(x_o)) = hg^n(x_o) = 1/(n+1) = f(1/n)$$

$$hgh^{-1}(0) \quad = h(g(0)) = h(0) = f(0).$$

Therefore, g is topologically equivalent to f.

Hereafter when we speak of a transitive function on X we will mean the function f defined on page 29.

Now let $I = [0, 1]$ and let X be the space defined on page 29.

Consider $X \times I$. I allows transitive functions and X allows transitive functions, but it will now be shown that there is no transitive function from $X \times I$ to $X \times I$.

Suppose $h:X \times I \to X \times I$ is a transitive function. Let $I_n = X \times (1/n)$, then $h(I_n) \subset I_k$ for some integer k because h is continuous and because the continuous image of a connected set is connected. Define $\hat{g}(1/n) = 1/k$, and it is seen that $\hat{g}$ is transitive on X if h is transitive on X x I. Thus, $\hat{g} \approx f$.

Furthermore, if h is transitive at the point (1/n, x), then g is transitive at 1/n. But by the proof above g has only one transitive point, say p. Therefore, if h is transitive, then there exists x such that $0 \leq x \leq 1$ where h is transitive at (p, x).

Choose any point $(p, x) \in I_p$, then $\overline{\bigcup_{n=0}^{\infty} h^n(p,x)} \cap I_p = (p,x)$ because there exists no integer $n_o > 0$ such that $g^{n_o}(p) = p$ because then g would be periodic at p. Therefore, $\overline{\bigcup_{n=0}^{\infty} h^n(p,x)} \cap I_p = (p,x)$ which means that

$$\overline{\bigcup_{n=0}^{\infty} h^n(p,x)} \neq X \times I.$$

Hence, h is not transitive.

Now, as a last result in this chapter a theorem concerning the composite of two transitive functions is given.

Theorem 3.4: The composite of two transitive functions is not necessarily a transitive function.

Proof: What A. S. Besicovitch actually proved in the example mentioned above is that the following is a transitive homeomorphism from the plane into the plane.

$$T(re^{i\phi}) = re^{i\phi}e^{f(\phi)+\delta},$$

whenever f is periodic and continuous of period 2π such that $\int_0^{2\pi} f(\phi)d\phi = 0$, and δ is an irrational multiple of π (1).

Since T is a homeomorphism, then T^{-1} exists. Now

$T^{-1}(re^{i\phi}e^{f(\phi)+\delta}) = re^{i\phi} = re^{i\phi}e^{f(\phi)+\delta}(e^{-f(\phi)-\delta})$, thus, $T^{-1}(re^{i\phi}) = re^{i\phi}e^{-f(\phi)-\delta}$.

The function f is periodic of period 2π, hence, $f(k) = f(k+2\pi)$, therefore, $-f(k) = -f(k+2\pi)$ and it is seen that -f is periodic of period 2π. Now $\int_0^2 -f(\phi)d\phi = -\int_0^2 f(\phi)d\phi = 0$. Now δ is an irrational multiple of π, therefore, $-\delta$ is an irrational multiple of π. Therefore, -f and $-\delta$ satisfy the conditions of Besicovitch's example, and thus T^{-1} is a transitive function. Now $T \circ T^{-1} = I$ which is certainly not transitive.

CHAPTER IV

ON THE LIMIT OF ORBITS

Definition 4.1: A single valued continuous transformation, $f(M) = M$, of a compact space onto itself is said to be pointwise periodic if for each point x of M there is a positive integer n such that $f^n(x) = x$.

Remark: Such a transformation is necessarily one-to-one, hence a homeomorphism.

Definition 4.2: By the orbit of a point x under f, or more briefly a point orbit, we shall mean the set of all y in M such that $f^n(x) = f^m(y)$ for some integers $m,n \geq 0$.

The definition of component orbit under a pointwise periodic homeomorphism is given in Whyburn (7, 259) . The definition for an arbitrary function is given below.

Definition 4.3: If C is a component of an invariant set G, then $\bigcup_{n=0}^{k} f^n(C)$ is the component orbit of C relative to G under f.

Definition 4.4: Let $f(M) = M$, then f is componentwise periodic if for each component C of M there is an integer n such that $f^n(C) \subset C$.

The following theorem concerning the limit set of a convergent sequence of point orbits is found in (3, 720) .

Theorem A: If M is a compact metric space, $f(M) = M$, a pointwise periodic transformation, $\{G_i\}$ a convergent sequence of point orbits under f with limit L, and if there is in L a connected set B such that f(B) = B, then L is connected.

After looking at the problem from a different point of view, Schweigert was able to prove the following theorem from which Theorem A follows as a corollary (5, 964).

Theorem B: If M is a compact metric space, f(M) = M a pointwise periodic transformation and $\{G_i\}$ a convergent sequence of closed component-orbits under f with limit set L, then L is likewise a closed component orbit.

The first work with component-orbits was done by L. Whyburn (8). In her paper, she studies the set of points that remian fixed under a topological transformation of a set M into itself. It is shown that the components of the complement of such a set of fixed points in M fall into groups of two types.

Haddock generalized Whyburn's results by relaxing the condition that the transformation be topological (2).

The results of Theorem A were localized by A. D. Wallace (6, 65). He obtained the following results.

Theorem C: Let $f:X\rightarrow X$ be a pointwise periodic homeomorphism on the totally disconnected locally compact Hausdorff space X, let $a \in X$ and let A be an open set about a. If $f^m(a) = a$, then there is a compact open set V with

$$a \in V = f^m(V) \subset A.$$

Corollary: If $f:X\rightarrow X$ is a pointwise periodic homeomorphism on the locally compact totally disconnected Hausdorff space X and if A is a compact open set in X, then there is an integer n such that

$$f^n(A) = A.$$

It occurred to me that one should be able to generalize Schweigert's results also. The purpose of this chapter is to generalize Theorem B by dropping the condition that the space be metric and by relaxing the condition that the transformation be topological.

Theorem 4.1: If M is a compact Hausdorff space, $f(M) = M$ a continuous, componentwise periodic, monotone transformation, $\{G_i\}$ a convergent sequence of closed component orbits under f with limit L, then L is likewise a closed component orbit.

In order to prove the theorem, a sequence of lemmas will first be presented.

Lemma 1: Under the conditions of the theorem $f(L) = L$, and L is closed.

Proof: (a) $f(L) \subset L$. Let x be a point of L, then since L is a limit set, there is a sequence $\{x_i\}$ of points in $\{G_i\}$ such that the sequence $\{x_i\}$ converges to x. Now $f(x_i)$ is in G_i for each i and since f is continuous $\{f(x_i)\}$ converges to $f(x)$. Since $f(x)$ is the limit of a sequence of points of $\{G_i\}$, we have $f(x)$ is in L.

Therefore $f(L) \subset L$. (1)

(b) $L \subset f(L)$. Let x be a point of L. It will be shown that there is a point y in L such that $f(y) = x$.

Since x is in L, there is a sequence $\{x_i\}$ of points converging to x such that x_i is in G_i for each i. Now $f^{-1}(x_i)$ is a connected set contained in G_i since f is monotone and G_i is a component orbit. Consider the sequence $\{f^{-1}(x_i)\}$, this sequence of connected sets has a limit set, say B, since M is a compact space and B is contained in L since L is the sequential limit set of $\{G_i\}$. Therefore, B is non-empty and contained in L.

Choose a point y in B, we wish to show that f(y) = x. Since y is in B, there exists a sequence of points $\{y_j\}$ in the sequence of sets $\{f^{-1}(x_i)\}$ such that $\{y_j\}$ converges to y. Now y_j in $f^{-1}(x_i)$ implies that $f(y_j) \subset f(f^{-1}(x_i)) = x_i$, therefore, $f(y_j) = x_i$ which implies that $\{f(y_j)\}$ converges to x, but since the function is continuous $\{f(y_j)\}$ converges to f(y). Thus, f(y) = x since the limit is unique.

Hence $L \subset f(L)$. (2)

(1) and (2) imply that L = f(L).

Since L is the limit of a sequence of closed sets, it is closed.

Lemma 2: Under the conditions of the theorem if $L=L_1 \cup L_2$ is a separation of L into components, then $f^n(L_i) = L_i$, i=1, 2 for some n=1, 2, $\cdots$.

Proof: Assume the conclusion is false, then we may assume that $f^n(L_1) \neq L_1$ for all n. Therefore, $f^n(L_1) = L_2$ for all n since $f(L) \subset L$ by the proof of Lemma 1 and since f is monotone. This means that $f(L_2) = f(f^n(L_1)) = f^{n+1}(L_1) = L_2$. Therefore, L_1 is not the image of a component of L contrary to the proof of Lemma 1 that $L \subset f(L)$. This gives a contradiction.

Thus, $f^n(L_i) = L_i$, $i = 1, 2$ for some $n = 1, 2, \cdots$.

<u>Lemma 3:</u> Under the conditions of the theorem, let A_k be the orbit of a component k of M. If j is a component of M and j is in A_k, then $A_k = A_j$.

<u>Proof:</u> Since j is in A_k then $f^n(j) = f^m(k)$ for some integers m and n, therefore, k is in A_j by the definition of component orbit.

Let g be a component of A_j, then there are integers s and t such that $f^s(g) = f^t(j)$. But we have $f^n(j) = f^m(k)$ which implies that for all integers $i > 0$, $f^{n+i}(j) = f^{m+i}(k)$ so there is no loss of generality in supposing that $n > t$. Therefore,

$$f^{s+n-t}(g) = f^n(j) = f^m(k),$$

so that g is in A_k. Thus, $A_j \subset A_k$.

By choosing a component h of A_k and repeating the argument, we see that $A_k \subset A_j$. Thus $A_k = A_j$.

This means that the orbits, such as A_k, are disjoint when different.

<u>Proof of the theorem:</u> It has been shown that $f(L) = L$ and L is closed. It must now be shown that L is a component orbit.

(1) If L is connected, then L is already a component relative to itself and since $f(L) = L$, L is a component orbit.

(2) Assume L is not connected.

Let K_o be a component of L and let $f(K_o) = K_1$, $f(K_1) = K_2, \cdots, f(K_{m-1}) = K_m$, $f(K_m) = K_o$, so that $K_o, K_1, \cdots, K_m$ is a component orbit and let $K = \bigcup_{i=0}^{m} K_i$. It will be shown that K = L.

Assume $K \neq L$, then L is separated and closed. Hence, there exists a neighborhood V of K having the property that the boundary F(V) is disjoint with L and also V is disjoint with L-K, that is,

$$F(V) \cap L = \emptyset, \ (M-V) \cap L \neq \emptyset.$$

Since K_o is in L, there exists a sequence of components $\{A_i\}$ such that A_i is in G_i and $\{A_i\}$ converges to K_o. Since almost all A_i are in V, we may suppose all are in V. For each i, let B_i be the first component in the sequence $f(A_i), f^2(A_i), \cdots$ belonging to M-V. We may again suppose B_i exists for all i because almost all G_i must intersect M-V. A subsequence of $\{B_i\}$, which we may suppose is the whole sequence, converges to a component C of $L \cap (M-V)$. Now the sets $f^{-1}(B_i)$ are connected since f is monotone, and they are in V, furthermore, they are contained in a sequence of components because the components are the maximal connected sets. By an argument like that given in part (b) of the proof of Lemma 1, there is a subsequence $\{D_i\}$ of the sequence $\{f^{-1}(B_i)\}$ converging to a limit set D such that $D \cap f^{-1}(C) \neq \emptyset$ and D is in $(V \cap L)$-K because if D were in K, then $f(D) \subset C$ would be in K.

It will now be shown that $f^{-1}(C)$ is in $(V \cap L)$-K. Since f(L) = L by Lemma 1, we have $f^{-1}(C)$ is in L. Suppose

$f^{-1}(C)$ is not in V, then $f^{-1}(C) = D \cup E$ where D is in $V \cap L$ and E is not in V and since $F(V) \cap L = \emptyset$, we have $D \cap E = \emptyset$. Now if $D \cap \bar{E}$ and $\bar{D} \cap E$ are non-empty, their point of intersection must be in F(V) but each set is in L, hence their point of intersection is in L because L is closed. This contradicts $F(V) \cap L = \emptyset$.

Since $f^{-1}(C) \subset (V \cap L)-K$, there exists an integer N_1 such that $f^{-1}(B_i) \cap A_i = \emptyset$ for $i > N_1$ because if $f^{-1}(B_i) \cap A_i \neq \emptyset$ for all $i = 1, 2, \cdots$, we could find a subsequence of the $\{f^{-1}(B_i)\}$ that converges to K_o and $f(K_o) \neq C$ because $f(K_o)$ is in K.

In the same way, the connected sets $f^{-2}(B_i)$ are in V for $i > N_1$, and we may argue as above that $f^{-2}(C)$ is in $(V \cap L)-K$, and there exists an integer $N_2 > N_1$ such that $f^{-2}(B_i) \cap A_i = \emptyset$. Again, $f^{-3}(B_i)$ for $i > N_2$ are in V and $f^{-3}(C)$ is in $(V \cap L)-K$ so find $N_3 > N_2$ such that $f^{-3}(B_i) \cap A_i = \emptyset$.

If we continue in this manner, we will obtain $f^{-n}(C)$ is in V contrary to the periodicity at C which would require that $f^{-n}(C) \cap C \neq \emptyset$ for some n, where $C \subset M-V$.

Corollary: If L contains a fixed point or a fixed component, then L is connected.

The proof of the corollary follows immediately from the proof of the theorem.

The following example shows that Theorem 4.1 is actually a generalization of Theorem B. Although the example is trivial, the imaginative reader can easily see that more complicated examples can be constructed in a similar manner.

<u>Example 4.1:</u> Example of a continuous componentwise periodic, monotone transformation which is not pointwise periodic.

Let $M = [0,1]$ with the relative topology from the reals.

Define

$$f(x) = \begin{cases} 3/2x, & 0 \leq x \leq 1/3 \\ 1/2, & 1/3 \leq x \leq 2/3 \\ 3/2x-1/2, & 2/3 \leq x \leq 1 \end{cases}$$

Clearly f is a continuous monotone transformation. Since M has only one component, namely M itself, and $f(M) = M$, then f is componentwise periodic. However, f is not pointwise periodic because $f^n(1/3) = 1/2$ for all n.

CHAPTER V

SUMMARY

The primary objectives of this paper are to study the orbit of a point under a transitive continuous function, to construct some examples of transitive continuous functions in particular topological spaces and to consider the limit of a convergent sequence of components under a componentwise periodic monotone transformation.

The orbit of a point x and transitive function are defined, then in Chapter II some fundamental properties of transitive functions are proved. Included among these results is a theorem which states that a point x is transitive with respect to F if and only if, given two arbitrary non-empty open subsets of the space X, then $UF \cap V \neq \emptyset$. This result is used extensively in Chapter III to construct examples of transitive continuous functions. There a transitive continuous function is constructed from the closed unit interval into itself. A transitive continuous function from the open unit interval into itself is also constructed. It follows from this that there exists a transitive continuous function from E^1 into E^1.

One of the main results of this paper is given in Chapter IV. It states that if M is a compact Hausdorff space, $f(M) = M$ a continuous, componentwise periodic, monotone transformation $\{G_i\}$ a convergent sequence of closed component orbits under f with limit L, then L is likewise a closed component orbit. This is a generalization of a theorem due to Schweigert (5) .

The following are some questions for further study. What are necessary and sufficient conditions on a function to insure that it is transitive? What restrictions must be placed on the space X to insure that a transitive function exists on X? For any continuous mapping $f:X \to X$ let

$$T = \{x : x \text{ is transitive with respect to } x\},$$

$$A = \{x : \text{there is a } g \text{ in } F \text{ such that } g(x) = x\} \text{ and}$$

$$L = \{x : \text{the limit of } f^n(x) \text{ exists}\} .$$

Are there other points and can they be characterized? Given an everywhere dense set A is there an F such that A is the set of transitive points of F.

A SELECTED BIBLIOGRAPHY

(1) Besicovitch, A. S., "A Problem on Topological Transformations of the Plane", Fundamenta Mathematicae, 28, (1937) 61-65.

(2) Haddock, A. G., "Rotation Groups Under Monotone Transformations", Fundamenta Mathematicae, 53, (1964) 173-175.

(3) Hall, D. W. and G. E. Schweigert, "Properties of Invariant Sets Under Pointwise Periodic Homeomorphisms", Duke Mathematical Journal, 4, (1938) 719-724.

(4) Oxtobey, J. C., "Note on Transitive Transformations", Proceedings of the National Academy of Science, 23, (1937) 443-446.

(5) Schweigert, G. E., "A Note on the Limit of Orbits", American Mathematical Society Bulletin, 46, (1940), 963-969.

(6) Wallace, A. D., "A Local Property of Pointwise Periodic Homeomorphisms", Colloquium Mathematicum, 9, (1962), 63-65.

(7) Whyburn, G. T., Analytic Topology, American Mathematical Society Colloquium Publications, 28, (1963).

(8) Whyburn, L., "Rotation Groups About a Set of Fixed Points", Fundamenta Mathematicae, 28, (1937) 124-130.

VITA

W. David Moon

Candidate for the Degree of

Doctor of Philosophy

Thesis: A STUDY OF ORBITS AND TRANSITIVE CONTINUOUS FUNCTIONS

Major Field: Mathematics

Biographical:

Personal Data: Born in Rose Bud, Arkansas, June 2, 1933, the son of Jack and Della Moon.

Education: Attended elementary school in Rose Bud, Arkansas; was graduated from Rose Bud High School, Rose Bud, Arkansas, in 1951; received the Bachelor of Science degree from Arkansas State Teachers College, Conway, Arkansas, with a major in mathematics, in May, 1958; received the Master of Arts degree at the University of Tennessee, Knoxville, Tennessee, in May, 1962; was a participant in a National Science Foundation Research Participation Program in the summer of 1964 at the University of Oklahoma, Norman, Oklahoma; was the recipient of a National Science Foundation Science Faculty Fellowship and attended Oklahoma State University at Stillwater, Oklahoma, in 1965-66; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in July, 1967.

Professional Experience: Taught mathematics as a teaching assistant at the University of Tennessee, Knoxville, Tennessee, 1959-1960; Instructor in the Department of Mathematics, University of Tennessee, Knoxville, Tennessee, 1960-1962; Consultant, Oak Ridge National Laboratory, Oak Ridge, Tennessee, 1961-1962; Assistant Professor of Mathematics, Southern State College, Magnolia, Arkansas, 1962-1963; Associate Professor of Mathematics, Arkansas College, Batesville, Arkansas, 1963-1965; Associate Professor of Mathematics and Chairman of the

Science Division, Arkansas College, Batesville, Arkansas, 1966-1967; Part-time instructor in the Extension Division of the University of Arkansas, Fayetteville, Arkansas, 1966-1967; Visiting Scientist Lecturer for the Arkansas Academy of Science, 1966-1967.

Organizations: Member of the Mathematical Association of America, the Arkansas Council of Teachers of Mathematics, the Arkansas Academy of Science; institutional member of the American Mathematical Society.