THE FUNDAMENTAL STATE-VARIABLE
FREQUENCY MATRIX

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in partial fulfillment of the require-
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DOCTOR OF PHILOSOPHY
May, 1967

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ACKNOWLEDGMENT

All documents such as this which finalizes an important phase of the writer's life are the results of inspiration contributed by many individuals. To all those from whom I have received inspiration I extend my deepest appreciation.

My sincere thanks to Dr. E. C. Fitch for his constant encouragement and confidence during my entire doctorate program, to Dr. William A. Blackwell for acting as my technical adviser and for his sincere interest in my work, and to Dr. Karl N. Reid and Professor Paul A. McCollum for their contributions to my graduate program as members of my graduate committee.

To Frank H. Winters and the Caterpillar Tractor Company my thanks for the confidence and financial assistance. Also, thanks to the members of the Catepillar Project Messrs. Jerry G. Mrazek, James E. Bose, and M. Allen Ross for their invaluable assistance and friendship throughout the project.

To my wife, Naomi, and my son, Steve, both of whom made my years of graduate study possible as well as worthy, my sincere thanks.

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LIST OF SYMBOLS

$\underline{A}$ . . n by n - dimensional differential transition matrix.

a . . mode description integer.

a_n . . coefficients of homogeneous system mathematical model.

$\underline{B}$ . . n by p - dimensional input matrix.

b . . mode description integer.

b_m . . coefficient of system mathematical model forcing function.

$\frac{a_n \omega^n}{b_m \omega^m}$ . . transmission asymptote functions.

$\frac{b_{m-1} \omega^{m-1}}{b_m \omega^m}$ . . transmission bandwidth boundaries.

$\underline{C}$ . . q by n - dimensional output matrix.

C . . capacitance in electrical network.

c . . mode description integer.

c_n . . output matrix elements.

$\underline{D}$ . . q by p - dimensional transmission matrix.

D . . damping coefficient in mechanical system.

db . . decibels, -20Log_{10} T.

$\underline{F}$ . . Fundamental State-Variable Frequency Matrix.

$F_n(\omega)$ . . complex functions.

$I(\omega)_c$ . . imaginary part of the characteristic polynomial when written as a complex function.

LIST OF SYMBOLS (Continued)

$\underline{I}$	. . identity matrix whose main diagonal is all ones with all the other entries zeros.
i	. . integer 0,1,2,...,etc.
$\underline{J}$	. . Jordan Canonical Form of the differential transition matrix.
j	. . $\sqrt{-1}$.
K_n	. . constant.
k	. . spring coefficient in mechanical system.
L	. . inductance in electrical network.
l	. . integer 0,1,2,...,etc.
$\underline{M}$	. . modal matrix.
M	. . mass of mechanical system.
m	. . integer subscript 0,1,2,...,etc.
n	. . integer subscript 0,1,2,...,etc.
$\underline{P}$	. . (n+q) by (n+q) - dimensional coefficient matrix.
$\underline{Q}$	. . (n+q) by n - dimensional coefficient matrix.
$\underline{R}$	. . (n+q) by p - dimensional coefficient matrix.
$\underline{R}$	. . Rational Canonical Form of the differential transition matrix.
R	. . resistance in electrical network.
$R(\omega)_c$	. . real part of the characteristic polynomial when written as a complex function.
r_n	. . roots of the characteristic polynomial.
$\underline{S}_m$	. . Spectrum Band-pass Matrix displaying entries which characterize the frequency spectrum in a specific frequency range.
s	. . Laplace Transform complex variable.
T	. . system transmissibility or gain, $\left\lvert \frac{y}{v} \right\rvert$.

LIST OF SYMBOLS (Continued)

$\underline{v}$	. . p - dimensional input vector.
$\underline{w}$	. . n+q - dimensional column vector with components $\dot{x}_1$, $\dot{x}_2 \ldots \dot{x}_n$, y_1, $y_2 \ldots y_q$.
$\underline{x}$	. . n - dimensional state vector with components x_1, $x_2, \ldots x_n$.
$\dot{\underline{x}}$	. . n - dimensional derivative of the state vector.
$\underline{y}$	. . q - dimensional output matrix.
$\underline{z}$	. . n - dimensional frequency state vector with components z_1, $z_2, \ldots z_{n-1}$, $-z_n$.
$\underline{z}'$	. . n - dimensional derivative of the frequency state vector.
α	. . compensator lead time constant.
β	. . compensator lag time constant.
$\underline{\Gamma}$	. . Frequency Canonical Form input matrix.
$\underline{\Omega}$	. . Frequency Transformation Matrix.
ω	. . driving frequency.
ω_n	. . natural undamped mode frequency.
τ/ω	. . frequency transform time.
$\cdot$	. . differentiation with respect to time.
$'$	. . differentiation with respect to frequency transform time.
$\_$	. . matrix notation.
$[\;\;]^T$	. . transpose of a matrix.
$[\;\;]^{-1}$	. . inverse of a matrix.
$\doteq$	. . approximately equal.

CHAPTER I

INTRODUCTION

The state-variable approach to the analysis of scientific problems, which includes nearly all the engineering disciplines, is far from new.[1] For many years this basic approach has been used by physicists in the description of dynamic occurrences in classical mechanics and quantum mechanics. However, many of the more recent advancements in the topological mathematical theory which are directly applicable to state space analysis have not, as yet, been employed by either the physicist or the engineer. Also, there exist many concepts in both of these fields of applied mathematics which have not been put in a proper perspective with reference to state models. Due to the increased interest in the state-variable methods as applied to control system analysis, a great deal of emphasis is being placed on the integration of classical and modern control methods.[2]

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[1]The term "state-variable" infers the description of system models in finite dimensional state space. These models contain explicit mathematical descriptions of the system's state in a vector matrix form. Appendix D shows a method for modeling systems in state space.

[2]Modern control methods and modern control approach to system analysis are the techniques of control system analysis employing state-variable models.

Motivation for the increased emphasis on state-variable methods by systems engineers stems from two sources. The first source is the increased efforts being placed on optimization. Most optimization procedures have been developed from a state space standpoint. Thus, in order to apply these theories to physical systems consistency of model domain and optimization technique domain is a necessity. State-variable representations for physical systems provides this time domain match with the optimization theories.

The second source of motivation is the need for organizing the analysis methods from the systems viewpoint. For the analysis of linear time invariant systems classical control theory provides transform techniques which allow the system equations to be readily solved. However, when this class of systems is extended to include time variance, these transform techniques are no longer readily applicable. Further, when nonlinear terms are introduced classical theory employs a variety of techniques to obtain a system evaluation. Thus, classical theory does not possess any particular technique which can be extended to cover all classes of system models. The state-variable methods can be extended to model both stationary as well as nonstationary systems and linear as well as nonlinear systems coupled with the time variance or time invariance characters. This property of the state-variable approach offers the framework for a more unified organization of systems analysis.

The basic principle of the state-variable methods is the description of the system mathematical model as a system of first order differential equations. This system of equations can be expressed in a form to which mathematical topology is directly applicable.

The particular class of mathematical models to be considered herein are those describable by linear, time invariant (stationary), ordinary differential equations. The state-variable representations of this class of system models find themselves amenable to the application of many subtopics of mathematical topology. These particular subtopics include matrix algebra and finite dimensional vector spaces. Both of these have a very rigidly developed history which, when applied to system analysis, provides a well founded analytical basis.

State-Variable Models

For any particular system there are many forms in which the state-variable model can be written. The particular form of the state model is dependent on the technique employed to obtain the model. Appendix D describes a method for modeling dynamic systems in state space. This appendix shows how the two state models presented below can be derived.

The most general state-variable models are the unreduced models. The mathematical presentation of these models is described in the following manner (1).[3] A system of linear, first order, differential equations with constant coefficients can be expressed completely by the matrix equation

$$\underline{P}\ \underline{w} = \underline{Q}\ \underline{x} + \underline{R}\ \underline{v} \tag{1-1}$$

where

$\underline{w} = \underline{w}(t) = \begin{bmatrix} \dot{\underline{x}} \\ - \\ \underline{y} \end{bmatrix}$, which is an (n+q) - dimensional column vector with components $\dot{x}_1, \dot{x}_2, \ldots \dot{x}_n, y_1, y_2, \ldots y_q$.

$\underline{x} = \underline{x}(t)$ = the n - dimensional <u>state vector</u> with components $x_1, x_2, x_3, \ldots x_n$.

$\underline{y} = \underline{y}(t)$ = the q - dimensional <u>output vector</u>.

$\underline{v} = \underline{v}(t)$ = the p - dimensional <u>input vector</u> with components $v_1, v_2, v_3, \ldots v_p$.

$\dot{\underline{x}} = \dot{\underline{x}}(t)$ = the n - dimensional <u>derivatives of state vectors</u>.

$\underline{P}$ = a coefficient matrix of dimension (n+q) by (n+q).

$\underline{Q}$ = a coefficient matrix of dimension (n+q) by n.

$\underline{R}$ = a coefficient matrix of dimension (n+q) by p.

Reduction of the unreduced model to standard state space form gives the following two equations

$$\dot{\underline{x}} = \underline{A}\ \underline{x} + \underline{B}\ \underline{v} \tag{1-2}$$

$$\underline{y} = \underline{C}\ \underline{x} + \underline{D}\ \underline{v} \tag{1-3}$$

where $\underline{x}$, $\dot{\underline{x}}$, $\underline{v}$, and $\underline{y}$ are vectors as defined in the unreduced

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[3]Numbers appearing in parentheses within the text refer to references on pages 117 and 118.

model. The coefficient matrices are as follows

$\underline{A}$ = n by n <u>differential</u> <u>transition</u> <u>matrix</u>

$\underline{B}$ = n by p <u>input</u> <u>matrix</u>

$\underline{C}$ = q by n <u>output</u> <u>matrix</u>

$\underline{D}$ = q by p <u>transmission</u> <u>matrix</u>.

The elements of the coefficient matrices are constants. These constants are either real or complex and for any particular system one form of the state-variable model can have elements which are complex numbers while another form of the state-variable model for the same system may have elements which are all real numbers. Therefore, a single state-variable model for a system is not unique. As many state models can be written for a system as there are combinations of the significant constants associated with any particular system. Three of the more familiar forms of the differential transition matrix are the Jordan, Rational, and Phase-Variable Canonical forms. The system characteristics explicitly displayed by the elements of the Jordan Canonical Form are the system's characteristic modes (roots). The Rational Canonical and the Phase-Variable Canonical forms display the coefficients of the system's characteristic equation as elements in the differential transition matrix.

Fundamental State-Variable Frequency Canonical Form

Due to the lack of uniqueness of the state model forms including any of the Canonical forms discussed above, the analysis procedures used in classical control theory can be represented in one form or another by the coefficient matrices associated with the state-variable models. This fact permits modern control theory to provide the much needed unification of control theory from a system analysis approach.

One of the analysis procedures which produces most of the system characteristics employed in classical control theory is the system real frequency response. The frequency response method evaluates the system closed loop characteristics by investigating the open loop transfer function. The frequency response approach has not as yet been employed in the modern approach to system analysis. At the present time the frequency response transfer function of a system described by a state-variable model is obtained from the state model in exactly the same form as it appears in classical system analysis. Direct application of classical techniques is much easier. With presently available modern control techniques it is not advantageous to follow the same or similar paths followed in the development of classical control theory.

There are various available means for constructing a system's frequency response spectrum from the state-variable models as will be shown in the following chapter. These techniques are based on the parallel developments in classical control theory; however, since no additional information is obtained, no particular advantage is gained by employing the state-variable approach instead of the well known transfer function method.

The "Fundamental State-Variable Frequency Matrix" developed in this dissertation demonstrates that from a frequency response standpoint there is a definite advantage to system state-variable modeling. The need for this matrix will occur particularly when a system modeled in state space is to be analyzed through its frequency response spectrum. The "Fundamental State-Variable Frequency Matrix" displays explicitly the critical gains, asymptotes, and frequencies associated with the system frequency response. These fundamental characteristics are derived from the entries in the "Fundamental State-Variable Frequency Matrix".

The information derived from the basic theory of the "Fundamental State-Variable Frequency Matrix" has application to compensation and synthesis as well as analysis. This theory has a rather unique inverse in its application by direct utilization of experimental frequency response information. Since the frequency matrix contains critical gains, asymptotes, and frequencies, experimental frequency response data can be used to fill out the matrix. Thus, the state-variable model can be systematically derived from test data.

Since the state models are not unique, it is possible to obtain the "Fundamental State-Variable Frequency Matrix" from a system's mathematical model in one of two ways. First, the state model can be written by any one of the standard available means which results in a canonical form.

With this model a transformation is performed resulting in the frequency matrix. The transformation necessary to transform state models into the Frequency Canonical Form is developed in this dissertation. Second, the state model can be written directly in the Frequency Canonical Form by a direct programming method developed in this dissertation.

CHAPTER II

SIGNIFICANT HISTORICAL CONTRIBUTIONS

The use of state-variable models has only recently received a great deal of emphasis by systems engineers. Consequently, no work has been exerted to express the system's frequency response spectrum utilizing modern control theory state models. However, four methods are presented by which the frequency response spectrum can be constructed from state-variable models. The first of these methods was developed from Brockett's work (2). Brockett's work, as will be shown, develops the transfer function of the system in a matrix form from which the frequency spectrum can be constructed using Bode's Theorems. The remaining three methods presented employ work of other investigators which has been modified by the author to use state models instead of the frequency domain transfer function. The first of these methods to obtain the frequency response from a state model is restricted to those systems which have simple forcing functions. This method employs the Jordan Canonical Form of the state model. The other two methods of drawing the frequency response spectrum from a state model involves the combining together of individually developed techniques. One of these, developed by Smith (3), which has some limitations, uses a direct programming technique to produce the state model in canonical form and a graphical method for drawing the frequency response spectrum. The other one uses the same means of developing the state model but the frequency response spectrum is obtained by a method developed by Ausman (4). The direct programming techniques referred to are discussed in Appendix A.

Brockett's State Model Transfer Function

Brockett's work considered the class of linear, time-invariant systems which can be described by the reduced state-variable models shown in the following vector matrix equations.

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{v}(t) \qquad (2\text{-}1)$$

$$\underline{y}(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,\underline{v}(t) \qquad (2\text{-}2)$$

In these equations $\underline{v}(t)$ and $\underline{y}(t)$ are the system input and output matrices, respectively, and $\underline{x}(t)$ is the system state-variable matrix. In this work it is assumed that $\underline{v}(t)$ and $\underline{y}(t)$ are vectors of the same dimension, say q, and that $\underline{x}(t)$ is a vector of dimension n. Particular emphasis is placed on the class of systems which have a single input and a single output. This is reflected in the state model by setting the dimension q equal to one. Consequently, $\underline{v}(t)$ and $\underline{y}(t)$ are one by one column vectors and are written as v(t) and y(t). The particular state-variable model used by Brockett is shown below.

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,v(t) \qquad (2\text{-}3)$$

$$y(t) = \underline{C}\,\underline{x}(t) + \underline{D}\,v(t) \qquad (2\text{-}4)$$

Equations (2-3) and (2-4) can be transformed by use of Laplace Transformations. The result of this transformation is

$$s\,\underline{x}(s) - \underline{x}(0) = \underline{A}\,\underline{x}(s) + \underline{B}\,v(s) \tag{2-5}$$

$$y(s) = \underline{C}\,\underline{x}(s) + \underline{D}\,v(s) \tag{2-6}$$

Equation (2-5) must be solved for $\underline{x}(s)$ in order to obtain the transfer function.

$$s\,\underline{x}(s) - \underline{A}\,\underline{x}(s) = \underline{x}(0) + \underline{B}\,v(s) \tag{2-7}$$

or

$$(\underline{I}\,s - \underline{A})\,\underline{x}(s) = \underline{x}(0) + \underline{B}\,v(s) \tag{2-8}$$

where

$\underline{I}$ is the identity matrix.

By post multiplying Equation (2-8) by $(\underline{I}\,s - \underline{A})^{-1}$ the result is

$$\underline{x}(s) = (\underline{I}\,s - \underline{A})^{-1}\,\underline{x}(0) + (\underline{I}\,s - \underline{A})^{-1}\,\underline{B}\,v(s) \tag{2-9}$$

Substituting Equation (2-9) into Equation (2-6) produces an expression of $y(s)$ in terms of $\underline{x}(0)$ and $v(s)$.

$$y(s) = \underline{C}\,(\underline{I}\,s - \underline{A})^{-1}\,\underline{x}(0) + \underline{C}\,(\underline{I}\,s - \underline{A})^{-1}\,\underline{B}\,v(s) + \underline{D}\,v(s) \tag{2-10}$$

The column vector $\underline{x}(0)$ represents all the initial conditions imposed on the system. For zero initial conditions Equation (2-10) reduces to

$$y(s) = \left[\underline{C}\,(\underline{I}\,s - \underline{A})^{-1}\,\underline{B} + \underline{D}\right] v(s) \tag{2-11}$$

The system transfer function as derived from the state-variable model is as shown in Equation (2-12).

$$\frac{y(s)}{v(s)} = \left[\underline{C} \; (\underline{I} \; s - \underline{A})^{-1} \; \underline{B} + \underline{D} \right] \qquad (2\text{-}12)$$

Since the inverse of a matrix can be written as the adjoint of the matrix divided by the determinant of the matrix, Equation (2-12) can be rewritten as follows.

$$\frac{y(s)}{v(s)} = \left[\frac{\underline{C} \; \mathrm{adj}(\underline{I} \; s - \underline{A}) \; \underline{B}}{\left| \underline{I} \; s - \underline{A} \right|} + \underline{D} \right] \qquad (2\text{-}13)$$

With this system transfer function the frequency response spectrum can be drawn by use of Bode's Theorems (5).

This method presents no advantage over the standard transfer function methods other than the fact that the state model was involved. If the analysis was initiated from the basic system mathematical model, then there is a definite disadvantage of following Brockett's procedure. This disadvantage is in the evaluation of the vector matrix equation to obtain the transfer function.

System Frequency Response Using State Models in Jordan Canonical Form

The general form of the state-variable model in Jordan Canonical Form for systems with distinct eigenvalues or unrepeated roots is shown in Equation (2-14).

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} r_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & r_2 & 0 & 0 & \cdots & 0 \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & 0 \\ 0 & \cdot & \cdots & \cdot & 0 & r_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} v \qquad (2\text{-}14)$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} v \qquad (2\text{-}15)$$

The elements in the differential transition matrix are either real numbers or complex numbers. The real or nonimaginary entries are over-damped system modes for negative entries and undamped or unstable system modes for positive entries. Complex elements will always occur in conjugate pairs since the polynomials considered have real coefficients. Therefore, complex elements with negative real parts are underdamped oscillatory system modes while those with positive real parts represent undamped or unstable oscillatory system modes. The frequencies of the complex and the real entries are the corner frequencies for the system's lag frequency response spectrum.[1] With this information it is possible to construct the asymptotes of actual lag frequency response spectrum and, by using Bode's Theorems from classical control theory, to draw the actual lag spectrum.

By examining both Equations (2-14) and (2-15) it is evident that the coefficients of the system forcing function are not explicitly displayed. Herein occurs the limitation for the use of state models in Jordan Canonical Form to determine system frequency response spectrums. There is no explicit information displayed in the state model such as forcing function coefficients which will contribute to the construction of the lead frequency response spectrum for a system. This statement is correct with the exception of simple forcing functions, e.g. simple sinusoidal inputs. For very simple system inputs the forcing function coefficients appear explicitly in the input matrix. Hence, sufficient information is displayed to allow the construction of the total frequency response spectrum.

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[1] The lag frequency response spectrum is that part of a frequency response spectrum constructed by using only the numerator of the transfer function. Also the lead frequency spectrum is that part of the frequency response spectrum constructed by using only the denominator of the transfer function or characteristic equation.

System Frequency Response Using State Models in Rational Canonical Form

The methods presented in this section to obtain the system frequency response spectrum will involve state-variable models in the Rational Canonical Form. However, the Phase-Variable Canonical Form can also be used equally as well for finding the frequency response of a system from its state model.[2] The basic difference in the uses of these two canonical forms is in the representation of the system gain. Since the Rational Canonical form presents the total gain in a combined form the developments in the following sections will be much clearer by using this form of the state model. Methods by which the state model can be derived in either Phase-Variable Canonical Form or Rational Canonical Form from the system mathematical model are presented in Appendix A.

In general terms, the state model used in the following two sections will be as shown in Equations (2-16) and (2-17).

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[2]The differences in the Rational Canonical Form and the Phase-Variable Canonical Form are primarily the locations of the coefficients of the characteristic and forcing function polynomials in the coefficient matrices. The differential transition matrix of the two Canonical forms display explicitly the coefficients of the characteristic function and are transposes of one another. The coefficients of the forcing function appear in the input matrix in the Rational Canonical Form while these coefficients appear in the output matrix in the Phase-Variable Canonical Form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & -\frac{a_1}{a_n} \\ 0 & \cdot & & & & & \cdot & \cdot \\ \cdot & & \cdot & & & & \cdot & \cdot \\ \cdot & & & \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot & & \cdot & \cdot \\ \cdot & & & & & \cdot & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} \frac{b_0}{a_n} \\ \frac{b_1}{a_n} \\ \cdot \\ \cdot \\ \frac{b_m}{a_n} \\ 0 \\ \cdot \\ 0 \end{bmatrix} v \quad (2\text{-}16)$$

$$y = \begin{bmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} v \quad (2\text{-}17)$$

If the input matrix is normalized with respect to the b_m/a_n entry in the input matrix, which is equivalent to normalizing the system forcing function with respect to the highest order terms, the result is as shown in the following state model.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & 0 & \cdots & 0 & -\frac{a_1}{a_n} \\ 0 & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ \cdot & & \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 0 & \cdot \\ 0 & \cdots & & 0 & 1 & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \frac{b_m}{a_n} \begin{bmatrix} \frac{b_0}{b_m} \\ \frac{b_1}{b_m} \\ \cdot \\ \cdot \\ 1 \\ 0 \\ \cdot \\ 0 \end{bmatrix} v \quad (2\text{-}18)$$

The scalar multiplying the input matrix is the gain of the system which is the result of normalizing both the numerator and the denominator polynomials of the system's transfer function.

The power of the state-variable techniques lies in the analysis of more complex systems, e.g. systems whose mathematical model is simultaneous, coupled, differential equations. In order to simplify the presentation of state-variable techniques it is necessary to use examples of a rather simple nature, even though these examples may be handled equally well by classical system analysis techniques.

Smith's Method

Smith's method for constructing the frequency response spectrum of a dynamic system involves the use of templets (3) and can be applied to state models. The method developed from Bode's Theorems for constructing dynamic system frequency spectrums can also be performed by use of templets. However, the templets which Smith developed do not require that the numerator and denominator be factored to find the corner frequencies and asymptotes as do those based on Bode's Theorems. The basic technique for Smith's method involves the use of a series of templets to construct the frequency spectrum from the unfactored system transfer function. Smith's method is applicable to a rational algebraic function of a complex variable as well as to transfer functions when the transform variable s is replaced by $j\omega$.

In classical control analysis the denominator of the transfer function is the characteristic equation and the numerator is the forcing function. The state model used with Smith's method for constructing the frequency spectrum is the Rational Canonical Form. In this form the coefficients of both the characteristic polynomial or characteristic equation and the forcing function are displayed explicitly.

Initially Smith's method deals with the separate polynomials from the term-pair standpoint. A term-pair is the grouping of the real and imaginary parts of the particular polynomial under investigation into pairs of successive terms. The information required by Smith's method can be obtained from the entries in the Rational Canonical Form of the differential transition matrix.

$$\underline{A} = \begin{bmatrix} 0\ 0\ .\ .\ .\ .\ 0 & -\frac{a_0}{a_n} \\ 1\ 0\ .\ .\ .\ .\ 0 & -\frac{a_1}{a_n} \\ 0\ 1 & . \\ .\quad . & . \\ .\quad\ \ . & . \\ .\qquad . & . \\ .\qquad\ \ . & . \\ 0\ .\ .\ .\ .\ 0\ 1 & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{matrix} \text{---}\ \omega^0 \\ \text{---}\ \omega^1 \\ \text{---}\ . \\ \text{---}\ . \\ \text{---}\ . \\ \text{---}\ . \\ \text{---}\ . \\ \text{---}\ \omega^{n-1} \end{matrix} \tag{2-19}$$

The elements in the right-hand column are the negative of the coefficients of the characteristic polynomial with the coefficient of the highest power normalized. To construct the lag frequency spectrum from the state model employing Smith's method the real and imaginary part of the characteristic and forcing function polynomials are written in the following form.

$$R(\omega)_c = \left[\frac{a_0}{a_n} - \frac{a_2}{a_n}\omega^2\right] + \left[\frac{a_4}{a_n}\omega^4 - \frac{a_6}{a_n}\omega^6\right] + \ldots \tag{2-20}$$

$$I(\omega)_c = \left[\frac{a_1}{a_n}\omega - \frac{a_3}{a_n}\omega^3\right] + \left[\frac{a_5}{a_n}\omega^5 - \frac{a_7}{a_n}\omega^7\right] + \ldots \quad (2\text{-}21)$$

where

$R(\omega)_c$ = real part of the characteristic polynomial when written as a complex function.

$I(\omega)_c$ = imaginary part of the characteristic polynomial when written as a complex function.

ω = the driving frequency.

Each of the pairs of terms in the parentheses is a term-pair. Noting that each of the separate terms represent linear functions of ω when the logarithm of each terms is employed, Smith constructed a series of templets he calls term-pair contours or term-pair templets. Illustrations of these templets are shown in Figures 2-1 and 2-2. With these templets it is possible to construct separate spectrums for each term-pair included in the real and imaginary part of the characteristic polynomial. This construction is done by placing the appropriate term-pair templet at the intersection of the two linear representations for the proper term-pair and drawing the contour dictated by the coefficients involved. This procedure is repeated for all term-pairs in both the real and imaginary parts resulting in the term-pair spectrums.

The next step in the construction of the frequency response spectrum is to obtain the spectrum for the real part and also the spectrum for the imaginary part by adding the separate composite contours. This step is accomplished

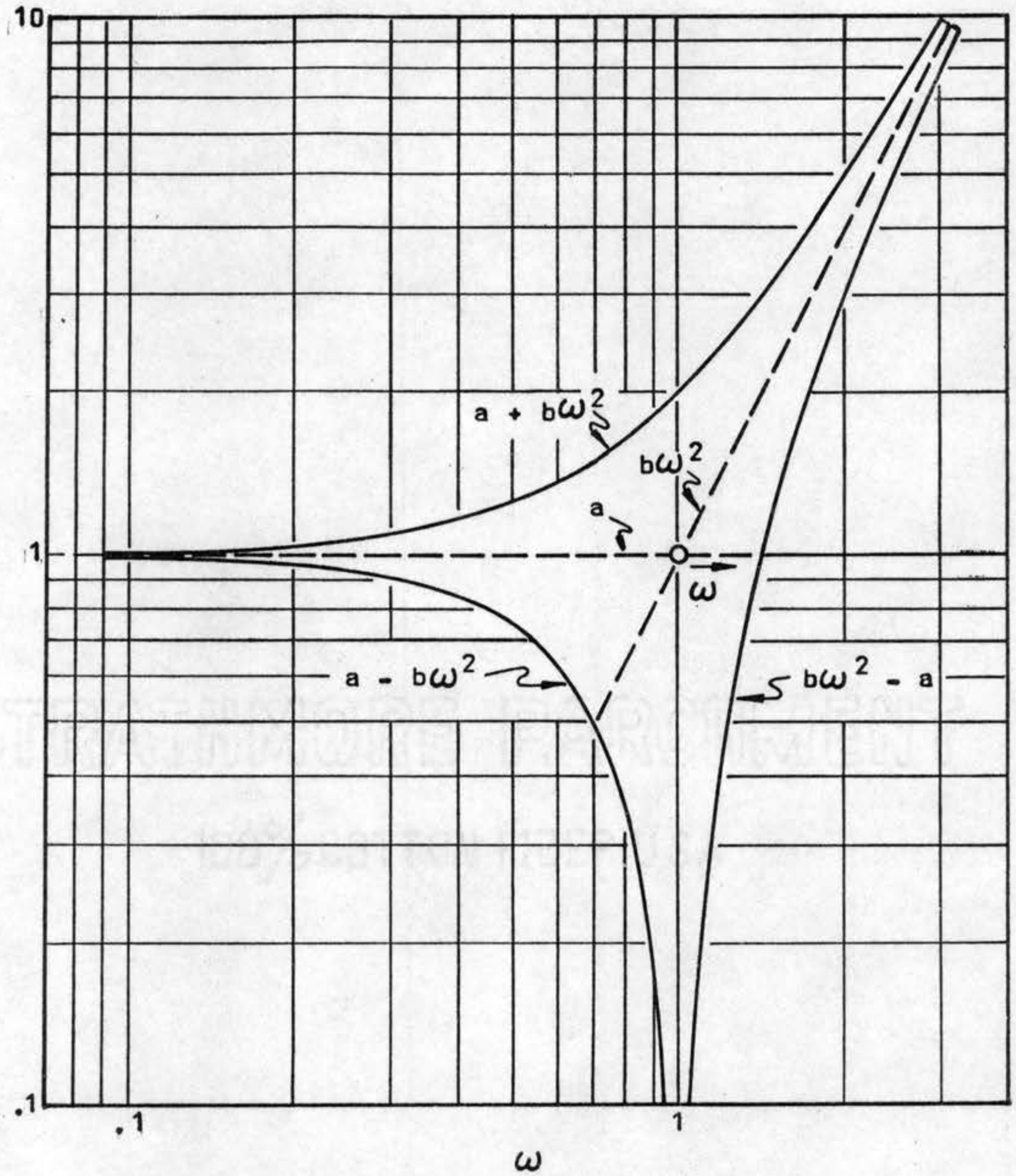


Figure 2-1. Contours of Term-pair Templets Representing Logarithmic Plots of $\pm a \pm b\omega^2$, Shown Oriented for $a = b = 1$.

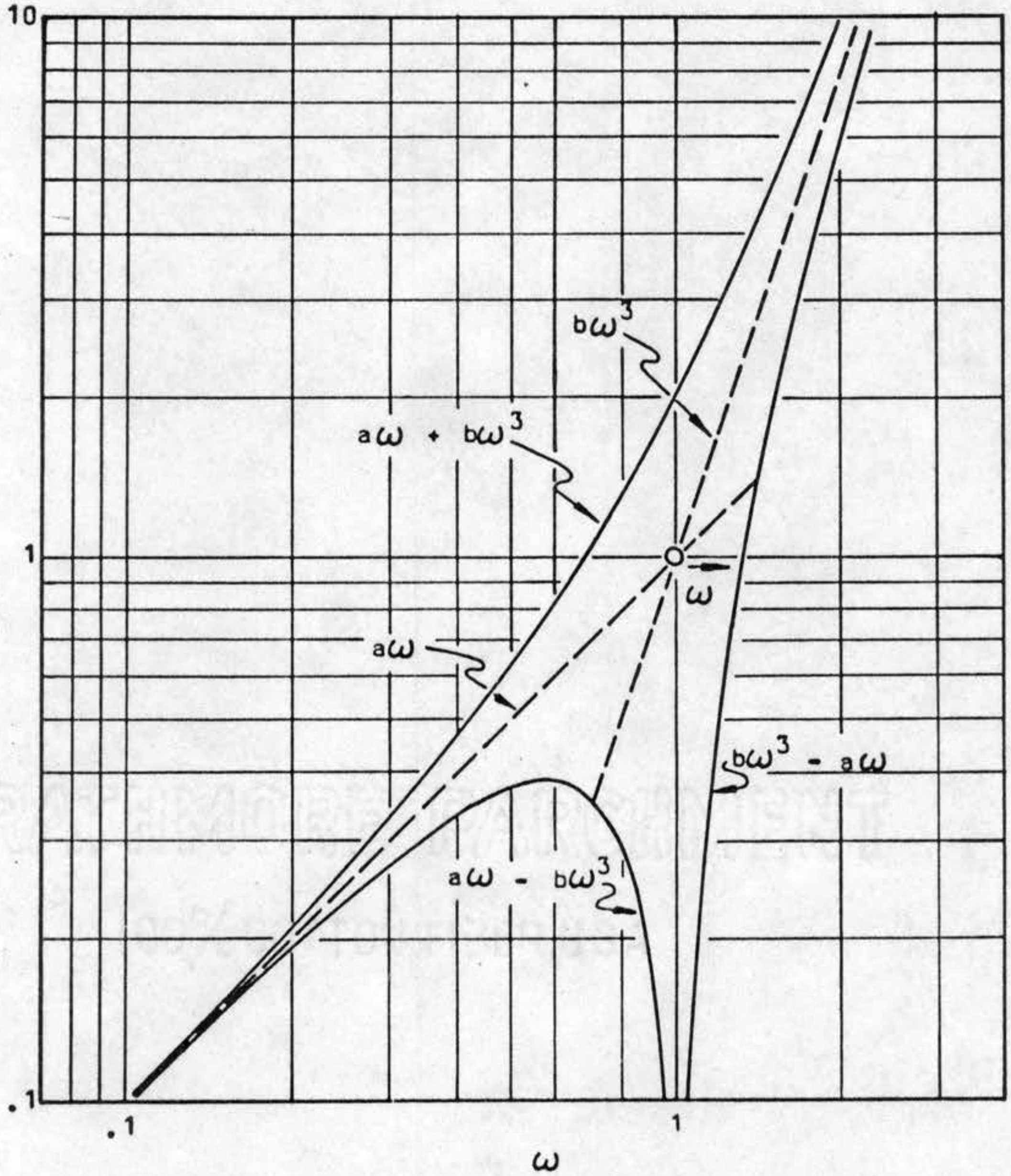


Figure 2-2. Contours of Term-pair Templets Representing Logarithmic Plots of $\pm a\omega \pm b\omega^3$, Shown Oriented for a = b = 1.

by using another templet which is shown in Figure 2-3. With this templet it is possible to either obtain the sum or difference of two term-pair contours. The summing of term-pairs is accomplished by placing the templet shown in Figure 2-3 over the contours with the reference point on the lower contour at the value of the driving frequency ω_1 at which the sum is to be found. The upper contour is set on curve a at ω_1 and the sum is read at the intersection of curve b and the driving frequency. This process is continued until all the contours for the term-pairs of the real part of the characteristic polynomial are summed. Finally, the term-pair contours for the imaginary part of the characteristic polynomial are summed. The results of these operations are two spectrum distributions, one for the real part and one for the imaginary part of the characteristic polynomial.

The next step is to obtain the amplitude and phase shift from these two spectrum distributions. This is accomplished by use of another templet which is shown in Figure 2-4. This templet is used in a similar manner as the previous templet shown in Figure 2-3. The lower of the real or imaginary spectrum distribution is placed at the reference point in Figure 2-4a at various driving frequencies and the upper spectrum distribution is placed under the curve e at the corresponding frequencies. At each frequency the amplitude is read under curve f. Figure 2-4b is used to obtain the phase angle in a similar manner. The sum total of all this manipulation is the lag frequency response spectrum for the system.

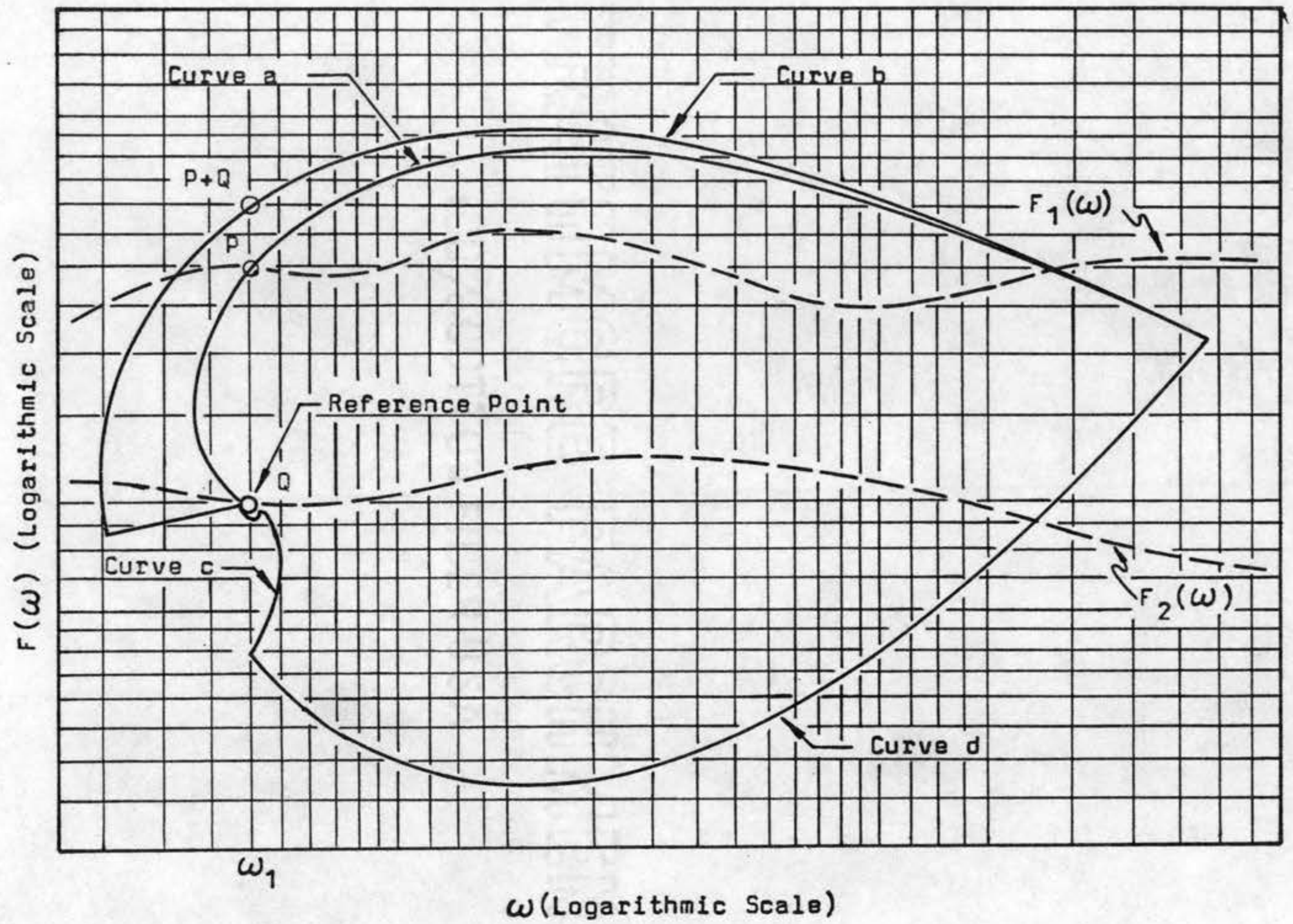
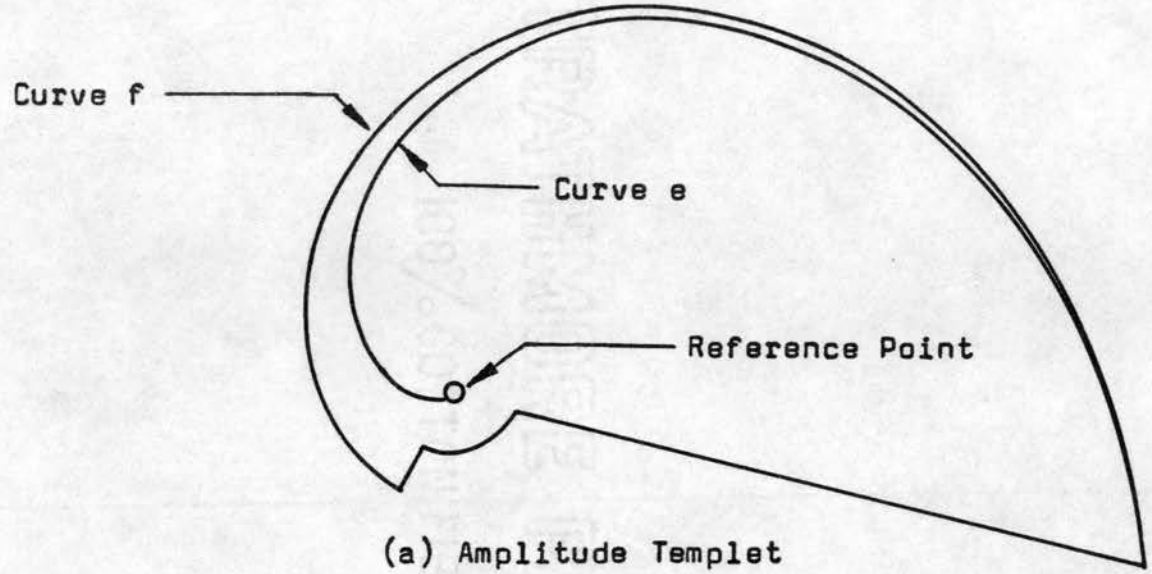
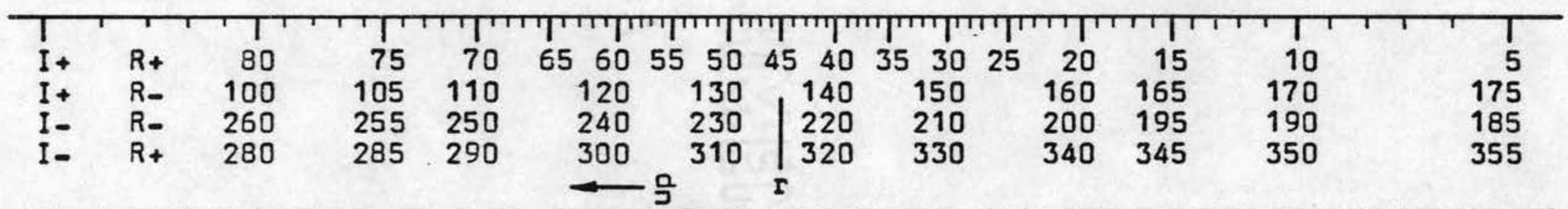


Figure 2-3. Summing Templet With Illustration of Its Use



(a) Amplitude Templet



(b) Angle Scale

Figure 2-4. Amplitude Templet and Angle Scale

To obtain the lead frequency response spectrum the coefficients of the forcing function appearing explicitly in the input matrix of the Rational Canonical state model are employed in exactly the same manner as the coefficients of the characteristic polynomial. Once the lead and lag frequency response spectrums have been constructed then the total frequency response spectrum is obtained. A mere point by point graphical subtraction of the lag spectrum from the lead spectrum can be employed to obtain the total spectrum since both are logarithmic functions. Also a similar point by point graphical subtraction can be used to obtain the phase shift.

Smith's method will provide the frequency response spectrum for a system; however, as the order of the system increases the number of templets necessary also increases. Specifically, it is necessary to have $(n-1)/2$ templets for an nth order system. Also, the templets employed are fashioned to one specific grid of logarithm paper. If frequency response data is to be constructed on logarithm paper of a different scale another set of templets must be used.

Smith's method is based directly on the information displayed explicitly in the Rational Canonical state model; however, the complexity and restrictions imposed deem practical utilization nearly impossible.

Ausman's Method

Ausman's method for the construction of a dynamic system's frequency response spectrum results primarily in the frequency versus amplitude response spectrum. For most practical applications the frequency versus amplitude response spectrum is a graphic description of the transmissibility or filter characteristics of the system. This fact is characteristic of time invariant linear systems. The time invariant linear filter will alter the amplitude and the phase relationships of the input signal, but the characteristic frequencies of the input as seen at the output are identical to those seen at the input. Ausman's method presents a technique for constructing the transmissibility or gain plot of a dynamic system without factoring the polynomials involved.

Since Ausman's method does not require that the characteristic polynomial or the forcing function polynomial be factored the Rational Canonical state model will provide sufficient information for direct application. This conclusion is substantiated by the illustrations which follow. The only value which the Rational Canonical state model does not display explicitly is the coefficient associated with the highest derivative. The coefficients of the characteristic polynomial appearing in the differential transition matrix of Rational Canonical state models reflect normalization with respect to the coefficient of the highest derivative. The coefficient of the highest derivative is, therefore, implicit within the differential transition matrix as well as the input matrix. The general form of linear time invariant system's mathematical model is shown in Equation (2-22).

$$a_n\frac{dy^n}{dt^n} + a_{n-1}\frac{dy^{n-1}}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y =$$

$$b_m\frac{dv^m}{dt^m} + b_{m-1}\frac{dv^{m-1}}{dt^{m-1}} + \dots + b_1\frac{dv}{dt} + b_0 v \qquad (2\text{-}22)$$

The Rational Canonical state model for the system represented dynamically by Equation (2-22) is shown in Equations (2-17) and (2-18).

Application of Ausman's method in conjunction with the Rational Canonical Form of the state model is demonstrated by the three illustrations below. The first two illustrations of system models consist of the two fundamental types of stable modes. The final illustration deals with a much more general dynamic system analogous to most physically realizable systems.

This first illustration involves a simple second order system which has dynamics characterized by two aperiodic (overdamped) modes. The general mathematical model for this type of system is shown below.

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = b_0 v \qquad (2\text{-}23)$$

The state-variable model written in Rational Canonical Form for this system is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -\frac{a_0}{a_2}\\ 1 & -\frac{a_1}{a_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \frac{b_0}{a_2}\begin{bmatrix}1\\ 0\end{bmatrix} v \qquad (2\text{-}24)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (2\text{-}25)$$

Ausman's method for the construction of system transmissibility spectrums works with separate combinations of all the individual terms in both the numerator and denominator of system transfer functions. The Rational Canonical Form of the state model displays the transfer function coefficients explicitly. Therefore, it is possible to construct system frequency response spectrums by using state models in conjunction with Ausman's Method. The construction of the frequency spectrum by Ausman's Method involves forming the ratio of successive numerator terms with the denominator terms of system transfer functions. The coefficients displayed in the Rational Canonical Form of the state model must, therefore, be multiplied by appropriate powers of the driving frequency ω so that when combinations of terms are made from state model entries, the result will be the same as those formed using the terms from the system's transfer function. The powers of the driving frequency ω used are equal to the subscript of the coefficient entry in the Rational Canonical Form of the state model, e.g. a_0/a_n must be multiplied by ω^0, a_m/a_n must be multiplied by ω^m, etc.

For the state model shown in Equations (2-24) and (2-25) the transmissibility characteristics are obtained by multiplying the entries in the differential transition coefficient matrix by the appropriate power of ω and then dividing each of these into the scalar gain b_0/a_2. Each of the quotients is an asymptote to system's transmissibility spectrum. This procedure produces all the asymptotes except the transmission asymptote at high frequencies. The high frequency asymptote is the gain scalar divided by ω raised to a power equal to the highest order derivative in the system model in differential form. This exact procedure can be used to construct transmissibility spectrums for nth order systems. When the forcing function polynomial contains many terms the dominance of each term is established by forming the ratio of successive terms. The ratios establish the frequencies where dominance changes from one forcing function term to the next. The application of the combination of Ausman's Method with the system state model shown in Equations (2-24) and (2-25) is shown in the following paragraphs.

When the driving frequency ω becomes very small, the coefficient a_0 predominates over the other two coefficients. The transmissibility takes the value shown below.

$$\underset{\omega \to 0}{\text{Limit}}\ T = \frac{(1)\frac{b_0}{a_2}}{\frac{a_0}{a_2}\omega^0} = \frac{b_0}{a_0} \qquad (2\text{-}26)$$

Equation (2-26) is represented by the dashed horizontal line in Figure 2-5 labeled "b_0/a_0". Similarly, at very high driving frequencies the coefficient a_2 predominates over the other terms. The transmissibility at this frequency is shown in Equation (2-27).

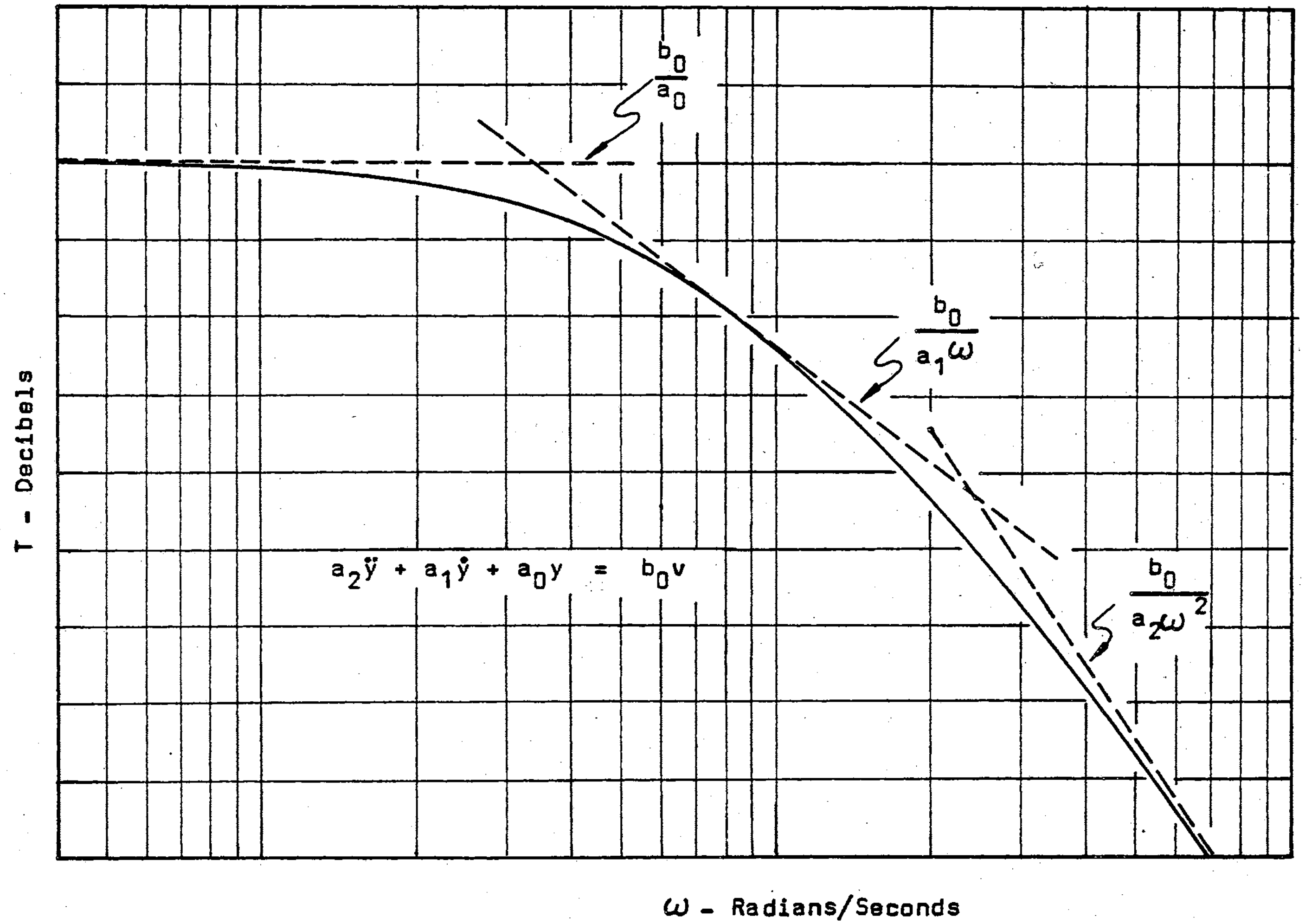


Figure 2-5. Simple Second Order System

$$\lim_{\omega \to \infty} T = \frac{b_0}{a_2 \omega^2} \tag{2-27}$$

Equation (2-27) is represented by the dashed line of slope minus 2 (40 db per decade) labeled "$b_0/a_2\omega^2$".

Assuming for the purpose of this illustration that the dynamic system has two aperiodic modes, the intermediate range of driving frequencies is dominated by the coefficient a_1. For this particular situation the transmissibility within this range of driving frequencies is

$$T = \frac{\frac{b_0}{a_2}}{\frac{a_1}{a_2}\omega^1} = \frac{b_0}{a_1 \omega} \tag{2-28}$$

Equation (2-28) is represented in Figure 2-5 by the dashed line labeled "$b_0/a_1\omega$" of slope 1 (20 db per decade).

With the three functions sketched on the graph as shown in Figure 2-5 it is apparent which coefficient predominates at any particular driving frequency. It also becomes apparent that at the driving frequencies where the dashed lines cross, the transmissibility has two equally weighted representations. For example, where b_0/a_0 crosses $b_0/a_1\omega$ the value of a_0 is exactly equal in magnitude to $a_1\omega$, but a_0 is 90 degrees different in phasing which makes a_0 and $a_1\omega$ equal to one another and the transmissibility is approximately

$$T_{a_0 = a_1\omega} = \frac{b_0}{\sqrt{a_0^2 + (a_1\omega)^2}} =$$

$$\frac{1}{a_0\sqrt{2}} = \frac{1}{a_1\omega\sqrt{2}} \tag{2-29}$$

Similarly at the crossing of $b_0/a_1\omega$ and $b_0/a_2\omega^2$

$$T_{a_1\omega} = a_2\omega^2 = \frac{b_0}{\sqrt{(a_1\omega)^2 + (a_2\omega^2)^2}} =$$

$$\frac{b_0}{a_1\omega\sqrt{2}} = \frac{b_0}{a_2\omega^2\sqrt{2}} \qquad (2\text{-}30)$$

The two points on the transmissibility curve and the three lines drawn previously permit the frequency response spectrum to be completely defined as shown by the solid line in Figure 2-5.

The second illustration involves a dynamic system whose mathematical model is the same as that for the previous illustration except that its characteristic mode is an under-damped oscillation. The application of Ausman's method to this type of system follows the exact procedure as that for the system with aperiodic modes. The exception to this first procedure is when the intermediate frequency range is considered.

To begin with, the two dashed lines b_0/a_0 and $b_0/a_2\omega^2$ are drawn as shown in Figure 2-6 for the large and small driving frequency ranges. Now when the intermediate range dominancy is evaluated by drawing the dashed line labeled "$b_0/a_1\omega$" it is found that this line is everywhere above at least one of the other two (b_0/a_0 and $b_0/a_2\omega^2$). For this situation the a_1 coefficient never dominates over a range of driving frequencies but rather at the driving frequency where b_0/a_0 and $b_0/a_2\omega^2$ cross $b_0/a_1\omega$ dominates. At this particular driving frequency b_0/a_0 is equal in magnitude to

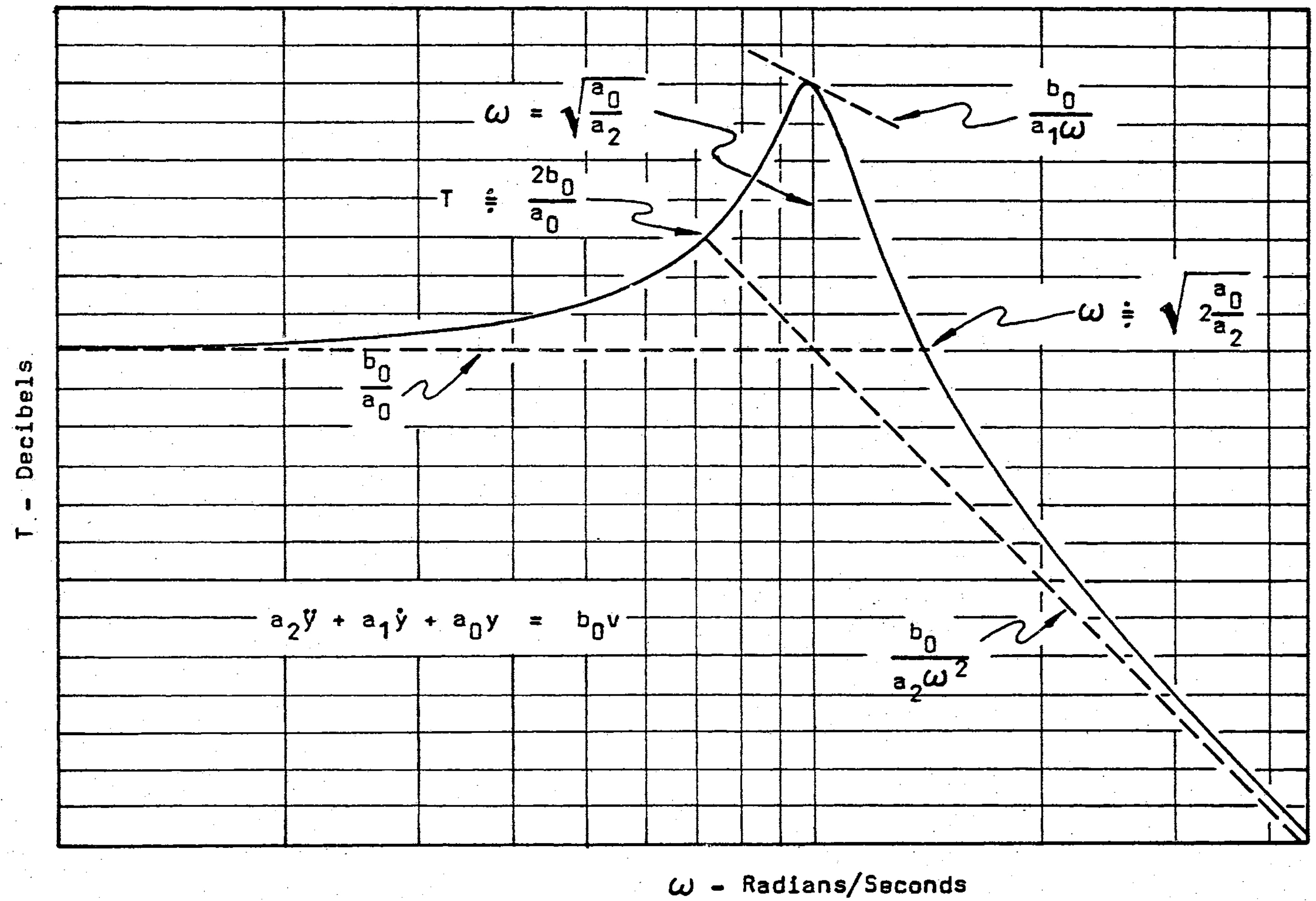


Figure 2-6. Underdamped Second Order System

$b_0/a_2\omega^2$, but b_0/a_0 is opposite in sign to $b_0/a_2\omega^2$ which indicates a phase shift of 180 degrees. Therefore, b_0/a_0 and $b_0/a_2\omega^2$ cancel leaving

$$T_{\omega = \omega_n} = \frac{b_0}{a_1\omega} \tag{2-31}$$

where

ω_n is the natural undamped frequency of the system. Equation (2-31) is represented by the dashed line labeled "$b_0/a_1\omega$" shown in Figure 2-6.

Two additional points which aid in defining the frequency response spectrum for highly underdamped systems are: 1) the point where the line $b_0/a_2\omega^2$ crosses the transmissibility curve which is approximately $2b_0/a_0$ and 2) the point where the b_0/a_0 line crosses the transmissibility curve which is at a frequency of $\sqrt{2a_0/a_2}$.

The lines and points which were calculated in this example are all shown in Figure 2-6. With this information the frequency response spectrum for an underdamped second order dynamic system is essentially completely defined. This frequency response spectrum is shown by the solid line in Figure 2-5.

The third illustration is the two degree-of-freedom mechanical system appearing schematically in the upper right-hand corner of Figure 2-7. The mathematical model for this system is two simultaneous linear coupled ordinary differential equations.

$$M\ddot{y}_1 + (\dot{y}_1 - \dot{y}_2)D + K_1y_1 = K_1v \tag{2-32}$$

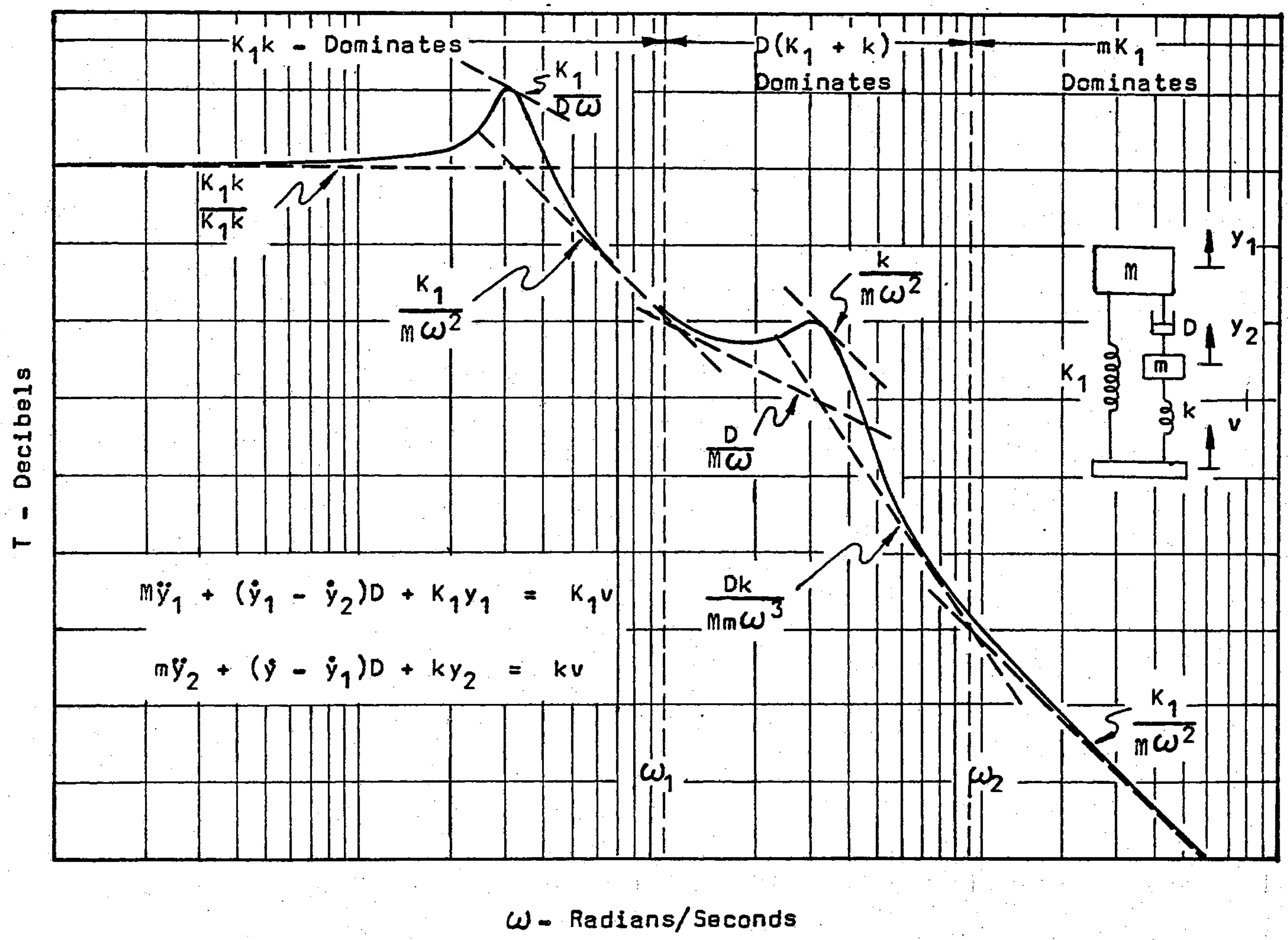


Figure 2-7. Two Degree-of-Freedom Mechanical System

$$m\ddot{y}_2 + (\dot{y}_2 - \dot{y}_1)D + ky_2 = kv \qquad (2\text{-}33)$$

The state-variable model written in Rational Canonical Form for this system considered as a single-input single-output system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\frac{kK_1}{mM} \\ 1 & 0 & 0 & -\frac{D(K_1 + k)}{mM} \\ 0 & 1 & 0 & -\frac{(mK_1 + kM)}{mM} \\ 0 & 0 & 1 & -\frac{D(M + m)}{mM} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \frac{mK_1}{mM} \begin{bmatrix} \frac{kK_1}{mK_1} \\ \frac{D(K_1 + k)}{mK_1} \\ 1 \\ 0 \end{bmatrix} v \qquad (2\text{-}34)$$

$$y_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad (2\text{-}35)$$

where

y_1 is the single-output

A similar state model can be written for this system considered as a single-input single-output with y_2 used as the output.

Construction of the frequency response spectrum for a general system as illustrated here involves the determination of the frequency bands over which the separate coefficients dominate. This is done by calculating the boundary frequencies where dominancy changes from one term to another. Specifically, the band boundaries are

$$\omega_1 = \frac{\dfrac{kK_1}{mK_1}}{\dfrac{D(K_1 + k)}{mK_1}} = \frac{kK_1}{D(K_1 + k)} \qquad (2\text{-}36)$$

and

$$\omega_2 = \frac{\dfrac{D(K_1 + k)}{mK_1}}{1} = \frac{D(K_1 + k)}{mK_1} \qquad (2\text{-}37)$$

These boundary frequencies which separate the three bands of dominance are shown in Figure 2-7 as vertical dashed lines. Also shown, are the dominating input matrix elements designating the frequency band dominated by each.

Once these bands are defined the construction of the transmissibility curve can begin. Starting at the low frequency end the transmissibility is represented approximately by

$$\lim_{\omega \to 0} T \doteq \frac{\dfrac{kK_1}{mK_1}\,\dfrac{K_1}{M}}{\dfrac{kK_1}{mM}} = \frac{kK_1}{kK_1} = 1 \qquad (2\text{-}38)$$

Continuing across the first frequency band

$$T \doteq \frac{\dfrac{kK_1}{mK_1}\,\dfrac{K_1}{M}}{\dfrac{D(K_1 + k)}{mM}\,\omega} = \frac{kK_1}{D(K_1 + k)\omega} \qquad (2\text{-}39)$$

or

$$T \doteq \frac{k}{D\omega} \qquad (2\text{-}40)$$

and finally in this frequency band

$$T \doteq \frac{\dfrac{kK_1}{mM}}{\dfrac{mK_1 + kM}{mM}\omega^2} = \frac{kk}{(mK_1 + kM)\omega^2} \qquad (2\text{-}41)$$

or

$$T \doteq \frac{k}{M\omega^2} \qquad (2\text{-}42)$$

Equations (2-38), (2-40), and (2-42) are shown in Figure 2-7 by dashed lines and are labeled appropriately. With these lines and applicable points calculated in a similar fashion as that shown in the previous two illustrations, the transmissibility curve is defined sufficiently within the first dominant band.

Now the second dominant input matrix element is used to calculate the transmissibility in the next dominant frequency band. The next three lines shown in Figure 2-7 are:

$$T \doteq \frac{\dfrac{D(K_1 + k)}{mK_1}\omega\,\dfrac{mK_1}{mM}}{\dfrac{(mK_1 + kM)}{mM}\omega^2} = \frac{D(K_1 + k)}{(mK_1 + kM)\omega} \qquad (2\text{-}43)$$

or

$$T \doteq \frac{D}{M\omega} \qquad (2\text{-}44)$$

Continuing on through this frequency band

$$T \doteq \frac{\dfrac{D(K_1 + k)}{mM}\omega}{\dfrac{D(M + m)}{mM}\omega^3} = \frac{K_1 + k}{(M + m)\omega^2} \qquad (2\text{-}45)$$

or

$$T \doteq \frac{k}{M\omega^2} \tag{2-46}$$

and

$$T \doteq \frac{D(K_1 + k)}{mM\omega^3} \tag{2-47}$$

or

$$T \doteq \frac{Dk}{mM\omega^3} \tag{2-48}$$

Again Equations (2-44), (2-46), and (2-48) are shown as dashed lines in Figure 2-7.

The final frequency band is the high frequency range. In this range the dominant coefficients are merely those which multiply the input matrix or the gain scalar.

$$\lim_{\omega \to \infty} T = \frac{mK\omega^2}{mK\omega^4} = \frac{k}{M\omega^2} \tag{2-49}$$

This equation is also shown in Figure 2-7 and labeled appropriately.

By using all the lines constructed as shown and also using the principles illustrated in the first two illustrations the frequency response spectrum for this system can be effectively constructed. The general principles demonstrated here in conjunction with state-variable models in Rational Canonical Form are readily adaptable to the most general case and, thus, provide a powerful tool for both analysis and synthesis since explicit information is utilized throughout.

On the whole little was mentioned about the phase shift associated with the frequency response spectrums. This places no hardship on the technique since once the amplitude spectrum is obtained the phase shift can be obtained (15). For example, lines whose slopes are plus 20 decibels per decade (+1) correspond to plus 90 degree phase shift, lines of zero slope correspond to zero phase shift, lines whose slopes are minus 20 decibels per decade (-1) correspond to minus 90 degrees phase shift, lines of slope minus 40 decibels per decade (-2) correspond to minus 180 phase shift, etc.

Summary

None of the procedures described in this chapter appear in literature in the manner specified herein. All the separate articles used have their own implication aside from that for which they were employed here using state models. Therefore, the author has contributed the mechanisms by which each of the methods presented in the articles could be used with state models. Some of the methods have an obvious application in relationship to the state-variable models; however, others, such as Smith's and Ausman's method, did not have obvious implications toward a combination with state models. The general use of the state model in Rational Canonical Form in conjunction with Ausman's techniques shows that a very general set of rules exists for the construction of frequency response spectrums from state-variable models.

These rules are outlined by the third illustration in the "Ausman's Method" sub-section of this chapter. However, these rules are quite complex and involved which require that the user be not only very familiar with frequency response techniques but also very familiar with state space modeling.

The development of the "Fundamental State-Variable Frequency Matrix" presented in the following chapter contains specific rules which dictate the important characteristics of the frequency response for physical systems. This particular Canonical Form of the state-variable model presents the basic characteristics of system frequency response spectrums explicitly in the matrices generated from the "Fundamental State-Variable Frequency Matrix". This explicit matrix display of a system's transmissibility or frequency response spectrum removes the requirement that the user be very familiar with frequency response techniques from classical control theory.

CHAPTER III

MATRIX DISPLAY OF SYSTEM FREQUENCY SPECTRUMS

The previous chapter presents several ways by which the frequency response spectrums for dynamic systems can be obtained through use of different forms of the system's state-variable model. The particular forms of the state model utilized most frequently were the Jordan Canonical Form and the Rational Canonical Form. Since these state model forms were developed to display particular characteristics other than those of the system frequency response, all of the methods developed thus far assume rather extensive knowledge of system analysis employing frequency response procedures. The result of the development of the "Fundamental State-Variable Frequency Matrix" contained in this chapter is an explicit display of system frequency response spectrum characteristics. With these spectrum characteristics the entire response spectrum can be drawn by merely examining the state-variable model in the Frequency Canonical Form. A summary of the analysis techniques for the use of the Frequency Canonical Form is presented in Chapter VI.

General Theory of the Fundamental State-Variable Frequency Matrix

The basic concept of the Fundamental State-Variable Frequency Matrix is the explicit disclosure of the frequency transmissibility possessed by all physical systems. Since it is not possible to display the entire transmission spectrum in a finite dimensional matrix, the entries in the Fundamental State-Variable Frequency Matrix consist of two fundamental frequency transmission or gain characteristics. These two characteristics are: 1) transmission bandwidth or frequency boundaries and 2) transmission spectrum asymptote functions. The transmission bandwidth boundaries establish the band-pass widths in which particular transmission-spectrum asymptote functions are applicable. The transmission-spectrum asymptote functions determine three characteristics of the system frequency response spectrum within a specific bandwidth range. These three characteristics are: 1) system gain at specific transmission frequencies, 2) system characteristic mode frequencies, and 3) spectrum asymptotes.

The spectrum characteristics as they appear as entries in the Fundamental State-Variable Frequency Matrix can be visualized by consideration of the individual functional contributions of each of the coefficients in the ordinary linear constant coefficient model in differential form. From this standpoint it is possible to consider the general modal form for either a lead or a lag mode. This general modal form for constant coefficients will be as shown below

$$r_{i,j} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4\,ac} \qquad (3\text{-}1)$$

where

$r_{i,j}$ = the general modes, which always appear as complex pairs if the mode is oscillatory.

a, b, and c = mode describing integers and are positive for stable modes which is the only type considered.

With this general modal description the total lead and lag modal spectrum can be visualized by superposition in the time domain. These spectrums are describable by the coefficients associated with the various derivatives in the characteristic and forcing functions of the system model. The model coefficients which describe these spectrums can be seen by considering the general form of time invariant linear ordinary differential equations.

$$a_n\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y =$$

$$b_m\frac{d^m v}{dt^m} + b_{m-1}\frac{d^{m-1}v}{dt^{m-1}} + \ldots + b_1\frac{dv}{dt} + b_0 v \qquad (3\text{-}2)$$

The coefficients of the forcing function establish the transmission bandwidth boundaries by forming the ratio of successive coefficients starting with the zeroth term and progressing to the higher order terms. The proof that these ratios do establish the band-pass boundaries is presented later in this chapter. For example, the band-pass boundaries are

$$\omega_0 = \frac{b_0}{b_1}, \ \omega_1 = \frac{b_1}{b_2}, \ \ldots, \ \omega_{m-1} = \frac{b_{m-1}}{b_m} \quad (3\text{-}3)$$

The coefficients of the homogeneous function combined with the coefficients of the forcing function are the transmission asymptote functions. The particular combination of coefficients is determined by the bandwidth under consideration. These functions are formed, as will be shown later, are

Bandwidth 0 to ω_0

$$\frac{b_0}{a_0}, \ \frac{b_0}{a_1\omega}, \ \frac{b_0}{a_2\omega^2}, \ \text{etc.} \quad (3\text{-}4a)$$

Bandwidth ω_0 to ω_1

$$\frac{b_1}{a_1}, \ \frac{b_1}{a_2\omega}, \ \frac{b_1}{a_3\omega^2}, \ \text{etc.} \quad (3\text{-}4b)$$

This generalization can be continued until the entire response spectrum is described. As will be demonstrated in the development of the frequency response spectrum from the Fundamental State-Variable Frequency Matrix, only a few specific transmission asymptote functions generated with respect to any specific bandwidth are directly applicable.

The Fundamental State-Variable Frequency Matrix

The discussion presented in the previous section points out the type of elements which are required to display system frequency response spectrums from system mathematical models. In a finite matrix it is not possible to display all the elements necessary to completely characterize a particular system's spectrum. However, by examining the general system model shown in Equation (3-2) it will be shown later that a transformation of the independent variable will introduce the necessary frequency variable into a finite matrix array so that the characteristics of the system's frequency spectrum are displayed explicitly. The matrix array which displays this explicit information is the Fundamental State-Variable Frequency Matrix. Since a system's state model is not unique it is possible to obtain the Frequency Canonical Form, which contains the Fundamental State-Variable Frequency Matrix, from other forms of the state model. This is done by use of the Frequency Transform Matrix. The following two sections show the particular mechanics involved in obtaining the Fundamental State-Variable Frequency Matrix from either the differential form or a state model form of a linear time invariant system mathematical model.

Frequency Canonical Form From Original System Models

The general form of time invariant linear mathematical models is shown in Equation (3-2). The procedure by which the state model Frequency Canonical Form is developed is initiated by "frequency-transforming" the independent variable. This transformation is $t = \tau/\omega$.[1] Application of this transformation to the general time invariant model results in Equation (3-5).

---

[1]This "frequency-transforming" is not to be confused with an operational transformation where the operation of differentiation is represented by an operator nor is it to be confused with the substitution of $j\omega$ for s.

$$a_n\omega^n\frac{d^ny}{d\tau^n} + a_{n-1}\omega^{n-1}\frac{d^{n-1}y}{d\tau^{n-1}} + \cdots + a_1\omega\frac{dy}{d\tau} + a_0y =$$

$$b_m\omega^m\frac{d^mv}{d\tau^m} + b_{m-1}\omega^{m-1}\frac{d^{m-1}v}{d\tau^{m-1}} + \cdots + b_1\omega\frac{dv}{d\tau} + b_0v \quad (3\text{-}5)$$

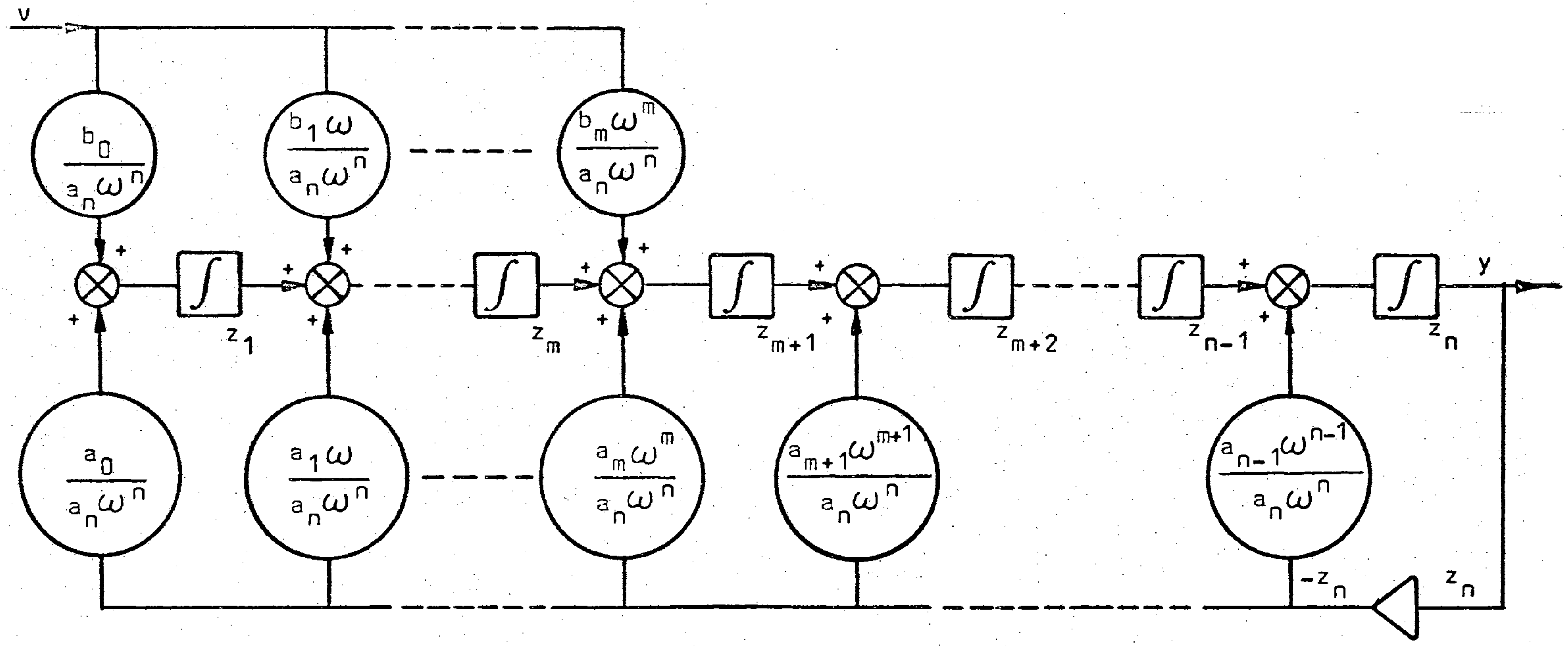
This frequency transformed model can be solved for the zeroth order terms which produces

$$b_0v - a_0y = a_n\omega^n\frac{d^ny}{d\tau^n} + \cdots + a_{m+1}\omega^{m+1}\frac{d^{m+1}y}{d\tau^{m+1}}$$

$$+ \left[a_m\omega^m\frac{d^my}{d\tau^m} - b_m\omega^m\frac{d^mv}{d\tau^m}\right] + \cdots + \left[a_1\omega\frac{dy}{d\tau} - b_1\omega\frac{dv}{d\tau}\right] \quad (3\text{-}6)$$

The state-variable diagram derived from Equation (3-6) is shown in Figure 3-1. The state-variable model in the Frequency Canonical Form can be derived from this diagram by considering the output of each integrator as a state-variable. Hence, each of the inputs to the integrators define a first order differential equation. This system of first order differential equations written in matrix form is as shown below

$$a_n\omega^n\begin{bmatrix} z_1' \\ z_2' \\ \cdot \\ z_m' \\ \cdot \\ z_{n-1}' \\ z_n' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & & 0 & a_0 \\ a_n\omega^n & 0 & & & 0 & a_1\omega \\ 0 & \cdot & & & \cdot & \cdot \\ \cdot & & \cdot & & 0 & a_m\omega^m \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 0 & a_{n-2}\omega^{n-2} \\ 0 & \cdots & & 0 & a_n\omega^n & a_{n-1}\omega^{n-1} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ z_m \\ \cdot \\ z_{n-1} \\ -z_n \end{bmatrix}$$



$$\frac{b_0}{a_n\omega^n}v - \frac{a_0}{a_n\omega^n}y = \frac{d^n y}{d\tau^n} + \dots + \frac{a_{m+1}}{a_n\omega^n}\omega^{m+1}\frac{d^{m+1}y}{d\tau^{m+1}}$$

$$+ \left[\frac{a_m\omega^m}{a_n\omega^n}\frac{d^m y}{d\tau^m} - \frac{b_m\omega^m}{a_n\omega^n}\frac{d^m v}{d\tau^m}\right] + \dots + \left[\frac{a_1\omega}{a_n\omega^n}\frac{dy}{d\tau} - \frac{b_1\omega}{a_n\omega^n}\frac{dv}{d\tau}\right]$$

Figure 3-1. State Diagram for Frequency Canonical Form Derived From the State Diagram for the Rational Canonical Form

$$+ \begin{bmatrix} b_0 & b_1\omega & . & b_m\omega^m & 0 & . & 0 \end{bmatrix}^T v \qquad (3\text{-}7)$$

$$y = \begin{bmatrix} 0 & . & . & . & . & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ . \\ . \\ . \\ z_{n-1} \\ -z_n \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} v \qquad (3\text{-}8)$$

This form of the state-variable model is the Frequency Canonical Form and the differential transition matrix is the Fundamental State-Variable Frequency Matrix. Rewriting Equations (3-7) and (3-8) in general state-variable form produces

$$a_n\omega^n \underline{z}' = \underline{F}\ \underline{z} + \underline{\Gamma}\ \underline{v} \qquad (3\text{-}9)$$

$$y = \underline{\Upsilon}\ \underline{z} \qquad (3\text{-}10)$$

where

$\underline{z}$ = the Frequency Canonical Form state variables.

$\underline{z}'$ = $d\underline{z}/d\tau$, differential of the Frequency Canonical Form state-variables with respect to τ/ω.

$\underline{F}$ = Fundamental State-Variable Frequency Matrix.

$\underline{\Gamma}$ = Frequency Canonical Form input Matrix.

v = input.

y = output.

$\underline{\Upsilon}$ = Frequency Canonical Form output matrix.

Frequency Canonical Form Derived From State Models

The Frequency Canonical Form and accompanying Fundamental State-Variable Frequency Matrix can be derived from other forms of the state models by linear transformations. Specifically, the Frequency Canonical Form is derived very readily by transformation of the state model from the Rational Canonical Form. The Rational Canonical Form is transformed into the Frequency Canonical Form by means of the transformation matrix $\underline{\Omega}$ shown in Equation (3-11).

$$\underline{\Omega} = \begin{bmatrix} \omega^{n-1} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \omega^{n-2} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 & \omega^{0} \end{bmatrix} \qquad (3\text{-}11)$$

The inverse of this transformation matrix is

$$\underline{\Omega}^{-1} = \frac{1}{\omega^{n}} \begin{bmatrix} \omega & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \omega^{2} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \omega^{n} \end{bmatrix} \qquad (3\text{-}12)$$

The reduced state model in Rational Canonical Form is

$$\underline{\mathring{x}} = \underline{R}\,\underline{x} + \underline{B}\,v \qquad (3\text{-}13)$$

$$y = \underline{C}\,\underline{x} \qquad (3\text{-}14)$$

where $\underline{R}$ is the differential transition matrix in Rational Canonical Form as shown below

$$\underline{R} = \begin{bmatrix} 0 & 0 & . & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & . & . & . \\ 0 & . & & . & . \\ . & & . & 0 & . \\ 0 & . & . & 1 & -\frac{a_{n-1}}{a_n} \end{bmatrix} \qquad (3\text{-}15)$$

and $\underline{B}$ is the input matrix the transpose of which is shown below

$$\underline{B}^T = \begin{bmatrix} \frac{b_0}{a_n} & . & . & \frac{b_m}{a_n} & 0 & . & 0 \end{bmatrix} \qquad (3\text{-}16)$$

By forming the transformation $\underline{x} = \underline{\Omega}\ \underline{z}$, the transformation for $\underline{\mathring{x}} = \omega\ \underline{\Omega}\ \underline{z}'$. These transformations can be substituted directly into Equations (3-13) and (3-14) producing

$$\omega\,\underline{\Omega}\ \underline{z}' = \underline{R}\ \underline{\Omega}\ \underline{z} + \underline{B}\ \underline{v} \qquad (3\text{-}17)$$

$$y = \underline{c}\ \underline{\Omega}\ \underline{z} \qquad (3\text{-}18)$$

Premultiplying Equation (3-17) by $1/\omega\ \underline{\Omega}^{-1}$ results in

$$\underline{z}' = \frac{1}{\omega}\ \underline{\Omega}^{-1}\ \underline{R}\ \underline{\Omega}\ \underline{z} + \frac{1}{\omega}\ \underline{\Omega}^{-1}\ \underline{B}\ v \qquad (3\text{-}19)$$

It is seen from Equation (3-9) that

$$\frac{1}{a_n\omega^n}\underline{F} = \frac{1}{\omega}\ \underline{\Omega}^{-1}\ \underline{R}\ \underline{\Omega}$$

$$\frac{1}{a_n\omega^n}\underline{\Gamma} = \frac{1}{\omega}\ \underline{\Omega}^{-1}\ \underline{B}$$

In general, if it is possible to transform the state model for a system into the Rational Canonical Form, then this transformation can be combined with the transform matrix $\underline{\Omega}$ to obtain the Frequency Canonical Form. For models with distinct eigenvalues the differential transition matrix $\underline{A}$ can be transformed into the Jordan Canonical Form by using the Modal Matrix $\underline{M}$ (6 and 9). The Rational Canonical Form of the state model can also be transformed into the Jordan Canonical Form by some transformation matrix $\underline{N}$ (9 and 10). The combination of the $\underline{M}$ and $\underline{N}$ matrices form a matrix which will transform $\underline{A}$ into the Rational Canonical Form $\underline{R}$.

$$\underline{R} = \underline{N}\,\underline{M}^{-1}\,\underline{A}\,\underline{M}\,\underline{N}^{-1} = \underline{P}^{-1}\,\underline{A}\,\underline{P} \qquad (3\text{-}20)$$

Spectrum Band-pass Matrices[2]

The Fundamental State-Variable Frequency Matrix is utilized to calculate the transmission asymptote functions for each of the bandwidths displayed in the input matrix. These bandwidths are determined by dividing the Frequency Canonical Form by the successive entries in the input matrix. Each of the state models formed by this division consists of a Spectrum Band-pass Matrix and an input matrix containing the bandwidth boundary applicable to the accompanying Spectrum Band-pass Matrix.

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[2]Spectrum Band-pass Matrices are matrices derived from the Fundamental State-Variable Frequency Matrix which display frequency spectrum characteristics within a specific bandwidth frequency range.

In general, bandwidth boundaries are determined by examining the generated input matrix and equating the unity entry to the entry immediately following. The result of equating these two input matrix entries is an equation explicit in ω. Solving this equation for ω produces the upper boundary of the frequency range for which the associated Spectrum Band-pass Matrix has been formed. The particular transmission asymptote function entries applicable within any two bandwidth boundaries are found by examining the Spectrum Band-pass Matrices for the frequency bands before and after the one under immediate consideration. One of the asymptote functions in the previous Spectrum Band-pass Matrix will be equal to an entry in the particular matrix under investigation at the bandwidth boundary frequency.

This equality of successive Spectrum Band-pass Matrix entries determines the first entry in the matrix which is applicable in this frequency range. A similar equality as that discussed above also exists between an entry in the Spectrum Band-pass Matrix describing the spectrum in the bandwidth following that under investigation and an entry in the presently considered Band-pass Matrix. Only this time the frequency where the two functions are equal is the next bandwidth boundary frequency. This process determines the last functional entry in the Band-pass Matrix considered which is applicable in this particular frequency bandwidth. This procedure can be conducted for the entire frequency range governed by the physical characteristics displayed in the mathematical state model.

In general, the frequency response spectrum for a system can be determined from the following matrices

$$\frac{1}{b_0}\underline{F} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{a_0}{b_0} \\ \frac{a_n\omega^n}{b_0} & 0 & \cdots & \cdot & \cdot \\ 0 & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdot & \frac{a_n\omega^n}{b_0} & \frac{a_{n-1}\omega^{n-1}}{b_0} \end{bmatrix} \qquad (3\text{-}21)$$

$$\frac{1}{b_0}\underline{\Gamma}^T = \begin{bmatrix} 1 & \frac{b_1\omega}{b_0} & \cdot & \frac{b_m\omega^m}{b_0} & 0 & \cdot & 0 \end{bmatrix} \qquad (3\text{-}22)$$

The transmission bandwidth boundary for the Spectrum Band-pass Matrix shown in Equation (3-22) is determined from

$$\frac{b_1\omega}{b_0} = 1 \quad \text{or} \quad \omega_0 = \frac{b_0}{b_1} \qquad (3\text{-}23)$$

The applicable entries in the matrix in Equation (3-21) include all the entries up to and including the entry which is equal to an entry in the $1/b_1\omega\ \underline{F}$ matrix at the frequency ω_0. The entry in the $1/b_1\omega\ \underline{F}$ matrix applicable in this equality is the first asymptote function applicable in the next bandwidth. The frequency response spectrum in the kth bandwidth is shown in Equation (3-24).

$$\frac{1}{b_k \omega^k} \underline{F} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{a_0}{b_k \omega^k} \\ \frac{a_n \omega^n}{b_k \omega^k} & 0 & \cdots & \cdot & \cdot \\ 0 & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdot & \frac{a_n \omega^n}{b_k \omega^k} & \frac{a_{n-1} \omega^{n-1}}{b_k \omega^k} \end{bmatrix} \qquad (3\text{-}24)$$

$$\frac{1}{b_k \omega^k} \underline{\Gamma}^T = \begin{bmatrix} \frac{b_0}{b_k \omega^k} & \cdot & 1 & \frac{b_{k+1} \omega^{k+1}}{b_k \omega^k} & \cdot & \frac{b_m \omega^m}{b_k \omega^k} & 0 & \cdot & 0 \end{bmatrix} \qquad (3\text{-}25)$$

The transmission bandwidth boundary for Equation (3-24) is

$$\omega_k = \frac{b_k}{b_{k+1}} \qquad (3\text{-}26)$$

Consider as an example of the formation of the Spectrum Band-pass Matrices the third order system described in Frequency Canonical Form as follows

$$a_3 \omega^3 \begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & a_0 \\ a_3 \omega^3 & 0 & a_1 \omega \\ 0 & a_3 \omega^3 & a_2 \omega^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \omega \\ 0 \end{bmatrix} v \qquad (3\text{-}27)$$

$$y = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} \qquad (3\text{-}28)$$

By dividing the Frequency Canonical Form input matrix by b_0 the transmission bandwidth boundary ω_0 is calculated. As will be shown later this is done by equating the unity entry which occurs to the next entry as shown below

$$1 = \frac{b_1 \omega}{b_0} \qquad (3\text{-}29a)$$

$$\omega_0 = \frac{b_0}{b_1} \qquad (3\text{-}29b)$$

The Spectrum Band-pass Matrices formed from the Fundamental State-Variable Frequency Matrix are

$$\underline{S}_0 = \frac{1}{b_0}\underline{F} = \begin{bmatrix} 0 & 0 & \frac{a_0}{b_0} \\ \frac{a_3\omega^3}{b_0} & 0 & \frac{a_1\omega}{b_0} \\ 0 & \frac{a_3\omega^3}{b_0} & \frac{a_2\omega^2}{b_0} \end{bmatrix} \qquad (3\text{-}30)$$

$$\underline{S}_1 = \frac{1}{b_1\omega}\underline{F} = \begin{bmatrix} 0 & 0 & \frac{a_0}{b_1\omega} \\ \frac{a_3\omega^3}{b_1\omega} & 0 & \frac{a_1\omega}{b_1\omega} \\ 0 & \frac{a_3\omega^3}{b_1\omega} & \frac{a_2\omega^2}{b_1\omega} \end{bmatrix} \qquad (3\text{-}31)$$

When $\omega = b_0/b_1$ is substituted into both of Equations (3-24) and (3-25) the entry $a_1\omega/b_0$ in Equation (3-24) is equal to the $a_1\omega/b_1\omega$ entry in Equation (3-25). These two transmission asymptote functions are equal at the transmission bandwidth boundary ω_0. At this frequency the dominance from $\underline{S}_0$ to $\underline{S}_1$ occurs. Therefore, the first part of the frequency spectrum is characterized by the entries a_0/b_0 and $a_1\omega/b_0$ in the $\underline{S}_0$ Spectrum Band-pass Matrix; and the second part of the spectrum is characterized by the entries $a_1\omega/b_1\omega$, $a_2\omega^2/b_1\omega$, and $a_3\omega^3/b_1\omega$ in the $\underline{S}_1$ Spectrum Band-pass Matrix.

The actual construction of the frequency response spectrum is performed by expressing the transmission asymptote functions in decibels. For use in the method described above involving the Fundamental State-Variable Frequency Matrix the transmission asymptote functions expressed in decibels are defined as shown below

$$\begin{matrix}\text{Transmission}\\ \text{Asymptote Functions}\end{matrix} = -20\text{Log}_{10}\frac{a_l\omega^l}{b_k\omega^k}\text{db} \qquad (3\text{-}32)$$

These functions are straight lines when plotted on semi-log graph paper.

$$\text{TAF} = 20\text{Log}_{10}\frac{b_k}{a_k} + (1-k)20\text{Log}_{10}\omega \qquad (3\text{-}33)$$

These straight lines determine the three characteristics of the frequency response spectrum discussed on page 44. The gains, natural frequencies, and asymptotes of the spectrum are displayed by these functionally straight lines. The actual utilization of these lines is illustrated in the next section where specific illustrations are used to develop the mathematics which substantiates the particular functional significance of the transmission asymptote functions and the transmission bandwidth boundaries.

Frequency Spectrum Data From Model Coefficients

Frequency response spectrums for physical systems represent steady-state system performance characteristics. For linear time invariant models the transmissibility spectrum is a measure of the amplification or attenuation in driving frequency magnitude which occurs at the output of the system. Since the output frequency is exactly equal to the input frequency the amplitude of the driving frequency and the phasing of the driving frequency are the only things altered. Consequently, the particular or the steady-state solution of the mathematical model excited by a sinusoidal input is all that need be considered for a complete description of system frequency response spectrums.

The steady-state frequency response spectrum for linear time invariant systems is described mathematically by the system transfer function with the complex transform variable s replaced by $j\omega$, where ω represents the sinusoidal driving frequency. The transfer function for a general nth order system is as follows.

$$\frac{y(s)}{v(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \qquad (3\text{-}34)$$

If the substitution discussed above is made into Equation (3-30), the transmissibility of the system is given by

$$T = \left|\frac{y}{v}\right|_{j\omega} = \left|\frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^n + \dots + a_1 s + a_0}\right|_{s = j\omega} \quad (3\text{-}35)$$

At very low frequencies the transmissibility approaches an asymptote function of b_0/a_0, i.e.

$$\underset{\omega \to 0}{\text{Limit}}\ T = \frac{b_0}{a_0} \quad (3\text{-}36)$$

Similarily at very high driving frequencies the transmissibility approaches an asymptote function of $b_m\omega^m/a_n\omega^n$. The result of these terms predominating the numerator and the denominator is

$$\underset{\omega \to \infty}{\text{Limit}}\ T = \frac{b_m\omega^m}{a_n\omega^n} \quad (3\text{-}37)$$

The intermediate range of the spectrum is determined by the successive use of the ratios of the terms in the numerator with all the terms in the denominator, such as $b_k\omega^k/a_l\omega^l$. This determination proceeds by first determining the frequency range of applicability for each of the numerator terms. The mathematical basis for establishing these dominant ranges is developed by considering the magnitude associated with the steady state part of the time solution (11). The kth and lth terms in the numerator of the steady state time solution can be written as follows in Equation (3-38).

$$\ldots + \left[(-1)^n a_n \omega^n + \ldots + a_0 \right] b_k \mathrm{Sin}\,\omega t$$

$$+ \left[(-1)^{n-1} a_{n-1} \omega^{n-1} + \ldots + a_1 \omega \right] b_k \omega \,\mathrm{Cos}\,\omega t$$

$$- \left[(-1)^n a_n \omega^n + \ldots + a_0 \right] b_l \omega \,\mathrm{Sin}\,\omega t$$

$$- \left[(-1)^{n-1} a_{n-1} \omega^{n-1} + \ldots + a_1 \omega \right] b_l \omega^2 \mathrm{Cos}\,\omega t + \ldots \quad (3\text{-}38)$$

where

$k = l + 1$

The boundary of dominance changes between the above numerator terms, with respect to the transmissibility, occurs where the magnitudes of the successive terms are equal. This equality between the successive terms in Equation (3-38) can be written as

$$b_k \omega^k = b_l \omega^l \quad (3\text{-}39)$$

Solving Equation (3-39) for ω produces

$$\omega_l = \frac{b_l}{b_k} \quad (3\text{-}40)$$

Equation (3-40) substantiates the use of the coefficients associated with the forcing function in establishing the transmission bandwidth boundaries in the frequency response spectrum.

The transmission asymptote functions which characterize the frequency spectrum can be established by considering Equation (3-31) written in the following way.

$$T = \left| \frac{b_m s^m}{a_n s^n + \dots + a_1 s + a_0} + \dots + \frac{b_0}{a_n s^n + \dots + a_1 s + a_0} \right|_{s=j\omega} \qquad (3\text{-}41)$$

If each of the terms of the transmissibility equation shown above are examined for the intermediate spectrum frequency range, it is seen that the numerator defines the dominant term contained in the forcing function. Each of the terms within the absolute value sign describe the frequency response spectrum within the defined band-pass range. For very low frequencies the asymptote function is b_0/a_0 since all the other terms are very nearly zero at low frequencies. The difference between the asymptote function and the actual spectrum is the contribution of all the other terms which have a finite value everywhere except at zero frequency. For very high frequency a similar description of the spectrum with respect to the asymptote function $b_m\omega^m/a_n\omega^n$ can be visualized as presented in Equations (3-32) and (3-33). As can be seen by the previous discussion the asymptote functions thus far substantiated are essentially first order approximations to the actual spectrum. The inclusion of more and more terms in the general model to describe any particular range of frequencies will assure closer and closer approximations to the actual response spectrum. However, the addition of terms to the first order approximations increases the complexity of calculation in an exponential manner.

The substantiation of the intermediate range transmission asymptotes is best presented by considering the general system modes (4). These modes are described by the following example.

$$T = \left| \frac{b_0}{a_2 s^2 + a_1 s + a_0} \right|_{s = j\omega} \qquad (3\text{-}42)$$

As has already been established the asymptote functions associated with the above example for the very low and very high frequencies are b_0/a_0 and $b_0/a_2\omega^2$, respectively. One of two things can occur in the intermediate frequency range. First, the eigenvalues of the characteristic equations can be distinct. If this is the case then by Bode's Theorems the asymptotic approximations have corner frequencies of

$$\omega_1 = + \frac{a_1}{2a_2} - \sqrt{\left(\frac{a_1}{2a_2}\right)^2 - \frac{a_0}{a_2}} \qquad (3\text{-}43a)$$

and

$$\omega_2 = + \frac{a_1}{2a_2} + \sqrt{\left(\frac{a_1}{2a_2}\right)^2 - \frac{a_0}{a_2}} \qquad (3\text{-}43b)$$

for $$\left(\frac{a_1}{2a_2}\right)^2 > \frac{a_0}{a_2} \qquad (3\text{-}43c)$$

The slope of the asymptote between these two corner frequencies is a minus 1 or a minus 20 decibels per decade. The corner frequencies and the asymptotic approximations based on Bode's Theorems are shown in Figure 3-2. The equation for this asymptotic approximation in the intermediate frequency range is shown in Equation (3-44).

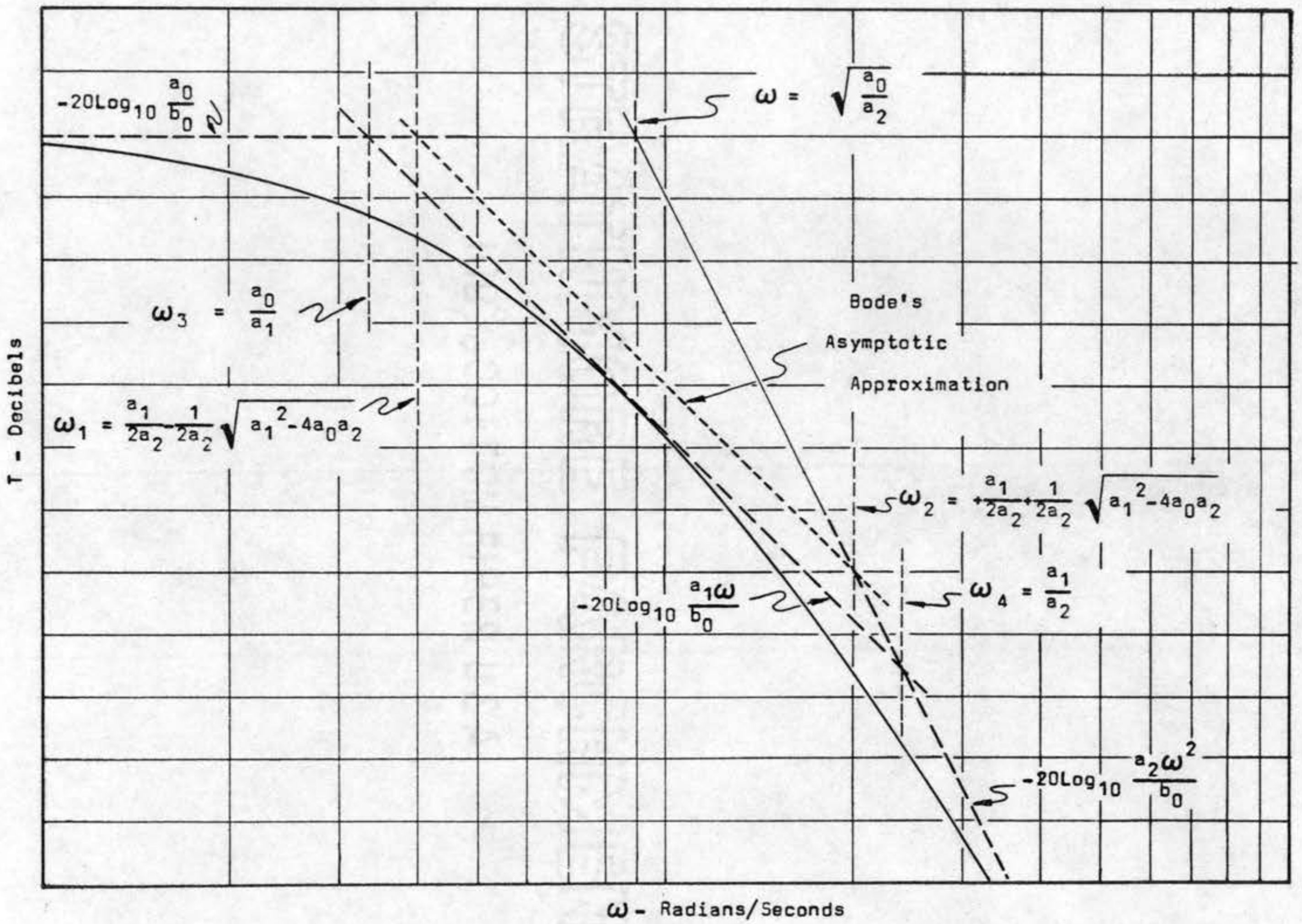


Figure 3-2. First Order Approximation for Simple Mode Spectrum

$$20\mathrm{Log}_{10}\ T = 20\mathrm{Log}_{10}\ \frac{b_0}{2a_0a_2}\left[a_1 - \sqrt{a_1^{\ 2} - 4a_0a_2}\right]$$
$$-20\mathrm{Log}_{10}\omega \qquad (3\text{-}44)$$

The equation for the asymptote function described herein is

$$20\mathrm{Log}_{10}\ T = 20\mathrm{Log}_{10}\ \frac{b_0}{a_1} - 20\mathrm{Log}_{10}\omega \qquad (3\text{-}45)$$

As shown by these two equations and also in Figure 3-2, the slopes of the two asymptotes are a minus one or a minus 20 decibels per decade. The actual equation for the frequency spectrum in decibels is

$$20\mathrm{Log}_{10}\ T = 20\mathrm{Log}_{10}\left|\frac{b_0}{-\ a_2\omega^2 + ja_2\omega + a_0}\right| \qquad (3\text{-}46)$$

The frequency where the low frequency range asymptote and the high frequency range asymptote are equal is $\omega = \sqrt{a_0/a_2}$ as shown in Figure 3-2. At this frequency the actual frequency spectrum has a transmissibility as shown below.

$$20\mathrm{Log}_{10}\ T = 20\mathrm{Log}_{10}\ \frac{b_0}{a_1}\sqrt{\frac{a_2}{a_0}} \qquad (3\text{-}47)$$

The asymptotic value of the frequency response spectrum from Bode's Theorems is

$$20\mathrm{Log}_{10}\ T = 20\mathrm{Log}_{10}\ \frac{b_0}{2a_0a_2}\sqrt{\frac{a_2}{a_0}}\left[a_1 - \sqrt{a_1^{\ 2} - 4a_0a_1}\right] \qquad (3\text{-}48)$$

The value for the asymptote function at $\omega = \sqrt{a_0/a_2}$ is

$$20\mathrm{Log}_{10}\ T = 20\mathrm{Log}_{10}\ \frac{b_0}{a_0}\sqrt{\frac{a_2}{a_0}} \qquad (3\text{-}49)$$

The value of the asymptote function at any other frequency is not equal to the actual spectrum since, if an increment of frequency is added to or subtracted from $\sqrt{a_0/a_2}$, Equation (3-49) is no longer equal to Equation (3-47). Therefore, the asymptote function is tangent to the actual spectrum at this frequency. As is evident from Figure 3-2 and Equation (3-49) the Spectrum Asymptote Functions for this type of system represents an excellent first order approximation. Several extremely important facts become evident upon close examination of Figure 3-2. These are: 1) at the corner frequencies of the asymptote functions the actual spectrum is 2.4 decibels down, 2) the corner frequencies are not the characteristic roots - they are $\omega_3 = a_0/a_1$ and $\omega_4 = a_1/a_2$, and 3) the asymptote function is truly tangent to the actual spectrum at $\omega = \sqrt{a_0/a_2}$. These facts aid in determining the actual frequency response spectrum much more accurately without having to factor the characteristic polynomial.

The second case in considering the intermediate frequency range is when the eigenvalues are complex conjugates or the system is underdamped. For this case the low and high frequency asymptote functions cross one another at the frequency shown below.

$$\frac{b_0}{a_0} = \frac{b_0}{a_2\omega^2}$$

or

$$\omega = \sqrt{\frac{a_0}{a_2}} \qquad (3\text{-}50)$$

At this frequency both terms are equal but $b_0/a_2\omega^2$ is 180 out of phase with the b_0/a_0 term so that the transmissibility is exactly

$$20\text{Log}_{10}\ T = 20\text{Log}_{10}\ \frac{b_0}{a_1}\sqrt{\frac{a_2}{a_0}} \qquad (3\text{-}51)$$

At $\omega = \sqrt{a_0/a_2}$ (the natural undamped frequency) the term $b_0/a_1\omega$ is completely dominant and this is the only frequency where the asymptote function $b_0/a_1\omega$ is dominant. Therefore, the value of this asymptote function at the frequency $\sqrt{a_0/a_2}$ is a point on the actual frequency spectrum. These facts are displayed in Figure 3-3. Curves from which the frequency of maximum transmissibility and a curve from which the difference in transmissibility can be obtained are contained in Appendix B.

The discussion presented above substantiates the fact that the spectrum asymptote functions will provide the basic characteristics of a system's frequency response spectrum employing the unfactored system model. The models used were the two basic modal forms which make-up the mathematical models for all linear time invariant models. The cascading or superposition of several of the basic models to form a complex system brings forth a limitation to the use of the entire range of spectrum asymptote functions. This limitation involves primarily the relative frequency locations of modes and particularly the determination of the transmissibility at the natural undamped modal frequencies. When the system modes are within a decade of one another the value for

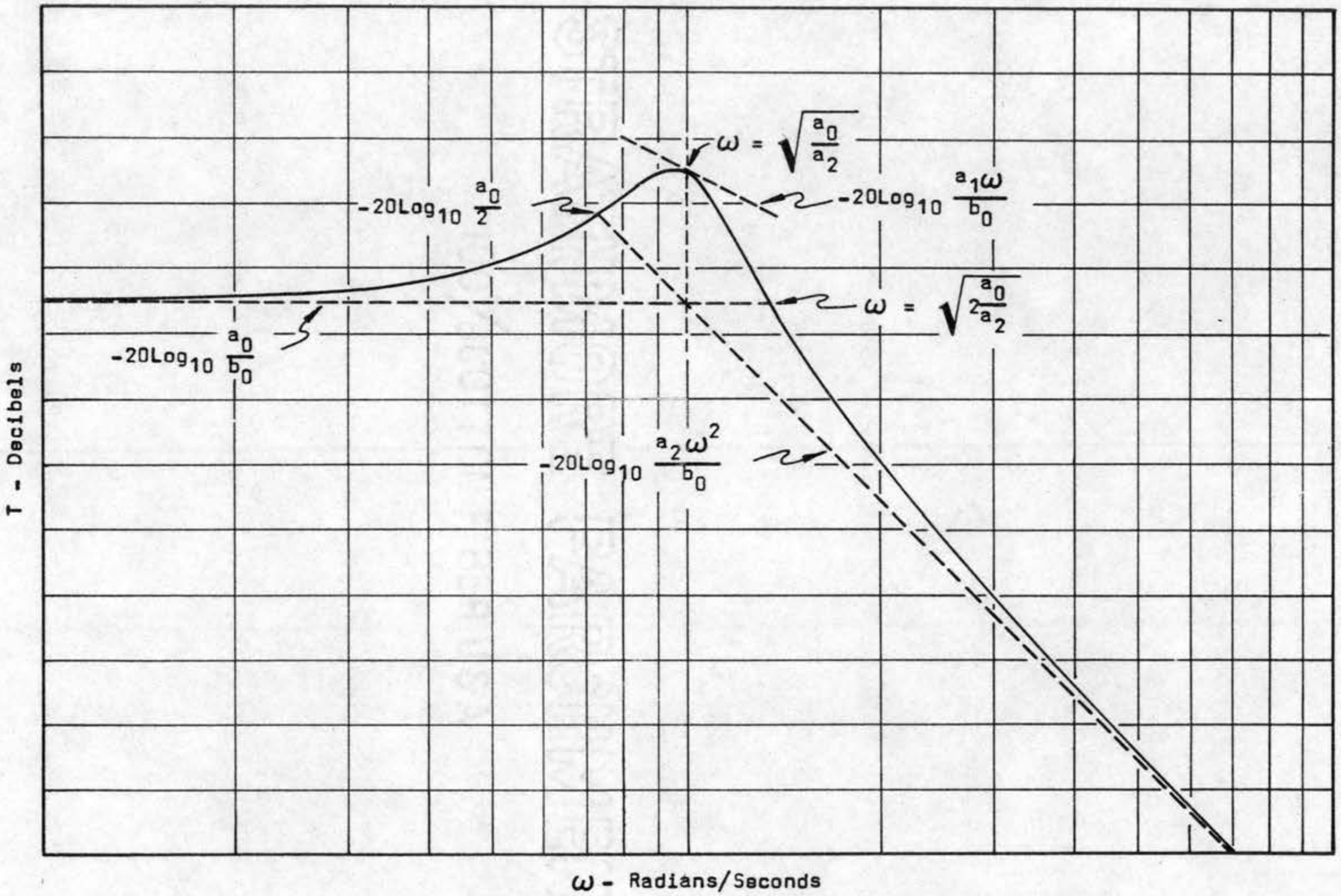


Figure 3-3. First Order Approximation for Underdamped Mode Frequency Response Spectrum

the transmissibility at the natural frequency of oscillatory modes determined by the appropriate asymptote function falls below the actual transmissibility characteristic of the system. However, the asymptote functions which approximate the general spectrum characteristics are still applicable. This situation is illustrated in Figure 3-4. The striking thing about the asymptote functions is that they will locate the undamped natural frequencies of the oscillatory modes and, as can be seen from Figure 3-4, produce a somewhat closer asymptote approximation than that constructed using Bode's Theorems, which is also shown in Figure 3-4.

Further illustrative substantiation of the basic principles involved in the construction of frequency response spectrums from the Fundamental State-Variable Frequency Matrix is shown in Figures 3-5, 3-6, and 3-7. These figures contain frequency response spectrums of systems whose characteristic polynomials are of order three.

Figure 3-5 shows the way the transmission asymptote functions adjust the first order approximation for the spectrum when nothing but the damping of an underdamped mode associated with the system is varied. Also in this figure the phenomena of mode interaction is evidenced by the asymptote function which is only dominate at the natural undamped mode frequency. The transmissibility approximated by these particular asymptote functions are consistantly lower than the actual transmissibility.

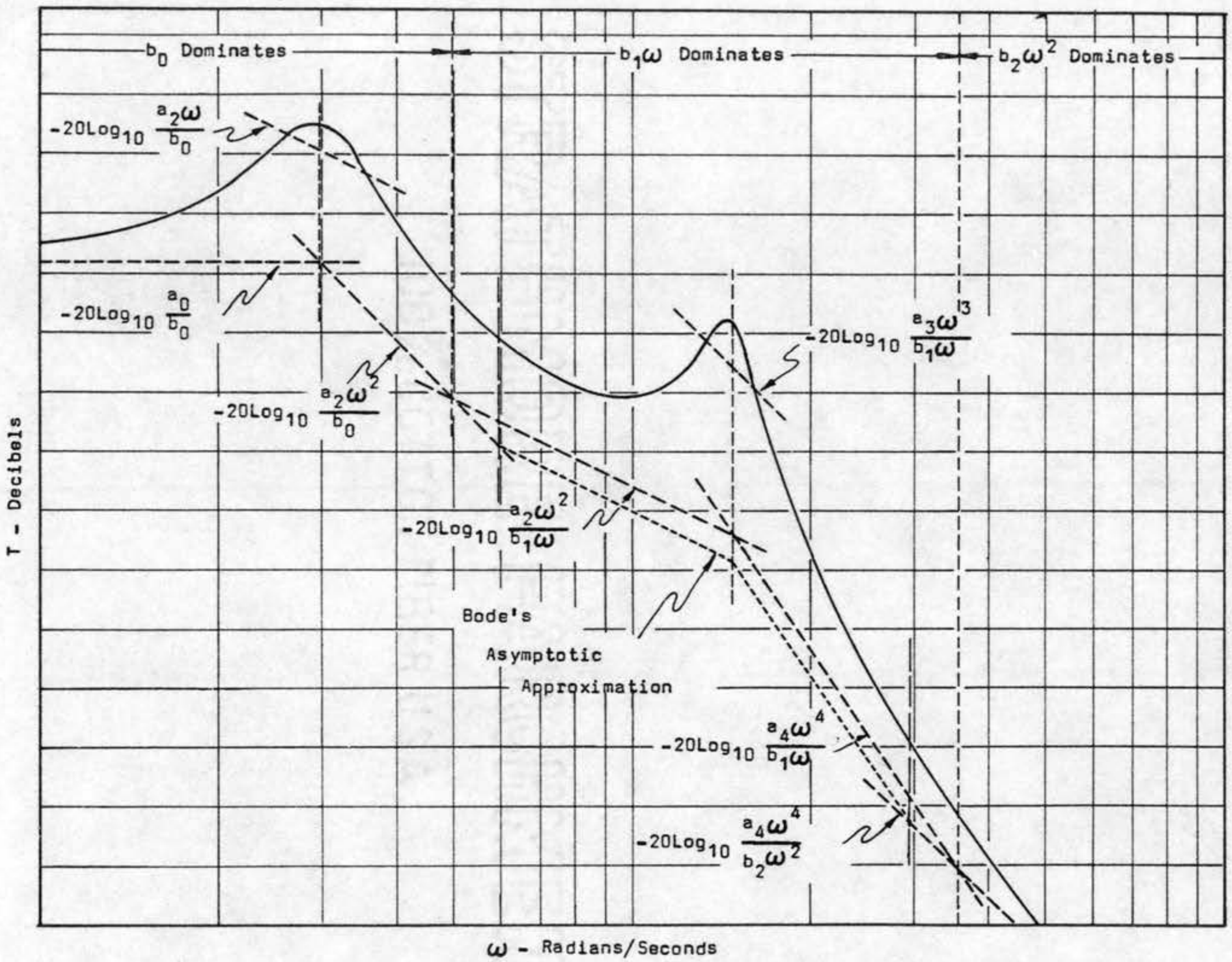


Figure 3-4. First Order Approximation for Multiple Mode Frequency Response Spectrum

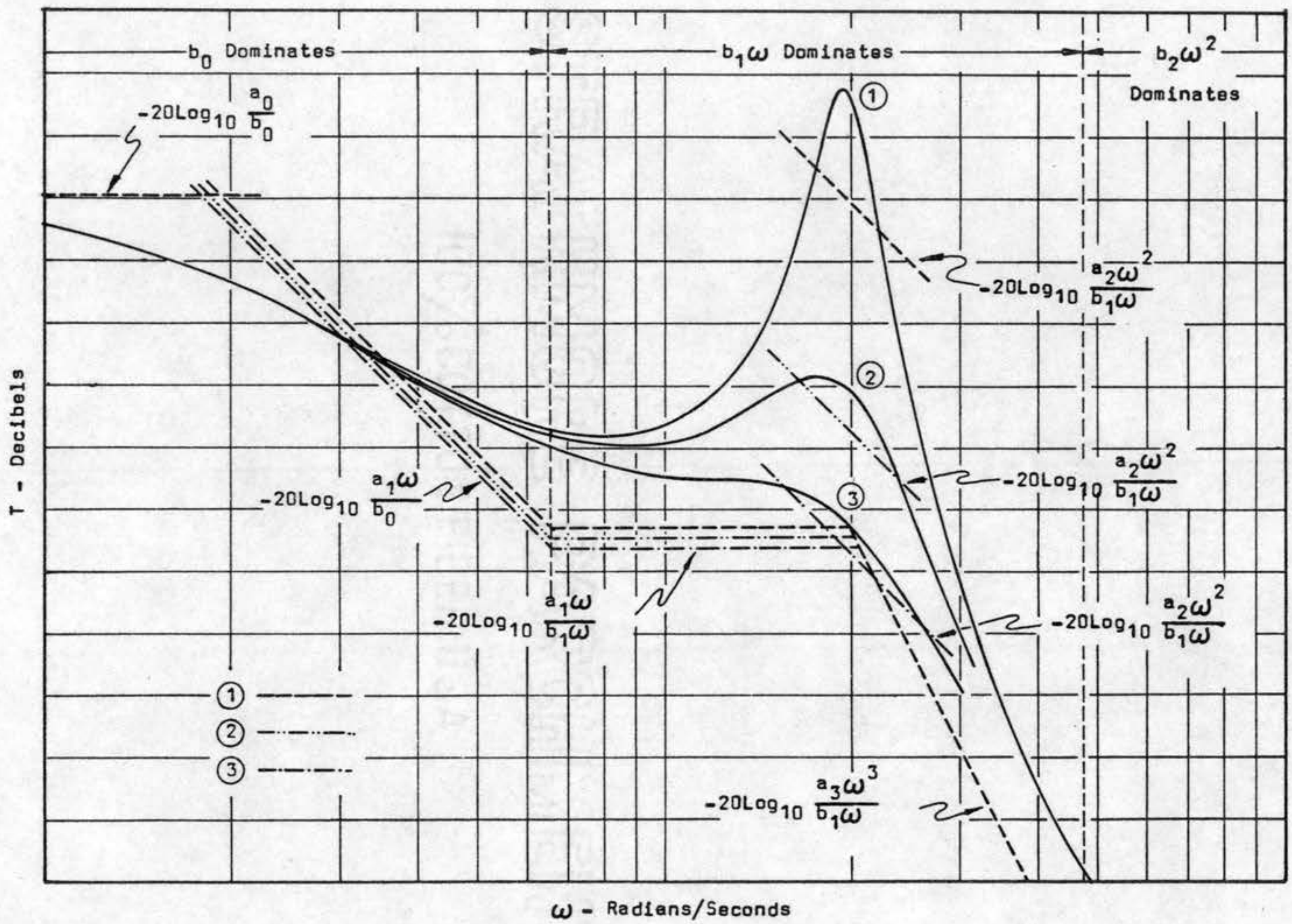


Figure 3-5. Transmission Function Representation for Frequency Response Spectrums With Damping Function Varied

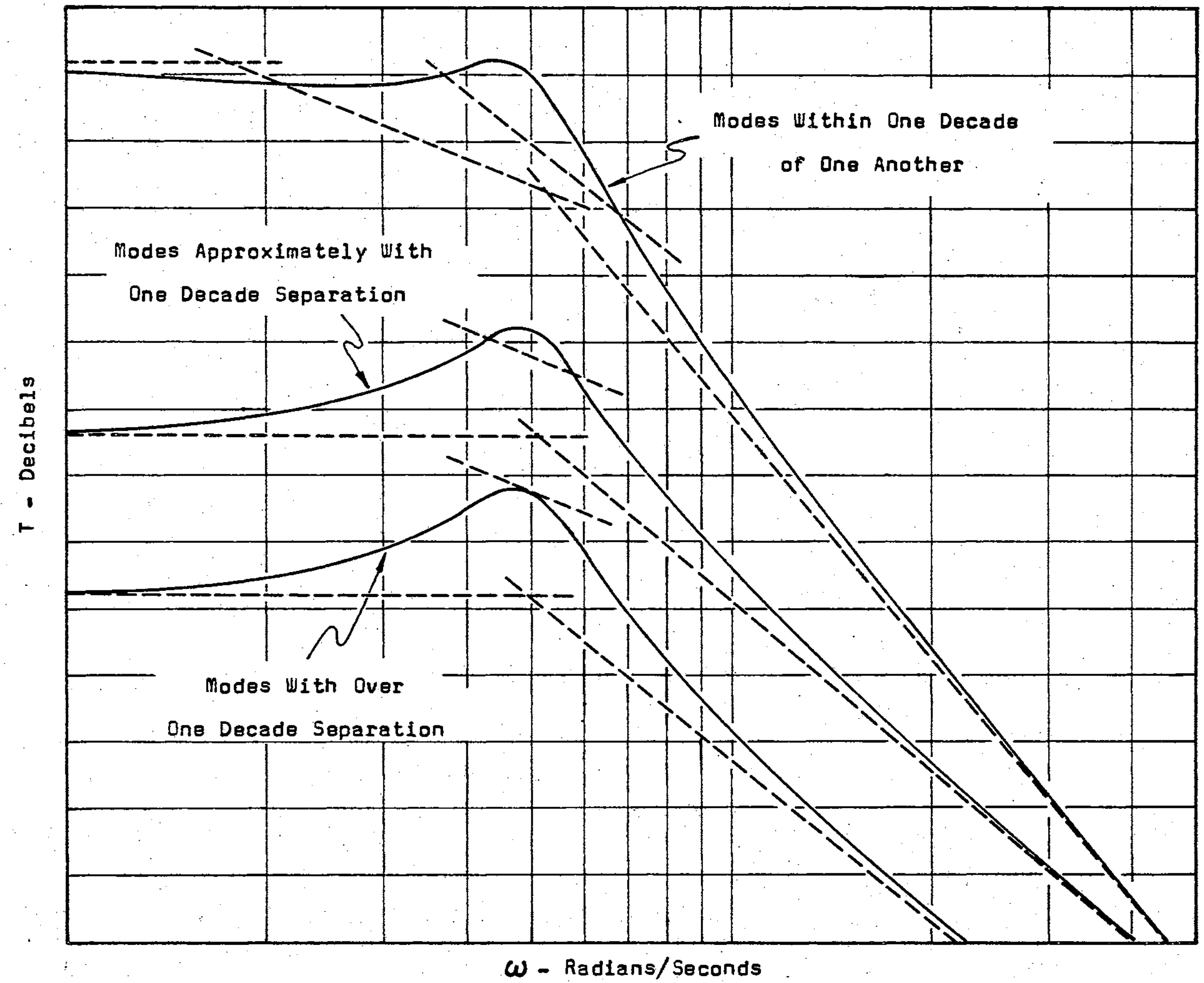


Figure 3-6. Transmission Function Representation for Frequency Response Spectrums for Various Mode Separations

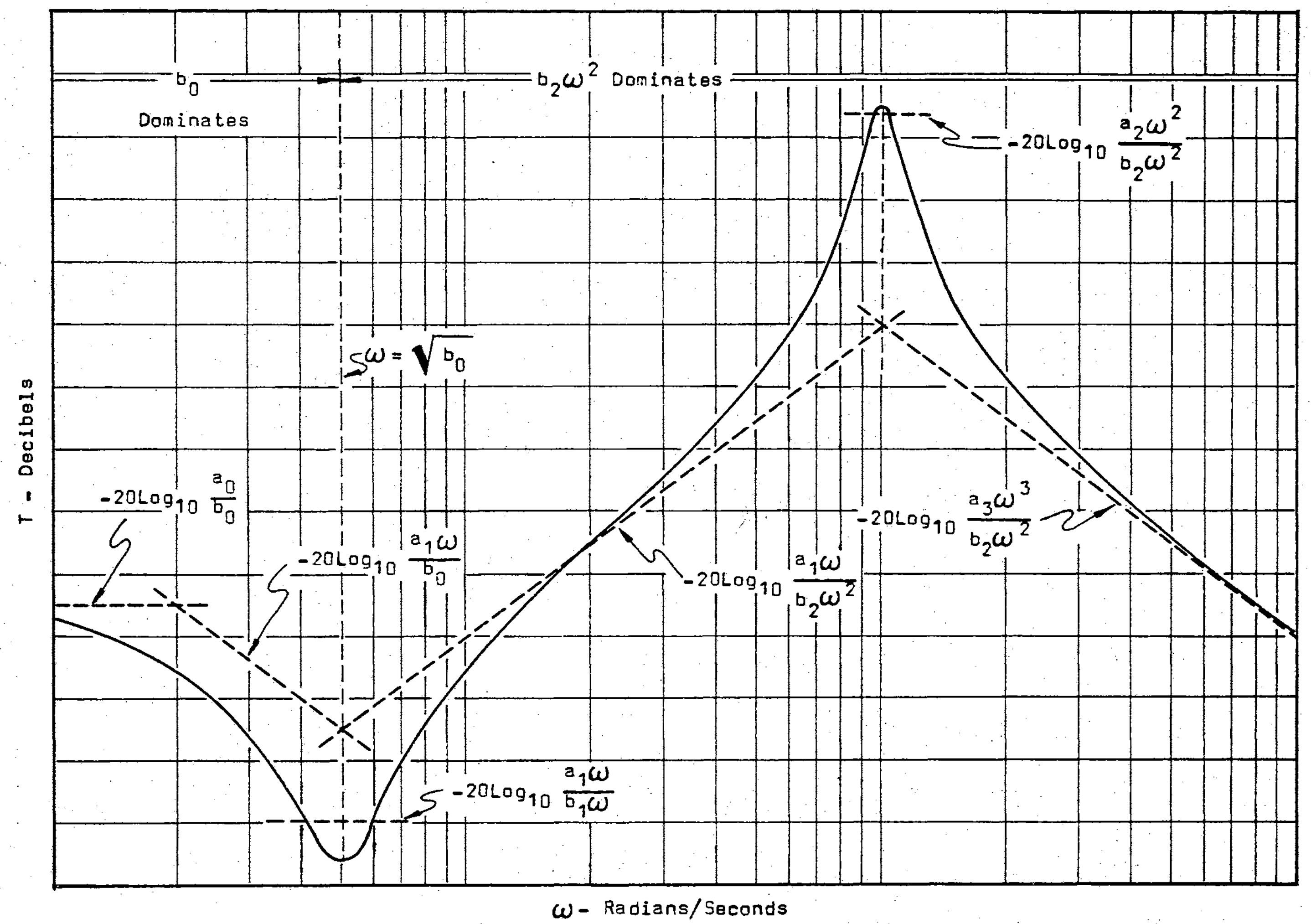


Figure 3-7. Transmission Function Representation for Frequency Response Spectrums With Coincident Transmission Bandwidth Boundaries

Figure 3-6 shows the manner in which the asymptote function which is dominate at the natural undamped mode frequency will approach the actual transmissibility as mode separation is effected.

Figure 3-7 shows what the response spectrum would look like if a transmission bandwidth were coincidental. This coincidental transmission bandwidth becomes evident when the Spectrum Band-pass Matrices are calculated. Normally, successive Spectrum Band-pass Matrices will have at least one entry which is equal to an entry in the previous Spectrum Band-pass Matrix and also an entry equal to an entry in the following Spectrum Band-pass Matrix as discussed before. Transmission band-pass boundaries which are coincident with the following boundaries are characterized by Spectrum Band-pass Matrices which have no entries equal to any entry in either the previous or following Spectrum Band-pass Matrix. Appendix C presents a means of dealing with very lightly damped third order systems and also a criterion for the stability of third and fourth order systems by the use of the transmission asymptote functions.

In general, the asymptote functions can be used to approximate the actual frequency response spectrum of an nth order system. This is also true for higher order systems since most stable, well behaved systems tend to be adequately represented by well separated break frequencies (4).

Summary

This chapter has substantiated that the implicit frequency response characteristics of physical systems can be displayed explicitly. This was accomplished through the use of the Fundamental State-Variable Frequency Matrix. This Frequency Canonical Form of the state-variable model is a natural addition to the more familiar Jordan and Rational Canonical Forms. As is demonstrated, the Frequency Canonical Form can be obtained from system mathematical models directly or by transformation from any form of the state model.

With the state model in the Frequency Canonical Form, Spectrum Band-pass Matrices can be calculated. These Spectrum Band-pass Matrices contain the transmission asymptote functions from which the frequency response spectrum is constructed. Also, during the generation of the Spectrum Band-pass Matrices, the transmission bandwidth boundaries are determined. The bandwidth boundaries establish the range of frequencies to which the associated Spectrum Band-pass Matrix is applicable.

This chapter also contains a rigorous mathematical substantiation of the entries in the Fundamental State-Variable Matrix as significant functions for characterizing system frequency response spectrums without having to factor polynomials. The transmission asymptote functions have been mathematically shown to be representative first order approximations to the frequency response spectrum. Also, proof is shown that under specified mode separation these asymptote functions provide the exact transmissibility for underdamped frequencies. Along with this proof the calculation of the transmission bandwidth boundaries from the entries in the Frequency Canonical Form input matrix has been mathematically substantiated. With these frequency spectrum characteristics it is possible to isolate all the characteristic modes of any time invariant linear system.

The basis of the construction of system frequency response spectrums by using the Fundamental State-Variable Frequency Matrix stems from the implicit characteristics of the coefficients of the unfactored characteristic and forcing function polynomials. From this basic system model mode representation the natural interaction of modes places a practical limitation on the application of the transmission asymptote functions. However, this limitation is only on the determination of the actual transmissibility at a mode's natural undamped frequency. This limitation involves maintaining modes at a minimum distance of one decade separation. In general, the first order approximation for a system's frequency response spectrum obtained by using the Fundamental State-Variable Frequency Matrix is closer to the actual spectrum than that obtained by using Bode's Theorems. Not only will this first order approximation obtained by using the Frequency Canonical Form be more representative but also the approximation can be obtained without factoring any polynomials.

CHAPTER IV

FREQUENCY CANONICAL FORM SYNTHESIS AND COMPENSATION APPLICATIONS

The use of frequency response spectrum information to represent the dynamic behavior of a system is especially valuable in system synthesis and compensation. Many times the excitation to which a system is subjected can be represented by a finite Fourier Series. With this means available to represent an excitation, system performance under this excitation can be obtained through the use of its frequency response spectrum. Along with the steady state performance much can be learned about the transient performance by examining the frequency response spectrum of a model. For these reasons the Frequency Canonical Form of state-variable models is very useful in the synthesis and compensation of dynamic systems.

The unique features provided by the Frequency Canonical Form for synthesis and compensation are the explicit display of the transmission bandwidths and the transmission asymptote functions. With this information it is possible to construct mathematical models for systems to meet predesignated frequency domain specifications. The construction process is done by superposition of model components described explicitly in the desired frequency spectrum. Also, systems whose dynamic characteristics are unacceptable can be compensated by direct adjustment of significant entries in the Spectrum Band-pass Matrices. These modified entries can then be reflected directly into the system model. This discussion leads to the topic of sensitivity of system parameters. Although nothing extensive will be presented, the sensitivity of system components can be examined on a very preliminary level through the use of the Frequency Canonical Form of system models.

Basic Feedback Control

The most fundamental characteristic of dynamic control systems analysis is that of feedback. The effect of unity feedback on the basic mathematical model is the increase of the coefficients in the open loop characteristic equation by an amount corresponding to particular coefficients in the forcing function. This increase of coefficients is done by adding the coefficients associated with equal orders of derivatives. This modification of the coefficients is illustrated below. The general nth order linear system with no feedback is described by

$$a_n \frac{d^n y}{dt^n} + \ldots + a_m \frac{d^m y}{dt^m} + \ldots + a_0 y = b_m \frac{d^m v}{dt^m} + \ldots + b_0 v \qquad (4\text{-}1)$$

When unity feedback is employed the system model becomes as that shown in Equation (4-2).

$$a_n\frac{d^n y}{dt^n} + \ldots + (a_m + b_m)\frac{d^m y}{dt^m} + \ldots + (a_0 + b_0)y = b_m\frac{d^m v}{dt^m} + \ldots + b_0 v \tag{4-2}$$

For physically realizable systems considered herein, unity feedback is interpreted in the Frequency Canonical Form as the addition of the input column matrix to the last column of the Fundamental State-Variable Frequency Matrix. The effect of unity feedback on the Frequency Canonical Form can be illustrated by the following. Consider the system whose dynamic characteristics are represented by

$$\frac{d^3y}{dt^3} + 10\frac{d^2y}{dt^2} + 416\frac{dy}{dt} + 800y = \frac{d^2v}{dt} + 105\frac{dv}{dt} + 500v \tag{4-3}$$

The state model for this system in Frequency Canonical Form can be written by inspection when Equations (3-2), (3-7), and (3-8) are employed as a comparison. The result is the following state model

$$\omega^3\begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 800 \\ \omega^3 & 0 & 416\omega \\ 0 & \omega^3 & 10\omega^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} + \begin{bmatrix} 500 \\ 105\omega \\ \omega^2 \end{bmatrix} v \tag{4-4}$$

$$y = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} \tag{4-5}$$

The state model for this system with unity feedback is

$$\omega^3 \begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1300 \\ \omega^3 & 0 & 521\omega \\ 0 & \omega^3 & 11\omega^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} + \begin{bmatrix} 500 \\ 105\omega \\ \omega^2 \end{bmatrix} v \quad (4\text{-}6)$$

This form of the state model is still the Frequency Canonical Form and the y equation is unchanged. The frequency response spectrum for this system with and without feedback is shown in Figure 4-1.

This simplified concept of feedback principles as applied to the Frequency Canonical Form of state models can be extended to more complex feedback loops. When the feedback loop contains other than unity gain the manner in which the different coefficients are added is dictated by the functional form of the dynamics contained in the feedback loop. The changes in the entries of the state model follow rather logically depending on the contents of the feedback loop. For other than simple gains in the feedback loop the entries in the input matrix will also be altered. The specific manner in which these entries in both the Fundamental State-Variable Frequency Matrix and the input matrix vary will be discussed in a following section where compensation is considered.

Synthesis in State Space

Synthesis in state space employing the Frequency Canonical State Model follows similarly to system analysis

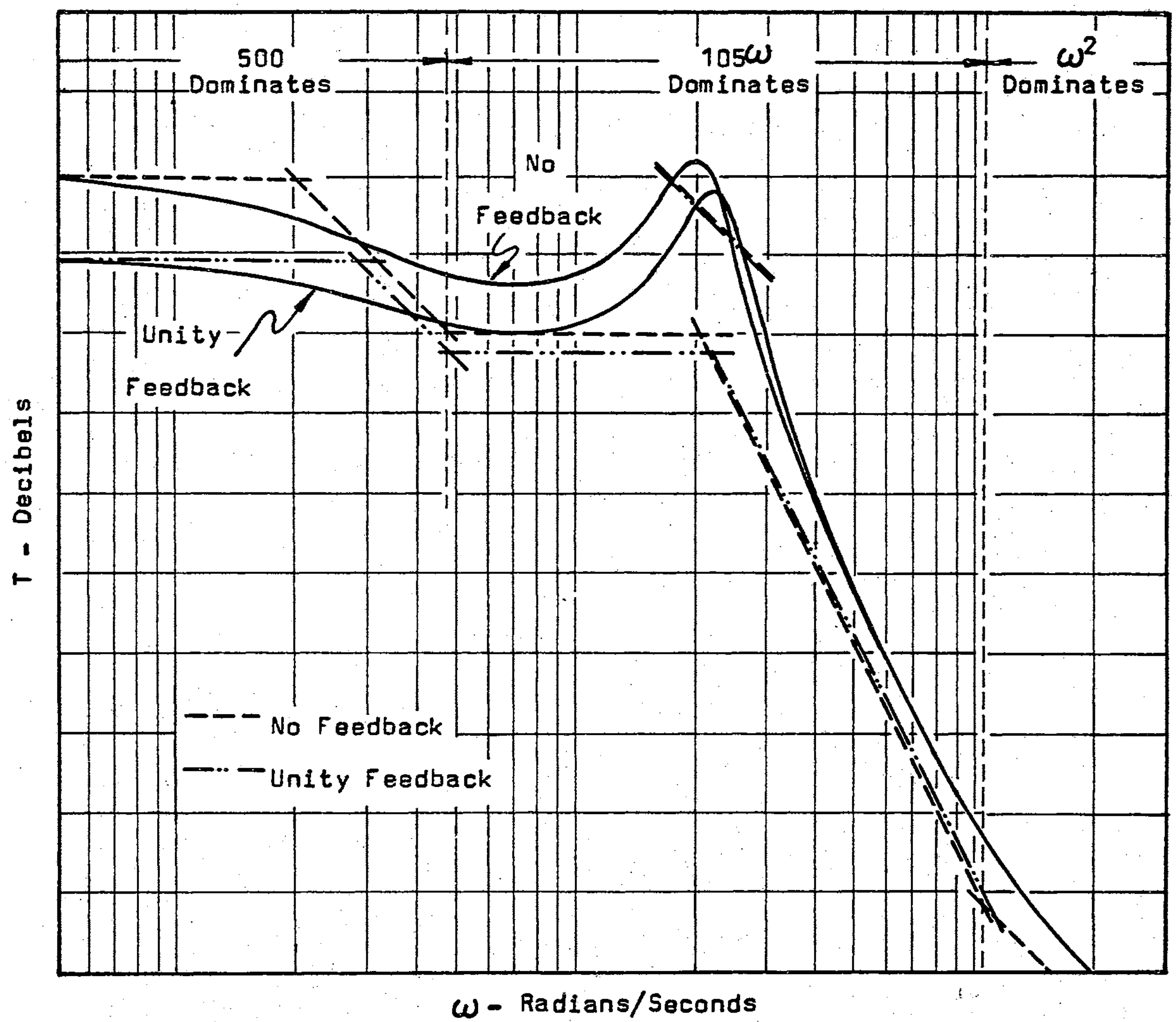


Figure 4-1. Comparison of Frequency Spectrums for a System With and Without Unity Feedback

employing this state model. The procedure of synthesis involves initially the construction of the specified frequency response spectrum by using straight lines whose slopes are multiples of 20 decibels per decade. This piecewise linear spectrum approximation describes the transmission asymptote functions. A typical illustration of these functions is shown in Figure 4-2. Every frequency where this first order approximation has a decrease in slope of 20 decibels per decade is a transmission bandwidth boundary. At frequencies where the slope decreases by 40 decibels per decade a coincidence of bandwidth boundary occurs. In other words, when a coincidence of bandwidth boundary occurs the dominance of input matrix elements skips an entry in the input matrix. The bandwidth frequency boundary where this occurs is the square root of the element in the input matrix which was skipped.

Figure 4-2 shows the frequency response spectrum of a system which was synthesized by first drawing the transmission asymptote functions. With these functional first order approximations for the frequency response of a desired system it is possible to write the state model in Frequency Canonical Form. By refering to Equations (3-7) and (3-8) the state model is written as shown in Equations (4-7) and (4-8).

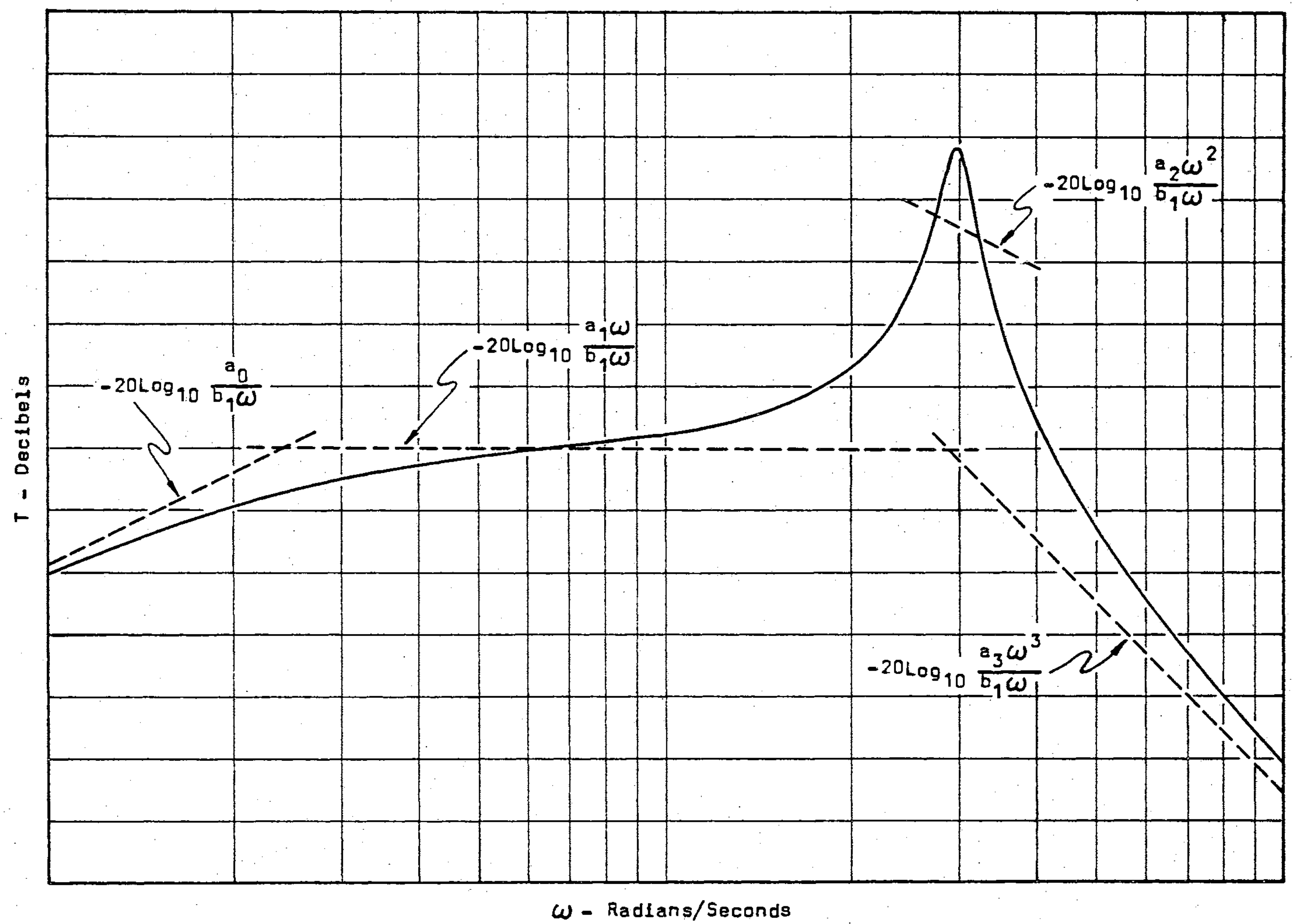


Figure 4-2. Synthesized System Frequency Response Spectrum by use of the Frequency Canonical State Model

$$0.176\omega^3 \begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 388.2 \\ 0.176\omega^3 & 0 & 158.5\omega \\ 0 & 0.176\omega^3 & 1.0\omega^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \omega & 0 \end{bmatrix}^T v \qquad (4\text{-}7)$$

$$y = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} \qquad (4\text{-}8)$$

The actual spectrum for the system represented by the above state model is also shown in Figure 4-2.

In many cases the modeled frequency response spectrum will not exactly meet the required frequency domain specifications. However, by using the methods discussed in the following section on compensation the modeled frequency response spectrum can be altered such that the specified requirements are adequately met.

Compensation in State Space

The specific characteristics of the Frequency Canonical Form possess features which permit the principle of compensation to be applied easily. The oscillatory mode peaks which always occur within transmission bandwidth boundaries are relatively independent of the boundaries involved. Therefore, it is possible to adjust this peak transmissibility by merely changing the entry in the Fundamental State-Variable Frequency Matrix which governs the associated transmissibility. This particular feature is demonstrated in Figure 4-3. As seen in this figure the transmissibility at the natural undamped frequency is easily adjusted to nearly any desired value.

Adjustment of break frequencies possesses the feature that any change in these frequencies also changes the transmission bandwidth boundaries. Since the actual break frequencies associated with the transmission asymptote function generally are not the system characteristic mode frequencies, this feature permits the shifting around of any bandwidth boundary to meet any specification to improve the frequency response of the represented system. In situations where oscillatory mode frequencies are altered the peak transmissibility associated with this mode will also be changed. Figure 4-4 illustrates the typical effect of changing the mode frequency with the normalized transmissibility function for the peak value unchanged. This operation places no limitation on the use of the Frequency Canonical Form in compensation since the peak transmissibility is virtually independent of the transmission bandwidth boundary and, thus, can be adjusted independently.

Often compensation infers the addition of equipment to obtain a desired system performance. Generally the addition of compensation elements involves increasing the order of either the characteristic and/or the forcing functions. The

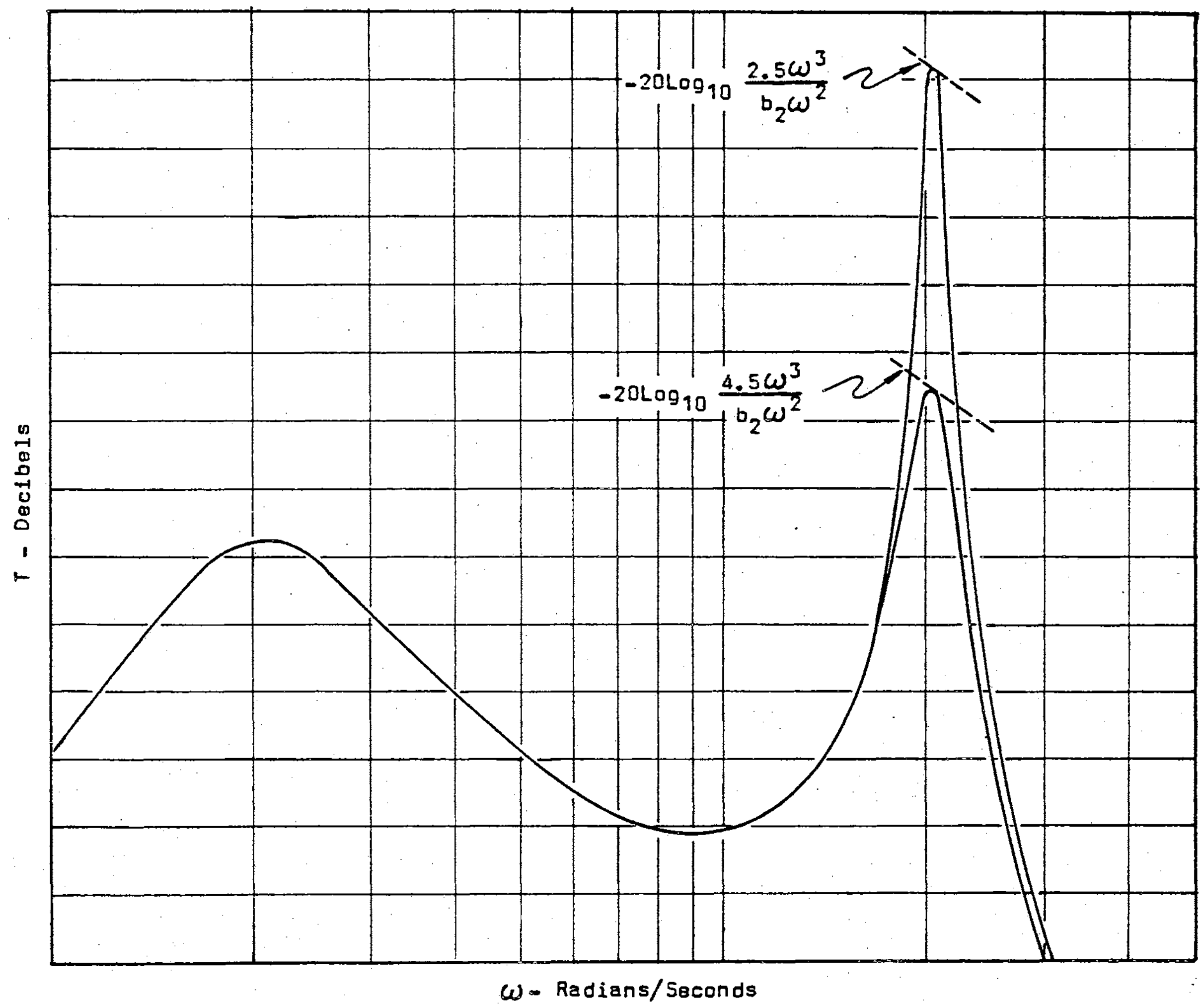


Figure 4-3. Resonant Peak Transmission Asymptote Function Variation

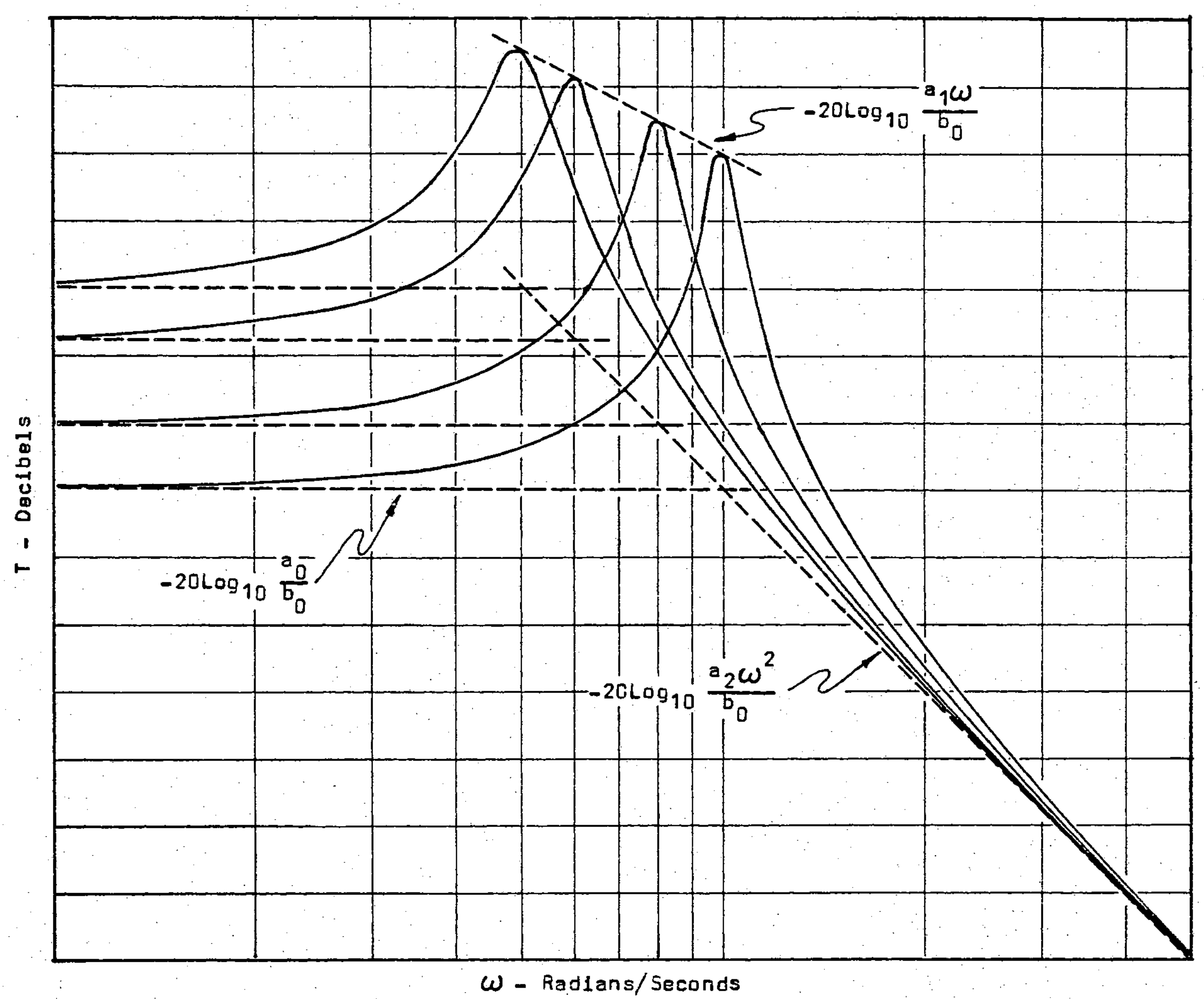


Figure 4-4. Effect of Zeroth Order Transmission Function on Frequency Response Spectrums

procedure for performing compensation from this standpoint entails merely the addition of elements in the Frequency Matrix and/or the input matrix. This operation is similar to the addition of feedback with other than a pure gain in the feedback loop. The following discussion demonstrates the procedure employed when compensation is performed using the Frequency Canonical Form in state space.

The classical compensation configurations used in present day control system theory are lag-lead and lead-lag. These can be used either in the feedback loop or in the forward loop. The particular details of the use to which either of these configurations in either the feedback or forward loops can be made are such that a single illustration will demonstrate the principles involved. In particular with the use of the Frequency Canonical State Model these principles can be demonstrated by considering the general model.

$$a_n\omega^n \begin{bmatrix} z_1' \\ z_2' \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_n' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ a_n\omega^n & 0 & \cdots & 0 & a_1\omega \\ 0 & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & 0 & \cdot \\ 0 & \cdots & 0 & a_n\omega^n & a_{n-1}\omega^{n-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -z_n \end{bmatrix}$$

$$+ \begin{bmatrix} b_0 & \cdot & \cdot & b_m\omega^m & 0 & \cdot & 0 \end{bmatrix}^T v \qquad (4\text{-}9)$$

$$y = \begin{bmatrix} 0 & . & . & . & . & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ . \\ . \\ . \\ . \\ -z_n \end{bmatrix} \qquad (4\text{-}10)$$

As illustrated previously, unity feedback is performed by adding the input column matrix to the last column of the Fundamental Frequency Matrix. For the general lag-lead or lead-lag configuration in the feedback loop the Frequency Canonical State Model can be developed as shown below.

$$a_n\omega^{n+1} \begin{bmatrix} z_1' \\ z_2' \\ . \\ . \\ . \\ . \\ z_{n+1}' \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 & . & . & . & 0 & \beta a_0 \\ a_n\omega^{n+1} & 0 & & & & . & . \\ 0 & . & & & & . & . \\ . & & . & & & 0 & \beta a_m\omega^m \\ . & & & . & & . & . \\ . & & & & . & 0 & . \\ 0 & . & . & . & . & a_n\omega^{n+1} & \beta a_n\omega^n \end{bmatrix} \right.$$

$$+ \begin{bmatrix} 0 & . & . & . & 0 \\ 0 & . & . & 0 & a_0\omega \\ . & & & . & . \\ . & & & . & . \\ 0 & . & . & 0 & a_{n-1}\omega^n \end{bmatrix}$$

$$+\alpha\begin{bmatrix} 0 & \cdots & 0 & b_0 \\ 0 & \cdots & 0 & b_1\omega \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & b_m\omega^m \\ \cdot & & \cdot & 0 \\ \cdot & & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\left. + \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \cdot & & \cdot & b_0 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & b_{m-1} \\ \cdot & & \cdot & 0 \\ \cdot & & \cdot & \cdot \\ 0 & \cdots & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -z_{n+1} \end{bmatrix}$$

$$+\left\{ \beta \begin{bmatrix} b_0\omega \\ b_1\omega^2 \\ \cdot \\ b_m\omega^{m+1} \\ 0 \\ \cdot \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \\ \cdot \\ b_{m-1} \\ 0 \\ \cdot \\ 0 \end{bmatrix} \right\} v \qquad (4\text{-}11)$$

$$y = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ -z_{n+1} \end{bmatrix} \qquad (4\text{-}12)$$

In frequency domain notation the configuration in the feedback loop reflected in Equation (4-11) is $(s+\alpha)/(s+\beta)$.

This type of compensation is characterized in the frequency spectrum by the addition of one transmission asymptote function and one transmission bandwidth boundary. In effect the state model representation shown in Equation (4-11) can be accomplished graphically by working with the spectrum representation.

Sensitivity Check in State Space

The system frequency performance sensitivity of various system contributions or components can also be investigated. This sensitivity investigation can be done by direct inspection of the elements of the Spectrum Band-pass Matrices. The transmission asymptote functions in these matrices describe the break frequencies of the oscillatory modes and, to some extent, also provide the actual transmissibility at these break frequencies. In the previous discussion on compensation it was pointed out how the various types of transmission asymptote functions affect the actual spectrum. At unity frequency the intercepts of the asymptote function are composites of all the components of the actual physical system.

Each of these components have a physical character which may change due to environmental changes. Thus, the system frequency response spectrum will have a time variant character as well. The effect of this possible time variance can be checked by examination of the sensitivity of the frequency spectrum described by the Fundamental State-Variable Frequency Matrix.

To illustrate this, the state model for the network shown in Figure 4-5 is as shown in Equation (4-13).

$$a_3\omega^3 \begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_0 \\ a_3\omega^3 & 0 & a_1\omega \\ 0 & a_3\omega^3 & a_2\omega^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ b_1\omega \\ 0 \end{bmatrix} e_i \qquad (4\text{-}13)$$

$$e_0 = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} \qquad (4\text{-}14)$$

where

$a_0 = R_1/C$

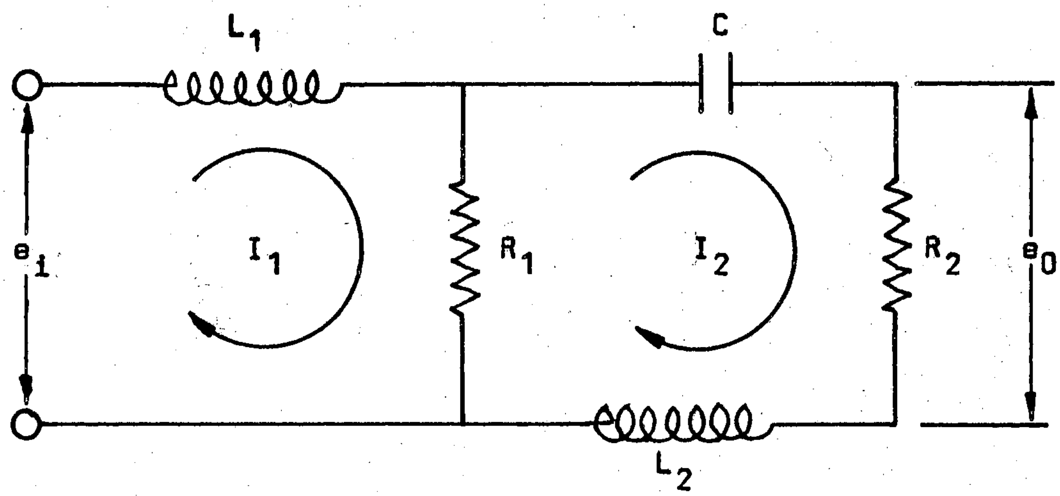
$a_1 = (L_1/C)+R_1R_2+R_1{}^2+R_2{}^2$

$a_2 = L_1R_1+L_1R_2+R_1L_2$

$a_3 = L_1L_2$

$b_1 + R_1R_2$

From this Frequency Canonical Form of the state model the entire frequency spectrum and variations in the spectrum



$$\omega_n = \sqrt{\frac{\frac{R_1}{C}}{L_1R_1 + L_1R_2 + R_1L_2}}$$

or

$$\omega_n = \sqrt{\frac{1}{L_1L_2}\left(\frac{L_1}{C} + R_1R_2 + R_1^2 + R_2^2\right)}$$

Figure 4-5. Electrical Network

can be visualized. Formation of the Spectrum Band-pass Matrices produces

$$\underline{S}_1 = \begin{bmatrix} 0 & 0 & \frac{a_0}{b_1\omega} \\ \frac{a_3\omega^3}{b_1\omega} & 0 & \frac{a_1\omega}{b_1\omega} \\ 0 & \frac{a_3\omega^3}{b_1\omega} & \frac{a_2\omega^2}{b_1\omega} \end{bmatrix} \qquad (4\text{-}15)$$

The entry $a_0/b_1\omega$ indicates that the spectrum will have a plus 20db per decade asymptote at low frequencies. The next entry $a_1\omega/b_1\omega$ shows an increase in asymptote slope to zero. The $a_2\omega^2/b_1\omega$ entry shows a minus 20db per decade asymptote slope and $a_3\omega^3/b_1\omega$ shows a minus 40db per decade asymptote slope. From this examination of matrix entries it is possible to show that an underdamped mode can occur at one of two frequencies. These frequencies are calculated in the following way. An underdamped mode may occur at the frequency where

$$\frac{a_0}{b_1\omega} = \frac{a_2\omega^2}{b_1\omega} \qquad (4\text{-}16)$$

which produces in terms of the system components a natural undamped frequency of

$$\omega_n = \sqrt{\frac{a_0}{a_2}} = \sqrt{\frac{R_1/C}{L_1R_1 + L_1R_2 + R_1L_2}} \qquad (4\text{-}17)$$

or the underdamped frequency could be where

$$\frac{a_1\omega}{b_1\omega} = \frac{a_3\omega^3}{b_1\omega} \qquad (4\text{-}18)$$

which is a natural frequency of

$$\omega_n = \sqrt{\frac{a_1}{a_3}} = \sqrt{\frac{1}{L_1L_2}\left(\frac{L_1}{C} + R_1R_2 + R_1^2 + R_2^2\right)} \quad (4\text{-}19)$$

A study of the components represented in Equations (4-17) and (4-19) will indicate the location sensitivity of the modal frequency spectrum within the spectrum with respect to any single component or combinations of components.

It is also possible to investigate the transmissibility at these frequencies in regards to the system component variations. These transmissibilities are determined by the asymptote functions $a_1\omega/b_1\omega$ and $a_2\omega^2/b_1\omega$, respectively. For the first possible natural undamped frequency calculated in Equation (4-17) the transmissibility at this frequency is

$$T_{\omega=\omega_n} = 20\text{Log}_{10}\frac{b_1}{a_1} =$$

$$20\text{Log}_{10}\frac{R_1R_2}{\frac{L_1}{C} + R_1R_2 + R_1^2 + R_2^2} \quad (4\text{-}20)$$

and for the natural undamped frequency calculated in Equation (4-19) the transmissibility is

$$T_{\omega=\omega_n} = 20\text{Log}_{10}\frac{b_1}{a_2\omega} =$$

$$20\text{Log}_{10}\frac{R_1R_2}{(L_1R_1 + L_1R_2 + R_1L_2)\omega_n} \quad (4\text{-}21)$$

where the ω_n in Equation (4-21) is that shown in Equation (4-19).

Sensitivity studies can be performed in a similar manner for most any system which is adequately described by linear time invariant ordinary differential equations.

Summary

The Frequency Canonical Form for state models of physical systems provides a powerful tool for control system designers and analysts. Its features are such that it can be used in many capacities to help the engineer solve day to day control problems. Many times the complexity of the system is such that a digital computer would be necessary to obtain the roots of the characteristic and forcing function polynomials. This form of the state model allows the engineer to perform the operations of analysis, synthesis, and compensation without reverting to the digital computer.

CHAPTER V

STATE MODELS DERIVED FROM EXPERIMENTAL DATA

An important feature which the Fundamental State-Variable Frequency Matrix provides is its capability of being derived from experimental frequency response data. Although most physical systems are not linear or stationary and, consequently, experimental frequency response data of these systems contain the effects of nonlinearities and time varying coefficients, many of these systems can be adequately represented by linear time invariant models. The linear, time invariant state model for this class of physical systems can be derived from its frequency response data through the use of the Fundamental State-Variable Frequency Matrix.

State Models From Frequency Response Data

The development of the Fundamental State-Variable Frequency Matrix showed that the entries in this matrix are first order approximations to the actual system frequency spectrum. This fact is the basis for the derivation of state space models from experimental frequency response spectrums.

To derive the Fundamental Frequency Matrix the entries in Spectrum Band-pass Matrices must first be obtained. This is accomplished by constructing a first order approximation to the experimental spectrum data. Since all the transmission asymptote functions have slopes which are multiples of 20 decibels per decade, this first order approximation must consist of straight line segments whose slopes are multiples of 20 decibels per decade. For simple breaks where the slope changes by only 20db per decade, these breaks are established by the approximation deviating from the actual spectrum by approximately 3 decibels.

Once a first order approximation is drawn, then the transmission bandwidth boundaries can be estimated. These boundaries occur at every frequency where the first order approximation has a decrease in slope. For decreases in slope of only 20 decibels the boundary dictates simply a change of dominance from one input matrix element to the next. However, if the decrease in slope is 40 decibels per decade then this dictates a coincidence of transmission bandwidth boundaries. The frequency associated with this coincidence is a boundary whose value is the square root of an input matrix element. Also, the change in dominance across this boundary is from the element before to the element after this particular element in the input matrix.

The results of this construction are the transmission bandwidth boundaries and the transmission asymptote functions. In order to evaluate the separate coefficients which make up these boundaries and functions, it is necessary to assume that the derived model is normalized with respect to

b_m. In other words, b_m is assumed to be equal to unity. No generality is sacrificed with this assumption since many operations in system analysis work with this normalized form. With $b_m = 1$ then all the entries in the input matrix can be calculated from the transmission boundaries. Once these coefficients are obtained then the remaining elements of the Fundamental Frequency Matrix and, finally, the Frequency Canonical State Model Form can be written.

The utilization of this procedure for obtaining state models from frequency spectrum data may involve the procedures discussed in the previous chapter under compensation to ascertain an adequate fit of the experimental data. This is due to the lack of concrete knowledge about the interaction of system modes.

To illustrate the utilization of this procedure consider the frequency response spectrum data shown in Figure 5-1. The actual state model represented in this figure in Frequency Canonical Form is

$$\omega^4 \begin{bmatrix} z_1' \\ z_2' \\ z_3' \\ z_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1600 \\ \omega^4 & 0 & 0 & 776\omega \\ 0 & \omega^4 & 0 & 411.6\omega^2 \\ 0 & 0 & \omega^4 & 5.6\omega^3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ -z_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 300\omega \\ 40\omega^2 \\ \omega^3 \end{bmatrix} v \qquad (5\text{-}1)$$

$$y = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 & z_2 & z_3 & -z_4 \end{bmatrix}^T \qquad (5\text{-}2)$$

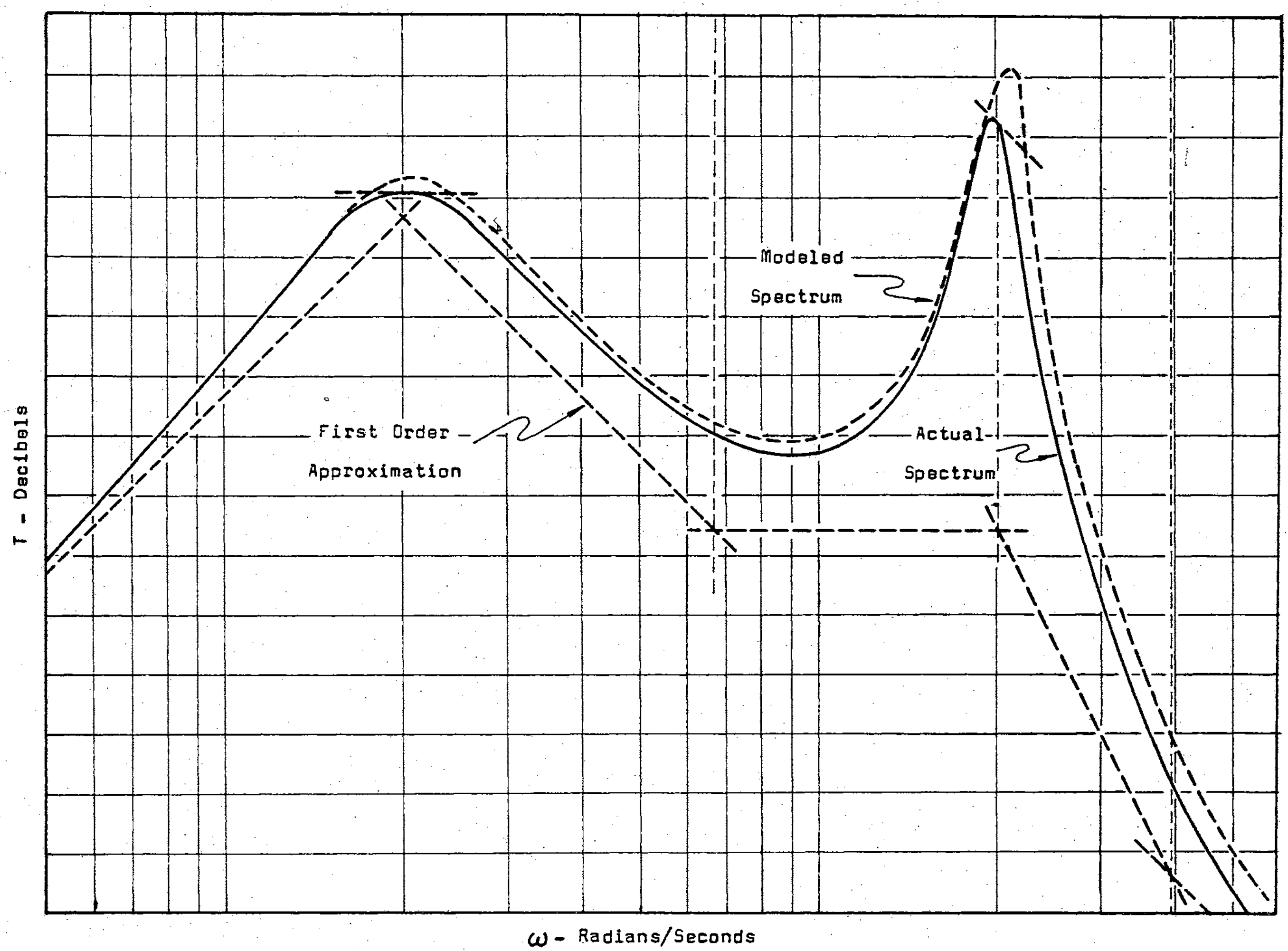


Figure 5-1. Frequency Canonical State Model Linear System Identification

The state model derived from this spectrum data is

$$.9\omega^4 \begin{bmatrix} z_1' \\ z_2' \\ z_3' \\ z_4' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1500 \\ .9\omega^4 & 0 & 0 & 675\omega \\ 0 & .9\omega^4 & 0 & 376\omega^2 \\ 0 & 0 & .9\omega^4 & 3.7\omega^3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ -z_4 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 275\omega & 39.5\omega^2 & \omega^3 \end{bmatrix}^T v \qquad (5\text{-}3)$$

$$y = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ -z_4 \end{bmatrix} \qquad (5\text{-}4)$$

The frequency response spectrum for the state model of Equations (5-3) and (5-4) are also shown in Figure 5-1. Once a state model has been fit improvements in this fit can be made by inspection and the appropriate matrix element can be adjusted to reflect the improved fit.

In order to illustrate the use of this procedure in a general situation the spectrum shown in Figure 5-2 was drawn in an arbitrary fashion using French Curves only to obtain a smooth spectrum. The resulting spectrum fit is also shown in this figure. No attempt has been made to improve the spectrum derived from the procedure developed in this chapter for obtaining the state model from experimental frequency

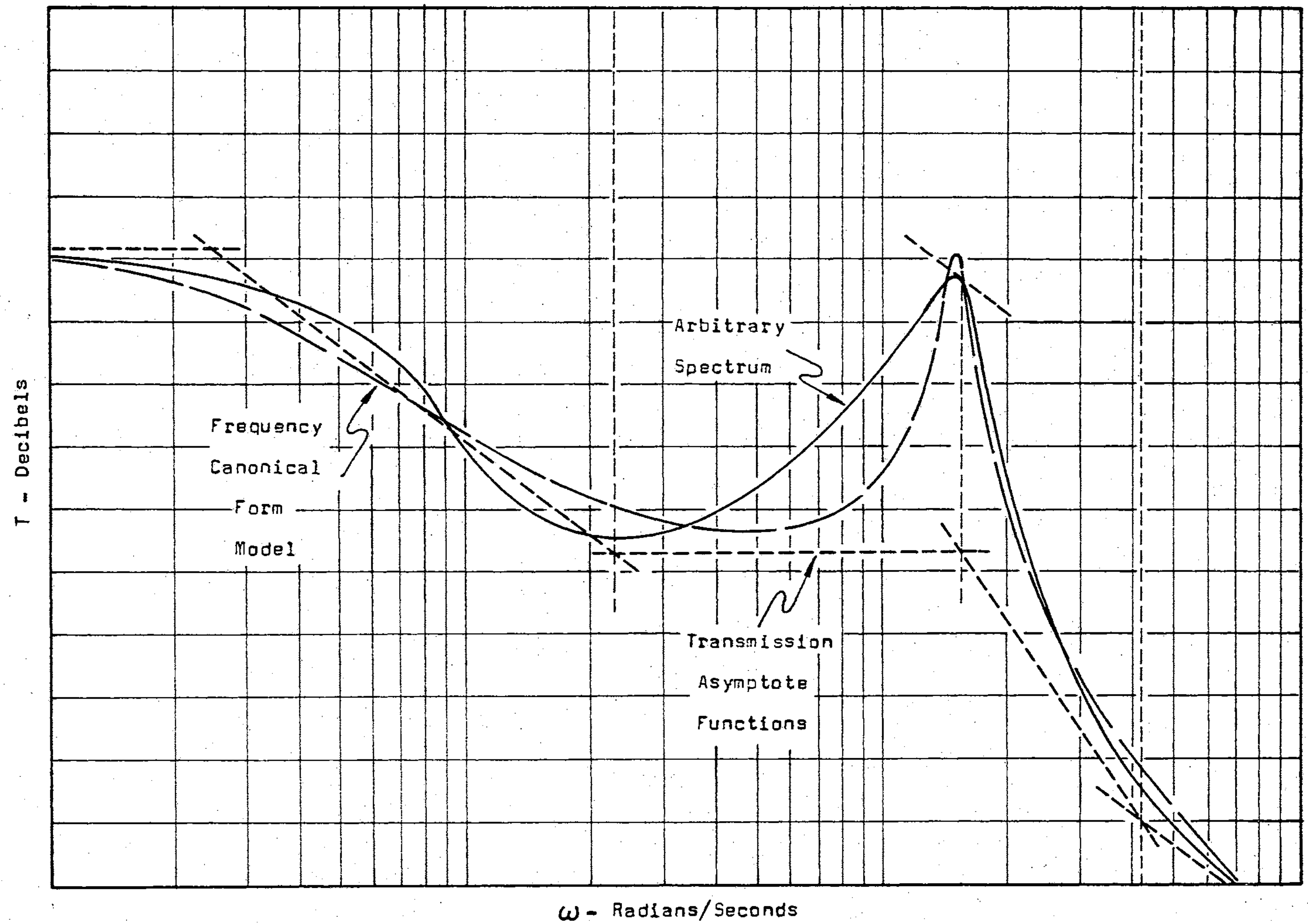


Figure 5-2. Frequency Canonical State Model Identification of Arbitrarily Drawn Spectrum

response data. It is possible to improve the linear time invariant representation if it is necessary. The Frequency Canonical Form of the state model representative of the derived or fitted spectrum is shown below.

$$10.4\omega^3 \begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 578.0 \\ 10.4\omega^3 & 0 & 2373\omega \\ 0 & 10.4\omega^3 & 20\omega^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} + \begin{bmatrix} 96.0 & 42.0\omega & \omega^2 \end{bmatrix}^T v \qquad (5\text{-}5)$$

$$y = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} \qquad (5\text{-}6)$$

The two illustrations presented above represent a rather general application of the Fundamental State-Variable Frequency Matrix method to state space modeling which can be derived from experimental frequency response data. The modeled spectrum shown in both Figures 5-1 and 5-2 were derived on the first modeling attempt. Consequently, no effort has been exerted to improve the model although it is possible to iterate and possibly obtain better spectrum fits. These figures demonstrate the closeness possible on the first try if care is taken in the selection of the asymptotes.

This modeling technique has several features which make its application rather simple. One feature is that the break frequencies are not necessarily the roots of the characteristic and forcing polynomial functions. This feature removes the task of having to try to guess at representative corner frequencies. Another is the ability to model peak transmissibilities without worrying about the mode representative damping ratio. Also, in effect, the coefficients generated are the coefficients of the linear time invariant model represented by its frequency response spectrum.

CHAPTER VI

SUMMARY OF SYSTEMS ANALYSIS AND SYNTHESIS VIA USE OF THE FREQUENCY CANONICAL STATE MODEL

The procedures outlined in this chapter summarize the detailed discussions regarding the application of the Frequency Canonical Form presented in the previous chapters. The type of systems to which the following procedures are applicable are those which are adequately modeled by linear, time invariant, ordinary differential equations. In general, these procedures are applied to the category of systems having a single input and a single output. The models for most physically dynamic systems can be adjusted to fit in this category by successively pairing the inputs and outputs.

Obtaining the Frequency Canonical Form

Initially the system mathematical representation can be in either the ordinary differential form or some state model form. If the model is in differential form the direct programming technique shown in Figure 3-1 can be applied. This procedure results directly in the Frequency Canonical Form. If the system model is initially in some state-variable form it can be transformed into the Frequency Canonical Form by first transforming the state model into Rational Canonical Form and then applying the Frequency Transformation. This transformation is

$$\underline{z}' = \frac{1}{\omega}\,\underline{\Omega}^{-1}\,\underline{R}\,\underline{\Omega}\,\underline{z} + \frac{1}{\omega}\,\underline{\Omega}^{-1}\,\underline{B}\,\underline{v} \qquad (6\text{-}1)$$

where

$\underline{z}' = \frac{1}{\omega}\,\underline{\Omega}^{-1}\,\dot{\underline{x}}$

$\underline{z} = \underline{\Omega}^{-1}\,\underline{x}$

$\underline{\Omega}$ = n by n Frequency Transformation Matrix (see Equation (3-11)).

$\underline{x}$ = state-variables for state model in Rational Canonical Form.

$\underline{R}$ = Rational Canonical Form of differential transition matrix.

$\underline{B}$ = input column matrix.

$\underline{v}$ = input vector.

also

$\underline{\Omega}^{-1}\,\underline{R}\,\underline{\Omega} = \underline{F}$ = Fundamental State-Variable Frequency Matrix.

Both of the procedures discussed above will result in the Frequency Canonical Form for the system's state model.

Analysis Using the Frequency Canonical Form

To construct the first order approximation for the system frequency response spectrum the following steps are suggested.

1. Divide the Fundamental State-Variable Frequency Matrix and the input matrix by the first entry in

the input matrix. This produces the first Spectrum Band-pass Matrix. The elements of this matrix are the transmission asymptote functions.

2. The range of frequencies to which this first Spectrum Band-pass Matrix is applicable is calculated by equating the first entry in the input matrix (after the division indicated above) to the second entry. This produces the first transmission bandwidth boundary.

3. The first two steps are repeated by using all successive entries in the input matrix starting with the Fundamental State-Variable Frequency Matrix in each instance. This produces m Spectrum Band-pass Matrices and m transmission bandwidth boundaries, where m is the order of the forcing function polynomial.

4. The first order approximation of the frequency response spectrum is constructed by starting at the low frequency using the first Spectrum Band-pass Matrix generated. The first entry in the right-hand column of this matrix is used as the first asymptote of the spectrum approximation. (All these functions are straight lines on log-log paper and they all have slopes which are multiples of 20 decibels per decade.)

5. The asymptote functions of the first Spectrum Matrix (in the right-hand column) are used successively

until the transmissibility calculated by an asymptote function in the first Spectrum Matrix is exactly equal to the transmissibility calculated by an asymptote function in the next Spectrum Matrix at the transmission boundary frequency. At this point the next Spectrum Band-pass Matrix is used to construct the first order approximation in the next frequency band.

6. The points where a change in Spectrum Band-pass Matrices to be used will always occur on the transmission boundary frequency and its occurrence will always be indicated by the equality of asymptote functions in successive matrices as discussed above.

For systems whose modes are separated by more than a decade this first order approximation will not only result in good straight line approximations for the system frequency response spectrums but it will also produce virtually the exact transmissibility for the oscillatory modes at the particular modes' undamped natural frequency. The undamped natural frequency is displayed explicitly in the Fundamental State-Variable Frequency Matrix for any arrangement of mode separations.

Synthesis Using the Frequency Canonical Form

Synthesis using the Frequency Canonical Form proceeds as follows.

1. Construct a first order approximation on log-log paper for the desired frequency response to be possessed by a system. These first order approximations must all have slopes which are some multiple of 20 decibels per decade.
2. At every frequency where the approximation has a decrease in slope calculate the elements of the input matrix. This is done by starting at the highest frequency where the approximation has a decrease in slope. This frequency is equal to b_{m-1}/b_m. Assuming $b_m = 1$ then b_{m-1} can be calculated. The next frequency where the slope decreases is equal to b_{m-2}/b_{m-1}, etc. Using the previously determined value all the elements of the input matrix can be determined.
3. The first order approximations drawn in step 1 are the transmission asymptote functions. From these straight lines all the entries in the Fundamental State-Variable Frequency Matrix can be calculated and, thus, the system state model in the Frequency Canonical Form is obtained.

Compensation Using the Frequency Canonical Form

System compensation employing the Frequency Canonical Form involves the knowledge of specific features of the transmission asymptote functions. These features are as shown below.

1. Asymptote functions which determine the transmissibility at oscillatory mode frequencies are virtually independent of the bandwidth boundaries but are sensitive to the closeness of other system modes.
2. All other asymptote functions govern the bandwidth boundaries and, therefore, adjustments in these functions will be propagated to higher frequencies in the spectrum.

In general, compensation can be performed either graphically and then interpreted into the state model or the state model can be worked with directly.

System Modeling From Experimental Data Using the Frequency Canonical Form

The procedure for state model identification from experimental frequency spectrums follows much the same channels as outlined for synthesis.

1. Initially, the experimental data must be approximated by straight lines on log-log paper. These straight lines must have slopes which are multiples of 20 decibels per decade.
2. Every frequency where this first order approximation decreases in slope is a transmission bandwidth boundary. These boundary frequencies are used to calculate the entries in the input matrix. By starting at the highest bandwidth frequency boundary which is equal to b_{m-1}/b_m where m is the number of

bandwidth boundaries, and setting $b_m = 1$, then b_{m-1} is determined. The next boundary frequency is b_{m-2}/b_{m-1} from which b_{m-2} is determined, etc. until all the elements in the input matrix are obtained.

3. The straight lines which make-up the first order approximation for the experimental data are the asymptote functions. These functions are ratios of the input matrix elements and the Frequency Matrix elements. Since all the input matrix elements have been obtained, then all the elements in the Frequency Matrix can also be obtained.

This procedure results in the state model representative of the first order approximation drawn initially. If, after checking the frequency response spectrum computed from the derived state model with that of the actual system, it is found that the derived model is not adequate, then some adjustments in the first order approximation can be performed easily in attempts to obtain a more adequate model. This procedure can be mechanized on a digital computer with an error criterion used to determine adequate first order approximations and, thus, model frequency response spectrums.

This chapter has summarized the various applications to which the Frequency Canonical State Model Form developed in this dissertation are amenable. Several details peculiar to specific situations are not included in this chapter. However adequate details as well as illustrations are provided in the chapters dealing with the particular application.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

It has been demonstrated in this dissertation that linear time invariant models for physical systems, whether in differential form or some state-variable form, can be expressed in the Frequency Canonical Form. This State Space Canonical Form can be used to display the frequency characteristics inherent in all dynamic systems in an explicit manner. The particular procedure developed herein produces the transmission bandwidths which are peculiar to the fundamental modes of the system as well as a transmissibility description of the modes. All this information is obtained without resorting to any sort of factorization of the system characteristic and forcing functions which, up to the present, has been the only way to ascertain this system characteristic data.

Based on an exhaustive literature survey and the experience of the author, to his knowledge no work has been done with system frequency response spectrums in conjunction with state space system models. Therefore, the following topics which are developed in this dissertation will contribute to extending the frontiers of knowledge in state space analysis of linear, time invariant dynamic systems.

1. Four procedures are developed by which presently known classical control techniques can be used in conjunction with state models to obtain the frequency response spectrums for dynamic systems from the representative mathematical models. These procedures necessitate a rather extensive knowledge and experience in frequency response techniques on the part of the user.
2. A new addition to the more familiar Jordan, Rational, and Phase-Variable Canonical Forms in state space is developed. This addition is the Frequency Canonical Form for state-variable models.
3. Along with the Frequency Canonical Form two methods are developed to obtain this particular Canonical Form. The first method uses a frequency transformation and a direct programming procedure to produce the Frequency Canonical Form from the system's mathematical model in differential form. The other method involves the development of a Frequency Transform Matrix. This matrix directly transforms a state model from Rational Canonical Form to Frequency Canonical Form. Since there are standard transformations which will transform any state model into Rational Canonical Form, this Frequency Transform Matrix will essentially transform any system state model into the Frequency Canonical Form.

4. The differential transition coefficient matrix of the Frequency Canonical Form is the "Fundamental State-Variable Frequency Matrix". Explicit in this matrix are first order approximations of the frequency response spectrum of the modeled system.

5. The first order approximations explicit in the "Fundamental State-Variable Frequency Matrix" are better approximations than the asymptotic approximations which Bode's Theorems produce. Also, it is not necessary to factor the polynomials involved as is the case in the use of Bode's Theorems.

6. The Frequency Canonical Form provides extremely useful applications of state models to the field of dynamic systems analysis. This dissertation demonstrates the usefulness of the Frequency Canonical Form in system analysis, synthesis, and compensation.

7. At present to the author's knowledge there has been no method developed by which state models can be derived from experimental data. The "Fundamental State-Variable Frequency Matrix" provides a means for the identification of state models from experimental frequency response data.

All the contributions contained in this dissertation which are listed above suggest much more research in the Frequency Canonical State Space. Therefore, the author recommends the following topics as areas which will produce significant contributions in state space analysis.

1. The phase shift associated with the frequency spectrums developed by the use of the Frequency Canonical Form of the state model can be obtained by Bode's Theorems (15). This phase shift spectrum can also be derived by an explicit matrix display of state model coefficient matrices.
2. Transformations for the state model which result in more canonical forms peculiar to engineering disciplines, for example state model forms specifically for stability evaluation which will provide obvious compensation possibilities, would contribute to the systems analysis field through the use of state model techniques.
3. The extension of the frequency response spectrum explicit display in state models for nonlinear systems will provide a major contribution to system analysis by state space techniques. The canonical form of the state model for this class of systems could conceivably be made up of coefficient matrices displaying explicitly all the jump frequencies peculiar to any specific system when excited by a sinusoidal excitation.
4. Sigma plots which are employed in classical control theory to determine the closed loop real roots from the open loop transfer function by substituting $-\sigma$

for s in the open loop transfer function can possibly be characterized by state model coefficient matrices. These matrices could then be transformed to display the closed loop poles explicitly without reverting to the construction of sigma plots (16).

SELECTED BIBLIOGRAPHY

(1) Gilbert, Elmer G. "Controllability and Observability in Multivariable Control Systems." Journal on Control of the Society for Industrial and Applied Mathematics (SIAM). Series A. Vol. 1. No. 2. (1963) 128-151.

(2) Brockett, Roger W. "Poles, Zeros, and Feedback: State Space Interpretation." IEEE Transactions on Automatic Controls. (April, 1965), 129-135.

(3) Smith, Earl F. "A Graphical Method for Plotting Amplitude and Phase Angle of Transfer Functions of Dynamic Systems Without Factoring Polynomials." NACA Technical Note 2522. (November, 1951).

(4) Ausman, J. S. "Amplitude Frequency Response Analysis and Synthesis of Unfactored Transfer Functions." Transactions of the ASME Journal of Basic Engineering. (March, 1964), 32-36.

(5) Chestnut, Harold and Robert W. Meyer. Servomechanisms and Regulating System Design, 2nd ed. John Wiley and Sons Inc., New York, N. Y. (1959).

(6) DeRusso, D. M., R. J. Roy, and C. M. Close. State Variables for Engineers. John Wiley and Sons Inc., New York, N. Y. (1965).

(7) Tou, J. T. Modern Control Theory. McGraw-Hill Book Company Inc., New York, N. Y. (1964).

(8) Dorf, R. C. Time-Domain Analysis and Design of Control Systems. Addison-Wesley Publishing Company Inc., Reading, Massachusetts (1965).

(9) Zadeh, L. A. and C. A. Desoer. Linear System Theory - The State Space Approach. McGraw-Hill Book Company Inc., New York, N. Y. (1963).

(10) Bellman, R. Introduction to Matrix Analysis. McGraw-Hill Book Company Inc., New York, N. Y. (1960).

(11) Kells, L. M. Elementary Differential Equations. McGraw-Hill Book Company Inc., New York, N. Y. (1954).

(12) Chidambara, M. R. "The Transformation to (Phase-Variable) Canonical Form." IEEE Transactions on Automatic Control. (October, 1965), 492-495.

(13) Johnson, C. D. and W. M. Wonham. "A Note on the Transformation to Canonical (Phase-Variable) Form." IEEE Transactions on Automatic Control. (July, 1964), 312-313.

(14) Mufti, I. H. "On the Reduction of a System to Canonical (Phase-Variable) Form." IEEE Transactions on Automatic Control. (April, 1965), 206-207.

(15) Bode, H. W. Network Analysis and Feedback Amplifier Design. D. Van Nostrand Company, Inc., New York, N. Y. (1945).

(16) McRuer, D. T., I. L. Ashkenas, and C. L. Guerre. "A Systems Analysis View of Longitudinal Flying Qualities." Flight Control Laboratory Wright-Patterson AFB, Ohio, WADD TR 60-43. (January, 1960).

APPENDIX A

STATE-VARIABLE DIAGRAMS FROM DIRECT PROGRAMMING METHODS

STATE-VARIABLE DIAGRAMS FROM DIRECT PROGRAMMING METHODS

There are two basic direct programming methods which produce vector state models in well known canonical forms. The first of these programming methods is Tou's direct programming method from which the resulting state model appears in Phase-Variable Canonical Form (12, 13, and 14). The Phase-Variable Canonical Form is merely the transpose of the Rational Canonical Form. The second direct programming method results in Rational Canonical Form state models.

The particular feature existing in these forms for the state models is the explicit display of the coefficients associated with the characteristic equation and the coefficients associated with the form of the forcing function. The basic difference in the state models produced by these direct programming methods, other than the rearrangement of the elements in the differential transition matrix, is the explicit display of the forcing function coefficients.

The general mathematical model to which these methods are applicable are single input - single output linear stationary ordinary differential equations. These system models can be expressed in the following form.

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0 y =$$

$$b_m \frac{d^m v}{dt^m} + b_{m-1} \frac{d^{m-1} v}{dt^{m-1}} + \ldots + b_0 v \qquad \text{(A-1)}$$

For a physically realizable system m is always equal to or less than n.

Tou's Direct Programming Method

Tou's direct programming method works with the system transfer function which can be found by transforming Equation (A-1) by using Laplace Transformations. The system transfer function resulting is

$$y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} v(s) \qquad \text{(A-2)}$$

Dividing both the numerator and denominator by s^{-n} produces

$$y(s) = \frac{b_m s^{m-n} + b_{m-1} s^{m-n-1} + \ldots + b_0 s^{-n}}{a_n + a_{n-1} s^{-1} + a_{n-2} s^{-2} + \ldots + a_0 s^{-n}} v(s) \qquad \text{(A-3)}$$

If the following substitution is made into Equation (A-3)

$$e(s) = \frac{v(s)}{a_n + a_{n-1} s^{-1} + \ldots + a_0 s^{-n}} \qquad \text{(A-4)}$$

the result is

$$y(s) = (b_m s^{m-n} + b_{m-1} s^{m-n-1} + \ldots + b_0 s^{-n})\, e(s) \qquad \text{(A-5)}$$

Equation (A-4) is equivalent to the following

$$e(s) = \frac{1}{a_n} v(s) - \frac{a_{n-1}}{a_n} s^{-1} e(s) - \ldots - \frac{a_0}{a_n} s^{-n} e(s) \qquad \text{(A-6)}$$

The state model diagram for this system can be drawn by using Equations (A-5) and (A-6). The state model diagram for this general linear stationary single input - single output system is shown in Figure A-1. The vector matrix state-variable model resulting from this figure is shown below for m = n-1.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & 0 \\ 0 & 0 & 0 & & & & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdot & \cdot & \cdot & \cdot & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \frac{1}{a_n} \end{bmatrix} v \quad \text{(A-7)}$$

$$y = \begin{bmatrix} b_{m-1} & b_{m-2} & \cdot & \cdot & \cdot & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} \quad \text{(A-8)}$$

The standard way the characteristic equation is expressed is in a normalized form. The normalization is, in general, performed with respect to the highest derivative. In other words the entries in the differential transition matrix in Equation (A-7) are the negative of the coefficients of the characteristic polynomial for the system. This form of the differential coefficient matrix is called

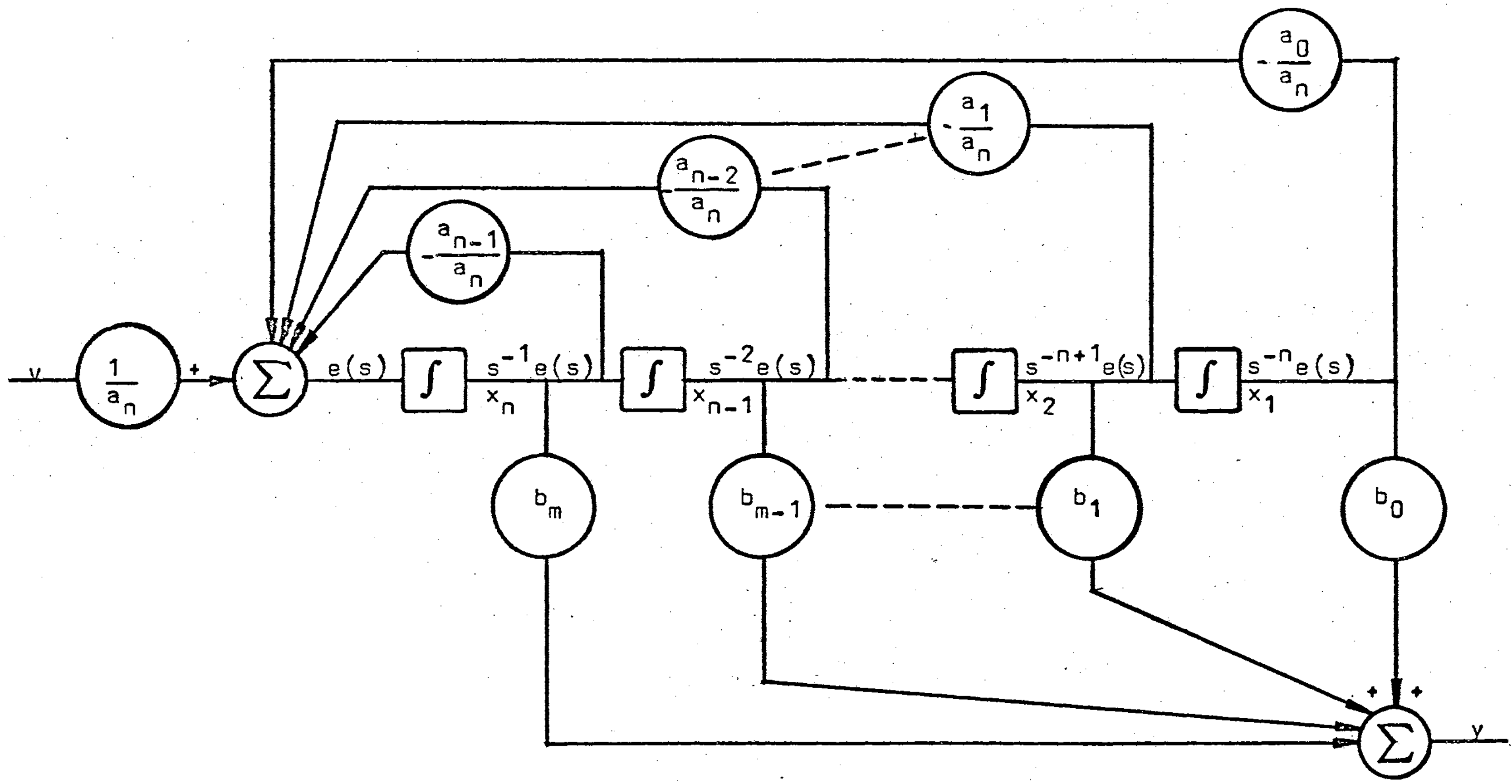


Figure A-1. State Model Diagram for Tou's Direct Programming Method the Phase-Variable Canonical Form. The entry in the column matrix multiplying v is the normalizing coefficient. As seen in Equation (A-8) the coefficients of the forcing function are displayed explicitly in the output matrix.

Rational Canonical Form Direct Programming Method

The Rational Canonical Form direct programming method works directly with the system mathematical model; however, for convenience, the differentials are usually replaced by the operator p where $p^n = d^n/dt^n$. Performing this substitution on Equation (A-1) produces

$$a_n p^n y + a_{n-1} p^{n-1} y + \ldots + a_1 p y + a_0 y = b_m p^m v + b_{m-1} p^{m-1} v + \ldots + b_1 p v + b_0 v \qquad \text{(A-9)}$$

If Equation (A-9) is solved for the zeorth order terms, the result is

$$b_0 v - a_0 y = (a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p) y - (b_m p^m + b_{m-1} p^{m-1} + \ldots + b_1 p) v \qquad \text{(A-10)}$$

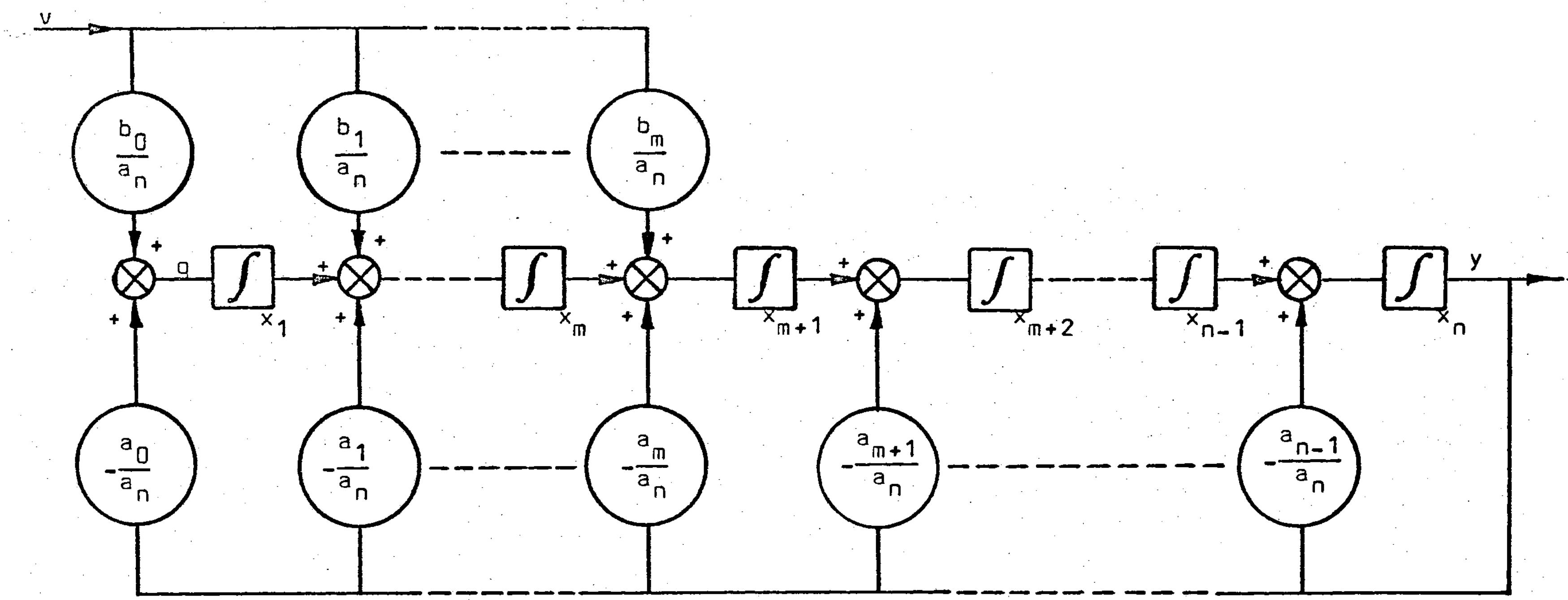
For $m = n$, Equation (A-10) can be rewritten as

$$b_0 v - a_0 y = p^n (a_n y - b_m v) + p^{n-1} (a_{n-1} y - b_{m-1} v) + \ldots + p (a_1 y - b_1 v) \qquad \text{(A-11)}$$

For $m \quad n$

$$b_0 v - a_0 y = a_n p^n y + a_{n-1} p^{n-1} y + \ldots + a_{m+1} p^{m+1} y + p^m (a_m y - b_m v) + \ldots + p^2 (a_2 y - b_2 v) + p (a_1 y - b_1 v) \qquad \text{(A-12)}$$

The state model diagram for Equation (A-12) is shown in Figure A-2.



$$q = \frac{b_0}{a_n}v - \frac{a_0}{a_n}y$$

$$= p^n y + \frac{a_{n-1}}{a_n}p^{n-1}y + \ldots + \frac{a_{m+1}}{a_n}p^{m+1}y + p^m\left[\frac{a_m}{a_n}y - \frac{b_m}{a_n}v\right]$$

$$+ \ldots + p^2\left[\frac{a_2}{a_n}y - \frac{b_2}{a_n}v\right] + p\left[\frac{a_1}{a_n}y + \frac{b_1}{a_n}v\right]$$

Figure A-2. State Model Diagram for the Rational Canonical Form Direct Programming Method

The state model derived from the diagram shown in Figure A-2 is shown below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ x_m \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & . & 0 & 0 & -\frac{a_0}{a_n} \\ 1 & 0 & 0 & . & 0 & 0 & -\frac{a_1}{a_n} \\ 0 & . & & & . & . & . \\ . & & . & & 0 & 0 & -\frac{a_m}{a_n} \\ . & & & . & . & . & . \\ . & & & & 1 & 0 & -\frac{a_{n-2}}{a_n} \\ 0 & . & . & . & 0 & 1 & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_m \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \frac{b_0}{a_n} \\ \frac{b_1}{a_n} \\ \cdot \\ \frac{b_m}{a_n} \\ 0 \\ \cdot \\ 0 \end{bmatrix} v \quad \text{(A-13)}$$

$$y = \begin{bmatrix} 0 & . & . & . & . & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_m \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{(A-14)}$$

The elements in the right hand column of the differential transition matrix in Equation (A-13) are the negative of the coefficients of the characteristic polynomial. As discussed previously, the division by a_n represents the normalizing of the characteristic polynomial with respect to the highest derivative. This form of the differential transition matrix is called the Rational Canonical Form. The coefficients of the forcing function appear implicitly in the input matrix. An explicit display of these coefficients is easily obtained by removing the normalizing coefficient used to normalize the characteristic polynomial.

APPENDIX B

RESONANT PEAK TRANSMISSIBILITIES

RESONANT PEAK TRANSMISSIBILITIES

In the development of system frequency response spectrums by the use of the Frequency Canonical Form the transmissibility at the natural undamped mode frequency is obtained explicitly. However, this value of transmissibility is not the maximum that will occur. The actual maximum occurs at a lower frequency. This frequency where the actual maximum occurs is a function of the damping existing as well as the natural undamped mode frequency. Specifically, the frequency where the maximum transmissibility occurs is

$$\omega_p = \omega_n \sqrt{1-2\zeta^2} \qquad \text{(B-1)}$$

where

ω_p = frequency of maximum transmissibility

ω_n = natural undamped mode frequency

ζ = damping ratio

The frequency of maximum transmissibility has been correlated with the natural undamped mode frequency and the ratio 1/a. This correlation is shown in Figure B-1. This figure can be utilized for the entire frequency range. The range of 1/a can be varied to encompass almost all possible values of 1/a which will occur. For ranges of 1/a from .17857 to 1.7857 the frequency range covers frequencies from

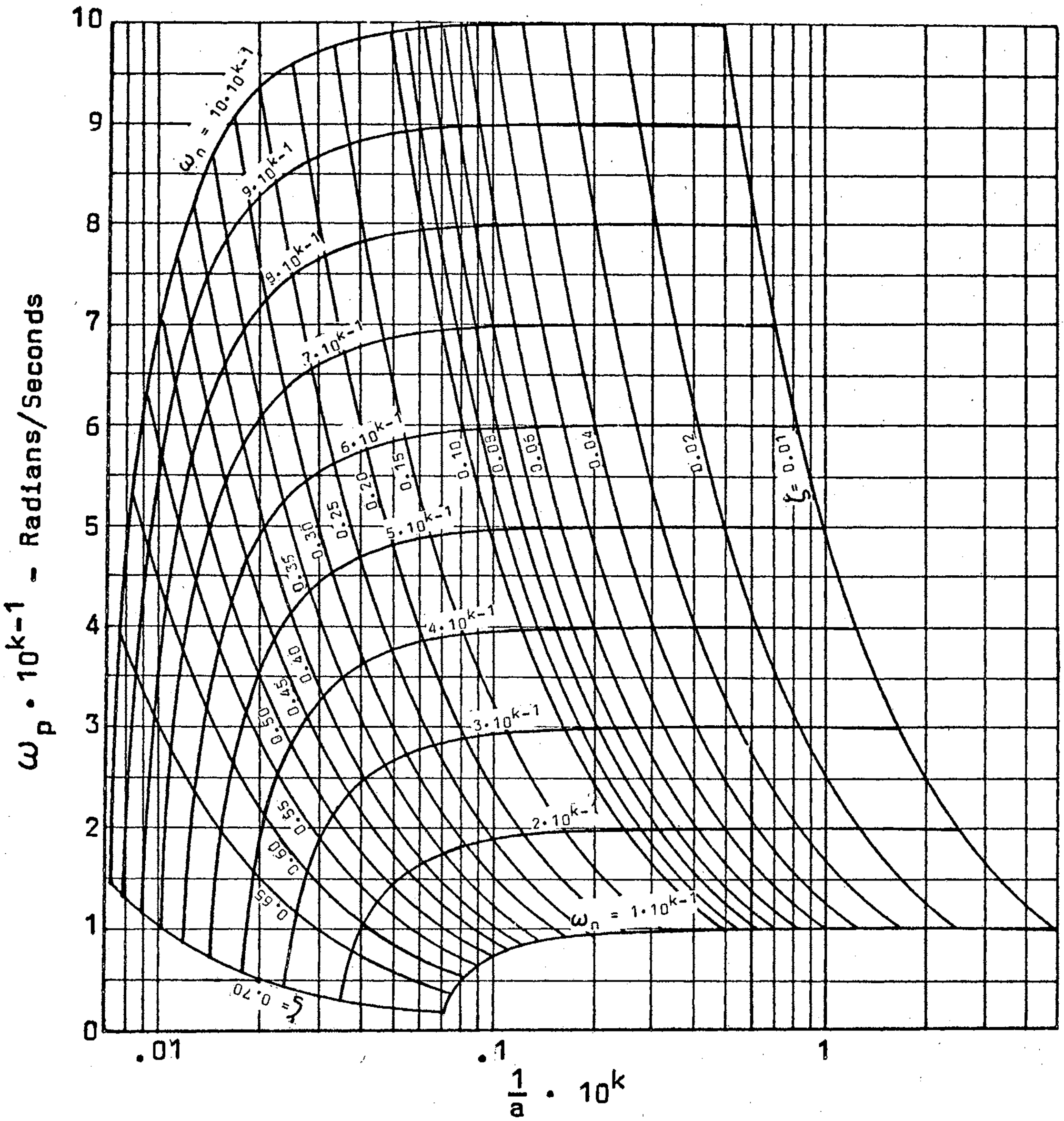


Figure B-1. Mode Damping Characteristics

0 to 10. A shift in the range of 1/a to the range 1.7857 to 17.857 shifts the frequency range to the range 0 to 1.0. Therefore, the scale designation on the abscissa is

$$\frac{1}{a} \cdot 10^{k} \qquad \text{(B-2)}$$

The scale designated on the ordinate is

$$\omega_p \cdot 10^{k-1} \qquad \text{(B-3)}$$

and the natural undamped mode frequency lines within the graph are

$$\omega_n \cdot 10^{k-1} \qquad \text{(B-4)}$$

Once the range of ω_n has been selected such that the required value of ω_n is within the range, then the appropriate value for k is used to establish the scale of the abscissa. The result of the use of Figure B-1 is the frequency of maximum transmissibility.

The next thing which must be obtained is the value of maximum transmissibility. This is possible by the use of Figure B-2 in conjunction with Figure B-1. Within the graph of Figure B-1 are lines of constant damping ratio ζ. Each point determined by the values 1/a and ω_n has an associated value of ζ. With the value of ζ from Figure B-1, Figure B-2 can be entered and a ΔT in decibels obtained. This increment in transmissibility is an increment which when added to the transmissibility at the natural undamped mode frequency will produce the maximum transmissibility. As indicated previously, the maximum occurs at ω_p.

$$\frac{d^2y}{dt^2} + 4\,\frac{dy}{dt} + 100y = 2v \qquad \text{(B-5)}$$

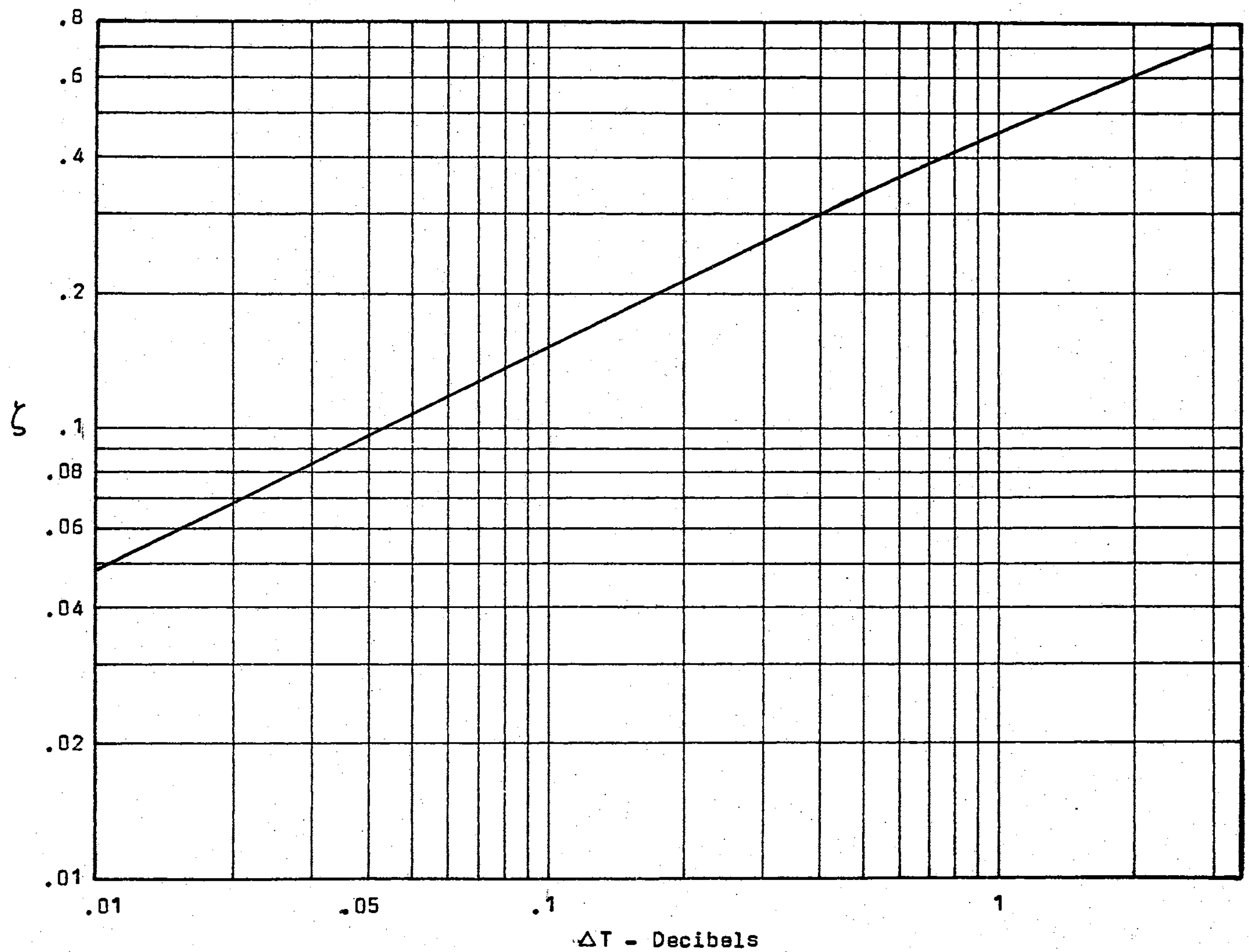


Figure B-2. Resonant Peak Transmissibility Differential

The vector matrix state model in Frequency Canonical Form is

$$\omega^2 \begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 0 & 100 \\ \omega^2 & 4\omega \end{bmatrix} \begin{bmatrix} z_1 \\ -z_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} v \qquad \text{(B-6)}$$

$$y = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ -z_2 \end{bmatrix} \qquad \text{(B-7)}$$

Then the transmission bandwidth boundaries are 0 and .

The Spectrum Band-pass Matrix is

$$\underline{S} = \begin{bmatrix} 0 & \frac{100}{2} \\ \frac{\omega^2}{2} & \frac{4\omega}{2} \end{bmatrix} \qquad \text{(B-8)}$$

From Equation (B-8) the transmission asymptote function for the low frequency range expressed in decibels is

$$-\ 20\mathrm{Log}_{10} \frac{100}{2} = 20\mathrm{Log}_{10}\ .02 = 33.99\mathrm{db} \qquad \text{(B-9)}$$

The transmission asymptote function for the high frequency range is

$$-\ 20\mathrm{Log}_{10} \frac{\omega^2}{2} = 20\mathrm{Log}_{10}\ 2 - 20\mathrm{Log}_{10}\ \omega^2$$

$$= 6.02 - 20\mathrm{Log}_{10}\ \omega^2 \qquad \text{(B-10)}$$

The intersection of these two transmission asymptote functions is

$$\frac{100}{2} = \frac{\omega^2}{2} \qquad \text{(B-11a)}$$

$$\omega_n = 10 \qquad \text{(B-11b)}$$

The magnitude of transmission asymptote function $-20\mathrm{Log}_{10}\ 4\omega/2$ at $\omega = 10$ is shown in Equation (B-12).

$$T = -20\mathrm{Log}_{10}\frac{40}{2} = 26.02\mathrm{db} \qquad (B-12)$$

The transmissibility spectrum as well as the asymptotic approximation generated from the transmission asymptotic functions are shown in Figure B-3.

The value of 1/4 is used to enter Figure B-1 along the abscissa and a frequency $\omega_n = 10$ is used within the graph. For $\omega_n = 10$ the value of k is 1. Employing this factor to obtain the correct abscissa location the value of frequency where maximum transmissibility occurs is $\omega_p = 9.59$. The damping ratio is $\zeta = 0.2$. With this value of damping ratio the incremental increase in transmissibility obtained from Figure B-2 is $\Delta T = 0.175$ decibels. The maximum transmissibility occuring at $\omega = 9.59$ radians per second is

$$T_{max} = 26.02 - 0.175 = 25.845\mathrm{db} \qquad (B-13)$$

as shown in Figure B-3.

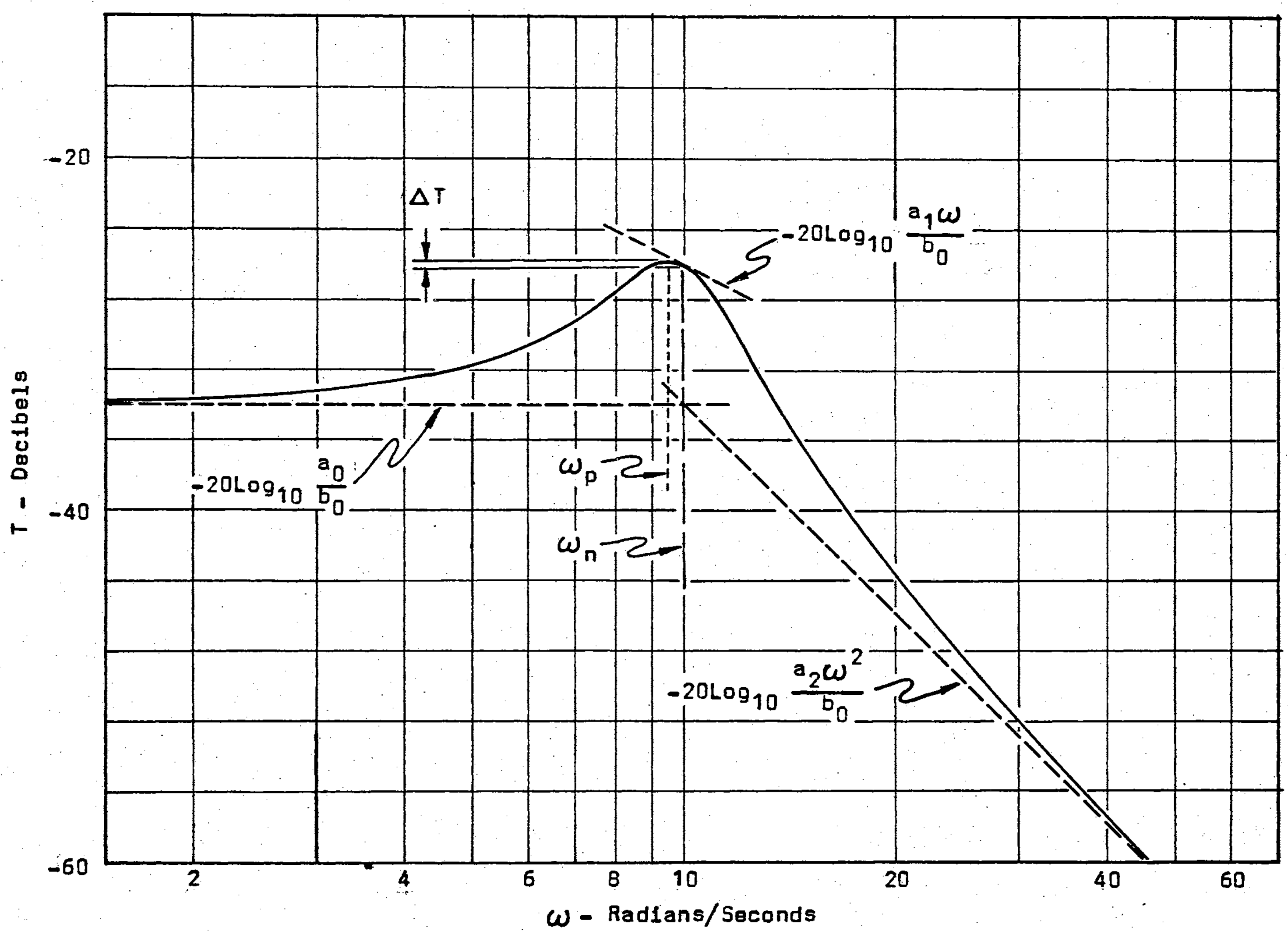


Figure B-3. Illustration of Resonant Peak Determination

APPENDIX C

STABILITY EVALUATION FOR THIRD AND FOURTH ORDER SYSTEMS (4)

STABILITY EVALUATION FOR THIRD AND FOURTH ORDER SYSTEMS (4)

The stability of systems whose characteristic polynonomials are third or fourth order can be determined by merely inspecting the Spectrum Band-pass Matrices. As is obvious to some extent the degree of stability can be estimated by applicable peak transmissibility calculations. This appendix deals primarily with absolute instability and the determination of peak transmissibility for third and fourth order systems which are very lightly damped.

To illustrate the principles involved in this stability evaluation the frequency spectrum for the following system state model in Frequency Canonical Form is shown in Figure C-1.

$$a_3\omega^3 \begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_0 \\ a_3\omega^3 & 0 & a_1\omega \\ 0 & a_3\omega^3 & a_2\omega^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ b_1\omega \\ 0 \end{bmatrix} v \qquad (C\text{-}1)$$

$$y = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ -z_3 \end{bmatrix} \qquad (C\text{-}2)$$

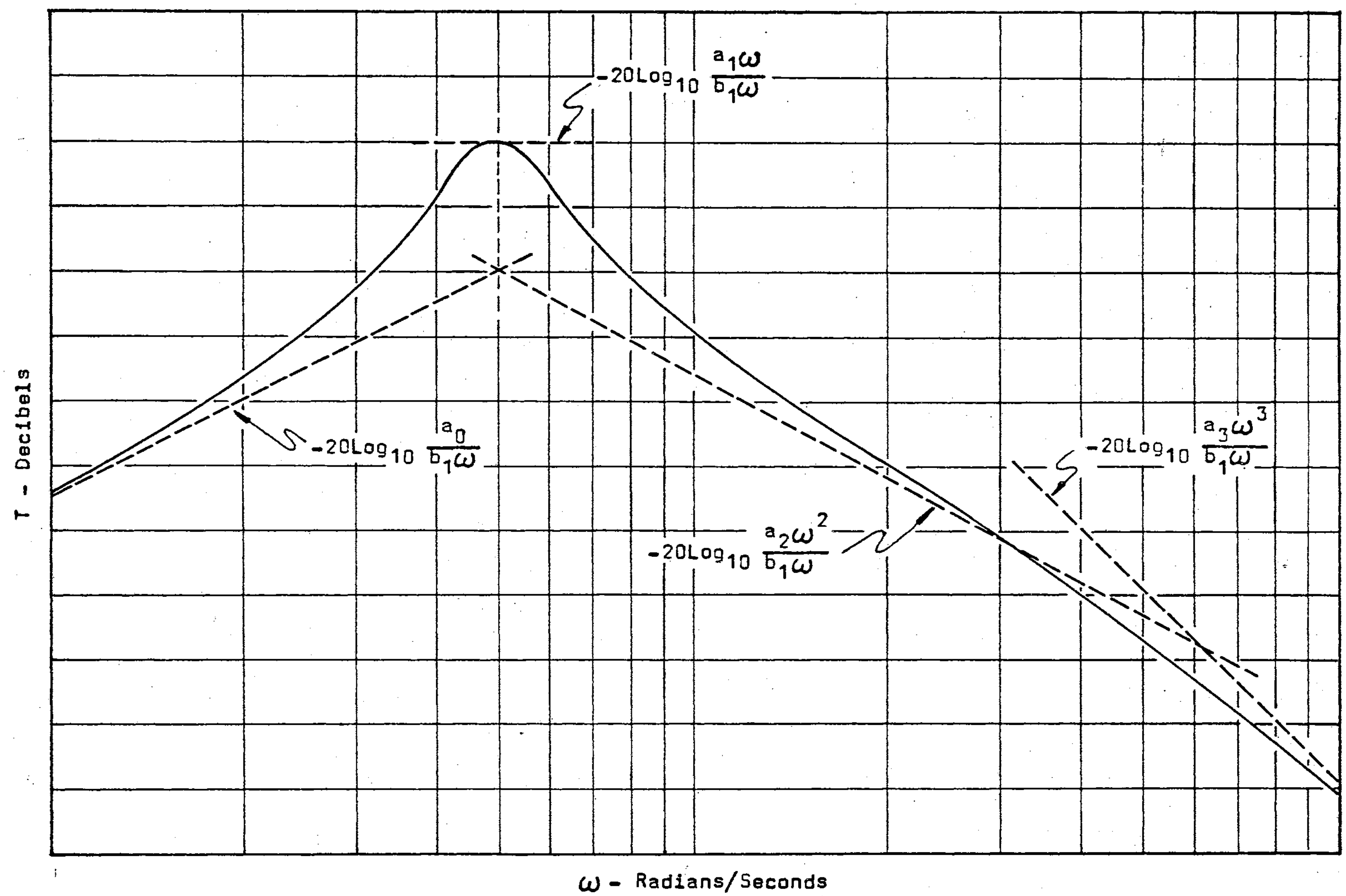


Figure C-1. Illustration of Stable System Frequency Response Spectrum Transmission Functions

The Spectrum Band-pass Matrix for this state model is shown in Equation (C-3).

$$\underline{S} = \begin{bmatrix} 0 & 0 & \frac{a_0}{b_1\omega} \\ \frac{a_3\omega^3}{b_1\omega} & 0 & \frac{a_1\omega}{b_1\omega} \\ 0 & \frac{a_3\omega^3}{b_1\omega} & \frac{a_2\omega^2}{b_1\omega} \end{bmatrix} \tag{C-3}$$

The natural undamped mode frequency associated with the oscillatory system mode occurs where

$$\frac{a_0}{b_1\omega} = \frac{a_2\omega^2}{b_1\omega} \tag{C-4}$$

or

$$\omega = \sqrt{\frac{a_0}{a_2}} \tag{C-5}$$

This frequency is determined by the intersection of these two transmission asymptote functions crossing as shown in Figure C-1.

Suppose that the asymptote function $a_3\omega^3/b_1\omega$ is greater than the asymptote function $a_1\omega/b_1\omega$ at the frequency $\omega = \sqrt{a_0/a_2}$. This condition is shown in Figure C-2. Mathematically, this indicates that

$$a_1\omega < a_3\omega^2 \quad \text{when} \quad \omega = \sqrt{\frac{a_0}{a_2}} \tag{C-6}$$

or

$$a_1 < \frac{a_3 a_0}{a_2} \tag{C-7}$$

which is the same as

$$a_1 a_2 - a_3 a_0 < 0 \tag{C-8}$$

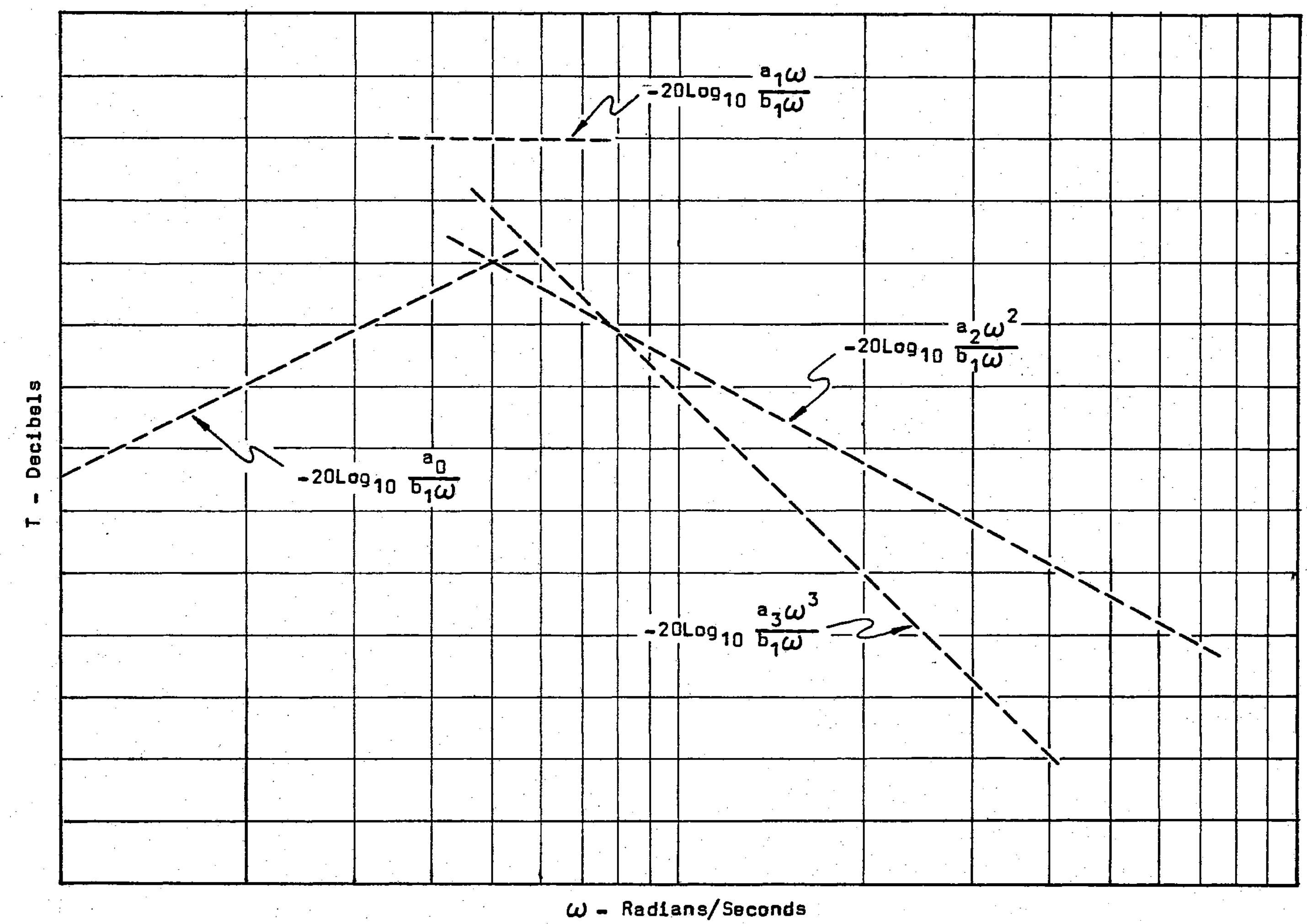


Figure C-2. Illustration of Unstable System Transmission Functions

However, one of the criterions for stability developed by Routh is that these coefficients of the characteristic polynomial in the above combination be greater than zero. This means that the system is a non-minimum phase system and, therefore, the frequency spectrum does not exist.

Systems which barely meet the stability criterion require special but quite logical consideration in order to estimate the resonance values of transmissibility. The frequency spectrum shown in Figure C-3 represents such a system. At the frequency $\omega = \sqrt{a_1/a_3}$ the asymptote functions $a_1\omega/b_1\omega$ and $a_3\omega^3/b_1\omega$ are equal but 180 degrees out of phase. Therefore, these two terms cancel one another producing the value of the transmissibility as shown below.

$$T = \frac{1}{\frac{a_0}{b_1\omega} - \frac{a_2\omega^2}{b_1\omega}} \qquad \text{(C-9)}$$

and specifically

$$(T)_{\omega = \sqrt{a_1/a_3}} = \frac{b_1\sqrt{\frac{a_1}{a_3}}}{a_0 - \frac{a_2 a_1}{a_3}} \qquad \text{(C-10)}$$

This equation indicates the boundary of stability since the denominator must be greater than zero. This special transmission resonance calculation is required only when two pairs of transmission asymptote functions are equal at the same frequency.

In general, most fourth order systems can be handled in exactly the same way as discussed in this appendix.

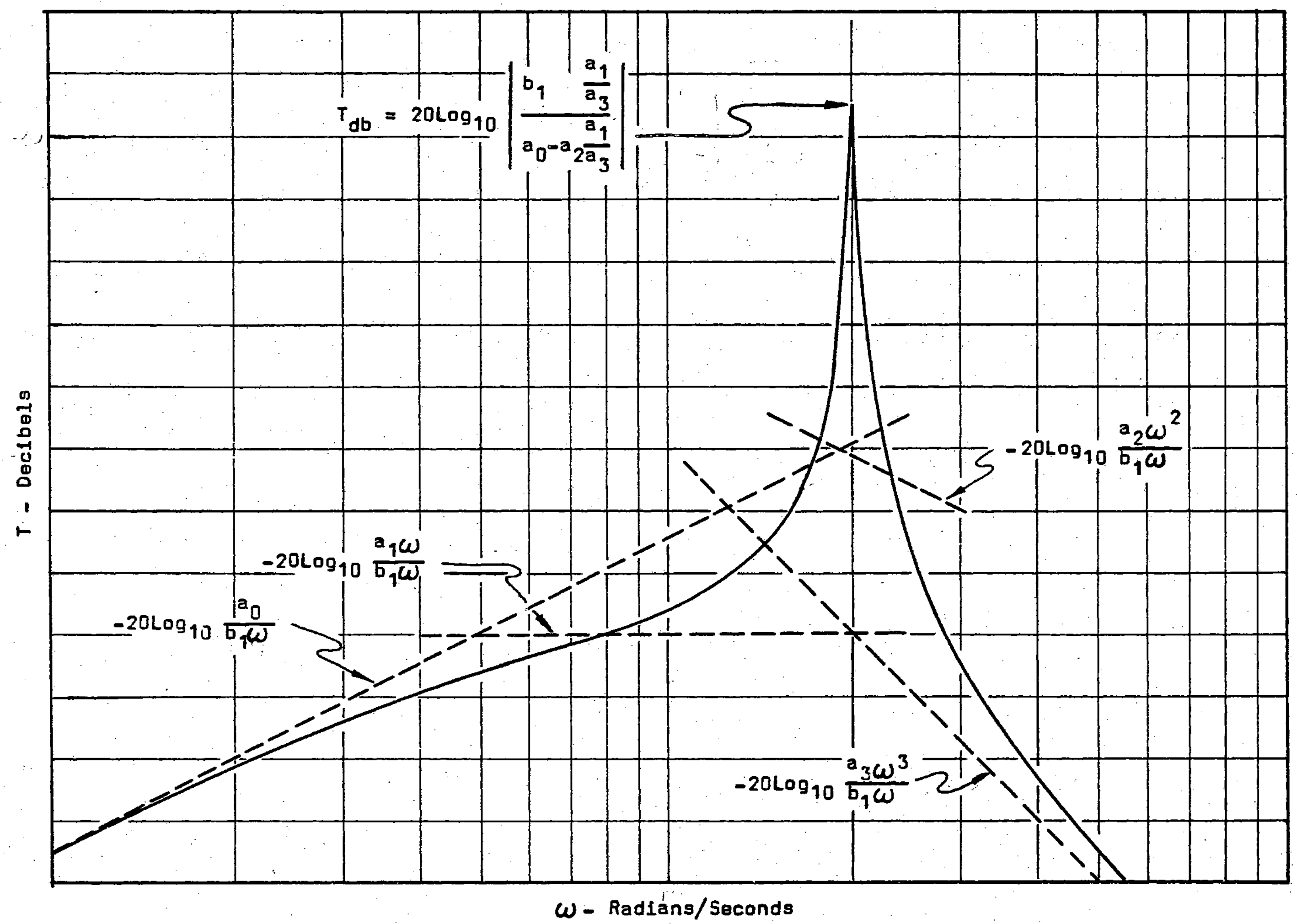


Figure C-3. Illustration of Very Lightly Damped System Transmission Functions and Calculation of Peak Transmissibility

APPENDIX D

PRINCIPLES OF STATE-VARIABLE MODELING

PRINCIPLES OF STATE-VARIABLE MODELING

Although state-variable analysis can be extended to encompass most systems describable by ordinary differential equations, the development contained in this appendix is for systems which fall into the class called linear time invariant. This class of ordinary differential equations can be illustrated by the following second order model in differential form.

$$a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = b_0v \tag{D-1}$$

The basis of state-variable modeling is the expression of Equation (D-1) as a system of first order ordinary differential equations. This expression of state models is obtained by making algebraic substitutions which consist of linear combinations of the zeroth, first, second, and so forth derivatives appearing in the differential form of the system model. These algebraic substitutions define the states or state-variables of the system. In the case of a single input - single output system, this substitution is performed as shown below.

$$x_1 = y \tag{D-2a}$$

$$x_2 = \dot{x}_1 = \frac{dy}{dt} \tag{D-2b}$$

$$x_3 = \dot{x}_2 = \frac{d^2y}{dt^2} \tag{D-2c}$$

If this algebraic substitution is performed on Equation (D-1) the following system of first order equations results

$$\mathring{x}_1 = 0x_1 + x_2 + 0v \qquad \text{(D-3a)}$$

$$a_2\mathring{x}_2 = -a_0x_1 - a_1x_2 + b_0v \qquad \text{(D-3b)}$$

$$y = x_1 + 0x_2 + 0v \qquad \text{(D-3c)}$$

Equations (D-3) can be written in the following unreduced state-variable form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathring{x}_1 \\ \mathring{x}_2 \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \\ 0 \end{bmatrix} v \qquad \text{(D-4)}$$

$$\underline{P}\ \underline{w} = \underline{Q}\ \underline{x} + \underline{R}\ \underline{v} \qquad \text{(D-5)}$$

Equation (D-5) is the generalized unreduced state space model for time invariant linear system with multiple inputs and multiple outputs.

Equations (D-3) can be grouped with the equations involving the derivatives of the state-variables in one group and the equations involving the outputs in the other. The result of this grouping written in state-variable form is the following reduced model.

$$\begin{bmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{a_0}{a_2} & -\frac{a_1}{a_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} v \qquad \text{(D-6a)}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} v \qquad \text{(D-6b)}$$

This system of equations is written in general form for a multiple input - multiple output system in the following state-variable notation.

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{v} \tag{D-7a}$$

$$\underline{y} = \underline{C}\,\underline{x} + \underline{D}\,\underline{v} \tag{D-7b}$$

The coefficient matrices shown in this appendix contain all constants. These constant matrices are characteristic of time invariant linear state space system models.

VITA

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