SENSITIVITY ANALYSIS OF
INVENTORY MODELS

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INVENTORY MODELS

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PREFACE

In the application of mathematical models of optimization by the industrial practitioner, significant decisions involve: 1) the degree of complexity and sophistication of complex and sophisticated mathematical models to be utilized for manipulation to obtain optimal relationships of the decision variables; 2) whether to acquire precise estimates of the value of the parameters to be utilized in the computation of the optimal numerical value of the decision variables; and 3) whether to actually use the optimal value of the decision variables if they are known relative to other manufacturing considerations.

The purpose of this research has been to develop some quantitative measures of error that might be incurred by: 1) the utilization of the wrong mathematical model of the inventory system; 2) the utilization of incorrect estimates of the parameter values; and 3) the utilization of incorrect values of the decision variables. It is this author's opinion that, in general, considerations such as scheduling problems and the utilization of maximum capacity, etc., far outweigh the cost consideration of optimization of inventories. This is a very critical problem area in the industrial environment since it is important for the production control manager to know the effect of not using the exact model or not acquiring precise estimates of parameters or not using the optimal decision values, etc., because of other manufacturing considerations. The findings of this research effort indicate that inventory models are relatively insensitive to variations from the "true" models, precise estimates of parameter values and optimal values of decision variables. Hence, the production control manager in the industrial environment can make control decisions based on other considerations than the minimization of inventory costs.

The successful completion of my Ph.D. program was accomplished with the generous financial support of a twelve-month Predoctoral National Science Foundation Faculty Fellowship and their cooperation in administration of the Fellowship for continuance of my program which allowed full-time devotion to graduate studies. General Motors Institute has been extremely cooperative relative to educational leave of absences during which this program has been completed. I am very grateful to both of these organizations.

This author is deeply indebted to many individuals for their invaluable assistance. In particular, sincere thanks and appreciation is expressed to Professor Wilson J. Bentley for his administrative and personal support when it was needed several times during this academic endeavor. Professor Bentley has been a constant source of inspiration that has compelled and assisted me to succeed far beyond my personal expectations and goals of my undergraduate days.

It is a pleasure to acknowledge indebtedness to Oklahoma State University for the opportunity to study under and to work with an outstanding and cooperative faculty. Special credit is due to each of this group for creating an atmosphere of inspiration and encouragement to allow an individual to achieve such an outstanding goal.

I also want to express personal gratitude to Dr. Jim Shamblin for the suggestion of this thesis topic and for his assistance while serving as research advisor. He has been an interested and stimulating consul. Appreciation is also extended to other members of Graduate Committee: Dr. E. Fergerson, Dr. G. T. Stevens, Dr. D. E. Bee, and Dean R. E. Venn and formerly Ph.D. Committee Chairman Dr. P. Torgersen, for their cooperation and their continual source of helpful suggestions. This committee provided a diversity of talents and viewpoints that I appreciated. Their thoughtful comments and assistance are reflected in many ways throughout my program of study and in this treatise.

Mrs. Margaret Estes deserves special credit and thanks for an excellent job of typing the tentative draft and the preparation of this treatise in its final form.

Finally, I want to express sincere gratefulness and appreciation to my wife, Tressa, and family, Jeffrey and Kimberly, who sacrificed their financial well-being and who forfeited many hours of leisure duing this graduate travail. They presented a special incentive for me to complete this effort as early as possible.

Complete acknowlegement of all the help I have received would be impossible, for countless hundreds have contributed indirectly. So in gratitude I shall shake the ethereal hand of those ancient pioneers in the field, and to those living contributors whom I have inadvertently overlooked, I can only say, "Take credit where credit is due."

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CHAPTER I

INTRODUCTION

During the last 50 years many articles and books have been written about inventory theory. In the last decade the field of inventory theory has been a fruitful area of research by economists, mathematicians, statisticians, and computer manufacturers. By the nature of the background of these researchers, there is, of course, a certain ambiguity in the choice of topics to be included under the heading of inventory theory. Broadly speaking, virtually any topic in operations research can be conceptualized as dealing with the economical management of stocks of commodities (inventories). In addition, large areas of economic theory are concerned with problems similar to problems of operations research. Any dynamic problem in economic theory is necessarily concerned with stocks, whether these be interpreted as capital, manufacturers inventories, bank reserves, or the accumulated savings and assets of an individual consuming unit.

In this vast technical literature relative to the topic of inventory theory, attempts have been made to define such terms as the "inventory problem", "inventory control", "inventory process", "inventory theory", "inventory management", and "inventory systems". It can be readily concluded that there does not presently exist any commonly accepted terms or definitions by the researchers in this area. Not only do we not have commonly accepted terms, but there are numerous views on what the inventory research field should encompass. Because of the diverse and wide range of points of view and use of definitions, it is necessary to establish definitions of some general terms to be used in this treatise. This author does not claim any originality in these definitions. However, these definitions have been modified and rephrased to fit a personal conceptual framework of the interrelation of these interacting concepts and definitions.

Definitions

Inventory Problem

An inventory problem involves decisions relative to (1) how much to manufacture (Q, economical lot size), and (2) when to replenish inventory (t, time interval between lot sizes). These two decisions answer the question of (3) how many economical lot sizes should be manufactured or purchased per some time period (N, number of orders placed). These three quantities (Q, t and N) are the decision variables of the inventory problem since they are subject to control of the decision maker.

Inventory situations are fundamentally alike, each involving some aspects of costs, service and usage. The objective in any given situation is to make a set of decisions which will satisfy some "optimizer" such as to minimize total costs while providing an acceptable service at the usage or demand rate. The problem is formulated and solved on an item basis with a given set of information about the environment of the particular item.

The total inventory problem is to ascertain specific values of these variables (Q, t and N) which will satisfy an optimization criterion such as minimize the total cost of the inventory system. Although finding the variables that give the minimum total cost is the main purpose of the inventory problem, there are other major questions of interest. In the industrial environment of the actual production and inventory control aspects of the manufacturing operation, there are several additional and pertinent questions that should be considered. For example: (a) a production control manager who can control the properties of the actual inventory system may want to know whether it is worthwhile to change the properties; (b) the optimal decision rules cannot be employed, or their employment may be more costly than some alternative rules; (c) the "accurate" estimation or determination of the parameters may be too costly.

Hence, the following additional problems are posed:

(1) What is the effect of an error in the economical lot size (Q) or the number of orders placed (N) or

the time between orders (t) on the total variable costs (TVC)?

(2) What is the effect of an error in estimating the true value of the parameters on the economical lot size (Q), the number of orders placed (N), or the time between orders (t) on the resulting total variable cost (TVC)?

(3) What is the effect of using the wrong model on the total variable cost?

Inventory System

An inventory system consists of the properties that describe the nature and the characteristics of the environment with respect to a particular inventory problem. These properties are essentially assumptions pertaining to the characteristics of the parameters and variables of the inventory environment. Naddor [1], suggests that all the properties of an inventory system can be classified into four types: (1) demand properties, (2) replenishment properties, (3) cost properties, and (4) constraint properties. Each of these classifications of properties are discussed in detail.

Demand properties. The demand properties involves information relative to the nature and characteristics of the demand pattern; for example, does demand occur at a known and uniform rate (deterministic), does demand occur at a known and non-uniform rate, is demand variable (probabilistic), etc.

Replenishment properties. The replenishment properties involve the quantity that is added to inventory or the time between when quantities are added to the inventory. In general, the replenishment properties can be controlled and hence can be "manipulated" to achieve minimal total costs of acquiring and holding inventories. The property of lead time which is defined as the time between placing an order and receiving the order is in general assumed to be known and constant or insignificant for deterministic inventory models. In probabilistic inventory models this property is extremely important.

Cost properties. The cost properties involve the measures of cost associated with inventory either positive or negative. In the specification of the properties of the inventory systems considered in this treatise, only these three kinds of costs are considered as significant and subject to control. Hence, all pertinent costs are defined and classified as (1) holding costs, (2) procurement costs, and (3) shortage costs. For a detailed discussion of these cost components, see Fabrycky [2].

(1) Procurement costs: alias; acquisition costs = replenishment of inventory costs. The procurement costs are the costs associated with replenishing inventories. However, this does not include the item or unit cost which is, in general, independent of the economical lot size.

(2) Holding costs: alias; inventory costs = carrying costs. The holding costs are the costs of carrying inventories which include the cost of capital and storage space costs.

(3) Shortage costs: alias; stock-out costs = back-order costs = out of stock costs. The shortage costs involve the costs of incurring an out of stock occurrence and not meeting a demand when it occurs.

Note that the cost properties of an inventory system are defined as being of only three types, namely, procurement, holding and shortage costs. Hence all the costs that are pertinent are defined and categorized into these three types. This limitation of the cost properties is necessary as a conceptual framework to define the scope of the models to which these results are valid.

The total variable cost of an inventory system will be the sum of these three costs. In any particular inventory system, any two or all three costs can be present. Obviously, these costs are closely related. When one cost is decreased, one of the other two costs, or sometimes, even both, may increase. The total variable cost is thus affected by the optimal inventory policy which affects the decisions relative to the solution of the inventory problem.

Constraint properties. The constraint properties involve placing limitations upon any pertinent factor of the system such as number of units in inventory or the capital invested in inventory, number of replenishments, etc.

Solution to an Inventory Problem

The solution to an inventory problem involves a set of specific values of the decision variables which minimize the sum of the cost associated with the properties of a specific inventory system.

Optimal Inventory Policy

The set of decision rules which minimizes these costs are referred to as optimal inventory policy. Optimal decisions policies are obtained by the use of models.

Models

A model is a mathematical representation of the inventory system's properties and interrelationships. In order to construct a model, the associated properties and resulting interrelationship must be specified or assumed.

Three Phases of Inventory Analysis

(1) The determination of the properties (specification of assumptions) of an inventory system.

(2) Mathematical model formulation and manipulation for an optimal solution.

(3) An analysis or evaluation of the solution. This analysis should evaluate what additional costs would be incurred if non-optimal inventory

policies are used. Also the range of the policies should be established.

It should be noted that this last phase will be defined and referred to as Sensitivity Analysis of the Inventory Model and is the major topic of the research reported in this treatise. Certain key questions and issues arise in the analysis of inventory problems which are of interest and significance both to the analyst and to the management consumers of his work. In addition management needs to have a general knowledge of the significance of the additional mathematical sophistication of the various models.

Thus, an essential step in any inventory analysis is the determination of how far one needs to go in using the refined mathematical methods available and where a more modified application will do. A thorough sensitivity analysis of the inventory model and its properties is necessary to help resolve this question.

An important, yet often overlooked, property of any decision model is the sensitivity of changes in parameter values. If one has constructed a model that appears to give reasonably reliable results in trial applications and that is relatively economical to solve, a whole new dimension has been added to the decision-making process if the variation to parameter values can be evaluated. There may be, for example, uncertainty about the value of a particular parameter. With a model one may explore various possible values of this parameter and observe the effect of parameter change on decisions and resulting economic outcome, either costs or profits. Thus, one could establish the sensitivity (or lack of it) of costs and decisions to error in parameter estimates.

An elementary inventory system is an inventory system of simple properties. The ascertaining of these properties of the particular inventory system of interest is the first step in the analysis of the inventory system. After specifying all the properties of the system, a model can be formulated which can be manipulated for an optimal solution.

There has been considerable research results reported in the technical literature concerning the assumptions of properties of various inventory systems and the formulation of the resulting model and its optimal solution phases of inventory analysis. It is not the intent of this research to review these results since they have been adequately developed and presented elsewhere in the technical literature. However, this author has "developed" a conceptual framework of the interrelation of the various models relative to their properties and their respective inventory systems which is presented in Chapter III. In addition, a complete derivation of the production with backorder model is presented in Appendix A. To this author's knowledge, this represents the first complete derivation of this model in this manner with the solution of the model reduced to recognizable form that can be related to the various properties that define the inventory system that this model represents.

Sensitivity

Sensitivity analysis evaluates the responsiveness of management decisions to various factors associated with such decisions. Sensitivity analysis can be used to evaluate the responsiveness of a model to changes in various controllable factors (parameters), and to evaluate the responsiveness of a model to various properties from which the model is derived. It can be used to measure the responsiveness of a model to non-optimal or incorrect values of decision variables. This information can be used to facilitate the appraisal of alternative courses of action. These measures of responsiveness can be obtained by a measurement of the change of the output of a model of a system based on a controlled change to an input to that model of the system.

The whole concept of inventory analysis is based on a conceptualized mathematical model of the inventory system's properties. Thus, as in using any mathematical model to represent a physical system as an aid in the decision process, it is of interest to know:

(1) The sensitivity of the model to the use of "incorrect" (non-optimal) values of decision variables.

(2) The sensitivity of the model to the use of "incorrect" estimates of the input parameters.

(3) The sensitivity of the model to the use of

"incorrect" properties of the system that define the model.

In regard to the sensitivity analysis of inventory models, the following questions are posed:

(1) What is the effect of an error in the decision variable on the total variable cost (TVC)?

(2) What is the effect of an error in estimating the true value of the parameters on the decision variable and the resulting total variable cost (TVC)?

(3) What is the effect of using the wrong model of the system on the total variable cost (TVC) which is in effect the consequences of assumming incorrect properties of the system.

These questions arise in the industrial environment for several real reasons: i.e., (a) a production control manager who can control the properties of the actual inventory system may want to know whether it is worthwhile to change the properties; (b) the optimal decision rules cannot be employed, or their employment may be more costly than some alternative rules; (c) the detailed estimation of the parameters may be too costly.

A measurement of the sensitivity of an inventory system can be calculated in terms of the change in the total variable cost of an inventory model. White [3] presents two different measures of the sensitivity of the total variable costs of an inventory model. However, he considers one of these, the ratio of the actual total variable costs to the optimal total variable costs, as being more important and considers only this one in his report. Alvarez [4] presents three types of sensitivity measures which are functions of the parameter values only. He considers that his model is correct and consequently does not pursue his definitions of at least three additional measures of sensitivity involving the combination of the incorrect parameter values and the wrong model.

In this treatise, the only measure of sensitivity which will be utilized is the ratio of the actual total variable costs to the optimal total variable costs. However, this measure will be calculated for three different problems.

(1) The effect of an error in the decision variable (Q) on the total variable costs.

(2) The effect of an error in the parameter values on the total variable costs.

(3) The effect of an error in the use of the wrong model (assuming incorrect properties).

By evaluation of the measure of sensitivity of these three separate problems, several of the sensitivity measures as proposed by White and Alvarez will be considered. However, like Alvarez, the combinatioral problem of two of the above three conditions will not be considered.

CHAPTER II

LITERATURE SEARCH

This chapter reviews the development of inventory theory and defines the bounds of existing knowledge in inventory models relevant to the study of the sensitivity of various inventory models.

The inventory problem has been a popular and fruitful field for researchers since the 1912-1915 derivation of the basic lot size relationship by Babcock (1912) and Harris (1915). Fairfield E. Raymond [5] summarized the work prior to 1931. In this classic treatise, the first attempt to deal with a large variety of inventory systems was presented along with the attempt to include all factors that might conceivable affect the economic lot size. This book is probably the most complete treatment of the subject of economic lot sizes, including the recent treatments by operations research oriented people. Very few of his recommended procedures for determining economic lot sizes were applied to production problems in the following years, possibly due to the economic conditions of the 1930's. Many of these topics have appeared recently in technical journals with little or no acknowledgement of Raymond's contributions.

Interest in the study of inventory systems has increased tremendously since World War II. Industrial engineers, economists, and mathematicians have all added to the basic models and systems with numerous publications being devoted to the subject. The publications have run the gamut from the extremely simple to highly developed abstract models. An excellent review and summary of the models and systems which were studied until 1951-1952 is presented by Whitin [6] in 1953. Whitin's bibliography contains 180 entries and covers the period from 1923 to 1951. The new edition of this book (1957) includes some recent developments as well, and contains 43 additional entries which cover publications up to 1956. However, most of the inventory control research has been conducted and resulting reports published in the last few years. These recent developments began with the attempt to provide procedures which are applicable in situations that had not been studied in previous models and systems.

A survey of the literature of the theory and concepts of economical lot sizes (ELS), alias; economical batch quantity (EBQ), economical order quantity (EOQ), run size, production order, etc., reveals several models proposed to optimize a particular criterion such as minimize variable costs. These models not only differ relative to the criteria being optimized but also relative to different assumptions and definitions used by the various authors. Some of these differences vanish when definitions are reduced to a common denominator, however, others are sometimes difficult to resolve. A review and compairson of the various models presented in the technical literature probably would be of interest, however, this is not the objective of this treatise.

The modern approach to inventory problems nearly always involves explicit models at some stage of the analysis. One's first encounter with the elaborate mathematical models presented in the technical literature can be a frustrating experience relative to judging the relative merit of the additional mathematical sophistication introduced to solve an inventory problem.

In these recent developments, powerful mathematical and statistical techniques are utilized in determining optimal policy decisions under specific conditions. These articles are highly theoretical and mathematical and have not been translated into any form suitable for application to the manufacturing environment. These articles emphasize the analysis of hypothetical inventory systems resulting in explicit decision policies valid only for the hypothetical system studied. Unsuccessful attempts have been made to apply the results to corresponding inventory systems encountered in practice since, in general, the actual and hypothetical inventory systems do not correspond and adjustments (which invalidate the mathematical optimality) are necessary in order to apply the results.

Considering the numerous articles on the topic of optimality of inventory systems and models, it is quite surprising that very few have considered the consequences of a non-optimal policy. The limited treatment of this topic is referred to as Sensitivity Analysis. There is a definite need for an investigation of the sensitivity of inventory systems and their models according to Fetter and Dalleck [7]

> Probably the most important single value to be derived from any model for use in a decision problem is in the area of sensitivity analysis.

It is this writer's opinion that information about the sensitivity of the inventory system and its corresponding model is as critical as knowing the optimal value of the inventory system and its model since, in general, the exact optimal value cannot be attained for practical considerations. As noted previously, the limited number of articles and publications on the topic of sensitivity of the inventory system and models is in marked contrast to the treatment afforded the broader subject of inventory control theory. There exists a paucity of information which extensively treats the topic of sensitivity analysis of inventory models.

The first two papers discussing the topic of Sensitivity Analysis of Inventory Models were published by Joel Levy [8, 9] in 1958 and 1959. In the 1958 paper, Mr. Levy considers the sensitivity of the purchase model. The consequences of using incorrect data in computing the decision variable Q is presented for the purchase inventory model.

The curve indicating the numerical consequences of the wrong decision variable on the total variable cost indicates that the error in the decision variable Q can vary from 0.53 to 1.85 and not increase total variable cost more than five per cent. This range does not coincide with findings of this research. Mr. Levy attempts to evaluate the sensitivity of probabilistic inventory models by considering Bellman's Dynamic Programming Recursive Relationship formulation of the probabilistic inventory model. Levy's major findings were that considering a probabilistic inventory model with demand as a stochastic variable, a convenient relationship of sensitivity of the model could not be found as was the case for the deterministic models.

In chronological order, the next article relative to sensitivity of inventory models was published in July, 1959, by Morris J. Solomon [10]. His major conclusion was that the total cost function is often very flat in the vicinity of the optimal value of the decision variable and hence relatively insensitive to the economical lot size value (decision variable) within a fairly wide range. Mr. Solomon then presented two approaches to defining an economical lot size range. One approach involves the purchase model inventory cost function. The second approach is derived for a quadratic cost function which is more general than the purchase model inventory cost function.

Samuel Eilon [11], the following year, presented a general approximation for the sensitivity of the function to be optimized by computing an optimal range for the purchase inventory model. He also presented results extending the optimal range to a general cost function consisting of two variable terms.

Ralph L. Disney [12] presented his results on sensitivity of errors in estimation on the true values of the parameters of the purchase inventory model. It is interesting to note that Disney's results is the same general expression as Levy derived in his deterministic analysis. However, Disney's curve of numerical values differs from Levy's curves of numerical values. The results of this research corresponds to Disney's results. Disney also presented results of the effect of error in parameters of the decision variable for the purchase model which corresponds to findings of this research for the purchase model.

Richard Withycombe [13] presented his findings relative to how much an arbitrary value of the decision variable (Q - order quantity) actually used may differ from a calculated optimal value of the decision variable. This paper, as previous authors, presented results only for the purchase inventory model. His findings were similar to Solomon's in that he concluded and presented numerical results to show that considerable latitude can be tolerated in deviating from the calculated optimal value of the decision variable.

Accardo and Kabus [14] presented a procedure to establish a flexibility interval for the decision variable (Q) based on an allowable percentage deviation from the minimum total variable cost. The procedure presented was based on the purchase inventory model. This article considered the case of a flexibility interval for the decision variable with no error in the parameters, and a case allowing for error in the parameters.

In the text by Flagle, Huggins and Roy [15], there is an extensive treatment of the sensitivity of the inventory models in the chapter on Elements of Inventory Systems written by Naddor. Although this treatment is rather extensive for various models of the sensitivity of the model to an error in the decision variable and to errors in the parameter values, the results are rather difficult to interpret because of the nature of the mathematic relationships that were developed.

In a text, "Inventory Systems", by Naddor [1], the results that were reported in Flagle, Huggins and Roy text [15] were expanded and discussed in additional detail. However, again the resulting mathematical relationships are not simplified and the results are somewhat difficult to evaluate relative to all the models considered.

Y. H. Rutenberg [16] presented a method of determining the value of the decision variable (EOQ) of the purchase inventory model for allowing a range for the setup cost parameter. In effect, this is an analysis of the sensitivity of the decision variable to an error in the setup cost parameter for this model.

Hadley and Whitin [17] provided a limited treatment of sensitivity. However, they present the problem of sensitivity analysis of the purchase model and leave the examination of this part to the student via problems.

Fulton [18], in a M.S. thesis, presents a rather extensive analysis of the sensitivity analysis of the parameters of several deterministic models. The mathematical relationship used in reporting his results are difficult to interpret since he did not attempt to indicate any relationship between the various models. However, Fulton did attempt to consider the sensitivity of one probabilistic inventory model by considering a power function of demand.

Mr. Alvarez [4] reported results of the sensitivity of system parameters of several deterministic and probabilistic models. The complete paper that was presented is unavailable and unpublished and hence the nature of the models considered is not known. However, the sensitivity relationships that are developed are quite similar to Fulton's and again are quite difficult to evaluate because of the complexity of the mathematical relationship.

A recent and more complete examination of sensitivity of inventory models is reported by White [3]. This treatment follows quite closely that of Naddor [1] and also that of Fulton [18] and Alvarez [4]. This treatise considers the sensitivity of several inventory models but fails to reduce the sensitivity relationship to simple relationships common to all the models. In fact, two measures of sensitivity are introduced which makes it difficult to evaluate the results.

White's paper did consider a probabilistic inventory model via expected values of the various parameters. However, his treatment was limited to only one expected value probabilistic model which did not greatly differ from the deterministic model that was considered.

In essence, there are a few isolated papers that have been published on the sensitivity of inventory models. In general these papers only considered the sensitivity of the decision variable (the economical order quantity) on the purchase inventory model. A few of the papers considered the establishment of a range for the decision variable for the purchase model which is, in effect, an analysis of the sensitivity of the decision variable. Several of the papers also consider the sensitivity of the parameters of the decision variable for the purchase model.

In general, there is not any correlation of sensitivity values of the various models. In fact, those authors that did deal with more than the purchase model, presented mathematical relationships of the sensitivity of various models that was extremely difficult to evaluate and did not correlate the findings to other models.

Also, most of the authors who consider both the sensitivity of the decision variable and the parameters did not recognize the relationship between the sensitivity of these two factors.

None of the references previously cited consider the sensitivity analysis of the properties of the model which is, in effect, an analysis of the error in using a wrong model or the assumptions that were used in the defining of the system that the model represented. It is this author's opinion that this is perhaps one of the most significant aspects of the sensitivity analysis of inventory models.

Only the Levy, Fulton and White papers attempted to evaluate sensitivity of probabilistic models. Obviously the mathematical relationship involving the probabilistic demand functions increase the degree of difficulty in sensitivity analysis of this type of models.

CHAPTER III

INVENTORY MODEL HIERARCHY

Methods of Analysis of Inventory Systems

E. Naddor, in the text by Flagle, Huggins, and Roy [15], indicates that essentially there exists three general approaches to the analysis of inventory systems. The first approach involves the detailed analysis of a general inventory system with hopes to apply the results to any specific inventory system. The second general approach involves the analysis of specific hypothetical inventory systems in a general manner. The third approach involves analyzing simple hypothetical inventory systems, then by the "building block" concept, analyze the more complex systems.

This writer, as does Naddor, prefers the latter approach since it involves methods of analysis rather than obtaining general results or results for specific systems encountered in practice. Hence, this can be conceptualized as a methodology for analyzing complex inventory systems as <u>extensions of elementary systems</u>.

An elementary system is a system with simple properties and an elementary system (model) can be extended to a different model by an additional assumption pertaining to one of the properties of the "elementary" system. For example, the purchase economical lot size model can be extended to the production model by the assumption of the property that the production rate is not infinity greater than the usage or demand rate. Thus the production model is an extension of the purchase model and the system it represents is an extension of the purchase inventory system. Hence, any model can be extended to another model by the nature of the properties that are assumed pertaining to the inventory systems that the models represent. Generally, one would expect that the solution of an inventory model and the extended models are closely related. These relationships form a basis of an inventory model hierarchy that has been observed and formulated and is presented as part of this research findings.

A survey of the literature on the theory and concepts of economical lot sizes (ELS): alias; economical batch quantity (EBQ), economical order quantity (EOQ), run size, production order, etc., reveal several models proposed to optimize a particular criterion such as minimize the total variable costs. These models not only differ relative to criteria being optimized but also relative to different assumptions of properties of the systems and definitions used by the various authors. Some of these differences vanish when definitions are reduced to a common denominator, however, others are sometimes difficult to resolve. A review and comparison of the various models presented in the technical literature probably would be of interest, however, this is not the objective of this treatise.

Most of the models can be reduced to the mathematical relationship:

TOTAL COSTS = TOTAL CONSTANT COSTS + TOTAL VARIABLE COSTS

$$TC = TCC + TVC$$

The constant costs per unit are those that are virtually independent of the economical lot quantity (i.e., the manufacturing or purchase cost consisting of the unit cost times the annual demand requirements). Obviously if price discounts or the manufacturing learning function is applicable, then these "constant" costs are not independent of the economical lot quantity. However, it is assumed that if total variable costs are minimized, total costs are minimized. It is important to note that the total constant costs consisting of the product of the unit value or unit cost and the demand requirements are normally quite large relative to the total variable costs.

The variable costs are a combination of a hyperbolic (procurement costs) and linear (holding costs, including interest and space charges) terms. It is emphasized here that, in general, all the models can be reduced to a combination of these two linear and hyperbolic terms. Now,

TOTAL VARIABLE COSTS = PROCUREMENT COSTS + HOLDING COSTS

$$TVC = PC + HC$$

In order to derive a particular model, the properties of the inventory system must be specified. As a building block for this inventory model hierarchy, a simple or elementary model will be discussed in detail.

Generally speaking, an elementary inventory system is an inventory system with simple properties which are essentially assumptions concerning the parameters of the inventory system. Note that the decision as to what is simple or complex properties is an arbitrary one. An elementary inventory system is defined as a system involving the balancing of two costs (cost of procurement of inventory and the cost of holding inventory). The properties of an elementary inventory system are defined in detail in the following discussion of the purchase model.

Purchase Model

Properties of System

The specific properties that define the purchase model are:

Demand Properties:

Demand is deterministic (known with uniform usage rate), total annual demand requirements will be denoted by D, demand rate will be denoted by d.

Replenishment Properties:

1. The rate of replenishment of inventory is infinite. This means that the rate of production which is denoted as p is infinitely greater than the rate of demand so that the replenishment of inventory is instantaneous. Symbolic $p \gg d$ or the total

production capacity denoted as P is infinitely greater than total demand requirements $P >> D$.

2. Lead time is deterministic (known and constant) or assumed to be insignificant or zero.

Cost Properties:

All costs will be defined as being of only three types: namely, procurement, holding, or shortage. It is implied that all other costs are not pertinent or are defined and categorized into these three types.

1. Procurement Cost — Costs associated with replenishing inventories. The cost of procurement occurs each time a procurement is made. Procurement costs is a linear function of the number of procurements—Denoted as N. Note that this is a hyperbolic cost function relative to the decision variable (Q). Total cost of procurement = (cost of procurement)(number of procurements).

2. Holding Cost — Costs associated with holding a unit in inventory, including interest and space charges. Holding costs is a linear function of the average number of units in inventory. Total holding costs = (average number of units in inventory)(holding cost/unit).

3. Shortage Cost — Costs associated with incurring a shortage of inventory items. For the purchase model, shortages are not allowed which is equivalent to the statement that the cost of a shortage

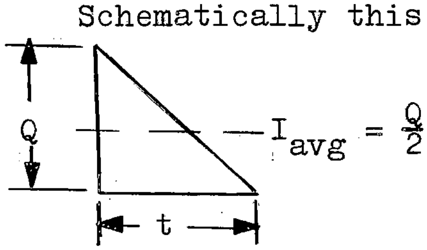
is infinite. Note that the replenishment properties of this system allows a shortage to be prevented. Replenishments are made whenever the inventory reaches a prescribed level so that shortages do not occur. Note that this could also be accomplished with proper adjustment of lead time and the known demand property.

Constraint Properties:

In the purchase model, there does not exist any constraining properties.

Formulation of Purchase Model

Schematically this system can be represented as:



t – time interval between Q, begins with Q units on hand and ends with zero units.

$$t = \frac{\text{1 planning period}}{\text{number of procurements/planning period}} = \frac{1}{N} = \frac{1}{D/Q} = \frac{Q}{D}$$

Procurement Costs = (Cost Per Procurement)(Number of Procurements)

$$PC \quad = \quad C_P \quad N$$

Holding Costs = (Holding Cost Per Unit Time)(Average Number Units Held)

$$HC \quad = \quad C_H \quad Q/2$$

$$TVC = PC + HC = C_P N + C_H Q/2$$

Manipulation of Model for Optimal Value of Decision Variable

$$TVC = C_P N + C_H Q/2 = C_P D/Q + C_H Q/2$$

Where N = D/Q

$$\frac{\partial\ TVC}{\partial\ Q} = 0 - \frac{C_P D}{Q^2} + \frac{C_H}{2} = 0$$

$$Q = \sqrt{\frac{2C_P D}{C_H}}$$

The above model is an excellent example of an operations research model insofar as the decision variables under control (only one in this model – Q) can be specified, a function to be optimized can be defined and then the value of the controlled decision variable which optimizes the desired function can be determined. In this model, the decision variable (Q – the economical lot quantity) which minimizes the sum of the total variable costs can be found by a simple mathematical method such as differentiation.

Production Model

Properties of System

Specific properties that define the production model are:

Demand Properties:

Same as for purchase model.

Replenishment Properties:

1. The rate of replenishment of inventory is not infinite. However, the production rate p is

greater than the demand rate, $p > d$. Note that the demand rate is such that during production of the lot size, consumption of the units occur to satisfy demand and the full lot size does not enter inventory.

2. Lead time is same as for purchase model.

Cost Properties:

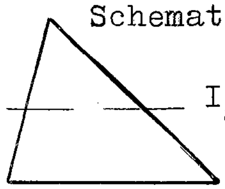
Same as for purchase model.

Constraint Properties:

Same as for purchase model.

Formulation of the Production Model

Schematically this system can be represented as:



$$I_{avg} = \frac{I_{max}}{2} \qquad I_{max} = Q\left(1 - \frac{D}{P}\right)$$

$$\begin{aligned} TVC &= PC + HC \\ &= C_P N + C_H Q/2 \left(1 - \frac{D}{P}\right) \\ &= C_P \frac{D}{Q} + C_H Q/2 \left(1 - \frac{D}{P}\right) \\ &= C_P \frac{D}{Q} + C_H Q/2 \left(\frac{P-D}{P}\right) \end{aligned}$$

Manipulation of Model for Optimal Value of Decision Variable

From which

$$Q = \sqrt{\frac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)}} = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{1}{1 - \frac{D}{P}}} = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{P}{P-D}}$$

Backorder Model

Properties of System

The specific properties that define the backorder model are:

Demand Properties:

Same as for purchase model.

Replenishment Properties:

Same as for purchase model.

Cost Properties:

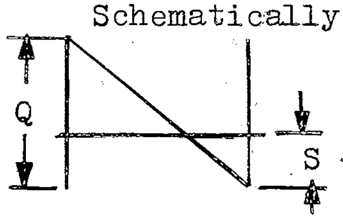
1. Procurement Cost — Same as for purchase model.
2. Holding Cost — Same as for purchase model.
3. Shortage Cost — Costs associated with incurring a shortage. This model allows a shortage to occur with the demand remaining captive such that it is supplied from future production. This cost is considered to be a linear function of average number units that are short and backordered. Total Shortage Cost = (Average number of units backordered)(Cost of shortage per unit).

Constraint Properties:

Same as for purchase model.

Formulation of the Backorder Model

Schematically this system can be represented as:



S - Number of units short

$$TVC = \text{Procurement Costs} + \text{Holding Costs} + \text{Shortage Costs}$$

$$= PC + HC + SC$$

$$= (C_P)(N) + (C_H)\frac{Q}{2} + C_S \text{ (Average number short)}$$

$$= C_P\frac{D}{Q} + C_H\frac{Q}{2} + C_S\frac{S}{2}$$

Manipulation of Model for Optimal Value of Decision Variable

From which (See Appendix A for complete derivation)

$$Q = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}$$

$$S = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}$$

Production With Backorder Model

Properties of System

Specific properties that define the production with backorder are:

Demand Properties:

Same as for the purchase model.

Replenishment Properties:

Same as for the production model.

Cost Properties:

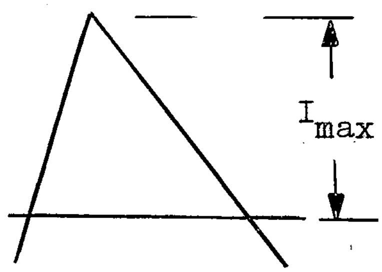
Same as for the backorder model.

Constraint Properties:

Same as for the purchase model.

Formulation of the Production With Backorder Model

Schematically this system can be represented as:



Manipulation of Model for Optimal Value of Decision Variable

From which the following is obtained (See Appendix A):

$$Q = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{1}{1-\frac{D}{P}}}\sqrt{\frac{C_H+C_S}{C_S}}$$

$$= \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}\sqrt{\frac{C_H+C_S}{C_S}}$$

$$S = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{P}{P-D}}\sqrt{\frac{C_H}{C_H+C_S}}$$

As previously noted, extensive work has been reported in various texts and technical literature relative to the derivation of the optimum inventory policy for various models based on different assumptions. However, as far as this writer can ascertain, this is the first presentation showing the interrelations of the various models, their respective properties, and resulting formula for the optimum value of the decision variables.

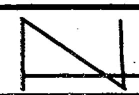
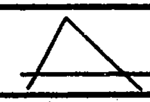
Of course, it should be noted that the value of the optimum decision variable can be used to answer the two important inventory questions: (1) how much to make or buy (Q), and (2) when to order (t)- the time between orders.

This hierarchy of inventory models that is related to the properties of the inventory system being defined is of considerable interest and value. Upon the inspection of Table I, which illustrates the relationships of the various properties and their effect on the decision variable formula, it appears that each property has the same effect on the decision variable regardless of the particular model. As an example the assumption that the cost of shortage property is not infinite has an effect of $\sqrt{\frac{C_S+C_H}{C_S}}$ on the decision variable in each model that this property is assumed. Likewise the assumption that the replenishment rate property is not infinite has an effect of $\sqrt{\frac{P}{P-D}}$ on the decision variable in each model for which this property is assumed. Hence, it appears that by knowing the effect of the various properties on the decision variables of these models, additional models' decision variables could be directly formulated without the mathematical manipulation of complex total variable cost functions that evolve when the combination of several properties are assumed to define a particular model of interest.

This "building block" concept of determining decision variable formula could be of considerable aid in the analysis of more complicated inventory systems simply by knowing the effect of various properties on the decision variable formula.

TABLE I

COMPARISON OF INVENTORY SYSTEM PROPERTIES THAT DEFINED THE VARIOUS INVENTORY MODELS

Properties \ Model	Purchase	Production	Backorder	Production With Backorder
Demand	Known and Constant	Known and Constant	Known and Constant	Known and Constant
Replenishment				
Replenishment Rate	Infinite(P>>D)	Finite (P > D)	Infinite(P>>D)	Finite (P > D)
Lead Time	Zero or Known and Constant	Zero or Known and Constant	Zero or Known and Constant	Zero or Known and Constant
Cost				
Procurement Cost	Linear Function of Number of Procurements	Linear Function of Number of Procurements	Linear Function of Number of Procurements	Linear Function of Number of Procurements
Holding Cost	Linear Function of Average Inventory	Linear Function of Average Inventory	Linear Function of Average Inventory	Linear Function of Average Inventory
Shortage Cost	Infinite	Infinite	Linear Function of Average Number Backordered	Linear Function of Average Number Backordered
Constraint	None	None	None	None
Schematic Representation of System				
Decision Variables Formula:				
Optimum Lot Size	$Q_o = \sqrt{\frac{2C_PD}{C_H}}$	$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}$	$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}$	$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}$
Optimum Number Backorder	$S_o = 0$	$S_o = 0$	$S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}$	$S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P}{P-D}}$

CHAPTER IV

SENSITIVITY ANALYSIS OF DECISION VARIABLES

There are three general decision variables in a deterministic inventory model: (1) the economical order quantity, (Q); (2) the time between orders, (t); and (3) the number of orders per year, (N). In essence, if a value for one of the decision variables is specified or known, the other two decision variable values can be determined from simple identities that exist as corollaries from the definitions of the model and the properties of the system that it represents.

This chapter deals primarily with the sensitivity analysis of the decision variable. A measure of the sensitivity of the economical lot size on the total variable cost function has been evaluated for the purchase, production, backorder, production with backorder, and restricted variables models. A measure of the sensitivity of the decision variable (t)- on the total variable cost function has been evaluated for the purchase model from which it follows that results will be the same for the other models. Likewise, a measure of the sensitivity of the decision variable (N)- on the total variable cost function has been evaluated for the purchase model from which it can be concluded that similar results exist for the other models. The sensitivity of a general inventory cost function consisting of a hyperbolic term and a linear term was evaluated with the same results as all the previous models. Hence, it is concluded that if the cost function consists of the sum of a hyperbolic and linear terms, its sensitivity to a change in the decision variable will be $\frac{1 + W^2}{2W}$ where W is the per cent deviation of the actual value of the decision variable to the optimal value of the decision variable. The measure of sensitivity that was evaluated was the ratio of total variable costs with the "actual" or "wrong" value of the decision variable to the total variable costs with the optimal value of the decision variable. The "actual" or "wrong" value of the decision variable was expressed as a percentage of the optimal value of the decision variable. Expressed mathematically

$$Q' = W\,Q_o$$

where

Q' = "actual" or "wrong" value of decision variable

Q_o = optimal value of decision variable

W = percentage deviation of the actual value of the decision variable to the optimal value of the decision variable.

Table II summarizes the results of the evaluation of the various inventory models and their sensitivity to an "error" in use of a "wrong" value for a decision variable. The derivations of the measure of sensitivity for the various models are presented in Appendix B.

Upon examination of Table II it is obvious that the sensitivity of any of the decision variables is the same for all the models. Further note that the measure of sensitivity is independent of the decision variable itself and only a function of W which is the percentage deviation of the actual value of the decision variable as related to the optimal value of the decision variable.

TABLE II

MEASURES OF SENSITIVITY FOR ALL MODELS AND ALL DECISION VARIABLES

	Sensitivity of Decision Variable Q	Sensitivity of Decision Variable t	Sensitivity of Decision Variable N
Purchase Model	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$
Production Model	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$
Backorder Model	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$
Production With Backorder	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$
Restricted Variable Model	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$	$\frac{1 + W^2}{2W}$

If $.73 \leq W \leq 1.37$ then the change in the TVC'/TVC_o ratio will be less than five per cent. Some specific numerical values of W and changes in the TVC'/TVC_o ratio are shown in Table III and Figure 1. This indicates that the cost consequences of an "error" in the decision variable greater than the "correct" value of decision variable is of less significance than having an incorrect value of the decision variable less than the "correct" value of the decision variable.

TABLE III

RANGE OF W FOR VARIOUS PER CENT CHANGES IN THE TVC'/TVC_o RATIOS

Change in TVC'/TVC_o	Range of W
≤ 1 %	$0.87 \le W \le 1.15$
≤ 2 %	$0.82 \le W \le 1.22$
≤ 3 %	$0.79 \le W \le 1.27$
≤ 4 %	$0.76 \le W \le 1.32$
≤ 5 %	$0.73 \le W \le 1.37$
≤ 6 %	$0.71 \le W \le 1.41$
≤ 7 %	$0.69 \le W \le 1.45$
≤ 8 %	$0.68 \le W \le 1.48$
≤ 9 %	$0.66 \le W \le 1.52$
≤ 10 %	$0.65 \le W \le 1.55$
≤ 25 %	$0.50 \le W \le 2.0$
≤ 68 %	$0.33 \le W \le 3.0$
≤ 112.5%	$0.25 \le W \le 4.0$

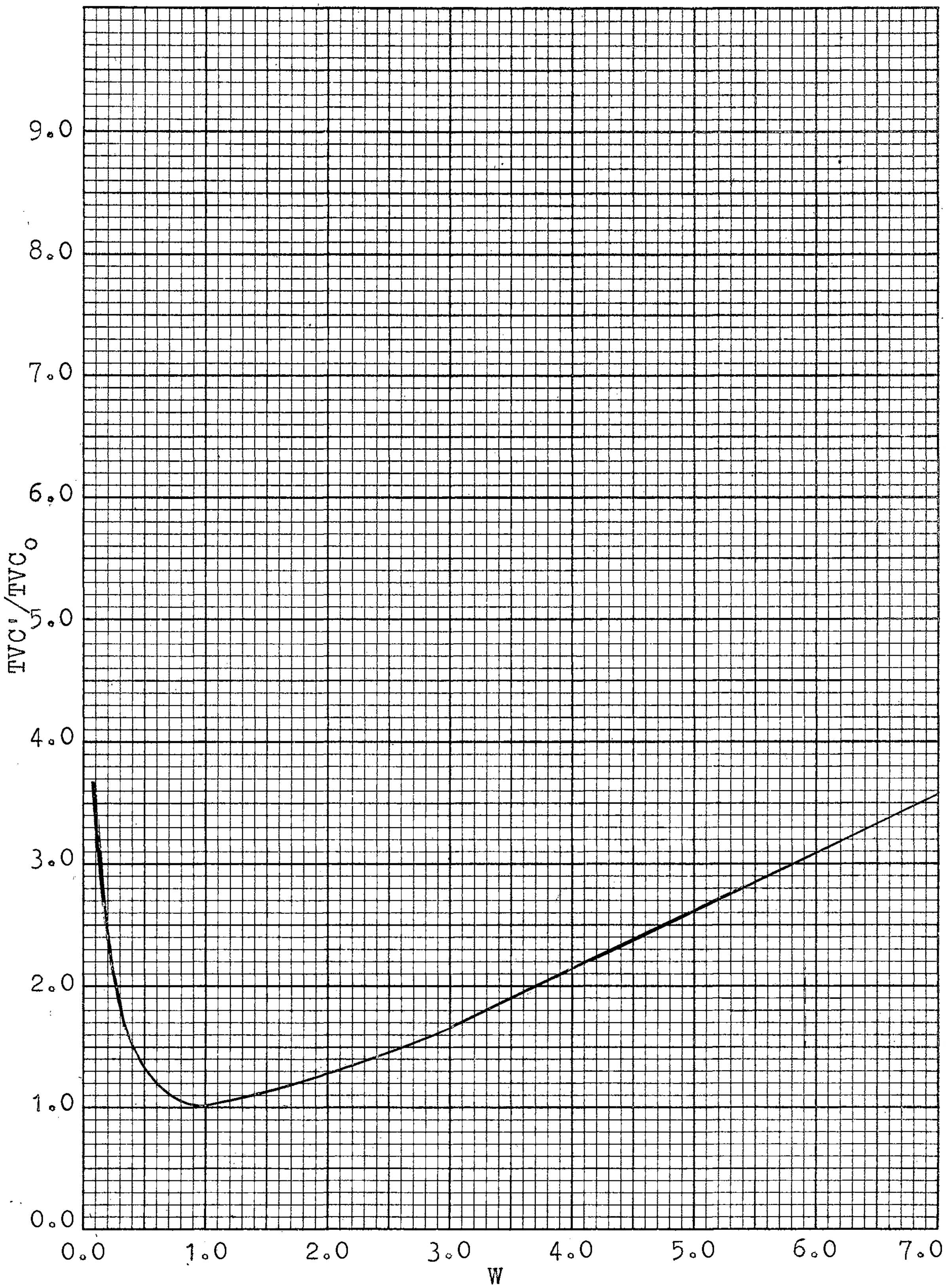


Figure 1. TVC'/TVC$_o$ Ratios Versus W Values

Note that

$$\frac{TVC'}{TVC_o} = \frac{1 + W^2}{2W} = \frac{1 + \left(\frac{1}{W}\right)^2}{2\left(\frac{1}{W}\right)}$$

which indicates that the same effect of W on the $\frac{TVC'}{TVC_o}$ ratio is also obtained from the inverse of W.

In the previous chapter it was noted that the total constant costs are normally quite dominant relative to the total variable costs. Hence, a "large" error could possibly occur in the total variable costs and yet the effect on the total costs could be quite small.

TOTAL COSTS = TOTAL CONSTANT COSTS + TOTAL VARIABLE COSTS

$$TC = TCC + TVC$$

$$TC' = TCC + TVC'$$

Measure of sensitivity of Total Costs (TC)

$$\frac{TC'}{TC_o} = \frac{TCC + TVC'}{TCC + TVC_o}$$

$$= \frac{TCC + \frac{1 + W^2}{2W} TVC_o}{TCC + TVC_o}$$

Hence, if the TCC term is large relative to the TVC term, then a large error in the TVC term will result in relatively small change in the total costs.

CHAPTER V

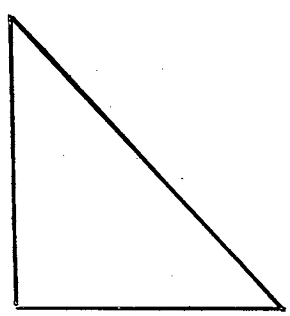
SENSITIVITY ANALYSIS OF PARAMETERS

An important consideration in the use of any model is its sensitivity to changes in the parameters of the model. The collection and analysis of information and data leading to the estimation of the following parameters; demand, procurement costs, holding costs and shortage costs are processes subject to error. Thus it is quite important to know the effect of erroneous values of parameters on the total variable cost since the refinement of the estimating procedure incurs additional time, effort and money. This chapter considers the problem of the effect of an "error" in the true value of the parameters on the economical lot quantity and the resulting total variable costs.

The sensitivity of the inventory systems to the use of "incorrect" estimates of the input parameters has been evaluated for the purchase, production, backorder and production with backorder models. The use of "incorrect" values of the parameters of the various models will effect and possibly result in an "incorrect" value of the decision variables. The results of Chapter IV indicated that the sensitivity of the various models to the effect of an "incorrect" value of a decision variable on the total variable cost was a function of only W - the per cent deviation of the actual value of the decision variable relative to the optimum value of the decision variable. Mathematically this was defined as $Q' = WQ_o$. In turn, $W = \frac{Q'}{Q_o}$ which allows W to be interpreted as the ratio of the "actual" decision variable to the optimum value of the decision variable. This can be further defined as the ratio of the parameters of the actual value of the decision variable to the parameters of the optimum value of the decision variable for any model of interest.

Denote errors in the estimates of parameters by primes i.e., C_H', C_P', D'.

Purchase Model

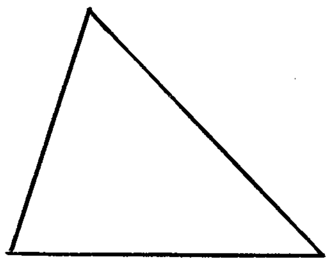


$$Q_o = \sqrt{\frac{2C_PD}{C_H}}$$

$$Q' = \sqrt{\frac{2C_P'D'}{C_H'}}$$

$$W = \frac{Q'}{Q_o} = \frac{\sqrt{\frac{2C_P'D'}{C_H'}}}{\sqrt{\frac{2C_PD}{C_H}}} = \frac{\sqrt{2C_P'D'C_H}}{\sqrt{2C_PD\ C_H}} = \sqrt{\frac{C_P'}{C_P}\ \frac{D'}{D}\ \frac{C_H}{C_H'}} = \sqrt{\frac{\frac{C_P'}{C_P}\ \frac{D'}{D}}{\frac{C_H'}{C_H}}}$$

Production Model

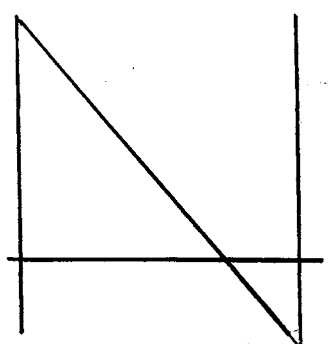


$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{1}{1 - D/P}}$$

$$Q' = \sqrt{\frac{2C_P'D'}{C_H'}}\sqrt{\frac{1}{1 - D'/P'}}$$

$$W = \frac{Q'}{Q_o} = \frac{\sqrt{\frac{2C_P'D'}{C_H'}}\sqrt{\frac{P'}{P'-D'}}}{\sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{P}{P-D}}} = \sqrt{\frac{\frac{C_P'}{C_P}\ \frac{D'}{D}}{\frac{C_H'}{C_H}}}\sqrt{\frac{\frac{1}{1-\frac{D'}{P'}}}{\frac{1}{1-\frac{D}{P}}}} = \sqrt{\frac{\frac{C_P'}{C_P}\ \frac{D'}{D}}{\frac{C_H'}{C_H}}}\sqrt{\frac{1-\frac{D}{P}}{1-\frac{D'}{P'}}}$$

Backorder Model



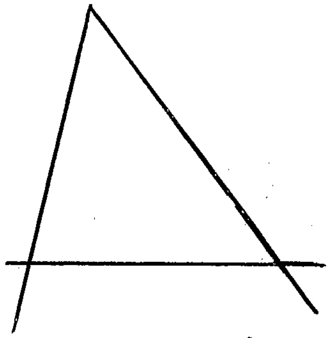
$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}$$

$$Q' = \sqrt{\frac{2C_P'D'}{C_H'}}\sqrt{\frac{C_H'+C_S'}{C_S'}}$$

$$W = \frac{Q'}{Q_o} = \frac{\sqrt{\frac{2C_P'D'}{C_H'}}\sqrt{\frac{C_H'+C_S'}{C_S'}}}{\sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}} = \sqrt{\frac{\frac{C_P'}{C_P}\ \frac{D'}{D}}{\frac{C_H'}{C_H}}}\sqrt{\left(\frac{C_H'}{C_S'} + 1\right)\left(\frac{C_S}{C_H} + 1\right)}$$

$$= \sqrt{\frac{C_P'}{C_P}\ \frac{D'}{D}\ \frac{C_H}{C_H'}}\qquad\sqrt{\left(\frac{C_H'}{C_S'} + 1\right)\left(\frac{C_S}{C_H} + 1\right)}$$

Production With Backorder Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{1}{1-D/P}}$$

$$Q' = \sqrt{\frac{2C_P'D'}{C_H'}}\sqrt{\frac{C_H'+C_S'}{C_S'}}\sqrt{\frac{1}{1-D'/P'}}$$

$$W = \frac{Q'}{Q_o} = \frac{\sqrt{\frac{2C_P'D'}{C_H'}}\sqrt{\frac{C_H'+C_S'}{C_H'}}\sqrt{\frac{P'}{P-D}}}{\sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H+C_S}{C_H}}\sqrt{\frac{P}{P-D}}}$$

$$= \sqrt{\frac{\frac{C_P'}{C_P}\;\frac{D'}{D}}{\frac{C_H'}{C_H}}}\sqrt{\left(\frac{C_H'}{C_S'}+1\right)\left(\frac{C_S}{C_H}+1\right)}\sqrt{\frac{1-\frac{D}{P}}{1-\frac{D'}{P'}}}$$

Change in TVC'/TVC_o	Range of W
≤ 1%	$0.87 \le W \le 1.15$
≤ 2%	$0.82 \le W \le 1.22$
≤ 3%	$0.79 \le W \le 1.27$
≤ 4%	$0.76 \le W \le 1.32$
≤ 5%	$0.73 \le W \le 1.37$
≤ 6%	$0.71 \le W \le 1.41$
≤ 7%	$0.69 \le W \le 1.45$
≤ 8%	$0.68 \le W \le 1.48$
≤ 9%	$0.66 \le W \le 1.52$
≤ 10%	$0.65 \le W \le 1.55$

For example, note that for any combination of values of ratios of the estimated to the true values of the parameters that are in the range of the W, from 0.73 to 1.37, then the change in the TVC'/TVC_o ratio $= \frac{1 + W^2}{2W} \leq 5$ per cent. Hence, if W is in this range then the change in total variable costs will be less than five per cent regardless of the model being considered. As an example, for the purchase model:

Change in $\frac{TVC'}{TVC_o} \leq 5\%$ if

$$0.73 \leq W \leq 1.37$$

$$0.73 \leq \frac{Q'}{Q_o} \leq 1.37$$

$$0.73 \leq \sqrt{\frac{\frac{C_P'}{C_P}\,\frac{D'}{D}}{\frac{C_H'}{C_H}}} \leq 1.37$$

for the production model:

Change in $\frac{TVC'}{TVC_o} \leq 5\%$ if

$$0.73 \leq W \leq 1.37$$

$$0.73 \leq \frac{Q'}{Q_o} \leq 1.37$$

$$0.73 \leq \sqrt{\frac{\frac{C_P'}{C_P}\,\frac{D'}{D}}{\frac{C_H'}{C_H}}}\sqrt{\frac{1 - \frac{D}{P}}{1 - \frac{D'}{P'}}} \leq 1.37$$

and so forth for the other models. The more complex the model, additional ratios of the estimated to true values of the parameters are involved in the W relationship. As a general observation, the errors in over and under-estimation of several parameters could well cancel the effect of each other. Also, the effect of estimating error is greater for under-estimation than for over-estimation. However, it seems logical that the likelihood of severe under-estimation is much less than of severe over-estimation.

This lack of sensitivity to errors in estimating parameters values is a saving grace since, if this was not true as discussed in this chapter, large periods of time and large quantities of money would have to be spent in gathering and refining estimates. Fortunately this is not necessary and "quick-and-dirty" estimates are generally sufficient to guide action and control decisions.

TABLE IV

SUMMARY OF EFFECT OF ERRORS IN PARAMETERS FOR THE VARIOUS MODELS

Model	Value of W in Terms of Parameters
Purchase Model	$W = \sqrt{\frac{C_P'}{C_P} \; \frac{D'}{D} \; \frac{C_H}{C_H'}}$
Production Model	$W = \sqrt{\frac{C_P'}{C_P} \; \frac{D'}{D} \; \frac{C_H}{C_H'}} \; \frac{\sqrt{\frac{1}{1 - D'/P'}}}{\sqrt{\frac{1}{1 - D/P}}}$
Backorder Model	$W = \sqrt{\frac{C_P'}{C_P} \; \frac{D'}{D} \; \frac{C_H}{C_H'}} \sqrt{\left(\frac{C_H'}{C_S'} + 1\right)\left(\frac{C_S}{C_H} + 1\right)}$
Production With Backorder Model	$W = \sqrt{\frac{C_P'}{C_P} \; \frac{D'}{D} \; \frac{C_H}{C_H'}} \sqrt{\left(\frac{C_H'}{C_S'} + 1\right)\left(\frac{C_S}{C_H} + 1\right)} \; \frac{\sqrt{\frac{1}{1 - D'/P'}}}{\sqrt{\frac{1}{1 - D/P}}}$

CHAPTER VI

SENSITIVITY ANALYSIS OF "WRONG" MODELS

The most significant problem in the consideration of inventory models is the selection of the appropriate model which is defined by the properties of the system being described. If the total variable costs of the various models are compared, then a measure of the sensitivity of using the models can be obtained. However, the difference between the models is, in essence, the difference of the properties that define the models. Hence, a measure of the sensitivity of the properties can be evaluated which is a measure of the assumptions concerning the properties of the system that the various models represent. In the industrial environment this problem is important to the production control manager who may be able to control the properties of the actual inventory system and wants to know if it is worthwhile to make a change in the properties. The quantitation of the significance of the difference of the various models is one of the more significant findings of this research activity.

Measure of Sensitivity Based on The "Actual" Model's Total Variable Cost Function

There are two measures of sensitivity that can be used in evaluation of the properties of an inventory system that define a particular inventory model. The first and simplest measure of sensitivity involves using the ratio of the total variable cost of the "actual" model with the "actual" model's decision variable. This measure of sensitivity assumes that the "true" model and its resulting decision variable is unknown while computing the total variable cost of the "actual" model. Hence, the total variable cost relationship and the decision variable associated with the "actual" model is utilized in the determination of the total variable cost of the "actual" model. Thus the measure of sensitivity is:

$$TVC'/TVC_O$$

where

TVC' - total variable cost of "actual" model with the "actual" model's decision variable — Q'

Q' - decision variable of "actual" model

TVC_O - total variable cost of "true" model with "true" model's decision variable — Q_O

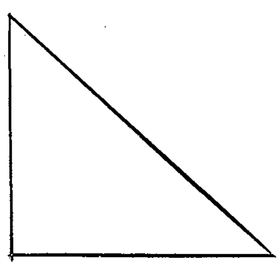
Q_O - decision variable of "true" model.

Using this measure of sensitivity, various combinations of the purchase, production, backorder, and production with backorder models are evaluated. It is noted that if a purchase model is the "actual" model and the production model is the "true" model a measure of the sensitivity of the purchase model versus the production model can be obtained.

Purchase Model Versus Production Model

The difference between the purchase model and the production model is the replenishment property concerning the rate of replenishment of the inventory, hence a measure of the sensitivity of this property or assumption is obtained when the ratio of these two models' total variable costs are compared. Further, note that the reciprocal of these two models is a measure of the sensitivity of the production model as the "actual" model as compared to the purchase model as the "true" model.

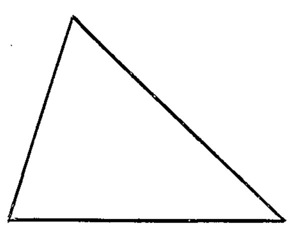
Actual Model



$$Q' = \sqrt{\frac{2C_P D}{C_H}}$$

$$TVC' = \frac{C_P D}{Q'} + \frac{C_H Q'}{2}$$

True Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{P}{P-D}}$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H Q_o}{2}\left(\frac{P-D}{P}\right)$$

Measure of Sensitivity is (See Appendix C, page 117):

$$\frac{TVC'}{TVC_o} = \sqrt{\frac{P}{P-D}} = \sqrt{\frac{1}{1-D/P}}$$

D/P	Change in TVC'/TVC_0
$\leq .093 \quad 0.10$	$\leq$ 5%
$\leq .17$	$\leq$ 10%

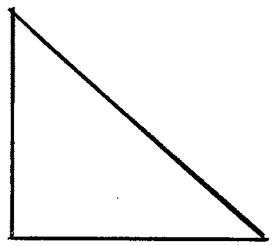
The interpretation of this measure of sensitivity of the purchase model versus the production model is that if the purchase model is the "actual" model used and the production model is the "true" model, then the total variable costs are over-estimated by the factor $\sqrt{\frac{P}{P-D}}$. This over-estimation of the holding costs is caused by the fact that average inventory of the purchase model is Q/2 whereas the average inventory of the production model is $Q/2\left(\frac{P-D}{P}\right)$. The $\sqrt{\frac{P}{P-D}}$ factor relates to the quantity of production (Q) which is used to satisfy the demand during manufacturing time and did not go into inventory.

If the D/P ratio is less than $0.093 \doteq 0.10$ then the change in TVC'/TVC_0 will be less than five per cent. Hence, any combination of values of D (Demand) and P (Production) that result in a ratio less than approximately 0.10, the resulting increase in cost due to using the "actual" model will be less than five per cent.

<u>Purchase Model Versus Backorder Model</u>

If the ratio of the total variable cost of the purchase model and the backorder model are compared a measure of the sensitivity of the cost property assuming infinite shortage costs is obtained since this is the only difference of the properties of the systems which the two models represent.

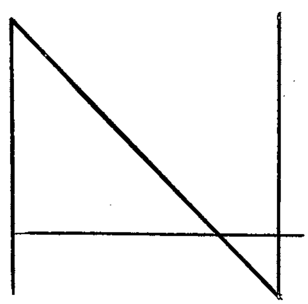
Actual Model



$$Q' = \sqrt{\frac{2C_P D}{C_H}}$$

$$TVC' = \frac{C_P D}{Q'} + \frac{C_H Q'}{2}$$

True Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}}$$

$$S_o = \sqrt{\frac{2C_P D}{C_S}}\sqrt{\frac{C_S}{C_H + C_S}}$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H Q_o}{Q_o} - C_H S_o + \frac{S_o^2 (C_H + C_S)}{2Q_o}$$

Measure of Sensitivity is (See Appendix C, page 118):

$$\frac{TVC'}{TVC_o} = \sqrt{\frac{C_H + C_S}{C_S}} = \sqrt{1 + \frac{C_H}{C_S}}$$

C_H/C_S	Change in TVC'/TVC_o
$\leq .10$	$\leq 5\%$
$\leq .20$	$\leq 10\%$

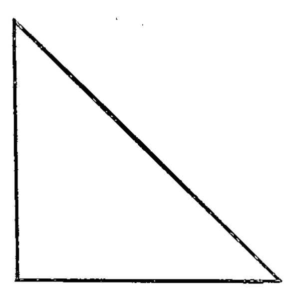
The interpretation of this measure of sensitivity of the purchase model versus the backorder model is that if the purchase model is the "actual" model used and the backorder model is the "true" model, then the total variable costs will be incorrectly estimated by the factor $\sqrt{\frac{C_H + C_S}{C_S}}$ or $\sqrt{1 + \frac{C_H}{C_S}}$. This over-estimates the total variable costs by this factor which is the difference of the respective decision variables of the two models.

If the C_H/C_S is less than approximately ten per cent, then the change in total variable costs will be less than five per cent. This means that if the cost of shortages is less than ten times the cost of holding a unit in inventory, then using an "actual" model ignoring the cost of shortage will increase the total variable costs less than five per cent.

Purchase Model Versus Production With Backorder Model

If the ratio of the total variable cost of the purchase model and the production with backorder model are compared, a measure of sensitivity of the combination of the cost property assuming infinite shortage cost and the replenishment property assuming that instantaneous replenishment of inventory is obtained.

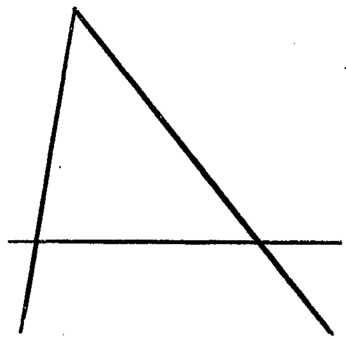
Actual Model



$$Q' = \sqrt{\frac{2C_P D}{C_H}}$$

$$TVC' = \frac{C_P D}{Q'} + \frac{C_H Q'}{2}$$

True Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}}\sqrt{\frac{P}{P - D}}$$

$$S_o = \sqrt{\frac{2C_P D}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}\sqrt{\frac{P - D}{P}}$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H Q_o}{2}\left(\frac{P-D}{P}\right) - C_H S_o + \frac{{S_o}^2(C_H + C_S)}{2Q_o\left(\frac{P-D}{P}\right)}$$

Measure of Sensitivity is (See Appendix C, page 119):

$$\frac{TVC'}{TVC_o} = \sqrt{\frac{C_H + C_S}{C_S}}\sqrt{\frac{P}{P-D}} = \sqrt{1 + \frac{C_H}{C_S}}\sqrt{\frac{1}{1 - D/P}} = \sqrt{1 + \frac{C_H}{C_S}}\sqrt{\frac{1}{1 - D/P}}$$

$\frac{C_H}{C_S} + \frac{D}{P}$	Change in TVC'/TVC_o
$\leq$ 1.10	$\leq$ 5%
$\leq$ 1.17	$\leq$ 10%

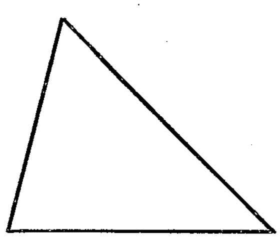
$\sqrt{\frac{P}{P - D}}\sqrt{\frac{C_H + C_S}{C_S}}$	Change in TVC'/TVC_o
$\leq$ 1.10	$\leq$ 5%
$\leq$ 1.20	$\leq$ 10%

The interpretation of this measure of sensitivity is that any combination of the square root of the product of the two terms $1 + \frac{C_H}{C_S}$ and $\frac{1}{1 - D/P}$ that are less than 1.10, then the change in the total variable costs will be less than five per cent. It has also been observed from numerical data that if the sum of the two ratios $(C_H/C_S + D/P)$ is less than ten per cent, then the change in the total variable cost will be equal to or less than five per cent.

Production Model Versus Backorder Model

If the ratio of the total variable costs of the production model and the backorder model are compared, a measure of sensitivity of the two properties that distinguish the two models is obtained. The properties involved are: (1) cost property, that cost of shortage is assumed infinite in the production model; (2) replenishment property, that the backorder model assumes instantaneous replenishment of inventory.

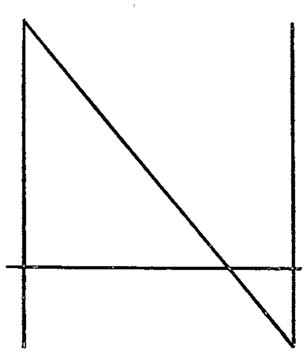
Actual Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P-D}{P}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}\,\frac{P-D}{P}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}}$$

$$S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2} - C_HS_o + \frac{S_o^{\,2}(C_H + C_S)}{2Q_o}$$

Measure of Sensitivity is (See Appendix C, page 120):

$$\frac{TVC'}{TVC_o} = \sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} = \sqrt{1+\frac{C_H}{C_S}}\sqrt{1-\frac{D}{P}}$$

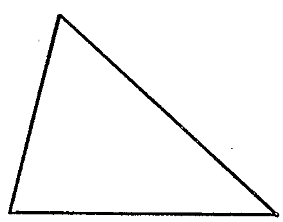
$\sqrt{\frac{P-D}{P}}\sqrt{\frac{C_H+C_S}{C_S}}$	Change in TVC'/TVC_o
$\leq$ 1.10	$\leq$ 5%
$\leq$ 1.20	$\leq$ 10%

In the comparison of these two models, the measure of sensitivity is quite large before a significant change of five per cent in total variable costs occurs. Note that the $TVC'/TVC_o = \sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} = \sqrt{1+\frac{C_H}{C_S}}\sqrt{1-\frac{D}{P}}$ upon inspection of this ratio, it becomes obvious that as C_H/C_S and D/P both become large, that the effect of the error is relatively small since they have opposite effect on the TVC'/TVC_o ratio. Note that the two distinguishing properties which are associated with the two models being evaluated, also differentiates the purchase model and the production with back-order model. However, in the previous evaluation, both properties that distinguished the two models were associated with only one model, namely, the production with backorder model. It is of interest to note the similarity of the TVC'/TVC_o ratios which satisfy the intuition of the nature of the two comparisons.

Production Model Versus Production With Backorder Model

A measure of the sensitivity of the cost shortage property which assumes cost of shortage is infinite can be obtained by comparing the ratio of the total variable costs of the production model to the production with backorder model. Notice that this result is the same as comparing the purchase model to the backorder model since both comparisons involve models that are distinguished only by the assumption concerning the cost property.

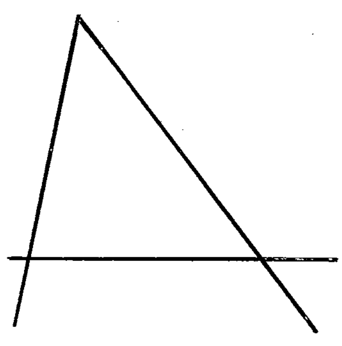
Actual Model



$$Q' = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{P}{P-D}}$$

$$TVC' = \frac{C_P D}{Q'} + \frac{C_H Q'}{2}\left(\frac{P-D}{P}\right)$$

True Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}}\sqrt{\frac{P}{P-D}}$$

$$S_o = \sqrt{\frac{2C_P D}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H Q_o}{2}\left(\frac{P-D}{P}\right) - C_H S_o + \frac{S_o^{2}(C_H + C_S)}{2Q_o\left(\frac{P-D}{P}\right)}$$

Measure of Sensitivity is (See Appendix C, page 122):

$$\frac{TVC'}{TVC_o} = \sqrt{\frac{C_H + C_S}{C_S}} = \sqrt{1 + \frac{C_H}{C_S}}$$

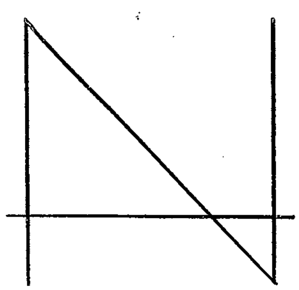
$\frac{C_H}{C_S}$	Change in TVC'/TVC_o
$\leq .10$	$\leq$ 5%
$\leq .20$	$\leq$ 10%

The interpretation of the table values are the same as for the purchase motel versus the backorder model measure of sensitivity.

Backorder Model Versus Production With Backorder Model

If the ratio of the total variable costs of the backorder model and the production with backorder model are compared, a measure of sensitivity is obtained of the replenishment property relative to the assumptions of instantaneous replenishment of inventory. It is noted that the comparison of these two models result in measuring the sensitivity of the replenishment property that is measured in the comparison of the purchase model to the production model. Again, it is reassuring to obtain the same results in both comparisons.

Actual Model

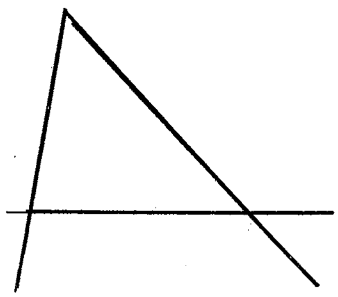


$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}}$$

$$S' = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}$$

$$TVC' = \frac{C_P D}{Q'} + \frac{C_H Q'}{2} - C_H S' + \frac{(S')^2 (C_H + C_S)}{2Q'}$$

True Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}} \sqrt{\frac{C_H + C_S}{C_S}} \sqrt{\frac{P}{P - D}}$$

$$S_o = \sqrt{\frac{2C_P D}{C_S}} \sqrt{\frac{C_H}{C_H + C_S}} \sqrt{\frac{P - D}{P}}$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H Q_o}{2} \left(\frac{P - D}{P}\right) - C_H S_o + \frac{S_o^{\,2} \, (C_H + C_S)}{2Q_o \left(\frac{P - D}{P}\right)}$$

Measure of Sensitivity is (See Appendix C, page 123):

$$\frac{TVC'}{TVC_o} = \sqrt{\frac{P}{P - D}} = \sqrt{\frac{1}{1 - D/P}}$$

$\frac{D}{P}$	Change in TVC'/TVC_o
$\leq .093$	$\leq$ 5%
$\leq .17$	$\leq$ 10%

This interpretation of this measure of sensitivity of these two models is the same as for the comparison of the purchase model and the production models.

Note that with the measure of sensitivity in Table V, the error of using the "actual" model, is the ratio of the "true" model's decision variable to the "actual" model's decision variable.

TABLE V

SUMMARY OF MEASURES OF SENSITIVITY FOR COMPARISON OF VARIOUS MODELS BASED ON "ACTUAL" MODEL'S TOTAL VARIABLE COST FUNCTION

Actual Model \ True Model	Purchase Model	Production Model	Backorder Model	Production With Backorder Model
Purchase Model	1	$\sqrt{\frac{P}{P-D}}$	$\sqrt{\frac{C_H+C_S}{C_S}}$	$\sqrt{\frac{P}{P-D}}\sqrt{\frac{C_H+C_S}{C_S}}$
Production Model	$\sqrt{\frac{P-D}{P}}$	1	$\sqrt{\frac{P-D}{P}}\sqrt{\frac{C_H+C_S}{C_S}}$	$\sqrt{\frac{C_H+C_S}{C_S}}$
Backorder Model	$\sqrt{\frac{C_S}{C_H+C_S}}$	$\sqrt{\frac{P}{P-D}}\sqrt{\frac{C_S}{C_H+C_S}}$	1	$\sqrt{\frac{P}{P-D}}$
Production With Backorder Model	$\sqrt{\frac{P-D}{P}}\sqrt{\frac{C_S}{C_H+C_S}}$	$\sqrt{\frac{C_S}{C_H+C_S}}$	$\sqrt{\frac{P-D}{P}}$	1

Measure of Sensitivity Based on The "True" Model's Total Variable Cost Function

The second and more difficult to analyze measure of sensitivity is defined as the ratio of the total variable cost of the "true" model with the non-optimal value of decision variable based on the "actual" model to the total variable cost of the "true" model using the optimal value of the decision variable. This measure of sensitivity assumes that the "true" model is known but that a decision variable of an erroneous model is used in ascertaining the total variable cost. Hence, the "error" in the total variable cost is only a function of the decision variable from the "actual" model and not a function of the total variable cost function of the "actual" model.

Thus the measure of sensitivity is:

$$TVC'/TVC_o$$

where

TVC' = total variable cost of "true" model with "actual" model's decision variable - Q'

Q' = decision variable of "actual" model

TVC_o = total variable cost of "true" model with "true" model's decision variable

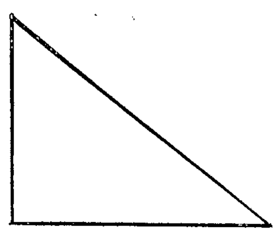
Q_o = decision variable of "true" model.

Using this measure of sensitivity, various combinations of the purchase, production, backorder, and production with backorder models are evaluated. As with the first measure of sensitivity, if the production model is the "true" model and the decision varialbe is based on the purchase as the "actual" model, a measure of sensitivity of the purchase model versus the production model can be obtained. This is in effect a measure of the sensitivity of the assumption concerning the instantaneous replenishment property of the purchase model. As will similarly be the case for all the comparisons, the reciprocal of these two models is a measure of the sensitivity of the production model as the "actual" model for the decision variable and the purchase model as the "true" model.

Purchase Model Versus Production Model

If the ratio of the total variable cost of the purchase model and the production model are compared a measure of the sensitivity of the replenishment property which assumes instantaneous replenishment of inventory is obtained. This measure of sensitivity evaluates only the effect of using a "wrong" value of a decision variable which may be based on a "wrong" model. This sensitivity value will be less than the previous computed measure since in both terms of the ratio, the "true" model is assumed to be known. Note again that the reciprocal of these two models is a measure of the sensitivity of the production model and its decision variable as the "actual" model as compared to the purchase model as the "true" model.

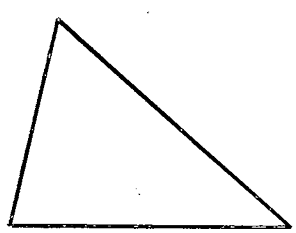
Actual Model



$$Q' = \sqrt{\frac{2C_P D}{C_H}}$$

$$TVC' = \frac{C_P D}{Q'} + \frac{C_H Q'}{2}\left(\frac{P-D}{P}\right)$$

True Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{P}{P-D}}$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H Q_o}{2}\left(\frac{P-D}{P}\right)$$

Measure of Sensitivity is (See Appendix C, page 124):

$$\frac{TVC'}{TVC_o} = \frac{1}{2}\left(\frac{2\ P - D}{\sqrt{P - D}}\right)$$

$$= \frac{1}{2}\left[\sqrt{\frac{P}{P - D}} + \sqrt{\frac{P - D}{P}}\right]$$

$$= \frac{1}{2}\left[\sqrt{\frac{1}{1 - D/P}} + \sqrt{1 - D/P}\right]$$

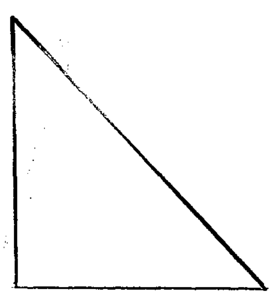
D/P	Change in TVC'/TVC_o
≤ 0.24	≤ 1%
≤ 0.32	≤ 2%
≤ 0.38	≤ 3%
≤ 0.43	≤ 4%
≤ 0.46	≤ 5%
≤ 0.49	≤ 6%
≤ 0.52	≤ 7%
≤ 0.54	≤ 8%
≤ 0.56	≤ 9%
≤ 0.58	≤ 10%

The interpretation of the table can be illustrated by considering the D/P ratio of 0.46 which has a change in the TVC'/TVC_o ratio of five per cent. This means that the D/P (demand requirements to production capacity) ratio can be as high as 0.46, and the total variable costs of using the decision variable based on the "actual" model will be less than five per cent. As previously discussed, this measure of sensitivity quantifies the effect of using the replenishment property which assumes instantaneous replenishment of inventory. The sensitivity of this property relative to the total variable costs is not critical since the D/P ratio can approach one-half and not exceed the total variable costs by six per cent. Hence, the "error" of using the purchase model's decision variable rather than the "true" production model's decision variable is less than six per cent if the demand requirements to production capacity ratio is less than 0.50.

Purchase Model Versus Backorder Model

A measure of sensitivity of the cost property that assumes that cost of shortage is infinite in the purchase model can be obtained by comparing the ratio of the total variable costs of the purchase model and the backorder model, since this is the only property that distinguishes these two models.

Actual Model



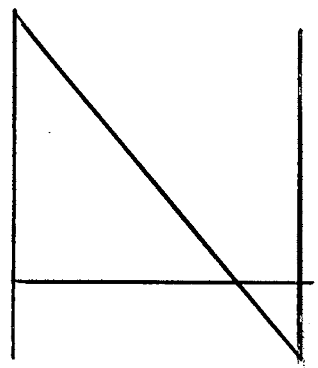
$$Q' = \sqrt{\frac{2C_PD}{C_H}}$$

$$S' = 0$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2} - C_H(S') + \frac{(S')^2(C_H+C_S)}{2Q'}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}$$

$$S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2} - C_HS_o + \frac{S_o^{\,2}(C_H+C_S)}{2Q_o}$$

Measure of Sensitivity is (See Appendix C, page 125):

$$\frac{TVC'}{TVC_o} = \sqrt{\frac{C_H + C_S}{C_S}} = \sqrt{1 + \frac{C_H}{C_S}}$$

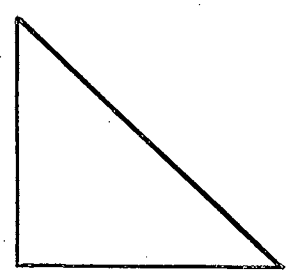
C_H/C_S	Change in TVC'/TVC_o
≤ 0.02	≤ 1%
≤ 0.04	≤ 2%
≤ 0.06	≤ 3%
≤ 0.08	≤ 4%
≤ 0.10	≤ 5%
≤ 0.12	≤ 6%
≤ 0.14	≤ 7%
≤ 0.16	≤ 8%
≤ 0.18	≤ 9%
≤ 0.20	≤ 10%

This measure of sensitivity of this cost property can be interpreted by an example. If the C_H/C_S ratio is less than 0.10, then the change in the total variable costs will be less than five per cent. This says that if the unit cost of shortage is less than ten times the cost of holding an unit in inventory, then the increase in the total variable cost will be less than five per cent, which indicates that this cost property can be ignored in practical considerations.

Purchase Model Versus Production With Backorder Model

A measure of sensitivity of the shortage cost property in combination with the replenishment property can be obtained by comparing the ratio of the total variable costs of the purchase model's decision variable based on the production model as the "true" to the production model with its decision variable as the "true" model.

Actual Model

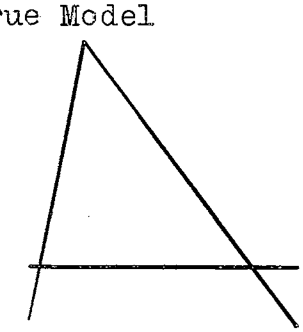


$$Q' = \sqrt{\frac{2C_P D}{C_H}}$$

$$S' = 0$$

$$TVC' = \frac{C_P D}{Q'} + \frac{C_H Q'}{2}\left(\frac{P-D}{P}\right) - C_H S' + \frac{(S')^2(C_H+C_S)}{2Q'\left(\frac{P-D}{P}\right)}$$

True Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}$$

$$S_o = \sqrt{\frac{2C_P D}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H Q_o}{2}\left(\frac{P-D}{P}\right) - C_H S_o + \frac{S_o^2(C_H+C_S)}{2Q_o\left(\frac{P-D}{P}\right)}$$

Measure of Sensitivity is (See Appendix C, page 126):

$$\frac{TVC'}{TVC_o} = \frac{1}{2}\sqrt{\frac{C_H+C_S}{C_S}}\left[\sqrt{\frac{P}{P-D}} + \sqrt{\frac{P-D}{P}}\right]$$

$$= \frac{1}{2}\sqrt{1 + \frac{C_H}{C_S}}\left[\sqrt{\frac{1}{1 - D/P}} + \sqrt{1 - \frac{D}{P}}\right]$$

C_H/C_S	D/P	Change in TVC'/TVC_o
≤ 0.10	≤ 0.10	≤ 5%
≤ 0.09	≤ 0.20	≤ 5%
≤ 0.08	≤ 0.25	≤ 5%
≤ 0.07	≤ 0.30	≤ 5%
≤ 0.06	≤ 0.33	≤ 5%
≤ 0.05	≤ 0.36	≤ 5%
≤ 0.04	≤ 0.38	≤ 5%

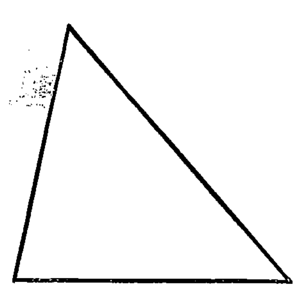
This measure of sensitivity can be interpreted as having several combinations of values of C_H/C_S and D/P ratios which increase the total variable costs less than five per cent. Specifically if the values of the C_H/C_S ratio are less than 0.10 and the D/P ratio is less than 0.10, then the change in the total variable costs will be less than five per cent. Note that other combinations of

C_H/C_S and D/P ratios will result in change in total variable costs less than five per cent. However, the largest the C_H/C_S ratio can be even for D/P ratio of 0.01, is 0.10. This is logical since if D/P approaches zero the $\sqrt{\frac{P}{P-D}} + \sqrt{\frac{P-D}{P}}$ term approaches the value of two which cancels the one-half factor. Hence, the largest the $\sqrt{1 + \frac{C_H}{C_S}}$ can be is 1.10, resulting in the largest possible value of the C_H/C_S ratio of 0.10 without exceeding the five per cent change in the total variable costs.

Production Model Versus Backorder Model

A measure of the sensitivity of the two properties that distinguish the production model and the backorder model, namely: 1) cost property which assumes that the cost of shortage is infinite for the production model, and 2) replenishment property which assumes instantaneous replenishment of inventory for the backorder model can be obtained if the ratio of the total variable costs of the two models are compared.

Actual Model

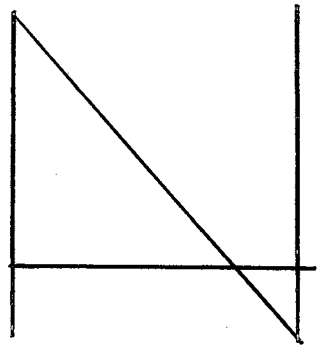


$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}$$

$$S' = 0$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2} - C_HS' + \frac{(S')^2(C_H+C_S)}{2Q'}$$

True Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}$$

$$S_o = \sqrt{\frac{2C_P D}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H Q_o}{2} - C_H S_o + \frac{S_o^2 (C_H+C_S)}{2Q_o}$$

Measure of Sensitivity is (See Appendix C, page 127):

$$\frac{TVC'}{TVC_o} = \frac{1}{2}\sqrt{\frac{C_H+C_S}{C_S}}\left[\sqrt{\frac{P}{P-D}} + \sqrt{\frac{P-D}{P}}\right]$$

$$= \frac{1}{2}\sqrt{1 + \frac{C_H}{C_S}}\left[\sqrt{\frac{1}{1 - D/P}} + \sqrt{1 - \frac{D}{P}}\right]$$

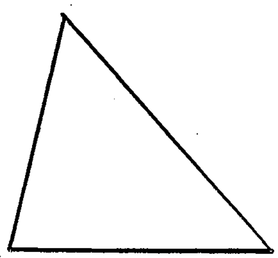
C_H/C_S	D/P	Change in TVC'/TVC_o
≤ 0.10	≤ 0.10	≤ 5%
≤ 0.09	≤ 0.20	≤ 5%
≤ 0.08	≤ 0.25	≤ 5%
≤ 0.07	≤ 0.30	≤ 5%
≤ 0.06	≤ 0.33	≤ 5%
≤ 0.05	≤ 0.36	≤ 5%
≤ 0.04	≤ 0.38	≤ 5%

Since the measure of sensitivity of these two models are the same as for the purchase model versus the production with backorder model, the same interpretation and comments are appropriate.

Production Model Versus Production With Backorder Model

A comparison of the ratio of the total variable costs of these two models will result in a measure of sensitivity of the cost property which assumes the cost of shortage is infinite. This comparison evaluates the same property as in the purchase model versus the backorder model comparison. It is reassuring to obtain the same measure of sensitivity for both comparisons.

Actual Model

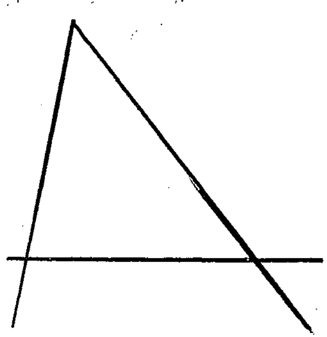


$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}$$

$$S' = 0$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}\left(\frac{P-D}{P}\right) - C_HS' + \frac{(S')^2(C_H+C_S)}{2Q'\left(\frac{P-D}{P}\right)}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}$$

$$S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2}\left(\frac{P-D}{P}\right) - C_HS_o + \frac{S_o^2(C_H+C_S)}{2Q_o\left(\frac{P-D}{P}\right)}$$

Measure of Sensitivity is (See Appendix C, page 128):

$$\frac{TVC'}{TVC_o} = \sqrt{\frac{C_H + C_S}{C_S}}$$

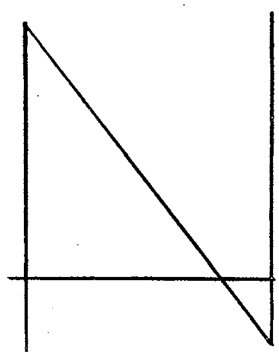
C_H/C_S	Change in TVC'/TVC_o
≤ 0.02	≤ 1%
≤ 0.04	≤ 2%
≤ 0.06	≤ 3%
≤ 0.08	≤ 4%
≤ 0.10	≤ 5%
≤ 0.12	≤ 6%
≤ 0.14	≤ 7%
≤ 0.16	≤ 8%
≤ 0.18	≤ 9%
≤ 0.20	≤ 10%

Since the measure of sensitivity of these two models is the same as the purchase model versus the production model, the same interpretation and comments are appropriate.

Backorder Model Versus Production With Backorder Model

The comparison of the ratio of these two models' total variable cost should result in a measure of sensitivity of the combination of the properties that distinguish the two models, namely: 1) replenishment property assuming that instantaneous replenishment of inventory is possible.

Actual Model

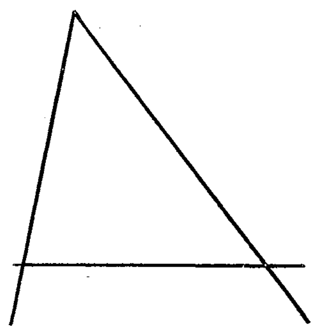


$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}$$

$$S' = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}\left(\frac{P-D}{P}\right) - C_HS' + \frac{(S')^2(C_H+C_S)}{2Q'\left(\frac{P-D}{P}\right)}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}$$

$$S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2}\left(\frac{P-D}{P}\right) - C_HS_o + \frac{S_o^{\,2}(C_H+C_S)}{2Q_o\left(\frac{P-D}{P}\right)}$$

Measure of Sensitivity is (See Appendix C, page 129):

So far this measure of sensitivity has resisted simplification that was possible with the other comparisons.

The second measure of sensitivity is less sensitive to the use of "actual" value of the decision variable. In other words, if the "true" model is known, the evaluation of the total variable costs using the "true" model and the "actual" value of the decision variable will result in less error than the evaluation of the total variable costs using the "actual" model and the "actual" value of the decision variable.

Table VII illustrates the numerical difference in the

TABLE VI

SUMMARY OF MEASURE OF SENSITIVITY FOR COMPARISON OF VARIOUS MODELS BASED ON "TRUE" MODEL'S TOTAL VARIABLE COST FUNCTION

"Actual" Model / "True" Model	Purchase Model	Production Model	Backorder Model	Production With Backorder Model
Purchase Model	1	$\frac{1}{2}\left\{\sqrt{\frac{P}{P-D}}+\sqrt{\frac{P-D}{P}}\right\}$	$\sqrt{\frac{C_H+C_S}{C_S}}$	$\frac{1}{2}\sqrt{\frac{C_H+C_S}{C_S}}\left\{\sqrt{\frac{P}{P-D}}+\sqrt{\frac{P-D}{P}}\right\}$
Production Model	$\frac{1}{2}\left\{\sqrt{\frac{P-D}{P}}+\sqrt{\frac{P}{P-D}}\right\}$	1	$\frac{1}{2}\sqrt{\frac{C_H+C_S}{C_S}}\left\{\sqrt{\frac{P}{P-D}}+\sqrt{\frac{P-D}{P}}\right\}$	$\sqrt{\frac{C_H+C_S}{C_S}}$
Backorder Model	$\sqrt{\frac{C_S}{C_H+C_S}}$	$\frac{1}{2}\sqrt{\frac{C_S}{C_H+C_S}}\left\{\sqrt{\frac{P-D}{P}}+\sqrt{\frac{P}{P-D}}\right\}$	1	
Production Model Backorder Model	$\frac{1}{2}\sqrt{\frac{C_S}{C_H+C_S}}\left\{\sqrt{\frac{P-D}{P}}+\sqrt{\frac{P}{P-D}}\right\}$	$\sqrt{\frac{C_S}{C_H+C_S}}$		1

two measures of sensitivity for the purchase versus production model comparison.

TABLE VII

COMPARISON OF MEASURES OF SENSITIVITY FOR PURCHASE VERSUS PRODUCTION MODELS

Purchase Versus Production Models		
D/P	$\frac{TVC'}{TVC_o} = \sqrt{\frac{P}{P-D}}$	$\frac{TVC'}{TVC_o} = \frac{1}{2}\left[\sqrt{\frac{P}{P-D}} + \sqrt{\frac{P-D}{P}}\right]$
.05	2.59%	0.03%
.10	5.40%	0.14%
.20	11.80%	0.62%
.30	19.52%	1.69%
.40	29.09%	3.28%
.50	41.42%	6.07%

In the evaluation of the combinations of the various models, some of the combinations resulted in measuring the effect of an "error" relative to the same assumption concerning a particular property that distinguished the models which resulted in the same measure of sensitivity.

CHAPTER VII

PROBABILISTIC INVENTORY MODELS WITH POWER DEMAND PATTERNS

In the consideration and evaluation of deterministic inventory models, the demand properties were specified such that demand was known and at a uniform usage rate. This property defined the demand pattern to be uniform throughout the period involved. Often, possibly most of the time in actual practice, the demand pattern will not be uniform, then this type of inventory models are classified as probabilistic inventory systems. It should be pointed out that, in general, the characteristic of the demand pattern is the property of the inventory system not subject to control of the decision maker.

Demand Patterns*

It is possible to have numerous ways in which inventory is depleted resulting in various demand patterns. The units may be withdrawn uniformly throughout the period as considered in deterministic models or, at one extreme, all the units may be withdrawn at the beginning of a period called a

*These concepts of demand patterns are "para-phrased" from Naddor's concept in his text [1].

instantaneous demand pattern. The other extreme would result if all the units are withdrawn at the end of the period. Obviously there exist many other withdrawal rates between the two extremes just cited. The different rates at which demand occurs during a period will be referred to as demand patterns. The following definitions will be used to define a class of general demand patterns.

Q - the amount in inventory at beginning of period ($t = 0$)

$Q(t')$ - the amount in inventory at time t'

x = total demand during period t

n - demand pattern index

t' - length of time from beginning of period ($t = 0$)

then

$$Q(t') = Q - x\,(t'/t)^{1/n} = Q - x\,\sqrt[n]{t'/t}$$

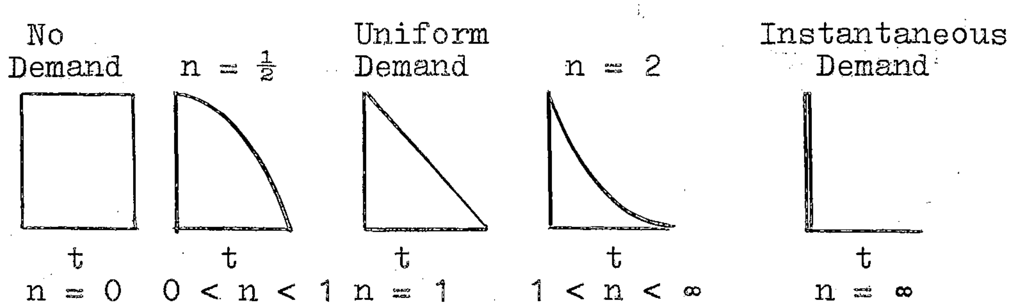


Figure 2. Demand Patterns

The demand patterns belonging to this class is referred to as power demand patterns. Their nature is entirely determined by n, the demand pattern index. When $n = 0$, one of the extreme values of n, there is no demand until the end of the period, when $n = \infty$, the other extreme value of n, the demand is instantaneous at beginning of the period. When the demand pattern index $n = 1$, the demand is uniform throughout the period resulting in the general class of deterministic inventory models.

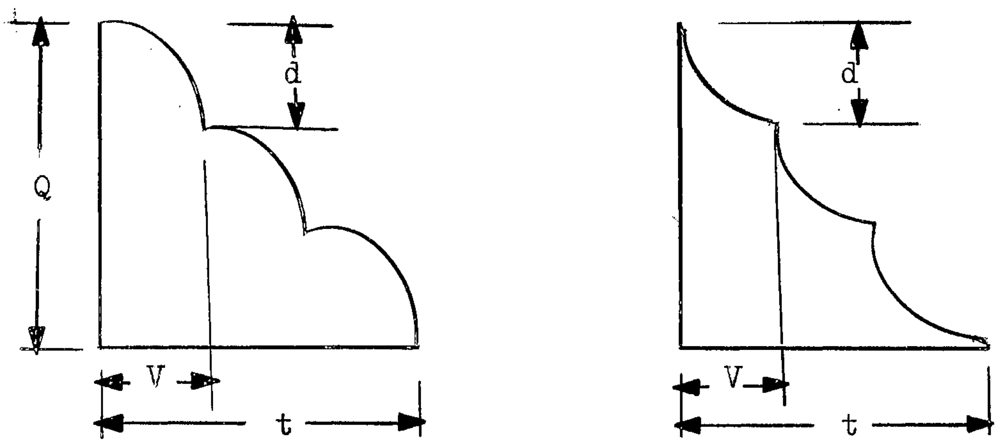


Figure 3. Power Demand Pattern Model*

V - defined period of time during which there is a power demand pattern with index n

d - known demand during V

m - be an integer number such that $Q = md$

$Q = d, 2d, 3d, \ldots, md$

$t = mV$

I_{avg} = average number of units in inventory

$$I_{avg} = \frac{Q}{2} + \frac{1}{t} m \int_0^V (Q_n - Q_1) dt'$$

where

$$Q_n = d - d \sqrt[n]{\frac{t'}{V}}$$

$$I_{avg} = \frac{Q}{2} + \frac{(d/2)(1-n)}{1+n} \qquad \text{where} \quad Q = md$$

$$= \frac{Q}{2} + \frac{Q(1-n)}{2m(1+n)} \qquad d = \frac{Q}{m}$$

*The derivation of this model and its results is taken from Naddor [1], pp. 53, 55.

$$TVC(Q_o) = C_H \left[\frac{Q}{2} + \frac{Q(1-n)}{2m(1+n)}\right] + C_P \frac{D}{Q}$$

where

$$Q = d,\ 2d,\ 3d,\ \ldots,\ md$$

$$TVC(Q_o - d) \leq TVC(Q_o) \leq TVC(Q_o + d)$$

Hence, the optimal lot size is independent of the demand index n, thus for even extreme values of n the same optimal lot size is obtained. However, the minimum cost will depend on the demand index n which appears in the term

$$I_{avg} = C_H \left[\frac{Q}{2} + \frac{Q(1-n)}{2m(1+n)}\right] = \frac{C_H Q}{2} + \frac{C_H Q}{2}\left(\frac{1-n}{m(1+n)}\right).$$

This term $\frac{C_H Q(1-n)}{2m(1+n)}$ can be interpreted as representing the resulting additional cost in a system with a power demand pattern as compared to uniform demand pattern. Note that the factor $\frac{Q}{2}\left(\frac{1-n}{m(1+n)}\right)$ is the inventory factor which indicates that the inventory is less or greater than the average inventory of the uniform demand pattern of $\frac{Q}{2}$.

Sensitivity Analysis of Probabilistic Inventory Models With Power Demand Patterns

This section presents the results of the evaluation of the sensitivity of a deterministic inventory model with a uniform power demand pattern index ($n = 1$) relative to a probabilistic inventory model with non-uniform power demand patterns ($n \neq 1$).

TVC' – total variable cost of system with a uniform demand pattern ($n = 1$)

TVC – total variable cost of system with a non-uniform demand pattern ($n \neq 1$)

n - power demand index

m - is an integer number representing the number of times that power index pattern occurs in cycle-t.

Then

$$\frac{TVC'}{TVC} = \frac{C_P \frac{D}{Q} + C_H \frac{Q}{2}}{C_P \frac{D}{Q} + C_H \left[\frac{Q}{2} + \frac{Q(1-n)}{2m(1+n)} \right]} \qquad \text{where } Q = \sqrt{\frac{2C_P D}{C_H}}$$

$$= \frac{\sqrt{\frac{C_P D \; C_H}{2}} + \sqrt{\frac{C_P D \; C_H}{2}}}{\sqrt{\frac{C_P D \; C_H}{2}} + C_H \frac{Q}{2} + \frac{C_H Q}{2}\left[\frac{(1-n)}{m(1+n)}\right]}$$

$$= \frac{\sqrt{\frac{C_P D \; C_H}{2}}\,[1+1]}{\sqrt{\frac{C_P D \; C_H}{2}} + \sqrt{\frac{C_P D \; C_H}{2}} + \sqrt{\frac{C_P D \; C_H}{2}}\left[\frac{(1-n)}{m(1+n)}\right]}$$

$$= \frac{\sqrt{\frac{C_P D \; C_H}{2}}\,[2]}{\sqrt{\frac{C_P D \; C_H}{2}}\left[1 + 1 + \frac{1-n}{m(1+n)}\right]}$$

$$= \frac{2}{2 + \frac{1-n}{m(1+n)}}$$

$$= \frac{2}{\frac{2m(1+n) + (1-n)}{m(1+n)}}$$

$$= \frac{2m(1+n)}{2m(1+n) + (1-n)}$$

The sensitivity of continuous as compared to discrete demand patterns can be considered and quantified since as m increases and n increases the demand pattern approaches the discrete stair-case pattern.

TABLE VIII

DEMAND PATTERN INDEX

m	n	Change in $\frac{TVC'}{TVC}$
1	$0.8 \leq n \leq 1.2$	5%
2	$0.6 \leq n \leq 1.4$	5%
3	$0.5 \leq n \leq 1.8$	5%
4	$0.4 \leq n \leq 2.3$	5%
5	$0.3 \leq n \leq 2.8$	5%
⋮		⋮
m	$0 \leq n \leq \infty$	5%

When m = 1, the TVC'/TVC ratio of costs is less than five per cent when the demand pattern index n is between 0.8 and 1.2. This can be interpreted as saying that the assumed total variable costs based on a uniform demand pattern (n = 1) results in a 20 per cent over-estimation of the actual total variable costs based on a demand pattern index (n > 1) and results in a 20 per cent under-estimation of the actual total variable costs based on a demand pattern index (n < 1). This satisfies the intuition of the model since a demand pattern index of n > 1 will result in the average inventory less than a model with a uniform demand pattern (n = 1). Hence, the assumed total variable costs will be greater than the actual total variable costs considering a smaller average inventory resulting in lower inventory holding costs. If the demand pattern index is such that $n < 1$, this will result in the average inventory greater than a model with a uniform demand pattern when $n = 1$. Thus the assumed total variable costs will be less than the actual total variable costs considering a larger average inventory resulting in higher inventory holding costs.

The difference between the actual and assumed inventory models is the inventory factor of $\frac{Q}{2}\left[\frac{1 - n}{m(1 + n)}\right]$ which can be positive or negative depending on the value of n. If $n > 1$, inventory holding costs are over-estimated, if $n < 1$, then inventory holding costs are under-estimated.

Hence, in general, the statement can be made that if the demand pattern index is greater than one, which is the uniform demand pattern, then the total variable cost function assuming a uniform demand pattern will result in an over-estimation of the actual total variable costs since the actual amount in inventory will be less than anticipated. Likewise, if the demand pattern index is less than one, the uniform demand pattern index, the total variable cost function assuming a uniform demand pattern will result in an under-estimation of the actual total variable costs since the actual amount in inventory will be greater than anticipated.

Further, it can be noted that the larger the value that m assumes, the less the error of using a total variable cost function with a uniform demand pattern. This seems obvious as the larger m is, the closer the uniform demand pattern becomes as an approximation to the actual demand pattern.

CHAPTER VIII

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

The purpose of this concluding chapter is to summarize the research effort, draw conclusions based on results, and make recommendations for future action. Hence, this chapter is concerned with three topics; the first will summarize the information presented by reviewing the contributions of each chapter, the second will draw conclusions relative to the results, and the third topic will present recommendations and proposals for further action and study.

Summary

Chapter I served to introduce the inventory problem. Due to the lack of commonly accepted definitions of terms in the area of inventory control, specific definitions were presented which allowed a conceptual framework to be developed and utilized to illustrate the interrelations of the various models. The classification of the inventory system properties into four categories of properties was utilized quite effectively in developing the various models and to emphasize the similarities of the decision variables formulas.

The three phases of inventory analysis were presented which emphasized the importance of the analysis and evaluation of a solution to an inventory problem. This last phase is the basis for this treatise, the Sensitivity Analysis of Inventory Models.

Chapter II reported on the bounds of knowledge of the topic of sensitivity analysis relative to inventory theory. The treatment of this topic in the technical literature has been extremely limited as contrasted to the treatment of the broader subject of optimization of inventory models. Also, those articles which considered the topic of sensitivity of inventory models were quite limited in models evaluated and, in general, considered only the sensitivity of the simple purchase model and its parameters in their analysis. In this author's opinion, this treatise is the most extensive treatment of the topic of sensitivity analysis of inventory models relative to additional models considered, numerical quantification of the measures of sensitivities obtained and the evaluation of "wrong" models, etc.

Chapter III presented an inventory model hierarchy based on the four classifications of properties which are assumed relative to the definition of a particular inventory system. The hierarchy illustrates the simple purchase model and its extensions to the other models by the assumptions concerning the properties that define each system that is described by the various models. The recognition of the inventory hierarchy and the relationships that existed between the decision variable formulas for the various models was one of the important concepts that enabled the sensitivity analysis to be extended beyond the simple purchase model to the extended models having additional properties. Knowledge of the interrelationship of the models greatly facilitated the extensive analysis of the effect of the use of "wrong" models.

Chapter IV involved the evaluation of a measure of sensitivity of the decision variables, Q - the economical lot size, t - the time interval between orders, and n - the number of orders, for the purchase, production, backorder, production with backorder, and the restricted variables models. The measure of sensitivity for all the models for all the decision variables was ascertained to be $\frac{1 + W^2}{2W}$ where if $0.73 \leq W \leq 1.37$, then the change in the total variable costs will be less than five per cent. Also, it was further ascertained that the measure of sensitivity is independent of the decision variable itself and only a function of W.

Chapter V evaluated the sensitivity of the inventory system to the use of "incorrect" values of the input parameters for the purchase, production, backorder and production with backorder models. The results of the previous chapter indicated that the effect of an incorrect value of the decision variable on the total variable cost was only a function of W. By definition, $W = \frac{Q'}{Q_o}$ which can be interpreted as the ratio of the parameters of the "actual" decision variable to the parameters of the optimal value of the decision variable for any model of interest. Again the range of W could be from 0.73 to 1.37 and the resulting change in the total variable cost would be less than five per cent. This can be interpreted to mean that as long as the estimates are not too far (the value of W specifies what "too far" means) from the true values of the parameters, then the model is not influenced much by the value of the parameters. This relative insensitivity of the parameters to errors is a saving grace and fortunately is true for all deterministic inventory models. If the models were extremely sensitive to changes in parameters, then large periods of time and large quantities of money would have to be spent merely in gathering data and refining estimates. Fortunately, sensitivity measures indicate that this is not necessary and "quick-and-dirty" estimates are generally sufficient to guide action without greatly influencing the total variable cost.

It was observed than an error in estimating the procurement cost parameter has the identical effect as errors in estimating the demand parameter. Also, errors of the same type in estimating the holding and procurement cost parameters will cancel each other due to their inverse relationship. Upon analysis of the

$$\frac{TVC'}{TVC_o} = \frac{1 + W^2}{2W}$$

relationship, it can be observed that the sensitivity is greater for under-estimation than over-estimation. However, in the practical situation it should be recognized that the likelihood of severe under-estimation is much less than severe over-estimation.

Chapter VI involved the sensitivity analysis of the use of "wrong" models under two different assumptions. These measures were quantified and tabulated relative to their effect on the change in the total variable costs.

If the measure of sensitivity is based on the "actual" model's total variable cost function, it can be concluded that the error of using the "wrong" model is a function of the difference of the "true" model's decision variable and the "actual" model's decision variable. In other words, the ratio of the "true" model's decision variable to the "actual" model's decision variable will be the effect on the total variable cost of using the "wrong" model.

If the "true" model is assumed to be known, the measure of sensitivity based on the "true" model's total variable cost function, will result in considerable less error for using the "wrong" model. This measure of sensitivity would appear to be the logical one to use even though it is computationally more involved since the actual total variable costs should be computed using the "true" model's total variable cost function.

Chapter VII considered the sensitivity analysis of a type of probabilistic inventory models. The effect of using the uniform demand function was evaluated and tabulated relative to a power demand function that would be the "true" demand pattern. In the class of probabilistic inventory models with power demand patterns, it has been ascertained that if the power index is in the range $0.8 \leq n \leq 1.2$, and $m = 1$, which is the extreme case, then the effect on the total variable costs will be less than five per cent. As the value of m increases, the range of n increases. In fact as m increases rapidly and n also increases, the sensitivity of the assumption relative to continuous versus the discrete withdrawal of demand is evaluated. If $m \leq 5$, and $0.03 \leq n \leq 2.8$, the change in total variable costs will be less than five per cent. In general, this indicates that the use of deterministic inventory models which assume a uniform demand pattern that has a power index of $n = 1$ does not significantly effect the total variable cost relative to probabilistic models which assumes non-uniform demand patterns.

Appendix A presents a complete derivation of the backorder model and the production with backorder model. As far as this author can ascertain, this is the first and complete derivation that has been published of the production with backorder model resulting in relationships for the decision variables which could be related to the other models and the properties that defined this model.

Conclusions

The first specific conclusion is that the development of the conceptual model hierarchy based on the framework of the four classifications of the properties was a valuable aid in illustrating the interrelations between the models and their respective decision variable formulas. This conceptual framework made it possible to obtain measures of sensitivities of models that previously had not been evaluated. This framework also greatly facilitated the evaluation of the "wrong" models since it was possible to anticipate the possible consequences of the use of "wrong" models from an examination of their respective decision variable's formulas.

The second specific conclusion is the measure of sensitivity for all the decision variables (Q, t, or N) is the same for the purchase, production, backorder, production with backorder and restricted variables models. Further it can be concluded that any model with a cost function consisting of the sum of a hyperbolic and linear terms will result in a measure of sensitivity of a change in the decision variable equal to $\frac{1 + W^2}{2W}$ where W is the per cent deviation of the value of the actual variable to the optimal value of that decision variable.

As a corollary to above conclusion, it can be stated that the sensitivity measure is independent of the decision variable itself and only a function of W. A second corollary of this conclusion involves the quantification of the sensitivity measures such that if the value of W is in the range, $0.73 \leq W \leq 1.37$, then the change in the total variable costs will be less than five per cent. This is true regardless of the model involved and the parameters of the models.

The next conclusion is that any error in the estimation of the parameters directly influence the value of W which in turn influences a change in the total variable cost since it is a function of W. Note that there exists several parameters in each decision variable, and that any error in estimation of any parameter has equal influence on the value of W. However, these errors would, in general, tend to counterbalance each other since errors of the same type in the estimation of the holding cost parameters and procurement cost parameter or the demand parameter influence the value of W in opposite directions or possibly errors of different types could be made in the estimation of the holding cost or demand parameter which would influence the value of W in opposite direction. As a corollary to this conclusion, the sensitivity is greater for under-estimation than for over-estimation of the parameters.

A general conclusion is that if the D/P and C_H/C_S ratios are less than 0.10 in the extreme comparison of models, which involves two properties, then the use of the simple purchase model will result in an increase of total variable costs of less than five per cent.

A specific conclusion is that if the cost of shortage property is involved in models being compared and if the C_H/C_S ratio is less than 0.10, then the increase in total variable cost will be less than five per cent if the model without the cost of shortage property is used.

A specific conclusion is that if only the replenishment property is involved between models being compared, and if the D/P ratio is less than 0.43, then the increase in the total variable cost will be less than five per cent if the model without the replenishment property assumption is used.

A specific conclusion relative to probabilistic inventory models is that unless the power demand index is extreme, $0.8 \leq n \leq 1.2$, the deterministic model with a uniform demand index is a good approximation to the probabilistic model since the increase in the total variable costs will be less than five per cent.

A general conclusion is that there exist many factors which contribute to the possible sources of error such as "wrong" model, errors in estimation of parameters, wrong values of decision variables and they tend to counterbalance each other relative to changes in the "true" total variable costs.

A second general conclusion is that if total constant costs are considered, the percentage error of total inventory costs would be less significant since the total constant costs are independent of the variable costs. Also, in many cases, the total constant costs would be significantly dominant relative to the variable costs which would allow a large error in the total variable costs and result in small percentage error in the total cost.

Recommendations

A general recommendation is that those in control of inventory management decisions use the findings of this research in their decision making process since many inventory control decisions could, in many cases, be made to facilitate scheduling problems and other considerations and not significantly influence the cost of inventory management.

The second recommendation involves the areas of further research. It is recommended that further research be devoted to the Sensitivity Analysis of Probabilistic Inventory Models. Specific topics that could be investigated are:

1. Specific measures of sensitivity could be possibly developed and evaluated for the effect of "wrong" values of decision variables.
2. Specific measures of sensitivity could be possibly developed and evaluated for the effect of "wrong" values of parameters.
3. Investigation could be pursued in developing a measure of sensitivity of the effect of "wrong" probabilistic demand distribution.
4. A measure of sensitivity of the "wrong" model could be investigated.
5. Also, it is conceivable that a measure of sensitivity of control inventory policies such as the fixed-order or fixed cycle systems could be formulated by extending the model formulation of additional properties to include variable

lead time.

6. In essence, measure of sensitivities could be formulated for the effect of the various combinations mentioned above.

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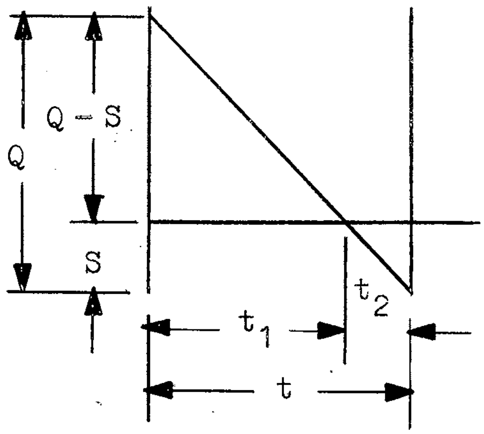
APPENDIX A

DERIVATION OF INVENTORY MODELS

Backorder Model



S - Number of units backordered

C_S- Unit cost of being short per some time period. (\$/unit-time)

TVC - Total Variable Cost

Identities from geometrical relationships:

$$\frac{t_1}{t} = \frac{Q - S}{Q} \qquad\qquad t = \frac{1}{N} = \frac{1}{D/Q} = \frac{Q}{D}$$

$$t_1 = \frac{Q - S}{Q}\, t = \frac{Q - S}{Q}\,\frac{Q}{D} = \frac{Q - S}{D}$$

$$\frac{t_2}{t} = \frac{S}{Q}$$

$$t_2 = \frac{S}{Q}\, t = \frac{S}{Q}\,\frac{Q}{D} = \frac{S}{D}$$

I_{avg} (avg. number of units in inventory during t_1) $= \frac{Q-S}{2}$

S_{avg} (avg. number of units short during t_2) $= \frac{S}{2}$

$$\begin{aligned}
TVC &= \text{Procurement Costs} + \text{Holding Costs} + \text{Shortage Costs} \\
&= [\quad C_P \quad + \quad C_H'\, t_1 \left(\frac{Q-S}{2}\right) + C_S'\, t_2\, \frac{S}{2}\ ]\ N \\
&= \quad C_P N \quad + C_H\, t_1\, N\left(\frac{Q-S}{2}\right) + C_S\, t_2\, N\left(\frac{S}{2}\right) \\
&= C_P\,\frac{D}{Q} + C_H \left(\frac{Q-S}{D}\right)\left(\frac{D}{Q}\right)\left(\frac{Q-S}{2}\right) + C_S \left(\frac{S}{D}\right)\left(\frac{D}{Q}\right)\left(\frac{S}{2}\right) \\
&= C_P\,\frac{D}{Q} + \frac{C_H}{2Q}\,(Q-S)^2 + \frac{C_S}{2Q}\,S^2
\end{aligned}$$

$$TVC = C_P \frac{D}{Q} + \frac{C_H}{2Q} Q^2 - \frac{C_H}{2Q} 2QS + \frac{C_H}{2Q} S^2 + \frac{C_S\ S^2}{2Q}$$

$$= C_P \frac{D}{Q} + \frac{C_H}{2} Q - C_H S + \frac{S^2}{2Q} [C_H + C_S]$$

$$\frac{\partial TVC}{\partial Q} = - \frac{C_P D}{Q^2} + \frac{C_H}{2}\ 0 - \frac{S^2}{2Q^2} [C_H + C_S] \qquad (1)$$

$$\frac{\partial TVC}{\partial S} = 0 + 0 - C_H + \frac{2S}{2Q} [C_H + C_S] \qquad (2)$$

Solve Equation (2) for S

$$\frac{S}{Q} [C_H + C_S] = C_H$$

$$S = \left(\frac{C_H}{C_H + C_S}\right) Q \qquad (3)$$

Solve Equation (1) for Q using value of S from Equation (3)

$$\frac{C_P D}{Q^2} = \frac{C_H}{2} - \frac{S^2}{2Q^2} [C_H + C_S]$$

$$= \frac{C_H}{2} - \frac{\left(\frac{C_H}{C_H + C_S}\right)^2 Q^2}{2Q^2} [C_H + C_S]$$

$$\frac{2C_P D}{Q^2} = C_H - \frac{C_H{}^2}{C_H + C_S}$$

$$\frac{2C_P D}{Q^2} = C_H \left[1 - \frac{C_H}{C_H + C_S}\right]$$

$$\frac{2C_P D}{C_H} = Q^2 \left[1 - \frac{C_H}{C_H + C_S}\right]$$

$$Q^2 = \frac{2C_P D}{C_H} \frac{1}{\left[1 - \frac{C_H}{C_H + C_S}\right]}$$

$$= \frac{2C_P D}{C_H} \frac{1}{\frac{C_H + C_S - C_H}{C_H + C_S}}$$

$$= \frac{2C_P D}{C_H} \frac{C_H + C_S}{C_S}$$

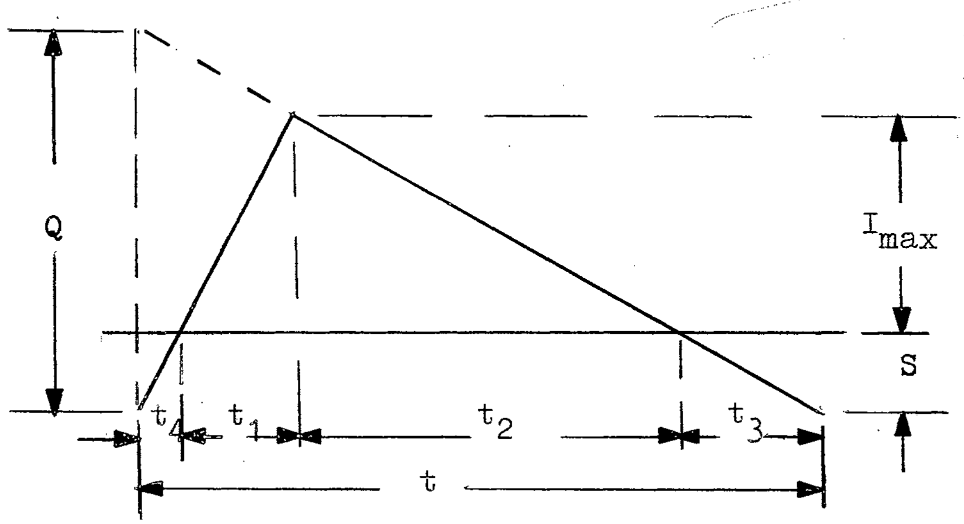
$$Q = \sqrt{\frac{2C_P D}{C_H}} \sqrt{\frac{C_H + C_S}{C_S}}$$

$$S = \frac{C_H}{C_H + C_S} \sqrt{\frac{2C_P D}{C_H}} \sqrt{\frac{C_H + C_S}{C_S}}$$

$$S = \sqrt{\frac{2C_P D}{C_H}} \sqrt{\frac{C_H + C_S}{C_S}} \sqrt{\frac{C_H{}^2}{(C_H + C_S)^2}}$$

$$= \sqrt{\frac{2C_P D}{C_S}} \sqrt{\frac{C_H}{C_H + C_S}}$$

Production With Backorder Model



I_{max} – maximum inventory

S – maximum shortage

D – demand (units/year)

C_P – cost of each procurement

C_H – cost of holding/unit-year

C_S – cost of shortage/unit-year

C_1 – cost of unit

P – production rate/year

<u>Some Identities</u>

$$I_{max} = t_1P - t_1D = t_1(P-D)$$

$$I_{max} = t_2D$$

Hence,

$$t_2D = t_1(P-D)$$

$$S = t_4P - t_4D = t_4(P-D)$$

$$S = t_3D$$

Hence, $t_3D = t_4(P-D)$

Add these two equations

$$(t_2+t_3)D = t_1+t_4(P-D)$$

$$Q = (t_1+t_4)P$$

Thus, $(t_1+t_4) = \frac{Q}{P}$

$$\begin{aligned} I_{max} + S &= (t_2+t_3)D \\ &= (t_1+t_4)\ (P-D) \\ &= \left(\frac{Q}{P}\right)\ (P-D) \\ &= Q\left(\frac{P-D}{P}\right) \\ &= Q\left(1-\frac{D}{P}\right) \end{aligned}$$

$$I_{max} = Q\left(1-\frac{D}{P}\right) - S$$

$$I_{avg} = \frac{Q}{2}\left(1-\frac{D}{P}\right) - \frac{S}{2}$$

$$TVC = [\underbrace{HC}_{\text{Holding Costs}} + \underbrace{PC}_{\text{Procurement Costs}} + \underbrace{SC}_{\text{Shortage Costs}}] \underbrace{N}_{\text{Number of Orders/Yr.}}$$

$$PC = C_P \qquad = \$/\text{procurement}$$

$$HC = \underbrace{(\text{Avg. Number on Hand})}_{I_{avg}}\underbrace{(\text{Time on Hand})}_{t_1 + t_2}\underbrace{(\text{Cost of Holding Unit})}_{C_H}$$

$$I_{avg} = \frac{Q}{2}\left(1 - \frac{D}{P}\right) - \frac{S}{2} \quad \text{(from above identities)}$$

$$\text{Time on Hand} = t_1 + t_2 \qquad \text{Note: } I_{max} = t_1(P-D)$$

$$= \frac{I_{max}}{P-D} + \frac{I_{max}}{D} \qquad \text{Hence } t_1 = \frac{I_{max}}{P-D}$$

$$= I_{max}\left[\frac{1}{P-D} + \frac{1}{D}\right] \quad \text{Note } I_{max} = t_2 D$$

$$= I_{max}\left[\frac{1}{D}\,\frac{1}{1-\frac{D}{P}}\right] \quad \text{Hence, } t_2 = \frac{I_{max}}{D}$$

$$HC = C_H\ (\text{Avg. Number on Hand})(\text{Time on Hand})$$

$$= C_H\left(\frac{I_{max}}{2}\right)(t_1 + t_2)$$

$$= \frac{C_H}{2}\left[Q\left(1-\frac{D}{P}\right) - S\right]\left[I_{max}\left(\frac{1}{D}\ \frac{1}{1-\frac{D}{P}}\right)\right]$$

$$= \frac{C_H}{2}\left[Q\left(1-\frac{D}{P}\right) - S\right]\left\{\left[Q\left(1-\frac{D}{P}\right) - S\right]\left[\left(\frac{1}{D}\ \frac{1}{1-\frac{D}{P}}\right)\right]\right\}$$

$$= \frac{C_H}{2}\left[Q\left(1-\frac{D}{P}\right) - S\right]^2\left[\frac{1}{D}\,\frac{1}{1-\frac{D}{P}}\right]$$

$$SC = (\text{Cost of Shortage})(\text{Avg. Number of Units Short})(\text{Time Short})$$

$$= C_S\ (\text{Avg. Number of Units Short})(\text{Time Short})$$

$$\text{Avg. Number of Units Short} = \frac{S}{2}$$

Time Short $= t_3 + t_4$ Note: $S = t_3 D$

$$= \frac{S}{D} + \frac{S}{P-D} \qquad t_3 = \frac{S}{D}$$

$$= S\left(\frac{1}{D} + \frac{1}{P-D}\right) \qquad S = t_4\ (P-D)$$

$$= S\left[\frac{1}{D}\left(\frac{1}{1-\frac{D}{P}}\right)\right] \qquad t_4 = \frac{S}{P-D}$$

$$SC = (C_S)\left(\frac{S}{2}\right)\left\{S\left[\frac{1}{D}\left(\frac{1}{1-\frac{D}{P}}\right)\right]\right\}$$

$$= \frac{C_S}{2}\ S^2\ \frac{1}{D}\left(\frac{1}{1-\frac{D}{P}}\right)$$

$$TVC = \left[PC + HC + SC\right] N$$

$$= \left\{C_P + \frac{C_H}{2}\left[Q\left(1-\frac{D}{P}\right) - S\right]^2 \left[\frac{1}{D}\left(\frac{1}{1-\frac{D}{P}}\right)\right] + \frac{C_S}{2} S^2 \left[\frac{1}{D}\ \frac{1}{1-\frac{D}{P}}\right]\right\} N$$

$$= C_P \frac{D}{Q} + \frac{C_H}{2}\left[Q^2\left(1-\frac{D}{P}\right)^2 - 2SQ\left(1-\frac{D}{P}\right) + S^2\right]\left[\frac{1}{D}\left(\frac{1}{1-\frac{D}{P}}\right)\right]\frac{D}{Q}$$

$$+ \frac{C_S}{2}\ S^2\left[\frac{1}{D}\left(\frac{1}{1-\frac{D}{P}}\right)\right]\frac{D}{Q}$$

$$TVC = \frac{C_P D}{Q} + \frac{C_H}{2}\left[Q\left(1-\frac{D}{P}\right) - 2S + \frac{S^2}{Q\left(1-\frac{D}{P}\right)}\right] + \frac{C_S}{2}\ \frac{S^2}{Q\left(1-\frac{D}{P}\right)}$$

$$\frac{\partial TVC}{\partial S} = 0 + 0 - \frac{2C_H}{2} + \frac{2S\ C_H}{2Q\left(1-\frac{D}{P}\right)} + \frac{C_S}{2}\ \frac{2S}{Q\left(1-\frac{D}{P}\right)}$$

$$= C_H + \frac{S\ C_H}{Q\left(1-\frac{D}{P}\right)} + \frac{C_S\ S}{Q\left(1-\frac{D}{P}\right)} = 0$$

$$S\left[\frac{C_H}{Q\left(1-\frac{D}{P}\right)} + \frac{C_S}{Q\left(1-\frac{D}{P}\right)}\right] = C_H$$

$$\frac{S}{Q\left(1-\frac{D}{P}\right)}\left[C_H + C_S\right] = C_H$$

$$S = \frac{C_H}{C_H + C_S} \quad Q\left(1 - \frac{D}{P}\right)$$

$$\frac{\partial TVC}{\partial Q} = \frac{C_P D}{Q^2} + \frac{C_H}{2}\left(1 - \frac{D}{P}\right) - 0 - \frac{C_H S^2}{2Q^2\left(1 - \frac{D}{P}\right)} - \frac{C_S S^2}{2Q^2\left(1 - \frac{D}{P}\right)}$$

$$\frac{C_P D}{Q^2} + \frac{C_H S^2}{2Q^2\left(1 - \frac{D}{P}\right)} + \frac{C_S S^2}{2Q^2\left(1 - \frac{D}{P}\right)} - \frac{C_H}{2}\left(1 - \frac{D}{P}\right) = 0$$

$$\frac{1}{Q^2}\left[C_P D + \frac{C_H S^2}{2\left(1 - \frac{D}{P}\right)} + \frac{C_S S^2}{2\left(1 - \frac{D}{P}\right)}\right] = \frac{C_H}{2}\left(1 - \frac{D}{P}\right)$$

$$\frac{C_P D + \frac{S^2}{2\left(1 - \frac{D}{P}\right)} \quad (C_H + C_S)}{\frac{C_H}{2}\left(1 - \frac{D}{P}\right)} = Q^2$$

$$Q^2 = \frac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)} + \frac{S^2}{2\left(1 - \frac{D}{P}\right)} (C_H + C_S)\ \frac{2}{C_H}\left(\frac{1}{1 - \frac{D}{P}}\right)$$

$$= \frac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)} + \frac{S^2}{\left(1 - \frac{D}{P}\right)^2} \quad \frac{C_H + C_S}{C_H}$$

$$= \left[\frac{2C_P D}{C_H} + \frac{S^2}{1 - \frac{D}{P}}\ \frac{C_H + C_S}{C_H}\right]\left(\frac{1}{1 - \frac{D}{P}}\right)$$

$$= \left\{\frac{2C_P D}{C_H} + \frac{\left[\frac{C_H}{C_H + C_S}\ Q\left(1 - \frac{D}{P}\right)\right]^2}{1 - \frac{D}{P}}\ \frac{C_H + C_S}{C_H}\right\}\left(\frac{1}{1 - \frac{D}{P}}\right)$$

$$= \left[\frac{2C_P D}{C_H} + \frac{\left(\frac{C_H}{C_H + C_S}\right)^2 Q^2\left(1 - \frac{D}{P}\right)^2}{1 - \frac{D}{P}}\left(\frac{C_H + C_S}{C_H}\right)\right]\left(\frac{1}{1 - \frac{D}{P}}\right)$$

$$= \frac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)} + \left(\frac{C_H}{C_H + C_S}\right)Q^2$$

$$Q^2 = \quad 1 - \frac{C_H}{C_H + C_S} = \frac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)}$$

$$Q^2 = \frac{\dfrac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)}}{1 - \dfrac{C_H}{C_H + C_S}} = \frac{\dfrac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)}}{\dfrac{C_H + C_S - C_H}{C_H + C_S}} = \frac{\dfrac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)}}{\dfrac{C_S}{C_H + C_S}}$$

$$Q^2 = \frac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)} \quad \frac{C_H + C_S}{C_S}$$

$$Q = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}}\sqrt{\frac{1}{1 - \frac{D}{P}}} = \sqrt{\frac{2C_P D}{C_H\left(1 - \frac{D}{P}\right)}}\sqrt{\frac{C_H + C_S}{C_S}}$$

$$= \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}}\sqrt{\frac{P}{P - D}}$$

Given Q

$$S = \sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}}\sqrt{\frac{1}{1 - \frac{D}{P}}}\sqrt{\left(1 - \frac{D}{P}\right)^2}\sqrt{\frac{C_H^{\,2}}{\left(C_H + C_S\right)^2}}$$

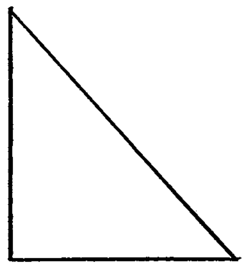
$$S = \sqrt{\frac{2C_P D}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}\sqrt{1 - \frac{D}{P}} = \sqrt{\frac{2C_P D}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}\sqrt{\frac{P - D}{P}}$$

APPENDIX B

DERIVATION OF MEASURES OF SENSITIVITY OF DECISION VARIABLES

Sensitivity Analysis of Decision Variable (Q)

Purchase Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}} \qquad Q' = WQ_o$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H}{2} Q_o$$

Measure of Sensitivity:

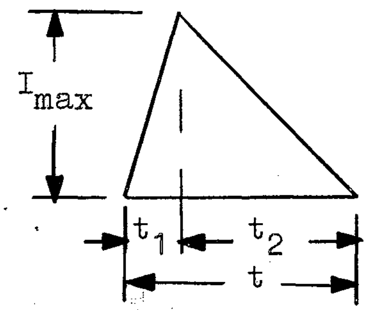
$$\frac{TVC'}{TVC_o} = \frac{\dfrac{C_P D}{Q'} + \dfrac{Q' C_H}{2}}{\dfrac{C_P D}{Q_o} + \dfrac{Q_o C_H}{2}} = \frac{\dfrac{C_P D}{WQ_o} + \dfrac{WQ_o C_H}{2}}{\dfrac{C_P D}{Q_o} + \dfrac{Q_o C_H}{2}} = \frac{\dfrac{C_P D}{W\sqrt{\dfrac{2C_P D}{C_H}}} + W\dfrac{C_H}{2}\sqrt{\dfrac{2C_P D}{C_H}}}{\dfrac{C_P D}{\sqrt{\dfrac{2C_P D}{C_H}}} + \dfrac{C_H}{2}\sqrt{\dfrac{2C_P D}{C_H}}}$$

$$= \frac{\dfrac{1}{W}\sqrt{\dfrac{C_P{}^2 D^2\ C_H}{2C_P D}} + W\sqrt{\dfrac{2C_P D\ C_H{}^2}{4\ C_H}}}{\sqrt{\dfrac{C_P{}^2 D^2\ C_H}{2C_P D}} + \sqrt{\dfrac{2C_P D\ C_H{}^2}{4\ C_H}}} = \frac{\dfrac{1}{W}\sqrt{\dfrac{C_P D\ C_H}{2}} + W\sqrt{\dfrac{C_P D\ C_H}{2}}}{\sqrt{\dfrac{C_P D\ C_H}{2}} + \sqrt{\dfrac{C_P D\ C_H}{2}}}$$

$$= \frac{\sqrt{\dfrac{C_P D\ C_H}{2}}\left[\dfrac{1}{W} + W\right]}{\sqrt{\dfrac{C_P D\ C_H}{2}}\ [1 + 1]} = \frac{\dfrac{1}{W} + W}{2}$$

$$= \frac{1 + W^2}{2W}$$

Production Model



$$I_{max} = Q_o\left(1 - \frac{D}{P}\right) = Q_o\left(\frac{P - D}{P}\right)$$

$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{1}{1 - \frac{D}{P}}} = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P - D}}$$

$$Q' = WQ_o$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_H}{2}\,Q_o\left(\frac{P - D}{P}\right)$$

Measure of Sensitivity:

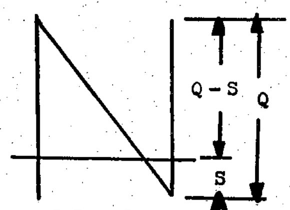
$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_H}{2}\,Q'\left(\frac{P-D}{P}\right)}{\frac{C_PD}{Q_o} + \frac{C_H}{2}\,Q_o\left(\frac{P-D}{P}\right)} = \frac{\frac{C_PD}{W\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}}\,W + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}\left(\frac{P-D}{P}\right)}{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_HC_P{}^2D^2}{2C_PD}}\sqrt{\frac{P-D}{P}} + W\sqrt{\frac{C_H{}^2}{4}\frac{2C_PD}{C_H}}\sqrt{\left(\frac{P}{P-D}\right)\left(\frac{(P-D)^2}{P^2}\right)}}{\sqrt{\frac{C_HC_P{}^2D^2}{2C_PD}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_H{}^2}{4}\frac{2C_PD}{C_H}}\sqrt{\left(\frac{(P-D)^2}{P^2}\right)\left(\frac{P}{P-D}\right)}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_HC_PD}{2}}\sqrt{\frac{P-D}{P}} + W\sqrt{\frac{C_HC_PD}{2}}\sqrt{\frac{P-D}{P}}}{\sqrt{\frac{C_HC_PD}{2}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_HC_PD}{2}}\sqrt{\frac{P-D}{P}}} = \frac{\sqrt{\frac{C_HC_PD}{2}}\sqrt{\frac{P-D}{P}}\left[\frac{1}{W} + W\right]}{\sqrt{\frac{C_HC_PD}{2}}\sqrt{\frac{P-D}{P}}\,[1 + 1]}$$

$$= \frac{\frac{1}{W} + W}{2} = \frac{1}{2W} + \frac{W}{2} = \frac{1}{2}\left(\frac{1}{W} + W\right) = \frac{1 + W^2}{2W}$$

Backorder Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} \qquad Q' = WQ_o$$

$$S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} \qquad S' = WS_o$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2} - C_HS_o + \frac{S_o^2\,[C_H+C_S]}{2Q_o}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\dfrac{C_PD}{Q'} + \dfrac{C_HQ'}{2} - C_HS' + \dfrac{S'^2\,[C_H+C_S]}{2Q'}}{\dfrac{C_PD}{Q_o} + \dfrac{C_HQ_o}{2} - C_HS_o + \dfrac{S_o^2\,[C_H+C_S]}{2Q_o}} = \frac{\dfrac{C_PD}{WQ_o} + \dfrac{C_HWQ_o}{2} - C_HWS_o + \dfrac{W^2S_o^2\,[C_H+C_S]}{2WQ_o}}{\dfrac{C_PD}{Q_o} + \dfrac{C_HQ_o}{2} - C_HS_o + \dfrac{S_o^2\,[C_H+C_S]}{2Q_o}}$$

$$= \frac{\dfrac{C_PD}{W\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}} + W\dfrac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - W\,C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} + \dfrac{W^2\sqrt{\frac{2^2C_P^2D^2}{C_S^2}}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\,[C_H+C_S]}{2W\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}}}{\dfrac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}} + \dfrac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} + \dfrac{\sqrt{\frac{2^2C_P^2D^2}{C_S^2}}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\,[C_H+C_S]}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}} + W\sqrt{\frac{2C_PD}{C_H}\frac{C_H^2}{4}}\sqrt{\frac{C_H+C_S}{C_S}} - W\sqrt{\frac{2C_PD\;C_H^2}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} + \frac{W^2}{2W}\sqrt{\frac{2^2C_P^2D^2C_H^2(C_H+C_S)^2\;C_H\;C_S}{C_S^2\;(C_H+C_S)^2\;2C_PD\;(C_H+C_S)}}}{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{2C_PD}{C_H}\frac{C_H^2}{4}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{2C_PD\;C_H^2}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} + \frac{1}{2}\sqrt{\frac{2^2C_P^2D^2\;C_H^2\;(C_H+C_S)^2\;C_H\;C_S}{C_S^2\;(C_H+C_S)^2\;2C_PD\;(C_H+C_S)}}}$$

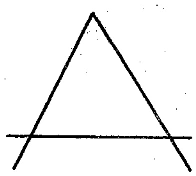
$$= \frac{\frac{1}{W}\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + W\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - W\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}} + \frac{W}{2}\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + W\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \frac{W}{2}\sqrt{\frac{2C_PD\;C_H}{4}}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{2C_PD\;C_H}{4}}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\left\{\frac{1}{W}\sqrt{\frac{C_S}{C_H+C_S}} + W\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}\right\}}{\sqrt{\frac{C_PD\;C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \underbrace{\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}}_{\sqrt{\frac{C_S}{C_H+C_S}}}\right\}} = \frac{\frac{1}{W}\sqrt{\frac{C_S}{C_H+C_S}} + W\sqrt{\frac{C_S}{C_H+C_S}}}{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_S}{C_H+C_S}}} = \frac{\sqrt{\frac{C_S}{C_H+C_S}}\left\{\frac{1}{W} + W\right\}}{\sqrt{\frac{C_S}{C_H+C_S}}\left\{1 + 1\right\}}$$

$$= \frac{\frac{1}{W} + W}{2} = \frac{1}{2W} + \frac{W}{2} = \frac{1 + W^2}{2W}$$

Production With Backorder Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} \qquad Q' = WQ_o$$

$$S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}} \qquad S' = WS_o$$

$$TVC = \frac{C_PD}{Q} + \frac{C_H}{2}Q\frac{P-D}{P} - C_HS + \frac{S^2C_H}{2Q\frac{P-D}{P}} + \frac{C_SS^2}{2Q\frac{P-D}{P}} = \frac{C_PD}{Q} + \frac{C_H}{2}Q\frac{P-D}{P} - C_HS + \frac{S^2(C_H+C_S)}{2Q\frac{P-D}{P}}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_H}{2}Q'\frac{P-D}{P} - C_HS' + \frac{(S')^2(C_H+C_S)}{2Q'\frac{P-D}{P}}}{\frac{C_PD}{Q_o} + \frac{C_H}{2}Q_o\frac{P-D}{P} - C_HS_o + \frac{(S_o)^2(C_H+C_S)}{2Q_o\frac{P-D}{P}}} = \frac{\frac{C_PD}{WQ_o} + \frac{C_H}{2}WQ_o\frac{P-D}{P} - C_HWS_o + \frac{(WS_o)^2(C_H+C_S)}{2WQ_o\frac{P-D}{P}}}{\frac{C_PD}{Q_o} + \frac{C_H}{2}Q_o\frac{P-D}{P} - C_HS_o + \frac{S_o^2(C_H+C_S)}{2Q_o\frac{P-D}{P}}}$$

$$= \frac{\frac{C_PD}{W\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}W\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} - C_HW\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \frac{W\sqrt{\frac{2^2C_P^2D^2}{C_S^2}}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\sqrt{\frac{(P-D)^2}{P^2}}(C_H+C_S)}{2W\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}\frac{P-D}{P}}}{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \frac{\sqrt{\frac{2^2C_P^2D^2}{C_S^2}}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\sqrt{\frac{(P-D)^2}{P^2}}(C_H+C_S)}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}\frac{P-D}{P}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + W\sqrt{\frac{2C_PD\;C_H}{4\;C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}\frac{(P-D)^2}{P^2}} - W\sqrt{\frac{2C_PD\;C_H^2}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}}{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD\;C_H}{4\;C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}\frac{(P-D)^2}{P^2}} - \sqrt{\frac{2C_PD\;C_H^2}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}}$$

$$\frac{+\frac{W}{2}\sqrt{\frac{2^2C_P^2D^2}{C_S^2}\frac{C_H}{2C_PD}\frac{C_H^2}{(C_H+C_S)^2}\frac{C_S}{C_H+C_S}\frac{(C_H+C_S)^2(P-D)^2}{P^2}\frac{P-D}{P}\frac{P^2}{(P-D)^2}}}{+\frac{1}{2}\sqrt{\frac{2^2C_P^2D^2}{C_S^2}\frac{C_H}{2C_PD}\frac{C_H^2}{(C_H+C_S)^2}\frac{C_S}{C_H+C_S}\frac{(C_H+C_S)^2(P-D)^2}{P^2}\frac{(P-D)}{P}\frac{P^2}{(P-D)^2}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + W\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - W\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}\sqrt{\frac{C_H}{C_S}} + \frac{W}{2}\sqrt{\frac{2C_PD\;C_H}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}} + \frac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + W\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - W\sqrt{\frac{2C_PD\;C_H}{4}}\sqrt{\frac{C_H}{C_H+C_S}\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{\frac{2C_PD\;C_H}{4}}\sqrt{\frac{C_H}{C_H+C_S}\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{P-D}{P}}\left\{\frac{1}{W}\sqrt{\frac{C_S}{C_H+C_S}} + W\sqrt{\frac{C_H+C_S}{C_S}} - W\sqrt{\frac{C_H}{C_H+C_S}\frac{C_H}{C_S}}\right\}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{P-D}{P}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}\frac{C_H}{C_S}}\right\}} = \frac{\frac{1}{W}\sqrt{\frac{C_S}{C_H+C_S}} + W\sqrt{\frac{C_H+C_S}{C_S}} - W\sqrt{\frac{C_H}{C_H+C_S}\frac{C_H}{C_S}}}{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}\frac{C_H}{C_S}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_S}{C_H+C_S}} + W\sqrt{\frac{C_S}{C_H+C_S}}}{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_S}{C_H+C_S}}} = \frac{\sqrt{\frac{C_S}{C_H+C_S}}\left\{\frac{1}{W} + W\right\}}{\sqrt{\frac{C_S}{C_H+C_S}}\left\{1 + 1\right\}} = \frac{\frac{1}{W} + W}{2} = \frac{1 + W^2}{2W}$$

Restricted Variables Model

Two Types:

1) Restrictions on available

a) Space
b) Money

} Modifies C_H

2) Restrictions on available

a) Procure Time
b) Procurement Cost
c) Manufacturing Capacity

} Modifies C_P

1st Case - Restriction on Space

S — maximum space available

U_S — unit space requirement (i.e.-space req'd/unit)

$S - U_S Q \leq 0$ — However for simplicity will use

$S - U_S \frac{Q}{2} \leq 0$

If: $S - U_S Q \leq 0$ is used, then

Need $Q = \sqrt{\frac{C_P D}{C_H}}$ where C_H is function of Q instead of $\frac{Q}{2}$

TVC = PC + HC + Restricted space costs

Where restriction space costs is a restriction on space embedded in the TVC function with a LaGrange multiplier.

$$TVC = C_P \frac{D}{Q} + C_H \frac{Q}{2} + \lambda\left(S - U_S \frac{Q}{2}\right)$$

$$\frac{\partial TVC}{\partial Q} = C_P \frac{D}{Q} + C_H \frac{Q}{2} + \lambda S - \lambda U_S \frac{Q}{2}$$

$$= -\frac{C_P D}{Q^2} + \frac{C_H}{2} + 0 - \frac{\lambda U_S}{2}$$

$$Q_o = \sqrt{\frac{2C_P D}{C_H - \lambda U_S}} \qquad Q' = WQ_o$$

$$TVC_o = C_P \frac{D}{Q} + C_H \frac{Q_o}{2} - \lambda U_S \frac{Q_o}{2}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_P D}{Q'} + \frac{C_H Q'}{2} - \lambda U_S \frac{Q'}{2}}{\frac{C_P D}{Q_o} + \frac{C_H Q_o}{2} - \lambda U_S \frac{Q_o}{2}} = \frac{\frac{C_P D}{WQ_o} + \frac{C_H W Q_o}{2} - \frac{\lambda U_S W Q_o}{2}}{\frac{C_P D}{Q_o} + \frac{C_H Q_o}{2} - \frac{\lambda U_S Q_o}{2}}$$

$$= \frac{\frac{1}{W}\frac{C_P D}{\sqrt{\frac{2C_P D}{C_H - \lambda U_S}}} + \frac{WC_H}{2}\sqrt{\frac{2C_P D}{C_H - \lambda U_S}} - \frac{W\lambda U_S}{2}\sqrt{\frac{2C_P D}{C_H - \lambda U_S}}}{\frac{C_P D}{\sqrt{\frac{2C_P D}{C_H - \lambda U_S}}} + \frac{C_H}{2}\sqrt{\frac{2C_P D}{C_H - \lambda U_S}} - \frac{\lambda U_S}{2}\sqrt{\frac{2C_P D}{C_H - \lambda U_S}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_P{}^2 D^2 (C_H - \lambda U_S)}{2C_P D}} + W\sqrt{\frac{2C_P D\ C_H{}^2}{4\,(C_H - \lambda U_S)}} - W\sqrt{\frac{2C_P D\ \lambda^2 U_S{}^2}{4\,(C_H - \lambda U_S)}}}{\sqrt{\frac{C_P{}^2 D^2 (C_H - \lambda U_S)}{2C_P D}} + \sqrt{\frac{2C_P D\ C_H{}^2}{4(C_H - \lambda U_S)}} - \sqrt{\frac{2C_P D\ \lambda^2 U_S{}^2}{4\,(C_H - \lambda U_S)}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_P D}{2}}\sqrt{C_H - \lambda U_S} + W\sqrt{\frac{C_P D}{2}}\sqrt{\frac{C_H{}^2}{C_H - \lambda U_S}} - W\sqrt{\frac{C_P D}{2}}\sqrt{\frac{\lambda^2 U_S{}^2}{C_H - \lambda U_S}}}{\sqrt{\frac{C_P D}{2}}\sqrt{C_H - \lambda U_S} + \sqrt{\frac{C_P D}{2}}\sqrt{\frac{C_H{}^2}{C_H - \lambda U_S}} - W\sqrt{\frac{C_P D}{2}}\sqrt{\frac{\lambda^2 U_S{}^2}{C_H - \lambda U_S}}}$$

$$= \frac{\sqrt{\frac{C_P D}{2}}\left(\frac{1}{W}\sqrt{C_H - \lambda U_S} + W\left[\sqrt{\frac{C_H{}^2}{C_H - \lambda U_S}} - \sqrt{\frac{\lambda^2 U_S{}^2}{C_H - \lambda U_S}}\right]\right)}{\sqrt{\frac{C_P D}{2}}\left(\sqrt{C_H - \lambda U_S} + \left[\sqrt{\frac{C_H{}^2}{C_H - \lambda U_S}} - \sqrt{\frac{\lambda^2 U_S{}^2}{C_H - \lambda U_S}}\right]\right)}$$

Where:

$$\sqrt{\frac{C_H{}^2}{C_H - \lambda U_S}} - \sqrt{\frac{\lambda^2 U_S{}^2}{C_H - \lambda U_S}} = \sqrt{C_H - \lambda U_S}$$

Square both sides:

$$\frac{C_H^{\ 2}}{C_H - \lambda U_S} - 2\sqrt{\frac{C_H^{\ 2}\ \lambda^2 U_S^{\ 2}}{(C_H - \lambda U_S)^2}} + \frac{\lambda^2 U_S^{\ 2}}{C_H - \lambda U_S} = C_H - \lambda U_S$$

$$\frac{C_H^{\ 2}}{C_H - \lambda U_S} - 2\ \frac{C_H \lambda U_S}{C_H - \lambda U_S} + \frac{\lambda^2 U_S^{\ 2}}{C_H - \lambda U_S} = C_H - \lambda U_S$$

$$\frac{C_H^{\ 2} - 2\ C_H \lambda U_S + \lambda^2\ U_S^{\ 2}}{C_H - \lambda U_S} = C_H - \lambda U_S$$

$$\frac{(C_H - \lambda U_S)^2}{C_H - \lambda U_S} = C_H - \lambda U_S$$

$$\sqrt{C_H - \lambda U_S} = \sqrt{C_H - \lambda U_S}$$

Hence:

$$= \frac{\frac{1}{W}\sqrt{C_H - \lambda U_S} + W\sqrt{C_H - \lambda U_S}}{\sqrt{C_H - \lambda U_S} + \sqrt{C_H - \lambda U_S}}$$

$$= \frac{\sqrt{C_H - \lambda U_S}\ \left[\frac{1}{W} + W\right]}{\sqrt{C_H - \lambda U_S}\ [1 + 1]}$$

$$= \frac{\frac{1}{W} + W}{2} = \frac{1}{2W} + \frac{W}{2}$$

$$= \frac{1 + W^2}{2W}$$

General Inventory Cost Model

$$TVC = C_P \frac{D}{Q} + C_H \frac{Q}{2} = \underbrace{\frac{A}{Q}}_{\substack{\text{Hyperbolic}\\ \text{Term}}} + \underbrace{BQ}_{\substack{\text{Linear}\\ \text{Term}}}$$

Where: $\left.\begin{array}{l} A = C_P\ D \\ B = C_H/2 \end{array}\right\}$ parameters

Q = Decision Variable

$Q_o = \sqrt{\frac{A}{B}}$ — Optimum value of decision variable (Q) $Q' = WQ_o$

Measure of Sensitivity:

$$\frac{TVC(Q')}{TVC(Q_O)} = \frac{\frac{A}{Q'} + BQ'}{\frac{A}{Q_o} + BQ_o}$$

$$= \frac{\frac{A}{WQ_o} + BWQ_o}{\frac{A}{Q_o} + BQ_o}$$

$$= \frac{\frac{A}{W\sqrt{\frac{A}{B}}} + BW\sqrt{\frac{A}{B}}}{\frac{A}{\sqrt{\frac{A}{B}}} + B\sqrt{\frac{A}{B}}}$$

$$= \frac{\frac{1}{W}\sqrt{AB} + W\sqrt{AB}}{\sqrt{AB} + \sqrt{AB}}$$

$$= \frac{\sqrt{AB}\left[\frac{1}{W} + W\right]}{\sqrt{AB}\ [1 + 1]}$$

$$= \frac{\frac{1}{W} + W}{2} = \frac{1 + W^2}{2W}$$

Sensitivity Analysis of Decision Variable (t)

Purchase Model

$$t_o = \frac{1}{N} = \frac{1}{\frac{D}{Q}} = \frac{\sqrt{\frac{2C_PD}{C_H}}}{D} = \sqrt{\frac{2C_PD}{C_HD^2}} = \sqrt{\frac{2C_P}{C_HD}} \qquad t' = Wt_o \ , \ W = \frac{t'}{t_o}$$

$$TVC_o = \frac{C_PD}{tD} + \frac{C_H}{2}t_oD = \frac{C_P}{t_o} + \frac{C_H}{2}t_oD$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_P}{t'} + \frac{C_H}{2}Dt'}{\frac{C_P}{t_o} + \frac{C_H}{2}Dt_o} = \frac{\frac{C_P}{Wt_o} + \frac{C_H}{2}DWt_o}{\frac{C_P}{t_o} + \frac{C_H}{2}Dt_o}$$

$$= \frac{\frac{C_P}{W\sqrt{\frac{2C_P}{C_HD}}} + \frac{C_H}{2}DW\sqrt{\frac{2C_P}{C_HD}}}{\frac{C_P}{\sqrt{\frac{2C_P}{C_HD}}} + \frac{C_H}{2}D\sqrt{\frac{2C_P}{C_HD}}} = \frac{\frac{1}{W}\sqrt{\frac{C_P^2C_HD}{2C_P}} + W\sqrt{\frac{2C_PC_H^2D^2}{4\ C_H\ D}}}{\sqrt{\frac{C_H\ D\ C_P^2}{2C_P}} + \sqrt{\frac{2C_P\ C_H^2\ D^2}{4\ C_H\ D}}}$$

$$= \frac{\frac{1}{W}\sqrt{\frac{C_P^2\ C_HD}{2}} + W\sqrt{\frac{C_P\ C_H\ D}{2}}}{\sqrt{\frac{C_H\ D\ C_P}{2}} + \sqrt{\frac{C_P\ C_H\ D}{2}}} = \frac{\sqrt{\frac{C_P\ C_H\ D}{2}}\left[\frac{1}{W} + W\right]}{\sqrt{\frac{C_P\ C_H\ D}{2}}\ [2]}$$

$$= \frac{1 + W^2}{2W}$$

Sensitivity Analysis of Decision Variable (N)

Purchase Model

$$N_o = \frac{D}{Q} = \frac{D}{\sqrt{\frac{2C_P D}{C_H}}} = \sqrt{\frac{C_H D^2}{2C_P D}} = \sqrt{\frac{C_H D}{2C_P}} \qquad N' = WN_o \qquad W = \frac{N'}{N_o}$$

$$TVC_o = C_P N_o + \frac{C_H}{2}\frac{D}{N}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{C_P N' + \frac{C_H}{2}\frac{D}{N'}}{C_P N_o + \frac{C_H}{2}\frac{D}{N_o}} = \frac{C_P W N_o + \frac{C_H}{2}\frac{D}{WN_o}}{C_P N_o + \frac{C_H}{2}\frac{D}{N_o}}$$

$$= \frac{W C_P\sqrt{\frac{C_H D}{2C_P}} + \frac{1}{W}\frac{C_H D}{2}\frac{1}{\sqrt{\frac{C_H D}{2C_P}}}}{C_P\sqrt{\frac{C_H D}{2C_P}} + \frac{C_H D}{2}\frac{1}{\sqrt{\frac{C_H D}{2C_P}}}} = \frac{W\sqrt{\frac{C_H D C_P^2}{2C_P}} + \frac{1}{W}\sqrt{\frac{2C_P C_H^2 D^2}{C_H D 4}}}{\sqrt{\frac{C_H D C_P^2}{2C_P}} + \sqrt{\frac{2C_P C_H^2 D^2}{4 C_H D}}}$$

$$= \frac{W\sqrt{\frac{C_P D C_H}{2}} + \frac{1}{W}\sqrt{\frac{C_P D C_H}{2}}}{\sqrt{\frac{C_P D C_H}{2}} + \sqrt{\frac{C_P D C_H}{2}}} = \frac{\sqrt{\frac{C_P D C_H}{2}}\left[W + \frac{1}{W}\right]}{\sqrt{\frac{C_P D C_H}{2}}\ [2]}$$

$$= \frac{1 + W^2}{2W}$$

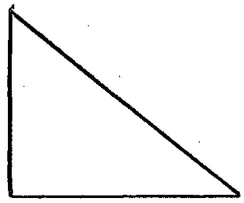
APPENDIX C

DERIVATION OF MEASURES OF SENSITIVITY OF USING "WRONG" MODELS

Measures of Sensitivity Based on the "Actual" Model's Total Variable Cost Function

Purchase Model Versus Production Model

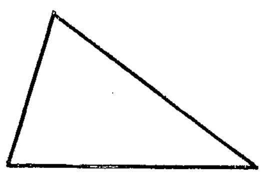
Actual Model



$$Q' = \sqrt{\frac{2C_P D}{C_H}}$$

$$TVC' = \frac{C_P D}{Q'} + \frac{C_H}{2} Q'$$

True Model



$$Q_o = \sqrt{\frac{2C_P D}{C_H}} \sqrt{\frac{P}{P - D}}$$

$$TVC_o = \frac{C_P D}{Q_o} + \frac{C_H}{2} Q_o \left(\frac{P - D}{P} \right)$$

Measure of Sensitivity:

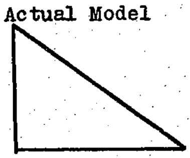
$$\frac{TVC'}{TVC_o} = \frac{\frac{C_P D}{Q'} + \frac{C_H}{2} Q'}{\frac{C_P D}{Q_o} + \frac{C_H}{2} Q_o \left(\frac{P - D}{P}\right)} = \frac{\frac{C_P D}{\sqrt{\frac{2C_P D}{C_H}}} + \frac{C_H}{2}\sqrt{\frac{2C_P D}{C_H}}}{\frac{C_P D}{\sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{P}{P - D}}} + \frac{C_H}{2}\sqrt{\frac{2C_P D}{C_H}}\sqrt{\frac{P}{P-D}}\sqrt{\frac{(P-D)^2}{P^2}}}$$

$$= \frac{\sqrt{\frac{C_P^{\,2}\, D^2\, C_H}{2C_P D}} + \sqrt{\frac{2C_P D\, C_H^{\,2}}{4C_H}}}{\sqrt{\frac{C_P^{\,2} D^2 C_H}{2C_P D}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_P D\, C_H^{\,2}}{4C_H}}\sqrt{\frac{P-D}{P}}} = \frac{\sqrt{\frac{C_P D\, C_H}{2}} + \sqrt{\frac{C_P D\, C_H}{2}}}{\sqrt{\frac{C_P D\, C_H}{2}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_P D\, C_H}{2}}\sqrt{\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_P D\, C_H}{2}}\,[\,1 + 1]}{\sqrt{\frac{C_P D\, C_H}{2}}\sqrt{\frac{P - D}{P}}\,[\,1 + 1]} = \frac{2}{\sqrt{\frac{P - D}{P}}\,[2]} = \sqrt{\frac{P}{P - D}}$$

Purchase Model Versus Backorder Model

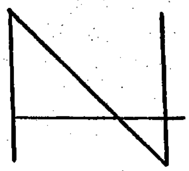
Actual Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}} \qquad S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2} - C_HS_o + \frac{S_o^2\,[C_H + C_S]}{2Q_o}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_HQ'}{2}}{\frac{C_PD}{Q_o} + \frac{C_HQ_o}{2} - C_HS_o + \frac{S_o^2\,[C_H + C_S]}{2Q_o}}$$

$$= \frac{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}}{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} + \frac{\sqrt{\frac{2^2C_P^2D^2}{C_S^2}}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\sqrt{(C_H+C_S)^2}}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}}}$$

$$= \frac{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}} + \sqrt{\frac{2C_PD\;C_H^2}{4C_H}}}{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{2C_PD\;C_H^2}{4C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{\frac{2^2C_P^2D^2C_H}{C_S^2\;2C_PD}}\sqrt{\frac{C_H^2C_S}{C_H+C_S}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}} + \sqrt{\frac{C_PD\;C_H}{2}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}}$$

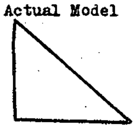
$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\,[1 + 1]}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{2C_PD\;C_H}{4C_H}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\,[2]}{\sqrt{\frac{C_PD\;C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \underbrace{\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}}_{\sqrt{\frac{C_S}{C_H+C_S}}}\right\}} = \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_S}{C_H+C_S}}} = \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}}\,[1 + 1]} = \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}}\,[2]}$$

$$= \sqrt{\frac{C_H + C_S}{C_S}}$$

Purchase Model Versus Production With Backorder Model

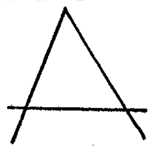
Actual Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_H}{2}Q'$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} \qquad S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2}\,\frac{P-D}{P} - C_HS_o + \frac{S_o^2[C_H+C_S]}{2Q_o\,\frac{P-D}{P}}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_H}{2}Q'}{\frac{C_PD}{Q_o} + \frac{C_HQ_o}{2}\,\frac{P-D}{P} - C_HS_o + \frac{S_o^2[C_H+C_S]}{2Q_o\,\frac{P-D}{P}}}$$

$$= \frac{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}}{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\,\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \frac{\sqrt{\frac{2^2C_P{}^2D^2}{C_S{}^2}}\sqrt{\frac{C_H{}^2}{(C_H+C_S)^2}}\sqrt{\frac{(P-D)^2}{P^2}}\sqrt{(C_H+C_S)^2}}{2\,\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}}}$$

$$= \frac{\sqrt{\frac{C_P{}^2D^2C_H}{2C_PD}} + \sqrt{\frac{2^2C_PD\;C_H{}^2}{4\quad C_H}}}{\sqrt{\frac{C_P{}^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD\;C_H{}^2}{4\quad C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{(P-D)^2}{P^2}\,\frac{P}{P-D}} - \sqrt{\frac{2C_PD\;C_H{}^2}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \frac{1}{2}\sqrt{\frac{2^2C_P{}^2D^2}{C_S{}^2}\,\frac{C_H}{2C_PD}}\sqrt{\frac{C_H{}^2}{(C_H+C_S)^2}\,\frac{C_S}{(C_H+C_S)}(C_H+C_S)^2}\sqrt{\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}} + \sqrt{\frac{C_PD\;C_H}{2}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{P-D}{P}\,\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}\,\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\,[2]}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{\frac{2C_PD\;C_H}{4}}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\,[2]}{\sqrt{\frac{C_PD\;C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}\right\}}$$

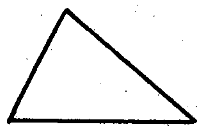
$$= \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \frac{P-D}{P}\left\{\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_S}{C_H+C_S}\,\frac{C_H}{C_S}}\right\}} = \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{P-D}{P}}\sqrt{\frac{C_S}{C_H+C_S}}}$$

$$= \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}}\,[1+1]}$$

$$= \sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}$$

Production Model Versus Backorder Model

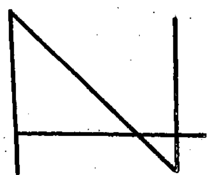
Actual Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}\ \frac{P-D}{P}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} \qquad S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_H\ Q_o}{2} - C_H\ S_o + \frac{S_o^2\ [C_H + C_S]}{2Q_o}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_HQ'}{2}\ \frac{P-D}{P}}{\frac{C_PD}{Q_o} + \frac{C_H\ Q_o}{2} - C_H\ S_o + \frac{S_o^2\ [C_H + C_S]}{2Q_o}}$$

$$= \frac{\dfrac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}\sqrt{\frac{(P-D)^2}{P^2}}}{\dfrac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} + \dfrac{\sqrt{\frac{2^2C_P^2D^2}{C_S^2}}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\sqrt{(C_H+C_S)^2}}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}}}$$

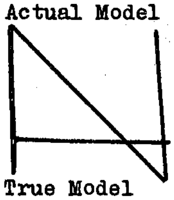
$$= \frac{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD\ C_H^2}{4C_H}}\sqrt{\frac{P}{P-D}\ \frac{(P-D)^2}{P^2}}}{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{2C_PD\ C_H^2}{4C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{2C_PD\ C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{2C_PD\ C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}}$$

$$= \frac{\sqrt{\frac{C_PD\ C_H}{2}}\sqrt{\frac{P-D}{P}}\,[1 + 1]}{\sqrt{\frac{C_PD\ C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \underbrace{\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}}_{\sqrt{\frac{C_S}{C_H+C_S}}}\right\}} = \frac{\sqrt{\frac{P-D}{P}}\,[2]}{\sqrt{\frac{C_S}{C_H+C_S}}\,[1 + 1]} = \frac{\sqrt{\frac{P-D}{P}}}{\sqrt{\frac{C_S}{C_H+C_S}}}$$

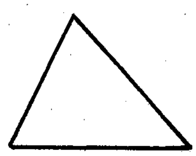
$$= \sqrt{\frac{P-D}{P}}\sqrt{\frac{C_H+C_S}{C_S}}$$

Backorder Model Versus Production Model

Actual Model



True Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}} \qquad S' = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_S}{C_H + C_S}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2} - C_H\,S' + \frac{(S')^2\,[C_H + C_S]}{2Q'}$$

$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_H\,Q_o}{2}\;\frac{P-D}{P}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\dfrac{C_PD}{Q'} + \dfrac{C_HQ'}{2} - C_HS' + \dfrac{(S')^2\,(C_H + C_S)}{2Q'}}{\dfrac{C_PD}{Q_o} + \dfrac{C_HQ_o}{2}\;\dfrac{P-D}{P}}$$

$$= \frac{\dfrac{C_PD}{\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{C_H + C_S}{C_S}}} + \dfrac{C_H}{2}\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{C_H + C_S}{C_S}} - C_H\sqrt{\dfrac{2C_PD}{C_S}}\sqrt{\dfrac{C_H}{C_H + C_S}} + \dfrac{\sqrt{\dfrac{2^2C_P{}^2D^2}{C_S{}^2}}\sqrt{\dfrac{C_H{}^2}{(C_H + C_S)^2}}\sqrt{(C_H + C_S)^2}}{2\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{C_H + C_S}{C_S}}}}{\dfrac{C_PD}{\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{P}{P-D}}} + \dfrac{C_H}{2}\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{P}{P-D}}\;\dfrac{(P-D)^2}{P^2}}$$

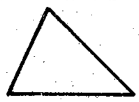
$$= \frac{\sqrt{\dfrac{C_P{}^2D^2C_H}{2C_PD}}\sqrt{\dfrac{C_S}{C_H + C_S}} + \sqrt{\dfrac{2C_PD\;C_H{}^2}{4C_H}}\sqrt{\dfrac{C_H + C_S}{C_S}} - \sqrt{2C_PD\;C_H}\sqrt{\dfrac{C_H}{C_H + C_S}}\sqrt{\dfrac{C_H}{C_S}} + \dfrac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\dfrac{C_H}{C_H + C_S}}\sqrt{\dfrac{C_H}{C_S}}}{\sqrt{\dfrac{C_P{}^2D^2C_H}{2C_PD}}\sqrt{\dfrac{P-D}{P}} + \sqrt{\dfrac{2C_PD\;C_H{}^2}{4C_H}}\sqrt{\dfrac{P}{P-D}}\;\dfrac{(P-D)^2}{P^2}}$$

$$= \frac{\sqrt{\dfrac{C_PD\;C_H}{2}}\left\{\sqrt{\dfrac{C_S}{C_H + C_S}} + \sqrt{\dfrac{C_H + C_S}{C_S}} - \sqrt{\dfrac{C_H}{C_H + C_S}}\sqrt{\dfrac{C_H}{C_S}}\right\}}{\sqrt{\dfrac{C_PD\;C_H}{2}}\sqrt{\dfrac{P-D}{P}}\,[1 + 1]} = \frac{\sqrt{\dfrac{C_S}{C_H + C_S}} + \sqrt{\dfrac{C_S}{C_H + C_S}}}{\sqrt{\dfrac{P-D}{P}}\,[2]} = \frac{\sqrt{\dfrac{C_S}{C_H + C_S}}\;[1 + 1]}{\sqrt{\dfrac{P-D}{P}}\,[2]}$$

$$= \sqrt{\frac{C_S}{C_H + C_S}}\sqrt{\frac{P}{P-D}}$$

Production Model Versus Production With Backorder Model

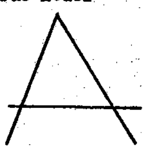
Actual Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_H}{2}Q' \quad \frac{P-D}{P}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} \qquad S_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_H}{2}Q_o \quad \frac{P-D}{P} - C_HS_o + \frac{(S_o)^2(C_H+C_S)}{2Q_o \; \frac{P-D}{P}}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_H}{2}Q' \quad \frac{P-D}{P}}{\frac{C_PD}{Q_o} + \frac{C_H}{2}Q' \quad \frac{P-D}{P} - C_HS_o + \frac{(S_o)^2\,(C_H+C_S)}{2Q_o \; \frac{P-D}{P}}}$$

$$= \frac{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}\sqrt{\frac{P}{P-D}}}} + \frac{C_H}{2}\quad \frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}}{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\,\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{\frac{2C_P^2D^2}{C_S^2}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\sqrt{\frac{(P-D)^2}{P^2}}\sqrt{(C_H+C_S)^2}}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}\;\frac{P-D^2}{P^2}}}}$$

$$= \frac{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD\;C_H^2}{C_H\;4}}\sqrt{\frac{P}{P-D}\;\frac{(P-D)^2}{P^2}}}{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD\;C_H^2}{4\;C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}\;\frac{(P-D)^2}{P^2}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}\;\frac{P-D}{P}} + \frac{1}{2}\sqrt{\frac{2C_P^2D^2C_H^2(P-D)^2C_HC_SP}{2C_PD(C_H+C_S)C_S^2(P-D)^2P^2}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{P-D}{P}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}\;\frac{P-D}{P}} + \frac{1}{2}\sqrt{C_PD\;C_H\;\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}\;\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{P-D}{P}}\,[1 + 1]}{\sqrt{\frac{P-D}{P}}\left\{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{2C_PD\;C_H}{4}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}\right\}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\,[2]}{\sqrt{\frac{C_PD\;C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}\right\}}$$

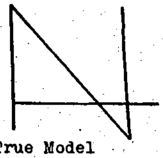
$$= \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_S}{C_H+C_S}}}$$

$$= \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}}\,[1 + 1]}$$

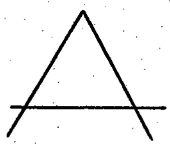
$$= \sqrt{\frac{C_H + C_S}{C_S}}$$

Backorder Model Versus Production With Backorder Model

Actual Model



True Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}} \qquad S' = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2} - C_HS' + \frac{(S')^2[C_H + C_S]}{2Q'}$$

$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}}\sqrt{\frac{P}{P-D}} \qquad S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2}\;\frac{P-D}{P} - C_HS_o + \frac{S_o^2[C_H + C_S]}{2Q_o\;\frac{P-D}{P}}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_HQ'}{2} - C_HS' + \frac{(S')^2(C_H + C_S)}{2Q'}}{\frac{C_PD}{Q_o} + \frac{C_HQ_o}{2}\;\frac{P-D}{P} - C_HS_o + \frac{(S_o)^2(C_H + C_S)}{2Q_o\;\frac{P-D}{P}}}$$

$$= \frac{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} + \frac{\sqrt{\frac{2^2C_P^2D^2}{C_S^2}}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\;(C_H+C_S)^2}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}}}{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\;\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \frac{\sqrt{\frac{2^2C_P^2D^2}{C_S^2}}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\;(C_H+C_S)^2\sqrt{\frac{(P-D)^2}{P^2}}}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}\;\frac{(P-D)^2}{P^2}}}}$$

$$= \frac{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{2C_PD}{4}\;\frac{C_H^2}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{\frac{2^2C_P^2D^2C_H\;C_H^2\;C_S}{2C_PD\;C_S^2\;C_H+C_S}}}{\sqrt{\frac{C_PD\;C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD}{4}\;\frac{C_H^2}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}\;\frac{(P-D)^2}{P^2}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}} + \frac{1}{2}\sqrt{\frac{2^2C_P^2D^2C_H\;C_H^2\;C_S\;(P-D)^2P}{2C_PD\;C_S^2(C_H+C_S)P^2\;(P-D)}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}} + \frac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{2C_PD\;C_H}{4}}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{\frac{2C_PD\;C_H}{4}}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}\right\}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{P-D}{P}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}}\right\}}$$

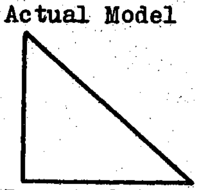
$$= \frac{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_S}{C_H+C_S}}}{\sqrt{\frac{P-D}{P}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_S}{C_H+C_S}}\right\}} = \frac{\sqrt{\frac{C_S}{C_H+C_S}}\;(2)}{\sqrt{\frac{P-D}{P}}\sqrt{\frac{C_S}{C_H+C_S}}\;(2)}$$

$$= \sqrt{\frac{P}{P-D}}$$

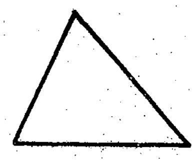
Measure of Sensitivity Based on the "True" Model's Total Variable Cost Function

Purchase Model Versus Production Model

Actual Model



True Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_H}{2} Q' \; \frac{P-D}{P}$$

$$Q_o = \sqrt{\frac{2C_PD}{C_H}} \sqrt{\frac{P}{P-D}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_H}{2} Q_o \; \frac{P-D}{P}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_HQ'}{2}\,\frac{P-D}{P}}{\frac{C_PD}{C_o} + \frac{C_HQ_o}{2}\,\frac{P-D}{P}} = \frac{\dfrac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}} + \frac{C_H}{2}\,\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}}{\dfrac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\,\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}}$$

$$= \frac{\sqrt{\frac{C_P{}^2D^2C_H}{2C_PD}} + \sqrt{\frac{2C_PD}{C_H}\,\frac{C_H{}^2}{4}}\sqrt{\frac{(P-D)^2}{P^2}}}{\sqrt{\frac{C_P{}^2D^2C_H}{2C_PD}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD}{C_H}\,\frac{C_H{}^2}{4}}\sqrt{\frac{(P-D)^2}{P^2}}\;\frac{P}{P-D}} = \frac{\sqrt{\frac{C_PD\,C_H}{2}} + \sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{(P-D)^2}{P^2}}}{\sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{P-D}{P}}}$$

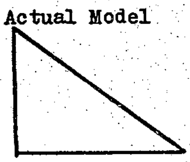
$$= \frac{\sqrt{\frac{C_PD\,C_H}{2}}\left\{1 + \sqrt{\frac{(P-D)^2}{P^2}}\right\}}{\sqrt{\frac{C_PD\,C_H}{2}}\left\{\sqrt{\frac{P-D}{P}} + \sqrt{\frac{P-D}{P}}\right\}} = \frac{1 + \sqrt{\frac{(P-D)^2}{P^2}}}{\sqrt{\frac{P-D}{P}} + \sqrt{\frac{P-D}{P}}}$$

$$= \frac{1}{2}\;\frac{2P-D}{\sqrt{P-D}\,\sqrt{P}}$$

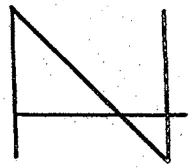
$$= \frac{1}{2}\sqrt{\frac{P}{P-D}} + \frac{1}{2}\sqrt{\frac{P-D}{P}} = \frac{1}{2}\left\{\sqrt{\frac{P}{P-D}} + \sqrt{\frac{P-D}{P}}\right\}$$

Purchase Model Versus Backorder Model

Actual Model



True Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}$$

$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}} \qquad S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2} - C_HS_o + \frac{S_o^2\,[C_H + C_S]}{2Q_o}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_HQ'}{2}}{\frac{C_PD}{Q_o} + \frac{C_HQ_o}{2} - C_HS_o + \frac{S_o^2\,[C_H + C_S]}{2Q_o}}$$

$$= \frac{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}}{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} + \frac{\sqrt{\frac{2^2C_P^2D^2}{C_S^2}}\sqrt{\frac{C_H^2}{(C_H+C_S)^2}}\sqrt{(C_H+C_S)^2}}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}}}$$

$$= \frac{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}} + \sqrt{\frac{2C_PD\;C_H^2}{4C_H}}}{\sqrt{\frac{C_P^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{2C_PD\;C_H^2}{4C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{\frac{2^2C_P^2D^2C_H}{C_S^2\;2C_PD}}\sqrt{\frac{C_H^2C_S}{C_H+C_S}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}} + \sqrt{\frac{C_PD\;C_H}{2}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}}$$

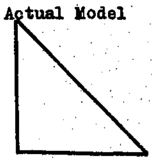
$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\,[1 + 1]}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{2C_PD\;C_H}{4C_H}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\,[2]}{\sqrt{\frac{C_PD\;C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \underbrace{\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}}_{\sqrt{\frac{C_S}{C_H+C_S}}}\right\}} = \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_S}{C_H+C_S}}} = \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}}\,[1 + 1]} = \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}}\,[2]}$$

$$= \sqrt{\frac{C_H + C_S}{C_S}}$$

Purchase Model Versus Production With Backorder Model

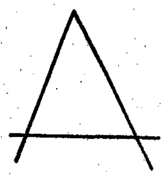
Actual Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}} \qquad S' = 0$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}\,\frac{P-D}{P} - C_HS' + \frac{(S')^2(C_H+C_S)}{2Q'\,\frac{P-D}{P}} = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}\,\frac{P-D}{P}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} \qquad S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2}\,\frac{P-D}{P} - C_HS_o + \frac{S_o{}^2[C_H+C_S]}{2Q_o\,\frac{P-D}{P}}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\dfrac{C_PD}{Q'} + \dfrac{C_HQ'}{2}\,\dfrac{P-D}{P}}{\dfrac{C_PD}{Q_o} + \dfrac{C_HQ_o}{2}\,\dfrac{P-D}{P} - C_HS_o + \dfrac{S_o{}^2[C_H+C_S]}{2Q_o\,\frac{P-D}{P}}}$$

$$= \frac{\dfrac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}} + \dfrac{C_H}{2}\,\dfrac{P-D}{P}\sqrt{\dfrac{2C_PD}{C_H}}}{\dfrac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}} + \dfrac{C_H}{2}\,\dfrac{P-D}{P}\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{C_H+C_S}{C_S}}\sqrt{\dfrac{P}{P-D}} - C_H\sqrt{\dfrac{2C_PD}{C_S}}\sqrt{\dfrac{C_H}{C_H+C_S}}\sqrt{\dfrac{P-D}{P}} + \dfrac{\sqrt{\frac{2^2C_P{}^2D^2}{C_S{}^2}}\sqrt{\frac{C_H{}^2}{(C_H+C_S)^2}}\sqrt{\frac{(P-D)^2}{P^2}}\sqrt{(C_H+C_S)^2}}{2\,\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}}}$$

$$= \frac{\sqrt{\dfrac{C_P{}^2D^2C_H}{2C_PD}} + \sqrt{\dfrac{2C_PD\;C_H{}^2}{C_H\;4}}\,\dfrac{P-D}{P}}{\sqrt{\dfrac{C_P{}^2D^2C_H}{2C_PD}}\sqrt{\dfrac{C_S}{C_H+C_S}}\sqrt{\dfrac{P-D}{P}} + \sqrt{\dfrac{2C_PD}{4}\,\dfrac{C_H{}^2}{C_H}}\sqrt{\dfrac{C_H+C_S}{C_S}}\sqrt{\dfrac{P}{P-D}\,\dfrac{(P-D)^2}{P^2}} - \sqrt{2C_PD\;C_H}\sqrt{\dfrac{C_H}{C_H+C_S}\,\dfrac{C_H}{C_S}}\sqrt{\dfrac{P-D}{P}} + \dfrac{1}{2}\sqrt{\dfrac{2^2C_P{}^2D^2}{C_S{}^2}\,\dfrac{C_H}{2C_PD}\,\dfrac{C_H{}^2}{(C_H+C_S)^2}\,\dfrac{C_S}{C_H+C_S}\,(C_H+C_S)^2}\sqrt{\dfrac{(P-D)^2}{P^2}\,\dfrac{P^2}{(P-D)^2}\,\dfrac{P-D}{P}}}$$

$$= \frac{\sqrt{\dfrac{C_PD\;C_H}{2}} + \sqrt{\dfrac{C_PD\;C_H}{2}}\,\dfrac{P-D}{P}}{\sqrt{\dfrac{C_PD\;C_H}{2}}\sqrt{\dfrac{C_S}{C_H+C_S}}\sqrt{\dfrac{P-D}{P}} + \sqrt{\dfrac{C_PD\;C_H}{2}}\sqrt{\dfrac{C_H+C_S}{C_S}}\sqrt{\dfrac{P-D}{P}} - \sqrt{2C_PD\;C_H}\sqrt{\dfrac{C_H}{C_H+C_S}\,\dfrac{C_H}{C_S}}\sqrt{\dfrac{P-D}{P}} + \dfrac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\dfrac{C_H}{C_H+C_S}\,\dfrac{C_H}{C_S}}\sqrt{\dfrac{P-D}{P}}}$$

$$= \frac{\sqrt{\dfrac{C_PD\;C_H}{2}}\left\{1 + \dfrac{P-D}{P}\right\}}{\sqrt{\dfrac{C_PD\;C_H}{2}}\sqrt{\dfrac{C_S}{C_H+C_S}}\sqrt{\dfrac{P-D}{P}} + \sqrt{\dfrac{C_PD\;C_H}{2}}\sqrt{\dfrac{C_H+C_S}{C_S}}\sqrt{\dfrac{P-D}{P}} - \sqrt{\dfrac{2C_PD\;C_H}{4}}\sqrt{\dfrac{C_H}{C_H+C_S}\,\dfrac{C_H}{C_S}}\sqrt{\dfrac{P-D}{P}}}$$

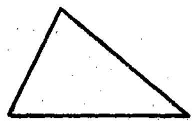
$$= \frac{\sqrt{\dfrac{C_PD\;C_H}{2}}\left\{1 + \dfrac{P-D}{P}\right\}}{\sqrt{\dfrac{C_PD\;C_H}{2}}\sqrt{\dfrac{P-D}{P}}\left\{\sqrt{\dfrac{C_S}{C_H+C_S}} + \underbrace{\sqrt{\dfrac{C_H+C_S}{C_S}} - \sqrt{\dfrac{C_H}{C_H+C_S}\,\dfrac{C_H}{C_S}}}_{\sqrt{\frac{C_S}{C_H+C_S}}}\right\}} = \frac{1 + \frac{P-D}{P}}{\sqrt{\frac{P-D}{P}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_S}{C_H+C_S}}\right\}} = \frac{1 + \frac{P-D}{P}}{\sqrt{\frac{P-D}{P}}\sqrt{\frac{C_S}{C_H+C_S}}} \quad (2)$$

$$= \frac{1}{2}\left\{\left(1 + \frac{P-D}{P}\right)\sqrt{\frac{P}{P-D}}\sqrt{\frac{C_H+C_S}{C_S}}\right\}$$

$$= \frac{1}{2}\sqrt{\frac{C_H+C_S}{C_S}}\left\{\sqrt{\frac{P}{P-D}} + \sqrt{\frac{P-D}{P}}\right\}$$

<u>Production Model Versus Backorder Model</u>

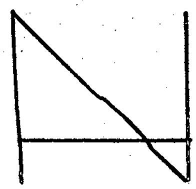
Actual Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}} \qquad S' = 0$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2} - C_HS' + \frac{(S')^2(C_H + C_S)}{2Q'}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H + C_S}{C_S}} \qquad S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H + C_S}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2} - C_HS_o + \frac{S_o^{\,2}[C_H + C_S]}{2Q_o}$$

<u>Measure of Sensitivity:</u>

$$\frac{TVC'}{TVC_o} = \frac{\dfrac{C_PD}{Q'} + \dfrac{C_HQ'}{2}}{\dfrac{C_PD}{Q_o} + \dfrac{C_HQ_o}{2} - C_HS_o + \dfrac{S_o^{\,2}\,[C_H + C_S]}{2Q_o}}$$

$$= \frac{\dfrac{C_PD}{\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{P}{P-D}}} + \dfrac{C_H}{2}\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{P}{P-D}}}{\dfrac{C_PD}{\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{C_H + C_S}{C_S}}} + \dfrac{C_H}{2}\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{C_H + C_S}{C_S}} - C_H\sqrt{\dfrac{2C_PD}{C_S}}\sqrt{\dfrac{C_H}{C_H + C_S}} + \dfrac{\sqrt{\dfrac{2^2C_P^{\,2}D^2}{C_S^{\,2}}}\sqrt{\dfrac{C_H^{\,2}}{(C_H + C_S)^2}}\sqrt{(C_H + C_S)^2}}{2\sqrt{\dfrac{2C_PD}{C_H}}\sqrt{\dfrac{C_H + C_S}{C_S}}}}$$

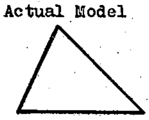
$$= \frac{\sqrt{\dfrac{C_P^{\,2}D^2C_H}{2C_PD}}\sqrt{\dfrac{P-D}{P}} + \sqrt{\dfrac{2C_PD\;C_H^{\,2}}{4C_H}}\sqrt{\dfrac{P}{P-D}}}{\sqrt{\dfrac{C_P^{\,2}D^2C_H}{2C_PD}}\sqrt{\dfrac{C_S}{C_H + C_S}} + \sqrt{\dfrac{2C_PD\;C_H^{\,2}}{4C_H}}\sqrt{\dfrac{C_H + C_S}{C_S}} - \sqrt{2C_PD\;C_H}\sqrt{\dfrac{C_H}{C_H + C_S}}\sqrt{\dfrac{C_H}{C_S}} + \dfrac{1}{2}\sqrt{2C_PD\;C_H}\sqrt{\dfrac{C_H}{C_H + C_S}}\sqrt{\dfrac{C_H}{C_S}}}$$

$$= \frac{\sqrt{\dfrac{C_PD\;C_H}{2}}\left\{\sqrt{\dfrac{P-D}{P}} + \sqrt{\dfrac{P}{P-D}}\right\}}{\sqrt{\dfrac{C_PD\;C_H}{2}}\left\{\sqrt{\dfrac{C_S}{C_H + C_S}} + \underbrace{\sqrt{\dfrac{C_H + C_S}{C_S}} - \sqrt{\dfrac{C_H}{C_H + C_S}}\sqrt{\dfrac{C_H}{C_S}}}_{\sqrt{\frac{C_S}{C_H + C_S}}}\right\}} = \frac{\sqrt{\dfrac{P-D}{P}} + \sqrt{\dfrac{P}{P-D}}}{\sqrt{\dfrac{C_S}{C_H + C_S}} + \sqrt{\dfrac{C_S}{C_H + C_S}}} = \frac{\sqrt{\dfrac{P-D}{P}} + \sqrt{\dfrac{P}{P-D}}}{2\sqrt{\dfrac{C_S}{C_H + C_S}}}$$

$$= \frac{1}{2}\sqrt{\frac{C_H + C_S}{C_S}}\left\{\sqrt{\frac{P}{P-D}} + \sqrt{\frac{P-D}{P}}\right\}$$

Production Model Versus Production With Backorder Model

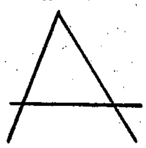
Actual Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_H}{2}\,Q'\;\frac{P-D}{P}$$

True Model



$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} \qquad S_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_H}{2}\,Q_o\;\frac{P-D}{P} - C_HS_o + \frac{(S_o)^2(C_H+C_S)}{2Q_o\;\frac{P-D}{P}}$$

Measure of Sensitivity:

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_H}{2}\,Q'\;\frac{P-D}{P}}{\frac{C_PD}{Q_o} + \frac{C_H}{2}\,Q'\;\frac{P-D}{P} - C_HS_o + \frac{(S_o)^2\,(C_H+C_S)}{2Q_o\;\frac{P-D}{P}}}$$

$$= \frac{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\;\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{P}{P-D}}}{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\,\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \frac{\sqrt{\frac{2C_P{}^2D^2}{C_S{}^2}\sqrt{\frac{C_H{}^2}{(C_H+C_S)^2}}\sqrt{\frac{(P-D)^2}{P^2}}\sqrt{(C_H+C_S)^2}}}{2\sqrt{\frac{2C_PD}{C_H}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}\;\frac{P-D^2}{P^2}}}}$$

$$= \frac{\sqrt{\frac{C_P{}^2D^2C_H}{2C_PD}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD}{C_H}\frac{C_H{}^2}{4}}\sqrt{\frac{P}{P-D}\;\frac{(P-D)^2}{P^2}}}{\sqrt{\frac{C_P{}^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD}{4}\frac{C_H{}^2}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}\;\frac{(P-D)^2}{P^2}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}\;\frac{P-D}{P}} + \frac{1}{2}\sqrt{\frac{2C_P{}^2D^2C_H{}^2(P-D)^2C_HC_SP}{2C_PD(C_H+C_S)C_S{}^2(P-D)^2P^2}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{P-D}{P}}}{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{2C_PD\;C_H}\sqrt{\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}\;\frac{P-D}{P}} + \frac{1}{2}\sqrt{C_PD\;C_H\;\frac{C_H}{C_H+C_S}\;\frac{C_H}{C_S}\;\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{P-D}{P}}\,[1+1]}{\sqrt{\frac{P-D}{P}}\left\{\sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\;C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{2C_PD\;C_H}{4}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}\right\}}$$

$$= \frac{\sqrt{\frac{C_PD\;C_H}{2}}\,[2]}{\sqrt{\frac{C_PD\;C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_H+C_S}{C_S}} - \sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{C_H}{C_S}}\right\}}$$

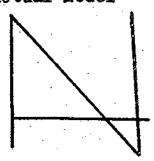
$$= \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_S}{C_H+C_S}}}$$

$$= \frac{2}{\sqrt{\frac{C_S}{C_H+C_S}}\,[1+1]}$$

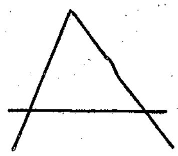
$$= \sqrt{\frac{C_H+C_S}{C_S}}$$

<u>Backorder Model Versus Production With Backorder Model</u>

Actual Model



True Model



$$Q' = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} \qquad S' = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}$$

$$TVC' = \frac{C_PD}{Q'} + \frac{C_HQ'}{2}\,\frac{P-D}{P} - C_HS' + \frac{(S')^2[C_H+C_S]}{2Q'\,\frac{P-D}{P}}$$

$$Q_o = \sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}} \qquad S_o = \sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}$$

$$TVC_o = \frac{C_PD}{Q_o} + \frac{C_HQ_o}{2}\,\frac{P-D}{P} - C_HS_o + \frac{(S_o)^2(C_H+C_S)}{2Q_o\,\frac{P-D}{P}}$$

<u>Measure of Sensitivity:</u>

$$\frac{TVC'}{TVC_o} = \frac{\frac{C_PD}{Q'} + \frac{C_HQ'}{2}\,\frac{P-D}{P} - C_HS' + \frac{(S')^2(C_H+C_S)}{2Q'\,\frac{P-D}{P}}}{\frac{C_PD}{Q_o} + \frac{C_HQ_o}{2}\,\frac{P-D}{P} - C_HS_o + \frac{(S_o)^2(C_H+C_S)}{2Q_o\,\frac{P-D}{P}}}$$

$$= \frac{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}} + \frac{C_H}{2}\,\frac{P-D}{P}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}} + \frac{\sqrt{\frac{2^2C_P{}^2D^2}{C_S{}^2}}\sqrt{\frac{C_H{}^2}{(C_H+C_S)^2}}\,(C_H+C_S)^2}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\,\frac{P-D}{P}}}{\frac{C_PD}{\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}} + \frac{C_H}{2}\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}\,\frac{(P-D)^2}{P^2} - C_H\sqrt{\frac{2C_PD}{C_S}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \frac{\sqrt{\frac{2^2C_P{}^2D^2}{C_S{}^2}}\sqrt{\frac{C_H{}^2}{(C_H+C_S)^2}}\,(C_H+C_S)^2\,\frac{(P-D)^2}{P^2}}{2\sqrt{\frac{2C_PD}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P}{P-D}}\,\frac{(P-D)^2}{P^2}}}$$

$$= \frac{\sqrt{\frac{C_P{}^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{2C_PD}{4}\,\frac{C_H{}^2}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\,\frac{P-D}{P} - \sqrt{2C_PD\,C_H}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{\frac{2^2C_P{}^2D^2\,C_H{}^2\,C_H\,C_S\,P^2}{2C_PD\,(C_H+C_S)\,C_S{}^2\,(P-D)^2}}}{\sqrt{\frac{C_P{}^2D^2C_H}{2C_PD}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{2C_PD}{4}\,\frac{C_H{}^2}{C_H}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{2C_PD\,C_H}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}} + \frac{1}{2}\sqrt{\frac{2^2C_P{}^2D^2C_H\,C_H{}^2C_S(P-D)^2\,P}{2C_PD\,C_S{}^2\,P^2\,(P-D)}}}$$

$$= \frac{\sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\,\frac{P-D}{P} - \sqrt{2C_PD\,C_H}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}} + \frac{1}{2}\sqrt{2C_PD\,C_H}\sqrt{\frac{C_H}{C_H+C_S}}\,\frac{P}{P-D}}{\sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{2C_PD\,C_H}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}} + \frac{1}{2}\sqrt{2C_PD\,C_H}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\,\frac{P-D}{P} - 2\sqrt{\frac{2C_PD\,C_H}{2}}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}} + \frac{2}{2}\sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\,\frac{P}{P-D}}{\sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - 2\sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}} + \frac{2}{2}\sqrt{\frac{C_PD\,C_H}{2}}\sqrt{\frac{C_H}{C_H+C_S}}\sqrt{\frac{P-D}{P}}}$$

$$= \frac{\sqrt{\frac{C_PD\,C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_H+C_S}{C_S}}\,\frac{P-D}{P} - 2\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}} + \sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\,\frac{P}{P-D}\right\}}{\sqrt{\frac{C_PD\,C_H}{2}}\left\{\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - 2\sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}\right\}}$$

$$= \frac{\sqrt{\frac{C_S}{C_H+C_S}} + \sqrt{\frac{C_H+C_S}{C_S}}\,\frac{P-D}{P} - \sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\left\{2 - \frac{P}{P-D}\right\}}{\sqrt{\frac{C_S}{C_H+C_S}}\sqrt{\frac{P-D}{P}} + \sqrt{\frac{C_H+C_S}{C_S}}\sqrt{\frac{P-D}{P}} - \sqrt{\frac{C_H}{C_H+C_S}\,\frac{C_H}{C_S}}\sqrt{\frac{P-D}{P}}\left\{2 - 1\right\}}$$

VITA

Ned Gene Jones

Candidate for the Degree of

Doctor of Philosophy

Thesis: SENSITIVITY ANALYSIS OF INVENTORY MODELS

Major Field: Engineering

Biographical:

Personal Data: Born in Vinland, Kansas, September 12, 1935, the son of Jim H. and Mary Jones.

Education: Attended grade school in Decatur, Arkansas; graduated from Tahlequah Senior High School, Tahlequah, Oklahoma, in May, 1953; attended Coffeyville College, Coffeyville, Kansas, Fall, 1953; entered Oklahoma State University in January, 1956, and received the Bachelor of Science degree with major in Industrial Engineering, May, 1959; re-entered Oklahoma State University in September, 1959, and received the Master of Science degree with major in Industrial Engineering, August, 1960; attended University of Michigan from September, 1960, to August, 1962 (part-time); recipient of National Science Faculty Fellowship and attended Georgia Institute of Technology from June, 1965, to January, 1966; re-entered Oklahoma State University in January, 1966, and completed requirements for the degree of Doctor of Philosophy in May, 1967.

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Professional Activities: Member of; American Institute of Industrial Engineers, The Institute of Management Science, American Society of Engineering Education: Registered Professional Engineer (Oklahoma): Alpha Pi Mu: Co-authored, "Confidence Intervals for Systematic Activity Sampling," Journal of Industrial Engineering, May-June, 1964; Authored Oklahoma State University, Report No. OR-64-1, "The Job-Shop Scheduling and Sequencing Problem", Spring, 1964: Chairman, Southwest Area AIIE Student Chapters 1963-1964: Recipient of Dupont Summer Faculty Research Award; Recipient of National Science Faculty Fellowship.