SMSG GEOMETRY AS A REAL VECTOR SPACE

By

ALONZO FRANKLIN JOHNSON

Bachelor of Science
Morehead State College
Morehead, Kentucky
1958

Master of Arts in Education
Morehead State College
Morehead, Kentucky
1961

Master of Science
University of Notre Dame
Notre Dame, Indiana
1965

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
July, 1967

SMSG GEOMETRY AS A REAL VECTOR SPACE

Thesis Approved:

Thesis Adviser

Dean of the Graduate College

ACKNOWLEDGMENTS

I would like to express my appreciation to all who assisted me in the preparation of this dissertation. In particular, I thank Dr. Gerald Goff, my dissertation adviser, for his guidance and assistance in the formulation and preparation of this dissertation. Additional thanks go to Dr. H. S. Mendenhall, my committee chairman, and Dr. Vernon Troxel, a member of my advisory committee. Each of these gentlemen was most helpful in the preparation of the dissertation.

Many on the Mathematics faculty have been of assistance to me in my mathematical training. Special thanks go to Dr. L. Wayne Johnson. Dr. Johnson's main considerations, in determining my teaching assignments, were my mathematics training and the well being of the students under my direction. He is truly a great educator.

I also express my gratitude to my wife, Helen, and sons, Mike and Jeff. Without their help and patience this dissertation would not have been possible. Thanks also to the typist, Miss Velda Davis.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	LINEAR ALGEBRA BACKGROUND	8
	Introduction and Notational Devices	8
	Definitions and Theorems on Vector Spaces	9
	Geometrical Vectors	19
III.	LINES, PLANES, COORDINATE SYSTEMS, AND SEPARATION	21
	Lines and Planes	21
	Convex Sets and Separation	30
IV.	ANGLES IN A EUCLIDEAN SPACE	38
V.	CONGRUENCE AND PARALLEL LINES	50
	S.A.S. and Parallel Postulates	51
	Isometries	57
	Congruence by Isometries	64
VI.	AREA AND VOLUME	69
	Topics From Topology	71
	Jordan Content in R_2	76
	Area in V	92
	Volume in V	100
VII.	SUMMARY AND EDUCATIONAL IMPLICATIONS	109
	Summary	109
	Educational Implications	110
SELECTED BIBLIOGRAPHY		112
APPENDIX A		114

CHAPTER I

INTRODUCTION

The history of mathematics in the nineteenth century is punctuated by three significant events. [7: 366][1] The first of these was the discovery, about 1829, of self consistent geometries other than Euclidean geometry. The postulates of geometry became, for the mathematician, mere hypothesis whose truth or falsity need not concern him. The geometries other than Euclidean have found many applications in the sciences.

The second of the three events occurred in algebra beginning about 1843. At this time non-commutative algebras were discovered. William Rowan Hamilton, after struggling with a physical problem for years, invented the quaternion algebra in which the commutative law of multiplication does not hold. Until this time algebra was thought of as generalized arithmetic; that is letters were used for arbitrary numbers. After 1843 mathematicians began noticing the structure properties of number systems such as the associative and commutative laws.

By weakening postulates, replacing postulates, or adding new postulates for the real number system new abstract systems were studied. Some of the mathematical systems studied were groups, rings,

---

[1]Numerals included in brackets will refer to items listed in the Bibliography. If two numerals separated by a colon are used, the first is the item in the Bibliography and the second is the page number.

integral domains, lattices, division rings, fields, and vector spaces. Mathematicians to date have studied over 200 such abstract algebraic structures.

The third profound mathematical event of the nineteenth century was the so-called arithmetization of analysis. Significant contributions were the developing of an acceptable theory of limits by Cauchy in 1821, and the derivation of real numbers from natural numbers in the last half of the nineteenth century.

Most mathematics was closely tied to geometry before Decartes published his first book on analytic geometry in 1873. Meserve states:

> Previously, (before Descartes work) linear terms such as x or 2y had been considered as line segments, quadratic terms such as x^2 or xy had been considered as areas, and cubic terms such as x^3 or x^2y had been considered as volume. The old interpretations were restrictive in the sense that only like quantities could be added. For example, it was permissible to add x^2 and xy (areas), but it was not permissible to add x^2 and x (i.e., an area and a line segment). [12: 377]

At this time polynomials with degree greater than three were not considered since there was no geometrical significance to x^4. Not until Dedekind gave the first definition of $\sqrt{2}$ in 1872 could the domination by geometry be said to have ended.

Before the events in the nineteenth century, mathematics was studied in small bits and pieces without the advantages of studying the generalizing concepts. Since these events the objectives in mathematics have shifted to the studying of structure. About this shift in objectives, Bell states:

> The shift of objective is typical of modern abstract mathematics. Specimens are no longer prized for their own curious sake as they were in the nineteenth century. ... Their interesting but somewhat meaningless collections would simplify themselves in an unsuspected coherence. [5: 236]

As stated above, one of the unifying concepts of mathematics is that of a vector space. In the last thirty years, this concept has assumed a major role in college undergraduate instruction and possibly will play a greater role in the future. As Athen has stated:

> More and more it is becoming evident that in the modernization of mathematical instruction, through the use of sets and structures, the topic of vectors and vector spaces will play a major role. [3: 382]

Some reasons given for the introduction of vectors and algebraic methods in secondary mathematics are:

1. The following mathematical topics offer great opportunities for the use of vectors: affine geometry, plane and spherical trigonometry, analytic geometry of linear manifolds and circles, conic sections, geometrical mappings and transformations and descriptive geometry. In teaching all these topics at the secondary school level, vector spaces would be limited to real vector spaces (linear vector algebra and the scalar product). [3: 383]

2. The fewer axioms in a mathematical system the better. In mathematics axioms are essential but in large numbers they become troublesome. For example, by the use of vectors the number of axioms in geometry can be greatly reduced.[2]

3. The use of coordinates is more related to the mathematics high school students will encounter later.

4. The concept of function is central in the algebraic approach.

5. Congruence is a unified concept in a Euclidean vector space where in some present geometry courses congruence is defined separately for different point sets.

6. In addition to the applications of vectors, the concept of a vector space can serve to relate the study of algebra and geometry in the secondary school mathematics curriculum. In many instances

---

[2]The points listed in items 2-5 are essentially those given by Professor G. P. Johnson in a talk at the 50th Annual Meeting of The Mathematical Association of America; Houston, Texas, January 28, 1967.

the study of geometry turns out to be merely a strange interlude between the study of algebra in the ninth grade and the continued study of algebra in the eleventh grade. [20: 218]

7. Vectors in the beginning years of mathematics will probably be taught by intuitive or informal methods. Since vector theory can easily be axiomatized, it serves as a topic which the teacher can use to present a mathematical structure.

Algebraic methods and the use of linear algebra is gaining in acceptance in the high school curriculum. A text in analytic geometry and a text in geometry by coordinates have been published by the School Mathematics Study Group (SMSG). At least two texts in linear algebra for high school students have been published. A text for secondary school geometry presenting Euclidean geometry through vectors is now being prepared by the University of Illinois Committee on School Mathematics (UICSM). [20] The Committee on the Undergraduate Program in Mathematics (CUPM) has recommended courses in both abstract algebra and linear algebra for secondary mathematics teachers. [6]

In 1959 the Report of the Commission on Mathematics (College Entrance Board), Program for College Preparatory Mathematics listed proposals for the high school geometry course. [15] One proposal was an introduction of coordinate geometry and, once coordinate geometry has been introduced, the use of analytic (algebraic) as well as synthetic methods in proving geometric theorems and exercises.

Some twenty years ago, the teachers of mathematics of the German Gymnasium instituted a serious study of the role that vector theory should play in the mathematics curriculum. [3] In 1965 the German Association for the Advancement of Mathematics and Science Instruction published recommendations for the teaching of vectors from the primary grades through the Gymnasium. Their point of view is that vector theory should not be introduced into the syllabus as a new isolated topic, it should penetrate all the mathematics instruction. In the lower years of elementary school they recommend that vectors be thought of as simple translations (directed line segments) with the formalization of a definition of a real vector space omitted until the latter years of the upper grades. The vector concept penetrates the whole of mathematics instruction, not as an exclusive method but as one that gives simplicity, clarification, and unification to the study when it is applied.

> During the last ten years the teaching of vectors, in German schools, has been introduced into the middle years (school years 8 through 10) with complete success. In the last few years experimental studies have been carried out in teaching vectors informally in the lower years (school years 5 to 7) with encouraging results. [3: 382]

The 1963 Report of The Cambridge Conference on School Mathematics, Goals for School Mathematics, recommends the study of linear algebra in the curriculum for grades seven through twelve. [8] In their recommended program linear spaces are encountered in two rounds.

> The first encounter is in connection with motions of Euclidean space and the presentation is restricted to finite dimensional linear spaces composed of pairs, triples, or perhaps n-tuples of real numbers. The second round takes up the general study of linear spaces. [8: 49]

One topic usually covered when studying vectors is an Euclidean vector space. The use of the word Euclidean suggests Euclidean geometry. But, a Euclidean vector space is formally defined as a real vector space with an inner product defined on it. (Each of these terms will be defined in Chapter II.) One assumption of this dissertation is that college students preparing to be secondary mathematics teachers should understand, especially in the three dimensional case, the relationship between these two uses of the word Euclidean.

The main purpose of this dissertaion will be to show that a Euclidean vector space is Euclidean in the geometric sense. A three dimensional vector space with inner product will be assumed. The undefined terms of SMSG geometry [17: 10] will be defined in terms of vectors in this vector space. With these definitions, the twenty-two postulates of SMSG geometry will be proved, thus making a three dimensional vector space a model for Euclidean geometry.

The reasons for doing this dissertation are twofold:

1. Although much of the material covered has been developed in segments in the literature before, in a search of the literature the writer could not find where these ideas were tied together to give a complete proof that a Euclidean vector space was Euclidean in the sense of high school geometry.

2. The material should have application for the prospective secondary mathematics teachers in their preparation both in geometry and algebra. It is expected that in the future high school teachers will be called upon to teach a course in linear algebra. The eminent algebraist E. Artin, in his book <u>Geometric Algebra</u> [2: 13] warns against using purely algebraic techniques in teaching such a course. Geometry sometimes helps a student "see" what is happening in the algebra. It is hoped that this material, relating vector spaces and high school geometry, will provide a teacher with materials useful in teaching a course in linear algebra or a course in geometry by vectors.

The development of the succeeding chapters will be as follows: Chapter II will give the assumptions on the background of the reader together with other preliminary considerations needed in the development. In Chapter III the assumptions of a three dimensional real vector space with inner product will be given. In the subsequent four chapters, the undefined terms of SMSG geometry will be defined, in terms of vectors in the vector space, and the twenty-two postulates of SMSG geometry will be proved in this vector space setting.

The topics in Chapter III were adapted from a doctoral dissertation at Oklahoma State University. This dissertation is entitled Foundations in Geometry for High School Teachers, authored by James Smith, 1963. [19] Chapter VI will cover area and volume in three space. Some analysis is used in Chapter VI.

CHAPTER II

LINEAR ALGEBRA BACKGROUND

Introduction and Notational Devices

This chapter will include the information about vector spaces necessary to develop the material in subsequent chapters. Also, there will be a brief discussion on how vectors are thought of geometrically.

In the material that follows, unless specifically stated otherwise, the capital letter R will designate the real number system. The capital letter V will designate a real vector space and the capital letters U and W will be subspaces of V. Vectors will be designated by capital letters other than U, V, and W. Lower-case English letters will be used to designate real numbers. 0 will be used to denote the real number zero exclusively.

Theorems and definitions, except for the twenty-two theorems which are postulates of SMSG Geometry, will be numbered consecutively by chapter. For example, Theorem 2.7 will be the seventh theorem in Chapter II. The twenty-two theorems which are postulates of SMSG Geometry will follow the numbering of SMSG Geometry. For example, Theorem VIII will be Postulate 8 of SMSG Geometry.

It is assumed that the reader has had some experience with real vector spaces. Therefore, a statement of some of the theorems in this chapter has been included for completeness but the proofs have been omitted. Most of these proofs can be found in any text on linear algebra, such as Baumont's text [4], or in a text on beginning abstract algebra such as Mostow, Sampson and Meyer's. [14]

Definitions and Theorems on Vector Spaces

<u>Definition 2.1.</u> A real vector space V is a non-empty set of elements, called vectors, and two operations, addition and multiplication by a real number (scalar multiplication), which satisfy the following axioms: For all $A, B, C \in V$; $r, s \in R$:

1. $A + B$ is a unique element in V, called the sum of A and B.
2. rA is a unique element in V, called the scalar product of r and A.
3. $A + B = B + A$.
4. $(A + B) + C = A + (B + C)$
5. There exist a vector $\theta \in V$ such that $\theta + A = A$ for each $A \in V$.
6. For each $A \in V$, there exists a vector $-A \in V$ such that $A + (-A) = \theta$, Notation: $A + (-B) = A - B$.
7. $r(sA) = (rs)A$.
8. $(r + s)A = rA + sA$.
9. $r(A + B) = rA + rB$.
10. $1A = A$.

The term vector space in this dissertation will always refer to a real vector space. One example of a vector space, probably the most often encountered in elementary mathematics, is

$$R_n = \{(x_1, x_2, \ldots, x_n) | x_i \in R, i = 1, 2, \ldots, n\}$$

with $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) =$ $(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and $r(x_1, x_2, \ldots, x_n) =$ $(rx_1, rx_2, \ldots, rx_n)$. The reader will probably profit by keeping in mind the space R_3. Examples from time-to-time will come from this space.

The following three computational theorems will be useful.

Theorem 2.2. $r \circ \theta = \theta$ and $0A = \theta$ [4: 41]

Theorem 2.3. $r(-A) = (-r)A = -(rA)$. In particular $(-1)A = -A$. [4: 41]

Theorem 2.4. If $rA = \theta$, then $r = 0$ or $A = \theta$ [4: 41]

A set of vectors, $\{A_1, A_2, \ldots, A_n\}$, in a vector space V are linearly dependent if there exists scalars $r_1, r_2, \ldots, r_n$, not all zero, such that $r_1A_1 + r_2A_2 + \ldots + r_nA_n = \theta$. If no such set of scalars exists, the set $\{A_1, A_2, \ldots, A_n\}$ is said to be linearly independent.

A set of vectors $\{A_1, A_2, \ldots, A_n\} \subset V$ is said to span the vector space V if for all $A \in V$ there exist scalars $r_1, r_2, \ldots, r_n$ such that $A = r_1A_1 + r_2A_2 + \ldots + r_nA_n$. The set $\{A_1, A_2, \ldots, A_n\}$ is a basis for V if it spans V and is linearly independent. A vector space is finite dimensional if it has a finite basis. In this chapter all vector spaces will be assumed to be finite dimensional. The dimension of a vector space V, written dim V, is the number of elements in a basis.

Theorem 2.5. All bases of a finite dimensional vector space contain the same number of elements. [4: 49]

Theorem 2.6. If $\{A_1, A_2, \ldots, A_i, \ldots, A_n\}$ is a basis for a vector space V and $r \neq 0$, then $\{A_1, A_2, \ldots, rA_i, \ldots, A_n\}$ is a basis for V. [4: 59]

Theorem 2.7. If A is a non-zero vector in a vector space V then $\{A\}$ is linearly independent. [4: 46]

Theorem 2.8. Any linearly independent subset of a vector space V can be completed to a basis for V. [4: 51]

Definition 2.9. An inner product on a real vector space V is a function mapping V X V into R satisfying:

1. $A \circ A \geq 0$, and $A \circ A = 0$ if and only if $A = 0$.

   Note: $A \circ A$ is the image of the pair (A, A) under this map.

2. $r(A \circ B) = (rA) \circ B$.
3. $A \circ B = B \circ A$.
4. $A \circ (B + C) = A \circ B + A \circ C$.

Easy consequences of this definition are:

Theorem 2.10. $\theta \circ A = 0$ for all $A \in V$. [4: 53]

Theorem 2.11. $A \circ B = 0$ if and only if, for each $r \neq 0$, $(rA) \circ B = 0$. [4: 52]

Definition 2.12. In terms of this inner product a norm, which is a map with domain the vector space V and range the reals, is defined by $|A| = \sqrt{A \circ A}$. Since $A \circ A \in R$ and $A \circ A \geq 0$, then $\sqrt{A \circ A} \in R$. Note: $|A|$ is the norm of a vector. If $r \in R$, then $|r|$ is the absolute value of r.

An example of an inner product on the space R_3 is

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = x_1y_1 + x_2y_2 + x_3y_3.$$

This inner product is called the usual or dot product on R_3. In terms of this inner product the norm becomes, $|(x_1, x_2, x_3)| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, the usual norm.

Let V be a vector space with inner product $A \circ B$ and norm defined $|A| = \sqrt{A \circ A}$. Four theorems concerning this norm follow.

Theorem 2.13. $|A| \geq 0$, and $|A| = 0$ if and only if $A = \theta$. This theorem follows directly from condition one on the inner product.

Theorem 2.14. $|rA| = |r||A|$.

Theorem 2.15. a) $|A \circ B| \leq |A||B|$ (Schwartz inequality)

b) $|A \circ B| = |A||B|$ if and only if one vector is a scalar multiple of the other.

Proof: a) If $A = \theta$, then $A \circ B = 0$ and $|A||B| = 0$. Thus $|A \circ B| = |A||B|$. Suppose $A \neq \theta$. Then, $|A| > 0$. For any real number t $|tA + B|^2 = (tA + B) \circ (tA + B) = t^2A \circ A + 2tA \circ B + B \circ B = |A|^2 t^2 + (2A \circ B)t + |B|^2 \geq 0$. Since the quadratic polynomial in t is always non-negative, its discriminant is non-positive. Thus, $(2A \circ B)^2 - 4|A|^2 |B|^2 \leq 0$ or $(A \circ B)^2 \leq |A|^2 |B|^2$. Thus, $|A \circ B| \leq |A||B|$.

b) If $rA = B$, then $A \circ B = A \circ rA = rA \circ A = r|A||A| = \pm |A||rA| = \pm |A||B|$ the sign being + if r is positive and - if r is negative. Thus, $|A \cdot B| = |A||B|$.

If $|A \circ B| = |A||B|$ and $A = \theta$, then $A = 0B$. If $A \neq \theta$, then $|A| > 0$ and the quadratic equation $|A|^2t^2 + 2A \circ Bt + |B|^2 = 0$ has a multiple root r since $(2A \circ B)^2 - 4|A|^2|B|^2 = 0$. Since $(tA + B) \circ (tA + B) = |A|^2t^2 + 2A \circ Bt + |B|^2 = 0$ has a root r, $(rA + B) \circ (rA + B) = 0$. Thus, $rA + B = \theta$ or $B = (-r)A$.

Two vectors, A and B, in an inner product space are orthogonal if $A \circ B = 0$. A vector A is said to be normalized if $|A| = 1$. A basis $\{A_1, A_2, \ldots, A_n\}$ of an inner product space V is an orthonormal basis if $|A_i| = 1$, $i = 1, 2, \ldots, n$ and $A_i \circ A_j = 0$ if $i \neq j$.

Theorem 2.16. Any subset of orthogonal vectors in V is linearly independent. In particular any set of n orthogonal vectors in an n dimensional vector space is a basis. [4: 53]

A non-empty subset W of a vector space V is a subspace of V if and only if W, with the operations of V, is a vector space.

Theorem 2.17. A non-empty subset of a vector space V is a subspace of V if and only if $rA + sB \in W$, for all $A, B \in W$, $r, s \in R$. [4: 42]

Theorem 2.18. If W is a subspace of a vector space, then $\theta \in W$. [4: 42]

Theorem 2.19. If U and W are subspaces of a vector space V, then $U \cap W$ is a subspace of V. [4: 43]

If W is a subspace of V with basis $\{A_1, A_2, \ldots, A_n\}$, then the notation $W = [A_1, A_2, \ldots, A_n]$ will be used. That is, $[A_1, A_2, \ldots, A_n]$ is the subspace of V with $\{A_1, A_2, \ldots, A_n\}$ as a basis.

The next theorem will be used frequently.

Theorem 2.20. Each subspace $W \neq \{\theta\}$ of an inner product vector space V has an orthonormal basis. If $\{A_1, A_2, \ldots, A_n\}$ is an orthonormal basis of W and $B = r_1A_1 + r_2A_2 + \ldots + r_nA_n$, $C = s_1A_1 + s_2A_2 + \ldots + s_nA_n$, then $B \cdot C = r_1s_1 + r_2s_2 + \ldots + r_ns_n$.

Proof: The proof will be given for a three dimensional subspace of V. The method will generalize to any finite dimension. The constructive process used in the proof is known as the Gram-Schmidt orthogonalization process.

Let $W = [B_1, B_2, B_3]$. Let $A_1 = \frac{1}{|B_1|} B_1$. Then, $A_1 \cdot A_1 = \left(\frac{1}{|B_1|} B_1\right) \cdot \left(\frac{1}{|B_1|} B_1\right) = \frac{1}{|B|^2} B_1 \circ B_1 = \frac{1}{|B_1|^2} |B_1|^2 = 1$.

Let $C_2 = B_2 - (A_1 \circ B_2)A_1$. Then $A_1 \circ C_2 = A_1 \circ B_2 - (A_1 \circ B_2)(A_1 \circ A_1) = A_1 \circ B_2 - (A_1 \circ B_2)1 = 0$. Let $A_2 = \frac{1}{|C_2|} C_2$. Then, by Theorem 2.11, $A_1 \circ A_2 = 0$ and $A_2 \circ A_2 = 1$.

Let $C_3 = B_3 - (A_1 \circ B_3)A_1 - (A_2 \circ B_3)A_2$. Then, $A_1 \circ C_3 = A_1 \circ B_3 - (A_1 \circ B_3)(A_1 \circ A_1) - (A_2 \circ B_3)(A_2 \circ A_1) = A_1 \circ B_3 - (A_1 \circ B_3)1 - (A_2 \circ B_3)0 = 0$. Similarly, $A_2 \circ C_3 = 0$. Thus, letting $A_3 = \frac{1}{|C_3|} C_3$, $A_1 \circ A_3 = 0$, $A_2 \circ A_3 = 0$ and $A_3 \circ A_3 = 1$.

For the second part of the theorem

$B \circ C = (r_1A_1 + r_2A_2 + \ldots + r_nA_n) \circ (s_1A_1 + s_2A_2 + \ldots + s_nA_n) = [r_1s_1A_1 \circ A_1 + r_1s_2A_1 \circ A_2 + \ldots + r_1s_nA_1 \circ A_n] + [r_2s_1A_2 \circ A_1 + r_2s_2A_2 \circ A_2 + \ldots + r_2s_nA_2 \circ A_n] + \ldots + [r_ns_1A_n \circ A_1 + r_ns_2A_n \circ A_2 + \ldots + r_ns_nA_n \circ A_n] = [r_1s_1\ 1 + r_1s_2\ 0 + \ldots + r_1s_n\ 0] + [r_2s_1\ 0 + r_2s_2\ 1 + \ldots + r_2s_n0] + \ldots + [r_ns_1\ 0 + r_ns_2\ 0 + \ldots + r_ns_n\ 1] = r_1s_1 + r_2s_2 + \ldots + r_ns_n$.

One of the concepts of linear algebra, which had its founding in geometry, is that of a coset or translate of a subspace. Let W be a subspace of a vector space V. Define a relation, $\sim$, on V by $A \sim B$ if and only if $A - B \in W$. It is seen that $\sim$ is an equivalence relation since :

1. $A - A = \theta \in W$. Therefore, $A \sim A$, for each A in V.

2. If $A \sim B$, then $A - B \in W$. Thus, $(-1)(A - B) =$
$B - A \in W$ or $B \sim A$.

3. If $A \sim B$ and $B \sim C$, then $A - B$ and $B - C$ are in W.
Therefore, $(A - B) + (B - C) = A - C \in W$ or $A \sim C$.

This equivalence relation partitions V into disjoint subsets called equivalence classes. The equivalence class containing A is denoted $A + W$. If A and B are two distinct vectors in V, then $A + W$ and $B + W$ are either identical or disjoint. The union of the equivalence classes is V.

Let $A + W$ be the equivalence class containing A. What other vectors are in $A + W$? If $B \in A + W$, then $B \sim A$. Thus, $B - A \in W$ or $B - A = C$, $C \in W$ or $B = A + C$, $C \in W$. Also, if $B = A + C$, $C \in W$, then $B - A = C$ or $B - A \in W$. Therefore, $B \sim A$ or $B \in A + W$. Thus, $A + W = \{A + C \mid C \in W\}$. This is the reason the notation $A + W$ is used for the equivalence class containing A. The equivalence class $A + W$ is called a coset of W.

Cosets will be used frequently and the preceding discussion constitutes a proof of the following theorem.

Theorem 2.21. If W is a subspace of a vector space V, then V is partitioned into disjoint cosets of the form $A + W = \{A + C \mid C \in W\}$ where $A + W = B + W$ if and only if $A - B \in W$.

It is possible to show that the set of cosets of a subspace W is a vector space by defining $(A + W) + (B + W) = (A + B) + W$ and $r(A + W) = rA + W$.

The following is a geometrical example for Theorem 2.21: Let

$V = R_3$. (Figure 1) Let $W = [(1,0,0),(0,1,0)]$. That is, W is the subspace of R_3 spanned by the vectors (1,0,0) and (0,1,0). Thus, W is the x_1x_2 plane.

The coset (0,0,1) + W is the set $\{(0,0,1) + (x,y,0)\} = \{(x,y,1)\}$. Thus, (0,0,1) + W is the plane parallel to the x_1x_2 plane and one unit above. Similarily, it is seen that (a,b,c) + W is the plane parallel to the x_1x_2 plane and c units from the x_1x_2 plane.

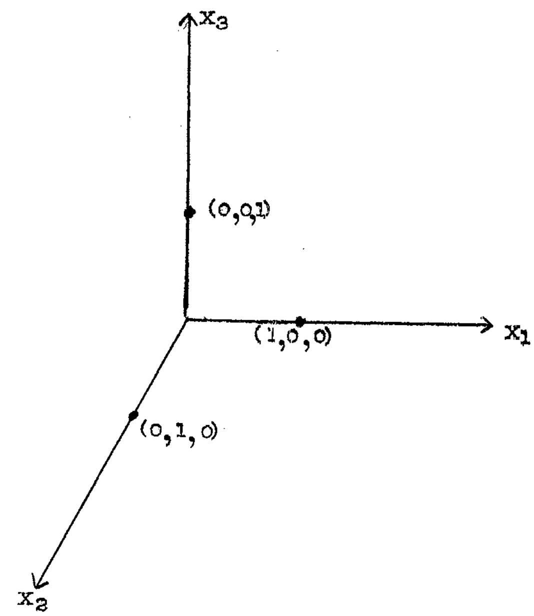

Figure 1.

<u>Theorem 2.22</u>. If W and U are subspaces of a vector space V and A + W, B + U are cosets of W and U, respectively, then $(A + W) \cap (B + U)$ is either empty or $(A + W) \cap (B + U)$ is a coset of the subspace $W \cap U$.

Proof: Suppose $(A + W) \cap (B + U) \neq \emptyset$. Then, there exists $P \in (A + W) \cap (B + U)$. Thus, $P = A + C = B + D$, $C \in W$, $D \in U$ or $A = P - C$, $B = P - D$.

$(A + W) \cap (B + U) = P + (W \cap U)$ because: $T \in (A + W) \cap (B + U)$ implies $T = A + X = B + Y$, $X \in W$, $Y \in U$. Therefore, $T = P + (X - C) = P + (Y - D)$. Thus, since $X - C = Y - D \in W \cap U$, $T \in P + (W \cap U)$.

If $Q \in P + (W \cap U)$, then $Q = P + E$, $E \in W$ and $E \in U$. Therefore, $Q = A + (C + E) = B + (D + E)$ and since $C + E \in W$, $D + E \in U$, $Q \in A + W$ and $Q \in B + U$. Thus, the statement is proved.

Illustrating this theorem, consider again $W = [(1,0,0), (0,1,0)] \subset R_3$. Let $U = [(1,0,0), (0,0,1)]$. $(0,0,1) + W$, as in the last example, is a plane one unit above the x_1x_2 plane. $(0,2,0) + U = \{(0,2,0) + (x,0,z)\} = \{(x,2,z)\}$ is a plane two units from the x_1x_3 plane. The intersection of these two cosets is the coset $(0,2,1) + [(1,0,0)]$ since $W \cap U = [(1,0,0)]$ and $(0,2,1) \in ((0,0,1) + W) \cap ((0,2,0) + U)$.

A linear transformation on a vector space V into a vector space W is a map f from V into W satisfying $f(rA + sB) = rf(A) + sf(B)$, for all $r, s \in R$, and for all $A, B \in V$. That is, f preserves addition and scalar multiplication.

Some theorems about linear transformations such as $f\colon V \to W$ follow.

Theorem 2.23. If $f\colon V \to W$ is a linear transformation, then $f(\theta) = \theta$. [4: 80]

Theorem 2.24. $f(V)$ is a subspace of W. ($f(V) = \{B \in W \mid B = f(A)$ for some $A \in V\}$.) [4: 80]

Theorem 2.25. $f^{-1}(\theta)$ is a subspace of V. ($f^{-1}(\theta) = \{A \in V \mid f(A) = \theta\}$.) [4: 80]

Theorem 2.26. Let V and W be vector spaces. Let $\{A_1, A_2, \ldots, A_n\}$ be a basis for V and $B_1, B_2, \ldots, B_n$ be any n vectors in W. Then, there exists one and only one linear transformation f from V into W such that $f(A_1) = B_1, f(A_2) = B_2, \ldots, f(A_n) = B_n$. [14: 225]

Theorem 2.27. $\dim V = \dim f(V) + \dim f^{-1}(\theta)$. [4: 85]

For a fixed vector A in an inner product space V, the map $f\colon V \to R$ defined by $f(B) = A \circ B$ is a linear transformation. This follows from properties two and four of the definition of inner product. That is, $f(rB + sC) = A \cdot (rB + sC) = r(A \cdot B) + s(A \cdot C) = rf(B) + sf(C)$. If $A \neq \theta$, then f is onto R because if $r \in R$, then $f\left(\frac{r}{|A|^2} A\right) = A \cdot \left(\frac{r}{|A|^2} A\right) = \frac{r}{|A|^2} A \cdot A = \frac{r}{|A|^2} |A|^2 = r$. Since R is a one dimensional vector space over itself, by Theorem 2.27:

<u>Theorem 2.28</u>. $\dim f^{-1}(\theta) = \dim V - \dim R = \dim V - 1$.

As an example of this theorem consider the vector space R_3 with the usual inner product. Let $A = (1,1,1)$. Then, $f^{-1}(0) = \{(x_1, x_2, x_3) \mid 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 = 0\}$ which is the two dimensional subspace $W = [(1,-1,0), (1,0,-1)]$. That is $f^{-1}(0)$ is the plane containing the origin which is spanned by the two vectors $(1,-1,0)$ and $(1,0,-1)$.

<u>Definition 2.29</u>. An isomorphism between two vector spaces V and W is a linear transformation from V into W which is one-to-one and onto. If there exists an isomorphism between V and W, then V and W are said to be isomorphic.

<u>Theorem 2.30</u>. If $f: V \to W$ is an isomorphism and U is a subspace of V, then $f(U) = \{B \in W \mid B = f(A) \text{ for some } A \in U\}$ is a subspace of W. The dimension of f(U) equals the dimension of U. [4: 83]

Let $V = [A_1, A_2, A_3]$ be a three dimensional vector space. Then each vector $A \in V$ can be written uniquely as $A = r_1A_1 + r_2A_2 + r_3A_3$. Using this fact, an isomorphism $f: V \to R_3$ can be defined by $f(A) = (r_1, r_2, r_3)$. Thus, under this isomorphism, $f(A_1) = (1,0,0)$, $f(A_2) = (0,1,0)$ and $f(A_3) = (0,0,1)$. Such an isomorphism f is called a coordinate system for V with respect to the basis $\{A_1, A_2, A_3\}$. $f(A) = (r_1, r_2, r_3)$ is called the coordinate of A relative to the basis $\{A_1, A_2, A_3\}$.

A particular type of linear transformation, which will be used in Chapter V, is an orthogonal linear transformation.

<u>Definition 2.31</u>. A linear transformation f, from a Euclidean vector space V into itself, is an orthogonal linear transformation if and only if $f(A) \cdot f(B) = A \cdot B$ for all A and B in V.

<u>Theorem 2.32</u>. Let $\{A_1, A_2, \ldots, A_n\}$ be an orthonormal basis for a Euclidean vector space V. Then, a linear transformation $f: V \to V$ is an orthogonal linear transformation if and only if $\{f(A_1), f(A_2), \ldots, f(A_n)\}$ is an orthonormal basis for V. In particular, if $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ are orthonormal basis for V then there exists exactly one orthogonal linear transformation $f: V \to V$ such that $f(A_1) = B_1, f(A_2) = B_2, \ldots, F(A_n) = B_n$. [4: 180]

<u>Theorem 2.33</u>. An orthogonal linear transformation $f: V \to V$ is an isomorphism. [4: 172]

<u>Theorem 2.34</u>. Let $f: V \to V$ be an orthogonal linear transformation. Then, $f^{-1}: V \to V$ is an orthogonal linear transformation. [4: 180]

Geometrical Vectors

In the literature there are essentially two ways of considering vectors in R_3. The first is as "directed line segments" and the second as "points."

The approach used in this dissertation will be to consider points as elements of a vector space. In R_3 the vector (1,1,1) would simply be a point. If the line segment with endpoints (0,0,0) and (1,1,1) was being considered this would be the set $\{(x_1, x_2, x_3) \mid (x_1, x_2, x_3) = t(1,1,1), 0 \leq t \leq 1\}$.

The "directed line segment" approach is sometimes presented with an assumed geometric background. In this case vectors are equivalence classes of line segments having the same length and direction. The interested reader is referred to items [4] and [14] in the Bibliography.

CHAPTER III

LINES, PLANES, COORDINATE SYSTEMS AND SEPARATION

Lines and Planes

The preliminary results about vector spaces have been included in Chapter II. This chapter will include material on the first ten postulates of SMSG geometry.

The basic assumption throughout the development will be the existence of a three dimensional inner product vector space V. The inner product space R_3 could have been assumed as the vector space V. The reason this is not done is that R_3 would give more information about the vectors than is needed to do the proofs (i.e., the vectors in R_3 are 3-tuples).

Definitions of terms used in the SMSG geometry text will be taken verbatim from the text to keep from distorting their meanings. The terms point, line, and plane are defined in terms of vectors in this development. In the SMSG development they are undefined terms. After some of the SMSG definitions, a short comment will be included converting the definitions of SMSG to vector terms.

Definition 3.1. A point is defined to be a vector in V.

Since V is a three dimensional vector space, the subspaces of V will be either three, two, one, or zero dimensional. Clearly, if W is a three dimensional subspace of V, then V = W. The trivial subspace, containing only the zero vector, is the only subspace of dimension zero.

Definition 3.2. A line is defined to be a coset of a one dimensional subspace of V. Equivalently, if W = [B] is a one dimensional subspace of V, then a line is a set of points

$$A + W = A + [B] = \{A + tB \mid t \in R\}.$$

Definition 3.3. A plane is defined to be a coset of a two dimensional subspace V. That is, if W = [B, C] is a two dimensional subspace of V, then $A + W = A + [B, C] = \{A + tB + sC \mid t, s \in R\}$ is a plane.

Definition 3.4. The set of all points is called space. [17: 53] Thus, space is the set V.

The purpose of the first eight theorems in this chapter is to show that the one and two dimensional cosets in a three dimensional space are analogous to planes and lines of Euclidean three dimensional geometry. As defined in Chapter II, the subspace W = [B, C] is the set

$$\{P \mid P = rB + sC, r, s \in R\}.$$

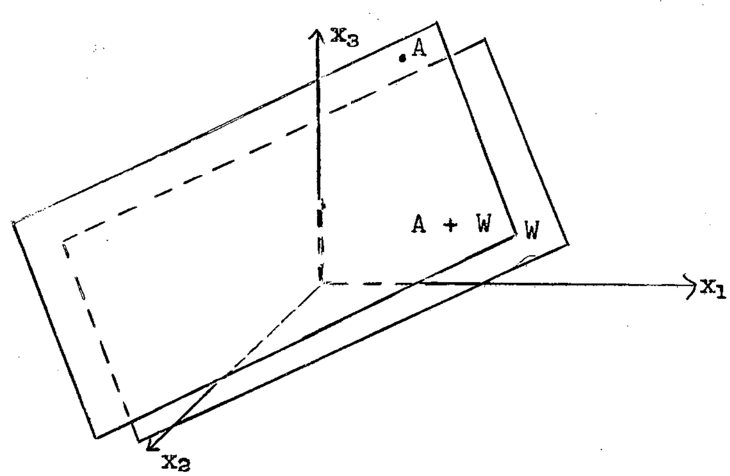

Figure 2.

Intuitively, the subset $W = \theta + W$ is a plane containing the origin. The coset $A + W = A + [B, C] = \{P | P = A + rB + sC, r, s \in R\}$ is the plane W, containing the origin, "translated" parallel to W and containing the point A. Two cosets $A + [B, C]$ and $D + [B, C]$ are equal if and only if $D - A \in [B, C]$. That is, D is A added to some vector in $[B, C]$. Similar remarks would hold for cosets of one dimensional subspaces and lines.

The first postulate of SMSG geometry is proved as Theorem I in the vector space V.

Theorem I. Given any two different points there is exactly one line which contains them.

Proof: Let A and B be two different points in V. Then $A - B \neq \theta$. Thus, $\{A - B\}$ is independent and $[A - B]$ is a one dimensional subspace of V. Thus, $A + [A - B]$ is a line and, since $A - A = \theta \in [A - B]$, $A \in A + [A - B]$. Similarly, $A - B \in [A - B]$ implies $B \in A + [A - B]$. Thus, $A + [A - B]$ is a line containing A and B.

Let $D + [C]$ be a line containing A and B. Then $A - B \in [C]$ and this implies, since $A - B \neq \theta$, that $[A - B] = [C]$. Since $A \in D + [C]$, $D + [C] = A + [C] = A + [A - B]$.

The next three postulates of SMSG geometry are concerned with distance and coordinate systems. An inner product on V has been assumed thus giving a norm on V defined by $|A| = \sqrt{A \circ A}$. This norm will be used in proving the next theorem.

Theorem II. (The Distance Postulate) To every pair of different points there corresponds a unique positive number.

Proof: If $A, B \in V$ are different points, then $A - B \neq \theta$.

$|A - B| = |-1||A - B| = |(-1)(A - B)| = |B - A|$. Let the positive number corresponding to A and B be $|A - B|$. Since $|A - B| = |B - A|$, the number does not depend on the order of the points. Since $|A - B| > 0$, this number satisfies the requirements of the theorem.

Definition 3.5. The distance between two points is the positive number given by the Distance Postulate (Theorem II). If the points are P and Q, then the distance is denoted by PQ. [17: 34]

Thus, the number $|P - Q| = |Q - P|$ is denoted PQ.

Postulate three of SMSG geometry is proved next.

Theorem III. The points of a line can be placed in correspondence with the real numbers in such a way that

(1) To every point of the line there corresponds exactly one real number,

(2) To every real number there corresponds exactly one point of the line, and

(3) The distance between two points is the absolute value of the difference of the corresponding numbers.

Proof: The first two requirements of this theorem are that there exists a one-to-one correspondence, f, between the points on the line and the real numbers. The third is that if A and B are on the line, then $AB = |f(A) - f(B)|$.

Let $A + [B]$ be the given line. Since $[B] = \left[\frac{1}{|B|} B\right]$ one may assume that $|B| = 1$.

$P \in A + [B]$ implies $P = A + tB$ where $A + tB$ is the only representation of P. That is, if $P = A + t_1 B = A + t_2 B$, then $t_1 = t_2$. Thus, define a map $f: A + [B] \to R$ by $f(P) = t$. $f(P) = f(Q)$ implies $P = A + f(P)B = A + f(Q)B = Q$. Thus, f is one-to-one. $r \in R$ implies $P = A + rB \in A + [B]$ and $f(P) = r$. Thus, f is onto R and, consequently, f is a one-to-one correspondence.

If $P = A + tB$ and $Q = A + sB$ are on $A + [B]$, then $PQ = |A + tB - (A + sB)| = |(t - s)B| = |t - s||B| = |t - s| \cdot 1 = |t - s| = |f(P) - f(Q)|$.

<u>Definition 3.6</u>. A correspondence of the sort described in Postulate 3 (Theorem III) is called a coordinate system for the line. The number corresponding to a given point is called the <u>coordinate</u> of the point. [17: 37]

A line (coset) $A + [B]$ has many names. In the last theorem the coordinate system was defined in terms of a particular name. If C and D are two distinct points on $A + [B]$, then $D - C \in [B]$. Since $D - C \neq \theta$, $[D - C] = [B]$. Thus, $A + [D - C] = A + [B]$. Since $C \in A + [B]$, $C + [D - C] = A + [B]$. Thus, the following theorem has been proved.

<u>Theorem 3.7</u>. If C and D are distinct elements of coset $A + [B]$, then $C + [D - C] = A + [B]$.

<u>Theorem IV</u>. (The Ruler Placement Postulate) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

Proof: Let $P, Q \in A + [B]$. Then, $A + [B] = P + [Q - P] = P + \left[\frac{1}{|Q - P|}(Q - P)\right]$. Since $P = P + 0\left(\frac{1}{|Q - P|}(Q - P)\right)$ and $Q = P + |Q - P|\left(\frac{1}{|Q - P|}(Q - P)\right)$, by the definition of a coordinate system and Theorem III, the coordinate of P is zero and the coordinate of Q is $|Q - P| > 0$.

<u>Definition 3.8</u>. B is <u>between</u> A and C if (1) A, B and C are distinct points on the same line and (2) AB + BC = AC. [17: 41]

B is between A and C will be denoted by $A \circ B \circ C$.

<u>Theorem 3.9</u>. If on the line $A + [C - A]$ a coordinate is chosen so that the coordinate of A is zero and the coordinate of C is $|C - A|$, then the line $A + [C - A]$ is partitioned into four disjoint sets given by:

i) $\{B|A \circ B \circ C\} = \{B|B = A + t(C - A), 0 < t < 1\}$.

ii) $\{B|A \circ C \circ B\} = \{B|B = A + t(C - A), 1 < t\}$.

iii) $\{B|B \circ A \circ C\} = \{B|B = A + t(C - A), t < 0\}$.

iv) $\{B|B = A \text{ or } B = C\} = \{B|B = A + t(C - A), t = 0 \text{ or } t = 1\}$.

Proof: Let B be a point on the line $A + [C - A] = A + \left[\frac{1}{|C - A|}(C - A)\right]$, $B \neq C$ and $B \neq A$. Then, $B = A + s\left(\frac{1}{|C - A|}(C - A)\right)$ for some real number s, $s \neq 0$ and $s \neq |C - A|$. By Definition 3.6, the coordinate of A is zero, the coordinate of C is $|C - A|$ and the coordinate of B is s.

By the definition of betweeness for points on a line, $A \circ B \circ C$ if and only if AB + BC = AC. This is equivalent to $|s - 0| + ||C - A| - s| = ||C - A|| - 0|$ or $|s| + ||C - A| - s| = |C - A|$. This implies that $0 < s < |C - A|$ or $0 < \frac{s}{|C - A|} < 1$. Thus, $B = A + t(C - A)$ where $t = \frac{s}{|C - A|}$ and $0 < t < 1$. Similar arguments would show $A \circ C \circ B$ if and only if $1 < \frac{s}{|C - A|}$ and $B \circ A \circ C$ if and only if $\frac{s}{|C - A|} < 0$.

<u>Definition 3.10</u>. For any two points A and B the <u>segment</u> $\overline{AB}$ is the

set whose points are A and B, together with all points that are between A and B. The points A and B are called the <u>endpoints</u> of $\overline{AB}$. [17: 45]

Thus, in vector space V, $\overline{AB} = \{C | C = A + t(B - A),\ 0 \leq t \leq 1\}$.

<u>Definition 3.11</u>. The distance AB is called the length of the segment $\overline{AB}$. [17: 45]

<u>Definition 3.12</u>. Let A and B be points of a line L. The <u>ray</u> $\overrightarrow{AB}$ is the set which is the union of (1) the segment $\overline{AB}$ and (2) the set of all points C for which B is between A and C. [17: 46]

<u>Definition 3.13</u>. If A is between B and C, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called opposite rays. [17: 46]

Thus, $\overrightarrow{AB} = \{C | C = A + t(B - A),\ 0 \leq t\}$. If $B \circ A \circ C$, then $\{P | P = A + t(C - A),\ 0 \leq t\}$ and $\{P | P = A + t(C - A),\ t \leq 0\} = \{P | P = A + t(B - A),\ t \geq 0\}$ are opposite rays.

<u>Definition 3.14</u>. A set of points is collinear if there is a line which contains all the points of the set. [17: 54]

<u>Definition 3.15</u>. A set of points is <u>coplanar</u> if there is a plane which contains all the points of the set. [17: 54]

The next theorem, which is postulate 5 in SMSG geometry, is based upon the definition of a plane as a coset of a <u>two</u> dimensional subspace and the assumption that space is a <u>three</u> dimensional vector space.

<u>Theorem V</u>. (a) Every plane contains at least three non-collinear points. (b) Space contains at least four non-coplanar points.

Proof: (a) Let $A + [B, C]$ be a plane. Then $A + \theta = A$, $A + 1B = A + B$ and $A + 1C = A + C$ are in $A + [B, C]$.

Suppose $D + [E]$ is a line containing each of these points. Then, $(A + B) - A = B \in [E]$ and $(A + C) - A = C \in [E]$. But since $[E]$ is a one dimensional subspace and $\{B, C\}$ is linearly independent, this is a contradiction. Thus, A, $A + B$ and $A + C$ are non-collinear and in the plane $A + [B, C]$.

(b) Since V is three dimensional, $V = [A_1, A_2, A_3]$ for some A_1, A_2 and A_3. $A_1 + A_2$, $A_1 + A_3$, A_1 and A_2 are four points of V.

Suppose $B + [C, D]$ is a plane containing each of these points. Then $(A_1 + A_2) - A_1 = A_2$, $(A_1 + A_3) - A_1 = A_3$ and $(A_1 + A_2) - A_2 = A_1$ are in $[C, D]$ contradicting the fact that $[C, D]$ is two dimensional. Thus, $A_1 + A_2$, $A_1 + A_3$, A_1, and A_3 are four non-coplanar points in V.

Using Theorem 2.22 of Chapter II, which states that the non-empty intersection of two cosets is a coset, the next theorem is proved.

Theorem VI. If two points lie in a plane, then the line containing these points lies in the same plane.

Proof: Let D and E be two points in plane $A + [B, C]$. Since $D \in A + [B, C]$, $A + [B, C] = D + [B, C]$. Since $D, E \in D + [B, C]$, $D - E \in [B, C]$. Thus, $[D - E] \subset [B, C]$. Therefore, $(D + [D - E]) \cap (D + [B, C]) = D + ([D - E] \cap [B, C] = D + [D - E]$ or the line, $D + [D - E]$, is contained in the plane $D + [B, C]$ containing D and E.

Theorem VII. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.

Proof: Let A, B, and C be three points of space. If A, B, and C are collinear, then they are in some line $A + [D]$. Therefore, $B - C \in [D]$ or $[B - C] = [D]$. Thus, the three points are in $A + [B - C] = A + [D]$. Since V is three dimensional, there exists a point E such that [B - C, E] is two dimensional. Thus, A, B and C are in A + [B - C, E].

If A, B and C are non-collinear, then {A - C, B - C} is linearly independent. If not, then A - C = r(B - C) implies A = C + r(B - C) or $A \in C + [B - C]$, which is a line containing B and C. Thus, since C = C + 0(A - C) + 0(B - C), A = C + 1(A - C) + 0(B - C), and B = C + 0(A - C) + 1(B - C), C + [A - C, B - C] is a plane containing A, B and C.

If D + [E, F] is another plane containing A, B, and C, then A - C and B - C are linearly independent and in [E, F]. Thus, [E, F] = [A - C, B - C] and, since $C \in D + [E, F]$, D + [E, F] = C + [A - C, B - C].

In the next theorem the assumption that V is three dimensional assumes a paramount role. The next theorem requires that the intersection of two different cosets of two dimensional subspaces, whose intersection is not empty, is a coset of a one dimensional subspace. This statement is not necessarily true in higher dimensional subspaces. This can be seen by looking at the cosets (1,1,1,1) + [(1,0,0,0), (0,1,0,0)] and (1,1,1,1) + [(0,0,1,0), (0,0,0,1)] of the two subspaces [(1,0,0,0), (0,1,0,0)] and [(0,0,1,0), (0,0,0,1)] of R_4. The intersection of these two cosets is {(1,1,1,1)} since (1,1,1,1) + r_1(1,0,0,0) + r_2(0,1,0,0) = (1,1,1,1) + r_3(0,0,1,0) + r_4(0,0,0,1) implies $r_1 = r_2 = r_3 = r_4 = 0$.

It is proved, in the next theorem, that in three dimensional space this intersection must be a line.

<u>Theorem VIII</u>. If two different planes intersect, then their intersection is a line.

Proof: Let A + [B, C] and D + [E, F] be two different planes. Since these planes intersect there is a point P in the intersection.

Then, by Theorem 2.22, (A + [B, C]) ∩ (D + [E, F]) = P + ([B, C] ∩ [E, F]). However, [B, C] ∩ [E, F] cannot be the zero subspace because if so then $r_1B + r_2C + r_3E + r_4F = \theta$ would imply that $r_1B + r_2C = -r_3E - r_4F$. This gives $r_1 = r_2 = r_3 = r_4 = 0$ or {B, C, E, F} is linearly independent. This contradicts V being three dimensional. [B, C] ∩ [E, F] cannot be either of the two subspaces since different cosets of the same subspace do not intersect. Thus, [B, C] ∩ [E, F] is a one dimensional subspace and P + ([B, C] ∩ [E, F]) is a line.

Convex Sets and Separation

Definition 3.16. A set A is called convex if for every two points P and Q of A, the entire segment $\overline{PQ}$ lies in A. [17: 62]

In a vector space a set A is convex if for every two points P and Q of A, the set $\{B | B = Q + t(P - Q), 0 \leq t \leq 1\}$ is a subset of A.

Theorem 3.17. A line, a plane, and space is each a convex set.

Proof: (a) If C, D ∈ A + [B], then C - D ∈ [B]

Therefore, $A + [B] = C + [C - D] = \{C + t(C - D) \mid t \in R\}$.

Therefore, $\{C + t(C - D) \mid 0 \leq t \leq 1\} \subset A + [B]$.

(b) If E, F ∈ A + [B, C], then E + [E - F] ⊂ A + [B, C]. Therefore, $\overline{EF} \subset A + [B, C]$.

(c) Space contains all points.

The next two theorems will be proved in reverse order from their presentation in SMSG geometry. Before proving them, and example will be given (Figure 3).

Example:

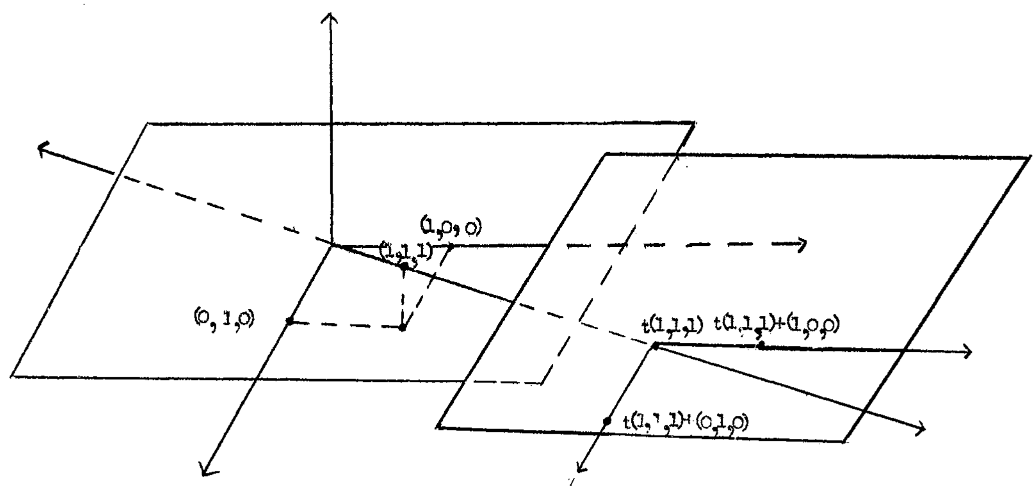

Figure 3

Consider R_3 with its usual inner product, $(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = x_1y_1 + x_2y_2 + x_3y_3$. The next theorem requires that a plane, such as $t(1,1,1) + [(1,0,0)(0,1,0)]$, $t \in R$, "separates" space into three parts, the plane and the two "half-spaces," such that each "half-space" is convex. How can these half-spaces be defined?

The basis $\{(1,0,0), (0,1,0)\}$ for the subspace $[(1,0,0), (0,1,0)]$ can be completed to an orthogonal basis for R_3. Since the basis for $[(1,0,0), (0,1,0)]$ was chosen so nicely it is seen that an orthogonal basis for R_3 could be the set, $\{(1,0,0), (0,1,0), (0,0,1)\}$.

Define a map $f: R_3 \to R$ by $f(x_1, x_2, x_3) = (x_1, x_2, x_3) \circ (0,0,1)$ where $(0,0,1)$ is the vector used to complete the basis. By Theorem 2.28, f is a linear transformation and $f^{-1}(0)$ is a two dimensional subspace of R_3. Since $(1,0,0) \circ (0,0,1) = 0$ and $(0,1,0) \circ (0,0,1) = 0$, this two dimensional space must be $[(1,0,0), (0,1,0)]$. Thus,

$(x_1, x_2, x_3) \in [(1,0,0), (0,1,0)]$ if and only if $(x_1, x_2, x_3) \circ (0,0,1) = 0$.

Consider the cosets $t(1,1,1) + [(1,0,0) (0,1,0)]$, $t \in R$. Two different t give different cosets for if $t_1(1,1,1) - t_2(1,1,1) = (t_1 - t_2)(1,1,1) \in [(1,0,0), (0,1,0)]$, then $t_1 - t_2 = 0$ or $t_1 = t_2$. If (x_1, x_2, x_3) is a given point, then $(x_1, x_2, x_3) - t(1,1,1) = r(1,0,0) + s(0,1,0)$ always has solutions in t, r, s. Therefore, every coset $(x_1, x_2, x_3) + [(1,0,0), (0,1,0)]$ has a vector of the form $t(1,1,1)$ in it. Thus, as the subspace $[(1,0,0), (0,1,0)]$ is translated parallel to itself along the line $\{t(1,1,1)\}$ every coset of $[(1,0,0), (0,1,0)]$ is obtained.

One can define the coset $t(1,1,1) + [(1,0,0), (0,1,0)]$ in terms of the inner product. Since, if $(x_1, x_2, x_3) \in t(1,1,1) + [(1,0,0), (0,1,0)]$, then $(x_1, x_2, x_3) = t(1,1,1) + r(1,0,0) + s(0,1,0)$, for some r, $s \in R$. Therefore, $(x_1, x_2, x_3) \circ (0,0,1) = (t(1,1,1) + r(1,0,0) + s(0,1,0)) \circ (0,0,1) = t(1,1,1) \circ (0,0,1)$. Also, if $(x_1, x_2, x_3) \circ (0,0,1) = t(1,1,1) \circ (0,0,1)$, then $((x_1, x_2, x_3) - t(1,1,1)) \circ (0,0,1) = 0$. This means, $((x_1, x_2, x_3) - t(1,1,1)) \in [(1,0,0), (0,1,0)]$ or $(x_1, x_2, x_3) + [(1,0,0), (0,1,0)] = t(1,1,1) + [(1,0,0), (0,1,0)]$. Thus, the coset $t(1,1,1) + [(1,0,0), (0,1,0)] = \{(x_1, x_2, x_3) \mid (x_1, x_2, x_3) \circ (0,0,1) = t(1,1,1) \circ (0,0,1)\}$.

For a fixed t, such as $t = 1$, the plane $(1,1,1) + [(1,0,0), (0,1,0)] = \{(x_1, x_2, x_3) \mid (x_1, x_2, x_3) \circ (0,0,1) = (1,1,1) \circ (0,0,1)\}$ is obtained. If P is a point in space, then P is in some coset $t(1,1,1) + [(1,0,0), (0,1,0)]$. Therefore, $P \circ (0,0,1) = t(1,1,1) \circ (0,0,1)$. It seems reasonable to expect that the points in cosets with $t < 1$ will be in one "half-space" determined by $(1,1,1) + [(1,0,0), (0,1,0)]$ while the points in cosets with $t > 1$ will be in the other "half-space". This is what the proof of Theorem X proves. Before proving this theorem a lemma is proved first.

Lemma 3.18. If $A + [B, C]$ is a plane with $\{B, C\}$ an orthogonal set and $\{B, C, D\}$ an orthogonal basis of V, then $A + [B, C] = \{P | P \circ D = A \circ D\}$.

Proof: The proof is similar to the example. Define a map $f: V \to R$ by $f(P) = P \cdot D$. f is a linear transformation having $f^{-1}(0)$ a two dimensional subspace of V. Since by assumption $f(B) = B \cdot D = 0$ and $f(C) = C \circ D = 0$, $f^{-1}(0) = [B, C]$. That is, $[B, C] = \{Q | Q \circ D = 0\}$.

To prove $A + [B, C] = \{P | P \circ D = A \circ D\}$, let $P \in A + [B, C]$. Then, $P = A + sB + tC$ for some $s, t \in R$. Therefore, $P \circ D = A \circ D + sB \circ D + tC \circ D = A \circ D$. Thus, $P \in \{P | P \circ D = A \circ D\}$. If $P \in \{P | P \circ D = A \circ D\}$, then $(P - A) \circ D = 0$. Therefore, $P - A \in [B, C]$ or $P \in A + [B, C]$. Thus, the sets are equal by definition of equality of sets.

Theorem X. (The Space Separation Postulate.) The points of space that do not lie in a given plane form two sets such that

(1) each of the sets is convex and

(2) if P is in one set and Q is in the other, then the segment $\overline{PQ}$ intersects the plane.

Proof: (1) Let $A + [B, C]$ be the plane and $\{B, C, D\}$ be a basis for V as in Lemma 3.18. Then $A + [B, C] = \{P | P \circ D = A \circ D\}$. Let $H_1 = \{P | P \circ D > A \circ D\}$ and $H_2 = \{P | P \circ D < A \circ D\}$.

H_1 is convex for if $P, Q \in H_1$ and $E \in \overline{PQ}$, then $E = P + t(Q - P) = (1 - t)P + tQ$, $0 \leq t \leq 1$. Therefore, $E \circ D = (P + t(Q - P)) \circ D =$

$(1 - t)P \cdot D + tQ \cdot D > (1 - t)A \cdot D + tA \cdot D = A \cdot D$. Thus, $E \in H_1$ or $\overline{PQ} \subset H_1$. Therefore, H_1 is convex. Similarly, H_2 is convex.

(2) If $Q \in H_1$ and $P \in H_2$, then $P \cdot D < A \cdot D$ and $Q \cdot D > A \cdot D$. Therefore, $P \cdot D < A \cdot D < Q \cdot D$ or $0 < A \cdot D - P \cdot D < Q \cdot D - P \cdot D$. Thus, $0 < \frac{A \cdot D - P \cdot D}{Q \cdot D - P \cdot D} < 1$. Consider $E = \left(1 - \left(\frac{A \cdot D - P \cdot D}{Q \cdot D - P \cdot D}\right)\right)P + \left(\frac{A \cdot D - P \cdot D}{Q \cdot D - P \cdot D}\right)Q$. Now $E \in \overline{PQ}$ and $E \cdot D = \left(1 - \left(\frac{A \cdot D - P \cdot D}{Q \cdot D - P \cdot D}\right)\right)(P \cdot D) + \left(\frac{A \cdot D - P \cdot D}{Q \cdot D - P \cdot D}\right)(Q \cdot D) = \frac{(Q \cdot D)(P \cdot D) - (A \cdot D)(P \cdot D)}{Q \cdot D - P \cdot D} + \frac{(A \cdot D)(Q \cdot D) - (P \cdot D)(Q \cdot D)}{Q \cdot D - P \cdot D} = A \cdot D$. Thus, $E \in A + [B, C]$ and the segment $\overline{PQ}$ intersects the plane. The proof is complete.

The sets H_1 and H_2 of Theorem X are independent of the name, $A + [B, C]$, of the plane. Because, if $A + [B, C] = E + [B, C]$, then $A - E \in [B, C]$. Therefore, $(A - E) \cdot D = 0$ or $A \cdot D = E \cdot D$. Thus, H_1 and H_2 are independent of the point in the coset chosen to name it. Also, there are exactly two unit vectors, D and $-D$, orthogonal to both B and C. Therefore, if $-D$ is chosen instead of D, then
$H_1 = \{P | P \cdot D > A \cdot D\} = \{P | P \cdot (-D) < A \cdot (-D)\}$ and
$H_2 = \{P | P \cdot D < A \cdot D\} = \{P | P \cdot (-D) > A \cdot (-D)\}$. Thus, H_1 and H_2 are independent of the vector D chosen to complete $\{B, C\}$ to a basis for V. If $[B, C] = [B_1, C_1]$, then D is also orthogonal to both B_1 and C_1. Therefore, H_1 and H_2 are independent of the basis chosen for $[B, C]$.

<u>Definition 3.19</u>. The two sets determined by Postulate 10 (Theorem X) are called half-spaces, and the given plane is called the <u>face</u> of each of them. [17: 66]

The next theorem is similar to the last. It requires that a line in a given plane separate the plane into two "half-planes." The idea of the proof is to characterize the line as the intersection of another plane with the given plane. Then, the two half-spaces of the plane are used to determine the two "half-planes" of the line. Before proving the main theorem, a lemma is proved.

Lemma 3.20. Let $A + [B]$ be a line in the plane $A + [B, C]$. If B is orthogonal to C, then $A + [B] = \{P | P \in A + [B, C] \text{ and } P \cdot C = A \cdot C\}$.

Proof: First it should be noted that if B is not orthogonal to C, then a vector C_1 can be chosen such that $A + [B, C] = A + [B, C_1]$ with B orthogonal to C_1.

$\{Q | Q \circ C = 0\}$ is a two dimensional subspace of V. Since $B \circ C = 0$, $\{B\}$ can be completed to a basis $\{B, D\}$ for this space. Thus, $\{Q | Q \circ C = 0\} = [B, D]$.

$(A + [B, C]) \cap (A + [B, D])$ is a coset of $[B, C] \cap [B, D]$. Since $C \circ C > 0$, $C \notin [B, D]$. Therefore, $[B, C] \cap [B, D] \neq [B, C]$. Thus, $[B, C] \cap [B, D] = [B]$ and $(A + [B, C]) \cap (A + [B, D]) = A + [B]$. Therefore, $P \in A + [B]$ implies $P \in A + [B, D]$ or $P = A + Q$, $Q \in [B, D]$. Thus, $P \circ C = A \circ C + Q \circ C = A \circ C$. Also, if $P \in A + [B \circ C]$ and $P \circ C = A \circ C$, then $(P - A) \circ C = 0$. Therefore $P - A \in [B, D]$ or $P \in A + [B, D]$. Thus, $P \in (A + [B, C]) \cap (A + [B, D]) = A + [B]$. Since both inclusions have been shown, $A + [B] = \{P | P \in A + [B, C] \text{ and } P \circ C = A \circ C\}$.

Theorem IX. (The Plane Separation Postulate.) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that

(1) each of the sets is convex and

(2) if P is in one set and Q is in the other, then the

segment $\overline{PQ}$ intersects the line.

Proof: Let the line be $A + [B]$ and the plane containing it be $A + [B, C]$ where C is orthogonal to B. By Lemma 3.20, $A + [B] = \{P \mid P \in A + [B, C] \text{ and } P \cdot C = A \cdot C\}$.

Let $H_1 = \{P \mid P \in A + [B, C] \text{ and } P \cdot C > A \cdot C\}$ and $H_2 = \{P \mid P \in A + [B, C] \text{ and } P \cdot C < A \cdot C\}$. Let P and Q be in H_1. Then, $P \cdot C > A \cdot C$ and $Q \cdot C > A \cdot C$. The segment $\overline{PQ} = \{D \mid D = P + t(Q - P), 0 \leq t \leq 1\} = \{D \mid D = (1 - t)P + tQ, 0 \leq t \leq 1\}$.

If $E \in \overline{PQ}$, then $E = (1 - t)P + tQ$. Therefore, $E \cdot C = (1 - t)P \cdot C + t(Q \cdot C) > (1 - t)A \cdot C + tA \cdot C = A \cdot C$. Since $P, Q \in A + [B, C]$, by Theorem VI, $\overline{PQ} \subset A + [B, C]$. Thus, $E \in A + [B, C]$ and $E \cdot C > A \cdot C$. Therefore, $E \in H_1$. Thus, $\overline{PQ} \subset H_1$ and H_1 is convex. Similarly, H_2 is convex.

If $P \in H_1$ and $Q \in H_2$, then $P \cdot C > A \cdot C > Q \cdot C$. Therefore, as in the proof of Theorem X, $1 > \frac{A \cdot C - Q \cdot C}{P \cdot C - Q \cdot C} > 0$. Let $t = \frac{A \cdot C - Q \cdot C}{P \cdot C - Q \cdot C}$. Then, $E = tP + (1 - t)Q$ is in $\overline{PQ}$. $E \cdot C =$

$$tP \cdot C + (1 - t)Q \cdot C = \left(\frac{A \cdot C - Q \cdot C}{P \cdot C - Q \cdot C}\right) P \cdot C +$$

$$\left(1 - \frac{A \cdot C - Q \cdot C}{P \cdot C - Q \cdot C}\right) Q \cdot C = \left(\frac{A \cdot C - Q \cdot C}{P \cdot C - Q \cdot C}\right) P \cdot C +$$

$$\left(\frac{P \cdot C - A \cdot C}{P \cdot C - Q \cdot C}\right) Q \cdot C = \frac{(A \cdot C)(P \cdot C) - (A \cdot C)(Q \cdot C)}{P \cdot C - Q \cdot C} = A \cdot C.$$

Therefore, $E \in \overline{PQ} \cap A + [B]$. Thus, the sets H_1 and H_2 satisfy the requirements of the theorem. By arguments similar to those at the end of Theorem X, it can be shown that the sets H_1 and H_2 are independent of the names of the plane and line.

<u>Definition 3.21</u>. Given a line L and plane E containing it, the two sets determined by Postulate 9 (Theorem IX) are called <u>half-planes</u>, and L is called the <u>edge</u> of each of them. It is said that L <u>separates</u>

E into the two half-planes. If two points P and Q of E lie in the same half-plane, it is said that they lie on the same side of L; if P lies in one of the half-planes and Q in the other, they lie on opposite sides of L. [17: 64]

The constructive steps used to determine the half-planes determined by a line will be used so often in the next chapter that they are listed below:

1. If A + [B] is a line in plane D + [E, F], then change the name of the plane D + [E, F] to A + [B, C] where C is orthogonal to B.

2. The half-planes of the line are then

$$H_1 = \{P \mid P \in A + [B, C] \text{ and } P \circ C > A \circ C\} \text{ and}$$

$$H_2 = \{P \mid P \in A + [B, C] \text{ and } P \circ C < A \circ C\}.$$

CHAPTER IV

ANGLES IN A EUCLIDEAN SPACE

This chapter will include the definition of an angle and the defining of a measure for the angles that will satisfy the postulates of SMSG geometry.

In Chapter III the ray with endpoint A and containing B was defined. This definition was shown to be equivalent to $\overrightarrow{AB} =$ $\{P \mid P = A + t(B - A),\ t \geq 0\}$.

<u>Definition 4.1</u>. An <u>angle</u> is the union of two rays which have the same end-point but do not lie in the same line. The two rays are called the <u>sides</u> of the angle, and their common end-point is called the <u>vertex</u>. [17: 71]

The notation for the angle whose sides are rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is $\measuredangle$ BAC or $\measuredangle$ CAB. It is noted that an angle is a set of points. "Sensed" angles are not considered in the geometry text published by SMSG. That is, there is no distinction made in the sides of the angles such as the initial or terminal side. Also note that the definition of angle does not allow a "straight" angle or a "zero" angle.

<u>Definition 4.2</u>. Let $\measuredangle$ BAC be an angle lying in plane E. A point P of E lies in the <u>interior</u> of $\measuredangle$ BAC if (1) P and B are on the same side of the line $\overleftrightarrow{AC}$ and also (2) P and C are on the same side of the line $\overleftrightarrow{AB}$. The <u>exterior</u> of $\measuredangle$ BAC is the set of all points of E that do

not lie in the interior and do not lie on the angle itself. [17: 73]

A measure for angles will be a set function, m, defined on the set of all angles with range contained in the real numbers. Postulates eleven through fourteen of SMSG geometry determine the other properties this measure should have. They include: (1) the range of m should be the real numbers between 0 and 180, (2) m should be additive, (3) angles with two sides collinear and a common vertex should have measures whose sum is 180, and (4) a ray on the edge of a half-plane should be a side of exactly two angles in this plane with a given measure.

The cosine function will be used to define the measure of an angle. Facts about this function which will be assumed are:

(1) Cosine is a function whose domain is the set of all real numbers and range is the closed interval $[-1, 1]$.

(2) Cosine is periodic with period 2π.

(3) Cosine has minimum value of negative one at real numbers of the form $(2n + 1)\pi$, $n = \ldots, -1, 0, 1, \ldots$. and maximum values of one at real numbers of the form $2n\pi$, $n = \ldots -1, 0, 1, \ldots$.

(4) Cosine is continuous and one-to-one on the open interval $(0, \pi)$ taking on every real number in the interval $(-1,1)$.

(5) Cosine $(r + s) = \cos r \cos s - \sin r \sin s$.

(6) If $a^2 + b^2 = 1$, $-1 < a < 1$ and $a = \cos r$, then $b = \pm \sin r$.

(7) The sine function is positive on the interval $(0, \pi)$ and negative on the interval $(\pi, 2\pi)$.

(8) $\sin (r + s) = \sin r \cos s + \cos r \sin s$.

Consider an angle $\measuredangle$ BAC. By Definition 4.1, $\measuredangle$ BAC = $\{A + t(B - A),\ 0 \leq t\} \cup \{A + t(C - A),\ 0 \leq t\}$ where A, B, and C are distinct non-collinear points.

Definition 4.3. The measure of angle $\measuredangle$ BAC, written m $\measuredangle$ BAC, is the real number r, with $0 < r < \pi$, where $\cos r = \frac{(B - A) \circ (C - A)}{|B - A|\ |C - A|}$.

Two things should be noticed about this definition. The first is that cos r only takes on values between -1 and 1. By the Schwartz inequality, Theorem 2.15, $|(B - A) \circ (C - A)| \leq |B - A||C - A|$ with equality if and only if one of the vectors B - A or C - A is a scalar multiple of the other. Since A, B, and C are non-collinear, equality cannot hold. Thus, $|(B - A) \circ (C - A)| < |B - A||C - A|$. Therefore, $-1 < \frac{(B - A) \circ (C - A)}{|B - A|\ |C - A|} < 1$ and the number r referred to in the definition always exists. Since cosine is one-to-one on $(0, \pi)$ there is exactly one r satisfying the conditions of the definition.

The second thing to notice about the definition is that the number r is defined in terms of a name, $\measuredangle$ BAC, for the angle. But if $\measuredangle$ BAC = $\measuredangle$ DAF, then D is on ray $\overrightarrow{AB}$ and F is on ray $\overrightarrow{AC}$, or F is on $\overrightarrow{AC}$ and D is on $\overrightarrow{AB}$. Suppose the first case occurs. Then, $D = A + t(B - A)$ and $F = A + s(C - A)$ for some $t > 0$ and $s > 0$. Therefore,

$$\frac{(B - A) \circ (C - A)}{|B - A|\ |C - A|} = \frac{t(B - A) \circ s(C - A)}{|t(B - A)|\ |s(C - A)|} = \frac{(D - A) \circ (F - A)}{|D - A|\ |F - A|},$$

and the measure of the angle is independent of the name used for the angle.

Theorem XI. (The Angle Measurement Postulate). To every angle $\measuredangle$ BAC there corresponds a real number between 0 and 180.

Proof: In the remarks preceding this theorem, it was shown that for every angle $\measuredangle$ BAC there exists a unique real number m $\measuredangle$ BAC where

$\cos(m \measuredangle BAC) = \frac{(B - A) \cdot (C - A)}{|B - A| \; |C - A|}$ and $0 < m \measuredangle BAC < \pi$. The inequality involving $m \measuredangle BAC$ can be multiplied by $\frac{180}{\pi}$ to get $0 < \frac{180}{\pi} m \measuredangle BAC < 180$. Define a new measure, m', for angles by $m' = \frac{180}{\pi} m$. This new measure would give the desired result. This proves the theorem.

The limits on the measure of an angle are quite arbitrary. The measure m, defined in Definition 4.3, has the measure of an angle between zero and π. Postulate 11, of SMSG, requires that the SMSG measure of an angle be between zero and 180. The measure $m' = \frac{180}{\pi} m$ gives the correct limits. Hereafter, in each theorem requiring
$0 < m' \measuredangle ABC < 180$, the writer will demonstrate that
$0 < m \measuredangle ABC < \pi$. The required result would be immediate.

The point $P = A + \frac{1}{|B - A|}(B - A)$ is the unique point in ray $\overrightarrow{AB}$ with $|P - A| = 1$. This fact is used in the following definition.

<u>Definition 4.4</u>. Let $\overrightarrow{AB}$ be a ray. The vector $P - A$ with $|P - A| = 1$ and P in ray $\overrightarrow{AB}$ is called the direction vector for ray $\overrightarrow{AB}$. Note: The direction vector is not necessarily a point in ray $\overrightarrow{AB}$.

<u>Theorem 4.5</u>. Let $\measuredangle BAC$ and $\measuredangle DEF$ be two angles. If the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have the same direction vectors as the rays $\overrightarrow{ED}$ and $\overrightarrow{EF}$, then $m \measuredangle BAC = m \measuredangle DEF$.

Proof: By hypothesis $\overrightarrow{AB} = \{A + tQ,\ t \geq 0\}$, $\overrightarrow{AC} = \{A + tP,\ t \geq 0\}$, $\overrightarrow{ED} = \{E + tQ,\ t \geq 0\}$ and $\overrightarrow{EF} = \{E + tP,\ t \geq 0\}$ where P and Q are direction vectors. $A + Q \in \overrightarrow{AB}$, $A + P \in \overrightarrow{AC}$, $E + Q \in \overrightarrow{EF}$ and $E + P \in \overrightarrow{EF}$. Therefore, by definition of the measure of an angle, $\cos(m \measuredangle BAC) =$

$$\frac{(A + Q - A) \circ (A + P - A)}{|A + Q - A| \; |A + P - A|} = \frac{Q \circ P}{|Q|\,|P|} = \frac{(E + Q - E) \circ (E + P - E)}{|E + Q - E| \; |E + P - E|} =$$

$\cos(m \measuredangle DEF)$. Thus $m \measuredangle BAC = m \measuredangle DEF$.

Example: Let $V = R_3$ with usual inner product. Let $A = (1,1,1)$, $B = (1 + 2\sqrt{3}, 3, 1)$ and $C = (4, 1 + 3\sqrt{3}, 1)$. $\frac{(B - A)}{|B - A|} \cdot \frac{(C - A)}{|C - A|} =$ $\frac{(2\sqrt{3}, 2,0) \cdot (3, 3\sqrt{3}, 0)}{|(2\sqrt{3}, 3,0)||(3, 3\sqrt{3}, 0)|} = \frac{\sqrt{3}}{2}$. Therefore, since $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $m\,(\measuredangle\, BAC) = \frac{\pi}{6}$.

The direction vector on ray $\overrightarrow{AB}$ is $B_1 = \frac{1}{|B - A|}(B - A) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$ and the direction vector on ray $\overrightarrow{AC}$ is $C_1 = \frac{1}{|C - A|}(C - A) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right)$. Any angle $\measuredangle\, DEF$ with $E = (x_1, x_2, x_3)$, $D = E + B_1$ and $F = E + C_1$ would have the same measure as $\measuredangle\, BAC$.

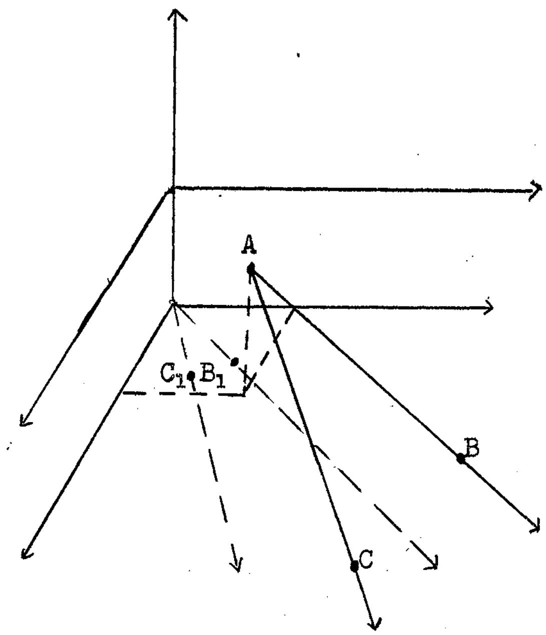

Figure 4.

$\measuredangle\, BAC$ is in plane $(1,1,1) + [(1,0,0), (0,1,0)]$. Since $A + B_1$ is in this plane $(1,1,1) + [(1,0,0), (0,1,0)]$, B_1 is in $[(1,0,0), (0,1,0)]$. Therefore, $\{B_1\}$ can be completed to an orthonormal basis for the plane $[(1,0,0), (0,1,0)]$. That is $[(1,0,0), (0,1,0)] = [B_1, D]$. In this example, since $B_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$, the possible choices for D are $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}, 0\right)$ and $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0\right)$. Suppose $D = \left(\frac{1}{2}, \frac{-\sqrt{3}}{2}, 0\right)$ is chosen.

Since $C_1 \in [B_1, D]$, there are scalars a and b such that $C_1 = aB_1 + bD$ or $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) = a\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right) + b\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}, 0\right)$. It is seen that $a = \frac{\sqrt{3}}{2}$ and $b = \frac{-1}{2}$. Therefore, $a = \frac{\sqrt{3}}{2} = \cos m \measuredangle\, BAC = \cos \frac{\pi}{6}$ and $-b = \frac{1}{2} = \sin m \measuredangle\, BAC$. If the vector $-D$ were chosen to complete the basis instead of D, then the results would have been that $\sin \measuredangle\, BAC = b$.

The last part of the example serves as an illustration of the following two general theorems.

Theorem 4.6. If $\measuredangle$ BAC is an angle with B_1 and C_1 as direction vectors for rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, respectively, then $\measuredangle$ BAC is in plane $A + [B_1, D]$ and $C_1 \in [B_1, D]$ where D is a vector used to complete $\{B_1\}$ to an orthonormal basis for the subspace $[B - A, C - A]$.

Proof: Since A, B, and C are three non-collinear points, there exists exactly one plane, $A + [B - A, C - A]$, containing them. Since $B_1 = \frac{1}{|B - A|}(B - A)$, $B_1 \in [B - A, C - A]$. Therefore, $\{B_1\}$ can be completed to an orthonormal basis $\{B_1, D\}$ for $[B - A, C - A]$ or $[B - A, C - A] = [B_1, D]$. Therefore, A, B, and C are in $A + [B_1, D] = A + [B - A, C - A]$. Since two points of each of the rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are in $A + [B_1, D]$, $\measuredangle$ BAC is contained in $A + [B_1, D]$. Also, $C_1 = \frac{1}{|C - A|}(C - A)$ implies $C_1 \in [B_1, D] = [B - A, C - A]$.

Theorem 4.7. With the information given in Theorem 4.6, if $C_1 = aB_1 + bD$, then $\cos m \measuredangle BAC = a$ and $\sin m \measuredangle BAC = \pm b$.

Proof: Since $\{B_1, D\}$ is an orthonormal basis for $[B_1, D]$, by Theorem 2.19, $C_1 \circ C_1 = a^2 + b^2$. Since $|C_1| = 1$, this implies $a^2 + b^2 = 1$. Now $\overrightarrow{AB} = \overrightarrow{AE}$ and $\overrightarrow{AC} = \overrightarrow{AF}$ where $E = A + B_1$ and $F = A + C_1$. Therefore, $\cos (m \measuredangle BAC) = \frac{B_1 \circ C_1}{|B_1||C_1|} = B_1 \circ (aB_1 + bD) = aB_1 \circ B_1 + bB_1 \circ D = a$. Also, since $a^2 + b^2 = 1$, $b = \pm \sin (m \measuredangle BAC)$.

Theorem XII. (The Angle Construction Postulate) Let $\overrightarrow{AB}$ be a ray on the edge of half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$ with P in H, such that $m \measuredangle PAB = r$.

Proof: For some D in V, the plane containing ray $\overrightarrow{AB}$ and half-plane H is $A + [B_1, D]$ where B_1 is the direction vector for ray $\overrightarrow{AB}$ and $\{B_1, D\}$ is an orthonormal basis for $[B_1, D]$. IF $P \in A + [B_1, D]$ and P is not collinear with A and B, then ray $\overrightarrow{AP}$ has a direction vector $P_1 \in [B_1, D]$ with $P_1 = aB_1 + bD$ where $b \neq 0$. Since $|P_1| = 1$, $a^2 + b^2 = 1$. By Theorem 4.7, cos (m ∡ PAB) = a and since $b \neq 0$, $-1 < a < 1$.

For the given r there is exactly one real number a such that cos r = a. There are exactly two b's such that $a^2 + b^2 = 1$. That is, $b = \pm\sqrt{1 - a^2}$.

From Chapter III the two half-planes of plane $A + [B_1, D]$ determined by line $\overleftrightarrow{AB}$ are $H_1 = \{P | P \circ D > A \circ D \text{ and } P \in A + [B_1, D]\}$ and $H_2 = \{P | P \circ D < A \circ D \text{ and } P \in A + [B_1, D]\}$. Now $(A + P_1) \circ D = (A + aB + bD) \circ D = A \circ D + aB \circ D + bD \circ D = A \circ D + b$. Therefore, if $b = \sqrt{1 - a^2}$, then $(A + P_1) \circ D > A \circ D$ and $A + P_1 \in H_1$. If $b = -\sqrt{1 - a^2}$, then $A + P_1 \in H_2$. Thus, there is exactly one ray $\overrightarrow{AP}$, where $A + P_1 \in \overrightarrow{AP}$, in each of the half-planes H_1 and H_2 such that m ∡ PAB = r.

A plane has been defined as a coset of a two dimensional subspace of V. If A, B, and C are three non-collinear points, then the coset $A + [B - A, C - A]$ is the unique coset (plane) containing these points. The following theorem gives another characterization of this coset.

<u>Theorem 4.8</u>. If A, B, and C are three non-collinear points, then the set $\{aA + bB + cC \mid a + b + c = 1\}$ is the plane containing A, B, and C.

Proof: Since it is known that the unique plane containing A, B, and C is $A + [B - A, C - A]$, the theorem will be proved if it is shown that $\{aA + bB + cC \mid a + b + c = 1\} = A + [B - A, C - A]$.

Let $P \in \{aA + bB + cC \mid a + b + c = 1\}$. Then, $P = aA + bB + cC = (1 - b - c)A + bB + cC = A + b(B - A) + c(C - A)$. Therefore, $P \in A + [B - A, C - A]$.

If $P \in A + [B - A, C - A]$, then there exists real numbers s and t such that $P = A + s(B - A) + t(C - A) = (1 - s - t)A + sB + tC$. Letting $a = 1 - s - t$, $b = s$ and $c = t$, $a + b + c = 1 - s - t + s + t = 1$. Consequently, $P \in \{aA + bB + cC \mid a + b + c = 1\}$. Therefore, the sets are equal.

<u>Theorem 4.9</u>. Let $\measuredangle BAC$ be an angle in plane $A + [B - A, C - A]$. The interior of $\measuredangle BAC$ is the set of points

$T = \{aA + bB + cC \mid a + b + c = 1, b > 0, c > 0\}$.

Proof: The half-planes determined by lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are:

$$H_1 = \{P \mid P \circ E > A \circ E \text{ and } P \in A + [B - A, C - A]\}$$

$$H_2 = \{P \mid P \circ E < A \circ E \text{ and } P \in A + [B - A, C - A]\}$$

$$H_1' = \{P \mid P \circ F > A \circ F \text{ and } P \in A + [B - A, C - A]\}$$

$$H_2' = \{P \mid P \circ F < A \circ F \text{ and } P \in A + [B - A, C - A]\}.$$

In the above E is chosen so that $[B - A, C - A] = [B - A, E]$ with E orthogonal to $B - A$. Also, F is chosen so that $[B - A, C - A] = [F, C - A]$ with F orthogonal to $C - A$.

The interior of $\measuredangle BAC$ is the intersection of the half-plane, H_1 or H_2, containing C and the half-plane, H_1' or H_2', containing B. The proof will be given for one case. The other three cases are similar.

Suppose the interior of $\measuredangle BAC$ is $H_1 \cap H_2'$. Then $C \in H_1$ and $B \in H_2'$. Therefore, $C \circ E > A \circ E$ and $(C - A) \circ E > 0$ with $B \circ F < A \circ F$ and $(B - A) \circ F < 0$. Let $P \in H_1 \cap H_2'$. By Theorem 4.8, $P = aA + bB + cC$ with $a + b + c = 1$. Therefore, $P = (1 - b - c)A + bB + cC$ or $P - A = b(B - A) + c(C - A)$. Thus, $(P - A) \circ E = b(B - A) \circ E + c(C - A) \circ E = b \circ 0 + c(C - A) \circ E = c(C - A) \circ E$. Since $P \in H_1$, $(P - A) \circ E > 0$. Since $(C - A) \circ E > 0$, it follows that $c > 0$. Also, $(P - A) \circ F = b(B - A) \circ F + c(C - A) \circ F =$

$b(B - A) \circ F + c \circ 0 = b(B - A) \circ F$. Since $P \in H_2'$, $(P - A) \circ F < 0$. Therefore, since $(B - A) \circ F < 0$, it follows that $b > 0$. Thus, $P \in T = \{aA + bB + cC \mid a + b + c = 1, b > 0, c > 0\}$.

Let $P \in T$. Then $P = aA + bB + cC$, $a + b + c = 1$, $b > 0$, $c > 0$. Thus, $P - A = b(B - A) + c(C - A)$. As before, $(P - A) \circ E = c(C - A) \circ E$ and $(P - A) \circ F = b(B - A) \circ F$. Since $(C - A) \circ E > 0$ and $c > 0$, $(P - A) \circ E > 0$ or $P \in H_1$. Since $(B - A) \circ F < 0$ and $b > 0$, $(P - A) \circ F < 0$ or $P \in H_2'$. Thus, $P \in H_1 \cap H_2'$. Therefore, $T = H_1 \cap H_2'$ and the theorem is proved for this case.

In a beginning course in geometry it is usually assumed that the bisector of an angle of a triangle intersects the opposite side. In synthetic geometry the proof is rather difficult. For example, see [13]. The previous theorem provides means for a rather simple proof of this statement in vector geometry. The next theorem will also imply that if a ray $\overrightarrow{AD}$ is in the interior of $\measuredangle BAC$, then B and C are on opposite sides of line $\overleftrightarrow{AD}$. This fact will be used in Theorem XIII.

<u>Theorem 4.10.</u> If D is a point in the interior of $\measuredangle BAC$, then segment $\overline{BC}$ intersects ray $\overrightarrow{AD}$.

Proof: $\overrightarrow{AD} = \{A + t(D - A), t \geq 0\}$. The interior of $\measuredangle BAC$ is the set, $\{aA + bB + cC \mid a + b + c = 1, b > 0, c > 0\}$. $\overline{BC} = \{B + r(C - B), 0 \leq r \leq 1\}$. The existence of real numbers t and r such that $t \geq 0$, $0 \leq r \leq 1$ and

$$A + t(D - A) = B + r(C - B) \tag{1}$$

is sufficient to prove the theorem.

Since D is in the interior of $\measuredangle BAC$, $D = aA + bB + cC$ with $b > 0$ and $c > 0$. Therefore, $A + t(D - A) = B + r(C - B)$ implies $A + t(aA + bB + cC - A) = B + r(C - B)$ and since $a = 1 - b - c$

$A - tbA - tcA + tbB + tcC = B + r(C - B)$ or $A - B + tb(B - A) + tc(C - A) - r(C - B) = \theta$. Since $C - B = (C - A) - (B - A)$ it follows that, $-1(B - A) + tb(B - A) + tc(C - A) - r(C - A) + r(B - A) = \theta$ or $(tb + r - 1)(B - A) + (tc - r)(C - A) = \theta$. Since $\{B - A, C - A\}$ is linearly independent, this implies that $tb + r - 1 = 0$ and $tc - r = 0$. Thus, $t = \frac{1}{b + c} > 0$ and $r = \frac{c}{b + c}$ with $0 < r < 1$.

A direct substitution in Equation (1) yields:

$$\begin{aligned} A + \frac{1}{b + c}(D - A) &= A + \frac{1}{b + c}(aA + bB + cC - A) \\ &= \frac{1}{b + c}(bA + cA + aA + bB + cC - A) \\ &= \frac{1}{b + c}(bB + cC) \\ &= B + \frac{1}{b + c}(bB + cC - (bB + cB)) \\ &= B + \frac{c}{b + c}(C - B). \end{aligned}$$

Therefore, $\frac{1}{b + c}$ and $\frac{c}{b + c}$ are solutions and the theorem is proved.

Theorem XIII. (The Angle Addition Postulate) If D is a point in the interior of $\measuredangle BAC$, then $m \measuredangle BAC = m \measuredangle BAD + m \measuredangle DAC$.

Proof: The measure of an angle does not depend upon the points used to name the angle. Therefore, it can be assumed that $|B - A| = |C - A| = |D - A| = 1$. A plane containing $\measuredangle BAC$ can be chosen, $A + [D - A, E]$, where $\{D - A, E\}$ is a orthonormal basis for $[D - A, E]$.

Thus, there are real numbers a, b, c and d such that $B - A = a(D - A) + bE$ and $C - A = c(D - A) + dE$. By Theorem 4.7, $\cos m \measuredangle BAD = a$, $\sin m \measuredangle BAD = \pm b$, $\cos m \measuredangle CAD = c$ and $\sin m \measuredangle CAD = \pm d$. The half-planes of $A + [D - A, E]$ determined by line $\overleftrightarrow{AD}$ are $H_1 = \{P | P \circ E > A \circ E \text{ and } P \in A + [D - A, E]\}$ and $H_2 = \{P | P \circ E < A \circ E \text{ and } P \in A + [D - A, E]\}$.

By Theorem 4.10, B and C are in the different half-planes determined by line $\overleftrightarrow{AD}$. Since $(B - A) \cdot E = (a(D - A) + bE) \cdot E =$ $a(D - A) \cdot E + bE \cdot E = b$ and $(C - A) \cdot E = (c(D - A) + dE) \cdot E = d$, b and d are of opposite signs. Since $0 < m \measuredangle BAD < \pi$ and $0 < m \measuredangle DAC < \pi$, both $\sin m \measuredangle BAD$ and $\sin m \measuredangle CAD$ are positive. Thus, $\sin m \measuredangle BAD = -b$ and $\sin m \measuredangle DAC = d$ or $\sin m \measuredangle BAD = b$ and $\sin m \measuredangle DAC = -d$.

Consider the case in which $\sin m \measuredangle BAD = -b$ and $\sin m \measuredangle DAC = d$. $\operatorname{Sin}(m \measuredangle BAD + m \measuredangle DAC) = \sin m \measuredangle BAD \cos m \measuredangle DAC +$ $\cos m \measuredangle BAD \sin m \measuredangle DAC = -bc + ad$. Since $B - A = a(D - A) + bE$ and $C - A = c(D - A) + dE$, $(ad - bc)D = dB - bC + (b - d + ad - bc)A$. If $ad - bc \neq 0$, then $D = \frac{b - d + ad - bc}{ad - bc} A + \frac{d}{ad - bc} B + \frac{-b}{ad - bc} C$. Since D is in the interior of $\measuredangle ABC$, by Theorem 4.9, $\frac{d}{ad - bc} > 0$. Since $d > 0$, $ad - bc > 0$. Since $0 < m \measuredangle BAD < \pi$, $0 < m \measuredangle DAC < \pi$ and sin is negative on $(\pi, 2\pi)$, $m \measuredangle BAD + m \measuredangle DAC < \pi$. This is all under the condition that $ad - bc \neq 0$. If $ad - bc = 0$ then, since on $(0, 2\pi)$ sin is only zero at π, $m \measuredangle BAD + m \measuredangle DAC = \pi$. In either case, $m \measuredangle BAD +$ $m \measuredangle DAC \leq \pi$.

Now $\cos m \measuredangle BAC = (a(D - A) + bE) \cdot (c(D - A) + dE) = ac + bd =$ $ac - (-b)d = \cos m \measuredangle BAD \cos m \measuredangle CAD - \sin m \measuredangle BAD \sin m \measuredangle CAD =$ $\cos (m \measuredangle BAD + m \measuredangle CAD)$. Since the cosine is one-to-one on $(0, \pi)$, $m \measuredangle BAC = m \measuredangle BAD + m \measuredangle CAD$ as was to be proved.

The case where $\sin m \measuredangle BAD = b$ and $\sin m \measuredangle DAC = -d$ would be proved in a similar manner.

<u>Definition 4.11.</u> If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, and $\overrightarrow{AD}$ is another ray, then $\measuredangle BAD$ and $\measuredangle DAC$ form a <u>linear pair</u>. [17: 82]

Definition 4.12. If the sum of the measures of two angles is 180, then the angles are called supplementary, and each is called a supplement of the other. [17: 82]

Theorem XIV. (The Supplement Postulate) If two angles form a linear pair, then they are supplementary.

Proof: Let $\measuredangle$ BAD and $\measuredangle$ CAD be two angles of a linear pair where $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays. It is assumed, without loss of generality, that $|B - A| = |C - A| = |D - A| = 1$. The rays are in the plane $A + [B - A, E]$ where $\{B - A, E\}$ is an orthonormal basis for the subspace $[B - A, E]$.

There exist real numbers a and b such that $D - A = a(B - A) + bE$. Since $\overrightarrow{AC}$ is opposite to ray $\overrightarrow{AB}$, $D - A = (-a)(C - A) + bE$. Therefore, $\cos m \measuredangle BAD = (D - A) \circ (B - A) = (a(B - A) + bE) \circ (B - A) =$ $a(B - A) \circ (B - A) + bE \circ (B - A) = a$. Similarly $\cos m \measuredangle CAD = -a$.

Since $|B - A| = 1$, $a^2 + b^2 = 1$ and since $|C - A| = 1$, $(-a)^2 + b^2 = 1$. Thus, $b = \pm \sin m \measuredangle BAD$ and $b = \pm \sin m \measuredangle CAD$. Since $0 < \sin x \leq 1$ for all x such that $0 < x < \pi$, if $b < 0$, then both signs are minus. If $b > 0$, then both signs are plus. Thus, $b^2 =$ $\sin m \measuredangle BAD \sin m \measuredangle CAD$ in either case. Therefore, $\cos (m \measuredangle BAD + m \measuredangle CAD) = \cos m \measuredangle BAD \cos m \measuredangle CAD -$ $\sin m \measuredangle BAD \sin m \measuredangle CAD = a(-a) - b^2 = -(a^2 + b^2) = -1$. This implies that $m \measuredangle BAD + m \measuredangle CAD = \pi$ which was to be proved.

CHAPTER V

CONGRUENCE AND PARALLEL LINES

One of the fundamental concepts of Euclidean geometry is congruence. From the time of Euclid this concept has carried with it the idea of "motion". That is, two point sets were considered congruent if one can be "moved", by a "rigid motion", so as to coincide with the other. It has always seemed incredible to some mathematicians and philosophers that geometry, a creation of the mind, should be so tied up with the physical and concrete idea of motion.

The SMSG geometry program excluded the concept of "motion" by defining congruence separately for different types of point sets. The definitions of congruence of angles, segments, and triangles are included in Definition 5.1.

Definition 5.1. i) Angles are congruent if they have the same measure, ii) Segments are congruent if they have the same length [17: 109], iii) Given a correspondence, ABC ↔ DEF (A corresponds to D; B corresponds to E; and C corresponds to F), between the vertices of two triangles. If every pair of corresponding sides are congruent, and every pair of corresponding angles are congruent, then the correspondence ABC ↔ DEF is a congruence between the two triangles. [17: 111]

The first part of this chapter will include the proof of Postulate 15 of SMSG geometry, using the definitions of congruence of SMSG geometry. The parallel postulate, Postulate 16, will also be proved in this section. This statement could have been proved following the definitions of a plane and a line. However, following the numbering of the SMSG postulates, it is included here. After parallel lines are introduced, a short discussion of parallelograms and rectangles will follow.

The second part of the chapter will deal with isometries of V. An isometry is a mathematical formulation of the physical notion of "rigid motion". Point sets of SMSG geometry will be shown to be congruent if and only if there is an isometry of V mapping one point set onto the other. This formulation of congruence will be used in Chapter VI to show that congruent point sets have the same "area".

S.A.S. and Parallel Postulates

Definition 5.2. If A, B, and C are any three non-collinear points, then the union of segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called a triangle, and is denoted by $\triangle ABC$; the points A, B, and C are called its vertices, and segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are called its sides. ... $\triangle ABC$ determines the angles, $\measuredangle$ BAC, $\measuredangle$ ABC, and $\measuredangle$ ACB, which are called the angles of $\triangle ABC$. [17: 72]

Using the definition of the measure of an angle given in Chapter IV and the definition of a triangle, the following theorem, called the law of cosines, is proved.

Theorem 5.3. Let $\triangle ABC$ be a triangle in V. Then, $(BC)^2 = (AB)^2 + (AC)^2 - 2(AB)(AC) \cos (m \measuredangle BAC)$.

Proof: $\triangle ABC = \{A + t(B - A),\ 0 \leq t \leq 1\} \cup \{A + t(C - A),\ 0 \leq t \leq 1\} \cup \{B + t(C - B),\ 0 \leq t \leq 1\}$.

$2(AB)(AC)\cos(m\angle BAC) = 2(AB)(AC)\dfrac{(B - A)\circ(C - A)}{(AB)(AC)}$

$= 2(B - A)\circ(C - A) = 2(B\circ C - A\circ C - B\cdot A + A\circ A)$ and

$2\,B\circ C = 2(AB)(AC)\cos(m\angle BAC) + 2\,A\circ C + 2\,B\circ A - 2\,A\circ A.$

Thus, $(BC)^2 = (C - B)\cdot(C - B) = C\cdot C - 2\,B\cdot C + B\cdot B$

$= C\cdot C - (2(AB)(AC)\cos(m\angle BAC) + 2A\cdot C + 2B\circ A - 2A\cdot A) + B\cdot B$

$= (A\cdot A - 2A\cdot C + C\cdot C) + (A\cdot A - 2A\circ B + B\circ B) -$

$2(AB)(AC)\cos(m\angle BAC)$

$= (A - C)\cdot(A - C) + (A - B)\circ(A - B) - 2(AB)(AC)\cos(m\angle BAC)$

$= (AC)^2 + (AB)^2 - 2(AB)(AC)\cos(m\angle BAC).$

This theorem is used in proving the SAS congruence postulate of SMSG geometry.

Theorem XV. (The S.A.S. Postulate.) Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Proof: Let $ABC \leftrightarrow DEF$ be the given correspondence between $\triangle ABC$ and $\triangle DEF$ with $\angle A \cong \angle D$, $\overline{AB} \cong \overline{DE}$ and $\overline{AC} \cong \overline{DF}$. Then, by the definitions of congruence, $m\angle A = m\angle D$, $AB = DE$ and $AC = DF$. By Theorem 5.3,

$(BC)^2 = (AB)^2 + (AC)^2 - 2(AB)(AC)\cos m\angle A = (DE)^2 + (DF)^2 -$

$2(DE)(DF)\cos m\angle D = (EF)^2.$

Since BC and EF are positive, $BC = EF$. Thus, by the definition of congruence, $\overline{BC} \cong \overline{EF}$.

By Theorem 5.3, $\cos m\angle B = \dfrac{(AC)^2 - (BC)^2 - (AB)^2}{2(BC)(AB)} =$
$\dfrac{(DF)^2 - (EF)^2 - (DE)^2}{2(EF)(DE)} = \cos m\angle E$. Thus, $m\angle B = m\angle E$ and, by definition of congruent angles, $\angle B \cong \angle E$. Similarly, $\angle C \cong \angle F$. Thus, since corresponding sides and angles of the two triangles are congruent,

$\Delta ABC \cong \Delta DEF$ as was to be proved.

Definition 5.4. Two lines are parallel if they are coplanar and do not intersect. [17: 241] Two planes, or a plane and a line, are parallel if they do not intersect. [17: 291]

In the SMSG development of geometry it is possible to prove, using the first fifteen postulate, that if C is not in line $\overleftrightarrow{AB}$ then there exists at least one line through C which is parallel to $\overleftrightarrow{AB}$. A vector proof is included here.

Theorem 5.5. Let C be a point not in line $\overleftrightarrow{AB}$. Then, there exists a line $\overleftrightarrow{CD}$ parallel to line $\overleftrightarrow{AB}$.

Proof: Since A, B, and C are distinct vectors, A, B, and C are in the plane $A + [B - A, C - A]$. $\overleftrightarrow{AB} = \{A + t(B - A), t \in R\}$. Let $\overleftrightarrow{CD} = \{C + t(B - A), t \in R\}$. If $P \in \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ then there exists real numbers t and s such that $P = A + t(B - A) = C + s(B - A)$. Therefore, $C - A = (t - s)(B - A)$. Since $\{B - A, C - A\}$ is a basis for $[B - A, C - A]$, this is a contradiction. Therefore, $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is empty. Also, each line is in plane $A + [B - A, C - A] = C + [B - A, C - A]$. Therefore, line $\overleftrightarrow{CD}$ is parallel to line $\overleftrightarrow{AB}$.

The next theorem, which is Postulate 16 of SMSG geometry shows that $\overleftrightarrow{CD}$ is the only line containing C and parallel to line $\overleftrightarrow{AB}$.

Theorem XVI. (The Parallel Postulate.) Through a given external point, there is at most one line parallel to a given line.

Proof: Let $\overleftrightarrow{AB}$ be a given line and C an external point. That is, C is not in line $\overleftrightarrow{AB}$. $\overleftrightarrow{AB} = \{A + t(B - A), t \in R\}$. As in the proof of Theorem 5.5, $\overleftrightarrow{CD} = \{C + t(B - A), t \in R\}$ is a line containing C and parallel to line $\overleftrightarrow{AB}$. Let $\overleftrightarrow{CE} = \{C + t(E - C), t \in R\}$ be a line containing C and parallel to line $\overleftrightarrow{AB}$. Since C and E are each in plane $A + [B - A, C - A]$, $E - C$ is in the subspace $[B - A, C - A]$. Therefore, there exists scalars s and t such that $E - C = s(B - A) + t(C - A)$.

Since $E \neq C$, $s \neq 0$ or $t \neq 0$. Suppose $t \neq 0$. Then, the vector $C - \frac{1}{t}(E - C) = A - \frac{s}{t}(B - A)$ would be on both line $\overleftrightarrow{CE}$ and $\overleftrightarrow{AB}$. But these lines are parallel. Therefore, $t = 0$ and $s \neq 0$. Thus, $E - C = s(B - A)$ and line $\overleftrightarrow{CD} = \{C + t(B - A), t \in R\} = \{C + t(E - C), t \in R\} = \overleftrightarrow{CE}$. That is, $\overleftrightarrow{CD}$ is the only line containing C and parallel to line $\overleftrightarrow{AB}$.

The definition of a rectangle is needed in Chapter V. Before defining a rectangle, three preliminary definitions are given.

<u>Definition 5.6</u>. Let A, B, C, and D be four points lying in the same plane, such that no three of them are collinear, and such that the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ intersect only in their end-points. Then the union of these four segments is a <u>quadrilateral</u>. [17: 263]

<u>Definition 5.7</u>. A parallelogram is a quadrilateral in which both pair of opposite sides are parallel. [17: 265]

Let ABCD be a parallelogram with sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Since C is not collinear with A and B, C is not on $\overleftrightarrow{AB} = \{A + t(B - A) | t \in R\}$. As in the proof of Theorem 5.5, the line through C parallel to $\overleftrightarrow{AB}$ is $\{C + t(B - A) | t \in R\}$. Similarly, the line through A parallel to line $\overleftrightarrow{BC}$ is $\{A + t(C - B) | t \in R\}$. D being on both of these lines implies $D = A + a(C - B) = C + b(B - A)$ for some a and b in R. Thus, $D - A = a(C - B)$ and $D - C = b(B - A)$. Now, since A, B, and C are non-collinear $B - A$ and $C - B$ are linearly independent. Thus, since $A + a(C - B) = c + b(B - A)$ implies $A - C + a(C - B) - b(B - A) = 0$ or $(a - 1)(C - B) + (-1 - b)(B - A) = 0$, $a - 1 = 0$ and $1 - b = 0$. Thus, $a = 1$ and $b = -1$.

Therefore, since $D - A = a(C - B) = (C - B)$ and $D - C = b(C - B) = -1(C - B)$, $|D - A| = a|C - B| = |C - B|$ and $|D - C| = |b||B - A| = |B - A|$. Therefore, $AB = CD$ and $BC = DA$.

<u>Definition 5.8</u>. A <u>right</u> <u>angle</u> is an angle of 90° (measure $\frac{\pi}{2}$). [17: 86]

Let $\measuredangle ABC$ be a right angle. Since $\cos (m \measuredangle ABC) = \frac{(A - B) \cdot (C - B)}{|A - B| \quad |C - B|}$ and $\cos (x) = 0$, $0 < x < \pi$, if and only if $x = \frac{\pi}{2}$, $\measuredangle ABC$ is a right angle if and only if $(A - B) \cdot (C - B) = 0$. Thus, the equivalent vector space definition of a right angle is, $\measuredangle ABC$ is a right angle if and only if $(A - B) \cdot (C - B) = 0$.

<u>Definition 5.9</u>. A <u>rectangle</u> is a parallelogram all of whose angles are right angles. [17: 268]

Let ABCD be a rectangle with sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$. Then, since ABCD is a parallelogram, $\overline{AB} = \{A + t(B - A) | 0 \leq t \leq 1\} = \{A + s \frac{(B - A)}{|B - A|} | 0 \leq s \leq |B - A|\}$, $\overline{BC} = \{B + t(C - B) | 0 \leq t \leq 1\} = \{B + s \frac{(C - B)}{|C - B|} | 0 \leq s \leq |C - B|\}$, $\overline{CD} = \{D + t(B - A) | 0 \leq t < 1\} = \{D + s \frac{(B - A)}{|B - A|} | 0 \leq s \leq |B - A|\}$ and $\overline{DA} = \{A + t(C - B) | 0 \leq t \leq 1\} = \{A + s \frac{(C - B)}{|C - B|} | 0 \leq s \leq |C - B|\}$. Since $\measuredangle ABC$ is a right angle, $(A - B) \cdot (C - B) = 0$. By Theorem 2.16, $\{B - A, C - B\}$ is linearly dependent. Since A, B, and C are in the plane $P + [Q,S]$ containing the rectangle, this implies $B - A$ and $C - B$ are in $[Q,S]$ and $P + [Q,S] = A + \left[\frac{B - A}{|B - A|}, \frac{C - B}{|C - B|}\right]$. Since $B - A$ and $C - B$ are orthogonal, $\left\{\frac{B - A}{|B - A|}, \frac{C - B}{|C - B|}\right\}$ is an orthonormal basis for $[Q,S]$. Thus, if

ABCD is a rectangle, then ABCD is in the plane

$A + \left[\frac{B - A}{|B - A|}, \frac{C - B}{|C - B|}\right]$ where $\left\{\frac{B - A}{|B - A|}, \frac{C - B}{|C - B|}\right\}$ is an orthonormal basis for the subspace $\left[\frac{B - A}{|B - A|}, \frac{C - B}{|C - B|}\right]$. A rectangular region is investigated next.

Definition 5.10. A point lies in the interior of a triangle if it lies in the interior of each of the angles of the triangle. A point lies in the exterior of a triangle if it lies in the plane of the triangle but is not a point of the triangle or its interior. [17:74].

Definition 5.11. i) A triangular region is the union of a triangle and its interior; ii) A polygonal region is the union of a finite number of coplanar triangular regions, such that if any two of these intersect the intersection is either a segment or a point. [17: 317]

By the definition of a polygonal region, if ABCD is a rectangle, then the rectangular region is the union of triangular regions ABC and ADC. The intersection of these two triangular regions is segment $\overline{AC}$. By Theorem 4.9, the interior of $\measuredangle$ ABC = $\{aA + bB + cC \mid a + b + c = 1,$ $a > 0, b > 0\}$. Similar remarks hold for the interior of the other angles of ΔABC. If $P = aA + bB + cC$, $a + b + c = 1$, and a, b, or c equals zero, say $c = 0$, then $P = aA + bB = (1 - b)A + bB =$ $A + b(B - A)$. Thus, P would be in the triangle. Thus, any triangular region ABC is the set $\{aA + bB + cC \mid a + b + c = 1, a \geq 0, b \geq 0,$ $c \geq 0\}$.

Theorem 5.12. Let ABCD be a rectangle. Then, the rectangular region ABCD is the set $\left\{A + t\frac{(B - A)}{|B - A|} + s\frac{(C - B)}{|C - B|} \mid 0 \leq t \leq |B - A|,\right.$

$0 \leq s \leq |C - B|\}$.

Proof: It will be shown that the rectangular region is the set $T = \{A + t(B - A) + s(C - B) | 0 \leq t \leq 1, 0 \leq s \leq 1\}$. This implies the theorem.

By definition, the rectangular region is the union of triangular regions ABC and ADC. Denote these by $\overline{ABC}$ and $\overline{ADC}$. Then, $\overline{ABC} = \{aA + bB + cC | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\}$ and $\overline{ADC} = \{aA + dD + cC | a + b + c = 1, a \geq 0, d \geq 0, c \geq 0\}$. Since ABCD is a rectangle, $D = A + (C - B)$ or $B = A + C - D$.

If $P \in \overline{ABC}$ then $P = aA + bB + cC = (a + b + c)A + (b + c)(B - A) + c(C - B) = A + (b + c)(B - A) + c(C - B)$ which is in T. If $P \in \overline{ADC}$, then $P = aA + dD + cC$ and $aA + dD + cC = aA + b(A + (C - B)) + cC =$ $(a + b)A + b(C - B) + cC = (a + b)A + (b + c)(C - B) + cB$ $= (a + b + c)A + (b + c)(C - B) + c(B - A)$ which is in T. Thus, $\overline{ABC} \cup \overline{ADC} \subset T$.

If $P \in T$, then $P = A + t(B - A) + s(C - B)$, $0 \leq t \leq 1$ and $0 \leq s \leq 1$. Therefore, $P = (1 - t)A + (t - s)B + sC$. If $t - s \geq 0$, then $P \in \overline{ABC}$. If $t - s < 0$, then $P = (1 - t)A + (t - s)(A + C - D) + sC = (1 - t)A + (s - t)D + (t - s)A + (t - s)C + sC = (1 - s) A + (s - t)D + tC$, which is in $\overline{ADC}$. Thus, the theorem is proved.

Isometries

This section will deal with transformations of V into V. A transformation from V into V is a function, f, assigning to each vector A in V a unique vector f(A) in V. An isometry is a particular type of transformations.

<u>Definition 5.13</u>. A transformation f from V into V is an isometry

if and only if $|f(A) - f(B)| = |A - B|$ for all A and B in V.

Theorem 5.14. If $f:V \to V$ is an isometry, then $(f(A) - f(B)) \cdot (f(C) - f(D)) = (A - B)(C - D)$ for all A, B, C, and D in V. In particular $(f(A) - f(\theta)) \cdot (f(B) - f(\theta)) = A \circ B$.

Proof: Since f is an isometry, for each A and B in V, i) $|f(A) - f(B)|^2 = |A - B|^2$. This implies that ii) $f(A) \cdot f(B) = -\frac{1}{2}(A \cdot A - 2A \circ B + B \circ B - f(A) \cdot f(A) - f(B) \circ f(B))$. Thus, $(f(A) - f(B)) \circ (f(C) - f(D)) = f(A) \cdot f(C) - f(A) \circ f(D) - f(B) \circ f(C) + f(B) \cdot f(D)$. Substituting for $f(A) \circ f(C)$, $f(A) \cdot f(D)$, $f(B) \circ f(C)$ and $f(B) \circ f(D)$ using ii) above and simplifying gives $(f(A) - f(B)) \circ (f(C) - f(D)) = (A - B) \circ (C - D)$.

The next two theorems describe the image and pre-image of an orthonormal basis under an isometry.

Theorem 5.15. Let $f:V \to V$ be an isometry. If $\{A_1, A_2, A_3\}$ is an orthonormal basis for V then $\{f(A_1) - f(\theta), f(A_2) - f(\theta), f(A_3) - f(\theta)\}$ is an orthonormal basis for V.

Proof: Since $\{A_1, A_2, A_3\}$ is an orthonormal basis, $A_i \circ A_j = 0$ if $i \neq j$, and $A_i \circ A_i = 1$. By Theorem 5.14, $(f(A_i) - f(\theta)) \circ (f(A_j) - f(\theta)) = A_i \circ A_j$. Thus, by Theorem 2.16, $\{f(A_1) - f(\theta), f(A_2) - f(\theta), f(A_3) - f(\theta)\}$ is an orthonormal basis for V.

Theorem 5.16. Let $f:V \to V$ be an isometry. If $\{A_1, A_2, A_3\}$ is an orthonormal basis for V with $f(B_i) = A_i$, $i = 1, 2, 3$, and $f(P) = \theta$, then $\{B_1 - P, B_2 - P, B_3 - P\}$ is an orthonormal basis for V.

Proof: $(B_i - P) \circ (B_j - P) = (f(B_i) - f(P)) \cdot (f(B_j) - f(P)) = A_i \circ A_j$, $i, j = 1, 2, 3$.

By Theorem 2.16, $\{\ B_i\ -\ P\ \ |i = 1, 2, 3\}$ is an orthonormal basis for V.

Theorem 5.17. Let $f: V \to V$ be an isometry. If $\{A_1, A_2, A_3\}$ is an orthonormal basis for V and $A = a_1A_1 + a_2A_2 + a_3A_3$ then $f(A) - f(\theta) = a_1(f(A_1) - f(\theta)) + a_2(f(A_2) - f(\theta)) + a_3(f(A_3) - f(\theta))$.

Proof: By Theorem 5.15, $\{f(A_1) - f(\theta), f(A_2) - f(\theta), f(A_3) - f(\theta)\}$ is an orthonormal basis for V. Thus, there are real numbers b_1, b_2, and b_3 such that $f(A) - f(\theta) = b_1(f(A_1) - f(\theta)) + b_2(f(A_2) - f(\theta)) + b_3(f(A_3) - f(\theta))$.

By Theorem 5.14, $(A - \theta) \circ (A_i - \theta) = (f(A) - f(\theta)) \circ (f(A_i) - (f(\theta))$ or $(a_1A_1 + a_2A_2 + a_3A_3) \circ A_i = (b_1(f(A_1) - f(\theta)) + b_2(f(A_2) - f(\theta)) + b_3(f(A_3) - f(\theta)) \circ (f(A_i) - f(\theta))$, $i = 1, 2, 3$. Since each of the bases is orthonormal, distributing the inner product implies that $a_i = b_i$, $i = 1, 2, 3$.

Theorem 5.18. If $f: V \to V$ is an isometry then f is one-to-one and onto V.

Proof: Let $\{A_1, A_2, A_3\}$ be an orthonormal basis for V. Let A, B be two distinct vectors of V. Then there exists real numbers a_1, a_2, a_3, b_1, b_2 and b_3 such that $A = a_1A_1 + a_2A_2 + a_3A_3$ and $B = b_1A_1 + b_2A_2 + b_3A_3$. Since $A \neq B$, at least one a_i does not equal the corresponding b_i. Now, by Theorem 5.17, $f(A) - f(\theta) = a_1(f(A_1) - f(\theta)) + a_2(f(A_2) - f(\theta)) + a_3(f(A_3) - f(\theta)) \neq b_1(f(A_1) - f(\theta)) + b_2(f(A_2) - f(\theta)) + b_3(f(A_3) - f(\theta)) = f(B) - f(\theta)$. Thus, $f(A) \neq f(B)$ or f is one-to-one.

Let $B \in V$. Then, $B - f(\theta)$ is in V. Since, by Theorem 5.15, $\{f(A_i) - f(\theta) | i = 1, 2, 3\}$ is a basis for V, there exists scalars a_1, a_2, and a_3 such that $B - f(\theta) = a_1(f(A_1) - f(\theta) + a_2(f(A_2) - f(\theta)) +$

$a_3(f(A_3) - f(\theta))$. Now, $A = a_1A_1 + a_2A_2 + a_3A_3$ is in V and, by Theorem 5.17, $f(A) - f(\theta) = B - f(\theta)$. Thus, $f(A) = B$. Therefore, f is onto V.

Definition 5.19. A transformation $f:V \to V$ is a translation if and only if there exists a fixed vector P in V such that $f(A) = A + P$ for all A in V.

In Theorem 5.15, $f:V \to V$ was an isometry with $\{A_1, A_2, A_3\}$ an orthonormal basis for V. f mapped the basis $\{A_1, A_2, A_3\}$ onto the set $\{f(A_1), f(A_2), f(A_3)\}$. To obtain an orthonormal basis, $f(\theta)$ was subtracted from each of these vectors, $f(A_i)$. That is, f followed by the translation $h(A) = A - f(\theta)$ mapped an orthonormal basis into an orthonormal basis.

The next four theorems will characterize an isometry as the composition of an orthogonal linear transformation and a translation. In the following, orthogonal linear transformations are used. The necessary information on these transformations was presented in Chapter II.

Theorem 5.20. If $h:V \to V$ is a translation, then h is an isometry.

Proof: $h(A) = A + P$ for some P in V. Therefore, $|A - B| = |A + P - (B + P)| = |f(A) - f(B)|$ for all A and B in V. Thus, f is an isometry.

Theorem 5.21. If $g:V \to V$ is an orthogonal transformation, then g is an isometry.

Proof: By definition of an orthogonal transformation, $A \circ B = g(A) \circ g(B)$ for all A and B in V. If A and B are in V then, since $g(A - B) = g(A) - g(B)$, $|A - B|^2 = (A - B) \circ (A - B) = g(A - B) \circ g(A - B) = (g(A) - g(B)) \circ (g(A) - g(B)) = |g(A) - g(B)|^2$. Thus, $|A - B| = |g(A) - g(B)|$, and g is an isometry.

Theorem 5.22. If $f:V \to V$ and $h:V \to V$ are isometries, then $hf:V \to V$ (i.e., $hf(A) = h(f(A))$) is an isometry. In particular, the composition of an orthogonal linear transformation and a translation or a translation and an orthogonal transformation is an isometry.

Proof: For all A and B in V, $|A - B| = |f(A) - f(B)|$. Since, f(A) and f(B) are in V and h is an isometry, $|f(A) - f(B)| = |hf(A) - hf(B)|$. Thus, $|A - B| = |hf(A) - hf(B)|$. Therefore, hf is an isometry.

This theorem states that the composition of an orthogonal transformation and a translation is an isometry. The next theorem states that these are the only isometries.

Theorem 5.23. A transformation $f:V \to V$ is an isometry if and only if $f = hg$ where h is a translation and g is an orthogonal linear transformation. Also $h(A) = A + f(\theta)$ and g maps an orthonormal basis $\{A_1, A_2, A_3\}$ onto an orthonormal basis $\{f(A_1) - f(\theta), f(A_2) - f(\theta), f(A_3) - f(\theta)\}$.

Proof: If g is an orthogonal transformation, then by Theorem 2.31, g maps an orthonormal basis $\{A_1, A_2, A_3\}$ onto an orthonormal basis $\{g(A_1), g(A_2), g(A_3)\}$. Let $h:V \to V$ be the translation defined by $h(A) = P$ for some fixed P in V. Let $f = hg$. By Theorem 5.22, f is an isometry. $f(\theta) = hg(\theta) = h(\theta) = P$. Also, $f(A_i) = hg(A_i) = g(A_i) + P = g(A_i) + f(\theta)$. Thus, $f(A_i) - f(\theta) = g(A_i)$ and $\{f(A_i) - f(\theta) | i = 1, 2, 3\}$ is an orthonormal basis for V.

Let $f:V \to V$ be an isometry. Define $h:V \to V$ by $h(A) = A + f(\theta)$. Let $\{A_1, A_2, A_3\}$ be an orthonormal basis for V. By Theorem 5.15, $\{f(A_i) - f(\theta), i = 1, 2, 3\}$ is an orthonormal basis for V. If $A \in V$ then $A = a_1A_1 + a_2A_2 + a_3A_3$ for some a_1, a_2, a_3 in R. Define

$g: V \to V$ by $g(A) = a_1(f(A_1) - f(\theta)) + a_2(f(A_2) - f(\theta)) + a_3(f(A_3) - f(\theta))$.

If $B = b_1A_1 + b_2A_2 + b_3A_3$ is in V, then $g(rA + sB) = (ra_1 + sb_1)(f(A_1) - f(\theta)) + (ra_2 + sb_2)(f(A_2) - f(\theta)) + (ra_3 + sb_3)(f(A_3) - f(\theta)) = rg(A) + sg(B)$. Thus, g is a linear transformation. Since $A_i = 1 \cdot A_i$, $i = 1, 2, 3$, $g(A_i) = f(A_i) - f(\theta)$. Thus, g maps an orthonormal basis onto an orthonormal basis of V. Therefore, by Theorem 2.31, g is an orthogonal linear transformation. By Theorem 5.17, $f(A) - f(\theta)) = a_1(f(A_1) - f(\theta)) + a_2(f(A_2) - f(\theta)) + a_3(f(A_3) - f(\theta))$. Therefore, $f(A) - f(\theta) = g(A)$. Thus, $hg(A) = g(A) + f(\theta) = f(A)$. This completes the proof.

If $f: V \to V$ is an isometry then, by Theorem 5.18, f is one-to-one and onto V. Thus, $f^{-1}: V \to V$ is a well defined transformation. If $A, B \in V$ then, since f is onto V, $A = f(C)$ and $B = f(D)$ for some $C, D \in V$. Therefore, $f^{-1}(A) = C$ and $f^{-1}(B) = D$. Since f is an isometry, $|f^{-1}(A) - f^{-1}(B)| = |C - D| = |f(C) - f(D)| = |A - B|$. Thus, Theorem 5.24 has been proved.

Theorem 5.24. If $f: V \to V$ is an isometry, then $f^{-1}: V \to V$ is an isometry.

Theorem 5.23 states that every isometry $f: V \to V$ is the composition of an orthogonal linear transformation and a translation. Since f^{-1} is an isometry $f^{-1} = h_1g_1$ where g_1 is an orthogonal linear transformation and h_1 is a translation. Since, $h_1(A) = A + P$, for some P in V, $h_1^{-1}(A) = A - P$ is also a translation. By Theorem 2.33, the inverse of an orthogonal linear transformation is an orthogonal linear transformation.

Therefore, since the inverse of the composition of two functions is the composition of their inverses in reverse order, $f = (f^{-1})^{-1} = (h_1 g_1)^{-1} = g_1^{-1} h_1^{-1}$. Thus, any isometry is also the composition of a translation and an orthogonal linear transformation.

Theorem 5.25. A transformation $f: V \to V$ is an isometry if and only if $f = gh$ where h is a translation and g is an orthogonal linear transformation.

Proof: In the remarks preceding the theorem, it was shown that if f is an isometry, then f has the desired form. The converse follows by Theorem 5.22.

Under an isometry planes and lines are mapped into planes and lines. In the following proof of this statement $h(A + W)$ is the set $\{P | P = h(A + Q) \text{ for some } Q \text{ in } W\}$.

Theorem 5.26. If $f: V \to V$ is an isometry and $A + W$ is a coset of a subspace W, then $f(A + W)$ is a coset of a subspace having the same dimension as W.

Proof: By Theorem 5.25, there exists a translation h and an orthogonal linear transformation g such that $f = gh$. Suppose $h(A) = A + P$. Then, $f(A + W) = gh(A + W) = g((P + A) + W) = g(P + A) + g(W)$. But, by Theorem 2.29, $g(W)$ is a subspace of V having the same dimension as W. Thus, $f(A + W)$ is the coset $g(P + A) + g(W)$ of the subspace $g(W)$.

Theorem 5.27. Let $f: V \to V$ be an isometry. Let $\measuredangle BAC$ be an angle with $f(B) = E$, $f(A) = D$, and $f(C) = F$. Then, $m \measuredangle BAC = m \measuredangle EDF$.

Proof: $\cos m \measuredangle BAC = \dfrac{(B - A) \cdot (C - A)}{|B - A| \quad |C - A|} = \dfrac{(f(B) - f(A)) \cdot (f(C) - f(A))}{|f(B) - f(A)| \quad |f(C) - f(A)|}$.

$$= \frac{(E-D) \cdot (F-D)}{|E-D| \quad |F-D|} = \cos m \measuredangle EDF.$$

Thus, $m \measuredangle BAC = m \measuredangle EDF$.

Congruence by Isometries

This section will include a definition of congruence using isometry. For triangles this definition will be shown to be equivalent to the SMSG definition of congruence.

Definition 5.28. Let W and U be point sets of V. Let $h: W \to U$ be a one-to-one correspondence between the points of W and the points of U. Then, h is a congruence between W and U if and only if there exists an isometry $f: V \to V$ such that f restricted to W equals h. W is said to be congruent to U if and only if there exists a congruence between W and U.

This is a much more general definition of congruence than that of SMSG geometry. This definition would include the definition of congruence of segments, angles, triangles, circles and three dimensional point sets.

It is possible for two point sets to have more than one congruence between them. For example, two isosceles triangles, ΔABC and ΔDEF, could be congruent with $ABC \leftrightarrow DEF$ and with $ABC \leftrightarrow EDF$.

Theorem 5.29. If ΔABC is congruent to ΔDEF by Definition 5.28, then the triangles are congruent by Definition 5.1.

Proof: Since ΔABC is congruent to ΔDEF by Definition 5.28, there exists a one-to-one correspondence h between ΔABC and ΔDEF. Let A, B, C correspond to D, E, F, respectively. h is also the restriction to ΔABC of an isometry f with $f(A) = D$, $f(B) = E$ and $f(C) = E$.

Therefore, $AB = |B - A| = |f(B) - f(A)| = |E - D| = DE$. Thus, $\overline{AB} \cong \overline{DE}$ by Definition 5.1. Similarly, $\overline{BC} \cong \overline{EF}$ and $\overline{AC} \cong \overline{DF}$. Also, $\cos m \measuredangle BAC = \frac{(B - A) \cdot (C - A)}{|B - A| \quad |C - A|}$. By Theorem 5.14, $(B - A) \cdot (C - A) = (f(B) - f(A)) \cdot (f(C) - f(A)) = (E - D) \cdot (F - D)$. Thus, $\cos m \measuredangle BAC = \frac{(B - A) \cdot (C - A)}{|B - A| \quad |C - A|} = \frac{(E - D) \cdot (F - D)}{|E - D| \quad |C - A|} = \cos m \measuredangle EDF$. Since cosine is one-to-one on $(0, \pi)$, $m \measuredangle BAC = m \measuredangle EDF$. Thus, by the Definition 5.1 of congruence of angles, $\measuredangle BAC \cong \measuredangle EDF$. Similarly, $\measuredangle BCA \cong \measuredangle EFD$ and $\measuredangle ABC \cong \measuredangle DEF$. Thus, by the SMSG definition of congruence of triangle $\triangle ABC \cong \triangle DEF$ under the correspondence $ABC \leftrightarrow DEF$.

The next theorems will be used to show that for triangles Definition 5.1 implies Definition 5.28. Only the condition that the two triangles have two sides and the included angles congruent is used. Since this is all that is used, at the same time, Postulate XV of SMSG geometry will be proved as a theorem for triangles in V.

<u>Definition 5.30</u>. Let $\{X, Y, Z\}$ be an orthonormal basis for V. An angle, $\measuredangle BAC$, is in standard position with respect to the basis $\{X, Y, Z\}$ if and only if $A = \theta$, B is on ray $\overrightarrow{\theta X}$ and C is in the half-plane containing Y of plane $[X, Y]$ determined by line $\overleftrightarrow{\theta X}$.

<u>Theorem 5.31</u>. Let $\{X, Y, Z\}$ be an orthonormal basis for V. Let $\measuredangle BAC$ be an angle in plane $A + [B - A, C - A]$. Then, there exists an isometry mapping $\measuredangle BAC$ into an angle $\measuredangle E\theta F$ in standard position with respect to the basis $\{X, Y, Z\}$.

Proof: Define the translation $h: V \to V$ by $h(P) = P - A$. Then, $h(A + [B - A, C - A]) = [B - A, C - A]$.

$\frac{1}{|B - A|}(B - A)$ is in $[B - A, C - A]$. Let $\frac{1}{|B - A|}(B - A) = B'$. Then, $\{B'\}$ can be completed to an orthonormal basis $\{B', C'\}$ for $[B - A, C - A]$.

The set $\{B', C'\}$ can be completed to an orthonormal basis $\{B', C', D'\}$ for V. Since $\{B', C', D'\}$ and $\{X, Y, Z\}$ are orthonormal bases for V, by Theorem 2.31, there exists an orthogonal linear transformation g such that $g(B') = X$, $g(C') = Y$ and $g(D') = Z$.

Let $f = gh$. Then, by Theorem 5.22, f is an isometry. $f(C) = gh(C) = g(C - A)$. Since $C - A$ is an element of $[B', C']$ and, by Theorem 2.29, g maps subspaces of V into subspaces of V, $f(C) = g(C - A) \in g[B', C'] = [X, Y]$. Also, $f(B) = gh(B) = g(B - A) = \frac{|B - A|}{|B - A|} g(B - A) = |B - A| g\left(\frac{1}{|B - A|}(B - A)\right) = |B - A| g(B') = |B - A| X$. $f(A) = gh(A) = g(\theta) = \theta$. By Theorem 5.26, $f(\overleftrightarrow{AB})$ is a line and $f(\overleftrightarrow{AC})$ is a line. Thus, $f(\overleftrightarrow{AB})$ must be the line containing θ and $\frac{1}{|B - A|} X$ and $f(\overleftrightarrow{AC})$ must be the line containing θ and $f(C)$. Since $AP + PB = AB$ or $|A - P| + |P - B| = |A - B|$ implies $|f(A) - f(P)| + |f(P) - f(B)| = |f(A) - f(B)|$, f preserves betweeness for sets of points on a line. Thus, $f(\overrightarrow{AB})$ is the ray with endpoint θ and containing $|B - A|X$ and $f(\overrightarrow{AC})$ is the ray with endpoint θ and containing $f(C)$.

Let $E = |B - A|X$ and $N = f(C)$. Since $f(C)$ is in $[X, Y]$ there are real numbers s and t such that $f(C) = sX + tY$. Since $f(B) \in \overrightarrow{\theta X}$ implies $f(C)$ is not an element of $\overrightarrow{\theta X}$, $t \neq 0$. The half-plane of $[X, Y]$, determined by line $\overleftrightarrow{\theta X}$, containing Y is the set $H = \{P | P \circ Y > 0 \text{ and } P \in [X, Y]\}$. Now $f(C) \circ Y = (sX + tY) \circ Y = t$. If $t > 0$, then $f(C)$ is in H and f is the desired isometry. If $t < 0$, then the orthogonal linear transformation $f_1(aX + bY + cZ) = aX - bY + cZ$ will map $f(C)$ into $sX - tY$. Thus, f followed by the orthogonal transformation f_1 is the required transformation.

<u>Theorem 5.32</u>. Let $ABC \leftrightarrow DEF$ be a one-to-one correspondence between the vertices of ΔABC and ΔDEF. Suppose $\measuredangle BAC \cong \measuredangle EDF$,

$\overline{BA} \cong \overline{DE}$ and $\overline{BC} \cong \overline{DF}$ according to Definition 5.1. Let $\{X, Y, Z\}$ be an orthonormal basis for V. Then, there is a triangle $\Delta M\theta N$ with $\measuredangle M\theta N$ in standard position and isometries f and f' such that f maps ΔABC into ΔMON and f' maps ΔDEF into ΔMON with $f(B) = f'(D) = M$ and $f(C) = f'(F) = N$.

Proof: By Theorem 5.31 there is an isometry f mapping $\measuredangle BAC$ into $\measuredangle X\theta R$ in standard position. Similarly, there is an isometry f' mapping $\measuredangle EDF$ into an angle $\measuredangle X\theta T$ in standard position. By Definition 5.1, since $\measuredangle BAC \cong \measuredangle EDF$, $m \measuredangle BAC = m \measuredangle EDF$. By Theorem XII, there is exactly one angle in the half-plane of $[X, Y]$ containing Y with one ray $\overrightarrow{OX}$ having measure $m \measuredangle BAC$. Thus, $\measuredangle X\theta R = \measuredangle X\theta T$.

By Definition 5.1, since $\overline{BA} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, $|B - A| = |E - D|$ and $|C - A| = |F - D|$. Since f(B) and $f'(E)$ are each on ray $\overrightarrow{\theta X}$, $f(A) = f'(D) = \theta$ and $|f(B)| = |f(B) - f(A)| = |B - A| = |E - D| = |f'(E) - f'(D)| = |f'(E)|$, $f(B) = f'(E)$. Similarly, $f(C) = f'(F)$.

Let $f(B) = M$ and $f(C) = N$. Then, since f and f' preserve betweeness, f maps ΔABC onto ΔMON and f' maps ΔDEF onto ΔMON as was to be proved.

<u>Theorem 5.33</u>. Let $A, B, C \leftrightarrow D, E, F$ be a correspondence between the vertices of triangles ΔABC and ΔDEF. If $\measuredangle ABC \cong \measuredangle DEF$, $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then ΔABC is congruent to ΔDEF using Definition 5.28 for congruence.

Proof: An isometry mapping ΔABC into ΔDEF is needed to prove the theorem. By Theorem 5.32, there is an isometry f_1 mapping ΔABC into a triangle $\Delta M\theta N$, with $\measuredangle M\theta N$ in standard position, and an isometry f mapping ΔDEF into $\Delta M\theta N$. Since f^{-1} is an isometry mapping $\Delta M\theta N$ into

ΔDEF, by Theorem 5.22, $f^{-1}f_1$ is the desired isometry establishing the congruence.

This theorem gives another proof of Theorem XV using congruence by similarities.

The last theorem in this chapter will be used in the proof of Theorem XVIII in Chapter VI.

<u>Theorem 5.34</u>. Let $\overline{ABC}$ be the triangular region of ΔABC and $\overline{DEF}$ the triangular region of ΔDEF. If $f: V \to V$ is an isometry mapping ΔABC into ΔDEF, then f maps the triangular region $\overline{ABC}$ into the triangular region $\overline{DEF}$.

Proof: By the remarks preceding Theorem 5.12, $\overline{ABC}$ = $\{aA + bB + cC | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\}$ and $\overline{DEF}$ = $\{aD + bE + cF | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\}$. Now, $f = gh$ where h is a translation and g an orthogonal linear transformation. Let $h(Q) = Q + P$. Thus, since $a + b + c = 1$,

$$h(\overline{ABC}) = \{aA + bB + cC + P | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\}$$
$$= \{a(A + P) + b(B + P) + c(C + P) | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\}.$$

Therefore, $f(\overline{ABC}) = gh(\overline{ABC}) =$

$\{g(a(A + P) + b(B + P) + c(C + P)) | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\} =$

$\{ag(A + P) + bg(B + P) + cg(C + P) | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\} =$

$\{af(A) + bf(B) + cf(C) | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\} =$

$\{aD + bE + cF | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\} =$

$\overline{DEF}$.

CHAPTER VI

AREA AND VOLUME

This chapter will include the definition of area for certain coplanar sets of points and volume for certain sets of points in V.

Area and volume are examples of set functions whose range is the set of extended real numbers. Ideally, a set function C such as area would have the following properties:

(1) The domain of definition of C would be the set of all coplanar sets of points.

(2) If I is a rectangle, then C(I) is the usual value, the length of the rectangle times its width.

(3) If U and W are disjoint coplanar sets such that C(U) and C(W) exist, then $C(U) + C(W) = C(U \cup W)$.

(4) If U_n, n = 1, 2, ..., is a sequence of sets such that $U_n \subset U_{n+1}$ and $\bigcup_{n=1}^{\infty} U_n = W$, then $\lim_{n\to\infty} C(U_n) = C(W)$. This is the property sometimes used in elementary geometry to find the area of a circle. The circle is approximated by polygons. The area of the circle is the limit of the areas of the polygons.

(5) If a point set U is congruent to a point set W, then C(U) = C(W).

If the axiom of choice is accepted, it has been shown that such a function does not exist. [16: 68]

In 1902, Herni Lebesgue constructed a measure for n dimensional vector spaces satisfying (2), (3), (4), and (5). [21] The measure was not defined for all subsets of the vector space. The domain of definition was a subset of the power set which included the vector space itself, and was closed under set theoretic differences and countable unions.

The measure Lebesgue defined would satisfy Postulates 16 through 22 of SMSG for area and volume. But, for the point sets that SMSG requires have area, a simpler set function can be used. For this purpose the writer will use Jordan content. This work was first done by Jordan about 1892. [16: 33] If a set has Jordan content, then it has Lebesgue measure and they are numerically equal. [10:82]

Jordan content will satisfy (2), (3), and (5). Only under restrictive conditions does it satisfy (4).

The area and volume of certain sets of points in V will be defined in terms of a coordinate system for V. In Chapter II a coordinate system for V was defined as an isomorphism between V and R_3. In this chapter R_2 and R_3 will be studied first. Then, the area and volume of point sets in V will be defined in terms of the area or volume of points of R_2 and R_3 which correspond under certain types of coordinate systems.

Since most of the postulates of SMSG concern area of coplanar sets of points, most of the theorems in this chapter will be proved for the two dimensional space R_2. They have obvious generalizations to R_3 and even to R_n. The usual inner product $(x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2$ and the usual norm $|(x_1, x_2)| = \sqrt{(x_1,x_2) \cdot (x_1,x_2)} = \sqrt{x_1^2 + x_2^2}$ will be used for R_2.

Topics From Topology

A few topics from topology will be used in the development that follows.

Definition 6.1. Let V be an inner product vector space. A set of points $S(P_0, r) = \{P \mid |P - P_0| < r\}$ is called an open sphere with center P_0 and radius r.

Definition 6.2. Let E be a set of points in V. E is a V-open set of points if and only if, for every $P \in E$, there exists an open sphere $S(P, r)$ such that $S(P, r) \subset E$. A set of points E is V-closed if and only if $V - E = \{P \mid P \notin E\}$ is V-open.

Theorem 6.3. The union of any collection of V-open sets is V-open. The intersection of any finite collection of V-open sets is V-open.

Proof: If P is a point in the union of a collection of V-open sets, then P is a point in at least one of the sets E. Since E is open, there exists an open sphere $S(P, r)$ such that $P \in S(P, r) \subset E$. Therefore, $S(P, r)$ is contained in the union of the sets of the collection. Thus, the union is V-open.

Let $E_1, E_2, \ldots, E_n$ be a finite collection of V-open sets. Suppose $P \in \bigcap_{i=1}^{n} E_i$. Thus, P is an element of each E_i. Since each E_i is V-open, there exists an open sphere $S(P, r_i)$ such that $P \in S(P, r_i) \subset E_i$ for each i. Let r be the minimum of the r_i. Then, $S(P, r) \subset S(P, r_i)$ for each $i = 1, 2, \ldots, n$. Thus, $P \in S(P, r) \subset \bigcap_{i=1}^{n} S(P, r_i) \subset \bigcap_{i=1}^{n} E_i$. Thus, $\bigcap_{i=1}^{n} E_i$ is V-open.

Theorem 6.4. The intersection of any collection of closed sets is closed. The union of any finite collection of closed sets is closed.

Proof: Let $\{F_i\}$ be a collection of closed sets. By the definition of closed sets, $V - F_i$ is V-open for each i. Thus, by Theorem 6.3, $\bigcup_i (V - F_i)$ is V-open. But, $V - (\bigcap_i F_i) = \bigcup_i (V - F_i)$. Thus, $\bigcap_i F_i$ is closed.

Let $F_1, F_2, \ldots, F_n$ be a finite collection of closed sets. Then $V - F_i$ is V-open. Thus, by Theorem 6.3, $V - \left(\bigcup_{i=1}^{n} F_i\right) = \bigcap_{i=1}^{n} (V - F_i)$ is V-open. Therefore, $\bigcup_{i=1}^{n} F_i$ is closed.

Definition 6.5. Let E be a set of points in V. A point P in E is an interior point of E if and only if there exists an open sphere containing P that is contained in E. The interior of E, denoted E°, is the set of all interior points of E.

Theorem 6.6. The interior of any subset E of V is V-open. If E_1 is a V-open set and $E_1 \subset E$, then $E_1 \subset E^\circ$.

Proof: Let $P \in E^\circ$. The first part of the theorem is proved if it is shown that there exists an open sphere containing P that is contained in E°. Since $P \in E^\circ$, there exists an open sphere S containing P that is contained in E. Since, by definition of E°, each point of S is a point of E°, S is contained in E°. Thus, E° is open.

If E_1 is V-open, then for each $P \in E_1$, there exists a sphere $S(P, r) \subset E_1$. Since $E_1 \subset E$, $S(P, r) \subset E$. Thus, by definition of E°, $P \in E^\circ$. Therefore, $E_1 \subset E^\circ$, and the theorem is proved.

Corollary 6.7. If E is V-open, then $E^\circ = E$.

Proof: By Theorem 6.6, since E is V-open, $E \subset E^\circ$. By definition of E°, $E^\circ \subset E$. Thus, $E = E^\circ$.

Definition 6.8. Let E be a subset of V. The closure of E, denoted $\bar{\bar{E}}$, is the intersection of all V-closed sets containing E.

Theorem 6.9. If E is a V-closed subset of V, then $\bar{\bar{E}} = E$.

Proof: By definition of $\bar{\bar{E}}$, $E \subset \bar{\bar{E}}$ since E is contained in each of the closed sets in the intersection. Since E is closed and $E \subset E$, $\bar{\bar{E}} \subset E$. Thus, $\bar{\bar{E}} = E$. The theorem is proved.

By the definition of $\bar{\bar{E}}$, if F is any closed set containing E, then $\bar{\bar{E}}$, the intersection of all closed sets containing E, is contained in F. Also, since arbitrary intersection of closed sets is closed, $\bar{\bar{E}}$ is closed.

Definition 6.10. Let E be a subset of V. A point P is a boundary point of E if and only if every open sphere with P as center contains a point of E and a point of V that is not in E. The set of all boundary points of E, denoted E_b, is called the boundary of E.

Theorem 6.11. Let E be a subset of V. Then, $\bar{\bar{E}} = E^\circ \cup E_b$.

Proof: If $P \in E^\circ \cup E_b$, then $P \in E^\circ$ or $P \in E_b$. If $P \in E^\circ$, then, since $E^\circ \subset E \subseteq \bar{\bar{E}}$, $P \in \bar{\bar{E}}$. Suppose $P \in E_b$ and $P \notin \bar{\bar{E}}$. Since $P \notin \bar{\bar{E}}$, $P \in V - \bar{\bar{E}}$. Since $\bar{\bar{E}}$ is closed, $V - \bar{\bar{E}}$ is open. Therefore, there exists an open sphere S which contains P and is contained in $V - \bar{\bar{E}}$. Thus, S contains no points of $\bar{\bar{E}}$ and, hence, no points of E. This contradicts that $P \in E_b$. Thus, if $P \in E_b$, then $P \in \bar{\bar{E}}$. Therefore, $E^\circ \cup E_b \subset \bar{\bar{E}}$.

To prove that $\bar{\bar{E}} \subset E^\circ \cup E_b$, the equivalent statement, $V - (E^\circ \cup E_b) \subset V - \bar{\bar{E}}$ will be proved. Let $P \in V - (E^\circ \cup E_b)$. Then, $P \notin E^\circ$ and $P \notin E_b$. Since $P \notin E_b$, there exists an open sphere S which contains P and only points of E or only points of V - E. If S contains only points of E, then $P \in E^\circ$. But $P \notin E^\circ$. Therefore, S contains only points of V - E. Since S is open, V - S is closed and contains E. Hence, $\bar{E} \subset V - S$ or $S \subset V - \bar{E}$. Thus, $P \in V - \bar{E}$. Therefore, $V - (E^\circ \cup E_b) \subset V - \bar{E}$. Thus, $\bar{E} = E^\circ \cup E_b$.

Theorem 6.12. If $E \subset F$ and F is closed, then $E_b \subset F$.

Proof: By definition of $\bar{E}$, $\bar{E} \subset F$. Since $\bar{E} = E^\circ \cup E_b$, $E_b \subset F$.

Definition 6.13. Two subsets E and F of V are non-overlapping if and only if they have at most boundary points in common.

A point, line, or plane is defined in the real vector space R_2 as they were for the general Euclidean space V. The half-planes, which were determined by Theorem IX in Chapter III, would be exactly the same for the space R_2 except that this plane would contain all points. This statement is used in the next theorem.

Theorem 6.14. Let A + [B] be a line in R_2. Let H_1 and H_2 be the half-planes determined by A + [B]. Then, H_1 and H_2 are R_2 - open, $(H_1)_b = A + [B]$ and $(H_2)_b = A + [B]$.

Proof: It may be assumed that $|B| = 1$. There exists a vector C such that $\{B, C\}$ is an orthonormal basis for R_2. By Definition 3.21, $H_1 = \{P | (P - A) \circ C > 0\}$ and $H_2 = \{P | (P - A) \circ C < 0\}$. If $P \in H_1$, then $P - A = bB + cC$ for some b and c in R. Since $(P - A) \circ C = c$, $c > 0$. If Q is an element of the sphere S(P, c), then $|P - Q| < c$. Now, $Q - A = b'B + c'C$ for some b′ and c′. Thus $|P - Q| = |P - A - (Q - A)| < c$ or $|(b - b')B + (c - c')C| < c$. That is, $\sqrt{(b - b')^2 + (c - c')^2} < c$. This implies that $\sqrt{(c - c')^2} = |c - c'| < c$. Thus, $-c < c - c' < c$ or $0 < c' < 2c$. Therefore, $(Q - A) \circ C = c' > 0$. Hence, $Q \in H_1$. Thus, $S(P, c) \subset H_1$. By definition of R_2-open sets, H_1 is R_2-open. Similarly, since S(P, -c) would be contained in H_2 if $P \in H_2$,

H_2 is R_2-open.

Now $H_1 \cup H_2 \cup (A + [B]) = R_2$ and the sets are disjoint. If $P \in H_2$, then, since H_2 is open, P is not a boundary point for H_1. If $P \in H_1$, since H_1 is open, P is not a boundary point for H_1. Therefore, $(H_1)_b \subset A + [B]$. Let $P \in A + [B]$ and $S(P, r)$ be any open sphere with center P. Let $Q_1 = P + \frac{r}{2} C$ and $Q_2 = P - \frac{r}{2} C$. Since $|Q_1 - P| = |\frac{r}{2} C| = \frac{r}{2} < r$, $Q_1 \in S(P, r)$. Similarly, $Q_2 \in S(P, r)$. Since $P \in A + [B]$, by Lemma 3.20, $(P - A) \cdot C = 0$. Thus, $(Q_1 - A) \cdot C = (P - A + \frac{r}{2}C) \cdot C = \frac{r}{2}C \cdot C = \frac{r}{2} > 0$. Similarly, $(Q_2 - A) \cdot C < 0$. Therefore, $Q_1 \in H_1$ and $Q_2 \notin H_1$. Thus, P is a boundary point for H_1 or $A + [B] \subset (H_1)_b$. Therefore, $A + [B] = (H_1)_b$. Similarly, it can be shown that $A + [B] = (H_2)_b$. This completes the proof of the theorem.

By Definition 5.10, the interior of a triangle is the intersection of the interior of the three angles determined by the triangle. The exterior is the set of points in the plane which are not in the interior of the triangle or in the triangle.

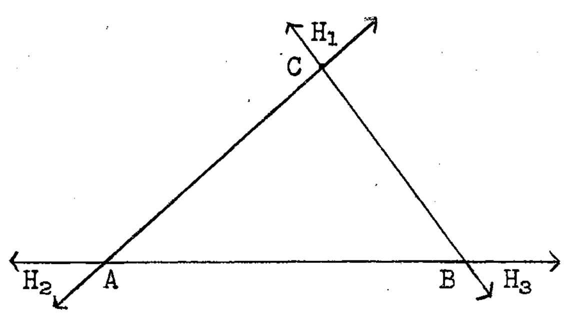

Figure 5.

Let ΔABC be a triangle in R_2. Let H_1 be the half-plane determined by line $\overleftrightarrow{AB}$ which contains C, H_2 the half-plane determined by line $\overleftrightarrow{BC}$ containing A, and H_3 the half-plane determined by line $\overleftrightarrow{AC}$ containing B.

Then, the interior of ΔABC is $H_1 \cap H_2 \cap H_3$. If H_1', H_2', and H_3' are the other half-planes determined by these lines, then the exterior of ΔABC is $H_1' \cup H_2' \cup H_3'$.

Theorem 6.15. The boundary of a triangular region in R_2 is the triangle.

Proof: Let the triangle be ΔABC. The interior of the triangle is the intersection of three half-planes. Since, by Theorem 6.14, the half-planes are open, the interior of the triangle is open. The exterior of the triangle is the union of three half-planes. Therefore, the exterior of the triangle is open. Thus, the exterior of the triangle and the interior of the triangle contain no boundary points of the triangle. Since, any point on the triangle is a boundary point for each of the half-planes determined by the segment containing the point, any point of the triangle is a boundary point for the triangular region. Hence, the triangle is the boundary of the triangular region.

Jordan Content in R_2

Definition 6.16. A closed interval $I = \langle(a_1, a_2), (b_1, b_2)\rangle$ in R_2 is the set $\{(x_1, x_2) | a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2\}$. An open interval $I = ((a_1, a_2), (b_1, b_2))$ is the set $\{P = (x_1, x_2) | a_1 < x_1 < b_1, a_2 < x_2 < b_2\}$. The measure of I is defined to be $(b_1 - a_1)(b_2 - a_2)$. The measure of I is denoted $\mu(I)$. A subset of E of R_2 is called bounded if and only if there exists a closed interval I such that $E \subset I$.

If $I = \langle(a_1, a_2), (b_1, b_2)\rangle$ is a closed interval and $P = (a, b)$ is not a point of I, then it is not true that $a_1 \leq a \leq b_1$ and $a_2 \leq b \leq b_2$. Thus, suppose $a < a_1$. Then, the sphere $S\left(a, \frac{a_1 - a}{2}\right)$ is contained in $R_2 - I$. Thus, a closed interval is R_2-closed. Similarly, an open interval is R_2-open.

If the interval I is degenerate, that is $a_1 = b_1$ or $a_2 = b_2$, then $\mu(I) = 0$. Otherwise $\mu(I) > 0$.

Intervals will be used to approximate other point sets in R_2 and, where possible, these approximations will be used to define the content of the set of points.

Definition 6.17. Let $I = \langle (a_1, a_2)(b_1, b_2) \rangle$ be a closed interval in R_2. A partition of I is a set $P = \{(x_i, y_j) | a_1 = x_0 < x_1 < \ldots < x_n = a_2$, and $b_1 = y_0 < y_1 < \ldots < y_m = b_2\}$. A partition determines mn intervals $I_{ij} = \langle (x_i, y_j), (x_{i+1}, y_{j+1}) \rangle$, $i = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, m-1$. These intervals are called subintervals of I determined by P. A partition P' of I is called finer than P if $P \subset P'$. The set of all partitions of an interval I is denoted $\Omega(I)$.

Definition 6.18. Let E be a bounded subset of R_2 and I be a closed interval containing E. For P a partition of I, denote the sum of the measures of all subintervals containing points of E by $\bar{J}(P, E)$. Denote the sum of the measures of all intervals which contain only points of E° by $\underline{J}(P, E)$. If $E^\circ = \emptyset$, then $\underline{J}(P, E)$ is taken to be zero. Then $\bar{c}(E) = \inf \{\bar{J}(P, E) | P \in \Omega(I)\}$ and $\underline{c}(E) = \sup \{\underline{J}(P, E) | P \in \Omega(I)\}$ are defined to be the outer (Jordan) content and inner (Jordan) content of E, respectively.

It is an exercise using the definition of infimum and supremum to show that $\bar{c}(E)$ and $\underline{c}(E)$ depend only on E and not on the interval I containing E.

Let E be a bounded set and $I = \langle (a_1, b_1), (a_2, b_2) \rangle$ be a closed interval containing E. Let P and P' be partitions of I with P' finer than P. Then $\bar{J}(P, E) \geq \bar{J}(P', E)$ and $\underline{J}(P', E) \geq \underline{J}(P, E)$. (6.18)

Let the partition $P = \{(x_i, y_j) | a_1 = x_0 < \ldots < x_n = b_1, a_2 < y_1 < \ldots < y_m = b_2\}$. To show that $\bar{J}(P, E) \geq \bar{J}(P', E)$ it is sufficient to take P' with one extra point, say x, such that $a_1 = x_0 < \ldots < x_i < x < x_{i+1} < \ldots < x_n = b$ and show $\bar{J}(P, E) \geq \bar{J}(P', E)$. The same proof could be done a finite number of times for any P'.

Now the interval $I_{ij} = \langle(x_i, y_j), (x_{i+1}, y_{j+1})\rangle$ is separated into two intervals $I'_{ij} = \langle(x_i, y_j), (x, y_{j+1})\rangle$ and $I''_{ij} = \langle(x, y_j), (x_{i+1}, y_{j+1})\rangle$ and $\mu(I'_{ij}) + \mu(I''_{ij}) = (x - x_i)(y_{j+1} - y_j) + (x_{i+1} - x)(y_{j+1} - y_j) = (x_{i+1} - x_i)(y_{j+1} - y_j) = \mu(I)$. Now, I_{ij} contains points of E. But not both of I'_{ij} and I''_{ij} need contain points of E. Thus, $\bar{J}(P, E) \geq \bar{J}(P', E)$. A similar argument shows $\underline{J}(P', E) \geq \underline{J}(P, E)$.

Clearly, since each $\bar{J}(P, E) \geq 0$ and each $\underline{J}(P, E) \geq 0$, $\bar{c}(E) \geq 0$ and $\underline{c}(E) \geq 0$. (6.19)

The outer and inner content of a bounded set will also be characterized by the use of interval unions.

Definition 6.20. An interval union S is a finite union of non-overlapping closed intervals $S = I_1 \cup I_2 \cup \ldots \cup I_n$.

Any finite set is an interval union. For, if $E = \{x_i, y_i)\}$ take I_i to be the degenerate interval $I_i = \langle(x_i, y_i), (x_i, y_i)\rangle$. The empty set is also considered an interval union.

Theorem 6.21. Let $S = I_1 \cup I_2 \cup \ldots \cup I_k$ be a union of non-degenerate closed intervals in R_2. Then, there exists a finite number of intervals $J_1, J_2, \ldots, J_q$ with the following properties:

(1) J_i and J_j have no interior points in common $i \neq j$,

(2) If s is an integer from 1 to k, then I_s contains all those intervals J_t that have an interior point in common with I_s, and

(3) $\mu(I_s)$ is the sum of the measure of those intervals that have interior points in common with I.

Proof: Suppose $I_r = \langle(a_{1r}, a_{2r}), (b_{1r}, b_{2r})\rangle, r = 1, 2, \ldots, k$. Let $x_0, x_1, \ldots, x_v$ be the set of all the numbers $a_{1r}, b_{1r}, r = 1, 2, \ldots, k$ arranged so that $x_0 < x_1 < \ldots < x_v$. Similarly, let $y_0, y_1, \ldots, y_w$ be the set of the a_{2r}, b_{2r} arranged so that $y_0 < y_1 < \ldots < y_w$. Define $J(m, r) = \langle(x_{m-1}, y_{r-1}), (x_m, y_r)\rangle$ and let $J_1, J_2, \ldots, J_q$ be the v w intervals so obtained. Clearly, J_i and J_j have no interior points in common if $i \neq j$. Let $I_s = \langle(a_{1s}, a_{2s}), (b_{1s}, b_{2s})\rangle$ be one of the intervals $I_1, I_2, \ldots, I_k$. Since I_s is not a degenerate interval, $x_i = a_{1s} < b_{1s} = x_j$ and $y_u = a_{2s} < b_{2s} = y_t$ for some i, j, u, and t. Then, I_s contains all those intervals J(m, r) for which $i + 1 \leq m \leq j$ and $u + 1 \leq r \leq t$. The sum of the contents of these intervals is

$$\sum_{m=i+1}^{j} \sum_{r=u+1}^{t} (x_m - x_{m-1})(y_r - y_{r-1}) = \left(\sum_{m=i+1}^{j}(x_m - x_{m-1})\right)\left(\sum_{r=u+1}^{t}(y_r - y_{r-1})\right)$$

$$= (x_j - x_i)(y_t - y_u) = (b_{1s} - a_{1s})(b_{2s} - a_{2s}) = \mu(I_s).$$

It remains to show that if I_s has interior points in common with J(m, r), then $i + 1 \leq m \leq j$ and $u + 1 \leq r \leq t$. Suppose $\langle(x_{m-1}, y_{r-1}), (x_m, y_r)\rangle \cap \langle(a_{1s}, a_{2s}), (b_{1s}, b_{2r})\rangle \neq \emptyset$. Then, there is an x and y such that $x_{m-1} < x < x_m$ with $a_{1s} < x < b_{1s}$ and $y_{r-1} < y < y_r$ with $a_{2s} < y < b_{2r}$. Therefore, $a_{1s} < x_m$ and $b_{1s} > x_{m-1}$. Since, $a_{1s} = x_i$ and $b_{1s} = x_j$, $x_i < x_m$ and $x_{m-1} < x_j$. Therefore, $i + 1 \leq m \leq j$. Similarly, $u + 1 \leq r \leq t$. This completes the proof.

An interval union $S = I_1 \cup I_2 \cup \ldots \cup I_n$ may be expressed as an interval union in many ways. If $\mu(S)$ is defined by $\mu(S) = \sum_{i=1}^{n} \mu(I_i)$, then $\sum_{i=1}^{n} \mu(I_i)$ would have to be independent of the particular interval union used to represent S.

If $S = I_1 \cup I_2 \cup \ldots \cup I_n = I_1' \cup I_2' \cup \ldots \cup I_m'$ are two representations of S as an interval union, then $S = I_1 \cup \ldots \cup I_n \cup I_1' \cup \ldots \cup I_m'$. By Theorem 6.21, there are non-overlapping intervals $J_1, J_2, \ldots, J_q$ such that each of the I_j and the I_j' have the property each contains all those interval J_i which have interior points in common with the interval and the content of the interval is the sum of the content of the sub-intervals. Therefore, $\sum_{i=1}^{n} \mu(I_i)$ and $\sum_{i=1}^{m} \mu(I_i')$ is each equal the sum of the $\mu(J_i)$ which have points in common with S. Thus, $\mu(S)$ may unambiguously be defined as $\mu(S) = \mu(I_1) + \ldots + \mu(I_n)$. (6.22)

If S_1 and S_2 are interval unions, then by considering the sub-intervals of $S_1 \cup S_2$ given by Theorem 6.21, it can be shown that $S_1 \cup S_2$ and $S_1 \cap S_2$ are interval unions and $\mu(S_1 \cup S_2) + \mu(S_1 \cap S_2) = \mu(S_1) + \mu(S_2)$. (6.23)

The boundary S_b of an interval union is itself a degenerate interval union and $\mu(S_b) = 0$. (6.24)

It follows from 6.23 and 6.24 that if S_1 and S_2 are non-overlapping interval unions, then $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$. It follows that if $S_1, S_2, \ldots, S_n$ are mutually non-overlapping interval unions then,

$$\mu(S_1 \cup S_2 \cup \ldots \cup S_n) = \mu(S_1) + \mu(S_2) + \ldots + \mu(S_n). \qquad (6.25)$$

The difference $S_1 - S_2$ of two interval unions need not be an interval union. The difference of two different intervals need not contain all of its boundary points. But $\overline{S_1 - S_2}$, the closure of

$S_1 - S_2$, is an interval union and S_2 and $\overline{S_1 - S_2}$ have at most boundary points in common, so that, by (6.25), $\mu(\overline{S_1 - S_2}) + \mu(S_2) = \mu(S_1)$ or

$$\mu(\overline{S_1 - S_2}) = \mu(S_1) - \mu(S_2). \tag{6.26}$$

Also, if S_1 and S_2 are interval unions with $S_2 \subset S_1$, then $\mu(S_2) \leq \mu(S_1)$. (6.27)

Let E be a bounded subset of R_2 and I an interval containing E. Then, by definition of a partition and an interval union, the union of the subintervals I_k determined by P and containing points of E is an interval union. Also, if $S = I_1 \cup I_2 \cup \ldots \cup I_n$ is an interval union, then, as in the proof of Theorem 6.21, I_s, $s = 1, 2, \ldots, n$, is the union of some of the intervals $J_1, J_2, \ldots, J_q$ where J_k is determined by the partition $x_0 < x_1 < \ldots < x_v$ and $y_0 < y_1 < \ldots < y_w$ of the interval $\langle (x_0, y_0), (x_v, y_w) \rangle$. Therefore, S is the union of intervals of a partition. Thus, the following theorem has been proved.

Theorem 6.28. Let E be a bounded set in R_2. Then $\bar{c}(E)$ is the infimum of the set of $\mu(S)$ where S is an interval union of non-overlapping intervals containing E.

Similarly, it can be shown that $\underline{c}(E)$ is the supremum of the set of $\mu(S)$ where S is an interval union of non-overlapping intervals containing only interior points of E.

Let E be a bounded set in R_2. Then $\underline{c}(E) \leq \bar{c}(E)$. (6.29)

This follows because:

Let I be an interval containing E. By definition of inner content, for each $t > 0$ there exists a partition P_1 of I such that $\underline{J}(P_1, E) > \underline{c}(E) - \frac{t}{2}$. Also, there exists a partition P_2 such that $\bar{J}(P_2, E) < \bar{c}(E) + \frac{t}{2}$. Now, $P_1 \cup P_2 = P$ is a partition of I which is at least as fine as P_1 or P_2. Therefore, by 6.18, $\underline{J}(P, E) \geq \underline{J}(P_1, E)$ and $\bar{J}(P, E) \leq \bar{J}(P_2, E)$. Since each addend of $\underline{J}(P, E)$ is an addend of $\bar{J}(P, E)$, $\underline{J}(P, E) \leq \bar{J}(P, E)$. Thus, $\underline{c}(E) - \frac{t}{2} < \underline{J}(P_1, E) \leq \underline{J}(P, E) \leq \bar{J}(P, E) \leq \bar{J}(P_1, E) < \bar{c}(E) + \frac{t}{2}$. Therefore, $\underline{c}(E) < \bar{c}(E) + t$. Since t is arbitrary, $\underline{c}(E) \leq \bar{c}(E)$ as was to be shown.

If E_1 and E_2 are bounded sets with $E_2 \subset E_1$, then $\bar{c}(E_2) \leq \bar{c}(E_1)$ and $\underline{c}(E_2) \leq \underline{c}(E_1)$. (6.30)

This follows from the fact that any interval union containing E_1 contains E_2 and any interval union contained in the interior of E_2 is contained in the interior of E_1.

If E is a bounded subset of R_2, then $\bar{c}(E) = \bar{c}(\bar{E})$ ($\bar{E}$ is the closure of E). (6.31)

This follows since $E \subset \bar{E}$ implies $\bar{c}(E) \leq \bar{c}(\bar{E})$. Also, since any interval union is closed, any interval union containing E contains $\bar{E}$. Therefore, $\bar{c}(\bar{E}) \leq \bar{c}(E)$. Thus, $\bar{c}(E) = \bar{c}(\bar{E})$.

Also, $\underline{c}(E^\circ) = \underline{c}(E)$. (6.32)

This follows since $E^\circ \subset E$ implies $\underline{c}(E^\circ) \leq \underline{c}(E)$. Also, any interval union containing only interior points of E contains only interior points of E°. Therefore, $\underline{c}(E) \leq \underline{c}(E^\circ)$. Thus, $\underline{c}(E) = \underline{c}(E^\circ)$.

<u>Theorem 6.33</u>. If E_1 and E_2 are bounded sets, then $\bar{c}(E_1 \cup E_2) + \bar{c}(E_1 \cap E_2) \leq \bar{c}(E_1) + \bar{c}(E_2)$.

Proof: Let $t > 0$ be given. By Theorem 6.28, there exists an interval union S_1 containing E_1 and an interval union S_2 containing E_2 such that $\bar{c}(E_1) + \frac{t}{2} > \mu(S_1)$ and $\bar{c}(E_2) + \frac{t}{2} > \mu(S_2)$. Now $S_1 \cup S_2$ and $S_1 \cap S_2$ are interval unions with $E_1 \cup E_2 \subset S_1 \cup S_2$ and $E_1 \cap E_2 \subset S_1 \cap S_2$. Therefore, $\bar{c}(E_1 \cup E_2) + \bar{c}(E_1 \cap E_2) \leq \mu(S_1 \cup S_2) + \mu(S_1 \cap S_2)$. By 6.23, $\mu(S_1 \cup S_2) + \mu(S_1 \cap S_2) = \mu(S_1) + \mu(S_2)$. Therefore,

$\bar{c}(E_1 \cup E_2) + \bar{c}(E_1 \cap E_2) \leq \mu(S_1) + \mu(S_2) < \bar{c}(E_1) + \frac{t}{2} + \bar{c}(E_2) + \frac{t}{2} =$
$\bar{c}(E_1) + \bar{c}(E_2) + t$. Since t is arbitrary,
$\bar{c}(E_1 \cup E_2) + \bar{c}(E_1 \cap E_2) \leq \bar{c}(E_1) + \bar{c}(E_2)$.

Theorem 6.34. If E_1 and E_2 are bounded sets, then $\underline{c}(E_1 \cup E_2) + \underline{c}(E_1 \cap E_2) \geq \underline{c}(E_1) + \underline{c}(E_2)$.

Proof: Let $t > 0$ be given. By Theorem 6.28, there exists interval unions S_1 and S_2 such that $S_1 \subset E_1^\circ$, $S_2 \subset E_2^\circ$, $\underline{c}(E_1) - \frac{t}{2} < \mu(S_1)$ and $\underline{c}(E_2) - \frac{t}{2} < \mu(S_2)$. $S_1 \cup S_2$ is an interval union contained in $(E_1 \cup E_2)^\circ$ and $S_1 \cap S_2$ is an interval union contained in $(E_1 \cap E_2)^\circ$. Thus, $\underline{c}(E_1 \cup E_2) \geq \mu(S_1 \cup S_2)$ and $\underline{c}(E_1 \cap E_2) \geq \mu(S_1 \cap S_2)$. Therefore,
$\underline{c}(E_1 \cup E_2) + \underline{c}(E_1 \cap E_2) \geq \mu(S_1 \cup S_2) + \mu(S_1 \cap S_2)$
$= \mu(S_1) + \mu(S_2) > \underline{c}(E_1) - \frac{t}{2} + \underline{c}(E_2) - \frac{t}{2}$
$= \underline{c}(E_1) + \underline{c}(E_2) - t$. Since t is arbitrary,
$\underline{c}(E_1 \cup E_2) + \underline{c}(E_1 \cap E_2) \geq \underline{c}(E_1) + \underline{c}(E_2)$.

An example will show that the less than or equal to in Theorems 6.33 and 6.34 cannot be replaced by equality. Let $E_1 = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1$ with x and y rational$\}$. Let $I = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Let $E_2 = I - E_1$. Now, E_1 and E_2 are dense in I. That is, every neighborhood of a point (x, y) in I contains infinitely many points of both E_1 and E_2. Thus, $\bar{E}_1 = \bar{E}_2 = I$. Therefore, $\bar{c}(E_1) = \bar{c}(\bar{E}_1) = 1$, and $\bar{c}(\bar{E}_2) = 1$. $E_1 \cap E_2 = \emptyset$. Therefore, $\bar{c}(E_1 \cap E_2) = 0$. $E_1 \cup E_2 = I$. Thus, $\bar{c}(E_1 \cup E_2) = 1$. Therefore, $\bar{c}(E_1 \cup E_2) + \bar{c}(E_1 \cap E_2) = 1 + 0 < 1 + 1 = \bar{c}(E_1) + \bar{c}(E_2)$.

In a similar fashion, $\underline{c}(E_1) = 0$, $\underline{c}(E_2) = 0$, $\underline{c}(E_1 \cap E_2) = 0$ and $\underline{c}(E_1 \cup E_2) = 1$. Therefore, $\underline{c}(E_1) + \underline{c}(E_2) = 0 + 0 < 1 + 0 = \underline{c}(E_1 \cup E_2) + \underline{c}(E_1 \cap E_2)$.

<u>Definition 6.35</u>. Let E be a bounded subset of R_2. Then E has content if and only if $\bar{c}(E) = \underline{c}(E)$. If E has content, then the content of E is denoted by $c(E)$ and $\bar{c}(E) = \underline{c}(E) = c(E)$.

<u>Theorem 6.36</u>. Let E be a bounded set in R_2 and let E_b denote its boundary. Then, $\underline{c}(E) + \bar{c}(E_b) = \bar{c}(E)$.

Proof: Let I be a closed interval containing E. Since I is closed, by Theorem 6.12, $E_b \subset I$. Let P be a partition of I. $\bar{J}(P, E_b)$ is the sum of the measures of those subintervals containing points of E_b, $\bar{J}(P, E)$ is the sum of the measures of those subintervals containing points of E, and $\underline{J}(P, E)$ is the sum of the measures of those subintervals containing only interior points of E.

Since $\bar{E} = E^{\circ} \cup E_b$, $\bar{J}(P, E_b) = \bar{J}(P, \bar{E}) - \underline{J}(P, E) = \bar{J}(P, E) - \underline{J}(P, E)$. Therefore, $\bar{J}(P, E_b) \geq \sup_{P \in \Omega(I)} \bar{J}(P, E) - \inf_{P \in \Omega(I)} J(P, E) = \bar{c}(E) - \underline{c}(E)$. Since this is true for each partition P, then $\bar{c}(P, E_b) = \sup_{P \in \Omega(I)} \bar{J}(P, E_b) \geq \bar{c}(E) - \underline{c}(E)$.

To prove the reverse inequality, let $t > 0$ be given. Choose a partition P_1 such that $\bar{J}(P_1, E) < \bar{c}(E) + \frac{t}{2}$ and choose a partition P_2 such that $\underline{J}(P_2, E) > \underline{c}(E) - \frac{t}{2}$. Let P be the partition $P_1 \cup P_2$. Since refinement of partitions increases inner sums $\underline{J}$ and decreases outer sums $\bar{J}$, $\bar{c}(E_b) \leq \bar{J}(P, E_b) = \bar{J}(P, E) - \underline{J}(P, E) \leq \bar{J}(P_1, E) - \underline{J}(P_2, E) < \bar{c}(E) - \underline{c}(E) + t$. Since t is arbitrary, $\bar{c}(E_b) \leq \bar{c}(E) - \underline{c}(E)$. Therefore, $\bar{c}(E_b) = \bar{c}(E) - \underline{c}(E)$, and the proof is complete.

<u>Corollary 6.37</u>. Let E be a bounded set in R_2 and E_b be the boundary of E. Then, E has content if and only if $\bar{c}(E_b) = 0$.

Proof: E has content if and only if $\bar{c}(E) = \underline{c}(E)$ or $\bar{c}(E) - \underline{c}(E) = 0$. But by Theorem 6.36, $\bar{c}(E) - \underline{c}(E) = \bar{c}(E_b)$. This proves the theorem.

By 6.29, $\underline{c}(E) \leq \bar{c}(E)$. Thus, to prove a set E has content it is sufficient to show that $\underline{c}(E) \geq \bar{c}(E)$. (6.38)

Also, since $\underline{c}(E) \geq 0$ and $\bar{c}(E) \geq 0$, for all bounded sets having content, $c(E) \geq 0$.

If $\bar{c}(E) = 0$, then, since $0 \leq \underline{c}(E) \leq \bar{c}(E) = 0$, E has content and $c(E) = 0$. (6.39)

Theorem 6.40. If E_1 and E_2 have content, then so have $E_1 \cup E_2$ and $E_1 \cap E_2$. Also, $c(E_1 \cup E_2) + c(E_1 \cap E_2) = c(E_1) + c(E_2)$.

Proof: By Theorem 6.34, $\underline{c}(E_1 \cup E_2) + \underline{c}(E_1 \cap E_2) \geq \underline{c}(E_1) + \underline{c}(E_2) = c(E_1) + c(E_2)$. By Theorem 6.33, $c(E_1) + c(E_2) = \bar{c}(E_1) + \bar{c}(E_2) \geq \bar{c}(E_1 \cup E_2) + \bar{c}(E_1 \cap E_2)$. Therefore, $\underline{c}(E_1 \cup E_2) + \underline{c}(E_1 \cap E_2) \geq \bar{c}(E_1 \cap E_2) + \bar{c}(E_1 \cap E_2)$. Since $\underline{c}(E_1 \cup E_2) \leq \bar{c}(E_1 \cup E_2)$ and $\underline{c}(E_1 \cap E_2) \leq \bar{c}(E_1 \cap E_2)$, this implies that $\underline{c}(E_1 \cup E_2) = \bar{c}(E_1 \cup E_2)$ and $\underline{c}(E_1 \cap E_2) = \bar{c}(E_1 \cap E_2)$. Thus, $E_1 \cup E_2$ and $E_1 \cap E_2$ have content.

Since $\underline{c}(E_1 \cup E_2) + \underline{c}(E_1 \cap E_2) \geq c(E_1) + c(E_2) \geq \bar{c}(E_1 \cup E_2) + \bar{c}(E_1 \cap E_2)$ and $\underline{c}(E_1 \cup E_2) + \underline{c}(E_1 \cap E_2) = \bar{c}(E_1 \cup E_2) + \bar{c}(E_1 \cap E_2) = c(E_1 \cup E_2) + c(E_1 \cap E_2)$, it follows that $c(E_1 \cup E_2) + c(E_1 \cap E_2) = c(E_1) + c(E_2)$.

Theorem 6.41. If $I = \langle (a_1, a_2), (b_1, b_2) \rangle$ is an interval, then I has content and $c(I) = \mu(I)$.

Proof: Since I is an interval containing I and for any partition P of I, $\bar{J}(P, I) = \mu(I)$, $\bar{c}(I) = \mu(I)$.

Let $t > 0$ be given. Let $t' = \min \left\{ \frac{t}{2(b_1 + b_2 - a_1 - a_2)}, \frac{(b_1 - a_1)}{3}, \frac{(b_2 - a_2)}{3} \right\}$.

Consider the partition P of I given by $a_1 = x_0 < x_1 = a_1 + t' <$

$x_2 = b_1 - t' < x_3 = b_1$ and $a_2 = y_0 < y_1 = a_2 + t' < y_2 = b_2 - t' < y_3 = b_2$. Now, $\underline{J}(P, I) = (b_1 - a_1 - 2t')(b_2 - a_2 - 2t') = (b_1 - a_1)(b_2 - a_2) - 2t'(b_1 - a_1 + b_2 - a_2) + 4(t')^2 > \mu(I) - t$. Since $\underline{c}(I) \geq \underline{J}(P, I)$, $\underline{c}(I) > \mu(I) - t$. Since t is arbitrary, $\underline{c}(I) \geq \mu(I)$. But $\mu(I) = \bar{c}(I)$. Therefore, $\underline{c}(I) \geq \bar{c}(I)$ and I has content. Since $\mu(I) = \bar{c}(I)$, $\mu(I) = c(I)$.

<u>Definition 6.42</u>. A one-to-one transformation $f:R_2 \rightarrow R_2$ is content preserving if and only if for each set E having content, f(E) has content and $c(E) = c(f(E))$.

If f and g are two content preserving maps, then fg is content preserving. That is, the composition of content preserving maps is content preserving. This follows because if E has content, then g(E) has content and $c(g(E)) = c(E)$. Therefore, since f is content preserving, fg(E) has content and $c(fg(E)) = c(g(E)) = c(E)$.

<u>Theorem 6.43</u>. If $f:R_2 \rightarrow R_2$ is a one-to-one transformation and f preserves the content of each interval, then f is content preserving.

Proof: Let E be any set having content. Then, there exists a closed interval I such that $E \subset I$. Therefore, $f(E) \subset f(I)$. Since f(I) has content, there exists an interval I′ such that $f(E) \subset f(I) \subset I'$. Therefore, f(E) is bounded. Since E has content, $\bar{c}(E) = \underline{c}(E)$. Thus, for $t > 0$ there exists an interval union $S = I_1 \cup I_2 \cup \ldots \cup I_n$ such that $E \subset I_1 \cup I_2 \cup \ldots \cup I_n$ and $\bar{c}(E) + t > c(I_1) + c(I_2) + \ldots + c(I_n)$. Since $f(E) \subset f(I_1 \cup I_2 \cup \ldots \cup I_n) = f(I_1) \cup f(I_2) \cup \ldots \cup f(I_n)$, $\bar{c}(f(E)) \leq \bar{c}(f(I_1) \cup f(I_2) \cup \ldots \cup f(I_n))$.

By Theorem 6.33, $\bar{c}(f(I_1) \cup f(I_2) \cup \ldots \cup f(I_n)) \leq \bar{c}(f(I_1)) + \bar{c}f((I_2)) + \ldots + \bar{c}(f(I_n))$. But, $\bar{c}(f(I_1)) + \bar{c}(f(I_2)) + \ldots + \bar{c}(f(I_n))$

$= c(I_1) + c(I_2) + \ldots + c(I_n)$. Therefore, $\bar{c}(f(I_1) \cup f(I_2) \cup \ldots \cup f(I_n)) \leq c(I_1) + c(I_2) + \ldots + c(I_n) \leq \bar{c}(E) + t$. Thus, $\bar{c}(f(E)) \leq \bar{c}(E) + t = c(E) + t$. Since t was arbitrary, $\bar{c}(f(E)) \leq c(E)$.

Similarly, there exists an interval union $S = I_1 \cup I_2 \cup \ldots \cup I_n$ such that $I_1 \cup I_2 \cup \ldots \cup I_n \subset E^\circ$ and $c(E) - t \leq \sum_{i=1}^{n} c(I_i)$. Since $f(I_1 \cup I_2 \cup \ldots \cup I_n) = f(I_1) \cup f(I_2) \cup \ldots \cup f(I_n) \subset f(E^\circ) \subset f(E)$, $\underline{c}(f(I_1) \cup f(I_2) \cup \ldots \cup f(I_n)) \leq \underline{c}(f(E))$. By Theorem 6.34, $\underline{c}(f(I_1)) + \underline{c}(f(I_2)) + \ldots + \underline{c}(f(I_n)) \leq \underline{c}(f(I_1) \cup \ldots \cup f(I_n)) + d$ where d is the content of the intersection of the image of some of the intervals $I_1, I_2, \ldots, I_n$. But the intersection of the image of any two of these intervals is the image of the intersection. Any two of these intervals intersect in a degenerate interval, hence, has content zero. Thus, since the image of an interval has the same content, $d = 0$. Therefore, $\underline{c}(f(I_1)) + \ldots + \underline{c}(f(I_n)) \leq c(f(I_1) \cup \ldots \cup f(I_n))$. Thus, $c(I_1) + c(I_2) + \ldots + c(I_n) = \underline{c}(f(I_1)) + c(f(I_2)) + \ldots + c(f(I_n)) \leq \underline{c}(f(E))$. Therefore, $\underline{c}(E) - t < \underline{c}(f(E))$ or $\underline{c}(E) = c(E) \leq \underline{c}(f(E))$. Therefore, since $\bar{c}(f(E)) \leq c(E) \leq \underline{c}(f(E))$, $\bar{c}(f(E)) = \underline{c}(f(E)) = c(E)$. Thus, $f(E)$ has content and $c(f(E)) = c(E)$.

<u>Theorem 6.44</u>. The transformations

0) $f_0(x_1, x_2) = (x_1, -x_2)$ or $(-x_1, x_2)$

a) $f_1(x_1, x_2) = (x_1, x_2) + (c, d)$

b) $f_2(x_1, x_2) = (x_2, x_1)$

c) $f_3(x_1, x_2) = (c\,x_1, \frac{1}{c}\,x_2)$ or $(\frac{1}{c}\,x_1, c\,x_2)$ and

d) $f_4(x_1, x_2) = (x_1 + dx_2, x_2)$ or $(x_1, x_2 + dx_1)$

are each content preserving.

Proof: Let $I = \langle(a_1, a_2), (b_1, b_2)\rangle = \{(x_1, x_2) | a_1 \leq x_1 \leq b_1$ and

$a_2 \leq x_2 \leq b_2\}$ be an interval in R_2. Then $c(I) = (b_1 - a_1)(b_2 - a_2)$. Now, $f_0(I) = \{(x_1, x_2) | a_1 \leq x_1 \leq b_1$ and $-b_2 \leq x_2 \leq -a_2\}$ or $\{(x_1, x_2) | -b_1 \leq x_1 \leq -a_1$ and $a_2 \leq x_2 \leq b_2\}$,

$f_1(I) = \{(x_1, x_2) | a_1 + c \leq x_1 \leq b_1 + c$ and $a_2 + d < x_2 \leq b_2 + d\}$,

$f_2(I) = \{(x_1, x_2) | a_2 \leq x_1 \leq b_2$ and $a_1 \leq x_2 \leq b_1\}$ and

$f_3(I) = \{(x_1, x_2) | ca_1 \leq x_1 \leq cb_1$ and $\frac{1}{c} a_2 \leq x_2 \leq \frac{1}{c} b_2\}$ or

$\{(x_1, x_2) | ca_1 \geq x_1 \geq cb_1$ and $\frac{1}{c} a_2 \geq x_2 \geq \frac{1}{c} b_2\}$.

In each case, $f_0(I)$, $f_1(I)$, $f_2(I)$, and $f_3(I)$ is an interval and has content $(b_1 - a_1)(b_2 - a_2) = c(I)$. Thus, each of f_0, f_1, f_2, and f_3 are content preserving by Theorem 6.43.

The proof that f_4 is content preserving is quite long. Therefore, the steps used in the proof will be outlined.

(1) If I is an interval, it will be shown that $c(I) \leq \bar{c}(f_4(I))$.

(2) The reverse inequality, $\bar{c}(f_4(I)) \leq c(I)$ will then be shown giving $c(I) = \bar{c}(f_4(I))$.

(3) Next, it will be shown that the boundary of $f_4(I)$ has content zero. This implies that $f_4(I)$ has content and $c(f_4(I)) = \bar{c}(f_4(I)) = c(I)$. This will complete the proof.

(1) Let $f_4(x_1, x_2) = (x_1 + dx_2, x_2) = (y_1, y_2)$. The other case for f_4 would be similar. Also, assume $d \geq 0$. The interval I is mapped by f_4 into the set of points $E = \{(y_1, y_2) | a_1 + dy_2 \leq y_1 \leq b_1 + dy_2$ and $a_2 \leq y_2 \leq b_2\}$. E is contained in the interval $I = \langle(a_1 + da_2, a_2), (b_1 + db_2, b_2)\rangle$. Hence, E is bounded.

Now, $a_1 \leq x_1 \leq b_1$ and $a_2 \leq x_2 \leq b_2$ implies, for each positive integer N and for some integer K, where $1 \leq K \leq N$, that

$a_2 + \frac{K-1}{N}(b_2 - a_2) \leq x_2 \leq a_2 + \frac{K}{N}(b_2 - a_2)$. Thus,

$d(a_2 + \frac{K-1}{N}(b_2 - a_2)) \leq dx_2 \leq d(a_2 + \frac{K}{N}(b_2 - a_2))$. Therefore,

$a_1 + d(a_2 + \frac{K-1}{N}(b_2 - a_2)) \leq x_1 + dx_2 \leq b_1 + d(a_2 + \frac{K}{N}(b_2 - a_2))$.

Thus, the set E is contained in the N intervals, $S_k = \{y_1, y_2)|a_1 + d(a_2 + \frac{K-1}{N}(b_2 - a_2)) \leq y_1 \leq b_1 + d(a_2 + \frac{K}{N}(b_2 - a_2))$ and $a_2 + \frac{K-1}{N}(b_2 - a_2) \leq y_2 \leq a_2 + \frac{K}{N}(b_2 - a_2)\}$. The content of each S_k is $(b_1 - a_1 + \frac{d}{N}(b_2 - a_2)) \cdot \frac{1}{N}(b_2 - a_2)$. Summing over the K intervals gives, $\sum_{K=1}^{N} c(S_k) = (b_1 - a_1) \cdot (b_2 - a_2) + \frac{d}{N}(b_2 - a_2)^2$. Since E =

$f_4(I) \subset \bigcup_{K=1}^{N} S_k$, $\bar{c}(f_4(I)) \leq (b_1 - a_1)(b_2 - a_2) + \frac{d}{N}(b_2 - a_2)^2$. Since, by choosing N large enough, $\frac{d}{N}(b_2 - a_2)^2$ may be made arbitrarily small, $\bar{c}(f_4(I)) \leq (b_1 - a_1)(b_2 - a_2) = c(I)$. The assumption that d was positive in the above argument affects only the inequalities on the interval S_k. If d is negative, then $c(S_k) = (b_1 - a_1)(b_2 - a_2) - \frac{d}{N}(b_2 - a_2)^2$. Thus, in either case, $\bar{c}(f_4(I)) \leq c(I)$.

(2) Now to show that $\bar{c}(f_4(I)) \geq c(I)$.

The inverse transformation of f_4 is the transformation f_5 where $f_5(X_1, X_2) = (X_1 - dX_2, X_2)$. This is the same type transformation as f_4. Thus, by what has been proved for f_4, $\bar{c}(f_5(I)) \leq c(I)$ for any interval I.

For any $t > 0$, there exist intervals $I_1, I_2, \ldots I_n$ such that

$f_4(I) \subset \bigcup_{i=1}^{n} I_i$ and $\bar{c}(f_4(I)) + t \geq c(I_1) + c(I_2) + \ldots + c(I_n)$.

For each $i = 1, 2, \ldots, n$, $\bar{c}(f_5(I_i)) \leq c(I_i)$. Thus, since

$f_4(I) \subset \bigcup_{i=1}^{n} I_i$, $I \subset \bigcup_{i=1}^{n} f_5(I_i)$. Therefore, $\bar{c}(I) \leq \bar{c}(\bigcup_{i=1}^{n} f_5(I_i)) \leq \sum_{i=1}^{n} \bar{c}(f_5(I_i)) \leq \sum_{i=1}^{n} c(I_i) \leq \bar{c}(f_4(I)) + t$. Therefore, $\bar{c}(I) \leq \bar{c}(f_4(I))$. Putting this together with $\bar{c}(f_4(I)) \geq \bar{c}(I)$, it follows that $c(I) = \bar{c}(f_4(I))$.

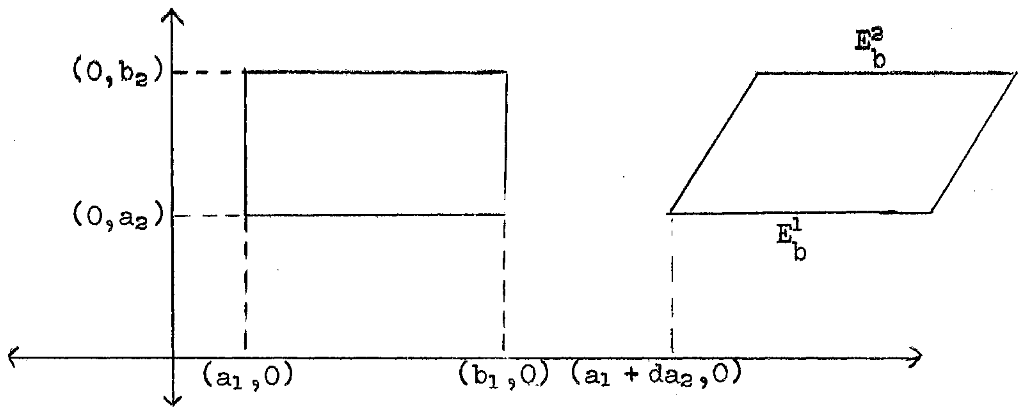

Figure 6.

(3) The set $f_4(I)$ is the set $E = \{(y_1, y_2) \mid a_1 + dy_2 \leq y_1 \leq b_1 + dy_2$ and $a_2 \leq y_2 \leq b_2\}$. (See Figure 6.) The boundary E_b of E is the union of the four sets $E_b^1 = \{(y_1, y_2) \mid a_1 + da_2 \leq y_1 \leq b_1 + da_2$ and $y_2 = a_2\}$, $E_b^2 = \{y_1, y_2) \mid a_1 + db_2 \leq y_1 \leq b_1 + db_2$ and $y_2 = b_2\}$, $E_b^3 = \{(y_1, y_2) \mid y_1 = a_1 + dy_2$ and $a_2 \leq y_2 \leq b_2\}$ and $E_b^4 = \{(y_1, y_2) \mid y_1 = b_1 + dy_2$ and $a_2 \leq y_2 \leq b_2\}$. Now, E_b^1 is the image of $I_1 = \{x_1, x_2) \mid a_1 \leq x_1 \leq b_1, x_2 = a_2\}$. I_1 is a degenerate interval and, hence, has content zero. By part (1), $\bar{c}(E_b^1) = c(I_1) = 0$. Similarly, $\bar{c}(E_b^2) = \bar{c}(E_b^3) = \bar{c}(E_b^4) = 0$. Thus, $\bar{c}(E_b) = 0$. Therefore, by Corollary 6.37, $E = f_4(I)$ has content. Thus, $c(f_4(I)) = \bar{c}(f_4(I)) = c(I)$. Therefore, by Theorem 6.43, f_4 is content preserving and the theorem is complete.

In Chapter V, Theorem 5.23, it was shown that any isometry of V was the composition of an orthogonal linear transformation and a translation. By a similar argument, it can be shown that any isometry f of R_2 is the composition of an orthogonal linear transformation g and a translation h. Let g be an orthogonal linear transformation from R_2 into R_2. Let $g(1, 0) = (a_{11}, a_{21})$ and $g(0,1) = (a_{12}, a_{22})$. Since g is a linear transformation, $g((x_1, x_2)) = g(x_1(1, 0) + x_2(0, 1)) = x_1 g(1, 0) + x_2 g(0.1) = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2)$. Since g is orthogonal, g maps the orthonormal bases $\{(1,0), (0,1)\}$ into an orthonormal basis $\{(a_{11}, a_{21}), (a_{12}, a_{22})\}$. Thus, $a_{11}^2 + a_{21}^2 = 1$, $a_{12}^2 + a_{22}^2 = 1$ and $a_{11}\ a_{12} + a_{21}\ a_{22} = 0$. Now, if $h(x_1, x_2) = (x_1, x_2) + (b, d)$ then $f(x_1, x_2) = hg(x_1, x_2) = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2) + (b, d)$.

These comments are used in the proof of the next theorem.

<u>Theorem 6.45</u>. If $f:R_2 \rightarrow R_2$ is an isometry, then f is content preserving.

Proof: $f = hg$ where $hg(x_1, x_2) = (a_{11}\ x_1 + a_{12}\ x_2, a_{21}\ x_1 + a_{22}\ x_2) + (b, d)$. By Theorem 6.44, h is content preserving. Therefore, if g is content preserving, the composition $f = hg$ is content preserving.

If $a_{11} = 0$, then, since $a_{11}^2 + a_{21}^2 = 1$, $a_{12}^2 + a_{22}^2 = 1$ and $a_{11}\ a_{12} + a_{21}\ a_{22} = 0$, $a_{12} \neq 0$. Thus, $g = g_1 g_2$ where $g_2(x_1, x_2) = (x_2, x_1)$ and $g_1(x_1, x_2) = (a_{12}\ x_1 + a_{11}\ x_2, a_{22}\ x_1 + a_{21}\ x_2)$. Since g_2 is a transformation of type b) in Theorem 6.44, g_2 is content preserving. Thus, g is content preserving if g_1 is content preserving. g_1 is a transformation like g except that the coefficient a_{12} of x_1 is non-zero. Therefore, one may assume $a_{11} \neq 0$ for g.

Since $a_{11} \neq 0$, $g = g_2 g_1$ where $g_1(x_1, x_2) = (a_{11} x_1, \frac{1}{a_{11}} x_2)$ and $g_2(x_1, x_2) = (x_1 + a_{11} a_{12} x_2, \frac{a_{21}}{a_{11}} x_1 + a_{22} a_{11} x_2)$. g_1 is a transformation of type c) in Theorem 6.44; thus, g_1 is content preserving. Therefore, g is content preserving if g_2 is content preserving.

Since $a_{11}^2 + a_{21}^2 = 1$, $a_{12}^2 + a_{22}^2 = 1$ and $a_{11} a_{12} + a_{21} a_{22} = 0$, multiplying the first two and squaring the last gives (1) $a_{11}^2 a_{12}^2 + a_{11}^2 a_{22}^2 + a_{21}^2 a_{12}^2 + a_{21}^2 a_{22}^2 = 1$ and (2) $a_{11}^2 a_{12}^2 + 2a_{11} a_{12} a_{21} a_{22} + a_{21}^2 a_{22}^2 = 0$. Thus, $a_{11}^2 a_{12}^2 + a_{21}^2 a_{22}^2 = -2a_{11} a_{12} a_{21} a_{22}$. Substituting in (1) gives $a_{11}^2 a_{22}^2 - 2a_{11} a_{12} a_{21} a_{22} + a_{21}^2 a_{12}^2 = 1$ or $(a_{11} a_{22} - a_{21} a_{12})^2 = 1$. Thus, $a_{11} a_{22} - a_{21} a_{12} = \pm 1$.

Now, $g_2 = g_4 g_3$ where $g_3(x_1, x_2) = (x_1 + a_{11} a_{12} x_2, x_2)$ and $g_4(x_1, x_2) = (x_1, \frac{a_{21}}{a_{11}} x_1 + (a_{22} a_{11} - a_{12} a_{21})x_2) = (x_1, \frac{a_{21}}{a_{11}} x_1 + (\pm x_2))$. If the sign on x_2 is +, then each of the transformations g_3 and g_4 is a transformation of type d) in Theorem 6.44. Thus, each is content preserving. If the sign is −, then $g_4 = g_6 g_5$ where $g_5(x_1, x_2) = (x_1, -x_2)$ and $g_6(x_1, x_2) = (x_1, \frac{a_{21}}{a_{11}} x_1 + x_2)$. Each of these is content preserving. Thus, g is content preserving, and the proof is finished.

Area in V

Let [B,C] be a two dimensional subspace in V with {B,C} a fixed orthonormal basis for [B,C]. Let $f:[B,C] \to R_2$ be a coordinate system for [B,C]. That is, $f(B) = (1,0)$, $f(C) = (0,1)$ and $f(x_1 B + x_2 C) = (x_1, x_2)$ are the coordinates of the points. Such a coordinate system, with {B,C} orthonormal, is called a cartesian coordinate system for the plane [B,C].

<u>Definition 6.46</u>. Let S be a set of points in a plane [B,C] such

that $f(S) = \{x_1, x_2) | (x_1\ B + x_2 C) \in S\}$ has content. The area of S is defined to be the content of f(S). In this chapter, the area of a set S will be denoted, A(S).

Does A(S) depend on the particular orthonormal basis {B,C} chosen for [B,C]? Let $\{B_1, C_1\}$ be any other orthonormal basis for [B,C]. Then, $B = a_{11}\ B_1 + a_{21}\ C_1$ and $C = a_{12}\ B_1 + a_{22}\ C_1$. Since $|B| = 1$, $a_{11}^2 + a_{21}^2 = 1$; since $|C| = 1$, $a_{12}^2 + a_{22}^2 = 1$, and since $B \cdot C = 0$, $a_{11}\ a_{12} + a_{21}\ a_{22} = 0$.

Let f be the cartesian coordinate system for [B,C] such that $f(x_1 B + x_2\ C) = (x_1, x_2)$. Let f_1 be the coordinate system for [B,C] such that $f_1(y_1\ B_1 + y_2\ C_1) = (y_1, y_2)$. Since $B = a_{11}\ B_1 + a_{21}\ C_1$, $f_1(B) = (a_{11}, a_{21})$ and, since $C = a_{12}\ B_1 + a_{22}\ C_1$, $f_1(C) = (a_{12}, a_{22})$.

Let $P = x_1\ B + x_2\ C = y_1\ B_1 + y_2\ C_1$ be in [B,C]. Then, $f(P) = (x_1, x_2)$ and $f_1(P) = (y_1, y_2)$. Now, $(y_1, y_2) = f_1(P) = f_1(x_1\ B + x_2\ C) = x_1\ f_1(B) + x_2\ f_1(C) = x_1(a_{11},\ a_{21}) + x_2(a_{12},\ a_{22}) = (a_{11}\ x_1 + a_{12}\ x_2,\ a_{21}\ x_1 + a_{22}\ x_2)$. Define the linear transformation $g: R_2 \to R_2$ by $g(x_1,\ x_2) = (a_{11}\ x_1 + a_{12}\ x_2,\ a_{21}\ x_1 + a_{22}\ x_2)$. Then, $f_1(P) = (y_1, y_2) = g(x_1, x_2) = gf(P)$. Since $g(1,0) = (a_{11}, a_{21})$ and $g(0,1) = (a_{12},\ a_{22})$, $|g(1,0)| = \sqrt{a_{11}^2 + a_{21}^2} = 1$, $|g(0,1)| = \sqrt{a_{12}^2 + a_{22}^2} = 1$ and $g(1,0) \cdot g(0,1) = a_{11}\ a_{12} + a_{21}\ a_{22} = 0$. Thus, g maps an orthonormal basis {(1,0), (0,1)} onto an orthonormal basis {g(1,0), g(0,1)}. Therefore, g is an orthogonal linear transformation. Thus, by Theorem 6.45, g preserves content of sets in R_2.

A(S) was defined to be the content of f(S). Since g preserves content, the content of gf(S) equals the content of f(S). But $gf = f'$. Therefore, the content of $f'(S)$ equals the content of f(S). In a similar fashion it can be shown that $f = g_1 f'$ where g_1 is an orthogonal linear transformation of R_2. Therefore, f(S) would have content if and only if $f'(S)$ has content and $c(f(S)) = c(f'(S))$. Thus, the area of S is independent of the particular orthonormal basis chosen in Definition 6.46 for the subspace [B,C]. This is listed as Theorem 6.47.

Theorem 6.47. The area of a set of points S in a plane [B,C] is independent of the orthogonal basis used for [B,C] in Definition 6.46.

If S is a set of points in a plane D + [B,C], then the set of points $S - D = \{P | P = Q - D, Q \in S\}$ is a subset of [B,C]. If D + [B,C] = E + [B,C], then $(D - E) \in [B,C]$. Thus, the set S - E = $\{Q - E | Q \in S\} = \{(Q - D) + (D - E) | Q \in S\} = (S - D) + (D - E)$. That is, the set S - E is the set S - D translated by the vector D - E. If f is the fixed coordinate system for [B,C] and $f(D - E) = (y_1, y_2)$ then $f(S - D) = \{(x_1, x_2) | (x_1 B + x_2 C) \in (S - D)\}$ and f(S - E) = $f[S - D) + (E - D)] = \{(z_1, z_2) | (z_1, z_2) = (x_1, x_2) + (y_1, y_2),$ $(x_1, x_2) \in f(S - D)\}$. Since, by Theorem 6.44, content is invariant under a translation, f(S - D) has content if and only if f(S - E) has content and $c(f(S - D)) = c(f(S - E))$. This shows that if the set S - D has area, the area of a set S in D + [B,C] may be defined as the area of the set S - D and the area does not depend upon the D used to name the plane D + [B,C].

Definition 6.48. Let S be a set in D + [B,C]. If the set S - D has content then, by definition, A(S) = A(S - D).

Technically, this gives two uses of the letter A for the coset [B,C]. But since $[B,C] = \theta + [B,C]$ and $S - \theta = S$, the uses are the same for subsets of [B,C].

Let $f: V \to V$ be an isometry. Then, by Theorem 5.25, f = gh where h is a translation and g an orthogonal linear transformation. Under h a coset $D + [B,C]$ is mapped into the coset $(D + h(\theta)) + [B,C] = h(D) + [B,C]$. For any set S in $D + [B,C]$ having area, $A(S) = A(S - D)$. Now, $h(S) = S + h(\theta)$ is in plane $h(D) + [B,C]$. Thus, $A(S + h(\theta)) = A(S + h(\theta)) - h(D)) = A(S + h(\theta) - (D + h(\theta))) = A(S - D)$. Thus $A(S) = A(h(S))$. Therefore, h preserves area.

Now, g, being an orthogonal linear transformation, maps $[B,C]$ into $[g(B), g(C)]$ where $\{g(B), g(C)\}$ is an orthonormal basis for $[g(B), g(C)]$. Let S be a subset of $[B,C]$ having area. Let f_1 be the coordinate system for $[B,C]$ such that $f_1(x_1B + x_2C) = (x_1,x_2)$ and f_2 be the coordinate system for $[g(B), g(C)]$ such that $f_2(y_1g(B) + y_2g(C)) = (y_1, y_2)$. If $S = \{P|P = x_1B + x_2C\}$, then $f_1(S) = \{(x_1,x_2)|(x_1B + x_2C) \in S\}$. Now, $g(S) = \{Q|Q = g(P), P \in S\} = \{Q|Q = x_1g(B) + x_2g(C), x_1B + x_2C = P \in S\}$. Therefore, $f_2(g(S)) = \{(x_1, x_2)|(x_1g(B) + x_2g(C)) \in g(S)\} = \{(x_1, x_2)|(x_1B + x_2C) \in S\}$. Therefore, $f_1(S) = f_2g(S)$. Thus, $A(S) = c(f_1(S)) = c(f_2g(S)) = A(g(S))$.

Now, if $S \subset D + [B,C]$ has area, then $g(S) \subset g(D) + [g(B), g(C)]$. Therefore, $A(S) = A(S - D)$ by definition of the area of S and $A(g(S)) = A(g(S) - g(D)) = A(g(S - D)$. Since $S - D$ is a subset of $[B,C]$ and $g(S) - g(D) = g(S - D)$ is a subset of $[g(B), g(C)]$, by the arguments in the last paragraph, $A(S) = A(g(S))$. Since the composition of area preserving maps is area preserving, the following theorem has been proved.

<u>Theorem 6.49</u>. If $f:V \to V$ is an isometry and S is a subset of the plane $D + [B,C]$ having area, then f(S) is a subset of the plane $f(D + [B,C])$ having area and $A(S) = A(f(S))$.

Theorem 6.50. Let $T = \{(a,b) + t(d,e) | 0 \le t \le 1\}$ be a segment in R_2. Then $c(T) = 0$.

Proof: The translation $h: R_2 \to R_2$, defined by $h(x_1, x_2) = (x_1, x_2) - (a,b)$, is a content preserving map and $h(T) = \{t(d,e) | \ 0 \le t \le 1\}$. The map $g: R_2 \to R_2$, defined by $g\langle x_1, x_2 \rangle =$

$$\left(\frac{d}{\sqrt{d^2 + e^2}} x_1 + \frac{e}{\sqrt{d^2 + e^2}} x_2, \frac{-e}{\sqrt{d^2 + e^2}} x_1 + \frac{d}{\sqrt{d^2 + e^2}} x_2 \right)$$ is a linear

transformation and $g(1,0) = \left(\frac{d}{\sqrt{d^2 + e^2}}, \frac{-e}{\sqrt{d^2 + e^2}} \right)$ with $g(0,1) =$

$\left(\frac{e}{\sqrt{d^2 + e^2}}, \frac{d}{\sqrt{d^2 + e^2}} \right)$. Now, $|g(1,0)| = 1$, $|g(0,1)| = 1$ and

$g(1,0) \circ g(0,1) = 0$. Thus, g is an orthogonal linear transformation. By Theorem 6.45, g is content preserving. Now $g(h(T)) =$

$\left\{ t\left(\frac{d^2 + e^2}{\sqrt{d^2 + e^2}}, 0 \right) 0 \le t \le 1 \right\} = \left\{ t | (d,e) | \ (1,0) | 0 \le t \le 1 \right\}$. Since this is a degenerate interval, $c(g(h(T))) = 0$. Since $c(T) = c(g(h(T)))$, $c(T) = 0$ as was to be proved.

Theorem 6.51. Every point and every segment in V has area zero.

Proof: Let $\overline{PQ} = \{P + t(Q - P) | 0 \le t \le 1\}$ be a segment in plane $P + [B,C]$. Then, $A(\overline{PQ}) = A(\overline{PQ} - P) = A(\{t(Q - P) | 0 \le t \le 1\} = c(f\{t(Q - P) | 0 \le t \le 1\})$. Let $f(Q - P) = (d,e)$. Then, $A(\overline{PQ}) = c\{t(d,e) | 0 \le t \le 1\}$. By Theorem 6.50, $A(\overline{PQ}) = 0$. Since the coordinate of any point is in a point in R_2, the area of a point is zero.

Theorem 6.52. Every triangular region has area.

Proof: Let ΔEFG be a triangle in a plane $D + [B,C]$ where $\{B,C\}$ is an orthonormal basis for $[B,C]$. By the remarks preceding Theorem 5.12, the triangular region EFG, denoted $\overline{EFG}$, is $\overline{EFG}$ = $\{aE + bF + cG | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\}$. Now, since $a + b + c = 1$, $\overline{EFG} - D = \{a(E - D) + b(F - D) + c(G - D) | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\}$. Let f be a coordinate system for [B,C] with respect to the orthonormal basis {B,C}. Suppose $f(E - D) = (x_1, x_2)$, $f(F - D) = (y_1, y_2)$ and $f(G - D) = (z_1, z_2)$. Then, $f(\overline{EFG} - D) = \{a(x_1, x_2) + b(y_1, y_2) + c(z_1, z_2) | a + b + c = 1, a \geq 0, b \geq 0, c \geq 0\}$. Thus, $f(\overline{EFG} - D)$ is a triangular region in R_2.

By Theorem 6.15, the boundary of the triangular region is the triangle or the union of three segments. By Theorem 6.50, the content of each segment is zero. Hence, the content of the union of the three segments is zero. Therefore, since the boundary of $f(\overline{EFG} - D)$ has content zero, by Corollary 6.37, $f(\overline{EFG} - D)$ has content. Thus, triangular region EFG has area.

Theorem 6.53. If S_1, S_2, ..., S_n are coplanar sets of points each having area, then $S_1 \cup S_2 \cup \ldots \cup S_n$ and $S_1 \cap S_2 \cap \ldots \cap S_n$ have area.

Proof: Since S_1, S_2, ..., S_n are coplanar, there exists a plane, D + [B,C] containing each of the sets. By definition, the area of each of these sets is the area of the sets $S_i - D \subset [B,C]$. Therefore, one may assume S_1, S_2, ..., S_n are in [B,C]. Let f be a cartesian coordinate system for [B,C]. Since S_1, S_2, ..., S_n have area, $f(S_1)$, $f(S_2)$, ..., $f(S_n)$ have content. Now, $f(S_1 \cup S_2 \cup \ldots \cup S_n) = f(S_1) \cup f(S_2) \cup \ldots \cup f(S_n)$. By Theorem 6.40, since $f(S_1)$, $f(S_2)$, ..., $f(S_n)$ have content, $f(S_1) \cup f(S_2) \cup \ldots \cup f(S_n)$ has content. Thus, $S_1 \cup S_2 \cup \ldots \cup S_n$ has area. The proof for $S_1 \cap S_2 \cap \ldots \cap S_n$ is similar.

Theorem XVII. To every polygonal region there corresponds a unique positive number.

Proof: A polygonal region is the union of a finite number of triangular regions. By Theorem 6.52, every triangular region has area. By Theorem 6.53, the union of a finite number of triangular regions has area. Since the set of coordinates of a triangular region contains a non-degenerate interval, the area of a polygonal region is positive. Thus, let the number required by the theorem be what has been called the area of a set of points in V.

SMSG at this point gives the following definition.

Definition 6.54. The area of a polygonal region is the number assigned to it by Postulate 17 (Theorem XVII). [17: 320]

Since the area function has been constructed in this chapter, the term area has been used previously. The two definitions of area, Definitions 6.48 and 6.54, are the same for polygonal regions.

Theorem XVIII. If two triangles are congruent, then the triangular regions have the same area.

Proof: By Theorem 5.33, there exists an isometry mapping one of the triangles into the other. By Theorem 5.34, the isometry maps the triangular region of one triangle onto the triangular region of the other triangle. By Theorem 6.49, isometries preserve area. Thus, the theorem is proved.

Theorem 6.55. Let S be the union of two coplanar point sets S_1 and S_2 each having area. If $A(S_1 \cap S_2) = 0$, then $A(S) = A(S_1) + A(S_2)$.

Proof: Without loss of generality, it is assumed that S_1 and S_2 are in some subspace [B,C]. Let f be a cartesian coordinate system for

$[B,C]$. By hypothesis, $c(f(S_1 \cap S_2)) = 0$. Since f is one-to-one, $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$ and $f(S) = f(S_1 \cup S_2) = f(S_1) \cup f(S_2)$. Thus, $c(f(S_1) \cap f(S_2)) = 0$. Therefore, by Theorem 6.40, $c(f(S_1) \cup f(S_2)) + c(f(S_1) \cap f(S_2)) = c(f(S_1)) + c(f(S_2))$ or $c(f(S)) = c(f(S_1)) + c(f(S_2))$. Thus, $A(S) = A(S_1) + A(S_2)$.

Theorem XIX. Suppose that the region R is the union of two regions R_1 and R_2. Suppose that R_1 and R_2 intersect at most in a finite number of segments and points. Then, the area of R is the sum of the areas of R_1 and R_2.

Proof: By Theorem 6.51, the area of a segment or a point is zero. Therefore, the area of a finite number of segments and points is zero. Thus, by Theorem 6.55, $A(R) = A(R_1) + A(R_2)$.

Theorem XX. The area of a rectangle is the product of the length of its base and the length of its altitude.

Proof: Let EFGH be a rectangle in plane $E + [B,C]$. Denote the rectangular region by $\overline{EFGH}$. By Theorem 5.12 and the remarks preceding it, $\left\{\frac{F-E}{|F-E|}, \frac{G-F}{|G-F|}\right\}$ is an orthonormal basis for $[B,C]$. Thus, $E + [B,C] = E + \left[\frac{F-E}{|F-E|}, \frac{G-F}{|G-F|}\right]$. Also, rectangular region $\overline{EFGH} = \left\{E + t\,\frac{F-E}{|F-E|} + s\,\frac{G-F}{|G-F|} \,\middle|\, 0 \le t \le |F-E|,\ 0 \le s \le |G-F|\right\}$. Thus, $\overline{EFGH} - E = \left\{t\,\frac{F-E}{|F-E|} + s\,\frac{G-F}{|G-F|} \,\middle|\, 0 \le t \le |F-E|,\ 0 \le s \le |G-F|\right\}$. Under the coordinate system f with respect to the basis $\left\{\frac{F-E}{|F-E|}, \frac{G-F}{|G-F|}\right\}$, $f(\overline{EFGH} - E) = \{(t,s) \mid 0 \le t \le |F - E|,\ 0 \le s \le |G-F|\}$. Since this is an interval, by Theorem 6.41, $c(f(\overline{EFGH} - E)) = |F-E|\ |G-F|$ which is the product of the length of the base of rectangle EFGH and its altitude. Thus, $A(\overline{EFGH}) = |F - E|\ |G - F|$ as was required.

Volume in V

SMSG Postulates 21 and 22 concern volume. The following quote is from the text.

> A vigorous treatment of volumes requires a careful definition of something analogous to polygonal regions in a plane (polyhedral regions is the name) and the introduction of postulates similar to the four area postulates. We will not give such a treatment, ... However, we will state explicitly the two numerical postulates we need. [17: 546]

An approach similar to this will be taken here. The definition of a rectangular parallelepiped will not be the one given by SMSG, but an equivalent definition. To prove the definition equivalent would be similar to the development used to prove Theorem 5.12 for rectangular regions.

The theorems about content in R_2 have there obvious generalizations to R_3. In R_3 an interval I would be set, $I = \langle (a_1, a_2, a_3), (b_1, b_2, b_3) \rangle$ $\{(x_1, x_2, x_3) | a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2 \text{ and } a_3 \leq x_3 \leq b_3\}$. Therefore, the proofs of most of the theorems about content in R_3 would only involve adding a third coordinate to the interval. For a general proof that isometries preserve content in R_n, see [21: 29]. All of the theorems about content in R_2 will be used for R_3.

Let $\{B_1, B_2, B_3\}$ be an orthonormal basis for V and $B = x_1 B_1 + x_2 B_2 + x_3 B_2$ be an element of V. A cartesian coordinate system for V is an isomorphism $f: V \to R_3$ with $f(B) = (x_1, x_2, x_3)$.

Definition 6.56. Let f be a cartesian coordinate system for V. If S is a subset of V, then S has volume if and only if f(S) has content and the volume of S is defined to be $c(f(S))$. The volume of S is denoted $v(S)$.

As in Theorem 6.47, the volume of a set S is independent of the orthonormal basis chosen for the coordinate system f. That is, if f' is another coordinate system, then $f' = gf$ and $f = g_1 f'$ where g and g_1 are orthogonal linear transformations on R_3. By the three dimensional analogue of Theorem 6.45, g and g_1 preserve content. Thus, $f(S)$ has content if and only if $f'(S)$ has content and $c(f(S)) = c(f'(S))$.

<u>Definition 6.57</u>. A rectangular parallelepiped is a set of points $S = \{A + r B_1 + s B_2 + t B_3 | 0 \leq r \leq a,\ 0 \leq s \leq b,\ 0 \leq t \leq c$ and $\{B_1, B_2, B_3\}$ is an orthonormal basis for $V\}$. Any of the six rectangular regions, $\{A + rB_1 + sB_2 | 0 \leq r \leq a,\ 0 \leq s \leq b\}$, $\{A + cB_3 + rB_1 + sB_2 | 0 \leq r \leq a,\ 0 \leq s \leq b\}$, ..., $\{A + sB_2 + tB_3 | 0 \leq s \leq b,\ 0 \leq t \leq b\}$ is called a base of S. The altitude corresponding to a base is, respectively, the length of one of the six segments $\{A + tB_3 | 0 \leq t \leq c\}$, $\{A + tB_3 | 0 \leq t \leq c\}$, ..., $\{A + rB_1 | 0 \leq r \leq a\}$

<u>Theorem XXI</u>. The volume of a rectangular parallelepiped is the product of the altitude and the area of the base.

Proof: The rectangular parallelepiped is a set S of the form $S = \{A + rB_1 + sB_2 + tB_3 | 0 \leq r \leq a,\ 0 \leq s \leq b,\ 0 \leq t \leq d$ and $\{B_1, B_2, B_3\}$ is an orthonormal basis for $V\}$. Let f be the coordinate system for V with respect to the basis $\{B_1, B_2, B_3\}$. The coordinate of A is some (m, n, p). Thus, $f(S) = \{(m, n, p) + (r, s, t) | 0 \leq r \leq a,\ 0 \leq s \leq b,\ 0 \leq t \leq d\}$. Since $h: R_3 \to R_3$ defined by $h(x_1, x_2, x_3) = (x_1, x_2, x_3) - (m, n, p)$ is content preserving, $c(f(S)) = c(f(S) - (m, n, p)) = c\{(r, s, t) | 0 \leq r \leq a,\ 0 \leq s \leq b,\ 0 \leq t \leq d\}$. But this is an interval and, by the three dimensional analogue of Theorem 6.41, $c(f(S)) = abd$.

Now the base of S is the rectangle $\{A + rB_1 + sB_2 | 0 \leq r \leq a, 0 \leq s \leq b\}$ and the altitude is the length of the segment $\{A + tB_3 | 0 \leq t \leq d\}$. By Theorem XX, the area of the rectangle is ab and the length of the segment is d. Thus, the theorem is proved.

To prove the next theorem, some theory of Riemann integration is used. The material on Riemann integration which follows is taken from [1]. In this text, the author uses E_n for what has been called R_n in this dissertation.

<u>Definition 6.58</u>. Let f be defined and bounded on a closed interval $I \subset E_n$. If P is a partition of I into m subintervals $I_1, \ldots, I_m$, let $m_k(f) = \inf\{f(x) | x \in I_k\}$, $M_k(f) = \sup\{f(x) | x \in I_k\}$. The numbers $U(P,f) = \sum_{k=1}^{m} M_k(f)\mu(I_k)$ and $L(P,f) = \sum_{k=1}^{m} m_k(f)\mu(I_k)$ are called upper and lower Riemann sums. The upper and lower Riemann integrals of f over I are defined as follows:

$$\overline{\int}_I f dX = \inf\{U(P,f) | P \text{ is a partition of } I\}$$

$$\underline{\int}_I f dX = \sup\{L(P,f) | P \text{ is a partition of } I\}. \quad [1: 254].$$

The function f has a Riemann integral on I if and only if $\underline{\int}_I fdX = \overline{\int}_I fdX$, and if $\underline{\int}_I fdX = \overline{\int}_I fdX$; the Riemann integral of f over I, denoted $\int_I fdX$, is $\underline{\int}_I fdX = \overline{\int}_I fdX$. [1: 255].

For S a subset of E_n (in this chapter E_2 and E_3 are being considered), define the function $\chi_S : E_n \to R$ by $\chi_S(X) = \begin{cases} 1 \text{ if } X \in S \\ 0 \text{ if } X \notin S \end{cases}$. χ_S is called the characteristic function of S.

<u>Theorem 6.59</u>. Let S be a bounded subset of E_2 or E_3 and I be an

interval containing S. S has Jordan content if and only if $\int_I \chi_S dX$ exists. Also, if S has content, then $\int_I \chi_S dX = c(S)$.

Proof: Let P be a partition of I and $I_1, \ldots, I_m$ the subintervals of I determined by P. Now, $U(P,\chi_S) = \sum_{k=1}^{m} M_k(\chi_S)\mu(I_k)$. If I_k contains points of S, then $M_k(\chi_S) = 1$. If I_k does not contain points of S, then $M_k(\chi_S) = 0$. Therefore, $U(P,\chi_S) = \sum \mu(I_k)$ where the sum is taken over all subintervals of I which contain points of S. Thus, $U(P,\chi_S) = \bar{J}(P,S)$ for each partition P of I. Therefore, $\bar{\int}_I \chi_S dX = \bar{c}(S)$.

Also, for any partition P of I, $\underline{J}(P,S)$ is the sum of the measures of those subintervals of I which contain only interior points of S. If I_k contains only interior points of S, then $m_k(\chi_S) = 1$. Thus, $\underline{J}(P,S) \leq L(P,\chi_S)$ for each partition P of I. Therefore, $\underline{c}(S) \leq \underline{\int}_I \chi_S dX$. How can $\underline{J}(P,S)$ be less than $L(P,\chi_S)$? If I_k contains only points of S, but not just interior points of S, then $m_k(f)$ is still one. Thus, $\mu(I_k)$ is included as an addend in $L(P,\chi_S)$ and not included as an addend in $\underline{J}(P,S)$. But, let $t > 0$ be given, then there exists a partition P of I such that $L(P,f) > \underline{\int}_I \chi_S dX - t$. $L(P,f) = \sum_{k=1}^{m} m_k(\chi_S)\mu(I_k)$. When $m_k(\chi_S) \neq 0$, then I_k contains only points of S. By Theorem 6.6, the open interval I_k (i.e., the interval excluding its boundary) is contained in the interior of S. Each of the intervals I_k can be approximated as closely as is needed to get closed intervals $I_k' \subset I_k$ and a partition P' of I which has the interval I_k' as subintervals with $J(P',S) > L(P,f) - t$. Thus, $\underline{c}(S) > L(P,f) - t > \underline{\int}_I \chi_S dX - 2t$. Since t is arbitrary, $\underline{c}(S) \geq \underline{\int}_I \chi_S dX$. Thus, $\underline{c}(S) = \underline{\int}_I \chi_S dX$.

Putting $\underline{c}(S) = \underline{\int}_I \chi_S dX$ together with $\bar{c}(S) = \bar{\int}_I \chi_S dX$, it follows that S has content if and only if $\int_I \chi_S dX$ exists and they are equal.

Let S be a subset of E_3. The coordinate hyperplane π_3 of E_3 is the subspace $\{(x_1,x_2,x_3)|x_3 = 0\}$. π_3 is isomorphic to E_2 where the isomorphism is the map mapping $(x_1,x_2,0)$ into (x_1,x_2). For all practical purposes, these two spaces, π_3 and E_2, are the same. If $I = \{(x_1,x_2,x_3)|\ a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2,\ a_3 \leq x_3 \leq b_3\}$ is an interval in E_3 then the projection of I onto the coordinate hyperplane π_3 is $I_3 = \{(x_1,x_2,0)|a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2\}$. The following theorem has been taken from [1: 264]. Modifications have been made in the notation to change from E_n to E_3. The interested reader is directed to the reference for a proof.

Theorem 6.60. Let f be defined on a closed interval $I = \{(x_1,x_2,x_3)|a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2,\ a_3 \leq x_3 \leq b_3\}$ in E_3. Assume that $\int_I f(x)dX$ exists. Then $\int_I f(x)dX = \int_{a_3}^{b_3}\left[\int_{I_3} f\ d(x_1,x_2)\right]dx_3$.

Before stating Theorem XXII, parallel planes in V will be discussed. Parallel planes were defined in Definition 5.4 as planes which do not intersect.

Planes were defined in Chapter III as cosets of two dimensional subspaces. Since two different cosets of the same subspace do not intersect, they are parallel. Suppose A + [B,C] and D + [E,F] are cosets of two different subspaces [B,C] and [E,F]. Then, at least one of E or F is not an element of [B,C]. Suppose E is not an element of [B,C]. Then, {E,B,C} is a linearly independent subset of V. Since V is three dimensional, {E,B,C} is a basis for V. Since D - A is in V, there exist real numbers e, b, and c such that D - A = eE + bB + cC. Thus, D - eE = A + bB + cC. Since D - eE is in D + [E,F] and A + bB + cC is in A + [B,C] these two planes intersect. Thus, the following theorem has been proved.

<u>Theorem 6.61</u>. Two planes are parallel if and only if they are different cosets of the same two dimensional subspace of V.

Let $A + [B_1, B_2]$ be a fixed plane in V with $\{B_1, B_2\}$ an orthonormal basis for $[B_1, B_2]$. $\{B_1, B_2\}$ can be completed to an orthonormal basis $\{B_1, B_2, B_3\}$ for V. Let f be the coordinate system with respect to this basis for V. Then, $A = a B_1 + b B_2 + c B_3$ for some a, b, and c. Thus, $A - c B_3 = a B_1 + b B_2$ or $A - c B_3 \in [B_1, B_2]$. Therefore, $A + [B_1, B_2] = c B_3 + [B_1, B_2]$. Now, $f(c B_3 + [B_1, B_2])$ is the set of coordinates of points in $cB_3 + [B_1, B_2]$. But, $cB_3 + [B_1, B_2] = \{cB_3 + x_1 B_1 + x_2 B_3 | x_1, x_2 \text{ are in } R\}$. Therefore, $f(cB_3 + [B_1, B_2]) = \{(x_1, x_2, c) | x_1, x_2 \text{ are in } R\}$. The following theorem has been proved.

<u>Theorem 6.62</u>. Let $A + [B_1, B_2]$ be a fixed plane in V and $\{B_1, B_2, B_3\}$ be an orthonormal basis for V. Then $A + [B_1, B_2] = cB_3 + [B_1, B_2]$ for some c in R. If f is the coordinate system for V with respect to the basis $\{B_1, B_2, B_3\}$, then the set of coordinates of the plane $A + [B_1, B_2]$ is the set $\{(x_1, x_2, c) | x_1, x_2 \text{ are in } R\}$.

Now, let D + W be a plane parallel to $A + [B_1, B_2]$. Then, by Theorem 6.61, D + W is a coset of $[B_1, B_2]$ or $D + W = D + [B_1, B_2]$. By Theorem 6.62, $D + [B_1, B_2] = dB_3 + [B_1, B_2]$ for some d in R. Since $dB_3 + [B_1, B_2]$ does not intersect $cB_3 + [B_1, B_2]$, $d \neq c$. The set of coordinates with respect to the basis $\{B_1, B_2, B_3\}$ for $dB_3 + [B_1, B_2]$ is the set $f(dB_3 + [B_1, B_2]) = \{(x_1, x_2, d) | x_1, x_2 \text{ are in } R\}$.

<u>Theorem 6.63</u>. Let $A + [B_1, B_2] = cB_3 + [B_1, B_2]$ be a fixed plane in V where $\{B_1, B_2, B_3\}$ be an orthonormal basis for V and f the coordinate system with respect to this basis. Then, the plane $D + [B_1, B_2]$ is parallel to the plane $cB_3 + [B_1, B_2]$ if and only if the coordinate set

for $D + [B_1, B_2]$ is of the form $\{(x_1, x_2, d) | x_1, x_2 \text{ in } R \text{ and } d \neq c\}$.

Proof: By the remarks preceding the theorem, if $D + W$ is parallel to $cB_3 + [B_1, B_2]$, then its coordinate set has the desired form. Conversely, if the coordinate set of $D + W$ is $\{(x_1, x_2, d) | x_1, x_2 \text{ are in } R \text{ and } d \neq c\}$, then for $P \in D + W$, $P = dB_3 + x_1B_1 + x_2B_2$. That is, P is in the coset $dB_3 + [B_1, B_2]$. If $P \in d\,B_3 + [B_1, B_2]$, then P has coordinate of the form (x_1, x_2, d). Thus, $P \in E + W$. Therefore, $D + W = d\,B_3 + [B_1, B_2]$. Since $c \neq d$, $dB_3 + [B_1, B_2] \neq cB_3 + [B_1, B_2]$ or the two cosets $D + W$ and $cB_3 + [B_1, B_2]$ are different cosets of the same subspace. Therefore, they are parallel. This completes the proof.

Theorem XXII (SMSG Postulate 22) is stated next. Some comments about the theorem follow the statement and precede the proof.

Theorem XXII. (Cavaliere's Principle.) Given two solids and a plane, If, for every plane which intersects the solids and is parallel to the given plane, the two intersections have equal areas, then the two solids have the same volume.

SMSG does not define solids. Therefore, in the proof, it will be assumed that a solid is a bounded set. Also, without some further restriction on the term solid content does not have this property. For example, let $S_1 = \{(x_1, x_2, x_3) | 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1,\ 0 \leq x_3 \leq 1 \text{ and } x_3 \text{ is rational}\}$ and $S_2 = \{(x_1, x_2, x_3) | 1 \leq x_1 \leq 2,\ 0 \leq x_2 \leq 1,\ 0 \leq x_3 \leq 1 \text{ and } x_3 \text{ is rational}\}$. Each plane parallel to the x_1x_2 plane intersects each of these "solids" in sets which have area zero or one. But each of the sets have outer Jordan content one and inner Jordan content zero. Hence, the sets have no volume as defined in this chapter. The added assumption that a solid has volume will be made in the proof.

Proof of Theorem XXII: Let the solids be S_1 and S_2 and the fixed plane be $A + [B_1, B_2]$ where $\{B_1, B_2, B_3\}$ is an orthonormal basis for V. Let f be the coordinate system with respect to the basis $\{B_1, B_2, B_3\}$. Now, $f(S_1)$ and $f(S_2)$ are contained in some interval. Let I = $\{a_1 \leq x_1 \leq b_1,\ a_2 \leq x_2 \leq b_2,\ a_3 \leq x_3 \leq b_3\}$ be an interval containing both of the sets. By Theorem 6.62, the coordinate set of every plane parallel to $A + [B_1, B_2]$ is of the form $\{(x_1, x_2, d) | d \text{ is fixed}\}$.

Let $S_1' = f(S_1)$ and $S_2' = f(S_2)$. By Theorem 6.59, $v(S_1) = \int_I \chi_{S_1'}\, dX$ and $v(S_2) = \int_I \chi_{S_2'}\, dX$. By Theorem 6.60,

$$v(S_1) = \int_I \chi_{S_1'}\, dX = \int_{a_3}^{b_3} \left[\overline{\int}_{I_3} \chi_{S_1'}\, d(x_1, x_2)\right] dx_3 \text{ and}$$

$$v(S_2) = \int_I \chi_{S_2'}\, dX = \int_{a_3}^{b_3} \left[\overline{\int}_{I_3} \chi_{S_2'}\, d(x_1, x_2)\right] dx_3.$$

Now $\overline{\int}_{I_3} \chi_{S_1'}\, d(x_1, x_2)$ is a function g_1 defined on the interval $[a_3, b_3]$ with range the reals. For each fixed d such that $a_3 \leq d \leq b_3$, $\chi_{S_1'}(x_1, x_2, d) = \begin{cases} 1 \text{ if } (x_1, x_2, d) \in S_1' \\ 0 \text{ if } (x_1, x_2, d) \notin S_1' \end{cases}$. If $dB_3 + [B_1, B_2]$ is a plane parallel to $A + [B_1, B_2]$, then the area of $(dB_3 + [B_1, B_2]) \cap S_1$ is the content of the coordinate set of the set $\{(dB_3 + [B_1, B_2]) \cap S_1) - dB_3\}$. This coordinate set is $E_1 = \{(x_1, x_2, d) - (0, 0, d) = (x_1, x_2, 0) | (x_1, x_2, d) \in f((dB_3 + [B_1, B_2]) \cap S_1)\}$. By Theorem 6.58, the area of the set $\{((dB_3 + [B_1, B_2]) \cap S_1) - dB_3\}$ is $\int_{I_3} \chi_{E_1}\, d(x_1, x_2)$. But for each fixed d, $\chi_{E_1} = \chi_{S_1'}$. Since the area of each plane intersected with S exists, $\int_{I_3} \chi_{E_1} d(x_1, x_2)$ exists. Since $\chi_{E_1} = \chi_{S_1'}$, $g_1(d) = \int_{I_3} \chi_{E_1}\, d(x_1, x_2) = \overline{\int}_{I_3} \chi_{S_1'}\, d(x_1, x_2)$ is the area of the plane $dB_3 + [B_1, B_2]$ intersected with S. Similarly, for each fixed d, $a_3 \leq d \leq b_3$, $g_2(d) = \overline{\int}_{I_3} \chi_{S_2'}\, d(x_1, x_2)$ is the area of the plane

$dB_3 + [B_1, B_2]$ intersected with S_2. Since it is given in the hypothesis that $g_1(x_3) = g_2(x_3)$ for each x_3 such that $a_3 \leq x_3 \leq b_3$ and since $v(S_1) = \int_{a_3}^{b_3} g_1 \, dx_3$ and $v(S_2) = \int_{a_3}^{b_3} g_2 dx_3$, it follows that $v(s_1) = v(S_2)$.

CHAPTER VII

SUMMARY AND EDUCATIONAL IMPLICATION

Summary

In this paper, Euclidean vector spaces were discussed. The writer has assumed an abstract three dimensional Euclidean vector space. In the vector space, the terms "point," "line," and "plane" were defined. With these definitions, the writer has shown that the twenty-two postulates of SMSG geometry are satisfied.

In Chapter I the statement of the problems, procedure, and scope of the paper were presented. In this chapter recent advances in the use of algebra in the high school curriculum were documented. Chapter II included an outline of the linear algebra background the reader of the paper would probably need. Particular emphasis was put on cosets of subspaces of a vector space.

In Chapter III lines and planes were defined as cosets of one and two dimensional subspaces. These cosets were shown to satisfy the incidence, coordinate system, and separation properties of postulates one through ten of SMSG geometry.

In Chapter IV the inner product, which the vector space was assumed to have, and the cosine function were used to define a measure on the set of angles. This measure was shown to satisfy Postulates eleven through fourteen of SMSG geometry. In the development of these theorems, a second vector formulation for a plane and the interior of an angle were presented.

In Chapter V the concept of congruence of triangles was investigated. Using SMSG's definition of congruent triangles and the law of cosines, the S.A.S. congruence postulate for triangles was proved. Because a broader definition of congruence was used by the writer to prove some area theorems in Chapter VI, a second formulation of congruence was presented in Chapter V. Congruence was defined in terms of isometries of the vector space. For triangles, the two definitions of congruence were shown to be equivalent.

In Chapter VI area and volume were defined using Jordan content in R_2 and R_3 and cartesian coordinate systems for V. It was shown that each of the point sets that has area in SMSG has area using this formulation of area. Cavaliere's principle connecting area and volume was proved for certain "solids."

Educational Implications

Algebraic methods are becoming more evident in the high school curriculum each year. A prospective high school teacher of mathematics will probably need more training in algebra than received by teachers who attended college a few years ago. This paper has presented a development of Euclidean vector spaces at a level within the domain of experience of a prospective high school teacher. Most of the theorems in the paper could have been proved in an n-dimensional Euclidean vector space. The writer chose a three dimensional vector space in light of the intended audience.

The reader, who is a potential high school teacher of mathematics, may find material in this paper that would help him in teaching Euclidean vector spaces at the high school level. This paper might also be of use in the training of teachers of geometry. CUPM, in its outline for the training of teachers, suggests the study of pure analytic geometry in its course on geometry. They further suggest the following be included in this course in geometry:

> Points, lines and so on may be defined and treated in terms of an algebraic model without the use of any synthetic postulates ... This is quite different from conventional analytic geometry wherein the synthetic postulates are used in proving that coordinate systems exist. The "purely analytic" treatment can be used to give a consistency proof for the synthetic postulates. [6: 22].

This paper has presented one such approach to Euclidean geometry.

Several colleges have seminars for undergraduate students in mathematics. The material in this paper could possibly be used in such a seminar for prospective high school teachers of mathematics.

Undoubtedly, the most immediate benefit of this paper is the experience gained by the writer in its preparation.

SELECTED BIBLIOGRAPHY

1. Apostol, Tom M., Mathematical Analysis, Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1957.

2. Artin, E., Geometric Algebra, New York: Interscience Publishers, 1957.

3. Athen, Herman, "The Teaching of Vectors in German Gymnasium," The Mathematics Teacher, Vol. 59 (1966), pp. 382-393.

4. Beaumont, Ross A., Linear Algebra, New York: Harcourt, Brace and World, Inc., 1965.

5. Bell, E. T., The Development of Mathematics, New York: McGraw-Hill Book Company, Inc., 1940.

6. Committee on the Undergraduate Program in Mathematics, Course Guides for the Training of Teachers of Junior High and High School Mathematics. Buffalo: Mathematical Association of America, June, 1961.

7. Eves, Howard, An Introduction to the History of Mathematics, New York: Holt, Rinehart and Winston, 1964.

8. Goals for School Mathematics, The Report of the Cambridge Conference on School Mathematics, Boston: Houghton Mifflin Company, 1963.

9. Hausner, Melvin, A Vector Approach to Geometry, Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1965.

10. Kestelman, H., Modern Theories of Integration, New York: Dover Publications, Inc., 1960.

11. Levi, Howard, Foundations of Geometry and Trigonometry, Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1960.

12. Meserve, Bruce E., "The Evolution of Geometry," The Mathematics Teacher, Vol. 49 (1956), pp. 372-382.

13. Moise, Edwin E., Elementary Geometry From an Advanced Standpoint, Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1963.

14. Mostow, George D., Sampson, Joseph H., and Meyer, Jean-Pierre, Fundamental Structures of Algebra, New York: McGraw-Hill Book Company, Inc., 1963.

15. Report of the Commission on Mathematics, Program for College Preparatory Mathematics, New York: College Entrance Examination Board, 1959.

16. Rogosinski, Werner W., Volume and Integral, New York: Interscience Publishers, Inc., 1962.

17. School Mathematics Study Group, Geometry, Unit 13, 14, New Haven: Yale University Press, 1960.

18. Schreier, O., and Sperner, E., Introduction to Modern Algebra and Matrix Theory, New York: Chelsea Publishing Company, English Edition, 1959.

19. Smith, James Lester, "Foundations in Geometry for High School Teachers," (unpub. Ed.D. dissertation, Oklahoma State University, 1960).

20. Szabo, Steven, "An Approach to Euclidean Geometry Through Vectors," The Mathematics Teacher, Vol. 59 (1966), pp. 218-235.

21. Von Neumann, John, Functional Operators, Volume I: Measures and Integrals, Princeton: Princeton University Press, 1950.

APPENDIX A

SMSG GEOMETRY POSTULATES

This Appendix is a listing of the postulates used by SMSG in the high school geometry text [17]. In the list, the numeral in parenthesis following the SMSG postulate number indicates the page on which the postulate is stated in the SMSG text. The numeral in parenthesis at the end of the postulate corresponds to the number of the page on which the proof of this postulate originates in this dissertation.

1. (P. 30) Given any two different points, there is exactly one line which contains both of them. (p. 23)

2. (P. 34) To every pair of different points there corresponds a unique positive number. (p. 23)

3. (P. 36) The points of a line can be placed in correspondence with the real numbers in such a way that

   (1) To every point of the line there corresponds exactly one real number,

   (2) To every real number there corresponds exactly one point of the line, and

   (3) The distance between two points is the absolute value of the difference of the corresponding numbers. (p. 24)

4. (P. 40) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive. (p. 25)

5. (P. 54) (a) Every plane contains at least three non-collinear points.

(b) Space contains at least four non-coplanar points. (p. 27)

6. (P. 56) If two points lie in a plane, then the line containing these points lies in the same place. (p. 28)

7. (P. 57) Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane. More briefly, any three points are coplanar, and any three non-collinear points determine a plane. (p. 28)

8. (P. 58) If two different planes intersect, then their intersection is a line. (p. 29)

9. (P. 64) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that

(1) each of the sets is convex and

(2) if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the line. (p. 35)

10. (P. 66) The points of space that do not lie in a given plane form two sets such that

(1) each of the sets is convex and

(2) if P is in one set and Q is in the other then the segment $\overline{PQ}$ intersects the plane. (p. 33)

11. (P. 80) To every angle $\angle BAC$ there corresponds a real number between 0 and 180. (p. 40)

12. (P. 81) Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $m\angle PAB = r$. (p. 43)

13. (P. 81) If D is a point in the interior of $\angle BAC$, then

m∢BAC = m∢BAD + m∢DAC. (p. 47)

14. (P. 82) If two angles form a linear pair, then they are supplementary. (p. 49)

15. (P. 115) Given a correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence. (p. 52)

16. (P. 252) Through a given external point there is at most one line parallel to a given line. (p. 53)

17. (P. 320) To every polygonal region there corresponds a unique positive number. (p. 98

18. (P. 320) If two triangles are congruent, then the triangular regions have the same area. (p. 98)

19. (P. 320) Suppose that the region R is the union of two regions R_1 and R_2. Suppose that R_1 and R_2 intersect at most in a finite number of segments and points. Then the area of R is the sum of the areas of R_1 and R_2. (p. 99)

20. (P. 322) The area of a rectangle is the product of the length of its base and the length of its altitude. (p. 99)

21. (P. 546) The volume of a rectangular parallelepiped is the product of the altitude and the area of the base. (p. 101)

22. (P. 548) Given two solids and a plane. If for every plane which intersects the solids and is parallel to the given plane the two intersections have equal areas, then the two solids have the same volume. (p. 106)

VITA

Alonzo Franklin Johnson

Candidate for the Degree of

Doctor of Education

Thesis: SMSG GEOMETRY AS A REAL VECTOR SPACE

Major Field: Higher Education - Mathematics

Biographical:

Personal Data: Born in Ashland, Kentucky, April 5, 1936, the son of Arnold E. and Myrtle Johnson.

Education: Attended public schools in Ashland, Kentucky; graduated from Ashland Senior High School in 1954; received the Bachelor of Science degree from Morehead State College, Morehead, Kentucky, in August, 1958, with a major in Mathematics; received the Master of Arts in Education degree from Morehead State College in August, 1961, with a major in Secondary Teaching; received the Master of Science degree from the University of Notre Dame, Notre Dame, Indiana, in May, 1965, with a major in Mathematics; attended Marshall College, Huntington, West Virginia during the summer of 1960 and Florida State University, Tallahassee, Florida during the summer of 1963; completed requirements for the Doctor of Education degree at Oklahoma State University in July, 1967.

Professional Experience: Taught high school mathematics at Boyd County, Kentucky the years 1957-1958 and 1959-1960; taught high school mathematics at Elliott County, Kentucky in 1958-1959; taught high school mathematics at Broward County, Florida from 1960 to 1964; attended a National Science Foundation, Academic Year Institute at Oklahoma State University in 1965-1966 teaching one class each semester; was a graduate assistant in Mathematics at Oklahoma State University, 1966-1967.