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GENERALIZATIONS OF CONNEXIONS ON

MANIFOLDS AND SUBMANIFOLDS

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CHAPTER I

INTRODUCTION

The principal object of study in the following work is a connexion on a differentiable manifold. This concept is a very old one, having its origin in the work of Levi-Civita on parallel displacement, and its roots in the geometry of Euclid. Loosely, a connexion is a structure on a manifold which permits one to compare tangent spaces to the manifold at different points. It has been employed in classical differential geometry in a two-fold manner: to define a parallelism of vector fields on a manifold, and to introduce a differentiation of tensor fields on a manifold with respect to which the derivative of a tensor field is another field of the same sort.

Suppose that M denote a differentiable manifold (of class C^∞), and that $\mathfrak{X}(M)$ be the totality of C^∞ vector fields on M. A connexion, in the broadest sense, is a mapping $D\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. One generally writes $D_X Y$ for the image under D of a pair (X,Y) of vector fields. Then a field $Y \in \mathfrak{X}(M)$ may be styled parallel with respect to a field X provided $D_X Y = 0$, where 0 is the zero field.

In order to obtain a tensor-differentiation, it is necessary to place restrictions on D. One class of such restricted connexions is that of affine connexions, the study of which has dominated connexion theory since the 1920's. A very pleasant discussion of these connexions, and the theory of connexions in fibre bundles, together with an historical overview and citations of relevant papers, may be found in Kobayashi's work [8].

In the present work, three connexion-related concepts are studies: Λ-congruences, (and their use in inducing connexions on submanifolds) in Chapter II, and non-linear and direction-dependent connexions in Chapter III.

The Λ-congruences are believed by the author to be a new structure, though the idea is based on the classical concept of a rigging of a submanifold, and is a simple generalization of the well-known normal distribution on a submanifold of a Riemannian space. The characterization of riggings and Λ-congruences as cross sections of certain fibre bundles is new, and the derivation of this characterization constitutes a large portion of Chapter II.

Non-linear connexions of a rather restricted sort have been studied recently by A. Kandatu [7] and by K. Yano and S. Ishihara [23]. Chapter III begins with an investigation of a more general non-linear connexion. The study of these leads quickly into questions about direction-dependent tensor fields on a manifold, and to a theory of connexions related to such objects. The connexion defined in this context is identical with an entity studied by Makoto Matsumoto [9, 10, 11, 12]. It is studied in quite different a fashion from his approach, however, and is given an invariant, coordinate-free characterization in Chapter III. Further, one observes that the rôle played by the tangent bundle in linear connexion theory is taken by the square of the tangent bundle in direction-dependent theory; the results on the square of the tangent bundle are all new.

Throughout, standard techniques of differential geometry are used. Notation is frequently heavy, and will be explained as it is encountered, but in general, the notation of Springer [19] will be employed for local analysis, and that of Hicks [5] for coordinate-free analysis. A few notational peculiarities should be noted. First, in a bundle (P,p,B), if a function $g:B \rightarrow M$ (M a manifold) is under discussion, the function $g \circ p$ will also be denoted by g. And in many equations of Chapter III, functions are denoted by their values at a (generic) point. For example, the expression $[(\partial x^i/\partial y^j) \circ \pi]\, \Gamma_s^p$ will be denoted by $\Gamma_s^p(m,X)\, \partial x^i/\partial y^j$. This is not logically proper, but it is convenient, and is occasionally done to emphasize domains. In a similar manner, pointwise expressions will sometimes fail to have the point of evaluation expressed, where this is obvious. Purists should have no especial difficulty in rewriting such statements in a more precise fashion, and it is to be hoped that less careful readers will not be lead into confusion by these devices.

The word "differentiable" will always mean C^∞. Occasionally, the useful phrase, "over a coordinate patch U on M", will be used prior to a piece of local analysis, in place of "over a coordinate patch (U,φ) on M with $x^i = u^i \circ \varphi$, u^i being the canonical coordinate maps of R^n". This will be done only where no confusion will result. The same phrase will be used if the analysis is to be done in $\pi^{-1}(U)$, where $\pi:P \to M$ is the projection map of a bundle with total space P over M.

Displayed equations or expressions are numbered serially in each section, as are lemmas and theorems. If Theorem 2 of Section 5, Chapter II, is referred to in the section in which it first appears, it is called "Theorem 2". If it is referred to in another section of the same chapter, it is cited as "Theorem 5.2", and if mentioned in a different chapter, as "Theorem II.5.2". The same conventions apply to displayed equations.

CHAPTER II

A-CONGRUENCES AND INDUCED CONNEXIONS

1. Introduction

Early in the development of differential geometry, it was recognized that inducing a connexion on a submanifold of a differentiable manifold with connexion requires an interplay between the tangent spaces of the submanifold and those of the ambient space. The standard technique of providing this interplay was to equip the submanifold with a global rigging (German: Einspannung) [15; p. 234 and p. 158].

Let M be a C^∞ m-dimensional manifold and N a C^∞ n-submanifold of M. A rigging of N in M is a set of $(m-n)$ C^∞, non-vanishing vector fields $X_1, \ldots, X_{m-n}$ of M defined over N with the property that, if $Y_1, \ldots, Y_n$ are vectors at $p \in N$ spanning the tangent space N_p, then $\{X_1, \ldots, X_{m-n}, Y_1, \ldots, Y_n\}$ form a basis of the tangent space to M at p.

Given a rigging of N in M, a connexion on M induces one on N by decomposition. Unfortunately, many manifolds do not support a global rigging; the Mobius band, imbedded in Euclidean three-space in the usual way, provides an example. But it is always possible to induce a Riemannian connexion on a submanifold N of a Riemannian space M by using, at each point p in N, the subspace of the tangent space to M at p consisting of vectors normal to N_p with respect to the Riemannian metric. For one may simply split the covariant derivative into its normal and tangential components [5; p. 75]. The Λ-congruences, defined in section 2, are an obvious extension of this normal splitting.

In section 3, existence and uniqueness of Λ-congruences are discussed, and section 4, the main part of this chapter, is devoted to a characterization of both riggings and Λ-congruences as cross-sections of certain fibre bundles.

Sections 5 and 6 are given to a consideration of the connexion induced by a Λ-congruence, and concepts related to it.

2. Definition of a Λ-congruence.

Let M be an m-dimensional C^∞ manifold, and N an n-dimensional C^∞ submanifold of M. Let Λ be a function which assigns to each point $p \in N$ a subspace Λ_p of M_p (the tangent space ot M at p) such that $M_p = \Lambda_p \oplus N_p$ (direct sum). Then Λ will be called a Λ-congruence. The Λ-congruence Λ is said to be C^∞ provided it has the C^∞ splitting property for vector fields, as follows:

Suppose X is a C^∞ field of vectors in M defined on a neighborhood U of N; that is, each point of U has a neighborhood $V \subseteq U$ such that X extends to a

C^∞ field on a neighborhood $\bar{V}$ in M with $\bar{V} \cap N \supseteq V$. For $p \in U$, the definition of Λ shows that

$$(1) \qquad X_p = A_p + B_p,$$

where $A_p \in \Lambda_p$ and $B_p \in N_p$. Then Λ has the C^∞ splitting property provided the vector fields A and B defined by (1) are C^∞ on their domains, for every such field X.

3. Existence and Uniqueness of Λ-congruences

The following theorem is trivial:

Theorem 1: If N is a C^∞ submanifold of a C^∞, paracompact, Hausdorff manifold M, then N supports a C^∞ Λ-congruence.

Proof: Since M is paracompact and Hausdorff, it supports a Riemannian metric tensor $\langle , \rangle$, and the normal distribution to N is defined. The normal distribution has the C^∞ splitting property [5; pp. 75-76], and is therefore a C^∞ Λ-congruence. Q.E.D.

A more interesting question is: to what extent does Theorem 1 characterize Λ-congruences? In other words, is every Λ-congruence on N the normal distribution over N defined by some Riemannian metric on M? In general (not to keep the reader in suspense) the answer is "No". But, for a large class of submanifolds, one may give an affirmative reply.

Call a C^∞ n-submanifold N of a C^∞ m-manifold M <u>widely imbedded</u> provided that, for each point $p \in N$, there exist special coordinate neighborhoods $\overline{U}$ and U about p, on M and N respectively, such that $\overline{U} \cap N = U$. Recall that a special coordinate pair $\overline{U}, U$ consists of a coordinate patch $(\overline{U}, \overline{\varphi})$ on M with $\overline{x}^i = u^i \circ \overline{\varphi}$, where u^i are the canonical coordinate maps on R^m, such that, if

$$U = \{q \in \overline{U} \mid \overline{x}^j (q) = 0, \ j = n + 1, \ldots, m\},$$

then U is a coordinate neighborhood on N with coordinate maps $\overline{x}^1|_U, \ldots, \overline{x}^n|_U$. The following easy lemma shows that the property of being widely imbedded characterizes submanifolds with the subspace topology:

<u>Lemma 1</u>: Let N be a C^∞ n-submanifold of the C^∞ m-manifold M. Then N is widely imbedded in M iff N has the subspace topology inherited from M.

<u>Proof</u>: The necessity of the condition is trivial. To show sufficiency, let $p \in N$, and let $(\overline{U}, \overline{\varphi})$ be a coordinate patch on M such that (U, φ) is a coordinate patch on N, where

$$U = \{q \in \overline{U} \mid u^j \circ \overline{\varphi}(q) = 0, \ j = n + 1, \ldots, m\}$$

and $\varphi : U \to R^n :: \varphi(q) = (u^1 \circ \overline{\varphi}(q), \ldots, u^n \circ \overline{\varphi}(q))$. These exist since N is an n-submanifold of M.

Now U is a neighborhood of p in N, and since N has the subspace topology, there exists an open set $\bar{V}_o$ in M such that $\bar{V}_o \cap N \subseteq U$. Let $\bar{V} = \bar{V}_o \cap \bar{U}$, and consider $(\bar{V}, \bar{\psi})$, where $\bar{\psi} = \bar{\varphi}|_{\bar{V}}$. Then since domain$(\bar{\psi})$ is a subset of domain$(\bar{\varphi})$, and $\bar{\psi}$ and $\bar{\varphi}$ are C^∞-related, $(\bar{V}, \bar{\psi})$ is a coordinate patch on M. For the same reasons, (V, ψ), where

$$V = \{q \in \bar{V} \mid u^j \circ \bar{\psi}(q) = 0,\ j = n+1, \ldots, m\}$$

and where $\psi = \varphi|_V$, is a coordinate patch on N. Note that $V = \bar{V} \cap N$. For that $V \subseteq \bar{V} \cap N$ is trivial, while, if $q \in \bar{V} \cap N$, one has from the definition of $\bar{V}$,

$$q \in \bar{V} \cap N = \bar{V}_o \cap \bar{U} \cap N \subseteq \bar{U}.$$

Thus $V \supseteq \bar{V} \cap N$, and $V = \bar{V} \cap N$ by double inclusion. Because p was an arbitrary point of N, N is widely imbedded in M. Q.E.D.

One can now prove:

__Theorem 2__: Let N be a C^∞ n-submanifold of the C^∞ m-manifold M, such that N is a closed topological subspace of M, and M is paracompact and Hausdorff. Then every C^∞ Λ-congruence to N is the normal distribution to N of some Riemannian metric on M.

__Proof__: Let $\bar{U}$ and U be special coordinate neighborhoods on M and N respectively, with $\bar{U} \cap N = U$. These exist about any point in N by Lemma 1. Let $\bar{\varphi}: \bar{U} \to R^m$ be the coordinate map of $\bar{U}$, write $\bar{x}^i = u^i \circ \bar{\varphi}$ for $i = 1, \ldots, m$, and let $x^\alpha = \bar{x}^\alpha|_U$ for $\alpha = 1, \ldots, n$. In what follows, lower case Roman letters will take on values from 1 to m; lower case Greek letters from α through λ will take on values from 1 to n; and lower case Greek letters from μ onward will take on values from n+1 to m.

Let $\frac{\partial}{\partial \bar{x}^i}$, $i = 1, \ldots, m$, be the coordinate vectors on $\bar{U}$, and for $p \in U$, let Λ_p be spanned by vectors

$$(1) \qquad L_\nu = g_\nu^\alpha(\bar{x}^1(p), \ldots, \bar{x}^n(p))\frac{\partial}{\partial \bar{x}^\alpha} + c_\nu^\mu(\bar{x}^1(p), \ldots, \bar{x}^n(p))\frac{\partial}{\partial \bar{x}^\mu},$$

so that g_ν^α and c_ν^μ are C^∞ R-valued maps on U. Introduce a mapping $\theta : \bar{\varphi}(\bar{U}) \to R^m$ by

$$(2) \qquad \theta(y^1,\ldots,y^m) = (y^1 + y^\mu g_\mu^1(y^1,\ldots,y^n), \ldots, y^n + y^\mu g_\mu^n(y^1,\ldots,y^n), y^\mu c_\mu^{n+1}(y^1,\ldots,y^n),\ldots,y^\mu c_\mu^m(y^1,\ldots,y^n)).$$

Write $z^j = u^j \circ \theta$, and notice that the determinant of the matrix $\left[\frac{\partial z^i}{\partial y^j}\right]$ is given at $(y^{n+1} = \ldots = y^m = 0)$ by

$$\det_{(y^{n+1} = \cdots = y^m = 0)} \begin{bmatrix} 1+y^\mu \dfrac{\partial g^1_\mu}{\partial y^1} & y^\mu \dfrac{\partial g^2_\mu}{\partial y^1} & \cdots & y^\mu \dfrac{\partial g^n_\mu}{\partial y^1} & y^\mu \dfrac{\partial c^{n+1}_\mu}{\partial y^1} & \cdots & y^\mu \dfrac{\partial c^m_\mu}{\partial y^1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ y^\mu \dfrac{\partial g^1_\mu}{\partial y^n} & y^\mu \dfrac{\partial g^2_\mu}{\partial y^n} & \cdots & 1+y^\mu \dfrac{\partial g^n_\mu}{\partial y^n} & y^\mu \dfrac{\partial c^{n+1}_\mu}{\partial y^n} & \cdots & y^\mu \dfrac{\partial c^m_\mu}{\partial y^n} \\ g^1_{n+1} & g^2_{n+1} & \cdots & g^n_{n+1} & c^{n+1}_n & \cdots & c^m_n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ g^1_m & g^2_m & \cdots & g^n_m & c^{n+1}_m & \cdots & c^m_m \end{bmatrix}$$

or

(3)
$$\det_{(y^{n+1} = \cdots = y^m = 0)} \left[\frac{\partial z^i}{\partial y^j}\right]$$

$$= \det \begin{bmatrix} I & [0] \\ [g^\alpha_\mu(y^1, \ldots, y^n)] & [c^\mu_\nu(y^1, \ldots, y^n)] \end{bmatrix}$$

$$= \det \left[c^\mu_\nu(y^2, \ldots, y^n)\right] \neq 0,$$

the final statement following since L_ν and $\partial/\partial x^\alpha$ form a linearly independent set of vectors at every point of U. It follows from (3) that in a neighborhood Q of the slab $\{(y^1, \ldots, y^m) \in \bar{\varphi}(\bar{U}) \mid y^{n+1} = \ldots = y^m = 0\}$, θ is invertible. Since θ is C^∞, θ^{-1} will be so also. One may assume $\theta(Q) \subseteq \bar{\varphi}(\bar{U})$, as this can always be arranged.

Now define $\bar{\Psi}: \bar{\varphi}^{-1}[\theta(Q)] \to R^m$ by $\bar{\Psi} = \theta^{-1} \circ \bar{\varphi}$, and write $\bar{y}^i = u^i \circ \bar{\Psi}$. One has

(4) $$\bar{y}^\alpha|_U = u^\alpha \circ \theta^{-1} \circ \bar{\varphi}|_U = x^\alpha, \quad a = 1, \ldots, n,$$

and

(5) $$\bar{y}^\mu|_U = u^\mu \circ \theta^{-1} \circ \bar{\varphi}|_U = 0, \quad \mu = n+1, \ldots, m.$$

Equations (4) and (5) show that the maps $\bar{y}^i$ form a special system of coordinates for M and N. Write $\bar{V} = \bar{\Psi}^{-1}(Q)$, so $(\bar{V}, \bar{\Psi})$ is a coordinate patch on M.

Since $\bar{\Psi}$ is a diffeomorphism of its domain onto its range, $\bar{\Psi}_*$ is non-singular. Suppose $p \in \bar{V} \cap N$; then one has

$$\left.\frac{\partial}{\partial \bar{y}^j}\right|_p = (\bar{\Psi}_*)^{-1} \left.\frac{\partial}{\partial u^j}\right|_{\bar{\Psi}(p)}$$

(6) $$= \frac{\partial}{\partial u^j}(u^i \circ \bar{\varphi} \circ \bar{\Psi}^{-1})_{\bar{\Psi}(p)} \left.\frac{\partial}{\partial \bar{x}^i}\right|_p$$

$$= \frac{\partial}{\partial u^j}(u^i \circ \theta)_{\bar{\Psi}(p)} \left.\frac{\partial}{\partial \bar{x}^i}\right|_p .$$

But

$$u^i \circ \theta = \begin{cases} u^i + u^\mu g^i_\mu(u^1,\ldots,u^n) & \text{if } i=1,\ldots,n \\ u^\mu c^i_\mu(u^1,\ldots,u^n) & \text{if } i=n+1,\ldots,m. \end{cases}$$

Therefore, in particular,

(8) $$\frac{\partial}{\partial u^\mu}(u^i \circ \theta) = \begin{cases} g^i_\mu(n^1,\ldots,u^n) & \text{if } i=1,\ldots,n, \\ c^i_\mu(u^1,\ldots,u^n) & \text{if } i=n+1,\ldots,m, \end{cases}$$

so that one has for $p \in \overline{V} \cap N$,

(9) $$\left.\frac{\partial}{\partial \overline{y}^\mu}\right|_p = g^\alpha_\mu(\overline{x}^1(p),\ldots,\overline{x}^n(p))\frac{\partial}{\partial \overline{x}^\alpha} + c^\nu_\mu(\overline{x}^1(p),\ldots,\overline{x}^n(p))\frac{\partial}{\partial \overline{x}^\nu},$$

or

(10) $$\left.\frac{\partial}{\partial \overline{y}^\mu}\right|_p = L_\mu(p).$$

It was precisely to obtain (10) that the definition (2) of θ was given. The effect of θ is to normalize the Λ-congruence over the patch $\overline{V}$.

Introduce a local metric tensor $\langle , \rangle_{\overline{V}}$, defined on $\overline{V}$ by requiring that $\overline{\Psi}$ be an isometry. Note that, from (10),

(11) $$\langle L_\mu, \frac{\partial}{\partial x^\alpha} \rangle_{\overline{V}} = 0,$$

so that Λ is normal to N in $\overline{V}$, re the metric $\langle , \rangle_{\overline{V}}$.

Cover N by coordinate neighborhoods $\{(\overline{V}_\gamma, \overline{\Psi}_\gamma)\}_{\gamma \in G}$, where G is an index set, and each pair $(\overline{V}_\gamma, \overline{\psi}_\gamma)$ is constructed as $(\overline{V}, \overline{\psi})$ above. Each $\overline{V}_\gamma$ may be taken to be open; hence, the union $\bigcup_{\gamma \in G} \overline{V}_\gamma$ is open. Now, N is closed, and M, being a Hausdorff, paracompact manifold, is normal, so that there exists an open neighborhood $\overline{S}$ of N in M such that

(12) $$\mathrm{Cl}(\overline{S}) \subseteq \bigcup_{\gamma \in G} \overline{V}_\gamma ,$$

where Cl denotes the topological closure. Then $M \sim \mathrm{Cl}(\overline{S})$ is an open subset of M and can be expressed as the union of coordinate patches $\{(\overline{U}_\delta, \overline{\varphi}_\delta)\}_{\delta \in G'}$, where G' is an index set disjoint from G. The totality of patches $\overline{V}_\gamma$ and $\overline{U}_\delta$ form an open cover of M which may be assumed locally finite, by the paracompactness of M. Let $\langle , \rangle_\gamma, \gamma \in G$, be local metric tensors defined on $\overline{V}_\gamma$ for each $\gamma \in G$, as $\langle , \rangle_{\overline{V}}$ was defined for $\overline{V}$. And let $\langle , \rangle_\delta$, for $\delta \in G'$, be a metric tensor defined on $\overline{U}_\delta$ by requiring that $\overline{\varphi}_\delta$ be an isometry, for each $\delta \in G'$.

Next, let $\{g_\gamma : \overline{V}_\gamma \to I\}_{\gamma \in G} \cup \{g_\delta : \overline{U}_\delta \to I\}_{\delta \in G'}$, be a C^∞ partition of unity subordinate to the cover $\{\overline{V}_\gamma\}_{\gamma \in G} \cup \{\overline{U}_\delta\}_{\delta \in G'}$ of M; here, I is the unit interval, $[0,1]$. Define

(13) $$\langle , \rangle = \sum_{\gamma \in G} g_\gamma \langle , \rangle_\gamma + \sum_{\delta \in G'} g_\delta \langle , \rangle_\delta .$$

Then $<,>$ is a Riemannian metric tensor for M. If $p \in N$, $X_p \in \Lambda_p$ and $Y_p \in N_p$, then for $\gamma \in G$ such that $p \in \overline{V}_\gamma$, one has $<X_p, Y_p>_\gamma = 0$ by the construction of $<,>_\gamma$, as in (11). And for $\gamma \in G$ such that $p \notin \overline{V}_\gamma$, $g_\gamma(p) = 0$. Since $p \in N$, $p \notin \overline{U}_\delta$ for every $\delta \in G'$, and $g_\delta(p) = 0$; therefore

$$<X_p, Y_p> = 0 , \tag{14}$$

and Λ is the normal distribution to N induced by the Riemannian metric $<,>$. Q.E.D.

Two examples will be presented to show that the hypotheses of Theorem 2 are critical. First, consider the case in which $M = R^2$ with the usual structure, and N is the open unit interval on the u^1-axis. Note that N has the subspace topology, but fails to be closed. For a Λ-congruence, take at the point $(u^1,0)$, $0 < u^1 < 1$, the subspace spanned by the vector

$$L_{(u^1,0)} = u^1 \frac{\partial}{\partial u^2} + (1-u^1) \frac{\partial}{\partial u^1}. \tag{15}$$

Suppose X is any C^∞ vector field on N; then X may be written

$$X_{(u^1,0)} = h(u^1) \frac{\partial}{\partial u^1} + k(u^1) \frac{\partial}{\partial u^2}, \tag{16}$$

and (15) and (16) yield

$$X_{(u^1,0)} = \frac{k(u^1)}{u^1} L_{(u^1,0)} + \left[h(u^1) - \frac{k(u^1)(1-u^1)}{u^1}\right] \frac{\partial}{\partial u^1}. \tag{17}$$

From (17), we see that the Λ-congruence so defined splits X into two C^∞ fields on N, since $(0,0) \notin N$. If $<,>$ be any Riemannian metric tensor on M such that Λ is normal to N under $<,>$, one must have

(18) $$0 = <L_{(u^1,0)}, \frac{\partial}{\partial u^1}> = (u^1-1) < \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}> .$$

Since $<,>$ is C^∞ on R^2, (18) must hold in the limit as u^1 tends to zero:

(19) $$\lim_{u^1 \to 0} <L_{(u^1,0)}, \frac{\partial}{\partial u^1}> = - <\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1}> = 0 .$$

But (19) contradicts the positive-definiteness of the Riemannian metric, and it is seen that Λ is not induced as the normal distribution to N of any Riemannian metric on M.

The next example is closed, but fails to have the subspace topology. Let M again be R^2 with the usual structure, and let N be the disjoint union of a denumerable set of real lines, imbedded in M as lines perpendicular to the u^1-axis, and crossing that axis at points $(1,0)$, $(\frac{1}{2},0)$, $(\frac{1}{3},0)$, etc., and at $(0,0)$. Define a Λ-congruence Λ by using as a basis at each point $p \in N$ the vector L_p given by

(20)
$$L_{(1/k,u^2)} = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^2},$$
$$L_{(0,u^2)} = \frac{\partial}{\partial u^1}.$$

That this is a C^∞ Λ-congruence on N is trivial.

If <,> be a Riemannian metric tensor for M with respect to which Λ is normal to N, one must have:

(21)
$$\langle L_{(1/k,0)}, \frac{\partial}{\partial u^2}\rangle = 0,$$

for every k. Then

(22)
$$\lim_{k\to\infty} \langle L_{(1/k,0)}, \frac{\partial}{\partial u^2}\rangle = 0,$$

while

(23)
$$\langle L_{(1/k,0)}, \frac{\partial}{\partial u^2}\rangle = \langle\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\rangle + \langle\frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2}\rangle.$$

From (20), $\langle\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\rangle_{(0,0)} = 0$, so that (22) and (23) together imply

(24)
$$\langle\frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2}\rangle_{(0,0)} = 0,$$

which again contradicts the positive-definiteness of the tensor <,>. Again, one concludes that Λ is not normal to N re <u>any</u> Riemannian metric on M.

It is to be noted that Theorem 2 applies to any compact submanifold of a paracompact Hausdorff manifold.

4. <u>Λ-congruences as Sections of a Fibre Bundle</u>

The objective of this section is to show how both riggings of, and Λ-congruences to, a submanifold of a differentiable manifold can be regarded as cross-sections of certain fibre bundles. The appropriate bundle for riggings is developed from the Stiefel bundle of k-frames over a manifold M, and that for Λ-congruences from the Grassmann bundle of k-planes over M. The Stiefel and Grassmann bundles and spaces will be described as they are encountered, primarily in order to fix the notation.

One begins with the Stiefel bundle; the exposition will follow, roughly, the treatment of Steenrod [20, p. 33].

A k-frame, v^k, in R^n is an ordered set of linearly independent vectors in R^n; one writes $v^k = (v_1, \ldots, v_k)$. Any fixed k-frame v_o^k can be transformed to any other by the action of some element of the full linear group on R^n, $Gl(n,R)$. Let $V_k'(R^n)$ be the set of all k-frames in R^n, and let $G_{n,k}$ be the subgroup of $Gl(n,R)$ leaving fixed each vector of a fixed k-frame v_o^k. Then $G_{n,k}$ is a closed subgroup (the <u>isotropy group</u> of v_o^k) of $Gl(n,R)$.

Suppose $v^k \in V_k'(R^n)$; associate with v^k an element $\sigma \in Gl(n,R)$ such that $\sigma(v_o^k) = v^k$. If

$\tau \in Gl(n,R)$ also maps v_o^k to v^k, then $\sigma^{-1}\tau$ leaves each vector of v_o^k fixed, and $\sigma^{-1}\tau \in G_{n,k}$. Also, if $\tau \in Gl(n,R)$ is such that $\sigma^{-1}\tau \in G_{n,k}$, then $\tau(v_o^k) = v^k$. The association

(1) $$v^k \to \sigma\, G_{n,k} \in Gl(n,R)/G_{n,k}$$

is a bijection, which will be denoted by φ; thus,

(2) $$\varphi\colon V_k'(R^n) \to Gl(n,R)/G_{n,k}.$$

Since the quotient space in (2) is a C^∞ manifold, $V_k'(R^n)$ inherits a C^∞ structure when it is required that φ be a diffeomorphism; then, $V_k'(R^n)$ becomes an nk-dimensional C^∞ manifold.

Let L denote the subspace of R^n spanned by the vectors $(\bar{e}_{k+1}, \ldots, \bar{e}_n)$, where $(\bar{e}_1, \ldots, \bar{e}_n)$ is the canonical basis of R^n. Henceforth, the first k vectors of this canonical basis will be taken as the standard reference frame v_o^k for $G_{n,k}$. Thus, a representative of a k-frame $(g_1^j\bar{e}_j, \ldots, g_k^j\bar{e}_j)$ is a matrix with the first k rows given by $g_\alpha^j (j = 1, \ldots, n; \alpha = 1, \ldots, k)$, and other entries arbitrary.

Notice that in the writing of matrices, the upper index has been taken as a row-counter, and the lower index as a column-counter. This convention will be adhered to unless it is specifically noted otherwise.

Now let $Q_k'(R^n)$ be the subspace (topological) of $V_k'(R^n)$ consisting of all k-frames which span subspaces complementary to L, that is, subspaces H such that R^n may be expressed as $L \oplus H$. One has the easy

<u>Lemma 1</u>: The set $Q_k(R^n)$ is open in $V_k'(R^n)$.

<u>Proof</u>: The natural projection $\eta: Gl(n,R) \to Gl(n,R)/G_{n,k}$ is an open map (see, e.g., [2; p. 37]). Say $\sigma \in Gl(n,R)$ represents an arbitrary, but fixed, point $v^k_{(0)} = (v_k^{(0)}, \ldots, v_k^{(0)}) \in Q_k(R^n)$. Then the subspace spanned by $v^k_{(0)}$ meets L trivially. Since L is itself closed in R^n, there is a neighborhood V of σ in $Gl(n,R)$ such that, if $\tau \in V$, then the subspace of R^n spanned by $\tau(v^k_{(0)})$ meets L trivially. Then $\varphi^{-1} \circ \eta(V)$ is a neighborhood of $v^k_{(0)}$ in $V_k'(R^n)$, each k-frame in which spans a space meeting L trivially, so $\varphi^{-1} \circ \eta(V) \subseteq Q_k(R^n)$. Hence $Q_k(R^n)$ is open in $V_k'(R^n)$ as claimed. Q.E.D.

<u>Corollary</u>: $Q_k(R^n)$ is an open submanifold of $V_k'(R^n)$.

This is immediate, since any open subset of a C^∞ manifold inherits a C^∞ structure. The manifold $Q_k(R^n)$ will be the <u>fibre</u> of the bundle whose cross-sections are riggings.

Because of the manner in which a C^∞ structure is defined for $V_k'(R^n)$, the group $Gl(n,R)$ acts to the left on $V_k'(R^n)$ in a C^∞ manner; to be precise, if $\sigma, \tau \in Gl(n,R)$, then $\sigma(\tau\, G_{n,k}) = \sigma\,\tau\, G_{n,k}$. This action is carried to $V_k'(R^n)$ via the diffeomorphism φ.

The bundle of bases over M, where M is a C^∞ $\underline{n}$-manifold, is a principal C^∞ fibre bundle over M with structure group $Gl(n,R)$; accordingly, one defines the Stiefel bundle of k-frames over M as the fibre bundle with fibre $V_k'(R^n)$ associated to the bundle of bases over M [1; pp. 45–49].

Recall that the bundle of bases over M has total space $B(M)$ given by

(3) $$B(M) = \{(m, e_1, \ldots, e_n) \mid m \in M, (e_1, \ldots, e_n) \text{ an ordered base of } M_m\}.$$

It also has projection $p: B(M) \to M :: p(m, e_1, \ldots, e_n) = m$, and right $Gl(n,R)$-action defined by

(4) $$(m, e_1, \ldots, e_n)\,\sigma = (m, \sigma_1^i e_i, \ldots, \sigma_n^i e_i),$$

where $\sigma = (\sigma_j^i) \in Gl(n,R)$. Over a coordinate patch U with coordinates x^i on M, a point $(m, e_1, \ldots, e_n) \in p^{-1}(U)$ has coordinates $(x^1(m), \ldots, x^n(m), \xi_1^1, \ldots, \xi_1^n, \xi_2^1, \ldots, \xi_2^n, \ldots, \xi_n^1, \ldots, \xi_n^n)$, where $e_i = \xi_i^j \frac{\partial}{\partial x^j}$.[1]

---

[1] Properly, one should say here that the point $b = (m, e_1, \ldots, e_n) \in p^{-1}(U)$ has coordinates $(\overline{x}^1(b), \ldots, \overline{x}^n(b), \overline{\xi}_1^1(b), \ldots, \overline{\xi}_n^n(b))$, where $\overline{x}^j : p^{-1}(U) \to R :: \overline{x}^j = x^j \circ p$, and $\overline{\xi}_j^i(b) = \xi_j^i$, where $e_i = \xi_i^j\, \partial/\partial x^j$. Thus the slot-functions of the coordinate map are maps on $p^{-1}(U)$, as is correct. However, for convenience, the functions $\overline{x}^i$ will be written x^i, the appropriate domain being clear from context, and similarly for $\overline{\xi}_j^i$.

<u>This convention will be employed whenever coordinates in a bundle are defined using lifts of coordinate domains in the base manifold.</u>

The total space of the Stiefel bundle, which will be denoted by $V_k'(M)$, is given by

(5) $$V_k'(M) = (B(M) \times V_k'(R^n))/Gl(n,R),$$

where $Gl(n,R)$ acts on $B(M) \times V_k'(R^n)$ by $\sigma(p,f) = (p\sigma, \sigma^{-1}f)$, for $\sigma \in Gl(n,R)$, $p \in B(M)$, and $f \in V_k'(R^n)$. Association of a point in space $V_k'(M)$ with a k-frame in the tangent space of M at the base of the fibre on which the point lies is accomplished by considering structure maps over coordinate patches on M. Let U be a coordinate patch on M as above, and let $\pi' : V_k'(M) \to M$ be the projection of the Stiefel bundle. Then the structure map $F_U' : (\pi')^{-1}(U) \to V_k'(R^n)$ is given by

$$F_U'[(m, h_1^i \frac{\partial}{\partial x^i}, \ldots, h_n^i \frac{\partial}{\partial x^i}), (g_1^j \bar{e}_j, \ldots, g_k^j \bar{e}_j))\, Gl(n,R)] =$$
$$(h_j^t\, g_1^j\, \bar{e}_t, \ldots, h_j^t\, g_k^j\, \bar{e}_t).$$

Over U, F_U' shows how the required association may be made; indeed, if one chooses as a representative of a point of $V_k'(M)$ the point

(7) $$b = ((m, h_1^i \frac{\partial}{\partial x^i}, \ldots h_n^i \frac{\partial}{\partial x^i}), (\bar{e}_1, \ldots, \bar{e}_k))$$

in $B(M) \times V_k'(R^n)$, then (6) shows that

$$F_U' (b\, Gl(n,R)) = (h_1^i\, \bar{e}_i, \ldots, h_k^i\, \bar{e}_i) .$$

Thus, one may associate the point $b\, Gl(n,R)$ with the k-frame $(h_1^i\, \partial/\partial x^i, \ldots, h_k^i\, \partial/\partial x^i)$ in M_m, and this correspondence is well-defined over U.

Now, in order to construct a fibre bundle with fibre

$Q_k(R^n)$, one must select an appropriate structure group. Let $G_{n,n-k}$ be the subgroup of $Gl(n,R)$ which leaves the subspace L spanned by the vectors $(\bar{e}_{k+1}, \ldots, \bar{e}_n)$ invariant. Then $G_{n,n-k}$ is a closed subgroup of $Gl(n,R)$, hence, a Lie group [2; pp. 123-125], and elements of $G_{n,n-k}$ send k-frames spanning complements to L in R^n to other "L-complementary" k-frames. Since $G_{n,n-k}$ acts on $Q_k(R^n)$ by restriction of the $Gl(n,R)$-action on $V_k'(R^n)$, the $G_{n,n-k}$-action is C^∞. It is to be noted that this action is <u>not</u> free.

The total space of the desired bundle for riggings can now be obtained, using the structure maps F_U'. Indeed, let N be a C^∞ (n-k)-dimensional submanifold of M. Let ξ denote the Stiefel bundle of k-frames over M, and $i: N \to M$ the inclusion map. Attention is now restricted to the bundle over N induced by i from ξ, which will be denoted by $i^*\xi$. This may be identified with the restriction of ξ to N [6; p. 19]. The total space $V_k^*(N)$ of $i^*\xi$ is given by

$$(9)\quad V_k^*(N) = \{((p, e_1, \ldots, e_n), (g_1^i\, \bar{e}_i, \ldots, g_k^i \bar{e}_i))\, Gl(n,R) \in V_k'(M) \mid p \in N\}.$$

Let $(\bar{U},\psi)$ be a special coordinate patch on M, with $x^i = u^i \circ \psi$, so that

$$(10)\qquad U = \{q \in \bar{U} \mid x^j(q) = 0 \quad \text{for } j = 1, \ldots, k\}$$

is a coordinate patch on N with coordinate maps

$x^{k+1}|_U, \ldots, x^n|_U$. Let N be covered with such special coordinate patches U. Let $\bar{\pi}$ be the projection of $i^*\xi$, and introduce structure maps $G'_U : \bar{\pi}^{-1}(U) \to V'_k(R^n)$ for $i^*\xi$ by $G'_U = F'_{\bar{U}} \circ i'$, where $i' : V^*_k(N) \to V'_k(M)$ is the inclusion map. Then,

(11)

$$G'_U[((p, h^i_1\,\partial/\partial x^i, \ldots, h^i_n\,\partial/\partial x^i), (g^i_1\,\bar{e}_i, \ldots, g^i_k\,\bar{e}_i))\,Gl(n,R)] = (h^i_j\, g^j_1\,\bar{e}_i, \ldots, h^i_j\, g^j_k\,\bar{e}_i).$$

Now, G'_U is C^∞, and hence continuous, on its domain. The manifold $Q_k(R^n)$ is open in $V'_k(R^n)$, so $G'^{-1}_U(Q_k(R^n))$ is open in $\bar{\pi}^{-1}(U)$, and therefore, since $\bar{\pi}^{-1}(U)$ is itself open, open in $V^*_k(N)$. Thus, if $Q_k(N)$ is defined by

$$Q_k(N) = \bigcup_U G'^{-1}_U(Q_k(R^n)), \tag{12}$$

where the union is taken over the special coordinate cover, one has that $Q_k(N)$ is an open submanifold of $V^*_k(N)$. A point of $Q_k(N)$ over the neighborhood U may be represented by

$$((p, \partial/\partial x^1, \ldots, \partial/\partial x^n), (g^i_1\,\bar{e}_i, \ldots, g^i_k\,\bar{e}_i))\,Gl(n,R). \tag{13}$$

Under G'_U, the point (13) goes to the point $(g^i_1\,\bar{e}_i, \ldots, g^i_k\,\bar{e}_i)$ in $Q_k(R^n)$, so one may identify (13) with the k-frame $(g^i_1\,\partial/\partial x^i, \ldots, g^i_k\,\partial/\partial x^i)$ in M_p. Since the map $\alpha : M_p \to R^n$ by $\alpha(\partial/\partial x^i) = \bar{e}_i$ is an isomorphism, and since N_p is spanned by $(\partial/\partial x^{k+1}, \ldots, \partial/\partial x^n)_p$ by the choice of the neighborhoods U, one has that the k-frame corresponding to (13) is indeed complementary to N_p.

The manifold $Q_k(N)$ is the required total space; write $\pi : Q_k(N) \to N$ for the obvious projection, and notice that $\pi = \bar{\pi}|_{Q_k(N)}$, so that π is C^∞.

A left $G_{n,n-k}$-action is defined on $Q_k(N)$ by first defining it over a special coördinate neighborhood $\bar{U}$, and then gluing these local actions together by means of transition functions [6; pp. 60-63]. Over a special patch U, if $\sigma \in G_{n,n-k}$, let

(14)

$$\sigma[((p, \partial/\partial x^1, \ldots, \partial/\partial x^n), (g_1^i \bar{e}_i, \ldots, g_k^i \bar{e}_i)) \, Gl(n,R)] = ((p, \partial/\partial x^1, \ldots, \partial/\partial x^n), (\sigma_j^i g_1^j \bar{e}_i, \ldots, \sigma_j^i g_k^j \bar{e}_i)) \, Gl(n,R),$$

where $\sigma = (\sigma_j^i)$. Since the $G_{n,n-k}$-action is C^∞ on $Q_k(R^n)$, this action is C^∞ on $\pi^{-1}(U)$ by local triviality, and by the fact that structure maps commute with actions.

Suppose now that $\bar{U}$ and $\bar{V}$ are two special coordinate neighborhoods on M with coordinates (x^i) and (y^i) respectively, and with U and V the associated patches on N. Let $p \in U \cap V$, and let $a \in \pi^{-1}(p)$ be any point of the fibre over p. Suppose a has the representations

(15)

$$a = ((p, \partial/\partial x^1, \ldots, \partial/\partial x^n), (g_1^i \bar{e}_i, \ldots, g_k^i \bar{e}_i)) \, Gl(n,R)$$

and

(16)

$$a = ((p, \partial/\partial y^1, \ldots, \partial/\partial y^n), (h_1^i \bar{e}_i, \ldots, h_k^i \bar{e}_i)) \, Gl(n,R)$$

over U and V respectively. Since $\partial/\partial y^j = (\partial x^i/\partial y^j)\, \partial/\partial x^i$, (16) yields

$$a = ((p, \frac{\partial x^i}{\partial y^1}\frac{\partial}{\partial x^i}, \ldots, \frac{\partial x^i}{\partial y^n}\frac{\partial}{\partial x^i}), (h_1^i\, \bar{e}_i, \ldots, h_k^i\, \bar{e}_i))\, Gl(n,R)$$

(17)

$$= ((p, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}), (\frac{\partial x^i}{\partial y^j} h_1^j\, \bar{e}_i, \ldots, \frac{\partial x^i}{\partial y^j} h_k^j\, \bar{e}_i))\, Gl(n,R).$$

From the uniqueness of the particular coordinate expression (13), (17) implies that

(18)
$$g_\alpha^i = \frac{\partial x^i}{\partial y^j} h_\alpha^j .$$

Accordingly, it appears natural to define the transition function $g_{UV} : U \cap V \to G_{n,n-k}$ by

(19)
$$g_{UV}(p) = \left[\frac{\partial x^i}{\partial y^j}\right]_p .$$

The matrix $[\partial x^i/\partial y^j]_p$ represents a member of $G_{n,n-k}$ by choice of the coordinate structure on U and V.

If $\bar{W}$ is yet another special patch on M with coordinates (z^i) and associated patch W on N, and if $p \in U \cap V \cap W$, then

$$\begin{aligned} g_{UV}(p) \circ g_{VW}(p) &= \left[\frac{\partial x^i}{\partial y^j}\right]_p \cdot \left[\frac{\partial y^r}{\partial z^s}\right]_p \\ &= \left[\frac{\partial x^i}{\partial y^j}\frac{\partial y^j}{\partial z^s}\right]_p \\ &= \left[\frac{\partial x^i}{\partial z^s}\right]_p , \end{aligned}$$

or

(20) $$g_{UV}(p) \circ g_{VW}(p) = g_{UW}(p),$$

and g_{UV} are acceptable transition functions. From (17), it is clear that these maps are compatible with the local $G_{n,n-k}$-actions defined in (14), so that one has

<u>Theorem 1</u>: The bundle with base N, total space $Q_k(N)$, fibre $Q_k(R^n)$, and projection π is a C^∞ fibre bundle η_N with structure group $G_{n,n-k}$. A C^∞ cross section of η_N is a rigging of N in M.

The last statement of the theorem is immediate from the construction of η_N. The bundle η_N may be styled the <u>Stiefel Tangent-Complement bundle</u>, or simply <u>STC-bundle</u>, over N.

The plan for obtaining a fibre bundle, cross sections of which will be $\bar{\Lambda}$-congruences to the submanifold N, consists of three major steps. First, the principal $G_{n,n-k}$-bundle associated to the STC-bundle is described. Second, a particular submanifold of the Grassmann manifold of k-planes in R^n is singled out, and a left $G_{n,n-k}$-action defined on it. Then, finally, the required bundle will be that whose fibre is this special Grassmann submanifold, and which is associated to the principal $G_{n,n-k}$-bundle over N.

The principal fibre bundle ξ_N associated to the

STC-bundle is constructed as follows (c.f. [6; pp. 62-63]): Let $G_{n,n-k}$ act on itself by right translation. Let $\{U_\gamma\}_{\gamma \in G}$, where G is an index set, be a special coordinate cover of the submanifold N as described by (10), and define

$$Z = \bigoplus_{\gamma \in G} U_\gamma \times G_{n,n-k}, \tag{21}$$

where $\oplus$ denotes the disjoint topological sum. A point $z \in Z$ may then be denoted by an indexed pair: $(m,\sigma)_\gamma$, where $\gamma \in G$, $m \in U_\gamma$, and $\sigma \in G_{n,n-k}$. An equivalence relation R is defined on Z by setting $(m,\sigma)_\gamma$ and $(p,\tau)_\delta$ R-related provided $m = p$, and $\tau = \sigma \cdot g_{U_\gamma U_\delta}(m)$. Then the total space E_N of ξ_N is given by

$$E_N = Z \bmod R. \tag{22}$$

The group $G_{n,n-k}$ acts on E_N to the right by

$$\overline{(m,\sigma)}_\gamma \, \tau = \overline{(m,\sigma\tau)}_\gamma , \tag{23}$$

where $\overline{(m,\sigma)}_\gamma$ is the R-equivalence class of $(m,\sigma)_\gamma \in Z$.

Let $q : Z \to E_N$ be the natural projection, and for each $\gamma \in G$, let $q_\gamma : U_\gamma \times G_{n,n-k} \to Z$ be the natural inclusion map. If one defines $h_\gamma : U_\gamma \times G_{n,n-k} \to E_N$ by $h_\gamma = q \circ q_\gamma$, then h_γ is a $G_{n,n-k}$-isomorphism of $U_\gamma \times G_{n,n-k}$ and $\xi_N|U_\gamma$, and requiring the local isomorphisms h_γ to be diffeomorphisms gives E_N the structure of a C^∞ manifold, and ξ_N becomes a C^∞ principal $G_{n,n-k}$-bundle over N.

Next, the Grassmann manifold $G_k(R^n)$ of k-planes in R^n will be examined. The approach used here is a mixture of the expositions of Steenrod [20; p. 35] and Husemoller [6; p. 13].

The Stiefel manifold of orthonormal k-frames in R^n, denoted by $V_k(R^n)$, is given by

(24) $V_k(R^n) = \{(v_1, \ldots, v_k) \in (S^{n-1})^k \mid \langle v_i, v_j \rangle = \delta_{ij}\}$,

where $\langle , \rangle$ is the usual inner product in R^n and S^{n-1} denotes the (n-1)-sphere. The space $V_k(R^n)$ has the relative topology as a subspace of $(S^{n-1})^k$. Let $\langle v_1, \ldots, v_k \rangle$ denote the subspace of R^n spanned by the k-frame $(v_1, \ldots, v_k) \in V_k(R^n)$. Then the set $G_k(R^n)$ is given the quotient topology from the map

(25) $\theta : V_k(R^n) \to G_k(R^n) :: \theta(v_1, \ldots, v_k) = \langle v_1, \ldots, v_k \rangle$.

Lemma 2: The mapping θ given in (25) is open.

Proof: Suppose U is an open set in $V_k(R^n)$, and consider $\theta^{-1}[\theta(U)]$. If $(v_1, \ldots, v_k) \in \theta^{-1}[\theta(U)]$, then $\langle v_1, \ldots, v_k \rangle \in \theta(U)$, and there exists a k-frame $(\bar{v}_1, \ldots, \bar{v}_k) \in U$ such that

(26) $$\langle v_1, \ldots, v_k \rangle = \langle \bar{v}_1, \ldots, \bar{v}_k \rangle .$$

Since U is open, there is a real number $\varepsilon > 0$ such that, if $O = (\bar{V}^1_\varepsilon \times \ldots \times \bar{V}^k_\varepsilon) \cap V_k(R^n)$, where $\bar{V}^\alpha_\varepsilon$ is an open ε-neighborhood of $\bar{v}_\alpha$ in S^{n-1} for $\alpha = 1, \ldots, k$, then $(\bar{v}_1, \ldots, \bar{v}_k) \in O \subseteq U$. It follows from (26) that $v_\beta = \sigma^\alpha_\beta \, \bar{v}_\alpha$, where the lower case Greek letters run over the range $(1, \ldots, k)$, and (σ^α_β) is an orthogonal matrix, hence, a length-preserving map of R^k. Define quantities $\bar\sigma^\alpha_\beta$ by

(27) $$\bar\sigma^\alpha_\beta \, \sigma^\beta_\gamma = \delta^\alpha_\gamma .$$

Next, take $\epsilon_1 > 0$ so that

(28) $$\epsilon_1 < \min \left[\epsilon / \left(\max_\alpha \sum_{\delta, \eta} | \bar\sigma^\delta_\alpha || \bar\sigma^\eta_\alpha | \right), \epsilon\right].$$

Then consider the neighborhood Q of $(v_1, \ldots, v_k)$ in $V_k(R^n)$ given by $Q = (V^1_{\epsilon_1} \times \ldots \times V^k_{\epsilon_1}) \cap V_k(R^n)$, where $V^\alpha_{\epsilon_1}$ is an open ϵ_1-neighborhood in S^{n-1} of v_α, for $\alpha = 1, \ldots, k$. And suppose that $(v_1 + a_1, \ldots, v_k + a_k) \in Q$, so that $|a_\beta|^2 < \epsilon_1$, for $\beta = 1, \ldots, k$. Then there exists an element $\tau \in Gl(n,R)$ such that $\tau(v_\beta) = v_\beta + a_\beta$. One has from (26) that

(29) $$<\tau(\bar v_1), \ldots, \tau(\bar v_k)> = <\tau(v_1), \ldots, \tau(v_k)> .$$

Write $\tau(\bar v_\beta) = \bar v_\beta + \bar b_\beta$; then, since τ is a linear map,

(30) $$\tau(\bar v_\beta) = \tau(\bar\sigma^\alpha_\beta \, v_\alpha) = \bar\sigma^\alpha_\beta \, \tau(v_\alpha) = \bar v_\beta + \bar\sigma^\alpha_\beta \, a_\alpha .$$

This shows that $\bar b_\beta = \bar\sigma^\alpha_\beta \, a_\alpha$, so one may compute:

$$|\bar b_\alpha|^2 \le \delta_{it} \, | \bar\sigma^\delta_\alpha || \bar\sigma^\eta_\alpha | \, a^{(i)}_\delta \, a^{(t)}_\eta \quad (\alpha \text{ not summed})$$

$$\le \sum_{\delta,\gamma} | \bar\sigma^\delta_\alpha || \bar\sigma^\gamma_\alpha | \, \{\max_\eta |a_\eta|^2\} \ (\alpha \text{ not summed})$$

or

$$|\bar b_\alpha|^2 < \epsilon ,$$

from (28). Here, $a_\delta = (a_\delta^{(1)}, \ldots, a_\delta^{(n)}) \in R^n$. It follows that the point $(\overline{v}_1 + \overline{b}_1, \ldots, \overline{v}_k + \overline{b}_k)$ is in U, so that, from (29), one has $(v_1 + a_1, \ldots, v_k + a_k) \in \theta^{-1}[\theta(U)]$. Thus, $Q \subseteq \theta^{-1}[\theta(U)]$, and therefore, $\theta^{-1}[\theta(U)]$ is open in $V_k(R^n)$, which means that $\theta(U)$ is open in $G_k(R^n)$, by the quotient topology, and θ is indeed an open map. Q.E.D.

Let $O_n \subseteq Gl(n,R)$ be the orthogonal subgroup, and note that O_n acts continuously and transitively on $G_k(R^n)$. Let $H_{n,k}$ be the isotropy group of the k-plane spanned by $(\overline{e}_1, \ldots, \overline{e}_k)$; then $H_{n,k}$ is a closed subgroup of O_n, and since O_n is compact, the natural map $G_k(R^n) \to O_n / H_{n,k}$ is a homeomorphism [20; p. 30]. The topological space $G_k(R^n)$ is made a C^∞ manifold by requiring this homeomorphism to be a diffeomorphism.

Next, let $H_k(R^n)$ denote the subset of $G_k(R^n)$ consisting of all k-planes in R^n complementary to $L = \langle\overline{e}_{k+1}, \ldots, \overline{e}_n\rangle$. The following lemma is readily established:

__Lemma 3__: The subspace $H_k(R^n)$ is open in $G_k(R^n)$.

__Proof__: The space $V_k(R^n)$ is homeomorphic to O_n/O_{n-k}, and the natural projection $O_n \to O_n/O_{n-k}$ is an open map [20; p. 34].[1] Say $(v_1, \ldots, v_k) \in V_k(R^n)$ spans

[1] The group O_n acts on $V_k(R^n)$, and O_{n-k} is the isotropy group of some fixed element of $V_k(R^n)$. The notation is used because this isotropy group is C^∞-isomorphic to the orthogonal subgroup of $Gl(n-k,R)$.

a complement to L, and suppose $\sigma \in O_n$ represents this element. Then, since $\langle v_1, \ldots, v_k \rangle \cap L = 0$ and L is closed, there is a neighborhood V of σ in O_n such that $\tau \in V$ implies that $\langle \tau(v_1), \ldots, \tau(v_k) \rangle \cap L = 0$. Then $[V]O_{n-k}$ is an open neighborhood of $(v_1, \ldots, v_k)$ in $V_k(R^n)$, each element of which spans a complement to L. Finally, Lemma 2 shows that $\theta\{[V]O_{n-k}\}$ is a neighborhood of $\langle v_1, \ldots, v_k \rangle$, each element of which is a k-plane complementary to L, so $\theta\{[V]O_{n-k}\} \subseteq H_k(R^n)$, and $H_k(R^n)$ is open as claimed. Q.E.D.

<u>Corollary</u>: $H_k(R^n)$ is an open submanifold of $G_k(R^n)$.

This is immediate.

The left $G_{n,n-k}$-action on $H_k(R^n)$ is defined by cutting down the $Gl(n,R)$-action on $G_k(R^n)$; if $\langle v_1, \ldots, v_k \rangle \in H_k(R^n)$, and $\sigma \in G_{n,n-k}$, then $\sigma\langle v_1, \ldots, v_k \rangle = \langle \sigma v_1, \ldots, \sigma v_k \rangle$. It is clear that this action is C^∞, since the $Gl(n,R)$-action and various inclusion maps are.

The C^∞ fibre bundle ζ_N with fibre $H_k(R^n)$ associated to the principal fibre bundle ξ_N may now be constructed. The total space $H_k(N)$ of this bundle is given by

$$H_k(N) = (E_N \times H_k(R^n)) / G_{n,n-k}, \tag{31}$$

so that a typical point of $H_k(N)$ is

$$(32) \qquad \overline{((m,(\sigma^i_j))_\gamma, \langle v_1, \ldots, v_k\rangle)\, G_{n,n-k}}\,.$$

Let U_γ be one of the patches of the special coordinate cover of N, with coordinates (x^i), so that for $p \in U_\gamma$, the tangent space N_p is spanned by $(\partial/\partial x^{k+1}, \ldots, \partial/\partial x^n)_p$. A complementary k-space in M_p is spanned by $(g^i_1\, \partial/\partial x^i, \ldots, g^i_k\, \partial/\partial x^i)_p$, with (g^i_j) a matrix in $G_{n,n-k}$. This k-plane may be identified with the point

$$(33) \qquad \overline{((m,(\delta^i_j))_\gamma, \langle g^i_1\, \bar{e}_i, \ldots, g^i_k\, \bar{e}_i\rangle)\, G_{n,n-k}}$$

in $H_k(N)$; the correspondence is clearly well-defined.

One may now state

<u>Theorem 2</u>: The bundle ζ_N with base N, total space $H_k(N)$, fibre $H_k(R^n)$, and structure group $G_{n,n-k}$, is a C^∞ fibre bundle over N. A C^∞ cross section of ζ_N is a C^∞ Λ-congruence to N in M.

<u>Proof</u>: This is complete when the final statement of the theorem has been established. From the construction of ζ_N, it is clear that a cross section is a Λ-congruence; it is necessary to check the C^∞ splitting property.

To that end, let $\Theta : N \to H_k(N)$ be a C^∞ cross section of ζ_N, and let Ψ be a C^∞ vector field along N in M. Thus, over a special coordinate patch $\bar{U}_\gamma$ on M with coordinates (x^i), and with associated patch U on N, one has

$$\Psi(m) = \psi^i(m) \frac{\partial}{\partial x^i} \tag{34}$$

for $m \in U$, where the R-valued functions ψ^i are C^∞ on U. The cross section Θ may be written

$$\Theta(m) = \overline{((m,(\delta^i_j))_\gamma}, \langle\theta^j_1(m)\bar{e}_j, \ldots, \theta^j_k(m)\bar{e}_j\rangle) G_{n,n-k} \tag{35}$$

over U. Because Θ is C^∞, the R-valued maps θ^j_α ($j = 1, \ldots, n$; $\alpha = 1, \ldots, k$) are C^∞ on U, and since, for each point $m \in U$, $(\theta^j_\alpha(m))$ is representative of a matrix in $G_{n,n-k}$, the matrix $(\theta^\alpha_\beta(m))$ is non-singular, where $\alpha, \beta = 1, \ldots, k$. Then $\bar{\theta}^\alpha_\beta : U \to R$ are defined and C^∞ on U, where

$$\theta^\alpha_\beta(m)\ \bar{\theta}^\beta_\gamma(m) = \delta^\alpha_\gamma . \tag{36}$$

Now $\Theta(m)$ is spanned by $(\theta^j_1(m)\ \partial/\partial x^j, \ldots, \theta^j_k(m)\ \partial/\partial x^j)_m$, so the decomposition

$$\Psi(m) = A(m) + B(m) \tag{37}$$

induced by $\Theta(m)$ and the tangent distribution over N permits one to write

$$A(m) = \lambda^\alpha(m)\ \theta^j_\alpha(m) \frac{\partial}{\partial x^j},$$

$$B(m) = \gamma^I(m) \frac{\partial}{\partial x^I}, \tag{38}$$

where lower case Greek letters range over $\{1, \ldots, k\}$, lower case Roman range over $\{1, \ldots, n\}$, and upper case

Roman over $\{k+1, \ldots, n\}$. The coefficients λ^α and γ^I are to be determined; this is straightforward. From (37) and (38), one has

$$\psi^I(m) = \gamma^I(m) + \lambda^\alpha(m)\ \theta^I_\alpha(m)\ ,$$

(39)

$$\psi^\beta(m) = \lambda^\alpha(m)\ \theta^\beta_\alpha(m)\ .$$

Hence

(40) $$\lambda^\gamma = \delta^\gamma_\alpha\lambda^\alpha = \lambda^\alpha\theta^\beta_\alpha\overline{\theta}^\gamma_\beta = \psi^\beta\overline{\theta}^\gamma_\beta\ ,$$

and

(41) $$\gamma^I = \psi^I - \lambda^\alpha\theta^I_\alpha = \psi^I - \psi^\beta\overline{\theta}^\alpha_\beta\theta^I_\alpha\ .$$

Since λ^γ and γ^I are thus seen to be C^∞ R-valued functions on U, it follows from (37) and (38) that Θ has the C^∞ splitting property, and is a C^∞ Λ-congruence. Q.E.D.

The bundle ζ_N will be called the <u>Grassmann Tangent-Complement bundle</u>, or simply the <u>GTC-bundle</u>, over N. The study of Λ-congruences per se will be terminated at this point, though it is to be remarked that one would hope the bundles constructed to prove of some value in making statements about submanifolds in the context of algebraic topology. In order to make such statements, it would be important to know much more about the topology of $Q_k(R^n)$ and $H_k(R^n)$. For example, what can be said about the homology and homotopy groups of these manifolds? This is a possible area for future work.

5. The Λ-connexion and Union Curves

It is, of course, the C^∞ splitting property of a C^∞ Λ-congruence that makes it a suitable vehicle for inducing connexions on submanifolds of manifolds with connexion. Indeed, let M be a C^∞ m-manifold upon which a linear connexion D is defined, and N a C^∞ n-submanifold of M, with Λ a C^∞ Λ-congruence on N. If X and Y are C^∞ vector fields tangent to N, one applies the decomposition induced by Λ to the field $D_X Y$:

(1)
$$D_X Y = \overset{\Lambda}{D}_X Y + \overset{\Lambda}{V}(X,Y)\ ,$$

where $(\overset{\Lambda}{D}_X Y)_p \in N_p$ for $p \in N$, and $\overset{\Lambda}{V}_p(X,Y) \in \Lambda_p$. It is straightforward to establish

Theorem 1: The function $\overset{\Lambda}{D}$ defined by (1) is a linear connexion on N, and $\overset{\Lambda}{V}$ is a covariant 2-tensor (c.f. [5; p. 75]).

Proof: Recall that, if $\mathfrak{X}(N)$ denote the set of all C^∞ vector fields defined on N, a linear connexion ∇ on N is a mapping $\nabla\colon \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}(N)$ satisfying the following axioms:

(2)
$$\begin{aligned} \nabla_{X+Y} Z &= \nabla_X Z + \nabla_Y Z\ , \\ \nabla_X (Y + Z) &= \nabla_X Y + \nabla_X Z\ , \\ \nabla_{fX} Z &= f\,\nabla_X Z\ , \\ \nabla_X fZ &= X(f)\,Z + f\,\nabla_X Z\ , \end{aligned}$$

where X, Y, and Z are in $\mathfrak{X}(N)$, and f is any C^∞ R-valued map on N.

Consider the first of axioms (2); since D is a linear connexion, if $X, Y, Z \in \mathfrak{X}(N)$, one has $D_{X+Y} Z = D_X Z + D_Y Z$, and the decomposition (1) yields

(3) $$D_{X+Y} Z = \hat{D}_X Z + \hat{V}(X,Z) + \hat{D}_Y Z + \hat{V}(Y,Z) .$$

Therefore, since N_p and Λ_p are linear spaces for each $p \in N$, and since the tangential and Λ-components given by the decomposition (1) are unique, (3) gives

(4) $$\hat{D}_{X+Y} Z = \hat{D}_X Z + \hat{D}_Y Z$$

and

(5) $$\hat{V}(X + Y,Z) = \hat{V}(X,Z) + \hat{V}(Y,Z) .$$

In precisely the same way, the rest of axioms (2) are established for $\hat{D}$, showing it to be a linear connexion on N, while $\hat{V}$ is an M-vector valued bilinear map on $\mathfrak{X}(N)$, where $\mathfrak{X}(N)$ is regarded as a module over the ring of C^∞ R-valued maps on N, and hence $\hat{V}$ is an M-vector valued 2-co tensor. Q.E.D.

Note that, in contradistinction to the case in which D is a Riemannian connexion and Λ the normal distribution to N, the connexion $\hat{D}$ and the tensor $\hat{V}$ are not generally symmetric.

Let attention now be turned to a class of curves in

N which have been studied rather widely in the case where D is Riemannian. Suppose that N is a surface in R^3, upon which a Λ-congruence (here, a congruence of lines) has been defined. Then a union curve on N is a curve on N having the property that the osculating plane at each point of the curve contains the line of the congruence passing through that point.[1]

The osculating plane to a curve $\sigma : I \to N$, where I is a compact real interval, is determined by the tangent $T = \sigma_*(d/dt)$, where t is the usual coordinate of R^1, and by the first curvature vector $D_T\, T = k_1\, N_1$ of σ, wherever $D_T\, T \neq 0$. Here, D is the usual connexion on R^3, and N_1 is a unit vector. It is possible to choose N_1 independently of $D_T\, T$, in which case, setting $D_T\, T = k_1\, N_1$ defines the function k_1 [5; p. 74].

One has the following result:

Theorem 2: A curve $\sigma : I \to N$, as above, is a union curve of the surface N re the Λ-congruence Λ iff

(6) $$\overset{\Lambda}{D}_T\, T = g\, T,$$

where g is a C^∞ R-valued function along σ.

Proof: If σ is a union curve, then at a point

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[1] Pauline Sperry [16]. Miss Sperry does not make the requirement that no line of the congruence be tangent to N, but this case is degenerate, and other authors have generally avoided it. See, e.g., C. E. Springer [17; p. 688].

$p \in \sigma[I]$, Λ_p lies in the plane determined by N_1 and T, so one may write $\hat{V}(T,T) = k\,T + k_2\,N_1$. From

(7) $$k_1\,N_1 = D_T\,T = \hat{D}_T\,T + \hat{V}(T,T)\,,$$

one has

(8) $$\hat{D}_T\,T = (k_1 - k_2)\,N_1 - k\,T\,.$$

Since $(\hat{D}_T\,T)_p \in N_p$, k_1 must equal k_2, and (6) holds with $g = -k$.

Suppose, on the other hand, that (6) holds; then one has from (7) that

(9) $$\hat{V}(T,T) = k_1\,N_1 - g\,T.$$

If $k_1 \neq 0$, then σ is a union curve, by (9). If, however, $k_1 = 0$, then $g = 0$ also, since $\hat{V}_p(T,T) \in \Lambda_p$, and (7) shows that σ is a straight line in R^3, so that the osculating plane is indeterminate. Let the convention be adopted that such rulings in N are union curves. Then the argument is complete. Q.E.D.

With Theorem 2 for motivation, return to the case of an m-manifold M with linear connexion D, and an n-submanifold N with Λ-congruence Λ, and <u>define</u> a <u>union curve</u> of N to be any curve $\sigma : I \to N$ with tangent T which satisfies

(10) $$\hat{D}_T\,T = g\,T\,,$$

for some C^∞ $\bar{R}$-valued function g along σ.[1]

In the case where D is Riemannian and Λ the normal distribution to N, $\overset{\Lambda}{V}(X,X)$ has been called the <u>normal</u> or <u>asymptotic curvature vector</u> of X, and a curve $\sigma : I \to N$ with tangent T such that $\overset{\Lambda}{V}(T,T) = 0$ along σ is an <u>asymptotic line</u> [5; p. 76]. Therefore, in the general case of D linear and Λ a Λ-congruence, $\overset{\Lambda}{V}(X,X)$ may be styled the <u>Λ-relative asymptotic curvature vector</u> of X, and a curve $\sigma : I \to N$ with tangent T such that $\overset{\Lambda}{V}(T,T) = 0$ along σ, a <u>Λ-relative asymptotic line</u>. If and only if $\overset{\Lambda}{D}_T\, T = 0$, σ will be named a <u>Λ-geodesic</u>.

The following theorem is immediate:

<u>Theorem 3</u>: Let N be a C^∞ n-submanifold of the C^∞ m-manifold M, where M has a linear connexion D, and N a C^∞ Λ-congruence Λ. Then a curve in N which is a geodesic in M is a union curve in N iff it is a Λ-geodesic and a Λ-relative asymptotic line in N. A curve in N which is not a geodesic of M is a Λ-geodesic iff $(D_T\, T)_p$ lies in Λ_p for each point p of the curve.

It is to be noted that Theorem 3 generalizes well-known properties of induced Riemannian connexions [5; p. 27 and p. 77].

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[1] This agrees with a generalization for hypersurfaces given by Kentaro Yano [22].

6. Union Connexions

In his doctoral dissertation [3], Roy B. Deal introduced a connexion on a metric surface in R^3, the geodesics of which are the union curves of the surface relative to some congruence of lines. In a later paper [18], C. E. Springer developed part of a theory of curvature based on Deal's connexion. This connexion was discovered by writing the differential equations of union curves in the form

$$\frac{d^2x^\alpha}{ds^2} + U^\alpha_{\beta\gamma}\left(x, \frac{dx}{ds}\right) \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0 \tag{1}$$

(where x^α are coordinates in N), and styling the functions $U^\alpha_{\beta\gamma}$ the "coefficients of the connexion".

In this section, a coordinate-free derivation of a union connexion for an n-submanifold N of a Riemannian m-manifold M is given. The local coefficients of this connexion will also be calculated.

Let M be endowed with a Riemannian metric tensor <,>, and associated Riemannian connexion D, while the submanifold N carries a C^∞ Λ-congruence Λ. Then one has

<u>Theorem 1</u>: A curve $\sigma : I \to N$ with tangent T is a union curve of N iff

$$\overset{\Lambda}{D}_T\, T = -\langle \overset{\Lambda}{V}(T,T), T\rangle T \,, \tag{2}$$

where σ is parametrized by arc-length.

Proof: By definition, σ is a union curve iff

$$\hat{D}_T\, T = g\, T\,, \tag{3}$$

where g is a C^∞ R-valued function along σ. From the definition of D one has also

$$D_T\, T = \hat{D}_T\, T + \hat{V}(T,T)\,. \tag{4}$$

Because the parametrization of σ is by arc-length, T is a unit vector, and since D is the Riemannian connexion associated with the metric $<,>$, one has

$$\langle D_T\, T, T\rangle = 0\,. \tag{5}$$

Then, from (3), (4), and (5),

$$\begin{aligned} 0 &= \langle \hat{D}_T\, T, T\rangle + \langle \hat{V}(T,T),T\rangle \\ &= g + \langle \hat{V}(T,T),T\rangle\,. \end{aligned} \tag{6}$$

The theorem follows immediately from (6). Q.E.D.

Accordingly, the union covariant derivative $\breve{D}_X\, Y$ of a C^∞ vector field Y in the direction X may be defined by

$$\breve{D}_X\, Y = \hat{D}_X\, Y + \langle \hat{V}(X,Y),Y\rangle Y\,. \tag{7}$$

The choice of which slots are to be filled by X and which by Y in (7) is governed by a desire to have first-slot linearity; other choices are, of course, possible. From

(7), one sees at once

<u>Theorem 2</u>: A curve in the submanifold N is a geodesic of the union connexion relative to the Λ-congruence Λ iff it is a union curve relative to the congruence, where N is a submanifold of a Riemannian space M, and the union connexion is defined by (7).

It is to be noted that definition (7) depends critically on (2), and (2) requires a metric of a fairly special sort to be defined on M for its derivation. It would not appear likely that a union connexion could be defined for a submanifold of an arbitrary manifold M with a linear connexion. The difficulty is, of course, that a union curve is a <u>parametrized</u> curve, rather than a point set, and that the function g in (3) can be chosen with great freedom, in general.

From equation (7), it is a simple matter to derive local expressions for the union connexion. The conventions regarding valuation of various indices mentioned in the proof of Theorem 3.2 will be followed. Attention is now restricted to a patch U on M with coordinates (x^i) and a patch $V \subseteq U$ on N with coordinates (u^α). Write

(8) $$a_{ij} = \langle \partial/\partial x^i, \partial/\partial x^j \rangle$$

and

(9) $$g_{\alpha\beta} = \langle \partial/\partial u^\alpha, \partial/\partial u^\beta \rangle .$$

It may be supposed that U and V are chosen so both Λ and the normal distribution to N have C^∞ bases over V, with the normal distribution spanned by unit vectors N_μ $(\mu = n+1, \ldots, m)$, with

(10) $$N_\mu = \xi^i_\mu \frac{\partial}{\partial x^i}$$

and Λ spanned at each point of V by λ_τ $(\tau = n+1, \ldots, m)$, where

(11) $$\lambda_\tau = \omega^\alpha_\tau \frac{\partial}{\partial u^\alpha} + c^\mu_\tau N_\mu$$

If a vector field X of M defined along V is given by

(12) $$X = \sigma^\alpha \frac{\partial}{\partial u^\alpha} + p^\mu N_\mu ,$$

then

(13) $$X = \tau^\alpha \frac{\partial}{\partial u^\alpha} + q^\mu \lambda_\mu ,$$

where τ^α and q^μ are to be determined. From (13) and (11), one obtains

(14) $$X = (\tau^\alpha + q^\mu \omega^\alpha_\mu) \frac{\partial}{\partial u^\alpha} + q^\mu c^\tau_\mu N_\tau .$$

Therefore

(15) $$\begin{cases} \tau^\alpha = \sigma^\alpha - q^\mu \omega^\alpha_\mu , \\ \\ p^\tau = q^\mu c^\tau_\mu . \end{cases}$$

Let C^∞ R-valued functions $\bar{c}^\nu_\tau$ on V be defined by the relation $\bar{c}^\nu_\tau \, c^\sigma_\nu = \delta^\sigma_\tau$. Then (15) may be written

$$(16) \qquad \begin{cases} q^\mu = \bar{c}^\mu_\tau \, p^\tau , \\ \\ \tau^\alpha = \sigma^\alpha - \bar{c}^\mu_\tau \, p^\tau \, \omega^\alpha_\mu , \end{cases}$$

which is what is desired.

Now let $\bar{D}$ be the induced Riemannian connexion on N. For vector fields X and Y tangent to N, one has

$$(17) \qquad D_X \, Y = \bar{D}_X \, Y + V^\mu \, (X,Y) \, N_\mu ,$$

where V^μ are certain symmetric 2-covariant tensors over V [5; p. 75].

From (16) and (17), it follows that

$$(18) \qquad \breve{D}_X \, Y = \bar{D}_X Y - \bar{c}^\mu_\tau \, \omega^\alpha_\mu \, V^\tau \, (X,Y) \frac{\partial}{\partial u^\alpha} + \bar{c}^\mu_\tau \, V^\tau \, (X,Y) \, \lambda_\mu .$$

Write

$$(19) \qquad V^\tau \, (X,Y) = \Omega^{\ \tau}_{\alpha\beta} \, X^\alpha \, Y^\beta ,$$

where $X = X^\alpha \, \partial/\partial u^\alpha$ and $Y = Y^\beta \, \partial/\partial u^\beta$. The quantities $\Omega^{\ \tau}_{\alpha\beta}$ are the second fundamental tensors over V with respect to the normals N_τ of classical differential geometry [21; p. 164]. One obtains from (18) and (19)

$$(20) \qquad \overset{\Lambda}{V}(\underset{\sim}{X},\underset{\sim}{Y}) = \bar{c}^\mu_\tau \, \Omega^{\ \tau}_{\alpha\beta} \, X^\alpha \, Y^\beta \, \lambda_\mu ,$$

which, together with (11), shows that

(21) $$<\overset{\Lambda}{V}(X,Y),Y> = \bar{c}^{\mu}_{\tau}\, g_{\gamma\delta}\, \Omega^{\tau}_{\alpha\beta}\, \omega^{\gamma}_{\mu}\, X^{\alpha}\, Y^{\beta}\, Y^{\delta}.$$

Therefore, from (18) and (21), one may write the required local expression:

(22) $$\breve{D}_X\, Y = X^{\alpha}\{\partial Y^{\gamma}/\partial u^{\alpha} + \bar{\Gamma}^{\gamma}_{\alpha\beta}\, Y^{\beta} - Y^{\beta}\,\Omega^{\tau}_{\alpha\beta}\,(\bar{c}^{\mu}_{\tau}\,\omega^{\gamma}_{\mu} - g_{\epsilon\delta}\,\bar{c}^{\mu}_{\tau}\,\omega^{\epsilon}_{\mu}\, Y^{\delta} Y^{\gamma})\}\,\partial/\partial u^{\gamma},$$

where $\bar{\Gamma}^{\gamma}_{\alpha\beta}$ are the coefficients of the induced Riemannian connexion $\bar{D}$. Write $q^{\gamma}_{\tau} = \bar{c}^{\mu}_{\tau}\,\omega^{\gamma}_{\mu}$; then the direction-dependent "coefficients of the union connexion" can be picked off from (22) and written in the form

(23) $$\breve{\Gamma}^{\gamma}_{\alpha\beta}(m,Y) = \bar{\Gamma}^{\gamma}_{\alpha\beta} - \Omega^{\tau}_{\alpha\beta}\,(q^{\gamma}_{\tau} - g_{\epsilon\delta}\, q^{\epsilon}_{\tau}\, Y^{\delta}\, Y^{\gamma}).$$

Note that the coefficients of the linear connexion $\overset{\Lambda}{D}$ are given by

(24) $$\overset{\Lambda}{\Gamma}{}^{\gamma}_{\alpha\beta} = \bar{\Gamma}^{\gamma}_{\alpha\beta} - \Omega^{\tau}_{\alpha\beta}\, q^{\gamma}_{\tau}.$$

These are to be compared with coefficients introduced by K. Yano [22; p. 55].

If, using (23), one writes the differential equations of union curves of N after the form of (1), he obtains

(25) $$\frac{d^2u^{\alpha}}{ds^2} + \bar{\Gamma}^{\alpha}_{\beta\gamma}\,\frac{du^{\beta}}{ds}\,\frac{du^{\gamma}}{ds} - \Omega^{\tau}_{\beta\gamma}\,\left(q^{\alpha}_{\tau} - g_{\epsilon\delta}\, q^{\epsilon}_{\tau}\,\frac{du^{\delta}}{ds}\,\frac{du^{\alpha}}{ds}\right)\frac{du^{\beta}}{ds}\,\frac{du^{\gamma}}{ds} = 0$$

which are precisely the differential equations of union curves of a subspace of a Riemannian space as derived by T. K. Pan, who begins from another definition than than employed here [13].

The introduction of the union connexion $\check{D}$ permits yet another decomposition of the vector field $D_X Y$, where X and Y are C^∞ fields on N:

(26) $$D_X Y = \check{D}_X Y + [\overset{\Lambda}{V}(X,Y) - \langle\overset{\Lambda}{V}(X,Y),Y\rangle Y].$$

Write $U(X,Y)$ for the quantity in the square brackets in the right member of (26). Then $U(X,X)$ may be styled the <u>relative curvature vector</u> of X, and $\langle U(X,X),U(X,X)\rangle^{\frac{1}{2}}$ may be called the <u>relative curvature</u> of X. Also, a curve in N with the property that the relative curvature of its tangent vector is zero at each point of the curve might be called a <u>union-asymptotic curve</u> of N; then (26) shows that a curve in N is simultaneously a union curve and a union-asymptotic curve of N iff it is a geodesic of M.

From (11), (20), and (21), one observes that $U(X,X)$ is given in local coordinates by

(27) $$U(X,X) = \Omega^{\tau}_{\beta\gamma} X^\beta X^\gamma \{q^\alpha_\tau - q^\epsilon_\tau g_{\epsilon\delta} X^\delta X^\alpha\} \partial/\partial u^\alpha$$
$$+ \Omega^{\tau}_{\beta\gamma} X^\beta X^\gamma N_\tau.$$

From (27), one may recognize that $U(X,X)$ coincides with an identically named vector described by Pan [13; p. 7].

It would be possible to continue with the discussion of the union connexion, but the writer prefers to pass now to a consideration of a class of connexions which will include the union connexion as a specific example.

CHAPTER III

NON-LINEAR AND DR-CONNEXIONS

1. Introduction

Recently, K. Yano and S. Ishihara, and A. Kandatu, have studied what they refer to as a non-linear connexion on a differentiable manifold [23, 7]. This is defined by a distribution on the tangent bundle, transversal to fibres over the base manifold, and invariant under the action of the group of non-zero real numbers. Much of what these workers derive is not actually dependent on this invariance condition, which is reflected in the homogeneity (of degree 1) of the connexion coefficients in their directional arguments. Since the union connexion of equation (II.6.7) provides a geometrically non-trivial example of a connexion whose coefficients are not homogeneous in the directional arguments in any degree, it appears reasonable to study connexions defined by fibre-transversal distributions on the tangent bundle which do not necessarily satisfy any invariance properties. This is done in the present chapter, in sections 2 and 3.

In the study of such non-linear connexions, those objects classically referred to as "direction-dependent" tensor fields will be encountered. A new definition of such entities is given, in the context of fibre bundle theory, in section 4. In the following sections, connexions for such objects are studied. Rather remarkably, and pleasantly, one is lead to a type of connexion introduced from quite different a launching point by T. Okada, and employed in an interesting series of papers by Makoto Matsumoto [9, 10, 11, 12].[1] In sections 5 through 7, these connexions are studied from rather different a standpoint from that adopted by these workers.

The remainder of the chapter is devoted to showing how, for direction-dependent connexions, the rôle played in the theory of linear connexions by the tangent bundle is taken by the square of the tangent bundle (in the terminology of Steenrod [20; p. 49]). The results of these sections (8 through 11) provide analogues of results known for linear connexions and tangent bundles, in particular, that of Yano and Ledger on linear connexions on the tangent bundle [24].

2. Definition of a Non-linear Connexion

A. Kandatu has offered the following coordinate-free characterization of a non-linear connexion: let M be a C^∞ n-manifold, and $\mathfrak{X}(M)$ the totality of C^∞ vector

[1] T. Okada, "A formulation of Finsler connections with the use of fibre bundles," (graduation thesis, University of Kyoto, Kyoto, Japan, no date), cited by M. Matsumoto [9; p. 1].

fields on M. Then a mapping $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ is a non-linear connexion provided it satisfies the following axioms:

(1) a) $\nabla_{Y+Z} X = \nabla_Y X + \nabla_Z X$,

b) $\nabla_{fY} X = f \nabla_Y X$,

c) $\nabla_Y fX = Y(f) X + f \nabla_Y X$,

d) $(\nabla_Y X)_p = (\mathring{\nabla}_Y X)_p$, if $X_p = 0$

e) $(\nabla_Y (X + Z))_p = (\nabla_Y X)_p + (\nabla_Y Z)_p$, if $X_p + Z_p = 0$,

where X, Y, and Z are in $\mathcal{X}(M)$, f is a C^∞ R-valued function on M, and $\mathring{\nabla}$ is an arbitrary linear connexion on M.[1]

Equivalently, such a connexion may be defined by an n-dimensional, fibre-transversal, C^∞ distribution on the tangent bundle T(M) of M, which is invariant under the action of the group of non-zero real numbers, and which may have singularities across the zero cross section of T(M) [23; p. 272].[2] If, indeed, the distribution does not possess singularities, a simple lemma of Peter Dombrowski shows that the connexion is actually linear [4; p. 76]. In the present work, however, no distribution will have singu-

[1]Kandatu [7; pp. 259-260]. Kandatu credits Professor S. Ishihara with suggesting this definition.

[2]The word "singularity" here refers to a point at which the distribution fails to be C^∞. In particular, it does not imply a dimensional change at any point.

larities, and the connexion described by (1) is linear. The direction of generalization will be rather different from that of Yano and Ishihara.

Let the union connexion (II.6.7) be tested against axioms (1); it is easily seen that (a), (b), (d), and (e) are satisfied, but instead of (c), D satisfies only the weaker condition

$$\nabla_Y(-X) = -\nabla_Y X. \tag{2}$$

Let mappings $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying (a), (b), (d), and (e) of (1), and (2), be referred to as <u>NH-connexions</u>, the prefix "NH" deriving from "non-homogeneous". Of course, linear connexions are included in the class of NH-connexions.

The local representation of an NH-connexion follows readily from its defining properties. Consider a coordinate patch U on M with coordinates (x^i); if $X = X^i\,\partial/\partial x^i$, write

$$(\nabla_{\partial/\partial x^i} X)_m = \{(\partial X^j/\partial x^i)_m + \Gamma_i^j\,(m,X)\}\,(\partial/\partial x^j)_m. \tag{3}$$

Then the functions Γ_i^j are well-defined functions on the tangent bundle $T(M)$, or, more precisely, on $\pi^{-1}(U)$, where $\pi\colon T(M) \to M$ is the projection map. For suppose $X_m = Z_m$ for two C^∞ vector fields X and Z on M, with $m \in M$. From (2) and (1.e) one has

$$(4)\qquad \left(\nabla_{\frac{\partial}{\partial x^i}}(X - Z)\right)_m = \left(\nabla_{\frac{\partial}{\partial x^i}} X\right)_m - \left(\nabla_{\frac{\partial}{\partial x^i}} Z\right)_m ,$$

while by (1.d),

$$\left(\nabla_{\frac{\partial}{\partial x^i}}(X - Z)\right)_m = \left(\mathring{\nabla}_{\frac{\partial}{\partial x^i}}(X - Z)\right)_m$$

$$(5)\qquad = \left(\frac{\partial X^j}{\partial x^i} - \frac{\partial Z^j}{\partial x^i}\right)\left(\frac{\partial}{\partial x^j}\right)_m .$$

From (3), (4), and (5), the conclusion follows:

$$(6)\qquad \Gamma_i^j\,(m,X_m) = \Gamma_i^j\,(m,Z_m)\,.$$

Using (3), (1.a) and (1.b), one may write the desired local expression:

$$(7)\qquad \nabla_Y X = Y^i\,\{\partial X^j/\partial x^i + \Gamma_i^j\,(m,X)\}\,\partial/\partial x^j\,.$$

From the fact that $\nabla_Y X$ is itself a C^∞ vector field, one concludes from (7) that the functions Γ_i^j transform according to the law

$$(8)\qquad \overline{\Gamma}_t^q(m,X) = \Gamma_i^j(m,X)\,\frac{\partial \overline{x}^q}{\partial x^j}\,\frac{\partial x^i}{\partial \overline{x}^t} + X^p\,\frac{\partial^2 x^j}{\partial \overline{x}^t\,\partial \overline{x}^k}\,\frac{\partial \overline{x}^q}{\partial x^j}\,\frac{\partial \overline{x}^k}{\partial x^p},$$

where $m \in U \cap \overline{U}$, $\overline{U}$ is another coordinate patch on M with coordinates $(\overline{x}^i)$, and $\overline{\Gamma}_t^q$ are functions to R on $\pi^{-1}(\overline{U})$ defined by the analogue of (7) that holds over $\overline{U}$.

Conversely, if, for each coordinate patch U in a cover of M, functions $\Gamma_j^i : \pi^{-1}(U) \to R$ are given so that

(8) holds, and such that $\Gamma^i_j(m,-X) = -\Gamma^i_j(m,X)$, then these functions define, via (7), a global NH-connexion.

Consider now the tangent bundle $T(M)$, and recall that if U is a coordinate patch on M with coordinates (x^i), then $\pi^{-1}(U)$ is a coordinate patch on $T(M)$, a point $(m,X) \in \pi^{-1}(U)$ having coordinates $(x^1(m), \ldots, x^n(m), \xi^1, \ldots, \xi^n)$, where $X = \xi^i\, \partial/\partial x^i$.[1] Suppose V is another coordinate patch on M with coordinates (y^i); then, if $X = \xi^i\, \partial/\partial x^i = \eta^i\, \partial/\partial y^i$, one has

(10) $$\eta^k = \xi^j\, (\partial y^i/\partial x^j)\ .$$

The tangent spaces of $T(M)$ are, over U, spanned by the coordinate vectors $(\partial/\partial x^1, \ldots, \partial/\partial x^n, \partial/\partial \xi^1, \ldots, \partial/\partial \xi^n)$. Suppose a vector field Q be given on $T(M)$, with representation

(11) $$Q = \alpha^i(m,X)\,\frac{\partial}{\partial x^i} + \beta^j(m,X)\,\frac{\partial}{\partial \xi^j}$$

over U. Then, if $m \in U \cap V$, (10) and (11) yield

(12) $$Q = (\alpha^i\,\frac{\partial y^j}{\partial x^i})\,\frac{\partial}{\partial y^j} + (\alpha^i \xi^j\,\frac{\partial^2 y^k}{\partial x^i\,\partial x^j} + \beta^j\,\frac{\partial y^k}{\partial x^j})\,\frac{\partial}{\partial \eta^k}\,.$$

Equation (12) is the transformation equation for vectors on $T(M)$.

If $X = X^j\, \partial/\partial x^j$ is a vector field on M, the field $\tilde{X}$ on $T(M)$ defined over U by

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[1] See footnote, page 21.

(13) $$\tilde{X}(m,Y) = X^j(m)\{\partial/\partial x^j - \Gamma_j^h(m,Y)\ \partial/\partial \xi^h\}$$

is called the <u>horizontal lift</u> of X (c.f. [7; p. 263]). Using (12) and (8), and writing $\bar{\Gamma}_i^j$ for the connexion coefficients over the patch $\bar{V}$, one sees that

$$\tilde{X}(m,Y) = X^j \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i} + \{X^i\ \xi^j \frac{\partial^2 y^k}{\partial x^i\ \partial x^j} - \Gamma_j^h(m,Y)\ X^i \frac{\partial y^k}{\partial x^h}\}\frac{\partial}{\partial \eta^k}$$

(14) $$= \bar{X}^i \frac{\partial}{\partial y^i} + \{X^i\ \xi^j \frac{\partial^2 y^k}{\partial x^i\ \partial x^j}$$

$$- \bar{\Gamma}_t^p \frac{\partial x^h}{\partial y^p}\frac{\partial y^t}{\partial x^j}\frac{\partial y^k}{\partial x^h} X^j - X^i \xi^p \frac{\partial y^s}{\partial x^p}\frac{\partial^2 y^t}{\partial x^i\ \partial x^q}\frac{\partial x^h}{\partial y^t}\frac{\partial x^q}{\partial y^s}\frac{\partial y^k}{\partial x^h}\}\frac{\partial}{\partial \eta^k}.$$

On using the fact that

$$\frac{\partial y^t}{\partial x^q}\frac{\partial x^q}{\partial y^s} = \delta_s^t,$$

one finds that the first and last terms in the braces in (14) subtract out, and what is left may be written

(15) $$\tilde{X}(m,Y) = \bar{X}^i\ \partial/\partial y^i - \bar{\Gamma}_t^k\ (m,Y)\ \bar{X}^t\ \partial/\partial \eta^k,$$

where $X = \bar{X}^i\ \partial/\partial y^i$. It follows that (13) defines a global lift of X to a field on $T(M)$.

Notice that the horizontal lifts to $T(M)$ of vector fields on M define a smooth n-dimensional distribution on $T(M)$ which is transversal to fibres. This distribution is spanned locally by horizontal lifts of coordinate vector fields on a coordinate patch U; thus, in $\pi^{-1}(U)$, it is spanned by the vectors B_i $(i = 1, \ldots, n)$, where

(16)
$$B_i = \frac{\partial}{\partial x^i} - \Gamma_i^h(m,X) \frac{\partial}{\partial \xi^h}.$$

Now turn the situation around, and suppose given a C^∞ fibre-transversal distribution H of dimension n on $T(M)$. The fibre-transversality implies that π_*, restricted to $H(m,Z)$, is an isomorphism. The unique vector $\tilde{X}$ in $H(m,Z)$ such that $\pi_*(\tilde{X}) = X$ for some $X \in M_m$, may be styled the <u>horizontal lift</u> of X, and if X is a C^∞ vector field, the lifts $\tilde{X}$ define a C^∞ field on $T(M)$, since H is C^∞. In particular, the horizontal lifts B_i of coordinate vector fields $\partial/\partial x^i$ over a patch U on M span H on $\pi^{-1}(U)$. Write

(17)
$$B_i = \alpha_i^t(m,X) \frac{\partial}{\partial x^t} + \beta_i^t(m,X) \frac{\partial}{\partial \xi^t},$$

where α_i^t and β_i^t are functions to be investigated.

Since $\pi_*(B_i) = \partial/\partial x^i$, it follows that $\alpha_i^t(m,X) = \delta_i^t$ for all $(m,X) \in \pi^{-1}(U)$. And the transformation equation (12) shows that over $U \cap V$, where V is a coordinate patch on M with coordinates (y^i), B_i is

(18)
$$\frac{\partial y^k}{\partial x^i}\left\{\frac{\partial}{\partial y^k} + \left[\frac{\partial x^t}{\partial y^k} \xi^s \frac{\partial^2 y^p}{\partial x^t\, \partial x^s} + \frac{\partial x^t}{\partial y^k} \frac{\partial y^p}{\partial x^s} \beta_t^s\right] \frac{\partial}{\partial \eta^p}\right\}.$$

If the horizontal lifts of the vectors $\partial/\partial y^i$ be denoted by $\bar{B}_i$, then (18) shows $\bar{B}_i$ to be the vector in braces in (18).

But if $\overline{\beta}_i^t$ be defined over V as β_i^t are defined over U by (17), one must have

(19)
$$\overline{\beta}_k^p(m,X) = \beta_t^s(m,X)\,\frac{\partial x^t}{\partial y^k}\,\frac{\partial y^p}{\partial x^s} + \frac{\partial x^t}{\partial y^k}\,\frac{\partial^2 y^p}{\partial x^t\,\partial x^s}\,\xi^s\,,$$

since "lifting" is an isomorphism.

Equations (19) show that $-\beta_t^s(m,X)$ transform like the Γ_j^i in (8), so a covariant derivative ∇ may be defined, after the pattern of equation (7), using these functions as coefficients. This operator is a reasonable sort of connexion.

Henceforward, connexions induced by C^∞, n-dimensional, fibre-transversal distributions H on T(M) will be called <u>non-linear connexions</u>. Notice that the covariant derivative ∇ defined by such a distribution need not satisfy (2); in order that this hold, one must require

(20)
$$\beta_t^s(m,-X) = -\beta_t^s(m,X)\,.$$

This means that H must satisfy a species of symmetry about the null cross section of T(M):

(21)
$$H(m,-X) = -H(m,X)\,.$$

The minus sign before H(m,X) in (21) denotes the mapping

(22)
$$Y^i\,B_i(m,-X) \mapsto -Y^i\,B_i(m,X)\,.$$

It seems reasonable to remark at this point why one cannot give an invariant, coordinate-free characterization like (1) for an arbitrary non-linear connexion. The connexion concept is a very broad one, and essentially any map $D:\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ may be called a connexion. If one is given such a mapping, and writes

(23) $$D_{\partial/\partial x^i} X = \partial X^h/\partial x^i \,(\partial/\partial x^h) + \Gamma_i^h(m,X)(\partial/\partial x^h) ,$$

where first-slot linearity is being assumed for convenience, he does not know that Γ_i^h are well-defined functions in the tangent bundle. For a trivial example, let

(24) $$D_{\partial/\partial x^i} X = \{\frac{\partial X^h}{\partial x^i} + \bar{\Gamma}_i^h(m,X)\}\, f(m,X)\, \frac{\partial}{\partial x^h} ,$$

where $\bar{\Gamma}_i^h$ are, say, coefficients of some linear connexion, and f is an arbitrary but fixed C^∞ R-valued map on $T(M)$. If one puts the covariant derivative (24) into the form (23), then the "functions" Γ_i^h are given by

(25) $$\Gamma_i^h = f(m,X)\, \bar{\Gamma}_i^h(m,X) + [f(m,X) - 1]\, \frac{\partial X^h}{\partial x^i} ,$$

and thus Γ_i^h depend on values of X in a neighborhood of m, and are not well-defined functions on $T(M)$.

If enough is included in the list of axioms characterizing a connexion to insure that the Γ_i^h in (23) are well-defined in $\pi^{-1}(U)$, enough is included to make the connexion rather special. Indeed, the axioms for an NH-connexion seem to this writer to be minimal in this respect.

3. Integrability Conditions

Consider now the distribution H of a non-linear connexion; conditions under which H is integrable are sought. By the Frobenius Theorem on the integrability of distributions (see, e.g., [1; p. 23]), H is integrable iff

(1) $$[X,Y]_{(m,Z)} \in H(m,Z) \quad \forall\ X,Y \in H(m,Z).$$

Since the fields B_i $(i = 1, \ldots, n)$ of (2.16) form a local spanning set of H, (1) holds iff

(2) $$[B_i,B_j] = \gamma_{ij}^k\, B_k \qquad (\gamma_{ij}^k \text{ R-valued maps on } \pi^{-1}(U)),$$

over the coordinate patch U. From (2.16), one obtains

$$[B_i,B_j] = [B_i,B_j](x^k)(\partial/\partial x^k) + [B_i,B_j]\,(\xi^k)(\partial/\partial\xi^k)$$

or

(3) $$[B_i,B_j] = \{\frac{\partial\Gamma_i^k}{\partial x^j} - \frac{\partial\Gamma_j^k}{\partial x^i} + \Gamma_i^h\,\frac{\partial\Gamma_j^k}{\partial\xi^h} - \Gamma_j^h\,\frac{\partial\Gamma_i^k}{\partial\xi^h}\}\frac{\partial}{\partial\xi^k}.$$

Thus one sees that $[B_i,B_j]$ is a vertical vector, that is, $\pi_*([B_i,B_j]) = 0$. In order that $[B_i,B_j]$ be horizontal, then, it is necessary and sufficient that $[B_i,B_j] = 0$; hence, one has

Theorem 1: The horizontal distribution H of a non-linear connexion, with coefficients $\Gamma_i^h(m,X)$ over a coordinate patch U on M, is integrable iff, over each such patch,

(4) $$R^{k}_{ij} = 0 ,$$

where R^{k}_{ij} denotes the quantity appearing in curly braces in equation (3).

This result is to be compared with that of Kandatu [7; p. 267].

The quantities $R^{k}_{ij}(m,X)$ appearing in (4) are $\underline{R}$-valued functions on $\pi^{-1}(U) \subseteq T(M)$, and the operators $\partial/\partial x^i$ and $\partial/\partial \xi^k$ involved in the definition of them are coordinate vectors in $\pi^{-1}(U)$, so that they transform in $\pi^{-1}(U \cap V)$ by

$$\frac{\partial}{\partial y^i} = \frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^k} + \eta^k \frac{\partial^2 x^j}{\partial y^i\, \partial y^k}\frac{\partial}{\partial \xi^j},$$

(5)

$$\frac{\partial}{\partial \eta^k} = \frac{\partial x^h}{\partial y^k}\frac{\partial}{\partial \xi^h},$$

where V is a coordinate patch on M with coordinates (y^i), and (y^i, η^i) are the induced coordinates in $\pi^{-1}(V)$. Using (5) and (2.8), one finds that, under such a change of canonical coordinates in $T(M)$, the quantities $R^{k}_{ij}(m,X)$ in U are related to similarly defined entities $\bar{R}^{k}_{ij}(m,X)$ in V by

(6) $$\bar{R}^{k}_{ij}(m,X) = R^{p}_{ts}(m,X)\, \frac{\partial x^t}{\partial y^i}\frac{\partial x^s}{\partial y^j}\frac{\partial y^k}{\partial x^p},$$

which follows by straightforward, if quite tedious, computation.

In spite of the appearance of (6), R^{k}_{ij} cannot be regarded as components of a tensor field on M, since they are functions in $\pi^{-1}(U)$. Nor can these functions be thought of as components of a tensor on $T(M)$; there are not even enough functions given to determine one! But it is possible to associate the functions R^{k}_{ij} with a tensor field on $T(M)$, using the distribution of the non-linear connexion. Let B^{i} $(i = 1, \ldots, n)$ be the basis dual to the basis B_{i} $(i = 1, \ldots, n)$ of $H(m,X)$ for $m \in U$. Then a tensor $\bar{R}$ is defined on $T(M)$ by

$$\bar{R} = R^{k}_{ij}(m,X)\ B^{i} \otimes B_{k} \otimes B^{j}\,.$$

This approach is due to Yano and Ishihara [23; pp. 281-283]. Classically, however, such collections of functions as R^{k}_{ij} have been referred to as "direction-dependent tensor fields on M", or as "tensor fields on M with coefficients in the tangent bundle". (See, for example, H. Rund's book [14], in which such objects abound.)

The approach employed here will be to retain as much as possible of the classical point of view, while attempting to put the concept on a more rigorous mathematical foundation. A detailed consideration of this subject begins in the next section; for now, consider the following purely local definition, the significance of which is more terminological than fundamental:

<u>Definition 1</u>: Let $\{(U_\gamma, \varphi_\gamma)\}_{\gamma \in G}$, G an index set, be a coordinate cover of the C^∞ n-manifold M, with $x^i_\gamma = u^i \circ \varphi_\gamma$. Let $T(M)$ be the tangent bundle over M, with projection π. If ${}_\gamma T^{i_1 \ldots i_m}_{j_1 \ldots j_k}$ are mk C^∞ R-valued functions defined on $\pi^{-1}(U_\gamma)$ for each $\gamma \in G$, these functions are said to define a <u>DR-tensor field of type (k,m)</u> on M, provided that in $\pi^{-1}(U_\gamma \cap U_\delta)$, the relationships

(7)
$$ {}_\delta T^{i_1 \ldots i_m}_{j_1 \ldots j_k} = {}_\gamma T^{h_1 \ldots h_m}_{t_1 \ldots t_k} \frac{\partial x_\gamma^{t_1}}{\partial x_\delta^{j_1}} \cdots \frac{\partial x_\gamma^{t_k}}{\partial x_\delta^{j_k}} \frac{\partial x_\delta^{i_1}}{\partial x_\gamma^{h_1}} \cdots \frac{\partial x_\delta^{i_m}}{\partial x_\gamma^{h_m}} $$

hold, for each $\gamma, \delta \in G$ such that $U_\gamma \cap U_\delta \neq \varphi$.

The name, DR-tensor, derives from "DiRection-dependent tensor". In particular, the DR-tensor defined by $R^{\;k}_{ij}$ will be referred to as the <u>curvature DR-tensor</u> of the non-linear connexion. Also, one other DR-tensor associated with a non-linear connexion might be mentioned, which appears, in particular, if one examines integrability conditions for the almost-complex structure induced on the tangent bundle by the distribution of a non-linear connexion. This is defined in $\pi^{-1}(U)$ by

(8)
$$ T^{\;i}_{jk}(m,X) = \frac{\partial \Gamma_j^i}{\partial \xi^k}(m,X) - \frac{\partial \Gamma_k^i}{\partial \xi^j}(m,X) , $$

and it will be called the torsion DR-tensor of the non-linear connexion (c.f. [7; p. 268]). Note that, upon differentiating equation (2.8) with respect to ξ^s, one sees that $\partial\Gamma_j^i/\partial\xi^s$ transform like ordinary Christoffel symbols of the second kind (see, e.g., [19; pp. 111-112]). From this, it follows at once that T_{ji}^k satisfy the transformation equation (7) and define globally a DR-tensor.

4. DR-vectors

A careful look at the local definition of DR-tensors given in section 3 suggests the following coordinate-free characterization of a DR-vector field: a DR-vector field on a C^∞ n-manifold M is a function which assigns to each point (m,X) of the tangent bundle $T(M)$ over M, a vector in M_m. Beyond its intrinsic simplicity and close similarity to the definition of an ordinary vector field on M, this definition has the advantage of leading at once to a formulation of DR-vectors in terms of fibre bundles, so that faintly artificial statements as to what is meant by the smoothness of a DR-vector field can be avoided. This is now described.

Let $T^2(M)$ be the set of all triples (m,X,Y), where $m \in M$, and $X, Y \in M_m$. The set $T^2(M)$ is made a C^∞ 3n-manifold as follows: suppose (U,φ) is a coordinate patch on M with $x^i = u^i \circ \varphi$. Let $\pi_2: T^2(M) \to M$ by $\pi_2(m,X,Y) = m$, and introduce a coordinate map

$\bar{\varphi} : \bar{U} = \pi_2^{-1}(U) \to R^{3n}$ so:

(1)

$$\bar{\varphi}(m,X,Y) = (x^1(m),\ \ldots,\ x^n(m),\ \xi^1,\ \ldots,\ \xi^n,\ \eta^1,\ \ldots,\ \eta^n),$$

where $X = \xi^i\ \partial/\partial x^i$ and $Y = \eta^i\ \partial/\partial x^i$. Suppose, too, that (V,ψ) is a coordinate patch on M with $y^i = u^i \circ \psi$, and $U \cap V \neq \varphi$. Define $\bar{\psi} : \bar{V} = \pi_2^{-1}(V) \to R^{3n}$ precisely in analogy to (1):

(2)

$$\bar{\psi}(m,X,Y) = (y^1(m),\ \ldots,\ y^n(m),\ \beta^1,\ \ldots,\ \beta^n,\ \gamma^1,\ \ldots,\ \gamma^n),$$

where $X = \beta^i \partial/\partial y^i$ and $Y = \gamma^i \partial/\partial y^i$. Then, since

$$X = \xi^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

and

$$Y = \eta^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j},$$

it follows that $\bar{\psi} \circ \bar{\varphi}^{-1} : \bar{\varphi}\{\pi_2^{-1}(U \cap V)\} \to R^{3n}$ is given by

$$\bar{\psi} \circ \bar{\varphi}^{-1} (x^1,\ \ldots,\ x^n,\ \xi^1,\ \ldots,\ \xi^n,\ \eta^1,\ \ldots,\ \eta^n) =$$

$$(3)\quad (y^1 \circ \varphi^{-1} (x^1,\ \ldots,\ x^n),\ \ldots,\ y^n \circ \varphi^{-1} (x^1,\ \ldots,\ x^n),$$

$$\xi^p \frac{\partial y^1}{\partial x^p},\ \ldots,\ \xi^p \frac{\partial y^n}{\partial x^p},\ \eta^p \frac{\partial y^1}{\partial x^p},\ \ldots,\ \eta^p \frac{\partial y^n}{\partial x^p}).$$

The expression for $\bar{\varphi} \circ \bar{\psi}^{-1}$ is similar. Thus one sees that the patches $(\bar{U},\bar{\varphi})$ and $(\bar{V},\bar{\psi})$ are C^∞-related. Since $T^2(M)$ is covered by such patches, this procedure endows $T^2(M)$ with a differentiable structure, making it a C^∞ 3n-manifold. It is clear that π_2 is C^∞ with respect to this structure.

Consider now the bundle $\xi_M = (T^2(M), \bar{\pi}, T(M))$, where $\bar{\pi}(m,X,Y) = (m,X)$. From the definition of the C^∞ structures of $T^2(M)$ and $T(M)$, it is trivial that $\bar{\pi}$ is C^∞. Further, since the fibre $\bar{\pi}^{-1}(m,X)$ has the structure of an n-dimensional vector space, the group $Gl(n,R)$ acts fibre-wise on $T^2(M)$ by

$$(4) \qquad \sigma(m,X,Y) = (m,X,\sigma Y)$$

for $\sigma \in Gl(n,R)$; this action is clearly C^∞. The local triviality of the bundle is immediately apparent from the coordinate structure (1). Therefore, ξ_M is a C^∞ fibre bundle over $T(M)$ with structure group $Gl(n,R)$ and fibre R^n.

It is to be remarked that ξ_M is just the square of the tangent bundle; that is, it is the bundle induced from the tangent bundle over M by the projection $\pi: T(M) \to M$. It would have been possible to begin the discussion of DR-vector fields with this notion, but the description given has the advantage of detail, displays coordinates, and fixes notation, which is useful, as much of the analysis which follows is local.

From the definition of a DR-vector field given at the beginning of this section, it is now clear that a <u>cross-section of the bundle</u> ξ_M <u>is a DR-vector field</u>. A DR-vector field will therefore be defined to be C^∞ iff it is a C^∞ cross-section. The usual techniques of multilinear algebra permit the construction of such bundles for DR-objects of all sorts.

Note that if $\sigma: T(M) \to T^2(M)$ is a C^∞ cross-section, then one may write $\sigma(m,X) = (m,X, \tau(m,X))$, so that

$$\tau(m,X) = \varphi^i(m,X) \; (\partial/\partial x^i) \tag{5}$$

over a coordinate patch U on M. Thus $\varphi^i(m,X) = \tau(m,X)(x^i)$; and, since σ is a C^∞ as a cross-section of ξ_M, φ^i are C^∞ maps on $\pi^{-1}(U)$. If V is a second coordinate patch on M with coordinates (y^i), then for $m \in U \cap V$,

$$\tau(m,X) = \varphi^i(m,X) \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}. \tag{6}$$

Equation (6) shows that τ (or σ) is a DR-vector in the sense of the local definition of section 3.

From the discussion above, one sees that the totality $\mathfrak{D}(M)$ of C^∞ DR-vector fields on M is a module over the ring $C^\infty(T(M),R)$ of R-valued differentiable maps on $T(M)$. Since a DR-vector at a point is just a vector on M, it cannot operate on $C^\infty(T(M),R)$, and therefore, the <u>Lie product cannot be defined on $\mathfrak{D}(M)$</u>.

5. The Bundle η_M and DR-connexions

Next, the C^∞ principal fibre bundle over $T(M)$ with structure group $Gl(n,R)$ associated to the bundle ξ_M of section 4 will be described. Let $B_A(M)$ be the set of all $(n + 2)$-tuples $(m, X, e_1, \ldots, e_n)$, where $(m,X) \in T(M)$ and $(e_1, \ldots, e_n)$ is an ordered base of M_m. And let $\pi_B : B_A(M) \to T(M)$ by $\pi_B(m, X, e_1, \ldots, e_n) = (m,X)$, while $p_B : B_A(M) \to M$ is defined by $p_B(m, X, e_1, \ldots, e_n) = m$. The diagram below should help in permitting one to visualize the relationships among the spaces under consideration.

$$\begin{tikzcd}
 & B_A(M) \arrow[dl, "\pi_B"'] \arrow[dr, "p_B"] & \\
T(M) \arrow[rr, "\pi"] & & M \\
 & T^2(M) \arrow[ul, "\bar{\pi}"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

The diagram is commutative.

The map p_B is used to lift coordinate patches on M to $B_A(M)$, to give $B_A(M)$ the structure of a C^∞ $(2n+n^2)$-manifold. Indeed, let U be a coordinate patch on M with coordinates (x^i), and let $\bar{U} = p_B^{-1}(U)$. Define $\bar{\varphi} : \bar{U} \to R^{2n+n^2}$ by

$$(1) \quad \bar{\varphi}(m, X, e_1, \ldots, e_n) = (x^1(m), \ldots, x^n(m), \xi^1, \ldots, \xi^n, \xi^1_1, \ldots, \xi^n_1, \ldots, \xi^1_n, \ldots, \xi^n_n),$$

where $X = \xi^i \, \partial/\partial x^i$ and $e_j = \xi^i_j \, \partial/\partial x^i$. Just as in the case of $T^2(M)$, one can see that the coordinate patches $(\bar{U}, \bar{\varphi})$ so defined on $B_A(M)$ are C^∞-related, and since these patches cover $B_A(M)$, it becomes a C^∞ $(2n + n^2)$-manifold. It is clear that both π_B and p_B are C^∞ with respect to this structure.

Let R_g denote the right action on $B_A(M)$ by an element $g = (g^i_j)$ of $Gl(n,R)$; then R_g is defined by

(2) $$R_g(m,X,e_1, \ldots, e_n) = (m,X,g^i_1 e_i, \ldots, g^i_n e_i).$$

This action is clearly C^∞ and free. For coordinate patches U with coordinates (x^i) on M, structure maps $F_U : p_B^{-1}(U) \to Gl(n,R)$ are defined by

(3) $$F_U(m,X,e_1, \ldots, e_n) = (\xi^i_j) \in Gl(n,R),$$

where $e_j = \xi^i_j\, \partial/\partial x^i$. Note that

(4) $$F_U \circ R_g(m,X,e_1,\ldots,e_n) = F_U(m,X,g^i_1 e_i,\ldots,g^i_n e_i) = (g^i_j\, \xi^k_i)$$

while

(5) $$R_g \circ F_U(m,X,e_1,\ldots,e_n) = R_g(\xi^k_i) = (g^i_j\, \xi^k_i).$$

Therefore,

(6) $$F_U \circ R_g = R_g \circ F_U.$$

Finally, define $S_U : \pi_B^{-1}(\overline{U}) \to \overline{U} \times Gl(n,R)$ by

(7) $$S_U(m,X,e_1,\ldots,e_n) = ((m,X), (\xi^k_j)),$$

where $\overline{U} = \pi^{-1}(U)$ for the coordinate patch U, and $e_j = \xi^i_j\, \partial/\partial x^i$. Then S_U is a diffeomorphism.

It follows from these observations that the bundle $\eta_M = (B_A(M), \pi_B, T(M))$ is a C^∞ principal fibre bundle with structure group $Gl(n,R)$. It is easy to see that η_M is, in fact, the principal $Gl(n,R)$-bundle associated to ξ_M.

Since the diagram

$$\begin{tikzcd} B_A(M) \arrow[r, "f"] \arrow[d, "\pi_B"'] & B(M) \arrow[d, "p"] \\ T(M) \arrow[r, "\pi"'] & M \end{tikzcd}$$

commutes, where $f(m,X,e_1, \ldots, e_n) = (m,e_1, \ldots, e_n)$ and p is the projection of the bundle of bases over M, one has that η_M is the bundle induced from the bundle of bases by the projection of the tangent bundle. Since the bundle of bases is principal, one has at once that η_M is principal. The more circuitous approach used above is employed for the same reasons as given in section 4 in the case of ξ_M.

Since η_M is a principal bundle, if $T(M)$ is paracompact (which it will be if M is paracompact), η_M will admit a connexion in the sense of fibre bundle theory [1; p. 83], that is, a 2n-dimensional C^∞ distribution H on $B_A(M)$ transversal to fibres over $T(M)$ and invariant under right actions:

$$(8) \qquad (R_g)_* H_p = H_{R_g(p)},$$

for $p \in B_A(M)$ and $g \in Gl(n,R)$. It follows from the fibre-transversality that the restriction of $(\pi_B)_*$ to H_p is an isomorphism onto the tangent space of $T(M)$ at $\pi_B(p)$.

It is interesting to note that, given a connexion H on η_M, there exists a well-defined concept of a horizontal lift of a curve in M to a curve in $B_A(M)$. Let

$\sigma : I \to M$ be a C^∞ curve; then one has the <u>natural lift</u> $\sigma_* : I \to T(M) :: \sigma_*(t) = (\sigma(t), \sigma_*(\frac{d}{dt}|_t))$. Then let $\tilde{\sigma} : I \to B_A(M)$ be the unique horizontal lift of σ_* through a point p in the fibre $\pi_B^{-1}(\sigma_*(0))$. Then $p_B \circ \tilde{\sigma} = \sigma$. The curve $\tilde{\sigma}$ will be called the <u>canonical lift</u> of σ through p.

Given this lift, one may define <u>canonical parallel translation</u> along curves in M to be a diffeomorphism of fibres in $B_A(M)$. Let $\sigma : I \to M$ again be any C^∞ curve, and let $p = (\sigma(0), \sigma_*(d/dt)_0, e_1, \ldots, e_n)$ be a point of the fibre over $\sigma_*(0) \in T(M)$. If $\tilde{\sigma}$ be the canonical lift of σ through p, define $T_\sigma(p) = \tilde{\sigma}(1)$ in $\pi_B^{-1}(\sigma_*(1))$, where, for convenience, I has been taken to be the unit interval. Then, as T_σ is simply ordinary parallel translation along σ_* in $T(M)$ by the connexion H, T_σ is a diffeomorphism, and $T_\sigma \circ R_g = R_g \circ T_\sigma$ [1; p. 78].

Suppose $X_o \in M_{\sigma(0)}$, and $X_o = a^i e_i$, where $(e_1, \ldots, e_n)$ is an arbitrarily chosen, but fixed, base of $M_{\sigma(0)}$. If $T_\sigma\,(\sigma(0), \sigma_*(d/dt)_0, e_1, \ldots, e_n)$ is $(\sigma(1), \sigma_*(d/dt)_1, \bar{e}_1, \ldots, \bar{e}_n)$, then the vector $X_1 \in M_{\sigma(1)}$, where $X_1 = a^i \bar{e}_i$, will be said to result from X_o by canonical parallel translation along σ. This translation is independent of the particular canonical lift chosen, that is, of the particular basis $(e_1, \ldots, e_n)$. For if $(\hat{e}_1, \ldots, \hat{e}_n)$ is another basis of $M_{\sigma(0)}$, and $\hat{e}_i = h_i^j e_j$, define quantities $\hat{h}_j^i$ by the equation

(9)
$$h_i^j\,\hat{h}_j^k = \delta_i^k\,;$$

then $X_0 = a^i\,\hat{h}_i^j\,\hat{e}_j$. Since $T_\sigma \circ R_g = R_g \circ T_\sigma$, one has

(10)
$$T_\sigma(\sigma(0),\sigma_*(d/dt)_0,\hat{e}_1,\ldots,\hat{e}_n) = (\sigma(1),\sigma_*(d/dt)_1,h_1^i\bar{e}_i,\ldots,h_n^i\bar{e}_i).$$

Thus, if X_1' is the vector resulting from canonical parallel translation of X_0 along σ using the basis $(\hat{e}_i)$, one has

$$X_1' = a^i\,\hat{h}_i^j\,h_j^k\,\bar{e}_k = a^i\bar{e}_i = X_1\,,$$

from (9).

If $X(t)$ is a vector field along σ, $X(t)$ will be said to be <u>canonically parallel</u> along σ provided that, for each $u, v \in I$, $X(u)$ result from $X(v)$ by canonical parallel displacement along $\sigma|_{[u,v]}$ (or $\sigma|_{[v,u]}$, as the case may be). In particular, a curve σ may be styled a <u>canonical path</u> in M iff $\sigma_*(d/dt)$ is canonically parallel along σ. Thus, in one sense, a canonical path is a direct generalization of a geodesic of a linear connexion.

Except for its naturality, there is nothing sacred about using the tangent to the curve σ to obtain the lift of σ to a curve in $T(M)$ which is so vital to the foregoing constructions. Thus, if $X(t)$ be any vector field along σ, one has the lift $\bar{\sigma} : I \to T(M)$ by $\bar{\sigma}(t) = (\sigma(t),X(t))$; the unique horizontal lift of $\bar{\sigma}$ to $B_{\underline{A}}(M)$ may be called the <u>$X(t)$-relative horizontal lift</u> of σ. Obviously, there is also a concept of $X(t)$-relative parallel translation along curves in M, and of $X(t)$-relative parallel fields along curves in M. A curve σ whose tangent vector field is $X(t)$-relative parallel along σ could be named an <u>$X(t)$-relative path</u> re the connexion H.

Not surprizingly, canonical parallel translation does not induce a covariant differentiation of tensor fields, nor of DR-tensor fields, on M. The reason for this is that, given a curve $\sigma : I \to M$, the tangent T_{σ_*} to $\sigma_* : I \to T(M)$, and therefore the tangent to the canonical lift $\tilde{\sigma}$, depend to the second order on σ:

(12)
$$T_{\sigma_*}(t) = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{d^2x^i}{dt^2}\frac{\partial}{\partial \xi^i},$$

where $x^i(t) = x^i \circ \sigma(t)$ in a coordinate patch U on M. Otherwise expressed, what is critical here is that the natural lifts σ_* do not determine a C^∞ n-distribution on $T(M)$.

Accordingly, let a non-linear connexion Γ be introduced on $T(M)$, with Γ spanned by the vectors B_i of (2.16) over a coordinate patch U. The connexion Γ will be referred to in the present context as a <u>non-linear support</u> for the connexion H in η_M.

The introduction of the non-linear support permits not only a unique lifting of tensor fields on M to tensor fields on $T(M)$, but also the association of DR-fields with fields on $T(M)$ after the fashion of Yano and Ishihara mentioned in section 3. In particular, suppose

$\tau : T(M) \to T^2(M)$ is a DR-vector field given in the coordinate patch U by $\tau(m,X) = \varphi^i(m,X)\, \partial/\partial x^i$; then one may associate with τ the "lift"

(13) $$\tilde{\tau}(m,X) = \varphi^i(m,X)\, B_i(m,X)\,.$$

This will be called the <u>Γ-lift</u> of τ. One also has the "lift"

(14) $$\bar{\tau}(m,X) = \varphi^i(m,X)\, B_{i*}(m,X)\,.$$

where $B_{i*} = \partial/\partial\xi^i$ over the patch U.[1] The field $\bar{\tau}$ will be called the <u>vertical lift</u> of τ. It is easily verified that each of (13) and (14) defines a global vector field on $T(M)$, given a global DR-vector field τ.

Now suppose $(W, F, Gl(n,R), T(M))$ is a vector bundle associated to the principal bundle η_M, with total space W, fibre F, and projection $p_W : W \to T(M)$. Let $\bar{U}$ be a neighborhood of a point $(m,X) \in T(M)$, and let $Q : \bar{U} \to W$ be a cross-section over $\bar{U}$. It is desired to define a covariant derivative $\overset{\Gamma}{\nabla}_{\tau(m,X)} Q$ to be an element of the fibre of W over (m,X), where $\tau(m,X)$ is the value at (m,X) of a (possibly local) cross-section of ξ_M.

Recall that the connexion H induces a C^∞ fibre-transversal distribution H' on W in the following way: By definition, $W = (B_A(M) \times F) / Gl(n,R)$, with $A_g(p,f) = (R_g(p), L_{g^{-1}}(f))$ for $p \in B_A(M)$, $f \in F$, $g \in Gl(n,R)$,

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[1] In discussions on the tangent bundle, the index $i*$, and others like it, take on values $n+1, \ldots, 2n$. The asterisk is used so that the summation convention may be applied with dissimilar index-domains, as in (14).

A_g the action by g on $B_A(M) \times F$, R_g the right action by g on $B_A(M)$, and L_g the left action by g on F. Let $K_p : F \to W$ by $K_p(f) = (p,f)Gl(n,R)$; then K_p is an isomorphism of the vector space F and the fibre $p_W^{-1}(\pi_B(p))$, and $K_{R_g(p)}(f) = K_p(L_g(f))$.

Now consider $b \in W$, $p \in B_A(M)$ such that $p_W(b) = \pi_B(p)$. The space H_b' is defined thus: let $\psi_f : B_A(M) \to B_A(M) \times F$ by $\psi_f(p) = (p,f)$, so that ψ_f is a diffeomorphism of $B_A(M)$ into $B_A(M) \times F$. If then, $f \in F$ be such that $K_p(f) = b$,

$$H_b' = \lambda_* \circ (\psi_f)_* H_p , \tag{15}$$

where $\lambda : B_A(M) \times F \to W$ is the natural projection [1; p. 84]. The function H' is called the <u>horizontal distribution</u> on W.

The horizontal distribution on W is spanned at each point b by tangents to horizontal lifts of curves in $T(M)$ through $p_W(b)$. These lifts will be described.

Let $b \in p_W^{-1}(\sigma(0))$, where $\sigma : I \to T(M)$ is a C^∞ curve. Choose $f \in F$ and $p \in B_A(M)$ so that $\pi_B(p) = \sigma(0)$ and $K_p(f) = b$. Let $\tilde{\sigma}$ be the horizontal lift of σ to $B_A(M)$ by H, passing through p. Then $\bar{\sigma} : I \to W$ is defined by

$$\bar{\sigma}(t) = K_{\tilde{\sigma}(t)}(f) . \tag{16}$$

The right invariance of H and the manner in which K_p behaves with respect to $Gl(n,R)$-action show $\bar{\sigma}$ to depend only on σ, b, and H. Now

$$\bar{\sigma}_*\left(\frac{d}{dt}\right) = \lambda_* \circ (\psi_f)_* \circ \tilde{\sigma}_* \left(\frac{d}{dt}\right)$$

since $K_{\tilde{\sigma}(t)}(f) = (\tilde{\sigma}(t), f)\, Gl(n,R) = \lambda \circ \psi_f(\tilde{\sigma}(t))$, and so $\bar{\sigma}$ is H'-horizontal.

The fibre-transversality of H' shows that the tangent space W_b of W at the point $b \in W$ decomposes into a direct sum:

$$W_b = H'_b \oplus V_b, \tag{17}$$

where V_b is the subspace of vertical vectors, that is, vectors $Z \in W_b$ such that $(p_W)_* Z = 0$. If $Z \in W_b$, write

$$Z = H'(Z) + V(Z), \tag{18}$$

where $H'(Z) \in H'_b$ and $V(Z) \in V_b$, for the decomposition of Z induced by (17).

One may now define $\overset{\Gamma}{\nabla}_{\tau(m,X)} Q$ by requiring that it measure how far Q fails to be horizontal in the "direction" τ:

$$\begin{aligned} \overset{\Gamma}{\nabla}_{\tau(m,X)} Q &= V(Q_*(\tilde{\tau}(m,X))) \\ &= Q_*(\tilde{\tau}(m,X)) - H'(Q_*(\tilde{\tau}(m,X))). \end{aligned} \tag{19}$$

Here, $\tilde{\tau}(m,X)$ is the Γ-lift of τ given by (13). Since the fibre of W over (m,X) is a vector space, the fibre may be identified with its tangent space, and (19) may be regarded as defining $\overset{\Gamma}{\nabla}_{\tau(m,X)} Q$ as an element of the fibre. Let $\overset{\Gamma}{\nabla}_{\tau(m,X)} Q$ be called the <u>horizontal covariant derivative</u> of Q in the direction τ.

One may define a second type of covariant derivative, independent of the non-linear support Γ, by employing the vertical lift of equation (14). Thus, one sets

$$\overset{\vee}{\nabla}_{\tau(m,X)} Q = V(Q_*(\bar{\tau}(m,X))), \tag{20}$$

where $\bar{\tau}(m,X)$ is the vertical lift of τ. This derivative will be referred to as the <u>vertical covariant derivative</u> of Q in the direction τ.

Since, with the aid of the non-linear support, one can define these covariant derivatives in strict analogy with the case of linear connexions (compare the foregoing with [1; p. 111]), a connexion H in η_M, <u>together with a non-linear support</u> Γ on $T(M)$, will be named a <u>DR-connexion</u> on M.[1]

6. Coordinate Description of DR-connexions

Let M be a C^∞ n-manifold with a DR-connexion (H,Γ), where H is a connexion in η_M and Γ a non-linear support on $T(M)$. Attention will now be restricted

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[1]As thus defined, the DR-connexion is identical with an entity studied by M. Matsumoto in, e.g., [9; pp. 1-6]. Matsumoto calls such structures "Finsler connections", and the spaces which carry them, "Finsler spaces", since the concept arose in the work of T. Okada on Finsler manifolds. Nonetheless, the terminology seems unfortunate, as "Finsler space" in the literature has signified a manifold with a Finsler metric. Note that a space with a linear connexion is not called a "Riemannian space", though such spaces stand in the same relationship with manifolds with Riemannian metric as do spaces with DR-connexion re Finsler spaces.

to neighborhoods over a coordinate patch U on M with coordinates (x^i).

Recall that the tangent spaces of $T(M)$ are spanned over $\pi^{-1}(U)$ by the vectors B_i and B_{i*} of (2.16) and (5.14). Consequently, the distribution H of the DR-connexion is spanned over $p_B{}^{-1}(U)$ by the horizontal lifts of B_i and B_{i*}. In terms of the coordinates (5.1) on $p_B{}^{-1}(U)$, write

$$(1)\qquad \begin{aligned} (\bar{B}_i)_p &= \frac{\partial}{\partial x^i} - \Gamma_i^h(m,X)\,\frac{\partial}{\partial \xi^h} - G_{ij}^h(p)\,\frac{\partial}{\partial \xi_j^h}, \\ (\bar{B}_{i*})_p &= \frac{\partial}{\partial \xi^i} - D_{ij}^h(p)\,\frac{\partial}{\partial \xi_j^h}, \end{aligned}$$

where $p = (m,X,e_1, \dots, e_n) \in p_B{}^{-1}(U) \subseteq B_A(M)$. The right invariance of H, and its fibre-transversality, imply that for $g = (g_j^i) \in Gl(n,R)$,

$$(2)\qquad (R_g)_*\,\bar{B}_i(p) = \bar{B}_i(R_g(p)).$$

From (1), and the definition (5.2) of R_g, (2) shows that

$$(3)\qquad G_{ij}^h(R_g(p)) = G_{is}^h(p)\; g_j^s.$$

Hence, the functions $G_{ij}^h : p_B{}^{-1}(U) \to R$ are linear in the basis argument, and one may write

$$(4)\qquad G_{ij}^h(p) = F_{is}^h(m,X)\;\xi_j^s(p).$$

A similar argument with the vectors B_{i*} shows that one may write

(5) $$D^h_{ij}(p) = C^h_{is}(m,X)\ \xi^s_j(p)\ ,$$

so that equations (1) become

$$(\bar{B}_i)_p = \frac{\partial}{\partial x^i} - \Gamma^h_i(m,X)\ \frac{\partial}{\partial \xi^h} - F^h_{is}(m,X)\ \xi^s_j(p)\ \frac{\partial}{\partial \xi^h_j}\ ,$$

(6)

$$(\bar{B}_{i*})_p = \frac{\partial}{\partial \xi^i} - C^h_{is}(m,X)\ \xi^s_j(p)\ \frac{\partial}{\partial \xi^h_j}\ .$$

The functions Γ^h_i, F^h_{is}, and C^h_{is} are called the <u>coefficients of the DR-connexion</u>. They coincide with those given by Matsumoto [9; p. 4].

From the definition of the coordinate structure in $p_B^{-1}(U)$, if a vector V_p tangent to $B_A(M)$ at $p=(m,X,e_1, \ldots, e_n)$ is given by

(7) $$V_p = \alpha^i(p)\ \frac{\partial}{\partial x^i} + \beta^i(p)\ \frac{\partial}{\partial \xi^i} + \gamma^i_j(p)\ \frac{\partial}{\partial \xi^i_j}\ ,$$

and if $p \in p_B^{-1}\ (U \cap W)$, where W is another coordinate patch on M with coordinates (y^i) and induced coordinates $(y^i,\ \eta^i,\ \eta^i_j)$ on $p_B^{-1}(W)$, then V_p is given by

$$V_p = \alpha^i(p)\ \frac{\partial y^j}{\partial x^i}\ \frac{\partial}{\partial y^j} + [\alpha^k(p)\xi^j\ \frac{\partial^2 y^i}{\partial x^k\ \partial x^j} + \beta^k(p)\ \frac{\partial y^i}{\partial x^k}]\frac{\partial}{\partial \eta^i}$$

(8)

$$+ [\alpha^k(p)\xi^t_j\ \frac{\partial^2 y^i}{\partial x^t\ \partial x^k} + \gamma^k_j(p)\ \frac{\partial y^i}{\partial x^k}]\frac{\partial}{\partial \eta^i_j}\ .$$

If, over $p_B^{-1}(W)$, the coefficients of the DR-connexion are given by $\bar{\Gamma}_i^h$, $\bar{F}_{is}^h$, and $\bar{C}_{is}^h$, the transformation equation (8) shows that one must have, for example:

(9)
$$\bar{F}_{is}^h(m,X)\ \eta_j^s = \frac{\partial x^t}{\partial y^i}[F_{tk}^q(m,X)\ \xi_j^k\ \frac{\partial y^h}{\partial x^q} - \xi_j^k\ \frac{\partial^2 y^h}{\partial x^k\ \partial x^t}]\,.$$

which follows upon applying (8) to the first of equations (6). Therefore,

(10)
$$\bar{F}_{is}^h(m,X)\ \frac{\partial y^s}{\partial x^q} = F_{tq}^k(m,X)\ \frac{\partial y^h}{\partial x^k}\ \frac{\partial x^t}{\partial y^i} - \frac{\partial^2 y^h}{\partial x^q\ \partial x^t}\ \frac{\partial x^t}{\partial y^i}\,.$$

Similarly, one has the transformation equations

(11)
$$\bar{C}_{is}^n(m,X) = C_{tq}^k(m,X)\ \frac{\partial x^t}{\partial y^i}\ \frac{\partial x^q}{\partial y^s}\ \frac{\partial y^h}{\partial x^k}$$

and

(12)
$$\bar{\Gamma}_q^i(m,X) = \Gamma_s^k(m,X)\ \frac{\partial x^s}{\partial y^q}\ \frac{\partial y^i}{\partial x^k} - \frac{\partial^2 y^i}{\partial x^j\ \partial x^s}\ \xi^j\ \frac{\partial x^s}{\partial y^q}\,.$$

Note that (11) shows C_{ij}^h to be the components of a DR-tensor, which will be denoted by C, and that (12) agrees with (2.8) — it would be quite upsetting if it did not!

Clearly, given functions Γ_i^j, F_{ik}^h, and C_{ik}^h on each patch U of a coordinate cover of M, with values in R, which satisfy the transformation equations (12), (10), and (11) respectively, a DR-connexion on M is uniquely determined.

Next, a coordinate expression for $\overset{\Gamma}{\nabla}_Y Q$ and $\overset{v}{\nabla}_Y Q$ will be constructed, where Y and Q are DR-vector fields on M. The local form of the horizontal lift of a vector $X = X^i B_i + X^{i*} B_{i*}$ in $(T(M))_{\pi_B(p)}$ to $\bar{X} \in (B_A(M))_p$ will be first determined. Since the lifting operation is an isomorphism, $\bar{X} = X^i \bar{B}_i + X^{i*} \bar{B}_{i*}$, so one has

$$\bar{X} = X^i \frac{\partial}{\partial x^i} + (X^{i*} - X^k \Gamma_k^i) \frac{\partial}{\partial \xi^i}$$

(13)

$$- (X^i F_{ij}^k + X^{i*} C_{ij}^k + X^i \Gamma_i^h C_{hj}^k) \xi_s^j \frac{\partial}{\partial \xi_s^k},$$

where functions on the tangent bundle are all evaluated at $\pi_B(p)$.

Now, recall from section 5 that the horizontal subspace at $q \in T^2(M)$ is spanned by tangents of horizontal lifts of curves; this fact will be used to obtain the local form of the horizontal lift of a vector $X = \alpha^i \partial/\partial x^i + \beta^i \partial/\partial \xi^i$ on $T(M)$ to $T^2(M)$. In fact, let $\gamma : I \to T(M)$ be a curve in $T(M)$ with tangent X at $\gamma(0)$. Choose $p = (\gamma^i(0), \gamma^{i*}(0), \partial/\partial x^1, \ldots, \partial/\partial x^n) \in B_A(M)$, so that $\pi_B(p) = \gamma(0)$. Then $K_p(f) = (p,f)\ Gl(n,R)$ in $T^2(M)$ for $f = (f^1, \ldots, f^n) \in R^n$; in local coordinates, one has

(14) $$K_p(f) = (\gamma^i(0), \gamma^{i*}(0), f^i).$$

Say $q = (\gamma^i(0), \gamma^{i*}(0), \eta^i) \in \bar{\pi}^{-1}(\gamma(0)) \subseteq T^2(M)$. Then by (14), $K_p(f) = q$ iff $f = (\eta^1, \ldots, \eta^n)$. So if $\tilde{\gamma}$ be the horizontal lift of γ to $B_A(M)$ through p, then the horizontal lift $\bar{\gamma}$ of γ to $T^2(M)$ through q is

$\bar{\gamma}(t) = K_{\tilde{\gamma}(t)}(f)$, from section 5, with $f = (\eta^1, \ldots, \eta^n)$. Write

(15) $$\tilde{\gamma}(t) = (\gamma^i(t),\ \gamma^{i*}(t),\ \theta_1^i(t)\ \frac{\partial}{\partial x^i}, \ldots, \theta_n^i(t)\ \frac{\partial}{\partial x^i}) .$$

Then

(16) $$K_{\tilde{\gamma}(t)}(f) = (\gamma^i(t),\ \gamma^{i*}(T),\ \eta^j\ \theta_j^i(t))$$

in local coordinates, from (14). The desired lift of X is $\bar{\gamma}_*(d/dt)_0$. From (16), one sees that

(17) $$\frac{d}{dt}(x^i \circ \bar{\gamma}) = \frac{d\gamma^i}{dt},$$
$$\frac{d}{dt}(\xi^i \circ \bar{\gamma}) = \frac{d\gamma^{i*}}{dt},$$
$$\frac{d}{dt}(\eta^i \circ \bar{\gamma}) = \eta^j\ \frac{d}{dt}\ \theta_j^i .$$

At $t = 0$, $d\gamma^i/dt = \alpha^i$, $d\gamma^{i*}/dt = \beta^i$, and $(d\theta_j^i/dt)$ are the last n^2 components of the horizontal lift to $p \in B_A(M)$ of the vector X; from (13):

(18) $$(d\theta_j^k/dt)_{t=0} = -[\alpha^i\ F_{ij}^k + \beta^i\ C_{ij}^k + \alpha^i\ \Gamma_i^h\ C_{hj}^k] ,$$

where the choice of p made set $\xi_j^s(p) = \delta_j^s$. Then from (17) and (18), it is seen that the required lift to $q = (m,Z,Y) \in T^2(M)$ of X at $(m,Z) \in T(M)$ is

(19) $$\overline{X}_q = \alpha^i \frac{\partial}{\partial x^i} + \beta^i \frac{\partial}{\partial \xi^i} - \eta^j(\alpha^i F^k_{ij} + \beta^i C^k_{ij} + \alpha^i \Gamma^h_i C^h_{hj}) \frac{\partial}{\partial \eta^k},$$

where all functions in the tangent bundle are evaluated at (m,Z).

In particular, (19) shows that the horizontal subspace at $q \in T^2(M)$ is spanned by the vectors:

(20) $$S_i = \frac{\partial}{\partial x^i} - \eta^j (F^k_{ij} + \Gamma^h_i C^k_{hj}) \frac{\partial}{\partial \eta^k},$$

$$S_{i*} = \frac{\partial}{\partial \xi^i} - \eta^j C^k_{ij} \frac{\partial}{\partial \eta^k},$$

where $q = (m,Z,Y)$, $Y = \eta^i \partial/\partial x^i$, and the tangent bundle functions are evaluated at (m,Z).

Now suppose $Q : \overline{U} \to T^2(M)$ is a DR-vector field, where $\overline{U} = \pi^{-1}(U) \subseteq T(M)$, and Q is given locally by

(21) $$Q(m,X) = (x^i, \xi^i, Q^i(m,X)),$$

while $Y = Y^i(m,X)\, \partial/\partial x^i$ is another such field. The Γ-lift of Y is given by

(22) $$\tilde{Y}(m,X) = Y^i(m,X) \frac{\partial}{\partial x^i} - Y^j(m,X)\, \Gamma^i_j(m,X) \frac{\partial}{\partial \xi^i}.$$

From (21) and (22) one sees that

$$(Q_*(\tilde{Y}))_{(m,X,Q)} = Y^i(m,X)\frac{\partial}{\partial x^i} - Y^j(m,X)\,\Gamma^i_j(m,X)\frac{\partial}{\partial \xi^i}$$

(23)

$$+ Y^j(m,X)\left[\frac{\partial Q^i}{\partial x^j} - \Gamma^k_j(m,X)\frac{\partial Q^i}{\partial \xi^k}\right]\frac{\partial}{\partial \eta^i}.$$

This may be expressed in terms of the basis $(S_i, S_{i*}, \partial/\partial\eta^j)$ of the tangent space to $T^2(M)$ at $(m,X,Q(m,X))$:

$$Q_*(\tilde{Y}) = Y^i S_i + Y^h\,\Gamma^i_h\,S_{i*}$$

(24)

$$+ Y^j\left[\frac{\partial Q^i}{\partial x^j} - \Gamma^k_j\frac{\partial Q^i}{\partial \xi^k} + Q^h\,F^{\;i}_{jh}\right]\frac{\partial}{\partial \eta^i},$$

where the tangent bundle functions are evaluated at (m,X). One can now pick off the vertical part of $Q_*(\tilde{Y})$ from equation (24), as this is simply the last term in the right member of (24). Making the identification of the vertical fibre and its tangent space, one has

(25) $$\overset{\Gamma}{\nabla}_{Y(m,X)}\,Q = Y^j(m,X)\left\{\frac{\partial Q^i}{\partial x^j} - \Gamma^k_j\frac{\partial Q^i}{\partial \xi^k} + Q^h\,F^{\;i}_{jh}\right\}\frac{\partial}{\partial x^i}.$$

The local expression (25) shows that $\overset{\Gamma}{\nabla}$ so defined coincides with the "absolute covariant derivative" of Matsumoto [11; p. 364].

If the vertical lift $\overline{Y}$ is used in place of the Γ-lift Y in the computations above, one obtains

(26) $$Q_*(\overline{Y}) = Y^i(m,X)\,S_{i*} + \left(Y^j\frac{\partial Q^i}{\partial \xi^j} + Y^j Q^k\,C^{\;i}_{jk}\right)\frac{\partial}{\partial \eta^i},$$

whence

(27) $$\overset{\vee}{\nabla}_{Y(m,X)} Q = Y^j(m,X)\{\frac{\partial Q^i}{\partial \xi^j} + Q^k(m,X)\ C^{\ i}_{jk}(m,X)\}\frac{\partial}{\partial x^i}.$$

7. Invariant Characterization, Curvature, and Torsion of $\overset{\Gamma}{\nabla}$ and $\overset{\vee}{\nabla}$.

Having derived the covariant derivatives $\overset{\Gamma}{\nabla}$ and $\overset{\vee}{\nabla}$ associated with a DR-connexion $(H,\underline{\Gamma})$ on a C^∞ n-manifold M, one finds it natural to ask to what extent specifying the values of such derivatives determines the connexion. It turns out that, provided the non-linear support be specified independently, the values of the derivatives determine the connexion completely.

From the local representation (6.25) of $\overset{\Gamma}{\nabla}$, one sees immediately that

(1) $$\overset{\Gamma}{\nabla}_{Y+Z}\, Q = \overset{\Gamma}{\nabla}_Y\, Q + \overset{\Gamma}{\nabla}_Z\, Q,$$

(2) $$\overset{\Gamma}{\nabla}_Y\,(P+Q) = \overset{\Gamma}{\nabla}_Y P + \overset{\Gamma}{\nabla}_Y\, Q,$$

and

(3) $$\overset{\Gamma}{\nabla}_{fY}\, Q = f\ \overset{\Gamma}{\nabla}_Y\, Q,$$

where P, Q, Y, and Z are C^∞ DR-vector fields on M and f is a C^∞ R-valued function on T(M). Also,

$$\overset{\Gamma}{\nabla}_Y fQ = Y^j[Q^i \frac{\partial f}{\partial x^j} + f \frac{\partial Q^i}{\partial x^j} - \Gamma_j^k f \frac{\partial Q^i}{\partial \xi^k} - \Gamma_j^k Q^i \frac{\partial f}{\partial \xi^k}$$

$$+ f\, Q^h F_{jh}^{\,i}]\frac{\partial}{\partial x^i},$$

so that

(4)
$$\overset{\Gamma}{\nabla}_Y fQ = f \; \overset{\Gamma}{\nabla}_Y Q + (\tilde{Y}f)Q\,,$$

where $\tilde{Y}$ is the Γ-lift of Y.

Conversely, suppose a map $\overset{\Gamma}{\nabla} : \mathcal{D}(M) \times \mathcal{D}(M) \to \mathcal{D}(M)$ be given, where $\mathcal{D}(M)$ denotes the totality of C^∞ DR-vector fields on M, so that $\overset{\Gamma}{\nabla}$ satisfies (1) – (4). Introduce C^∞ functions $F_{jk}^{\,i} : \pi^{-1}(U) \to R$, where U is a coordinate patch on M with coordinates (x^i), by

(5)
$$\left(\overset{\Gamma}{\nabla}_{\partial/\partial x^i}\left(\frac{\partial}{\partial x^j}\right)\right)_{(m,X)} = F_{ij}^{\,k}(m,X)\, \frac{\partial}{\partial x^k}.$$

Then if $Q = Q^i(m,X)\, \partial/\partial x^i$ and $Y = Y^i(m,X)\, \partial/\partial x^i$ are DR-vector fields, using (1) – (5) results in:

$$\overset{\Gamma}{\nabla}_Y Q = \overset{\Gamma}{\nabla}_{Y^i \frac{\partial}{\partial x^i}} Q^j \frac{\partial}{\partial x^j}$$

$$= Y^i[\frac{\delta}{\partial x^i}\,(Q^j)\, \frac{\partial}{\partial x^j} + Q^j \;\overset{\Gamma}{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}]$$

or

(6)
$$\overset{\Gamma}{\nabla}_Y Q = Y^i[\frac{\partial Q^k}{\partial x^i} - \Gamma_i^h \frac{\partial Q^k}{\partial \xi^h} + Q^j F_{ij}^{\,k}]\frac{\partial}{\partial x^k}.$$

Next, from the local representation (6.27) of $\overset{\vee}{\nabla}_Y Q$, one has immediately that

(7) $$\overset{\vee}{\nabla}_{Y+Z} Q = \overset{\vee}{\nabla}_Y Q + \overset{\vee}{\nabla}_Z Q,$$

(8) $$\overset{\vee}{\nabla}_Y (P+Q) = \overset{\vee}{\nabla}_Y P + \overset{\vee}{\nabla}_Y Q,$$

and

(9) $$\overset{\vee}{\nabla}_{fY} Q = f \overset{\vee}{\nabla}_Y Q,$$

where Y, Z, P, and Q are C^∞ DR-vector fields and $f: T(M) \to R$ is C^∞. Further,

$$\overset{\vee}{\nabla}_Y f Q = Y^j[f \frac{\partial Q^i}{\partial \xi^j} + Q^i \frac{\partial f}{\partial \xi^j} + f Q^k C^{\,i}_{jk}]\frac{\partial}{\partial x^i}$$

or

(10) $$\overset{\vee}{\nabla}_Y f Q = (\overline{Y}f) Q + f \overset{\vee}{\nabla}_Y Q,$$

where $\overline{Y}$ is the vertical lift of Y. Conversely, suppose that a map $\overset{\vee}{\nabla}: \mathfrak{D}(M) \times \mathfrak{D}(M) \to \mathfrak{D}(M)$ be given, satisfying (7) - (10). As above, introduce C^∞ maps $C^{\,i}_{jk} : \pi^{-1}(U) \to R$ by

(11) $$(\overset{\vee}{\nabla}_{\partial/\partial x^i} (\frac{\partial}{\partial x^j}))_{(m,X)} = C^{\,k}_{ij}(m,X) \frac{\partial}{\partial x^k}.$$

Then, if Q and Y are given as above, one uses (7) - (11) to compute

$$\overset{\vee}{\nabla}_Y Q = Y^i[\overline{\frac{\partial}{\partial x^i}}(Q^j)\frac{\partial}{\partial x^j} + Q^j \overset{\vee}{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}]$$

or

(12) $$\overset{\vee}{\nabla}_Y Q = Y^i[\frac{\partial Q^k}{\partial \xi^i} + Q^j C^k_{ij}]\frac{\partial}{\partial x^k}.$$

Further, equations (5) and (11) show F^k_{ij} and C^k_{ij} to have appropriate transformation properties, and comparing the results above with those of section 6, one has

<u>Theorem</u> <u>1</u>: If mappings $\overset{\Gamma}{\nabla}$, $\overset{\vee}{\nabla} : \mathfrak{D}(M) \times \mathfrak{D}(M) \to \mathfrak{D}(M)$ be given, satisfying (1) - (4) and (7) - (10) respectively, a non-linear support on $T(M)$ having been specified in advance, then there exists a unique DR-connexion on M with respect to which $\overset{\Gamma}{\nabla}$ is the horizontal, and $\overset{\vee}{\nabla}$ the vertical, covariant derivative.

Using this invariant characterization, it is a simple matter to define and test curvature and torsion DR-tensors for a DR-connexion. One has the <u>horizontal torsion DR-tensor</u> $\overset{\Gamma}{T}$, defined by

(13) $$\overset{\Gamma}{T}(X,Y) = \overset{\Gamma}{\nabla}_X Y - \overset{\Gamma}{\nabla}_Y X - [X,Y]$$

and the <u>vertical torsion DR-tensor</u> $\overset{\vee}{T}$, defined by

(14) $$\overset{\vee}{T}(X,Y) = \overset{\vee}{\nabla}_X Y - \overset{\vee}{\nabla}_Y X,$$

where X and Y are C^∞ vector fields on M. The tensor character of $\overset{\Gamma}{T}$ and $\overset{\vee}{T}$ is trivial to check, using the properties of $\overset{\Gamma}{\nabla}$ and $\overset{\vee}{\nabla}$. For example, if $f : M \to R$ is C^∞, then

$$\overset{\Gamma}{T}(fX,Y) = \overset{\Gamma}{\nabla}_{fX} Y - \overset{\Gamma}{\nabla}_Y fX - [fX,Y]$$

$$= f\,\overset{\Gamma}{\nabla}_X Y - f\,\overset{\Gamma}{\nabla}_Y X - (\tilde{Y} f)X + (Yf)X - f[X,Y]$$

$$= f\,\overset{\Gamma}{T}(X,Y)\,,$$

since $\tilde{Y} f = Y f$, as f is direction-independent.

It is important to be aware that the DR-vector valued DR-tensors $\overset{\Gamma}{T}$ and $\overset{\vee}{T}$ operate on vectors on M; in particular, (13) is meaningless if X and Y are (non-trivially) DR-vector fields, since $\mathcal{D}(M)$ is not a Lie algebra.

If one writes $\overset{\Gamma}{T}(\partial/\partial x^i, \partial/\partial x^j) = \overset{\Gamma}{T}{}^k_{ij}(m,X)(\partial/\partial x^k)$ over a coordinate patch U on M, then

$$\overset{\Gamma}{T}{}^k_{ij} = \overset{\Gamma}{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \overset{\Gamma}{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$$

or

(15) $$\overset{\Gamma}{T}{}^k_{ij}(m,X) = F^k_{ij}(m,X) - F^k_{ji}(m,X)\,.$$

Similarly, if $\overset{\vee}{T}(\partial/\partial x^i, \partial/\partial x^j) = \overset{\vee}{T}{}^k_{ij}(m,X)\,\partial/\partial x^k$ then

(16) $$\overset{\vee}{T}{}^k_{ij}(m,X) = C^k_{ij}(m,X) - C^k_{ji}(m,X)\,,$$

after the classical fashion.

One has also three curvature DR-tensors; these have values in the (linear) space of linear transformations on M_m.

The first of these is the <u>horizontal curvature DR-tensor</u> $\overset{\Gamma}{R}$, defined by

(17) $$\overset{\Gamma}{R}(X,Y)\,Z = \overset{\Gamma}{\nabla}_X \overset{\Gamma}{\nabla}_Y Z - \overset{\Gamma}{\nabla}_Y \overset{\Gamma}{\nabla}_X Z - \overset{\Gamma}{\nabla}_{[X,Y]} Z,$$

where X, Y, and Z are C^∞ vector fields on M. Also, there is the <u>vertical curvature DR-tensor</u> $\overset{\vee}{R}$ defined by

(18) $$\overset{\vee}{R}(X,Y)\,Z = \overset{\vee}{\nabla}_X \overset{\vee}{\nabla}_Y Z - \overset{\vee}{\nabla}_Y \overset{\vee}{\nabla}_X Z.$$

The third curvature DR-tensor involves both $\overset{\Gamma}{\nabla}$ and $\overset{\vee}{\nabla}$, and will be called the mixed curvature DR-tensor, denoted by $\overset{\circ}{R}$. Recall that the DR-vector valued DR-tensor C is given by

(19) $$C_{(m,Z)}\,(X,Y) = C^k_{ij}(m,Z)\,X^i\,Y^j\,\frac{\partial}{\partial x^k}.$$

Also, introduce a map $\hat{\nabla}$ which sends a pair of C^∞ vector fields X and Y on M to a DR-vector field by

(20) $$(\hat{\nabla}_Y X)_{(m,Z)} = Y^i\left\{\frac{\partial X^j}{\partial x^i} + \frac{\partial \Gamma^j_i}{\partial \xi^k}(m,Z)\,X^k\right\}\frac{\partial}{\partial x^j}.$$

Then the mixed curvature DR-tensor is given by

(21) $$\overset{\circ}{R}(X,Y)\,Z = \overset{\vee}{\nabla}_X \overset{\Gamma}{\nabla}_Y Z - \overset{\Gamma}{\nabla}_Y \overset{\vee}{\nabla}_X Z + C(\hat{\nabla}_Y X,\, Z).$$

It is a simple matter to check the multilinearity of $\overset{\Gamma}{R}$, $\overset{\vee}{R}$, and $\overset{\circ}{R}$ over the $C^\infty(M,R)$-module of C^∞ vector fields on M, using the properties of $\overset{\Gamma}{\nabla}$ and $\overset{\vee}{\nabla}$. The DR-tensor nature of these entities is also evidenced by their expression in local coordinates. By entirely straightforward, but remarkably tedious, computation, one has over a coordinate patch U on M the equations:

(22)
$$\overset{\Gamma}{R}(X,Y)\,Z = X^k Y^i Z^j \Big\{ \frac{\partial F_i{}^p{}_j}{\partial x^k} - \frac{\partial F_k{}^p{}_j}{\partial x^i} - \Gamma_k^t \frac{\partial F_i{}^p{}_j}{\partial \xi^t} + \Gamma_i^t \frac{\partial F_k{}^p{}_j}{\partial \xi^t} + F_{ij}^{\,t} F_{kt}^{\,p} - F_{kj}^{\,t} F_{it}^{\,p} \Big\} \frac{\partial}{\partial x^k},$$

(23)
$$\overset{\vee}{R}(X,Y)\,Z = X^p Y^j Z^k \Big\{ \frac{\partial C_j{}^i{}_k}{\partial \xi^p} - \frac{\partial C_p{}^i{}_k}{\partial \xi^j} \; C_{jk}^{\,t} C_{pt}^{\,i} - C_{pk}^{\,t} C_{jt}^{\,i} \Big\} \frac{\partial}{\partial x^i},$$

and

(24)
$$\overset{\circ}{R}(X,Y)\,Z = X^i Y^j Z^h \Big\{ \Gamma_{ji}^{\,t} C_{th}^{\,k} + \frac{\partial F_{jh}^{\,k}}{\partial \xi^i} - \frac{\partial C_{ih}^{\,k}}{\partial x^j} + \Gamma_j^s \frac{\partial C_{ih}^{\,k}}{\partial \xi^s} + C_{is}^{\,k} F_{jh}^{\,s} - C_{ih}^{\,s} F_{js}^{\,k} \Big\} \frac{\partial}{\partial x^k},$$

where $\Gamma^t_{ji} = \partial\Gamma^t_j/\partial\xi^i$. The quantities in curly brackets in (22), (23), and (24) will be denoted by $\overset{\Gamma}{R}{}^p{}_{kij}$, $\overset{\vee}{R}{}^i{}_{pjk}$, and $\overset{\circ}{R}{}^k{}_{ijh}$, respectively.

Some of the algebraic significance of classical curvature tensors is lost in the more general setting of DR-connexions, which is, of course, only to be expected. In particular, if D is a linear connexion on M, its curvature tensor measures how far D fails to be a Lie algebra homomorphism [1; p. 116]. No such interpretation can be placed on $\overset{\Gamma}{R}$, $\overset{\vee}{R}$, or $\overset{\circ}{R}$, since $\overset{\Gamma}{\nabla}$ and $\overset{\vee}{\nabla}$ are not Lie algebra valued maps.

8. DR-connexions as Distributions on $T^2(M)$

In this section begins what is the primary function of this chapter: the investigation of the rôle played by $T^2(M)$ in the theory of DR-connexions. In section 5, it was seen how a DR-connexion determines a C^∞, 2n-dimensional, fibre-transversal distribution on $T^2(M)$, just as a (linear) connexion in the bundle of bases determines a distribution in the tangent bundle. The converse is also true, provided that a non-linear support be specified, as is shown in this section.

Let M be, as usual, a C^∞ n-manifold, T(M) and $T^2(M)$ denoting the usual spaces. Let H be a C^∞ 2n-distribution on $T^2(M)$, transversal to fibres over T(M) and invariant under the action of Gl(n,R). The fibre-transversality implies that vector fields on $T(M)$ can be lifted to vector fields on $T^2(M)$ in a unique manner, via the inverse of the restriction to H of the Jacobian of the projection $\bar{\pi} : T^2(M) \to T(M)$.

The vertical spaces on $T^2(M)$ are spanned over a coordinate patch U on M by the vectors $\partial/\partial\eta^k$, $k = 1, \ldots, n$, as in section 4, and H is spanned over U by the lifts of the coordinate vectors $(\partial/\partial x^i)$ and $(\partial/\partial\xi^i)$ on $\pi^{-1}(U)$. These lifts may be written

$$L_i(m,X,Y) = \frac{\partial}{\partial x^i} - G_i^k(m,X,Y)\,\frac{\partial}{\partial\eta^k},$$

(1)

$$K_i(m,X,Y) = \frac{\partial}{\partial\xi^i} - D_i^k(m,X,Y)\,\frac{\partial}{\partial\eta^k}.$$

The requirement that H be invariant under left action is

(2) $$(L_g)_* \, H_{(m,X,Y)} = H_{(m,X,gY)}\,.$$

In terms of L_i and K_i, this means that

$$(L_g)_* \, L_i(m,X,Y) = L_i(m,X,gY)\,,$$

(3)

$$(L_g)_* \, K_i(m,X,Y) = K_i(m,X,gY)\,,$$

since "lifting" and $(L_g)_*$ are isomorphisms. Now

$$(L_g)_* \, G_i^k(m,X,Y) \frac{\partial}{\partial\eta^k} = G_i^k(m,X,Y) \frac{\partial}{\partial\eta^k} (\eta^t \circ L_g) \frac{\partial}{\partial\eta^t}$$

(4)
$$= G_i^k(m,X,Y) \frac{\partial}{\partial\eta^k} (g_s^t \, \eta^s) \frac{\partial}{\partial\eta^t},$$

$$= G_i^k(m,X,Y) \, g_k^t \frac{\partial}{\partial\eta^t}.$$

It follows from (1), (3), and (4) that

$$G_i^k(m,X,gY) = G_i^t(m,X,Y) \, g_t^k,$$

and the functions G_i^k are linear in the third slot, so one may write

(5)
$$G_i^k(m,X,Y) = H_{ij}^k(m,X) \, \eta^j$$

over U, where $Y = \eta^i \, \partial/\partial x^i$. Similarly, one arrives at linearity in the third slot of D_i^k, and writes

(6)
$$D_i^k(m,X,Y) = C_{ij}^k(m,X) \, \eta^j.$$

Consider now the lift of the vector B_i of (2.16), where a non-linear support Γ is supposed given with coefficients Γ_j^i over U. This lift is

(7)
$$L_i - \Gamma_i^h \, K_h = \frac{\partial}{\partial x^i} - \Gamma_i^h \frac{\partial}{\partial \xi^h} - (H_{ij}^k - \Gamma_i^h \, C_{hj}^k) \, \eta^j \frac{\partial}{\partial\eta^k}.$$

Define functions $F_{ij}^k : \pi^{-1}(U) \to R$ by

(8)
$$F_{ij}^k(m,X) = H_{ij}^k(m,X) - \Gamma_i^h(m,X) \, C_{hj}^k(m,X).$$

From the fact that L_i and K_i are vector fields over $\pi_2^{-1}(U)$, and from the defining equation (8), it follows that Γ_i^h, F_{ij}^k, and C_{ij}^k transform properly, and may be taken as the coefficients of a DR-connexion on M. Further, the horizontal distribution H' of this connexion is spanned over U by the vectors

$$S_i = \frac{\partial}{\partial x^i} - \eta^j(F_{ij}^k + \Gamma_i^h C_{hj}^k) \frac{\partial}{\partial \eta^k}$$

(9)

$$= \frac{\partial}{\partial x^i} - \eta^j H_{ij}^k \frac{\partial}{\partial \eta^k}$$

and

(10)
$$S_{i*} = \frac{\partial}{\partial \xi^i} - \eta^i C_{ij}^k \frac{\partial}{\partial \eta^k}$$

as given by equations (6.20). But, from (1), one sees that $L_i = S_i$ and $K_i = S_{i*}$, so $H' = H$. The explicit form of the DR-connexion which it was necessary to take shows it to be uniquely determined, and one has

<u>Theorem 1</u>: Given a non-linear support Γ on the tangent bundle $T(M)$ of a C^∞ n-manifold M, a DR-connexion on M is uniquely determined by a C^∞, 2n-dimensional, fibre-transversal distribution on $T^2(M)$, invariant under $Gl(n,R)$-action.

9. Paths Relative to a DR-connexion

Makoto Matsumoto has studied various curves, most of them curves in the tangent bundle, which are associated with a DR-connexion [10]. In this section are proved, for some of these curves, theorems analogous to the well-known result that a curve $\gamma : I \to M$ in a manifold with a linear connexion is a geodesic iff its natural lift $\gamma_* : I \to T(M) :: \gamma_*(t) = (\gamma(t), \gamma_*(d/dt))$ is horizontal (see, e.g., [23; pp. 290-291]). Also, the canonical paths introduced in section 5 are studied in greater detail. Indeed, the study begins at that point, with

<u>Theorem 1</u>: If a curve $\gamma : I \to M$ is a canonical path, then in a coordinate patch U on M with coordinates (x^i), and $x^i(t) = x^i \circ \gamma(t)$, the curve satisfies the differential equations:

$$(1) \quad \frac{d^2x^k}{dt^2} + \frac{d^2x^i}{dt^2}\frac{dx^j}{dt} C^{\,k}_{ij} + (F^{\,k}_{ij} + \Gamma^h_i\, C^{\,k}_{hj}) \frac{dx^i}{dt}\frac{dx^j}{dt} = 0 ,$$

where the tangent bundle functions are evaluated at $(\gamma(t), \gamma_*(d/dt))$.

<u>Proof</u>: Let $\gamma_* : I \to T(M)$ be the natural lift of γ to $T(M)$, as above, and let

$$\overline{\gamma}_* : I \to B_A(M) :: t \to (\gamma(t),\ \gamma_*(d/dt),$$

$$\theta_1^i(t)\ \frac{\partial}{\partial x^i},\ \ldots,\ \theta_n^i(t)\ \frac{\partial}{\partial x^i})$$

be a horizontal lift of γ_*. define C^∞ R-valued maps $\overline{\theta}_j^i$ by

(2) $$\overline{\theta}_j^i\ \theta_k^j = \delta_k^i .$$

Then one has, for $u \in I$,

(3) $$\gamma_*(\frac{d}{dt})\ (u) = \left.\frac{dx^i}{dt}\right|_{t=u}\ \overline{\theta}_i^j(u)\ \theta_j^k(u)\ \left.\frac{\partial}{\partial x^k}\right|_{\gamma(u)} .$$

Since γ is a canonical path, the vector $\gamma_*(d/dt)(v)$, for $v \in I$, must result from $\gamma_*(d/dt)(u)$ by canonical parallel translation:

(4) $$\gamma_*(\frac{d}{dt})\ (v) = \left.\frac{dx^i}{dt}\right|_{t=u}\ \overline{\theta}_i^j(u)\ \theta_j^k(v)\ \left.\frac{\partial}{\partial x^k}\right|_{\gamma(v)} .$$

But

(5) $$\gamma_*(\frac{d}{dt})(v) = \left.\frac{dx^k}{dt}\right|_{t=v}\ \left.\frac{\partial}{\partial x^k}\right|_{\gamma(v)} ,$$

so that from (4) and (5), one obtains

(6) $$\left.\frac{dx^k}{dt}\right|_{t=v} = \left.\frac{dx^i}{dt}\right|_{t=u}\ \overline{\theta}_i^j(u)\ \theta_j^k(v) .$$

Differentiating (6) with respect to v yields

$$\left.\frac{d^2x^k}{dt^2}\right|_{t=v} = \left.\frac{dx^i}{dt}\right|_{t=u} \bar{\theta}^j_i(u) \left(\frac{d}{dt}\theta^k_j\right)(v) . \tag{7}$$

Equation (7) holds in the limit as $v \to u$ in the form

$$\left.\frac{d^2x^k}{dt^2}\right|_{u} = \left.\frac{dx^i}{dt}\right|_{u} \bar{\theta}^j_i(u) \left(\frac{d}{dt}\theta^k_j\right)(u) , \tag{8}$$

since (7) must hold for all $u, v \in I$, and all the functions involved are C^∞.

Now the horizontal lift of $(\gamma_*)_*(d/dt)$ to $\bar{\gamma}_*(t)$ is given by the vector P_t:

$$P_t = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{d^2x^i}{dt^2}\frac{\partial}{\partial \xi^i} - \left[\frac{dx^i}{dt} F^k_{ij} + \frac{d^2x^i}{dt^2} C^k_{ij} + \frac{dx^h}{dt} C^k_{hj}\right] \theta^j_s(t) \frac{\partial}{\partial \xi^k_s}, \tag{9}$$

where the tangent bundle functions are evaluated at $\gamma_*(t)$. Since $\bar{\gamma}_*$ is horizontal, $(\bar{\gamma}_*)_* (d/dt) = P_t$, and since

$$(\bar{\gamma}_*)_* \left(\frac{d}{dt}\right) = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{d^2x^i}{dt^2}\frac{\partial}{\partial \xi^i} + \left(\frac{d}{dt}\theta^i_j\right)(t)\frac{\partial}{\partial \xi^i_j}, \tag{10}$$

one has, on using (9),

$$\left(\frac{d}{dt}\theta^k_s\right)(t) = -\left(\frac{dx^i}{dt} F^k_{ij} + \frac{d^2x^i}{dt^2} C^k_{ij} + \frac{dx^h}{dt} C^k_{hj}\right) \theta^j_s(t) ,$$

or

$$\text{(11)} \qquad \bar{\theta}^s_j(t)\,\left(\frac{d}{dt}\,\theta^k_s\right)(t) = -\left(\frac{dx^i}{dt}\,F^{\,k}_{ij} + \frac{d^2x^i}{dt^2}\,C^{\,k}_{ij} + \frac{dx^i}{dt}\,C^{\,k}_{ij}\right).$$

On substituting the expression (11) in (8), one obtains the desired equations (1). Q.E.D.

Unfortunately, the converse of Theorem 1 does not appear to hold, in general, because the rather complex involvement of the quantities d^2x^i/dt^2 in (1) indicates that solutions of (1) may not be unique (or even exist!), and therefore, the gap between equations (8) and (6) cannot be bridged by the usual sort of uniqueness argument. Also, one cannot conclude from geometric considerations that there is a unique canonical path through each point in M in each direction, and equations (1) provide no help in this regard. This is <u>not</u> at all surprizing, however; one encounters here once again the difficulties that prevent the use of canonical parallel translation in defining a covariant derivative.

A possible way out of some of these difficulties is suggested by

<u>Theorem</u> <u>2</u>: If a curve $\gamma : I \to M$ in a C^∞ n-manifold M is a canonical path, its natural lift

$$\tilde{\gamma} : I \to T^2(M) :: \tilde{\gamma}(t) = (\gamma(t),\ \gamma_*(d/dt),\ \gamma_*(d/dt))$$

is horizontal.

Proof: The tangent field $T_{\tilde{\gamma}}(t)$ to $\tilde{\gamma}$ is given by

(12)
$$T_{\tilde{\gamma}}(t) = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{d^2x^i}{dt^2}\frac{\partial}{\partial \xi^i} + \frac{d^2x^i}{dt^2}\frac{\partial}{\delta \eta^i},$$

over a coordinate patch U on M. In terms of the basis $(S_i, S_{i*}, \partial/\partial\eta^i)$ of the tangent space to $T^2(M)$ at $\tilde{\gamma}(t)$, (12) reads

$$T_{\tilde{\gamma}}(t) = \frac{dx^i}{dt} S_i + \frac{d^2x^i}{dt^2} S_{i*}$$

(13)
$$+ [\frac{dx^i}{dt}\frac{dx^j}{dt}(F^{\ k}_{ij} + \Gamma^h_i C^{\ k}_{hj})$$

$$+ \frac{d^2x^i}{dt^2}\frac{dx^j}{dt} C^{\ k}_{ij} + \frac{d^2x^k}{dt^2}]\frac{\partial}{\partial\eta^k}$$

Thus, $T_{\tilde{\gamma}}(t)$ is horizontal iff equations (1) are satisfied. Since a canonical path satisfies equations (1) by Theorem 1, $\tilde{\gamma}(t)$ is horizontal. Q.E.D.

The natural course suggested by Theorem 2 is to redefine a canonical path to be a curve γ whose natural lift $\tilde{\gamma}$ is horizontal. Again, this directly generalizes a usual property of geodesics of a linear connexion, and such canonical paths would include the canonically auto-parallel curves, though perhaps adding others. Under this course, both Theorems 1 and 2 would hold with "iff" in place of "if", but existence and uniqueness questions would remain open nonetheless.

Consider now other paths related to the DR-connexion.

Matsumoto defines a horizontal (re Γ) curve $\sigma : I \to T(M)$ to be a <u>horizontal path</u> provided it is the projection of an integral curve $\tilde{\sigma} : I \to B_A(M)$ of each of the vector fields

$$T_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial \xi^j} - F_{ij}^k \xi_s^j \frac{\partial}{\partial \xi_s^k},$$

and shows that σ is a horizontal path iff it satisfies the equations

$$(14) \qquad \begin{aligned} &\frac{d^2x^i}{dt^2} + F_{jk}^i(\sigma(t)) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0 , \\ &\frac{d\xi^i}{dt} + \Gamma_j^i(\sigma(t)) \frac{dx^j}{dt} = 0 , \end{aligned}$$

over a coordinate patch U on M [10; pp. 309-310]. Then one has

<u>Theorem 3</u>: A curve $\sigma : I \to T(M)$ by $\sigma(t) = (x^i(t), \xi^i(t))$ over a coordinate patch U is a horizontal path iff it is horizontal re Γ and its lift

$$\bar{\sigma} : I \to T^2(M) :: \bar{\sigma}(t) = (x^i(t), \xi^i(t), dx^i/dt)$$

is horizontal.

Proof: The tangent field $T_{\bar{\sigma}}(t)$ to $\bar{\sigma}$ is given by

$$T_{\bar{\sigma}}(t) = \frac{dx^i}{dt}\frac{\partial}{\partial x^i} + \frac{d\xi^i}{dt}\frac{\partial}{\partial \xi^i} + \frac{d^2x^i}{dt^2}\frac{\partial}{\partial \eta^i},$$

over the patch U. Relative to the basis $(S_i, S_{i*}, \partial/\partial\eta^i)$, this reads

$$T_{\bar{\sigma}}(t) = \frac{dx^i}{dt} S_i + \frac{d\xi^i}{dt} S_{i*} + \Big[\frac{dx^i}{dt}\frac{dx^j}{dt} F^{\,k}_{ij}$$

(15)

$$+ \frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma^h_i C^{\,k}_{hj} + \frac{d\xi^i}{dt}\frac{dx^j}{dt} C^{\,k}_{ij} + \frac{d^2x^k}{dt^2}\Big]\frac{\partial}{\partial\eta^k}.$$

The tangent field T_σ to σ is given by

(16)
$$T_\sigma = \frac{dx^i}{dt} B_i + \Big(\frac{d\xi^h}{dt} + \frac{dx^i}{dt}\Gamma^h_i\Big)\frac{\partial}{\partial\xi^h}$$

over U. In both (15) and (16), the tangent bundle functions are evaluated at $\sigma(t)$. By (16), σ is horizontal iff the second of equations (14) holds, and from (15), $\bar{\sigma}$ is horizontal iff

(17)
$$\frac{dx^i}{dt}\frac{dx^j}{dt}\big[F^{\,k}_{ij} + \Gamma^h_i C^{\,k}_{hj}\big] + \frac{d\xi^h}{dt}\frac{dx^j}{dt} C^{\,k}_{hj} + \frac{d^2x^k}{dt^2} = 0\,.$$

Upon substituting $-\Gamma^h_i(dx^i/dt)$ for $d\xi^h/dt$ in (17), from the second of (14), one obtains the first equation of (14), and the theorem follows. Q.E.D.

Matsumoto calls a curve $\sigma : I \to M$, given by $x^i(t) = x^i \circ \sigma(t)$ in a coordinate patch U, a <u>quasi-path</u>

iff it satisfies the differential equations

(18) $$\frac{d^2x^i}{dt^2} + F^{\,i}_{jk}\,(\sigma(t),\ \sigma_*(\frac{d}{dt}))\ \frac{dx^j}{dt}\,\frac{dx^k}{dt} = 0$$

over U [10; pp. 314–317]. It is to be noted, from (6.25), that if the tangent field to σ be T, equations (18) are just

(19) $$(\overset{\Gamma}{\nabla}_T\ T)_{(\sigma(t),T)} = 0\ ,$$

which gives a "geometrical" meaning to the curve. This discussion of paths relative to a DR-connexion will be concluded by noting the following two simple theorems, which give an indication of the relationship between canonical paths and paths defined using the helpful non-linear support.

Theorem 4: If the DR-tensor C of (7.19) is a zero DR-tensor, then every canonical path is a quasi-path.

This is immediate from equations (1) and (18). If the redefinition of canonical paths suggested by Theorem 2 is taken, then if C is a zero DR-tensor, every quasi-path is also a canonical path.

Call a geodesic of the non-linear connexion Γ a Γ-path. Then one has:

Theorem 5: A canonical path which is also a Γ-path is a quasi-path.

Proof: If $\sigma : I \to M :: \sigma(t) = (x^i(t))$ in a patch U is a canonical path, it satisfies

(20) $$\frac{d^2x^k}{dt^2} + F_{ij}^{\ k} \frac{dx^i}{dt}\frac{dx^j}{dt} + \Gamma_i^h C_{hj}^{\ k} \frac{dx^i}{dt}\frac{dx^j}{dt} + C_{ij}^{\ k} \frac{dx^j}{dt}\frac{d^2x^i}{dt^2} = 0 .$$

If it is also a Γ-path, then

(21) $$\frac{d^2x^i}{dt^2} = -\Gamma_j^i \frac{dx^j}{dt},$$

and replacing d^2x^i/dt^2 in the last term of (20) with the expression from (21), the last two terms of (20) cancel, yielding equations (18). Q.E.D.

10. Integrability Conditions for the DR-connexion

For the remainder of this chapter, the basis $(S_i, S_{i*}, \partial/\partial\eta^i)$ of the tangent spaces to $T^2(M)$ over a coordinate patch U will no longer be convenient to use. An obvious extension of the method of adapted frames of Yano and Ishihara [23; pp. 275-277] will be employed instead. An adapted frame for DR-theory in $T^2(M)$ is

$$A_i = \frac{\partial}{\partial x^i} - \Gamma_i^h \frac{\partial}{\partial \xi^h} - F_{ij}^{\ k}\, \eta^j \frac{\partial}{\partial \eta^k},$$

(1) $$A_{i*} = \frac{\partial}{\partial \xi^i} - C_{ij}^{\ k} \frac{\partial}{\partial \eta^k},$$

$$A_{i'} = \frac{\partial}{\partial \eta^i},$$

over U, where the coordinates in U are (x^i) and the induced coordinates in $\pi_2^{-1}(U)$ are (x^i, ξ^i, η^i), as in section 4. All tangent bundle functions are evaluated at a point (m,X); as this will be true throughout this section, the arguments of such functions will be omitted without comment. In this section and the next, lower case Roman indices will run over the range $\{1, \ldots, n\}$, such indices with an asterisk are valued from $n+1$ to $2n$, and lower case Roman letters with a prime take on values from $2n+1$ to $3n$. The summation convention will apply to such expressions as $a^i b_{i*}$. Lower case Greek letters will range from 1 to $3n$.

The formal advantages of the adapted frame (A_α) will become quite evident in section 11, where linear connexions on $T^2(M)$ are discussed. The goal of the present section is the determination of integrability conditions for the distribution in $T^2(M)$ corresponding to a DR-connexion (H,Γ) on M with local coefficients F^k_{ij}, C^k_{ij}, and Γ^i_j. One begins with

<u>Lemma 1</u>: With (A_α) as in (1),

(2) $$[A_i, A_j] = R^k_{ij}\, A_{k*} + (C^p_{kt}\, R^k_{ij} + \overset{\Gamma}{R}{}^p{}_{jit})\, \eta^t\, A_{p'}\,,$$

(3) $$[A_i, A_{j*}] = \Gamma^k_{ij}\, A_{k*} + \overset{\circ}{R}{}^p{}_{jit}\, \eta^t\, A_{p'}\,,$$

(4) $$[A_{i*}, A_{j*}] = \overset{\vee}{R}{}^k{}_{jit}\, \eta^t\, A_{p'}\,,$$

(5) $$[A_i, A_{j*}] = F^k_{ij}\, A_{k'}\,,$$

(6) $$[A_{i'}, A_{j*}] = -\, C^i_{jk}\, A_{k'}\,,$$

(7) $$[A_{i'}, A_{j'}] = 0\,,$$

where $R_i{}^k{}_j$ are the components of the curvature DR-tensor of the non-linear support relative to $(\partial/\partial x^i, \partial/\partial \xi^i)$; $\overset{\Gamma}{R}{}^p{}_{jit}$ the components of the horizontal curvature DR-tensor, from (7.22); $\overset{v}{R}{}^k{}_{jit}$ those of the vertical curvature DR-tensor, from (7.23); and $\overset{\circ}{R}{}^p{}_{jit}$ those of the mixed curvature DR-tensor, from (7.24). Also, $\Gamma^i_{jk} = \partial\Gamma^i_j/\partial\xi^k$.

<u>Proof</u>: This is straightforward computation. Equation (3) will be derived; the remaining equations follow from entirely similar arguments. For (3), compute

$$A_i\, A_{j*}(x^k) = A_{j*}A_i(x^k) = 0\ ,$$

while

$$A_iA_{j*}(\xi^k) = 0$$

and

$$A_{j*}\, A_i(\xi^k) = A_{j*}(-\Gamma^k_i) = -\,\Gamma^k_{ij}\ .$$

Finally,

$$A_i\, A_{j*}(\eta^k) = A_i\ (-C^{\,k}_{js}\ \eta^s)$$

$$= -\frac{\partial C^{\,k}_{js}}{\partial x^i} + \Gamma^h_i \frac{\partial C^{\,k}_{js}}{\partial \xi^h}\ \eta^s + F^{\,t}_{is}\, C^{\,k}_{jt}\eta^s$$

and

$$A_{j*}\, A_i(\eta^k) = A_{j*}\ (-F^{\,k}_{is}\ \eta^s)$$

$$= -\,\frac{\partial F^{\,k}_{is}}{\partial \xi^j}\ \eta^s + C^{\,t}_{js}\ F^{\,k}_{it}\ \eta^s\ .$$

Therefore,

(8)
$$[A_i, A_{j*}] = \Gamma_{ij}^k \frac{\partial}{\partial \xi^k} + [\frac{\partial F_{is}^k}{\partial \xi^j} - \frac{\partial C_{js}^k}{\partial x^i}$$
$$+ \Gamma_i^h \frac{\partial C_{js}^k}{\partial \xi^h} + F_{is}^t C_{jt}^k - C_{js}^t F_{it}^k]\, \eta^s \frac{\partial}{\partial \eta^k}.$$

From (7.24), equations (8) yield

(9)
$$[A_i, A_{j*}] = \Gamma_{ij}^k \frac{\partial}{\partial \xi^k} + (\overset{\circ}{R}{}^k{}_{jis} - C_{ts}^k \Gamma_{ij}^t)\, \eta^s \frac{\partial}{\partial \eta^k}.$$

If (9) be written in terms of the adapted frame, the result is equation (3). Q.E.D.

When one has Lemma 1 at his disposal, the following theorem

Theorem 1: The horizontal distribution on $T^2(M)$ corresponding to the DR-connexion (H,Γ) is integrable iff

$$\overset{\vee}{R}{}^k{}_{jis}\, \eta^s = 0$$

(10)
$$(C_{ks}^p R_{ij}^k + \overset{\Gamma}{R}{}^p{}_{jis})\eta^s = 0$$

$$\overset{\circ}{R}{}^p{}_{jis}\, \eta^s = 0\,,$$

over each coordinate patch U on M.

Proof: The distribution is integrable iff $[A_i, A_j]$, $[A_i, A_{j*}]$, and $[A_{i*}, A_{j*}]$ are horizontal, by the

Frobenius Theorem. Equations (10) are simply the statement that the vertical part of these vectors vanish, by Lemma 1. Q.E.D.

In particular, one has the

Corollary: Suppose either that the DR-tensor C of (7.19) is a zero DR-tensor, or that the non-linear support is flat. Then the horizontal distribution in $T^2(M)$ is integrable iff $\overset{\Gamma}{R}$, $\overset{\gamma}{R}$, and $\overset{\circ}{R}$ are zero DR-tensors.

11. Linear Connexions on $T^2(M)$

The object of this section is to prove for DR-connexion theory a theorem analogous to that of Yano and Ledger on the existence and uniqueness of a certain symmetric linear connexion on $T(M)$, induced by a linear connexion on M [24; p. 498]. The C^∞ n-manifold M is assumed to be endowed with a DR-connexion, with (A_α) as in (10.1) the adapted frame over a coordinate patch U on M.

Suppose that D is a linear connexion on $T^2(M)$, with coefficients $\Lambda^{\alpha}_{\beta\gamma}$ with respect to the adapted frame; that is,

(1)
$$D_{A_\alpha} A_\beta = \Lambda^{\gamma}_{\alpha\beta} A_\gamma .$$

Then one has

Lemma 1: The following subsets of the set $(\Lambda^{\gamma}_{\alpha\beta})$ of coefficients of a linear connexion on $T^2(M)$, relative to the adapted frame, transform like tensors on M with a change of adapted frame:

(2) $$\Lambda_{i*\beta}^{\gamma},\ \Lambda_{i'\beta}^{\gamma},\ \Lambda_{jk*}^{t},\ \Lambda_{jk}^{t*},\ \Lambda_{jk}^{t'},\ \Lambda_{jk'}^{t},\ \Lambda_{jk'}^{t*},\ \Lambda_{jk*}^{t'}.$$

The remaining coefficients

(3) $$\Lambda_{jk}^{t},\quad \Lambda_{jk*}^{t},\quad \text{and}\quad \Lambda_{jk'}^{t'},$$

transform like (linear) connexion coefficients on M. Conversely, of course, any collection of coefficients which transform in this manner determines a linear connexion on $T^2(M)$.

<u>Proof</u>: Suppose U and V are coordinate patches on M with coordinates (x^i) and (y^i) respectively, and $U \cap V \neq \varphi$. Let (A_α) denote the adapted frome over U, and $(\bar{A}_\alpha)$ that over V. Then, with induced coordinates $(x^i,\ \xi^i,\ \eta^i)$ in $\pi_2^{-1}(U)$, one has

(4) $$A_i = \delta_i^j \frac{\partial}{\partial x^j} - \Gamma_i^h(m,X) \frac{\partial}{\partial \xi^h} - F_{ij}^k(m,X)\, \eta^j \frac{\partial}{\partial \eta^k}.$$

Let $(y^i,\ \varphi^i,\ \chi^i)$ denote the induced coordinates in $\pi_2^{-1}(V)$. Then, if a vector Q on $\pi_2^{-1}(U \cap V)$ is given by

(5) $$Q = \alpha^i \frac{\partial}{\partial x^i} + \beta^i \frac{\partial}{\partial \xi^i} + \gamma^i \frac{\partial}{\partial \eta^i},$$

one finds

$$Q = \alpha^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} + (\alpha^j \xi^k \frac{\partial^2 y^i}{\partial x^j \partial x^k} + \beta^k \frac{\partial y^i}{\partial x^k}) \frac{\partial}{\partial \varphi^i}$$

(6)

$$+ (\alpha^j \eta^k \frac{\partial^2 y^i}{\partial x^j \partial x^k} + \gamma^k \frac{y^i}{x^k} \frac{\partial}{\partial \chi^i},$$

from the coordinate transformation (5.3). Applying (6) to (4) yields

$$A_i = \frac{\partial y^k}{\partial x^i} \frac{\partial}{\partial y^k} - \frac{\partial y^t}{\partial x^i} \overline{\Gamma}^h_t \frac{\partial}{\partial \varphi^h} - \frac{\partial y^t}{\partial x^i} \overline{F}^h_{tj} \chi^j \frac{\partial}{\partial \chi^h},$$

where the transformation equations (6.10) and (6.12) have been applied. Thus

(7)
$$A_i = \frac{\partial y^k}{\partial x^i} \overline{A}_k .$$

In an entirely similar fashion, one finds

(8)
$$A_{i*} = \frac{\partial y^k}{\partial x^i} \overline{A}_{k*} ,$$

and

(9)
$$A_{i'} = \frac{\partial y^k}{\partial x^i} \overline{A}_{k'} .$$

Equations (7), (8), and (9) are the reason for the introduction of the adapted frame. They may be summarized in the single statement

(10)
$$A_\alpha = G^\beta_\alpha \overline{A}_\beta ,$$

where (G^{β}_{α}) are given by the $(3n \times 3n)$-matrix

$$(11) \qquad [G^{\beta}_{\alpha}] = \begin{bmatrix} (\partial y^k/\partial x^i) & (0) & (0) \\ (0) & (\partial y^k/\partial x^i) & (0) \\ (0) & (0) & (\partial y^k/\partial x^i) \end{bmatrix} .$$

Define functions $\overline{G}^{\beta}_{\alpha}$ by $\overline{G}^{\beta}_{\alpha}\, G^{\gamma}_{\beta} = \delta^{\gamma}_{\alpha}$, so that

$$(12) \qquad [\overline{G}^{\beta}_{\alpha}] = \begin{bmatrix} (\partial x^i/\partial y^k) & (0) & (0) \\ (0) & (\partial x^i/\partial y^k) & (0) \\ (0) & (0) & (\partial x^i/\partial y^k) \end{bmatrix} .$$

In G^{β}_{α}, β is a column-counting index, and α a row-counting index. From (10), one has

$$D_{A_\alpha} A_\beta = G^{\gamma}_{\alpha}\, \overline{A}_{\gamma}\, (G^{\delta}_{\beta})\, \overline{A}_{\delta} + G^{\gamma}_{\alpha}\, G^{\delta}_{\beta}\, D_{\overline{A}_\gamma}\, \overline{A}_{\delta} ,$$

whence

$$(13) \qquad \Lambda^{\gamma}_{\alpha\beta}\, \overline{G}^{\sigma}_{\gamma} = A_{\alpha}\, (G^{\sigma}_{\beta}) + G^{\gamma}_{\alpha}\, G^{\delta}_{\beta}\, \overline{\Lambda}^{\sigma}_{\gamma\delta}$$

which is the basic transformation equation of the coefficients of the connexion D, under a change of adapted frame. But, from (11), the quantities

$$A_{j*}(G^{\alpha}_{\beta}),\ A_{j'}(G^{\alpha}_{\beta}),\ A_j(G^{i}_{k*}),\ A_j(G^{i*}_{k}),\ A_j(G^{i}_{k'}),$$
$$A_j(G^{i'}_{k}),\ A_j(G^{i*}_{k'}),\ \text{and}\ A_j(G^{i'}_{k*})$$

are all zero. Using this fact, together with equations (13), yields, for example,

(15) $$\Lambda_{i*\beta}^{\gamma}\ \overline{G}_{\gamma}^{\sigma} = G_{i*}^{k*}\ G_{\beta}^{\delta}\ \overline{\Lambda}_{k*\delta}^{\ \sigma} .$$

so that $\Lambda_{i*\beta}^{\gamma}$ transform in the tensor manner. That the same is true of each of the quantities in (2) follows in the same way, from (13) and (14). The second assertion of the lemma is clear from (11) and (13), while the final statement is trivial. Q.E.D.

Some notational conventions will now be established which will be convenient in what follows. First, if X is a C^∞ vector field on M, there exist three naturally defined lifts of X to a global field on $T^2(M)$. If, indeed, $X = X^i\ \partial/\partial x^i$, let

$$\lambda_1 X = X^i A_i ,$$

(16) $$\lambda_2 X = X^i A_{i*} ,$$

$$\lambda_3 X = X^i A_{i'} .$$

By equations (10), the form of equations (16) is independent of the particular coordinate system used in defining the lifts, so that (16) do define global vector fields on $T^2(M)$. Equations (16) may also be applied to a DR-vector field; for example, if $X = X^i(m,Y)\ \partial/\partial x^i$, then

(17) $$(\lambda_1 X)_{(m,Y,Z)} = X^i(m,Y)\ A_i(m,Y,Z) ,$$

and similarly for $\lambda_2 X$ and $\lambda_3 X$.

Next, suppose that K is a C^∞ p-covariant DR-tensor with values in the space of linear transformations on M_m.

Define h_1K to a C^∞, vector-valued, p-covariant tensor on $T^2(M)$, by

(18) $$(h_1K(Q_1, \ldots, Q_p))_{(m,X,Y)} = \lambda_1[K_{(m,X)}((\pi_2)_* Q_1, \ldots, (\pi_2)_* Q_p)(Y)],$$

where $(m,X,Y) \in T^2(M)$, and $Q_1, \ldots, Q_p$ are vectors in $(T^2(M))_{(m,X,Y)}$. Similarly, h_2K and vK are defined by

(19) $$(h_2K)_{(m,X,Y)}(Q_1, \ldots, Q_p) = \lambda_2[K_{(m,X)}((\pi_2)_* Q_1, \ldots, (\pi_2)_* Q_p)(Y)],$$

and

(20) $$(vK)_{(m,X,Y)}(Q_1, \ldots, Q_p) = \lambda_3[K_{(m,X)}((\pi_2)_* Q_1, \ldots, (\pi_2)_* Q_p)(Y)].$$

In addition, write

$$\overset{1}{R}{}^p{}_{jis} \equiv C^p_{ks} R^k_{ij} + \overset{\Gamma}{R}{}^p{}_{jis},$$

and define the linear-transformation valued 2-covariant DR-tensor $\overset{1}{R}$ by

(22) $$\overset{1}{R}(X,Y)Z = X^j Y^i Z^p \overset{1}{R}{}^s{}_{jip} \frac{\partial}{\partial x^s}$$

over each coordinate patch U on M.

Then one may prove

Lemma 2: If X and Y are C^∞ vector fields on M, M having a DR-connexion (H,Γ), then

(23) $$[\lambda_1 X, \lambda_1 Y] = \lambda_1[X,Y] + \lambda_2 R(X,Y) + v \overset{1}{R}(\lambda_1 X, \lambda_1 Y),$$

(24) $$[\lambda_1 X, \lambda_2 Y] = \lambda_2(\hat{\nabla}_X Y) + v \mathring{R}(\lambda_1 X, \lambda_1 Y),$$

(25) $$[\lambda_1 X, \lambda_3 Y] = \lambda_3(\overset{\Gamma}{\nabla}_X Y),$$

(26) $$[\lambda_2 X, \lambda_2 Y] = v \check{R}(\lambda_1 X, \lambda_1 Y),$$

(27) $$[\lambda_2 X, \lambda_3 Y] = v\, C(\lambda_1 X, \lambda_1 Y),$$

and

(28) $$[\lambda_3 X, \lambda_3 Y] = 0,$$

where $\hat{\nabla}$ is the operator of (7.20), R the curvature DR-tensor of the non-linear support, and C is given by (7.29).

Proof: The relationships of Lemma 2 follow from those of Lemma 10.1 by easy computation. For example,

$$\begin{aligned}[\lambda_1 X, \lambda_1 Y] &= [X^i A_i, Y^j A_j] \\ &= X^i Y^j [A_i, A_j] + X^j A_j(Y^k) A_k - Y^j A_j(X^k) A_k \\ &= X^i Y^j R^k_{ij} A_{k*} + X^i Y^j \overset{1}{R}{}^k{}_{ijs} \eta^s A_{k'} \\ &\quad + [X^j \frac{\partial Y^k}{\partial x^j} - Y^j \frac{\partial X^k}{\partial x^j}] A_k ,\end{aligned}$$

whence (23). The other computations are similar. Q.E.D.

One may now establish the desired theorem:

<u>Theorem 1</u>: Given a DR-connexion on M, there exists a unique symmetric linear connexion D on $T^2(M)$ such that

$$D_{\lambda_3 X}\, \lambda_3 Y = 0, \quad D_{\lambda_3 X}\, \lambda_2 Y = 0, \quad D_{\lambda_3 X}\, \lambda_1 Y = 0\,, \tag{29}$$

$$D_{\lambda_2 X}\, \lambda_1 Y = 0, \tag{30}$$

$$D_{\lambda_2 X}\, \lambda_2 Y = \tfrac{1}{2}\, v\, \check{R}\ (\lambda_1 X,\ \lambda_1 Y)\,, \tag{31}$$

$$D_{\lambda_1 X}\, \lambda_1 Y = \lambda_1 [\ \overset{\Gamma}{\nabla}_X\, Y - \tfrac{1}{2}\, \overset{\Gamma}{T}\, (X,Y)] + \tfrac{1}{2} \lambda_2 R(X,Y) \tag{32}$$

$$+ \tfrac{1}{2}\, v\, \overset{1}{R}\, (\lambda_1 X,\ \lambda_1 Y)\,.$$

<u>Proof</u>: A linear connexion D on $T^2(M)$ is completely determined by its values on the lifts $\lambda_1 X$, $\lambda_2 X$, and $\lambda_3 X$ of vector fields on M, since these provide spanning sets for the tangent spaces of $T^2(M)$. Therefore, D is symmetric iff

$$D_{\lambda_1 X}\, \lambda_1 Y - D_{\lambda_1 Y}\, \lambda_1 X = [\lambda_1 X,\ \lambda_1 Y] = \tag{33}$$

$$\lambda_1 [X,Y] + \lambda_2 R(X,Y) + v\ \overset{1}{R}(\lambda_1 X,\ \lambda_1 Y)\,,$$

$$D_{\lambda_1 X}\, \lambda_2 Y - D_{\lambda_1 Y}\, \lambda_1 X = [\lambda_1 X,\ \lambda_2 Y] = \tag{34}$$

$$\lambda_2 (\ \hat{\nabla}_X\ Y) + v\ \mathring{R}\ (\lambda_1 X,\ \lambda_1 Y)\,,$$

$$D_{\lambda_1 X}\,\lambda_3 Y - D_{\lambda_3 Y}\,\lambda_1 X = [\lambda_1 X,\ \lambda_3 Y] = \lambda_3(\overset{\Gamma}{\nabla}_X\,Y)\,, \tag{35}$$

$$D_{\lambda_2 X}\,\lambda_2 Y - D_{\lambda_2 Y}\,\lambda_2 X = [\lambda_2 X,\ \lambda_2 Y] = v\,\overset{\vee}{R}\,(\lambda_1 X,\ \lambda_1 Y)\,, \tag{36}$$

$$D_{\lambda_2 X}\,\lambda_3 Y - D_{\lambda_3 Y}\,\lambda_2 X = [\lambda_2 X,\ \lambda_3 Y] = v\,C\,(\lambda_1 X,\ \lambda_1 Y)\,, \tag{37}$$

and

$$D_{\lambda_3 X}\,\lambda_3 Y - D_{\lambda_3 Y}\,\lambda_3 X = [\lambda_3 X,\ \lambda_3 Y] = 0\,. \tag{38}$$

Now, setting $D_{\lambda_3 X}\,\lambda_3 Y = 0$ is consistent with (38); equation (31) is consistent with (36), since $\overset{\vee}{R}$ is skew-symmetric, and (32) is consistent with (33), since $[X,Y] = \overset{\Gamma}{\nabla}_X\,Y - \overset{\Gamma}{\nabla}_Y\,X - \overset{\Gamma}{T}\,(X,Y)$, and since both R and $\overset{1}{R}$ are skew-symmetric.

But setting $D_{\lambda_3 X}\,\lambda_2 Y = 0$ is consistent with (37) iff

$$D_{\lambda_2 X}\,\lambda_3 Y = v\,C\,(\lambda_1 X,\ \lambda_1 Y)\,, \tag{39}$$

and setting $D_{\lambda_3 X}\,\lambda_1 Y = 0$ is consistent with (35) iff

$$D_{\lambda_1 X}\,\lambda_3 Y = \lambda_3(\ \overset{\Gamma}{\nabla}_X\,Y)\,, \tag{40}$$

and finally, setting $D_{\lambda_2 X}\,\lambda_1 Y = 0$ is consistent with (34) iff

$$D_{\lambda_1 X}\,\lambda_2 Y = \lambda_2(\ \hat{\nabla}_X\,Y) + v\,\overset{\circ}{R}\,(\lambda_1 X,\ \lambda_1 Y)\,. \tag{41}$$

Thus, one sees that a linear connexion D on $T^2(M)$ which is symmetric and satisfies equations (29), (30), (31), and (32) is completely determined, and therefore unique, provided it exist at all.

The existence of D may be proved by displaying its coefficients, and applying Lemma 1. Let attention be restricted to $\pi_2^{-1}(U)$ for U a coordinate patch on M.

In terms of the adapted frame, let $D_{A_\alpha} A_\beta = \Lambda^\gamma_{\alpha\beta} A_\gamma$. Then, locally,

$$D_{\lambda_1 X} \lambda_1 Y = \left(X^i \frac{\partial Y^k}{\partial x^i} + \Lambda^k_{ij} X^i Y^j\right) A_k + X^i Y^j (\Lambda^{k*}_{ij} A_{k*} + \Lambda^{k'}_{ij} A_{k'}) . \tag{42}$$

But equation (32) reads, re the adapted frame,

$$D_{\lambda_1 X} \lambda_1 Y = \left(X^i \frac{\partial Y^k}{\partial x^i} + X^i Y^j F^k_{ij}\right) A_k + \tfrac{1}{2} X^i Y^j \left(R^k_{ij} A_{k*} + \overset{1}{R}{}^k{}_{ijs} \eta^s A_{k'}\right). \tag{43}$$

Equations (42) and (43) together show that one must have

$$\Lambda^k_{ij}(m,X,Y) = F^k_{ij}(m,X) , \tag{44}$$

$$\Lambda^{k*}_{ij}(m,X,Y) = \tfrac{1}{2} R^k_{ij}(m,X) , \tag{45}$$

and

$$\Lambda^{k'}_{ij}(m,X,Y) = \tfrac{1}{2} \overset{1}{R}{}^k{}_{ijs}(m,X) \eta^s . \tag{46}$$

Entirely similar reasoning leads to the remaining coefficients:

$$(47) \qquad \Lambda^{k*}_{ij*} = \Gamma^{k}_{ij}, \qquad \Lambda^{k'}_{ij*} = \mathring{R}^{k}{}_{ijs}\, \eta^{s}$$

$$\Lambda^{k'}_{ij'} = F^{k}_{ij}, \qquad \Lambda^{k'}_{i*j*} = \tfrac{1}{2}\, \check{R}^{k}{}_{ijs}\, \eta^{s},$$

$$\Lambda^{\underline{k}'}_{i*j'} = C^{\underline{k}}_{ij},$$

with the rest zero. Testing these coefficients against the requirements of Lemma 1, one sees that they do determine a linear connexion on $T^2(M)$, and D exists, as stated. Q.E.D.

This theorem may be employed, in the same manner as its analogue in the work of Yano and Ledger, to give geometric results connecting $T^2(M)$, $T(M)$, and M. The following two corollaries are examples; Corollary 1 is to be compared with Yano and Ledger's Corollary 2 [24; p. 499], and Corollary 2 below with their Corollary 3 [24; p. 500]. Throughout, D is the connexion of Theorem 1.

<u>Corollary 1</u>: Let V_1 and V_2 be C^∞ vector fields on $T^2(M)$ such that $\bar{\pi}_*(V_i) = k_1(X_i)$ for $i = 1, 2$, where X_1 and X_2 are C^∞ fields on M, and k_1 denotes the Γ-lifting operation. Then, if the non-linear support is flat and $\overset{\Gamma}{\nabla}$ is torsion-free,

$$(48) \qquad \bar{\pi}_*(D_{V_1} V_2) = k_1(\overset{\Gamma}{\nabla}_{X_1} X_2)$$

and

(49) $$(\pi_2)_*(D_{V_1} V_2)_{(m,X,Y)} = (\overset{\Gamma}{\nabla}_{X_1} X_2)_{(m,X)} .$$

<u>Proof</u>: Since $\overline{\pi}_*(V_i) = k_1(X_i)$, $V_i = \lambda_1 X_i + W_i$, where W_i are vectors on $T^2(M)$ which are vertical over $T(M)$. This permits the calculation:

(50) $$D_{V_1} V_2 = D_{\lambda_1 X_1} \lambda_1 X_2 + X_1^i W_2^j \lambda_3 (\overset{\Gamma}{\nabla}_{Z_i} Z_j) + X_1 (W_2^i) A_{i'} + W_1 (W_2^i) A_{i'} ,$$

where $Z_j = \partial/\partial x^j$. Upon applying Theorem 1 again, and noting that the last three terms in the right member of (50) are vertical, both over $T(M)$ and M, one obtains

$$\overline{\pi}_*(D_{V_1} V_2) = \overline{\pi}_*(D_{\lambda_1 X_1} \lambda_1 X_2)$$

or

(51) $$\overline{\pi}_*(D_{V_1} V_2) = \overline{\pi}_*\{\lambda_1[\overset{\Gamma}{\nabla}_{X_1} X_2 - \tfrac{1}{2} \overset{\Gamma}{T} (X_1,X_2)]\} + \overline{\pi}_*(\tfrac{1}{2} \lambda_2 R(X_1,X_2)).$$

By hypothesis, $\overset{\Gamma}{T} = 0$; since Γ is flat, $R = 0$. Therefore,

(52) $$\overline{\pi}_*(D_{V_1} V_2) = \overline{\pi}_* \circ \lambda_1 (\overset{\Gamma}{\nabla}_{X_1} X_2) .$$

Both statements of the corollary are immediate from (52). Q.E.D.

<u>Corollary 2</u>: Let $\sigma : I \to T^2(M)$ be a geodesic of the connexion D of Theorem 1 which is nowhere tangent to fibres over $T(M)$ and such that $\bar{\pi} \circ \sigma$ is Γ-horizontal on $T(M)$. Then $\bar{\pi} \circ \sigma$ is a horizontal path.

<u>Proof</u>: Since σ is nowhere tangent to fibres over $T(M)$, $\bar{\pi} \circ \sigma$ is a regular curve. Also, since $\bar{\pi} \circ \sigma$ is horizontal, the tangent T to σ is of the same type as the vectors V_i of Corollary 1, so that, from (51),

(53)
$$\bar{\pi}_*(D_T\, T) = \bar{\pi}_*\{\lambda_1[\overset{\Gamma}{\nabla}_T\, T - \tfrac{1}{2}\overset{\Gamma}{T}(T,T)] + \tfrac{1}{2}\lambda_2\, R(T,T)\}\,.$$

But $\overset{\Gamma}{T}$ and R are skew-symmetric, so (53) yields

(54)
$$(\bar{\pi}_*(D_T\, T))_{\bar{\pi}\circ\sigma(t)} = k_1\,(\overset{\Gamma}{\nabla}_T\, T)_{\bar{\pi}\circ\sigma(t)}\,.$$

Since σ is a geodesic of D, $D_T\, T = 0$, and

(55)
$$k_1\,(\overset{\Gamma}{\nabla}_T\, T) = 0$$

along $\bar{\pi} \circ \sigma$. The corollary follows from (55). Q.E.D.

The relative heaviness of the hypotheses in these two corollaries, as compared with the analogous results of Yano and Ledger, arises from the complexity of the interplay between DR-vector fields on M and Γ-lifts of these to vector fields on $T(M)$. A C^∞ DR-vector field is <u>not</u> a vector field on <u>any</u> manifold; the analogy between $T(M)$ and $T^2(M)$, for linear and DR-connexion theory respectively, cannot be stretched indefinitely for just this reason.

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APPENDIX

LIST OF SYMBOLS

The list below is designed to assist the reader who finds himself floundering under the deluge of symbols in this work. It is broken into two parts, one for Chapter II, and another for Chapter III. The reason for this is that the two chapters are nearly disjoint with respect to special notation, and a few symbols in the chapters are similar in appearance, while similarity in meaning is entirely fortuitous.

Symbols are listed in order of appearance.

CHAPTER II:

$\mathfrak{X}(M)$:	set of C^∞ vector fields on M	p.	1
$Gl(n,R)$:	full linear group on R^n	p.	18
$V_k'(R^n)$:	Stiefel manifold of k-frames in R^n	p.	18
$G_{n,k}$:	isotropy group of an element of $V_k'(R^n)$	p.	18
$Q_k(R^n)$:	manifold of L-complementary k-frames	p.	19
$B(M)$:	total space of the bundle of bases over M	p.	21
$V_k'(M)$:	total space of Stiefel bundle of k-frames over M	p.	22
$G_{n,n-k}$:	subgroup of $Gl(n,R)$ leaving L invariant	p.	23
ξ:	Stiefel bundle of k-frames over M	p.	23
$V_k^*(N)$:	total space of restriction to N of ξ	p.	23
$Q_k(N)$:	total space of STC-bundle	p.	24

η_N:	STC-bundle	p.	27
ξ_N:	principal bundle associated to STC-bundle	p.	27
E_N:	total space of ξ_N	p.	28
$G_k(R^n)$:	Grassmann manifold of k-planes in R^n	p.	28
$H_{n,k}$:	isotropy group in O_n of element in $G_k(R^n)$	p.	31
O_n:	orthogonal subgroup of $Gl(n,R)$	p.	31
$H_k(R^n)$:	manifold of L-complementary k-planes in R^n	p.	31
ζ_N:	GTC-bundle	p.	32
$H_k(N)$:	total space of GTC-bundle	p.	32
$\overset{\Lambda}{D}$:	Λ-connexion	p.	36
$\overset{\Lambda}{V}$:	Λ-component tensor	p.	36
$\breve{D}$:	union connexion	p.	42

projections of bundles:

$p: B(M) \to M$

$\pi': V_k'(M) \to M$

$\bar{\pi}: V_k^*(N) \to N$

$\pi: Q_k(N) \to N$

CHAPTER III:

$T(M)$:	total space of tangent bundle over M	p.	51
Γ_j^i:	coefficients of non-linear connexion	p.	52

$\tilde{X}$:	horizontal lift of X (Γ-lift)	pp. 55, 73
ξ_M:	square of tangent bundle over M	p. 65
$T^2(M)$:	total space of square of tangent bundle over M	p. 65
$B_A(M)$:	total space of augmented bundle of bases over M	p. 66
$\mathfrak{D}(M)$:	totality of C DR-vector fields on M	p. 66
η_M:	augmented bundle of bases over M	p. 68
σ_*:	natural lift of σ to $T(M)$	p. 70
$\tilde{\sigma}$:	canonical lift of σ to $B_A(M)$	p. 70
Γ:	non-linear support for a DR-connexion	p. 72
$\overset{\Gamma}{\nabla}$:	horizontal covariant derivative	p. 75
$\overset{v}{\nabla}$:	vertical covariant derivative	p. 76
F^k_{ij}:	coefficients of DR-connexion	p. 77
C^k_{ij}:	coefficients of DR-connexion	p. 78
$\overset{\Gamma}{R}$:	horizontal curvature DR-tensor	p. 89
$\overset{v}{R}$:	vertical curvature DR-tensor	p. 89
$\overset{\circ}{R}$:	mixed curvature DR-tensor	p. 89
$\hat{\nabla}$:	DR-vector valued operator	p. 89
λ_1, λ_2, λ_3:	lifting operators	p. 111
h_1, h_2, v:	lifting operators	p. 112

$\Lambda_{\alpha\gamma}^{\beta}$: coefficients of linear connexion on $T^2(M)$ p. 116

projections of bundles:

$\pi : T(M) \to M$

$\pi_2 : T^2(M) \to M$

$\bar{\pi} : T^2(M) \to T(M)$

$p_B : B_A(M) \to M$

$\pi_B : B_A(M) \to T(M)$