NONLINEAR ANALYSIS OF RECTANGULAR PLANE FRAMES BY MATRIX POLYGON

By

PHILIP NORCROSS ELDRED

Bachelor of Science
University of Vermont
Burlington, Vermont
1956

Master of Science
University of Vermont
Burlington, Vermont
1963

Submitted to the faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
July, 1967

NONLINEAR ANALYSIS OF RECTANGULAR PLANE

FRAMES BY MATRIX POLYGON

Ahmed Salama

Thesis Adviser

Donald E. Boyd

Jeanne Agnew

R L James

N N Durham

Dean of the Graduate College

PREFACE

The investigation into the problem of nonlinear behavior of structural frames presented in this thesis is the result of the author's attendance at the Conference on Plastic Design of Multi-Story Frames at Lehigh University in 1965. It became evident at the conference that this was a problem for which adequate methods of solution were not available.

The method of solution which was adopted is based on concepts presented by Professor Jan J. Tuma in his lectures during several courses and seminars at Oklahoma State University.

The author wishes to express his indebtedness and sincere appreciation to the following individuals and organizations:

To Professor Jan J. Tuma, the chairman of the advisory committee for his inspiration and encouragement;

To Professor Ahmed E. Salama, thesis advisor, for his advice and assistance in preparing this thesis;

To the other members of the advisory committee, Professors Donald E. Boyd, Robert L. Janes, and Jeanne L. Agnew for their guidance and help;

To the National Science Foundation and the School of Civil Engineering at Oklahoma State University for the invitation to attend the Summer Institutes for Teachers of Structures in 1960 and 1963;

To the National Aeronautics and Space Administration for making available a traineeship for one year of full time study;

To Messers C. H. Lindbergh, J. M. Hutt, and R. K. Munshi for their friendship and encouragement;

To Mrs. Joan Neal for her expert typing of the manuscript;

To his wife Kay for her understanding, encouragement, and faithful assistance.

July, 1967

Stillwater, Oklahoma

Philip N. Eldred

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

a (subscript)	End of beam-column
A	Area of cross-section
[A]	Linear transmission matrix
[A']	Linear transmission matrix for hinge points
[AFA]	Frame flexibility matrix
$\begin{matrix} AFA & A' \\ A'^T & 0 \end{matrix}$	Augmented flexibility matrix
[AFBM]	Displacements at origin due to basic moments
b	Flange width of WF beam
b (subscript)	End of beam-column
B	Width of frame
[BM]	Basic moment matrix
[BM']	Basic moment matrix for hinge points
d	Depth of member
E	Modulus of elasticity
[F]	Member flexibility matrix
h	Parameter of curvature equation
H	Height of frame
I	Moment of inertia of cross-section
L	Length of beam-column
[M]	Joint moment matrix
M_p	Plastic moment value
M_{pc}	Reduced plastic moment value

M_y	Moment at first yield
n	Parameter of curvature equation
N	Number of plastic hinges
P	Axial force on beam-column, load applied to frame
P_h	Horizontal wind load
P_v	Vertical column loads
Q	Axial force on pin-ended column equivalent to a beam-column
r	Radius of gyration
$[R]$	Redundant matrix
t	Flange thickness of WF beam
w	Web thickness of WF beam
α	Slope of column deflection curve
α_e	Slope at end of elastic portion of column deflection curve
α_o	Initial slope of column deflection curve
β	Angle between axis of beam-column and axis of column deflection curve
γ	Plastic hinge rotation
$\bar{\gamma}$	Permanent plastic hinge rotation
$[\delta]$	Frame displacements at origin
Δ	Sidesway deflection
θ	Total end rotation of beam-column
λ	End rotation of beam-column due to bowing and yielding
λ_e	End rotation of beam-column due to bowing
λ_p	End rotation of beam-column due to yielding
σ_y	Normal stress at yield point
ϕ	Curvature

CHAPTER I

INTRODUCTION

1.1 Statement of the Problem

In recent years efforts have been made to extend the plastic design method to more complex structures such as multi-story building frames (5, 6, 18). For these structures, the plastic method of analysis may be highly unconservative if the nonlinear effects are not taken into consideration. Most significant are the effects of reduction in plastic moment values and changes in frame geometry. Also of some influence are the effects of nonlinear moment-curvature relationship and "bowing" of individual members (beam-column effect). In this thesis is developed a method of accounting for these nonlinear effects by applying corrections to a conventional elastic flexibility analysis.

1.2 Assumptions

The development of the method is based on the following assumptions:

1. Torsional buckling and deformations out of the plane of the structure are prevented.
2. Axial and shearing deformations are small and may be neglected.
3. The members are steel wide-flange shapes for which the moment-curvature relationship is known.

4. The frame is subjected to horizontal wind loads and failure is by the formation of a collapse mechanism.
5. Strain hardening is neglected.

1.3 Background

Several investigators have presented methods for the nonlinear analysis of frames. Among these Goldberg and Richard (11, 25) considered the effect of a nonlinear moment-curvature relationship by treating the problem as an initial value problem. Gerstle and Zarboulas (10) determined deflections using an elastic-plastic and then a nonlinear moment-curvature relationship and compared the results. The buckling of inelastic portal frames has been studied by Chu and Parbarcius (3) and by Moses (21) using essentially a trial and error procedure to determine the relationship between sidesway deflection and the applied loads.

Ojalvo (22) analyzed elasto-plastic frames by constructing moment-rotation curves for the members and then finding the points of intersection of the curves to determine the moments at the joints. Lu (20) considered the inelastic buckling of symmetrically loaded frames using a modified moment distribution procedure. Nonlinear elastic frames were analyzed by Saafan (26) by relaxing imaginary external restraints at the joints. His method includes the effects of finite deflections, bowing, and changes in the stiffness of the members.

Lind (17, 18) has analyzed tall frames by an iteration procedure in which a state of deformation such as some plastic mechanism is assumed and then the corresponding loads are determined by reconstitution of the frame which initially has a hinge inserted at each joint.

Gauger (9) applied the string-polygon method to the problem of determining the inelastic joint rotations in elastic-plastic frames and Tuma (29) devised the matrix-polygon method as a matrix formulation of the string-polygon method.

Experimental work has been conducted by Augusti (1), Hrennikoff (14), Van Kuren and Galambos (30), and Yen, Lu, and Driscoll (32). Especially to be noted is the work of Ketter, Kaminski, and Beedle (15) in determining the moment-curvature relationship for wide-flange steel shapes.

The previous methods of nonlinear analysis have used either the stiffness method, a trial and error process for enforcing compatibility, or an initial-value approach. The method developed herein is a new approach to the problem in which the nonlinear effects are included as corrections to an elastic flexibility analysis using the matrix-polygon method.

CHAPTER II

FIRST ORDER ELASTIC-PLASTIC ANALYSIS

Both the elastic and the plastic concepts of structural behavior provide useful methods for the analysis of planar frames. With many conditions of loading, however, a frame cannot be considered as perfectly elastic or perfectly plastic. An example is the problem of finding the distribution of moments in a rigid frame in which some yielding has taken place. This chapter will describe a method for evaluating these moments taking into account the formation of plastic hinges and reduced plastic moment values.

2.1 Moment-Curvature Relationship

The first order elastic-plastic analysis is based on the assumption of a moment-curvature relationship of the type shown in Figure 1. It is assumed that the curvature at any cross-section of a member varies linearly with the moment acting at that section until the plastic moment value is reached after which no further increase in moment occurs. The full plastic moment value M_p is reduced to M_{pc} as the result of axial compression in the member.

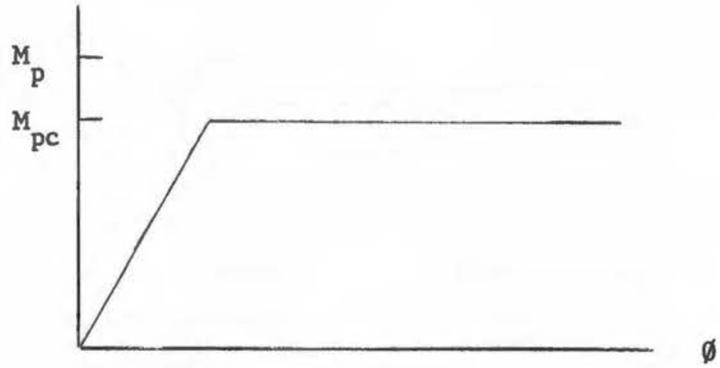


Figure 1. Elastic-Plastic Moment-Curvature Diagram

In accordance with the assumption of an elastic-plastic moment-curvature relationship, neglecting for the present the effects of geometry change and bowing of members, a structure will behave elastically with increasing load until at some point in the structure the moment reaches the value of M_{pc}. Thereafter with further increases in load, assuming no reversals of stress, the moment at this "plastic hinge" will remain equal to M_{pc}. It will be assumed that strain-hardening does not take effect so that M_{pc} may be further reduced if the compression in the member increases with increasing load on the structure. Eventually a second and a third plastic hinge will form until a sufficient number of hinges develop for the structure to become a mechanism at which time it is incapable of carrying any additional load.

2.2 Augmented Flexibility Matrix

Using the flexibility method to formulate the problem, a structure is considered to be jointed at the points of load application and at points of discontinuity in the structure. The moments at these joints are the unknowns and the angular deformations are expressed as functions of these moments. The angular deformations are referred to as "elastic weights" because they can be related by the principles of statics. A difficulty arises with this approach, however, as soon as a plastic hinge forms in that at the hinge location the moment becomes known and the angular deformation becomes unknown. This difficulty is resolved by augmenting the frame flexibility matrix with additional terms expressing the conditions that the moments at the hinges are equal to M_{pc}. One additional equilibrium equation is required for each hinge. Thus the augmented flexibility matrix for a one bay rigid frame will be of order 3+N where N is the number of hinges formed.

The basic matrix equation for the flexibility analysis of an elastic portal frame is

$$[A][F][A]^T[R]+[A][F][BM] = [\delta] = [0] \tag{1}$$

where the matrices are defined as follows:

$[A]$ = linear transmission matrix

$[F]$ = member flexibility matrix

$[R]$ = redundant matrix

$[BM]$ = basic moment matrix consisting of any set of moments that satisfy the conditions of equilibrium for the loaded frame

$[\delta]$ = matrix of displacements of the frame transferred to the origin of the coordinate system.

Certain abbreviations will be used subsequently:

$[AFA] = [A][F][A]^T$ = frame flexibility matrix

$[AFBM] = [A][F][BM]$ = displacements at origin due to basic moments.

The redundants are found by premultiplying both sides of equation (1) by the inverse of the frame flexibility matrix. Thus,

$$[R]+[AFA]^{-1}[AFBM] = [0]$$

or

$$[R] = -[AFA]^{-1}[AFBM] \tag{2}$$

Finally the moments $[M]$ at the joints are found:

$$[M] = [A]^T[R]+[BM] \tag{3}$$

When plastic hinges have formed in the structure, the condition that the moments at the hinge points are equal to M_{pc} is expressed by

$$[A']^T[R]+[BM'] = [M_{pc}] \tag{4}$$

where

$[A']$ = linear transmission matrix for the hinge locations (3xN)

$[BM']$ = matrix of basic moments at the hinge points (Nx1)

$[M_{pc}]$ = matrix of reduced plastic moment values at the hinge points with the signs of the moments at the hinge points (Nx1)

The inelastic rotations $[\gamma]$ of the plastic hinges are also of interest and may be found:

$$[AFA][R]+[AFBM]+[A'][\gamma] = [0] \tag{5}$$

Combining equations (4) and (5) the complete formulation becomes:

$$\begin{bmatrix} AFA & A' \\ A'^T & 0 \end{bmatrix} \begin{bmatrix} R \\ \gamma \end{bmatrix} + \begin{bmatrix} AFBM \\ BM' \end{bmatrix} = \begin{bmatrix} 0 \\ M_{pc} \end{bmatrix} \tag{6}$$

Solving for the unknowns, we obtain

$$\begin{bmatrix} R \\ Y \end{bmatrix} = \begin{bmatrix} AFA & A' \\ A'^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} -AFBM \\ M_{pc} - BM' \end{bmatrix} \tag{7}$$

The final joint moments are determined from equation (3).

The augmented flexibility matrix in equations (6) and (7) would appear to be ill-conditioned due to the zeros on the main diagonal. In the examples, however, (Chapter V) the inverse was found by the method of partitioning (12) and no difficulty was encountered using this method. The augmented flexibility matrix will become singular when a sufficient number of hinges have developed to form a collapse mechanism.

CHAPTER III

NONLINEAR BEAM-COLUMN ANALYSIS

3.1 Lambda Values

The ordinary flexibility functions are not quite adequate when the columns in a structure are subjected to relatively large axial compressive forces, e.g. the columns in the lower floors of a multi-story building. The presence of secondary moments resulting from the axial forces will sometimes call for a more exact analysis. In addition, any yielding that may occur at the column ends prior to the formation of a plastic hinge may also need to be accounted for in the analysis. The angular functions due to these effects have been lumped together and will be referred to as lambda values. Lambda values represent additional angular rotations at the ends of a beam-column and are included in the formulation as follows:

$$[AFA][R] + [AFBM] + [A][\lambda] = [0] \tag{8}$$

$$[R] = [AFA]^{-1}[-AFBM-A\lambda] \tag{9}$$

Lambda values may also be included when the augmented flexibility matrix is used to account for plastic hinges:

$$\begin{bmatrix} R \\ \lambda \end{bmatrix} = \begin{bmatrix} AFA & A' \\ A'^{T} & 0 \end{bmatrix}^{-1} \begin{bmatrix} -AFBM-A\lambda \\ M_{pc}-BM' \end{bmatrix} \tag{10}$$

In computing the lambda values, it will be assumed that the end moments and axial force acting on the beam-column are known and that no intermediate loads are applied. If intermediate loads were present, the beam-column would have to be considered in segments.

3.2 Column Deflection Curves

The basis of the nonlinear beam-column analysis is the column deflection curve which represents the shape of a pin-ended column having the same axial load, cross-sectional dimensions, and material properties as the beam-column under consideration (5, 24). Figure 2 shows a column deflection curve and related beam-columns. Any beam-column subjected only to end loads is equivalent to a portion of some column deflection curve.

If the initial slope α_o of the column deflection curve is known or can be easily determined, analysis of the beam-column becomes a relatively simple matter. We will consider a beam-column in double curvature, Figure 3, since in the examples to follow the columns will deform in this manner. The derivation is general in that either end moment may be positive and the other negative, however, a slight modification in the signs would be required for a beam-column in single curvature. It will be assumed that the angle β between the chord of the beam-column and the axis of the column deflection curve is sufficiently small so that Q can be taken equal to P without appreciable error.

The conditions relating the column deflection curve to the beam-column are:

$$x_b - x_a = L \tag{11}$$

$$Py_a = M_a \tag{12}$$

$$Py_b = M_b \tag{13}$$

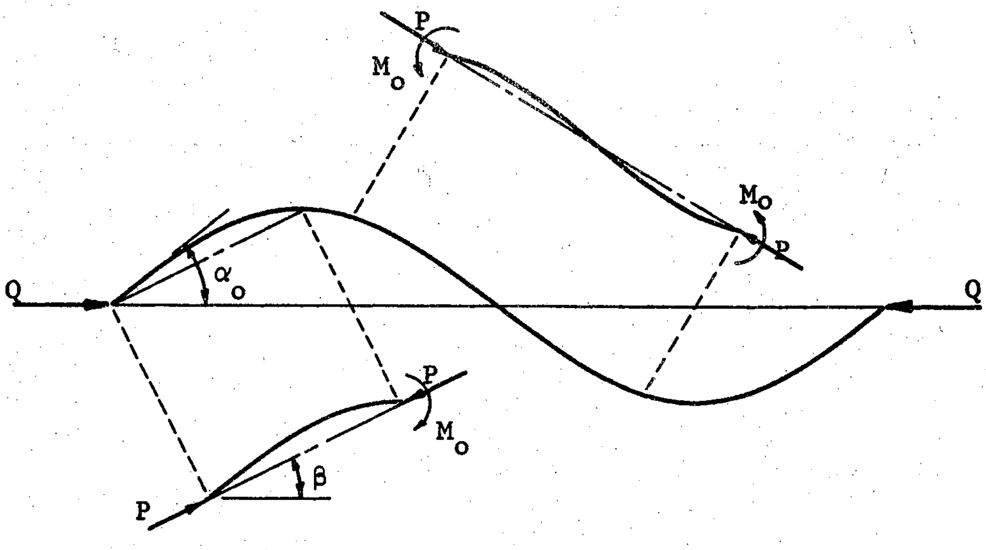


Figure 2. Column Deflection Curves and Related Beam-Columns

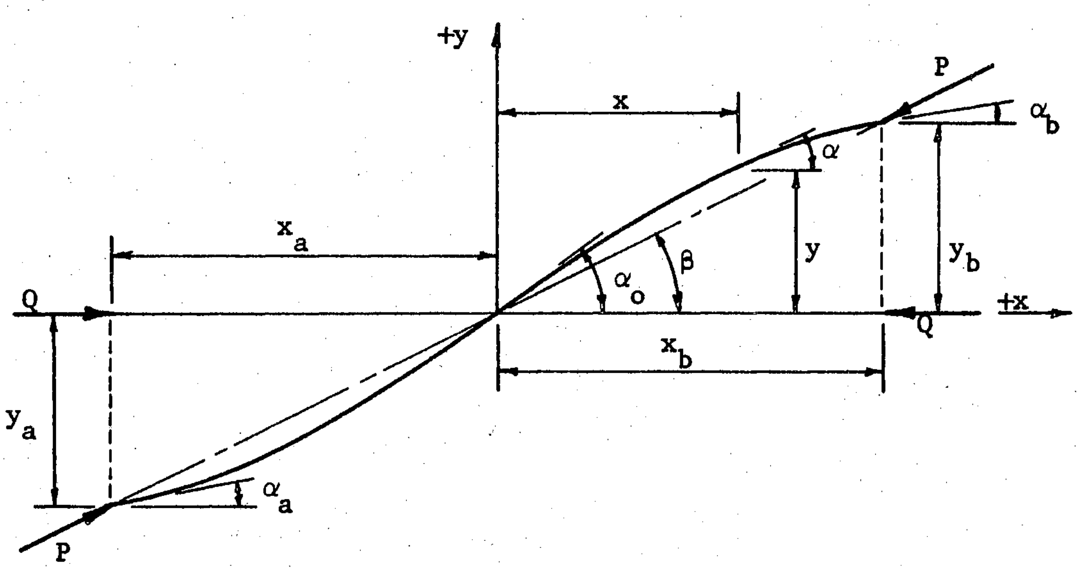


Figure 3. Beam-Column in Double Curvature

In accordance with these relationships, the angle α_o is found as follows. Assuming an elastic beam-column, the shape of the deflection curve is governed by the differential equation

$$Py + EI \frac{d^2y}{dx^2} = 0$$

Taking the origin at the inflection point, the boundary conditions are: $y = 0$ and $\frac{dy}{dx} = \alpha_o$ at $x = 0$. The solution then becomes a portion of a sine wave:

$$y = \frac{\alpha_o}{k} \sin (kx) \qquad (14)$$

and

$$\alpha = \frac{dy}{dx} = \alpha_o \cos(kx) \qquad (15)$$

where

$$k = \sqrt{\frac{P}{EI}}$$

With the solution of the differential equation it is now possible to solve for α_o. Rewriting equation (14) and substituting the end values of the beam column (Figure 3), the result is

$$\alpha_o = \frac{ky_a}{\sin(kx_a)} = \frac{ky_b}{\sin(kx_b)}$$

or

$$y_a \sin(kx_b) = y_b \sin(kx_a)$$

From equation (11) we write

$$y_a \sin(kx_b) = y_b \sin(kx_b - kL)$$

Substituting the trignometric identity for $\sin(kx_b - kL)$, we obtain

$$y_a \sin(kx_b) = y_b \sin(kx_b)\cos(kL) - y_b \cos(kx_b)\sin(kL)$$

Dividing by $\cos(kx_b)$ gives

$$y_a \tan(kx_b) = y_b \tan(kx_b)\cos(kL) - y_b \sin(kL)$$

Rearranging and solving for $\tan(kx_b)$ yields

$$\tan(kx_b) = \frac{y_b \sin(kL)}{y_b \cos(kL) - y_a}$$

Thus,

$$x_b = \frac{1}{k}\tan^{-1} \frac{\sin(kL)}{\cos(kL) - y_a/y_b} \tag{16}$$

or, utilizing equations (12) and (13)

$$x_b = \frac{1}{k}\tan^{-1} \frac{\sin(kL)}{\cos(kL) - M_a/M_b} \tag{17}$$

Finally

$$\alpha_o = \frac{ky_b}{\sin(kx_b)} \tag{18}$$

or in terms of M_a, M_b, and P

$$\alpha_o = \frac{kM_b}{P} \sqrt{1 + \left[\frac{\cos(kL) - M_a/M_b}{\sin(kL)}\right]^2} \tag{19}$$

Equation (19) is also valid with very little error for an inelastic beam-column with reversed curvature since yielding will only occur over a short distance near the ends and will not appreciably affect y_a and y_b. The y-displacements of the inelastic as well as the elastic column deflection curve are therefore correctly given by equation (14). The end slopes, α_a and α_b in Figure 3 are affected by yielding, however, and must be determined by a numerical integration procedure using an appropriate nonlinear moment curvature relationship. Such relationships have been derived theoretically and verified experimentally (15). Figure 4 shows a set of nondimensionalized M-Ø curves for an 8WF31 section and Figure 5 shows the comparison of one of the curves with those of other wide-flange sections. It is seen that there is very little variation in the shape of the curves for a wide range of sections.

It will be assumed that the moment-curvature relationship (Figure 4) becomes nonlinear when the extreme fiber stress in the beam-column reaches σ_y, i.e. when

$$\frac{P}{A} + \frac{Mc}{I} = \sigma_y$$

or when

$$M = M_y = \frac{r^2}{c} (P_y - P) \tag{20}$$

where

$$P_y = A\sigma_y$$

and r is the radius of gyration of the cross-section.

When the moment reaches M_{pc}, the curvature becomes indeterminate and a plastic hinge forms. M_{pc} is determined from the following equations:

$$M_{pc} = M_p - \frac{P^2}{4\sigma_y w} \qquad \text{if } 0 \le P \le \sigma_y w(d-2t) \tag{21}$$

or

$$M_{pc} = \frac{d}{2} (P_y - P) - \frac{1}{4b\sigma_y} (P_y - P)^2$$

$$\text{if } \sigma_y w(d-2t) \le P \le P_y \tag{22}$$

where w is the web thickness, d is the depth of the section, t is the flange thickness and b is the flange width (31).

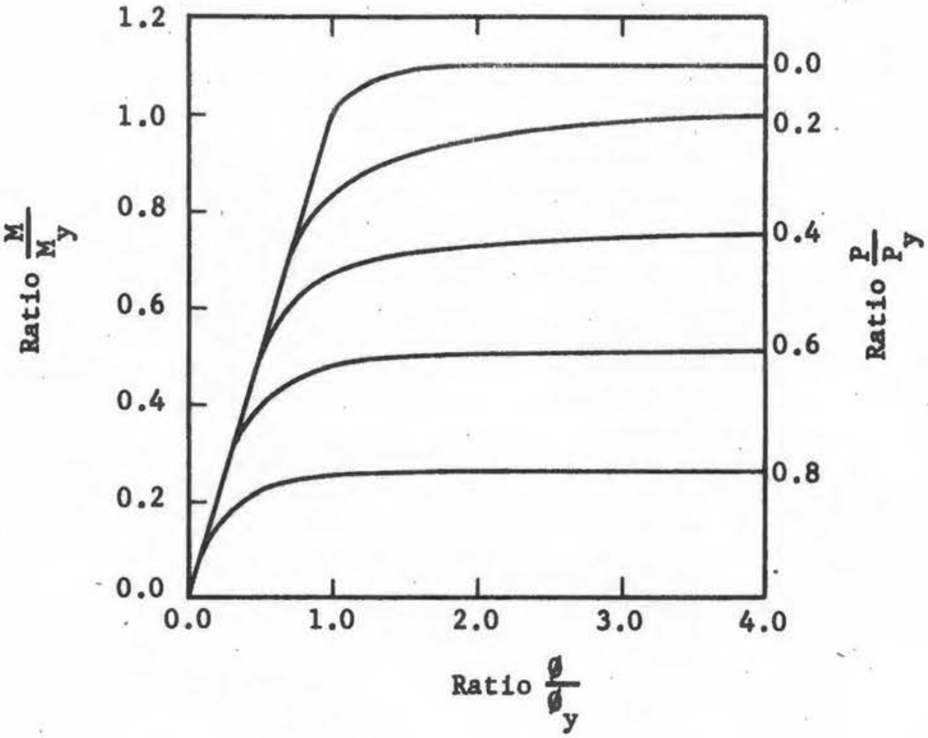


Figure 4. Nonlinear Moment-Curvature Diagram

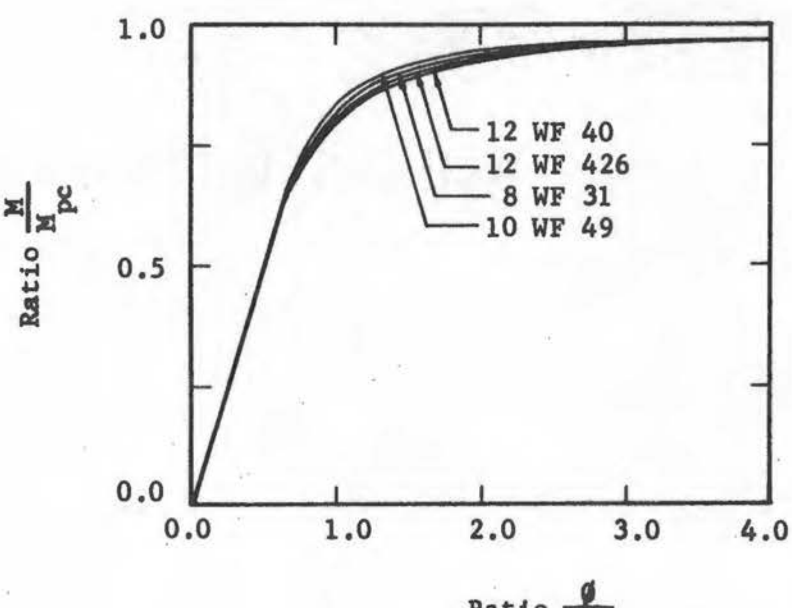


Figure 5. Comparison of M-ϕ Curves

In the present study the moment curvature relationship is expressed by

$$\phi = \frac{M}{EI} \qquad \text{if } M \le M_y$$
$$\phi = \frac{M}{EI} + \frac{hM_{pc}}{EI}\left(\frac{M-M_y}{M_{pc}-M_y}\right)^n \qquad \text{if } M_y \le M \le M_{pc} \tag{23}$$

Equation (23) consists of a parabola of degree n connecting the elastic and plastic regions of the moment-curvature diagram. The coefficient h represents the ratio of inelastic curvature to elastic curvature at formation of a plastic hinge. By varying n and h this equation can be made to approximate quite closely most moment curvature diagrams. In the example problems, values of four and three were used for n and h respectively to represent the curves in Figure 4.

3.3 Numerical Integration

The slope α of the elastic portions of the column deflection curves may be obtained directly from equation (15). When the moment M=Py exceeds M_y, the curvature is nonlinear and the slopes are obtained by integration using the curvatures given by equation (23). Referring to equations (14) and (15), we let

$$y_e = \frac{M_y}{P} \tag{24}$$

$$x_e = \frac{1}{k}\sin^{-1}\left(\frac{ky_e}{\alpha_o}\right) \tag{25}$$

$$\alpha_e = \alpha_o \cos(kx_e) \tag{26}$$

these are the starting values for the numerical integration (Figure 6). A suitable increment Δx, between one and four times the radius of gyration of the cross-section, is selected and the slope α_i is determined at $x_i = x_e + \Delta x$. The average moment in the interval is taken as

$$M_i = Py_i = \frac{P\alpha_o}{k} \sin \left[k(x_e + \frac{\Delta x}{2})\right] .$$

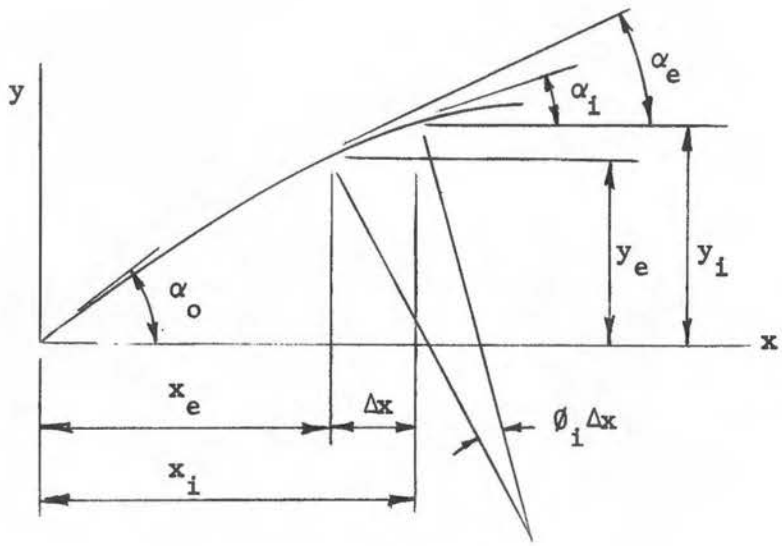


Figure 6. Numerical Integration

The curvature $\emptyset_i$ in the interval is assumed to be constant and is obtained from equation (23). The slope at x_i becomes

$$\alpha_i = \alpha_e - \emptyset_i \Delta x$$

In general

$$x_i = x_{i-1} + \Delta x \tag{27}$$

$$M_i = \frac{P\alpha_o}{k} \sin \left[k(x_{i-1} + \frac{\Delta x}{2})\right] \tag{28}$$

$$\alpha_i = \alpha_{i-1} - \emptyset_i \Delta x \tag{29}$$

The numerical integration proceeds until $x_i + \Delta x \geq x_a$ at which time $\Delta x' = x_a - x_i$. Finally

$$M_{i+1} = \frac{P\alpha_o}{k}\sin\left[k(x_i + \frac{\Delta x'}{2})\right] \tag{28a}$$

and

$$\alpha_a = \alpha_i + \phi_{i+1}\Delta x' \tag{29a}$$

The calculation for α_b is similar and may be carried out simultaneously with the calculation for α_a. It is convenient to perform the numerical integration using positive values, correcting the signs at the end. The sign of α_a and of α_b is the same as that of α_o which in turn is the same as M_b.

The end slopes of the beam-column may now be determined. The axis of the beam-column deviates from the axis of the column deflection curve by the angle β (Figure 3) where

$$\beta = \frac{y_b - y_a}{L} \tag{30}$$

and the end slopes are

$$\theta_a = \alpha_a - \beta \tag{31}$$

$$\theta_b = \beta - \alpha_b \tag{32}$$

These theta values may be broken down into the linear and nonlinear end rotations. Denoting by λ the nonlinear rotations ,

$$\theta_a = \frac{M_a}{3EI} + \frac{M_b}{6EI} + \lambda_a$$

$$\theta_b = \frac{M_b}{3EI} + \frac{M_a}{6EI} + \lambda_b$$

from which

$$\lambda_a = \theta_a - \left(\frac{M_a}{3EI} + \frac{M_b}{6EI}\right) \tag{33}$$

$$\lambda_b = \theta_b - \left(\frac{M_b}{3EI} + \frac{M_a}{6EI} \right) \tag{34}$$

The lambdas may be broken down into temporary (elastic) end rotations due to the axial force in the beam-column and permanent (plastic) end rotations due to yielding. The permanent end rotations λ_p are equal to the difference between the end slopes of the column deflection curve as calculated by equation (15) and by equation (29a):

$$\lambda_{ap} = \alpha_a - \alpha_o \cos(kx_a) \tag{35}$$

$$\lambda_{bp} = \alpha_o \cos(kx_b) - \alpha_b \tag{36}$$

The temporary end rotations λ_e are:

$$\lambda_{ae} = \lambda_a - \lambda_{ap} \tag{37}$$

$$\lambda_{be} = \lambda_b - \lambda_{bp} \tag{38}$$

CHAPTER IV

NONLINEAR FRAME ANALYSIS

4.1 Method of Nonlinear Analysis

Although a structure which is subjected to a large enough load will become nonlinear as the result of yielding or excessive deflection, in most cases it is still primarily elastic. In frames of the type considered in this thesis, subjected to horizontal forces, the moment gradients (change in moment along a member) are high so that yielding is confined to rather small areas. Rather than treating a nonlinear frame, then, as an assemblage of inelastic elements that must be joined together in some way so as to satisfy the conditions of equilibrium and compatibility, we will deal with a frame which is essentially elastic but which has finite discontinuities at the points of yielding.

The most severe discontinuity that can develop in a frame is the formation of a plastic hinge. The angular relationship of the members at the joint can no longer be determined by the application of Hooke's law. A major modification to the frame flexibility matrix is required. This was accomplished in Chapter II with the first order elastic-plastic analysis based on the augmented flexibility matrix.

The next most severe influence on the behavior of a flexible frame is the effect of sidesway deflection on the equilibrium of the frame. This is taken into account by formulating the equilibrium conditions in terms of the sidesway deflections which are then determined by iteration.

Unless the frame is on the verge of instability, the deflection should converge after one or two cycles of iteration.

Finally, small discontinuities will occur at the joints as the result of nonlinear behavior of individual members. These fall into two classes: (1) partial yielding at the ends of members when the moment is greater than M_y but less than M_{pc}, (2) bowing of the member when the axial force is sufficiently large to produce additional end rotations in a bent member. These effects are included also by iteration after the approximate end moments and axial forces become known.

4.2 Outline of Procedure

The steps in the nonlinear analysis are as follows:

(1) Perform a flexibility analysis assuming the frame to be elastic, equations (2) and (3), to determine the moments of the joints and the axial forces in the members.

(2) Compute M_{pc} for each member, equation (21) or (22), and compare with the joint moments. Place plastic hinges at joints where the moment is greater than M_{pc} by augmenting the frame flexibility matrix with rows and columns giving the locations of the hinges, equation (6). If no plastic hinges are found, go to step (4).

(3) Perform an elastic-plastic analysis, equation (7), and recalculate the joint moments, equation (3). Determine the axial forces in the members and repeat step (2) if the change in moments is appreciable.

(4) Compute the deflections of the joints and recalculate the basic moments using the deformed structure to formulate the equilibrium conditions. Go back to steps (1) or (3), depending on whether hinges have formed, if the change in deflection is appreciable.

(5) Compute M_y for each member, equation (20), and compare with the joint moments. If any moment is between M_y and M_{pc}, determine the lambda values, equations (33) and (34), for the affected member and perform a nonlinear analysis, equations (9) or (10). If there are no moments between M_y and M_{pc}, go to step (6).

(6) Iteration is continued until (a) the sidesway deflection and joint moments converge in which case the structure is stable, (b) the sidesway deflection diverges (elastic instability), or (c) a sufficient number of plastic hinges form to create a collapse mechanism.

(7) If the collapse load is desired it may be determined by a systematic procedure whereby a sequence of increasing loads is applied to the frame. An analysis is performed at each load until at some load a collapse mechanism develops or the sidesway deflection fails to converge. The maximum load lies between this load and the previous load.

4.3 Unloading

When unloading or load reversal takes place after partial yielding of the frame has occurred, permanent angular deformations will exist at the joints that have undergone yielding. If only partial yielding of a member has occurred ($M < M_{pc}$), the permanent deformations λ_p are given by equations (35) and (36). If full yielding has taken place, i.e. if plastic hinges have formed, the permanent angular deformations γ are determined from equation (10). These will be combined into a single term $\bar{\gamma}$ where

$$[\bar{\gamma}] = [\gamma] + [\lambda_p] \qquad (39)$$

Equation (10) becomes then

$$\begin{bmatrix} R \\ \gamma \end{bmatrix} = \begin{bmatrix} AFA & A' \\ A'^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} -AFBM-A\lambda-A\bar{\gamma} \\ M_{pc} - BM' \end{bmatrix} \tag{40}$$

Note that the sign of M_{pc} will change if the sign of the corresponding joint moment changes.

As the result of load reversal, a frame is likely to return to the elastic state with previously formed plastic hinges becoming inactive. Furthermore, new plastic hinges may form in different locations from the original ones. This means that it is necessary to start a new analysis assuming an elastic frame each time reversal of loading takes place. Thus for the first calculation the flexibility equation would have the form of equation (9) instead of equation (10):

$$[R] = [AFA]^{-1} [-AFBM-A\lambda-A\bar{\gamma}] \tag{41}$$

The permanent joint rotations $[\bar{\gamma}]$ must be included in all subsequent computations in accordance with equations (40) and (41).

CHAPTER V

NUMERICAL RESULTS

5.1 Example Frames

A number of frames have been analyzed using a computer program written for the IBM 7040 electronic computer at Oklahoma State University. A flow diagram of the program is included in Appendix A and some of the details of solution are given in Appendix B.

The frames which were analyzed are all similar to the configuration shown in Figure 7 with the exception that the second group (11-16) has pinned bases. Listed in Table I are the properties of the frames and the maximum loads according to the three most significant theories, simple plastic, elastic-plastic with reduced plastic moment, and nonlinear (including geometry change, bowing, and nonlinear moment-curvature relationship).

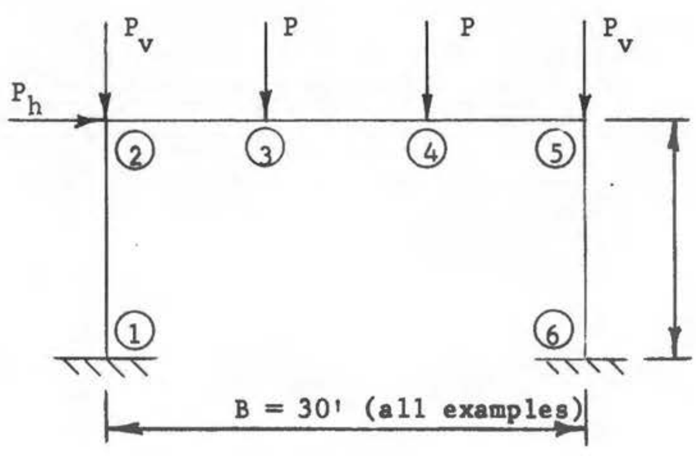


Figure 7. Example Frame

TABLE I

SUMMARY OF RESULTS

Frame No.	H, ft.	Beam		Columns		Loads		P_{max}, kips		
		Section	M_p, kip-ft.	Section	M_p, kip-ft.	P_h/P	P_v/P	Simple Plastic	Elastic Plastic	Nonlinear
1	15	21WF127	953.4	14WF119	632.7	1.5	0	97.2	95.0	95.0
2	15	21WF127	953.4	14WF119	632.7	1.5	4	97.2	82.5	77.5
3	15	10WF45	165.0	8WF40	119.7	1.5	4	17.8	16.5	15.0
4	15	10WF45	165.0	8WF40	119.7	2	6	16.0	13.5	12.0
5	20	10WF45	165.0	8WF40	119.7	2	6	12.0	11.0	9.5
6	15	8WF35	104.1	8WF20	57.3	1.5	4	10.2	9.0	7.6
7	15	8WF35	104.1	8WF20	57.3	2	6	7.64	6.8	5.8
8	20	8WF35	104.1	8WF20	57.3	2	6	5.73	5.4	4.6
9	20	8WF35	104.1	8WF20	57.3	2	10	5.73	5.0	4.0
10	30	8WF35	104.1	8WF20	57.3	2	10	3.82	3.5	2.7
11	15	21WF127	953.4	14WF119	632.7	1.5	4	56.3	50.0	45.0
12	15	8WF35	104.1	8WF20	57.3	1.5	4	5.10	4.8	3.8
13	15	8WF35	104.1	8WF20	57.3	2	6	3.82	3.6	3.0
14	20	8WF35	104.1	8WF20	57.3	2	6	2.86	2.8	2.2
15	20	8WF35	104.1	8WF20	57.3	2	10	2.86	2.7	1.9
16	30	8WF35	104.1	8WF20	57.3	2	10	1.91	1.8	1.2

Note: Frames 1-10 have fixed bases and frames 11-16 have pinned bases.

According to the simple plastic theory, the maximum load is determined by equating the external work of the loads to the internal work of the plastic hinges as the collapse mechanism is moved through a small displacement. The elastic-plastic method, as used in the examples, assumes a reduction in plastic moment values but includes no other nonlinear effects.

5.2 Graphs of Results

Figure 8 shows the variation in moments as the loads increase proportionally for Frame No. 1 according to the elastic-plastic method. The numbering of the moment curves corresponds to the numbering of the joints. The graph consists of segmented straight lines with breaks occurring at loads corresponding to the formation of plastic hinges. The column loads P_v are zero for this frame so there is only a slight reduction in the plastic moment values. Figure 9 shows the moment variation for the same frame when it is subjected to column loads and the nonlinear effects are included in the analysis.

The moment variation for a more slender frame is shown in Figure 10 and the relationship between load and deflection for this frame according to the three theories is shown in Figure 11. (Simple plastic theory assumes zero deflection until the collapse load P_p is attained, then unlimited deflection). The results of the analysis of frames with pinned bases are shown in Figures 12, 13 and 14.

To illustrate the versatility of the method of nonlinear analysis presented herein, the loading of frame No. 2 was modified to include a change in direction of the horizontal force P_h. The loads were increased proportionally until seven-eighths of the maximum load was

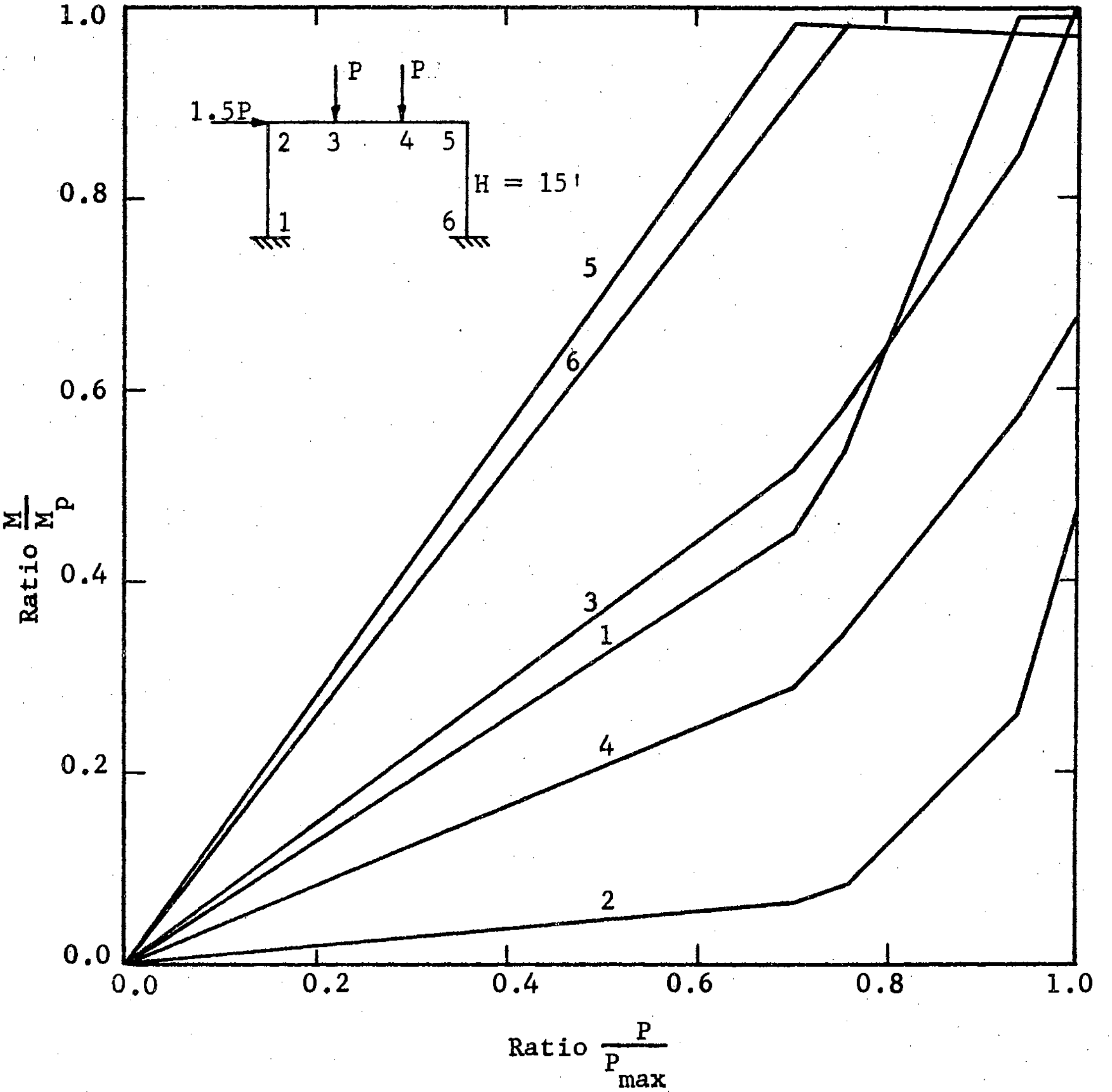


Figure 8. Elastic-Plastic Moments vs. Load - Frame No. 1

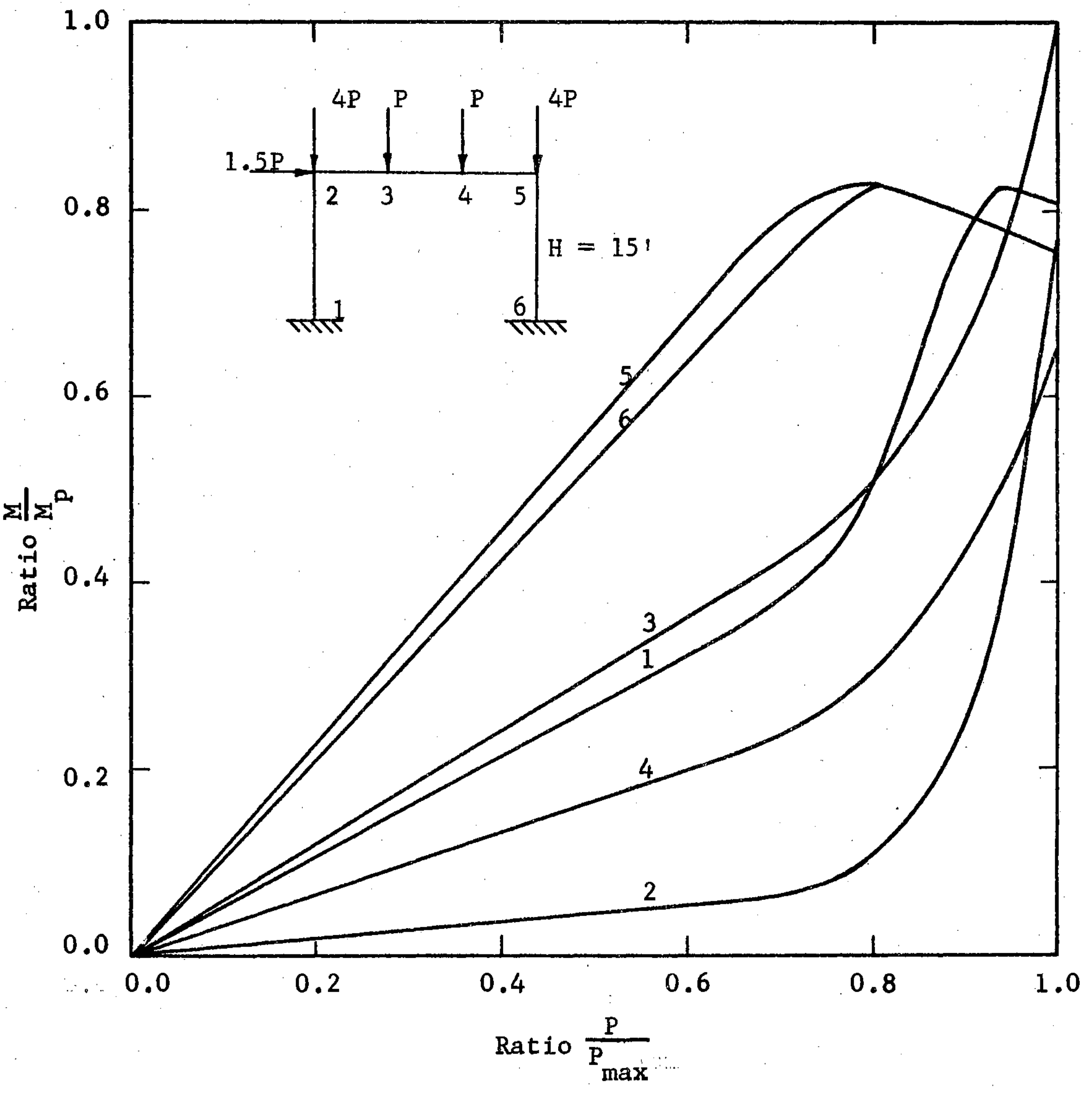


Figure 9. Nonlinear Moments vs. Load - Frame No. 2

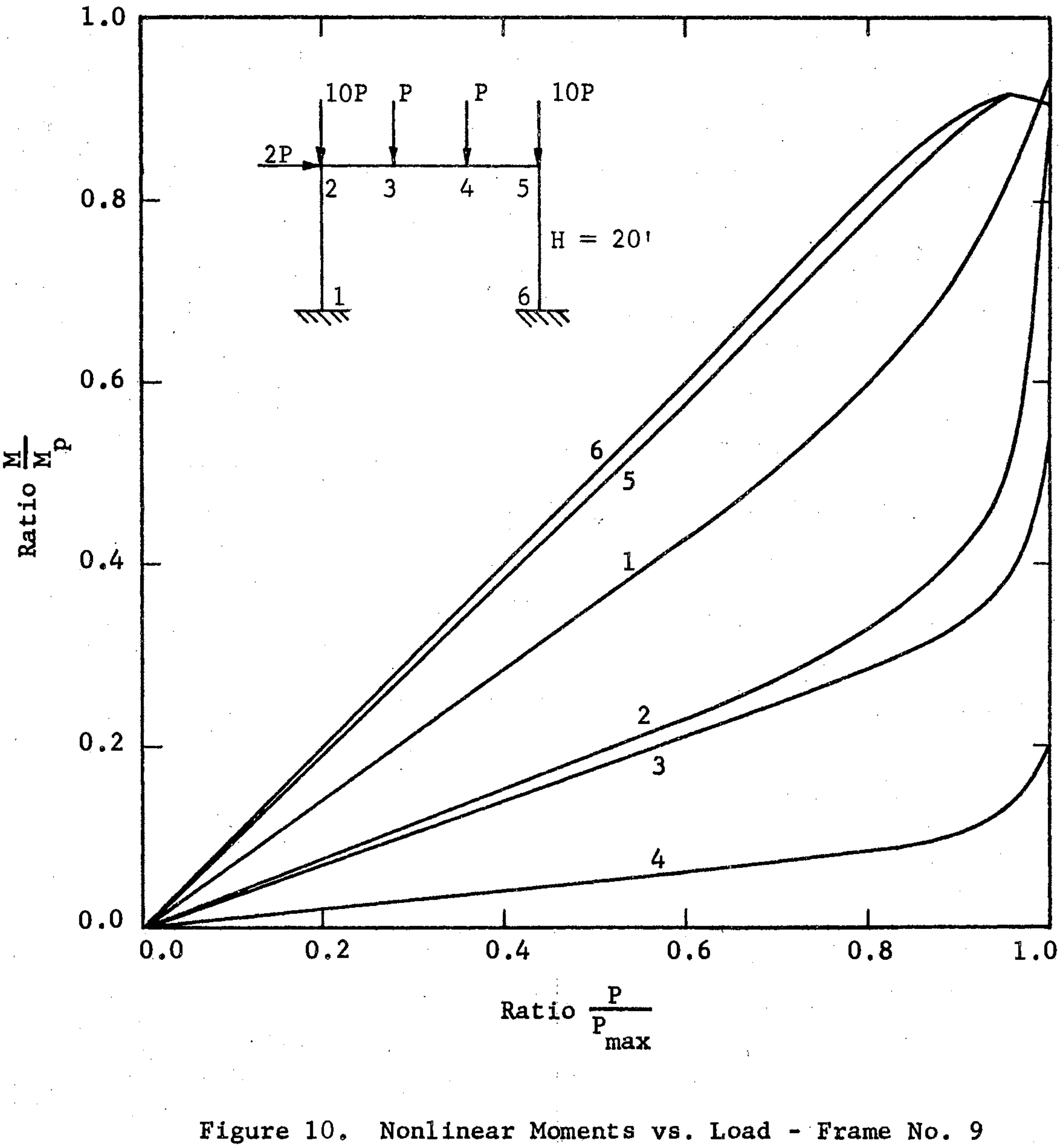


Figure 10. Nonlinear Moments vs. Load - Frame No. 9

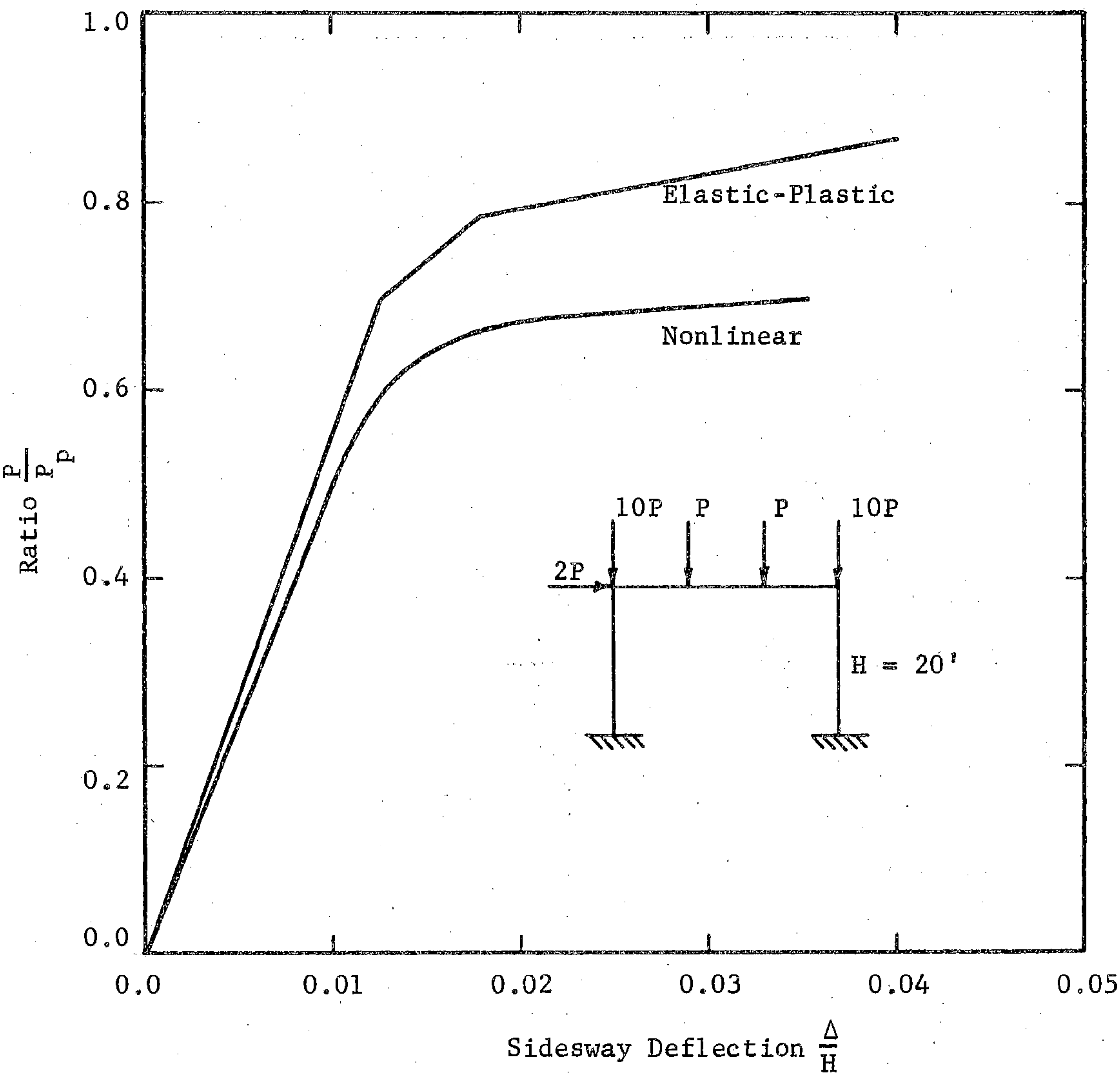


Figure 11. Load vs. Deflection - Frame No. 9

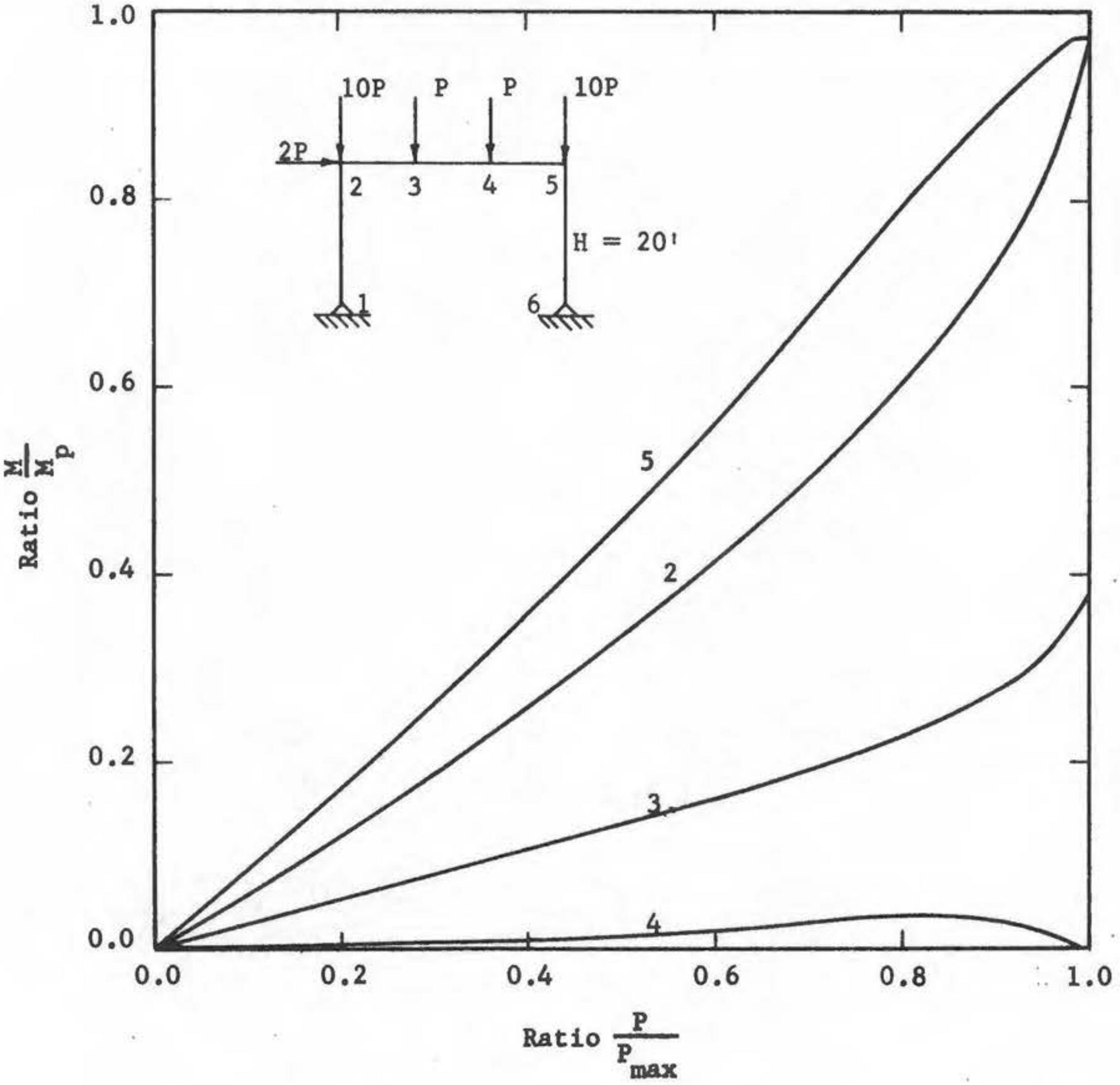


Figure 12. Nonlinear Moments vs. Load - Frame No. 15

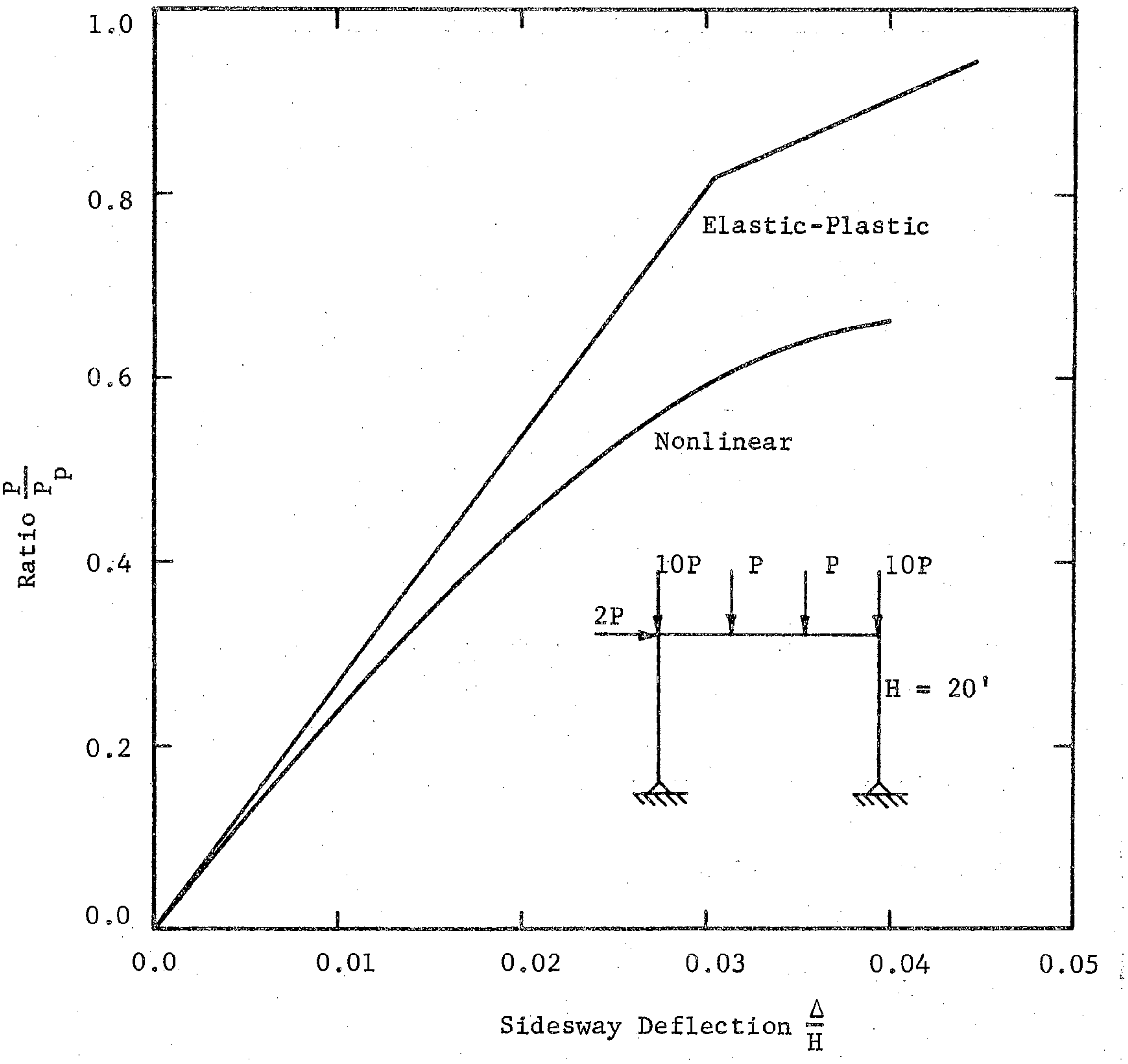


Figure 13. Load vs. Deflection - Frame No. 15

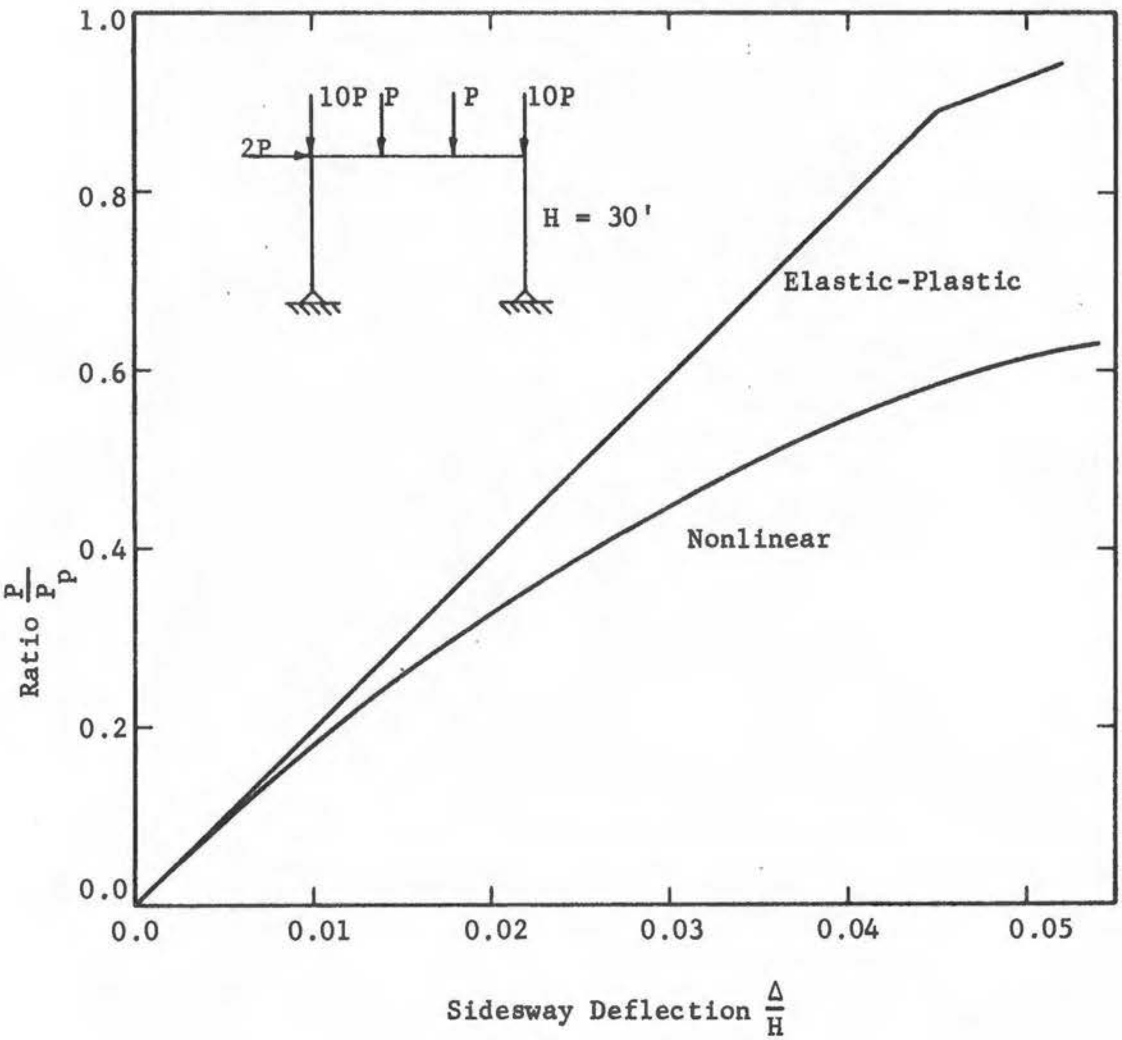


Figure 14. Load vs. Deflection - Frame No. 16

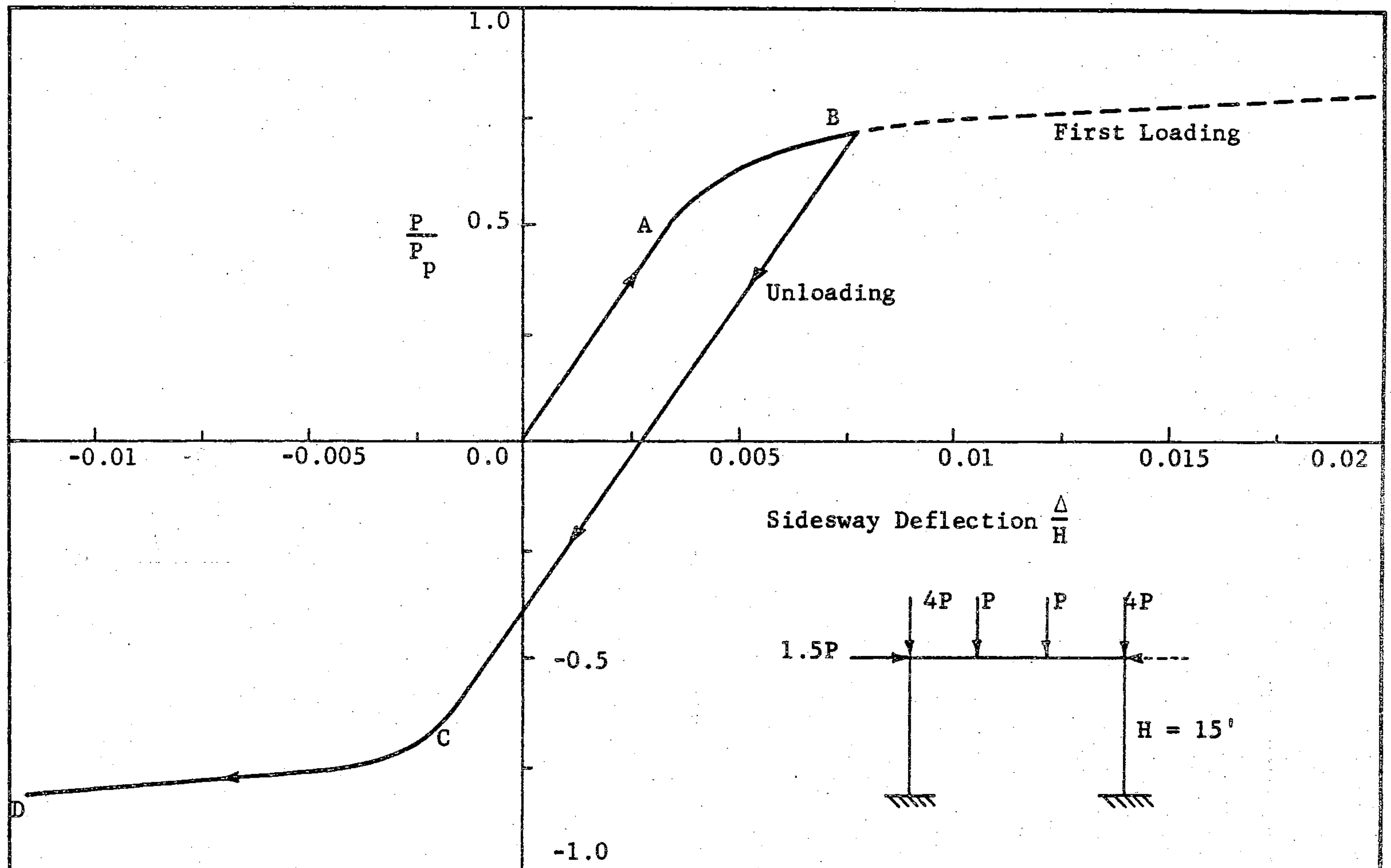


Figure 15. Load vs. Deflection Curve for Reversal of Loading

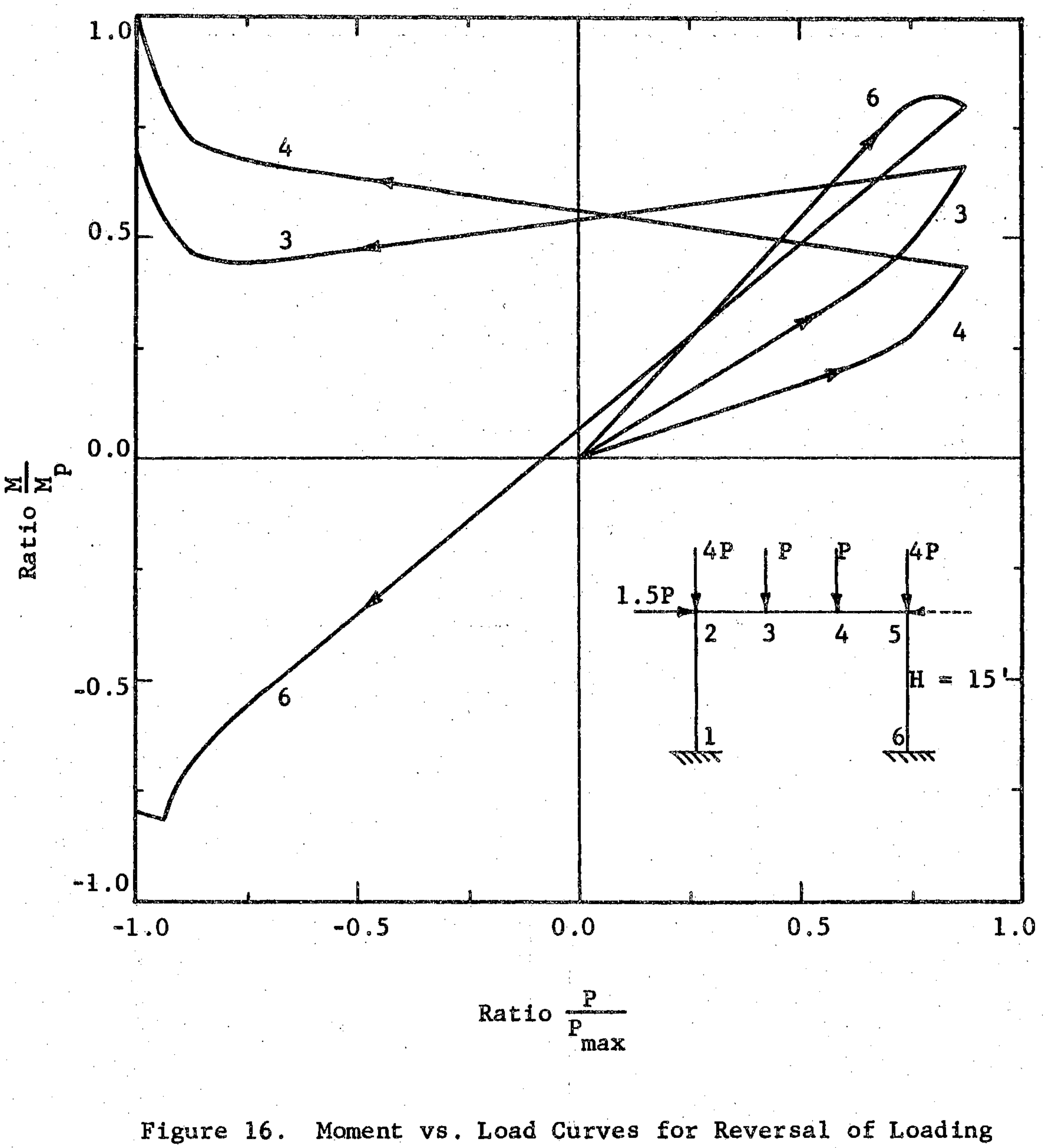


Figure 16. Moment vs. Load Curves for Reversal of Loading

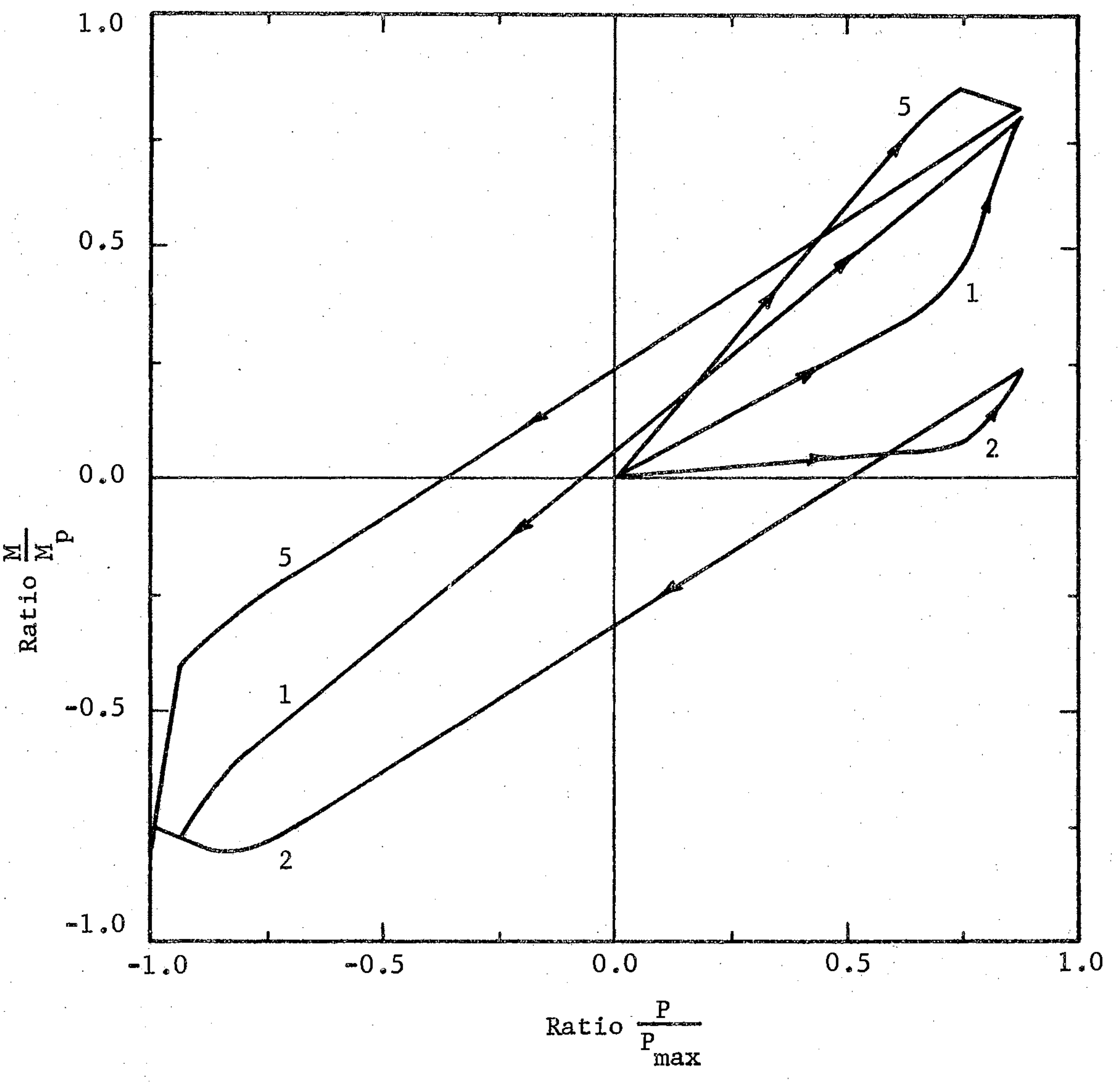


Figure 17. Moment vs. Load Curves for Reversal of Loading

reached at which time P_h was gradually reversed in direction. Thereafter the loads were again increased proportionally until failure. Figure 15 shows the resulting deflection curve with the start of yielding indicated at A and the reversal of load taking place at B. Yielding again occurs at C and failure comes at D. The moment curves, Figures 16 and 17, show the moments increasing uniformly until first yielding occurs, and then plastic hinges develop at joints 5 and 6. Upon reversal of the loading, the moments at joints 1, 2, 5, and 6 change sign while the signs of the moments at joints 3 and 4 remain unchanged.

5.3 Discussion of Results

Frame No. 1 is essentially the same frame that was analyzed by Richard and Goldberg (25) who included in their analysis only the effect of a nonlinear moment-curvature relationship. A comparison of Figure 8 with Figure 7 on page 46 of the reference will show that the primary effect of the nonlinear moment-curvature relationship is a rounding of the moment curves. In this example, the error introduced by neglecting the reduction in plastic moment values and the effect of geometry change (sidesway) is nearly negligible.

With the addition of column loads equal to 4P, however, the effects of reduced plastic moment and of sidesway are no longer negligible as is seen from Figure 9 (Frame No. 2). The reduction in plastic moment values is about 17 per cent of the full plastic moment at maximum load and the maximum load according to simple plastic theory is in error by about 25 per cent if these factors are not taken into account. Note that the plastic hinges form first at joints 5 and 6, then at joint 1, and finally at joint 3.

Frame No. 9 (Figures 10 and 11) is more flexible so the reduction in plastic moment values is not as great, only 8 per cent, and the effect of sidesway is more pronounced. The error in the maximum load if sidesway is neglected is 24 per cent and if sidesway and reduced plastic moment are both neglected the error is 43 per cent. In this case the last plastic hinge was formed at joint 2.

Frames with pinned bases naturally tend to be more flexible than frames with fixed bases. For these frames the effect of sidesway is more significant than is the effect of the reduction in plastic moment values as shown in Figures 13 and 14. The gradual deviation of the non-linear load-deflection curves from the elastic-plastic curves is the result of the sidesway effect. This is also evident in the gradual curving of the moment curves in Figure 12. The magnitude of the effect of the reduced plastic moment values is indicated by the difference between the maximum elastic-plastic and the maximum simple plastic loads.

The effect of reversing the direction of the horizontal force after plastic hinges have formed at two of the joints (joints 5 and 6) is shown in Figures 15, 16, and 17. It will be observed that the frame behaves elastically during most of the load reversal. Plastic hinges then form on the opposite side of the frame (joints 1 and 2) and eventually one of the original hinges (joint 6) yields in the opposite direction. The final hinge forms in the beam at joint 4.

Due to the permanent angular deformations of the joints, the sidesway deflections to the left of the vertical position are less than the deflections to the right at corresponding loads without load reversal. As a result, the sidesway effect is less severe and the maximum load (80 kips) is greater than in the case of loading in one direction.

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Summary

A method has been presented for the nonlinear analysis of rectangular plane frames. The method is based on an elastic flexibility analysis to which corrections are applied to account for the nonlinear effects.

The corrections applied to individual members include the beam-column effect due to axial force and the effect of yielding due to nonlinear moment-curvature relationship. These effects are calculated by numerical integration of the curvature function along the beam-column. An equation is derived for determining the initial slope of the column-deflection curve associated with the beam-column. This makes possible the direct determination of the end slopes of an inelastic beam-column with yielding at the ends.

The nonlinear corrections applied to the frame as a whole include the effects of geometry change and of the formation of plastic hinges. The effect of geometry change is accounted for by formulating the equilibrium equations on the deformed shape of the frame. Plastic hinges are accounted for by augmenting the flexibility matrix with additional rows and columns expressing the conditions that the moments at the hinges are known values and the hinge rotations are unknown. The reduction in plastic moment values due to axial compression is also included.

Real hinges in a frame may easily be accounted for in the analysis by simply setting the plastic moment values equal to zero at the hinges. Reversals of loading are taken care of by calculating the permanent angular deformations at the joints and then carrying these values along through the subsequent analysis.

The criteria for the determination of ultimate load is the formation of a sufficient number of plastic hinges to create a collapse mechanism or the nonconvergence of the sidesway deflection. When sufficient hinges have developed for the formation of a mechanism, the augmented flexibility matrix becomes singular. The largest load for which the flexibility matrix is nonsingular is taken as the ultimate load.

The essence of the method is that the linear and nonlinear effects are considered separately in the analysis. The elastic frame forms a reference to which the nonlinear effects are attached as small corrections. In this way the proper relationship between the linear effects and the nonlinear effects is maintained.

6.2 Conclusions

It has been demonstrated that the matrix-polygon method of structural analysis can be adapted to the nonlinear analysis of rectangular plane frames. Corrections in the form of inelastic weights are applied to the conjugate frame and the correct solution is obtained by iteration. Equilibrium is formulated on the deformed structure.

The method has several advantages over other existing methods. In addition to the collapse load, the method provides the moments at each joint in the frame for any load. The method is based on a systematic iteration procedure, rather than trial and error, for which convergence is rapid when the nonlinear effects are considered in the proper relationship.

It has been illustrated in the examples that in frames having high column loads such as the lower floors of tall buildings an elementary plastic analysis may be highly unconservative. The significant factors are the reduction in plastic moment capacity resulting from the axial force in the columns and the effect of geometry change on equilibrium.

It appears that the effect of residual stresses and shape factor on the moment-curvature relationship (causing rounding of the M-$\emptyset$ diagram) and the spread of plastification are significant only in determining the moments just prior to the formation of a plastic hinge. They have very little, if any, influence on the ultimate load which is dependent on the fully plastic moments.

The effect of bowing appears to be negligible for columns in double curvature. It is certainly the least significant of the nonlinear effects that were considered and probably could just as well have been ignored. With the pinned-base frames, however, the columns of which are in single curvature, this factor is of greater importance.

In analyzing each example frame, a series of incremented loads was applied in order to determine the behavior of the frame throughout the entire loading range up to the maximum load. The increments were between one-twentieth and one-fortieth of the maximum load. Using this approach satisfactory convergence of the joint moments and sidesway deflection was obtained after two cycles of iteration in most cases.

Convergence was not quite so good just prior to the formation of the plastic hinges when the lambda values would have the greatest effect. The deflection and the moment values had a tendency to oscillate and in those cases, the values from the first and second, or from the second and third cycles of iteration were averaged for plotting the curves. Although this was not serious, the tendency of the values to oscillate could be reduced by decreasing the values of the parameters n and h in the curvature expression, equation (23). This would increase the slope of the moment-curvature diagram at the point where a plastic hinge is assumed to form making the curvature less affected by small changes in moment.

If the frame becomes elastically unstable, the sidesway deflection will not converge. This was very nearly the case with example frame No. 16 (Figure 14) although failure was by the formation of plastic hinges due to the moments resulting from the large deflection.

The effect of geometry change was included in several ways with about the same results in each case. To be completely correct, the linear transmission matrix [A] should be adjusted for each change in sidesway deflection and the frame flexibility matrix [AFA] recalculated. This is unnecessary, however, with rectangular frames as long as the basic moments are reasonably close to the final moments so that the redundants are small. In all cases the basic moments are formulated on the deflected structure.

6.3 Possible Extensions

The possibilities for extending the matrix-polygon method of non-linear analysis are practically limitless. It should be possible to include the nonlinear effects in the analysis of any structure for which the string-polygon or matrix-polygon methods can be used. Further investigation is needed, however, before the method can be applied to highly redundant structures such as multi-story rigid frames and structures with several degrees of freedom.

Consideration also needs to be given to the effect of strain hardening (4, 14, 16). This factor could very well have a greater influence on frame behavior than the effects of bowing or of nonlinear moment curvature relationship.

SELECTED BIBLIOGRAPHY

(1) Augusti, Giuliano, "Experimental Rotation Capacity of Steel Beam-Columns," Journal of the Structural Division, ASCE, Vol. 90, No. ST6, Proc. Paper 4175, December, 1964, pp. 171-188.

(2) Bleich, F., Buckling Strength of Metal Structures, Chapters VI and VII, McGraw Hill, New York, 1952.

(3) Chu, Kuang-Han, and Algis Pabarcius, "Elastic and Inelastic Buckling of Portal Frames," Journal of the Engineering Mechanics Division, ASCE, Vol. 90, No. EM5, Proc. Paper 4094, October, 1964, pp. 221-249.

(4) Davies, Michael J., "Frame Instability and Strain Hardening in Plastic Theory," Journal of the Structural Division, ASCE, Vol. 92, No. ST3, Proc. Paper 4836, June, 1966, pp. 1-15.

(5) Driscoll, George C., et al., "Lecture Notes - Plastic Design of Multi-Story Frames," Fritz Engineering Laboratory Report, No. 273.20, Dept. of Civil Engineering, Lehigh University, Summer, 1965.

(6) Driscoll, George C., "Lehigh Conference on Plastic Design of Multi-Story Frames - A Summary", AISC Engineering Journal, April, 1966.

(7) Galambos, Theodore V. and Jagdish Prasad, "Ultimate Strength Tables for Beam-Columns," Welding Research Council Bulletin, No. 78, June, 1962.

(8) Galambos, Theodore V., and Maxwell G. Lay, "Studies of the Ductility of Steel Structures," Journal of the Structural Division, ASCE, Vol. 91, No. ST4, Proc. Paper 4444, August, 1965, pp. 125-151.

(9) Gauger, Fred N., "Plastic Deformation Analysis of Frames at Ultimate Load by the String Polygon Method," Master's Thesis Submitted to the Graduate School of the Oklahoma State University, August, 1961.

(10) Gerstle, Kurt H., and Vassilios Zarboulas, "Elastic-Plastic Deformations of Steel Structures," Journal of the Structural Division, ASCE, Vol. 89, No. ST1, Proc. Paper 3419, February, 1963, pp. 179-196.

(11) Goldberg, John E. and Ralph M. Richard, "Analysis of Nonlinear Structures," Journal of the Structural Division, ASCE, Vol. 89, No. ST4, Proc. Paper 3604, August, 1963, pp. 333-351.

(12) Hall, A. S. and R. W. Woodhead, Frame Analysis, John Wiley and Sons, Inc., New York, 1961.

(13) Horne, M. R. and K. I. Majid, "The Design of Sway Frames in Britain, Guest Lectures, the 1965 Summer Conference on Plastic Design of Multi-Story Frames," Fritz Engineering Laboratory Report, No. 273.46, Dept. of Civil Engineering, Lehigh University, 1966.

(14) Hrennikoff, Alexander D., "Importance of Strain Hardening in Plastic Design," Journal of the Structural Division, ASCE, Vol. 91, No. ST4, Proc. Paper 4424, August, 1965, pp. 23-34.

(15) Ketter, R. L., E. L. Kaminski, and L. S. Beedle, "Plastic Deformation of Wide Flange Beam Columns," Transactions, ASCE, Vol. 120, 1955, p. 1028.

(16) Lay, Maxwell G., and Paul D. Smith, "Role of Strain Hardening in Plastic Design," Journal of the Structural Division, ASCE, Vol. 91, No. ST3, Proc. Paper 4355, June, 1965, pp. 25-43.

(17) Lind, Niels C., "Analysis of Deflections in Elastic-Plastic Frames," Journal of the Structural Division, ASCE, Vol. 91, No. ST3, Proc. Paper 4381, June, 1965, pp. 197-218.

(18) Lind, Niels C., "Iterative Limit Load Analysis for Tall Frames," Journal of the Structural Division, ASCE, Vol. 90, No. ST2, Proc. Paper 3867, April, 1964, pp. 103-129.

(19) Lu, Le-Wu, "A Survey of Literature on the Stability of Frames," Welding Research Council Bulletin, No. 81, September, 1962.

(20) Lu, Le-Wu, "Inelastic Buckling of Steel Frames," Journal of the Structural Division, ASCE, Vol. 91, No. ST6, Proc. Paper 4577, December, 1965, pp. 185-214.

(21) Moses, Fred, "Inelastic Frame Buckling," Journal of the Structural Division, ASCE, Vol. 90, No. ST6, Proc. Paper 4169, December, 1964, pp. 105-121.

(22) Ojalvo, Morris and Le-Wu Lu, "Analysis of Frames Loaded into the Plastic Range," Journal of the Engineering Mechanics Division, ASCE, Vol. 87, No. EM4, Proc. Paper 2884, August, 1961, pp. 35-48.

(23) Ojalvo, Morris and Yuhshi Fukumoto, "Nomographs for the Solution of Beam-Column Problems," Welding Research Council Bulletin, No. 78, June, 1962.

(24) Ojalvo, Morris, "Restrained Columns," Journal of the Engineering Mechanics Division, ASCE, Vol. 86, No. EM5, Proc. Paper 2615, October, 1960, pp. 1-11.

(25) Richard, Ralph M. and John E. Goldberg, "Analysis of Nonlinear Structures: Force Method," Journal of the Structural Division, ASCE, Vol. 91, No. ST6, Proc. Paper 4553, December, 1965, pp. 33-48.

(26) Saafan, S. A., "Nonlinear Behavior of Structural Plane Frames," Journal of the Structural Division, ASCE, Vol. 89, No. ST4, Proc. Paper 3615, August, 1963, pp. 557-579.

(27) Smith and Sidebottom, Inelastic Behavior of Load-Carrying Members, John Wiley and Sons, New York, 1965.

(28) Timoshenko, Stephen P. and James M. Gere, Theory of Elastic Stability, McGraw Hill, New York, 1961.

(29) Tuma, Jan J., Seminar Notes, Spring, 1964.

(30) Van Kuren, Ralph C. and Theodore V. Galambos, "Beam-Column Experiments," Journal of the Structural Division, ASCE, Vol. 90, No. ST2, Proc. Paper 3876, April, 1964, pp. 223-256.

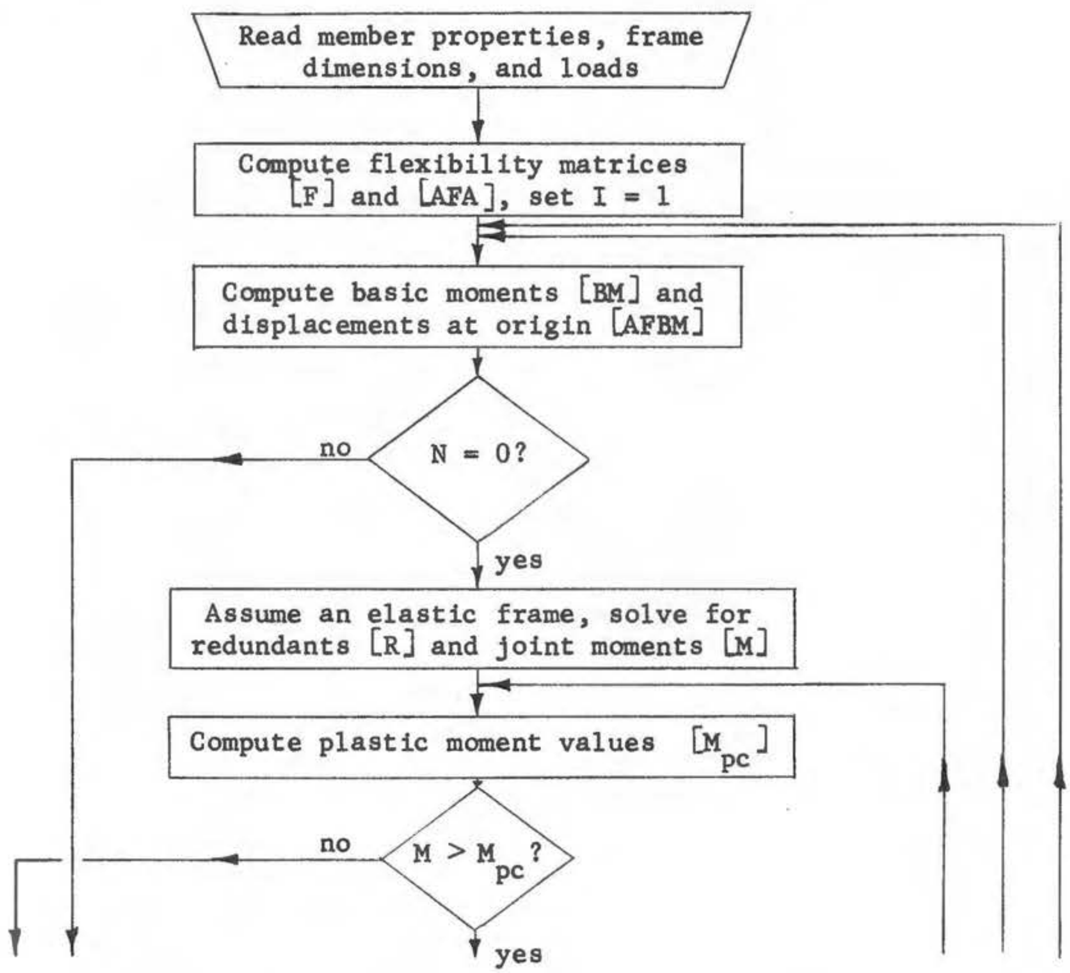
(31) Welding Research Council - ASCE Joint Committee, Commentary on Plastic Design in Steel, Chapter VII, Manual 41, ASCE, New York, 1961.

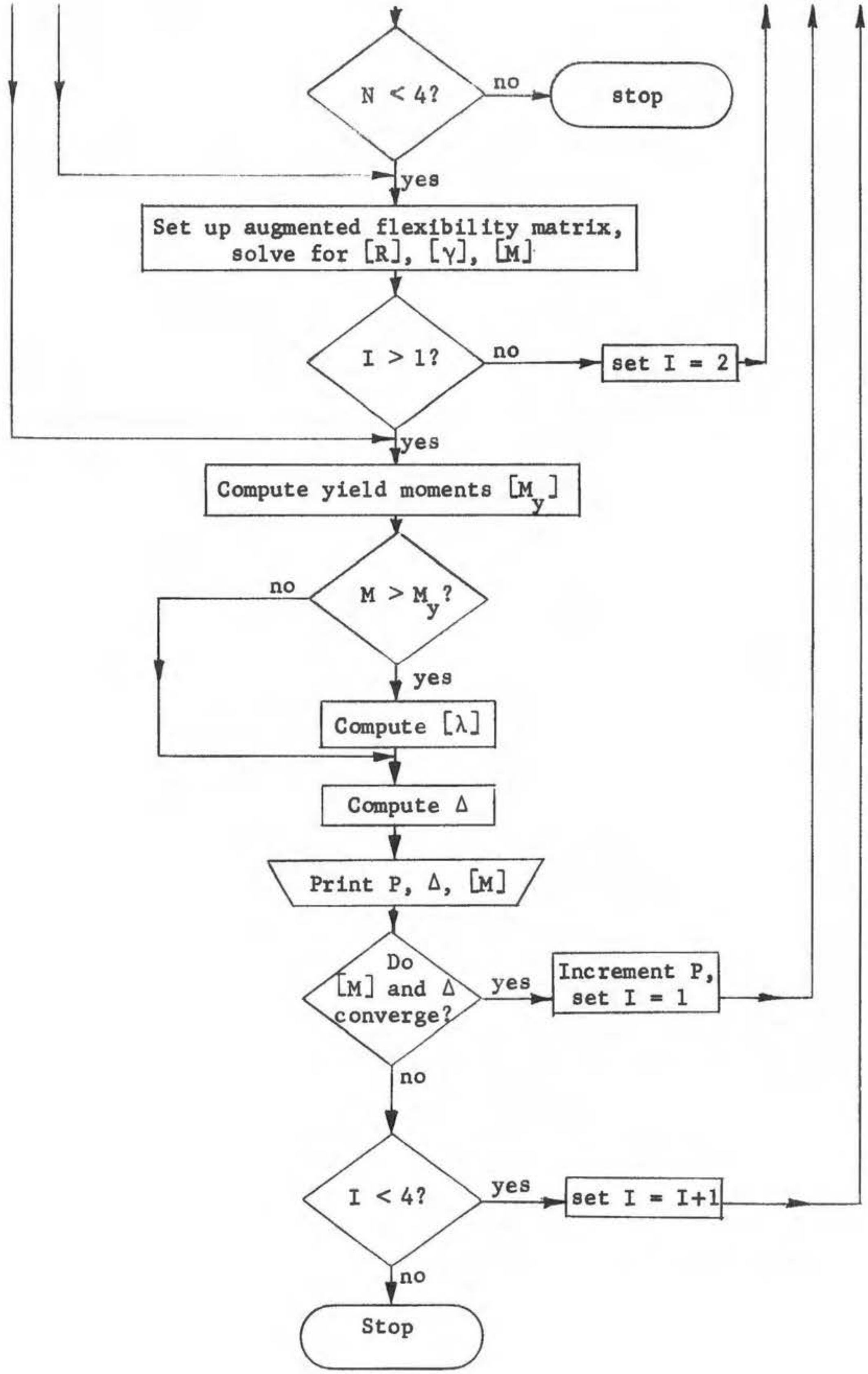
(32) Yen, Yu-Chin, Le-Wu Lu, and G. C. Driscoll, Jr., "Tests on the Stability of Welded Steel Frames," Welding Research Council Bulletin, No. 81, September, 1962.

APPENDIX A

COMPUTER FLOW DIAGRAM

The flow diagram which follows represents essentially the manner in which the computer program was set up for solving the example problems. On the diagram N indicates the number of hinges and I represents the number of cycles of iteration. It is assumed that the maximum load has been reached when four hinges have formed or when the sidesway deflection fails to converge after four cycles of iteration.





N < 4?
no
stop
yes
Set up augmented flexibility matrix, solve for [R], [γ], [M]
I > 1?
no
set I = 2
yes
Compute yield moments [M_y]
no
M > M_y ?
yes
Compute [λ]
Compute Δ
Print P, Δ, [M]
Do [M] and Δ converge?
yes
Increment P, set I = 1
no
I < 4?
yes
set I = I+1
no
Stop

APPENDIX B

AUGMENTED FLEXIBILITY MATRIX

The columns of frame No. 1 are 14 WF 119 sections with I = 1373.1 in^4 and M_p = 632.7 Kip-ft and the beam is a 21 WF 127 section with I = 3017.2 in^4 and M_p = 953.4 kip-ft. Steel having a yield point of 36 ksi is assumed for all the examples. The joints are numbered from one to six as shown in Figure 7.

The member flexibility matrix for frame No. 1 is

$$[F] = \begin{bmatrix} 17.48 & 8.74 & & & & \\ 8.74 & 22.78 & 2.65 & & & \\ & 2.65 & 10.60 & 2.65 & & \\ & & 2.65 & 10.60 & 2.65 & \\ & & & 2.65 & 22.78 & 8.74 \\ & & & & 8.74 & 17.48 \end{bmatrix} x10^{-6}$$

where the diagonal terms are the sum of the flexibilities of the members at each joint, e.g.

$$F_{22} = \sum \frac{L}{3EI} = \frac{15\ (144)}{3(30{,}000)\,(1373.1)} + \frac{10\ (144)}{3(30{,}000)\,(3017.2)} = 22.78 \times 10^{-6}$$

The off diagonal terms are the carry over values, e.g.

$$F_{12} = \frac{L}{6EI} = \frac{15\ (144)}{6(30{,}000)\,(1373.1)} = 8.74 \times 10^{-6}$$

Choosing the origin of coordinates at the center of the base, the linear transmission matrix is

$$[A] = \begin{bmatrix} y_i \\ x_i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 15 & 15 & 15 & 15 & 0 \\ -15 & -15 & -5 & 5 & 15 & 15 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The frame flexibility matrix is found by matrix multiplication:

$$[AFA] = [A][F][A]^T = \begin{bmatrix} 18.60 & 0 & 1.502 \\ 0 & 27.18 & 0 \\ 1.502 & 0 & 0.1526 \end{bmatrix} \times 10^{-3}$$

The first plastic hinge forms at joint five and the augmented flexibility matrix is written

$$[AFA_1] = \left[\begin{array}{ccc|c} 18.60 & 0 & 1.502 & 15 \\ 0 & 27.18 & 0 & 15 \\ 1.502 & 0 & 0.1526 & 1 \\ \hline 15 & 15 & 1 & 0 \end{array}\right]$$

For the second and third hinges, which form at joints six and one respectively, the augmented flexibility matrix becomes

$$[AFA_3] = \left[\begin{array}{ccc|ccc} 18.60 & 0 & 1.502 & 15 & 0 & 0 \\ 0 & 27.18 & 0 & 15 & 15 & -15 \\ 1.502 & 0 & 0.1526 & 1 & 1 & 1 \\ \hline 15 & 15 & 1 & 0 & 0 & 0 \\ 0 & 15 & 1 & 0 & 0 & 0 \\ 0 & -15 & 1 & 0 & 0 & 0 \end{array}\right]$$

The fourth hinge will develop at joint three. The addition of one more row and column to the flexibility matrix to represent this condition will result in a singular matrix, therefore this represents the collapse condition.

APPENDIX C

DERIVATION OF MATRIX POLYGON EQUATIONS

The matrix polygon method is based on the concept of representing the members of a frame as strings for which the angle changes are analogous to forces which are in equilibrium according to the equations of statics. The points of intersection of the "strings" are referred to as joints and are usually selected at points of discontinuity in the frame or at points of application of concentrated loads.

The moments at the joints are the sum of the basic moments and the moments due to redundants where the basic moments [BM] are defined as any set of joint moments which satisfy the conditions of equilibrium for the loads on the frame. Moments are considered positive when they produce compression on the outside of the frame. At joint i, Figure 18, the moment is

$$M_i = BM_i + R_x y_i + R_y x_i + R_z \tag{42}$$

In matrix form this is expressed

$$[M] = [BM] + [A]^T\,[R] \tag{43}$$

where

$$[A] = \begin{bmatrix} y_1 & y_2 & - - & y_i & - - & y_6 \\ x_1 & x_2 & - - & x_i & - - & x_6 \\ 1 & 1 & - - & 1 & - - & 1 \end{bmatrix} \quad \text{and} \quad [R] = \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}$$

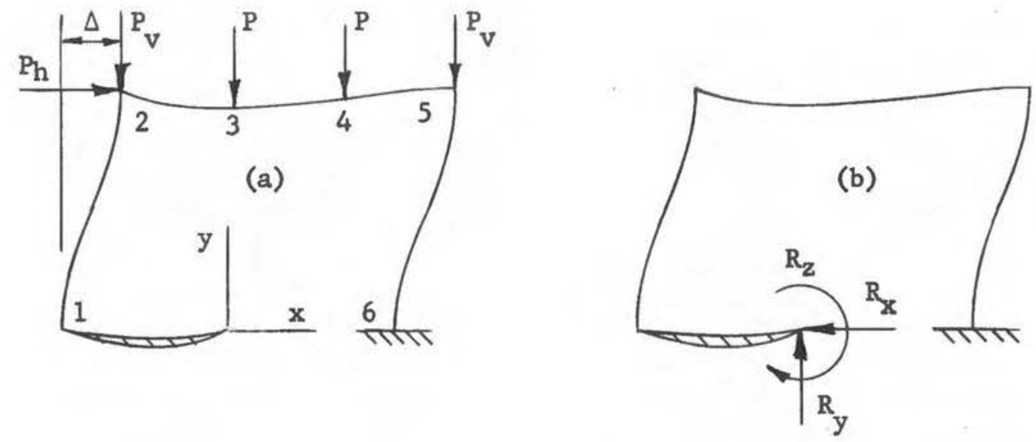


Figure 18. Basic Frame and Redundants

When the change in geometry is taken into account, the x-coordinates of joints 2-5 are functions of the sidesway deflection Δ, for example at joint 5, $x_5 = B/2 + \Delta$ where B is the width of the frame.

The angle changes at the joints (elastic weights), designated $\overline{P}$, are functions of the joint moments and the loads on the members between joints. For member ij, at end i:

$$\overline{P}_{ij} = f_{ij}M_i + g_{ij}M_j + \tau_{ij} \tag{44}$$

and at end j:

$$\overline{P}_{ji} = f_{ji}M_j + g_{ji}M_i + \tau_{ji} \tag{45}$$

in which f_{ij} is the end rotation of end i due to a unit moment at i; g_{ij} is the end rotation of end i due to a unit moment at j; and τ_{ij} and τ_{ji} are angular functions due to the loads on the member.

At joint j the elastic weight is the sum of the elastic weights at the joint for each member. Thus,

$$\overline{P}_j = g_{ji}M_i + f_{ji}M_j + f_{jk}M_j + g_{jk}M_k + \tau_{ji} + \tau_{jk} \tag{46}$$

or

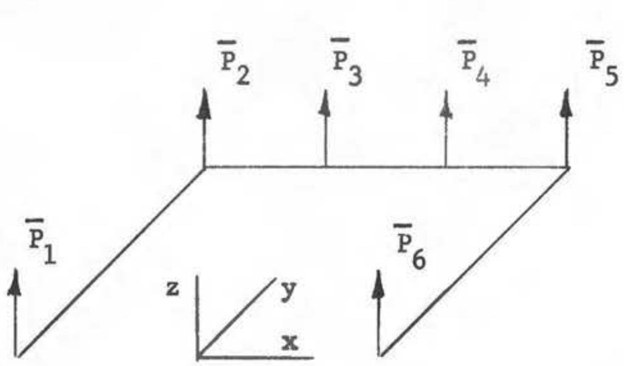


Figure 19. Elastic Weights Applied to Conjugate Frame

$$\overline{P}_j = F_{ji}M_i + F_{jj}M_j + F_{jk}M_k + \tau_j \tag{47}$$

where $F_{ji} = g_{ji}$, $F_{jj} = f_{ji} + f_{jk}$, and $\tau_j = \tau_{ji} + \tau_{jk}$. In matrix form equation (47) becomes

$$[\overline{P}] = [F][M] + [\tau] \tag{48}$$

In this thesis, it is assumed that all loads are applied at the joints, therefore, $[\tau] = [0]$. Additional angle changes are present, however, in the form of lambda values resulting from bowing and from yielding at the ends of members. Thus $[\tau]$ will be replaced by $[\lambda]$ and equation (48) becomes

$$[\overline{P}] = [F][M] + [\lambda] \tag{49}$$

Plastic hinge rotations $[\gamma]$ are included in a similar manner.

The condition that the displacements $[\delta]$ at the origin must be zero for a rigid frame is expressed by the following stereo-static equations (Figure 19):

$$\begin{aligned} \delta_x &= \Sigma\, \overline{P}_i y_i = 0 \\ \delta_y &= \Sigma\, \overline{P}_i x_i = 0 \\ \delta_z &= \Sigma\, \overline{P}_i = 0 \end{aligned} \tag{50}$$

In matrix form this may be expressed

$$[\delta\rfloor = [A]\lfloor\bar{P}] = [0] \tag{51}$$

where $[A]$ was defined previously.

Equations (49) and (51) may now be combined to give

$$[\delta] = [A][F][M] + [A]\lfloor\lambda] = [0] \tag{52}$$

Finally, substituting the expression for $[M]$ from equation (43), the matrix polygon equation may be written:

$$[\delta] = [A][F][BM] + \lfloor A][F][A]^T[R\rfloor + [A]\lfloor\lambda\rfloor = \lfloor 0] \tag{53}$$

APPENDIX D

COMPARISON WITH METHOD OF RICHARD AND GOLDBERG

Frame No. 1 (Table I) is similar to the one analyzed by Richard and Goldberg (25). Figure 20 shows the comparison of their results with the author's in the determination of the nonlinear moments.

The discrepancies between the two sets of curves are primarily due to the fact that Richard and Goldberg neglected the effects of sidesway and reduced plastic moment values. The maximum load determined by Richard and Goldberg's analysis (P_{max} = 114 kips) is based on steel having a yield point of 42 ksi and is the same as the simple plastic load. According to the author's analysis, using a yield point of 36 ksi, the maximum load is 95 kips whereas the simple plastic load is 97.2 kips. Since the curves are nondimensionalized, the influence of using different yield points is negligible except that the sidesway effect would be more pronounced with the higher yield point steel. Using the value of 95 kips rather than 97.2 kips for the maximum load accounts for a difference of about 2.3 per cent in the nondimensionalized loads. This is offset somewhat by the sidesway effect which in general tends to increase the moments at a given load.

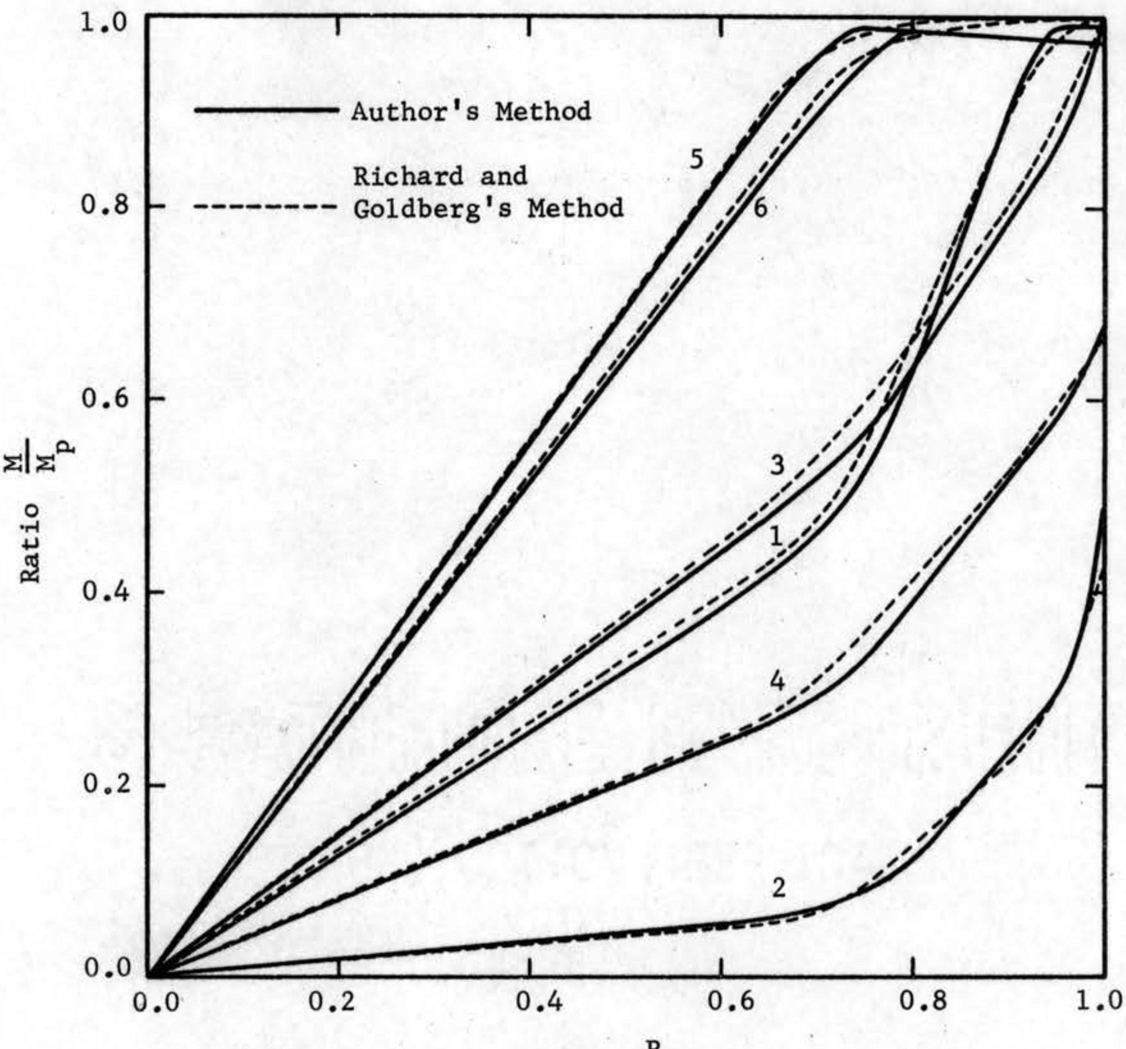


Figure 20. Comparison of Nonlinear Moment vs. Load Curves

VITA

Philip Norcross Eldred

Candidate for the Degree of

Doctor of Philosophy

Thesis: NONLINEAR ANALYSIS OF RECTANGULAR PLANE FRAMES BY MATRIX POLYGON

Major Field: Engineering

Biographical:

Personal Data: Born February 15, 1935, in Burlington, Vermont, the son of Horace and Helen Eldred.

Education: Graduated from Burlington High School, Burlington, Vermont, in June, 1952. Received the Bachelor of Science degree in Civil Engineering from the University of Vermont, Burlington, Vermont in June, 1956. Received the Master of Science degree from the University of Vermont in June, 1963. Attended the National Science Foundation Summer Institutes for Teachers of Structures at Oklahoma State University in 1960 and 1963. Attended the Conference on Plastic Design of Multi-Story Frames at Lehigh University in 1965. Member of Tau Beta Pi, Chi Epsilon and Sigma Xi. Completed the requirements for the degree of Doctor of Philosophy in July, 1967.

Professional Experience: Construction Engineer, U. S. Army, Corps of Engineers, 1956-1958. Instructor, University of Vermont, 1958-1963. Graduate Assistant and Instructor, Oklahoma State University, 1963-1967. Member of A.S.C.E., A.S.E.E., N.S.P.E., and Vermont Society of Professional Engineers. Registered Professional Engineer in Vermont.