SAMPLE SIZE DETERMINATION FOR BAYESIAN TOLERANCE INTERVALS

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TOLERANCE INTERVALS

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CHAPTER I

INTRODUCTION

The procedure used to obtain tolerance regions can be stated as follows: the result x of an experiment E is used to obtain a region $R(x)$ in which it is predicted that a proportion p of future replicates y of E will occur. For example, a manufacturer of transistors may wish to predict the lifetime which will be exceeded by 95 per cent of a batch of transistors. From a sample of the batch he obtains an interval, $R(x) = [r, \infty)$, in which it is predicted 95 per cent of the transistor lifetimes will occur. In slightly less general terms it is predicted that the interval obtained will cover the upper 95 per cent of the distribution of lifetimes.

The determination of tolerance intervals for distribution free variates was first developed by Wilks [9], who utilized the distribution of the order statistics. Others, including Bain [4] and Wald and Wolfowitz [8], obtained tolerance intervals for variables whose distribution belonged to some specific family of distributions. For the example of the previous paragraph, if the manufacturer had reason to believe that the lifetimes of each batch of transistors he obtained had a normal distribution, and that only the parameters changed from batch to batch, then he would use the results of the latter approach. If he could not assume a specific family of distributions, then he would use the results of Wilks.

Until 1964, when Aitchison [1] formulated a Bayesian approach, the field of tolerance regions had not attracted the interests of the Bayesians. This approach extends the parametric approach by assuming that the parameters of the family of distributions have some known distribution. It is with this approach that we shall be concerned and in particular it is the purpose of this thesis to investigate criteria of accuracy for Bayesian tolerance intervals and to relate these criteria to the determination of sample size. Since the Bayesian assumption of a prior distribution for the parameters of the family of densities of interest is a touchy one, we shall also investigate the sensitivity of tolerance intervals for the exponential distribution to inaccuracies in the assumption of the prior distribution.

Formulation of the Problem

Let $x_{1,n}$ represent the observations $x_1, x_2, \ldots, x_n$ from an experiment E, where the x_i come from the density $f(\cdot|\theta)$. Let y be a future observation from $f(\cdot|\theta)$ where the indexing parameter θ has the same value for $x_{1,n}$ and y. The coverage of a region, $R(x_{1,n})$, is defined as

$$C(R|\theta) = \int_{R(x_{1,n})} f(y|\theta)dy.$$

We shall consider two types of tolerance intervals. The first is a q tolerance interval for p coverage. The interpretation of this interval is that we have q confidence that the coverage will be at least p. This will be the topic of Chapter II. The second type, to be discussed in Chapter III is a p-expected coverage tolerance interval.

Throughout this thesis we shall be concerned with intervals of the form $[r, \infty)$, or in other words, we shall be interested in obtaining a lower tolerance limit, r. However, upper tolerance limits can be obtained by replacing q and p by $1 - q$ and $1 - p$, respectively.

The Bayesian Approach

Let $f'(\theta)$ represent the density of θ defined on the parameter space Ω, where θ is the parameter of $f(\cdot|\theta)$. Then, by Bayes' theorem, we obtain

$$f''(\theta|x_{1,n}) = \frac{f(x_{1,n}|\theta)f'(\theta)}{\int_{\Omega} f(x_{1,n}|\theta)f'(\theta)d\theta} ,$$

the posterior density of θ, conditional on $x_{1,n}$. This density will be used to obtain Bayesian tolerance limits and tolerance intervals.

We may be led to make the assumption of a prior density, $f'(\theta)$, if θ is actually a random variable or if θ is fixed, but unknown, and $f'(\theta)$ represents the "state of knowledge" about θ. The first reason would be applicable to the example at the first of this chapter. The average lifetime of a batch might be θ, and this would vary from batch to batch in accordance with $f'(\theta)$.

For the Bayesian approach we will be concerned with probabilities arising from $f''(\theta|x_{1,n})$. The non-Bayesian approach is concerned with probabilities arising from $f(x_{1,n}|\theta)$ and which are interpreted as the relative frequency arising from repeated experimentation, so that we shall refer to this as the frequentist approach.

Notation

Let U be a random variable which has the chi-square distribution with m degrees of freedom. Then we shall denote its density by $\chi^2(m)$ and its cumulative distribution function evaluated at u as $C(u; m)$. The solution for u of $C(u; m) = q$ will be $\chi_q^2(m)$ and will be referred to as the q probability point of U. If U has a noncentral t distribution with m degrees of freedom and noncentrality parameter c, then its density will be denoted by $t'(m, c)$ and its q probability point by $t_q'(m, c)$. These same quantities will be denoted by $t(m)$ and $t_q(m)$ for the central t distribution. The density for a normally distributed random variable with mean μ and variance σ^2 will be denoted by $N(\mu, \sigma^2)$. The q probability point of a standard normal variable will be denoted by z_q. The expression $\Pr\{u : \text{statement} \mid v\} = q$ will be interpreted as follows: Let $S(u)$ be the values of u which satisfy the statement. Then

$$q = \int_{S(u)} dF(u|v).$$

CHAPTER II

q TOLERANCE INTERVALS FOR p COVERAGE

Our objective is to obtain an interval $R(x_{1,n}) = [r(x_{1,n}), \infty)$ in which we have q confidence that $C(R|\theta)$ is at least p. The frequentist approach is to determine $R(x_{1,n})$ so that

$$\Pr\{x_{1,n} : C(R|\theta) > p|\theta\} = q \quad \text{for all} \quad \theta.$$

This is accomplished by solving $F(d_{1-p}(\theta)|\theta) = 1 - p$, where $F(d|\theta) = \int_{-\infty}^{d} f(y|\theta)dy$, for $d_{1-p}(\theta)$ and substituting a q confidence limit on θ for θ in $d_{1-p}(\theta)$. If $d_{1-p}(\theta)$ is a decreasing function of θ, then an upper q confidence limit is used, since we wish to have q confidence in obtaining a value less than $d_{1-p}(\theta)$. If $d_{1-p}(\theta)$ is an increasing function of θ, a lower q confidence limit is used. The interpretation of the interval obtained is that if samples of size n are repeatedly taken and $r(x_{1,n})$ obtained, then 100q per cent of the intervals $[r(x_{1,n}), \infty)$ will cover the interval $[d_{1-p}(\theta), \infty)$, regardless of what value θ has.

The Bayesian approach, as formulated by Aitchison [1], differs from the frequentist in that instead of substituting a q confidence limit on θ into $d_{1-p}(\theta)$, a q probability point of $f''(\theta|x_{1,n})$ is substituted. An upper q probability point is used when $d_{1-p}(\theta)$ is a decreasing function of θ and a lower q probability point when it is an increasing function of θ. To better see this, let $\delta_q(x_{1,n})$ be the q probability point of $f''(\theta|x_{1,n})$ and consider the case where $d_{1-p}(\theta)$ is a decreasing function of θ. Then

$$q = \Pr\{\theta : \theta \le \delta_q(x_{1,n})|x_{1,n}\}$$

$$= \Pr\{\theta : d_{1-p}(\delta_q(x_{1,n})) \le d_{1-p}(\theta)|x_{1,n}\}.$$

Thus if we take $r(x_{1,n}) = d_{1-p}(\delta_q(x_{1,n}))$, we will have q confidence that the interval $[r(x_{1,n}), \infty)$ covers the interval $[d_{1-p}(\theta), \infty)$. If $d_{1-p}(\theta)$ is an increasing function of θ, then

$$q = \Pr\{\theta : \theta \ge \delta_{1-q}(x_{1,n})|x_{1,n}\}$$

$$= \Pr\{\theta : d_{1-p}(\delta_{1-q}(x_{1,n})) \le d_{1-p}(\theta)|x_{1,n}\}$$

and $r(x_{1,n}) = d_{1-p}(\delta_{1-q}(x_{1,n}))$ will be the Bayesian lower tolerance limit for this case.

The interpretation of this approach is that if we consider the set of values of θ for which the coverage of our interval is at least p, then this set has probability measure, given $x_{1,n}$, of q. In other words, the probability is q that the value of the θ we drew was one of those for which the coverage of the interval was at least p. For the experimenter who just has one opportunity to obtain an interval of p coverage, and who is willing to make the assumption of a prior distribution, this interpretation may be more acceptable than the frequentist interpretation.

An Accuracy Criterion

Since we can obtain Bayesian q tolerance intervals for p

coverage for any sample size, the question arises as to what properties of the interval obtained depend on sample size and what properties measure accuracy in some sense. Since q is the probability, given the sample, that the θ we drew is one for which the coverage is at least p, we may want to consider the probability, given the sample, that the θ we drew is one for which the coverage is at least p', where p' is between p and 1. If we call this probability q', our tolerance interval would become more accurate as q' decreases in the sense that we have $q - q'$ confidence that the coverage is between p and p'. Thus as a measure of the accuracy of a Bayesian lower tolerance limit we shall consider

$$q' = \Pr\{\theta : r(x_{1,n}) \le d_{1-p'}(\theta) | x_{1,n}\}. \tag{2.1}$$

This is analogous to a frequentist accuracy criterion proposed by Goodman and Madansky [6] and investigated by Faulkenberry [5]. In order that (2.1) be used to predetermine the sample size necessary in order to obtain a limit $r(x_{1,n})$ with a specified accuracy q_0', q' should not be a function of $x_{1,n}$ and should be a decreasing function of n. As we shall see, these conditions are met for some familiar distributions.

The interpretation of a Bayesian tolerance limit and the proposed accuracy criterion (2.1) are illustrated in Figure 1. For a given sample $x_{1,n}$, $r(x_{1,n})$ is obtained, and the coverage of the interval $R(x_{1,n}) = [r(x_{1,n}), \infty)$ is obtained as a function of θ. This is the function which we defined in Chapter I as $C(R|\theta)$. We plot $C(R|\theta)$ and superimpose $f''(\theta|x_{1,n})$. (The ordinates for $C(R|\theta)$ and for $f''(\theta|x_{1,n})$ are not necessarily the same.) The Bayesian tolerance interval then has the property that the set of θ's for which $C(R|\theta)$

is at least p comprises $100q$ per cent of the density $f''(\theta|x_{1,n})$. Thus the value of θ for which $C(R|\theta) = p$ is $\delta_q(x_{1,n})$. If a different sample were obtained, $C(R|\theta)$ would change as would $f''(\theta|x_{1,n})$ but in such a way that the q probability point of $f''(\theta|x_{1,n})$ would still correspond to $C(R|\theta) = p$. The accuracy of the limit obtained is the probability of the set of θ's for which $C(R|\theta)$ is at least p' and so $C(R|\theta) = p'$ corresponds to the value $\theta = \delta_{q'}(x_{1,n})$. For a larger sample size we would expect the situation in Figure 2, that is, a smaller q'.

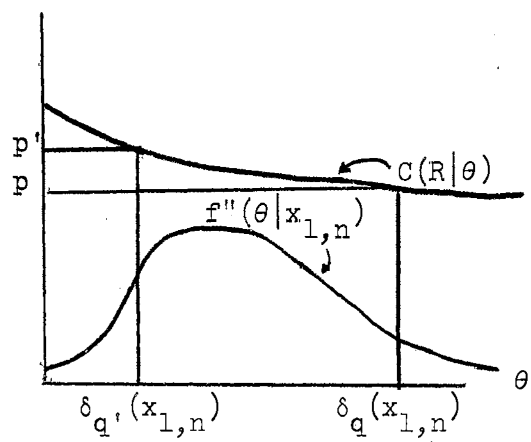

Figure 1

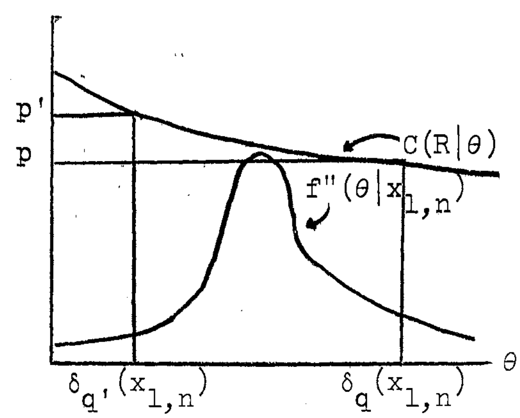

Figure 2

We now consider Bayesian tolerance limits and the accuracy criterion (2.1) for some specific distributions.

The Exponential Distribution

Let

$$f(x|\theta) = \theta e^{-\theta x} \qquad x > 0;\ \theta > 0.$$

We shall consider here a gamma prior density

$$f'(\theta) = b^a\theta^{a-1}e^{-b\theta}/\Gamma(a) \qquad \theta > 0;\ a > 0,\ b > 0.$$

Applying Bayes' theorem gives

$$f''(\theta|x_{1,n}) = (b + z)^{a+n}\theta^{a+n-1}e^{-(b+z)\theta}/\Gamma(a + n), \qquad \theta > 0,$$

where $z = \sum_1^n x_i$.

Now, $\int_d^\infty \theta e^{-\theta x}dx = e^{-\theta d}$ and setting this equal to p we obtain $d_{1-p}(\theta) = (-\ln p)/\theta$, a decreasing function of θ. Since $2\theta(b + z)$ has a $\chi^2(2a + 2n)$ distribution, $\delta_q(x_{1,n}) = \chi^2_q(2a + 2n)/2(b + z)$. Thus, substituting this for θ in $(d_{1-p}|\theta)$, we obtain

$$r(x_{1,n}) = d_{1-p}(\delta_q(x_{1,n})) = \frac{-2(b + z)\ln p}{\chi^2_q(2a + 2n)}\,. \tag{2.2}$$

We note that the frequentist solution for $r(x_{1,n})$ is obtained by taking $a = b = 0$.

To determine the accuracy of the limit $r(x_{1,n})$ we need to evaluate

$$\begin{aligned} q' &= \Pr\{\theta : r(x_{1,n}) \le d_{1-p'}(\theta)|x_{1,n}\} \\ &= \Pr\{\theta : \frac{-2(b + z)\ln p}{\chi^2_q(2a + 2n)} \le \frac{-\ln p'}{\theta}\ |x_{1,n}\} \\ &= \Pr\{\theta : 2(b + z)\theta \le \frac{\ln p'}{\ln p}\ \chi^2_q(2a + 2n)|x_{1,n}\} \\ &= \Pr\{u \le \frac{\ln p'}{\ln p}\ \chi^2_q(2a + 2n)\}, \end{aligned} \tag{2.3}$$

where u has a $\chi^2(2a + 2n)$ density. This is a decreasing function of

n, as we can see from Table I which tabulates values of q' for some combinations of q, $a + n$, p', and p. The corresponding frequentist accuracy is obtained by taking $a = 0$ and thus the Bayesian limit for $a \neq 0$ will be more accurate than the frequentist limit for the same sample size. Larger values of a will lead to a smaller sample size necessary to obtain a given accuracy. That this should be true is seen by considering the mean and variance of θ which are a/b and a/b^2, respectively. If we increase a and also increase b such that the mean remains constant, then the variance will decrease. This improving state of knowledge or decreasing variability of θ is reflected in the requirement of a smaller sample size. Since q' does not depend on $x_{1,n}$, we can determine the sample size necessary to obtain a q tolerance interval for p coverage and a given coefficient of accuracy, q'.

TABLE I

ACCURACY CRITERION FOR BAYESIAN LOWER q TOLERANCE LIMITS FOR p COVERAGE FOR THE EXPONENTIAL DENSITY

p	.90				.95			
q	.90		.95		.90		.95	
p'	.950	.975	.950	.975	.975	.990	.975	.990
a+n								
5	.350	.046	.460	.073	.361	.022	.471	.036
10	.161	.003	.241	.006	.171	.001	.253	.001
15	.073		.123		.080		.133	
20	.034		.061		.038		.068	
25	.015		.029		.017		.034	
30	.006		.014		.008		.016	

Suppose for a specific problem we wish to obtain a Bayesian .95 tolerance interval for .90 coverage and that the prior distribution has parameters $a = 3$ and $b = 2$. If we wish to determine this limit such that q' will be equal to .15 for $p' = .95$, we see from Table I that $a + n$ should be about 13 and thus we will need a sample of size 10. If the sample sum is 6, then the tolerance limit is

$$r(x_{1,n}) = \frac{-2(2 + 6) \ln .90}{\chi^2_{.95}(26)}$$

$$= .043.$$

The coverage of the interval $[.043, \infty)$ is

$$C(R|\theta) = \int_{.043}^{\infty} \theta e^{-\theta x} dx = e^{-.043\theta},$$

which is greater than .90 for θ less than 2.431, the .95 probability point of $f''(\theta|x_{1,n})$. Had the sample sum been 10 we would have obtained $C(R|\theta) = e^{-.065\theta}$ and the .95 probability point of $f''(\theta|x_{1,n})$ to be 1.621, and so for θ less than 1.621, $C(R|\theta)$ will be greater than .90.

The exponential distribution is often encountered in life-testing situations and a procedure often used in this context is to put m units on test and record the times of the first n failures. We now derive the lower Bayesian tolerance limit for this case.

Let $x_1^m, x_2^m, \ldots, x_n^m = x_{n,m}$ be the first n order statistics from a sample of size m. Then

$$f(x_{n,m}|\theta) = \frac{m!}{(m - n)!} \theta^n e^{-\theta(\sum_1^n x_i^m + (m - n)x_n^m)}.$$

Taking $f'(\theta) = b^a\theta^{a-1}e^{-b\theta}/\Gamma(a)$, as before, we obtain

$$f(x_{n,m}) = \int_0^\infty f(x_{n,m}|\theta)f'(\theta)d\theta$$

$$\propto \int_0^\infty \theta^{a+n-1}e^{-(b+z)\theta}d\theta,$$

where $z = \sum_1^n x_i^m + (m - n)x_n^m$. Thus the posterior distribution of θ is

$$f''(\theta|x_{n,m}) = \frac{(b + z)^{a+n}\theta^{a+n-1}e^{-\theta(b+z)}}{\Gamma(a + n)}, \qquad \theta > 0.$$

Hence, (2.2) and (2.3) give the Bayesian tolerance limit and accuracy, where z is defined as above for this case rather than the preceeding. To determine m, the number of units put on test, it would be necessary to relate this functionally to n(e.g. $2n = m$).

As an alternative to the gamma prior density for θ consider the uniform density

$$f'(\theta) = \frac{1}{b - a} \qquad 0 \le a < \theta < b,$$

$$= 0 \qquad \text{elsewhere.}$$

Since the posterior density, given the sample, is equal to the posterior density, given a sufficient statistic, we consider the density of $z = \sum_1^n x_i$ (or $z = \sum_1^n x_i^m + (m - n)x_n^m$, when order statistics are used).

$$f(z|\theta) = z^{n-1}\theta^n e^{-\theta z}/\Gamma(n)$$

$$f(z|\theta)f'(\theta) = \frac{z^{n-1}\theta^n e^{-\theta z}}{(b - a)\Gamma(n)}$$

$$= f''(\theta|z)h(z).$$

Thus

$$f''(\theta|z) \propto \theta^n e^{-\theta z} \qquad a < \theta < b.$$

This implies that

$$f''(\theta|z) = \frac{z^{n+1}\theta^n e^{-\theta z}/\Gamma(n+1)}{\int_a^b \frac{z^{n+1}\theta^n e^{-\theta z} d\theta}{\Gamma(n+1)}} \qquad a < \theta < b,$$

which is a truncated gamma distribution. In order to obtain $\delta_q(x_{1,n})$ we need to solve the equation

$$q = \int_a^{\delta_q(x_{1,n})} f''(\theta|z)d\theta.$$

Since $2\theta z$ has a truncated $\chi^2(2n + 2)$ distribution, this equation becomes

$$q = \frac{C(2z\delta_q(x_{1,n});\ 2n + 2) - C(2za;\ 2n + 2)}{C(2zb;\ 2n + 2) - C(2za;\ 2n + 2)},$$

and with the aid of chi-square or incomplete gamma tables this could be solved for $\delta_q(x_{1,n})$, and this in turn would be substituted into $d_{1-p}(\theta)$ to obtain $r(x_{1,n})$.

The accuracy is

$$q' = \Pr\{\theta : \frac{-\ln p}{\delta_q(x_{1,n})} \le \frac{-\ln p'}{\theta} \,|x_{1,n}\}$$

$$= \Pr\{\theta : \theta \le \frac{\ln p'}{\ln p}\, \delta_q(x_{1,n})|x_{1,n}\}$$

$$= \frac{C(2z\frac{\ln p'}{\ln p}\delta_q(x_{1,n});\ 2n+2) - C(2za;\ 2n+2)}{C(2zb;\ 2n+2) - C(2za;\ 2n+2)}.$$

Since this depends on $x_{1,n}$, we cannot use this to predetermine the sample size necessary for a specific value of q'. However, we can meet a specified accuracy q'_o if we sample sequentially, calculating q' at each step, until it becomes less than q'_o.

The Normal Distribution

For this example $\theta = (\mu, \sigma)$. We shall consider three cases, namely σ known, μ known, and μ and σ unknown. In all cases the prior distribution $f'(\theta)$ will be the natural conjugate prior. (For a discussion of natural conjugate priors, see Raiffa and Schlaifer [7].)

Case 1. σ known.

Let

$$f(x|\mu) = N(\mu,\ \sigma^2)$$

and let the prior density also be a normal density

$$f'(\mu) = N(a,\ \sigma^2/b).$$

Applying Bayes' theorem results in the normal density

$$f''(\mu|x_{1,n}) = N(\frac{ab + n\bar{x}}{b+n},\ \frac{\sigma^2}{b+n}),$$

where $\bar{x}$ is the sample mean. To determine the lower Bayesian tolerance limit we note first that $d_{1-p}(\mu)$ is $\mu - z_p\sigma$, an increasing function of μ. Thus we need $\delta_{1-q}(x_{1,n})$ which is $(ab + n\bar{x})/(b+n) - z_q\sigma/(b+n)^{\frac{1}{2}}$. Hence

$$r(x_{1,n}) = \frac{ab + n\bar{x}}{b + n} - \frac{z_q \sigma}{(b + n)^{\frac{1}{2}}} - z_p \sigma. \tag{2.4}$$

Taking $b = 0$ gives the frequentist result.

The accuracy criterion for this case is

$$q' = \Pr\{\mu : r(x_{1,n}) \leq d_{1-p'}(\mu) x_{1,n}\}$$

$$= \Pr\{\mu : \frac{ab + n\bar{x}}{b + n} - \frac{z_q \sigma}{(b + n)^{\frac{1}{2}}} - z_p \sigma \leq \mu - z_{p'} \sigma | x_{1,n}\}$$

$$= \Pr\{\mu : \frac{\mu - \frac{ab + n\bar{x}}{b + n}}{\sigma/(b + n)^{\frac{1}{2}}} \geq (z_{p'} - z_p)(b + n)^{1/2} - z_q | x_{1,n}\}$$

$$= \Pr\{z \geq (z_{p'} - z_p)(b + n)^{\frac{1}{2}} - z_q\}, \tag{2.5}$$

where z has a $N(0, 1)$ distribution. Hence q' is a decreasing function of n and does not depend on the sample $x_{1,n}$, so that we can predetermine the sample size necessary for a given accuracy. Table II tabulates (2.5) for some values of p, p', q, and $b + n$. We note that the entries in Table II are larger than the corresponding entries in Table I. That this might be expected is seen from the fact that more probability is in the left tail of the exponential density than is in the left tail of the normal density. Hence for the same sample size we would expect to be more accurate in obtaining a lower tolerance limit for the exponential than for the normal.

TABLE II

ACCURACY CRITERION FOR BAYESIAN LOWER q TOLERANCE LIMITS FOR p COVERAGE FOR THE NORMAL DENSITY WITH KNOWN VARIANCE

p	.90				.95			
q	.90		.95		.90		.95	
p'	.950	.975	.950	.975	.975	.990	.975	.990
b+n								
5	.682	.409	.798	.554	.719	.405	.827	.549
10	.555	.195	.691	.310	.613	.192	.742	.306
15	.453	.090	.596	.164	.536	.088	.665	.159
20	.368	.040	.510	.083	.450	.039	.593	.081
25	.299	.018	.434	.041	.385	.017	.528	.039
30	.242	.008	.368	.020	.329	.007	.468	.019
35	.195	.003	.310	.009	.281	.003	.413	.009
40	.157	.001	.259	.004	.239	.001	.364	.004
45	.126		.217	.002	.203		.320	.002
50	.101		.180		.173		.280	

Case 2. μ known.

Without loss of generality we can let μ be 0. Then

$$f(x|\sigma) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} e^{-x^2/2\sigma^2} .$$

The prior density is

$$f'(\sigma) \propto \left(\frac{1}{\sigma}\right)^{w+1} e^{-wv/2\sigma^2}, \qquad \sigma > 0,$$

and this results in the posterior density

$$f''(\sigma|x_{1,n}) \propto \left(\frac{1}{\sigma}\right)^{w+n+1} e^{-(wv+u)/2\sigma^2}, \qquad \sigma > 0$$

where $u = \sum_{1}^{n} x_1^2$. The prior density, $f'(\sigma)$, is the inverted gamma-2 density of Raiffa and Schlaifer [7]. To determine $\delta_q(x_{1,n})$ we first note that $(wv + u)/\sigma^2$ has a $\chi^2(w + n)$ distribution. Hence

$$q = \Pr\{\frac{wv + u}{\sigma^2} \le \chi_q^2(w + n)\}$$

$$= \Pr\{\sigma \ge \left[\frac{wv + u}{\chi_q^2(w + n)}\right]^{\frac{1}{2}}\}.$$

Thus

$$\delta_q(x_{1,n}) = \left[\frac{wv + u}{\chi_{1-q}^2(w + n)}\right]^{\frac{1}{2}}.$$

If we require that $p > .5$, then $d_{1-p}(\sigma) = -z_p\sigma$ is a decreasing function of σ, and hence the lower Bayesian tolerance limit is

$$r(x_{1,n}) = -z_p\left[\frac{wv + u}{\chi_{1-q}^2(w + n)}\right]^{\frac{1}{2}}.$$

Taking $w = 0$ gives the frequentist limit.

The accuracy criterion is

$$q' = \Pr\{\sigma : r(x_{1,n}) \le -z_{p'}\sigma|x_{1,n}\}$$

$$= \Pr\{\sigma : \sigma \le \frac{z_p}{z_{p'}}\left[\frac{wv + u}{\chi_{1-q}^2(w + n)}\right]^{\frac{1}{2}}|x_{1,n}\}$$

$$= \Pr\{\sigma : \frac{wv + u}{\sigma^2} \geq (\frac{z_{p'}}{z_p})^2 \chi^2_{1-q}(w + n)|x_{1,n}\}$$

$$= 1 - C\left((\frac{z_{p'}}{z_p})^2 \chi^2_{1-q}(w + n);\ w + n\right),$$

by the distribution of $(wv + u)/\sigma^2$. Since $\chi^2_{1-q}(w + n)$ is an increasing function of $w + n$, q' is a decreasing function of $w + n$ and since it is not a function of the sample $x_{1,n}$, it can be used to determine the sample size necessary for a specified accuracy. Some values of q' are presented in Table III.

TABLE III

ACCURACY CRITERION FOR BAYESIAN LOWER q TOLERANCE LIMITS FOR p COVERAGE FOR THE NORMAL DENSITY WITH KNOWN MEAN

p	.90				.95			
q	.90		.95		.90		.95	
p'	.950	.975	.950	.975	.975	.990	.975	.990
w+n								
5	.754	.585	.865	.750	.808	.666	.898	.808
10	.629	.331	.774	.514	.734	.465	.848	.641
15	.522	.174	.684	.353	.669	.314	.800	.487
20	.430	.087	.598	.190	.609	.206	.753	.358
25	.352	.042	.518	.106	.555	.133	.709	.256
30	.287	.020	.446	.058	.505	.085	.662	.178
40	.187	.004	.323	.015	.418	.027	.577	.083
50	.120	.001	.224	.004	.342	.012	.499	.036

Case 3. μ and σ unknown.

Aitchison [1] derives an upper tolerance limit for this case and we shall present here a parallel derivation for a lower tolerance limit. Two corrections must be made, however, in order that his results can be obtained. In equation (24) of [1] the exponent of $1/\sigma$ should be $w + 1$ rather than w and the change of variables in (29) should read $\eta = V^{\frac{1}{2}}/\sigma$.

For this case

$$f(x|\mu, \sigma) = N(\mu, \sigma^2)$$

and the prior density is just the product of the densities considered in the first two cases

$$f'(\mu, \sigma) \propto (\frac{1}{\sigma})e^{-\frac{b}{2}(\frac{\mu - a}{\sigma})^2} (\frac{1}{\sigma})^{w+1} e^{-wv/2\sigma^2}.$$

Applying Bayes' theorem yields

$$f''(\mu, \sigma|x_{1,n}) \propto (\frac{1}{\sigma})e^{-\frac{B}{2}(\frac{\mu - A}{\sigma})^2} (\frac{1}{\sigma})^{W+1} e^{-WV/2\sigma^2},$$

where

$$A = \frac{ab + n\bar{x}}{b + n}; \; B = b + n; \; W = \begin{cases} w + n, & b > 0; \\ w + n - 1, & b = 0; \end{cases}$$

$$V = (wv + ba^2 + \Sigma x_i^2 - BA^2)/W.$$

Since $d_{1-p}(\mu, \sigma) = \mu - z_p\sigma$, we wish to determine $r(x_{1,n})$ such that

$$\Pr\{(\mu, \sigma) : r(x_{1,n}) \leq \mu - z_p\sigma|x_{1,n}\} = q. \qquad (2.6)$$

Let

$$\eta = \frac{B^{\frac{1}{2}}(\mu - A)}{\sigma}, \qquad \nu = \frac{V}{\sigma^2}.$$

Then η has a $N(0, 1)$ distribution, ν has a $\{\chi^2(W)/W\}$ distribution, and η and ν are independent. In terms of η and ν, the inequality in (2.6) becomes

$$\frac{B^{\frac{1}{2}}(r(x_{1,n}) - A)}{V^{\frac{1}{2}}} \le \frac{(\mu - A - z_p\sigma)B^{\frac{1}{2}}}{V^{\frac{1}{2}}}$$

which is

$$\frac{B^{\frac{1}{2}}(r(x_{1,n}) - A)}{V^{\frac{1}{2}}} \le (\eta - z_p B^{\frac{1}{2}})/\nu^{\frac{1}{2}}. \tag{2.7}$$

From the distributions of η and ν, it follows that the right-hand side of the inequality has a noncentral t distribution with W degrees of freedom and noncentrality parameter $-z_p B^{\frac{1}{2}}$. Thus, in order that (2.6) holds, we must have

$$r(x_{1,n}) = A + \frac{V^{\frac{1}{2}}}{B^{\frac{1}{2}}} t'_{1-q}(W, -z_p B^{\frac{1}{2}}). \tag{2.8}$$

If we let $b = w = 0$, we obtain the frequentist result which is

$$r(x_{1,n}) = \bar{x} + s_{\bar{x}}\, t'_{1-q}(n - 1, -z_p n^{\frac{1}{2}}).$$

To evaluate the accuracy criterion for this case, we replace p by p' and substitute (2.8) into the inequality of (2.7) which yields

$$q' = \Pr\{(\mu, \sigma) : t'_{1-q}(W, -z_p B^{\frac{1}{2}}) \leq (\eta - z_{p'} B^{\frac{1}{2}})/\nu^{\frac{1}{2}} | x_{1,n}\}$$

$$= \Pr\{u \geq t'_{1-q}(W, z_p B^{\frac{1}{2}})\},$$

where u has the $t'(W, -z_{p'} B^{\frac{1}{2}})$ density. Since this is a function of p, p', q, $b + n$, and also $w + n$, we will not attempt to tabulate it. However, we note that this function does not depend on $x_{1,n}$ and that it could be used to predetermine the sample size necessary for a given accuracy.

In these examples we have seen that for the exponential and normal densities and for "nice" choices of a prior density the proposed accuracy criterion (2.1) is a decreasing function of the sample size and is not a function of the sample. Therefore, for these cases, the criterion can be used as a guide to obtaining Bayesian tolerance limits with a given accuracy. For cases where the accuracy criterion depends on the sample obtained, it can be used to measure the accuracy of the tolerance limit obtained and a decision made as to whether an additional sample is warranted.

Concluding Remarks

In this chapter we have considered a Bayesian approach to obtaining q tolerance intervals for p coverage. The Bayesian and frequentist approaches are essentially the same in that an upper or lower q "confidence limit" for θ is substituted into the lower p probability point of $f(\cdot|\theta)$. The difference is that the Bayesian q "confidence limit" is obtained from the posterior distribution of θ, $f''(\theta|x_{1,n})$, rather than from $f(x_{1,n}|\theta)$. We have also seen that by letting the parameters of the prior distribution become those for an improper density (i.e., infinite variance) the two approaches give the same result. This is not meant to imply that the frequentist theory is a special case of the Bayesian, but rather to provide the reader with a connection between the two approaches.

As a measure of accuracy we have adapted a frequentist measure of the probability of having coverage greater than that desired to the Bayesian approach. Faulkenberry [5] obtained "uniformly most accurate tolerance limits" from uniformly most accurate confidence limits, which are in turn obtained from Neyman-Pearson uniformly most powerful tests. In the absence of Bayesian uniformly most powerful tests it is not clear at this time whether an analogous theory of most accurate Bayesian tolerance limits can be developed.

In the next chapter we will consider a Bayesian approach to expected coverage tolerance intervals, as formulated by Aitchison and Sculthorpe [3], and a measure of accuracy for these.

CHAPTER III

p-EXPECTED COVERAGE TOLERANCE INTERVALS

We now consider a Bayesian approach to obtaining intervals of the form $[r(x_{1,n}), \infty)$ for which the expected coverage of the interval is p. The frequentist or classical approach is to determine $R(x_{1,n})$ such that $E[C(R|\theta)] = p$ for all θ, where the expectation is with respect to $f(x_{1,n}|\theta)$. For the Bayesian approach we shall also want to determine $R(x_{1,n})$ such that $E[C(R|\theta)] = p$, but instead the expectation will be with respect to $f''(\theta|x_{1,n})$. Thus to obtain a Bayesian lower tolerance limit for expected coverage p, we will need to solve the following equation for $r(x_{1,n})$:

$$p = \int_\Omega C(R|\theta) f''(\theta|x_{1,n}) d\theta$$

$$= \int_\Omega \int_{r(x_{1,n})}^{\infty} f(y|\theta) dy \; f''(\theta|x_{1,n}) d\theta.$$

By interchanging the order of integration, we obtain

$$p = \int_{r(x_{1,n})}^{\infty} \int_\Omega f(y|\theta) f''(\theta|x_{1,n}) d\theta \; dy$$

$$= \int_{r(x_{1,n})}^{\infty} h''(y|x_{1,n}) dy, \qquad (3.1)$$

where $h''(y|x_{1,n})$ is the posterior density of a future observation y, given $x_{1,n}$. Thus we see that $r(x_{1,n})$ is the $1 - p$ probability point of $h''(y|x_{1,n})$.

An interesting result of Aitchison [2] is that $r(x_{1,n})$ is equivalent to the value obtained by minimizing the expected cost, given $x_{1,n}$, for the following cost function:

$$C(r, y) = \begin{cases} r - y & \text{if } y \le r \\ \lambda(y - r) & \text{if } y > r. \end{cases}$$

A case where this type of cost function would be employed would be where y represents a demand, r the amount to be supplied, and λ the ratio of the cost per unit of having demand exceed supply to the cost per unit of having supply exceed demand. The Bayesian solution is to choose r so that $\int C(r, y)h''(y|x_{1,n})dy$ is minimized. Equating the partial derivative with respect to r of this function to zero yields, after some manipulation,

$$\int_r^\infty h''(y|x_{1,n})dy = \frac{1}{\lambda + 1} \tag{3.2}$$

Thus, letting $p = 1/(\lambda + 1)$, the solution to (3.1) is the same as that for (3.2). The frequentist approach to these two situations, that of expected coverage and that of a linear cost function, does not have this equivalence.

For a fixed value of θ the frequentist lower tolerance limit has expected coverage p, since this is how the limit is derived. However this does not hold for the Bayesian limit. That is

$$\int_{X_{1,n}} \int_{r(x_{1,n})}^{\infty} f(y|\theta)dy \; f(x_{1,n}|\theta)dx_{1,n} = p(\theta),$$

where $X_{1,n}$ is the product space of $x_{1,n}$, and $p(\theta)$ is not necessarily equal to p. However,

$$\begin{aligned}
\int_{\Omega} p(\theta)f'(\theta)d\theta &= \int_{\Omega}\int_{X_{1,n}} \int_{r(x_{1,n})}^{\infty} f(y|\theta)dy \; f(x_{1,n}|\theta)dx_{1,n}f'(\theta)d\theta \\
&= \int_{X_{1,n}} \int_{r(x_{1,n})}^{\infty} \int_{\Omega} f(y|\theta)f''(\theta|x_{1,n})d\theta dy f(x_{1,n})dx_{1,n} \\
&= \int_{X_{1,n}} \int_{r(x_{1,n})}^{\infty} h''(y|x_{1,n})dy \; f(x_{1,n})dx_{1,n} \\
&= \int_{X_{1,n}} p \; f(x_{1,n})dx_{1,n} \\
&= p.
\end{aligned}$$

Thus averaging over θ gives expected coverage p. In the context of the example of manufacturing batches of transistors, taking a sample and finding the Bayesian tolerance limit for p-expected coverage, this means that for any one batch of transistors the expected coverage may not be exactly p, but over the long run of batches it will be p. The frequentist limit gives expected coverage of p for each batch and thus also over the long run. This may be an advantage but, as we shall see in the next chapter, the coverage may vary more for the frequentist limit than for the Bayesian. Thus if the experimenter is willing to assume his choice of $f'(\theta)$ truly describes the situation, he may be better off in the long run by using a Bayesian tolerance limit.

There is also an important distinction in the interpretation of the two approaches. The frequentist obtains a certain expected coverage where the expectation is with respect to repeated sampling. The Bayesian considers the coverage obtained for each θ and then obtains a weighted average of these, where the weights are obtained from $f''(\theta|x_{1,n})$, such that this weighted average is equal to p.

An Accuracy Criterion

Bayesian tolerance intervals for p-expected coverage can be obtained for any sample size, and so the question again arises as to what should be used to measure accuracy and how might this measure be influenced by sample size. Since the coverage, on the average, will be p, but may vary considerably about it, we may wish to have some degree of confidence that the estimate $r(x_{1,n})$ is fairly close to $d_{1-p}(\theta)$, the lower p probability point of $f(y|\theta)$. Thus we are led to the following measure of accuracy which we will denote by q.

$$q = \Pr\{\theta : |r(x_{1,n}) - d_{1-p}(\theta)| \le \Delta | x_{1,n}\}. \tag{3.3}$$

The interpretation is that if we consider the set of θ's for which the limit we obtained is within Δ of the $1 - p$ probability point of $f(y|\theta)$, then the probability, given the sample, of the θ we drew being in this set is q. If this is actually to be a measurement of accuracy, then we would expect q to increase as the sample size increases, and if we are to utilize this to determine the sample size necessary to obtain a tolerance limit with a given accuracy, then q should not be a function of the sample, $x_{1,n}$. However, as we shall see, this latter condition is not met for some common densities.

If q is a function of $x_{1,n}$, then we may be able to modify (3.3) by replacing Δ by a multiple m of a function of $x_{1,n}$, say $k(x_{1,n})$, so that q will not be a function of the sample. Thus we will be able to predetermine the sample size necessary for having q confidence that the limit obtained is in the interval $[d_{1-p}(\theta) \pm m\cdot k(x_{1,n})]$ for specific values of p, q, and m.

If it is desired to obtain p-expected coverage tolerance limits of a specified accuracy q_o for an exact deviation, Δ, rather than a proportional deviation, m, then the following multi-stage sampling procedure will accomplish this.

1. Choose an initial value of m and determine the sample size necessary so that (3.3), with Δ replaced by $m\cdot k(x_{1,n})$, will hold for the specified values of p and q. If $k(0) \neq 0$, then a choice for the initial value of m might be $m_1 = \Delta/k(0)$.

2. For the sample size determined, n_1, take a sample of that size and calculate $m_1\cdot k(x_{1,n_1})$. If this is less than or equal to Δ, then our limit will have accuracy greater than or equal to the specified q_o and sampling will stop.

3. If $m_1\cdot k(x_{1,n_1})$ is greater than Δ, then a second and smaller value of m will be obtained by letting $m_2 = \Delta/k(x_{1,n_1})$, and the sample size determined for q_o and m_2. Letting this be n, the additional sample size necessary is $n_2 = n - n_1$.

4. The additional sample is obtained and from it and the previous sample $m_2\cdot k(x_{1,n_1+n_2})$ is calculated and compared with Δ as before. The process is then repeated until $m_i\cdot k(x_{1,n_1+n_2+\ldots+n_i})$ is less than or equal to Δ.

This algorithm will lead to a tolerance limit with accuracy q_o

for chosen p and Δ. We will now illustrate Bayesian p-expected coverage tolerance intervals and the accuracy criterion, (3.3), for the same distributions considered in Chapter II.

The Exponential Distribution

As in Chapter II, we have

$$f(y|\theta) = \theta e^{-\theta y}, \quad y > 0; \ \theta > 0$$

and

$$f''(\theta|x_{1,n}) = (b + z)^{a+n}\theta^{a+n-1}e^{-\theta(b+z)}/\Gamma(a + n), \ \theta > 0; \ z = \sum_1^n x_i.$$

Thus

$$h''(y|x_{1,n}) = \int_0^\infty \frac{(b + z)^{a+n}\theta^{a+n}e^{-\theta(b+z+y)}d\theta}{\Gamma(a + n)}$$

$$= \frac{(a + n)(b + z)^{a+n}}{(b + z + y)^{a+n+1}} .$$

To determine the Bayesian lower tolerance limit for an interval of p-expected coverage we need to determine r such that

$$p = \int_r^\infty \frac{(a + n)(b + z)^{a+n}}{(b + z + y)^{a+n+1}} dy$$

$$= \left[\frac{b + z}{b + z + r}\right]^{a+n}.$$

Solving this for $r(x_{1,n})$ yields

$$r(x_{1,n}) = (b + z)(p^{-\frac{1}{n+a}} - 1).$$

We note that taking $a = b = 0$ again gives the frequentist result and that for the linear cost function mentioned above the minimizing value of r is $(b + z)((\lambda + 1)^{1/n+a} - 1)$.

The coverage of the interval $[r(x_{1,n}), \infty)$ is $\int_{r(x_{1,n})}^{\infty} \theta e^{-\theta y} dy$ which is equal to $e^{-\theta r(x_{1,n})}$. The expected coverage for a given value of θ is

$$\begin{aligned} E\, e^{-\theta r(x_{1,n})} &= E\, e^{-\theta(b+z)(p^{-\frac{1}{n+a}} - 1)} \\ &= e^{-\theta b\gamma}\, E\, e^{-\theta z\gamma}, \end{aligned} \tag{3.4}$$

where $\gamma = p^{-\frac{1}{n+a}} - 1$. The expectation in (3.4) is just $m_z(-\theta\gamma)$, where $m_z(t)$ is the moment generating function of z, which in this case is $(1 - \frac{t}{\theta})^{-n}$, since z has the gamma distribution with parameters n and θ. Thus (3.4) becomes

$$\begin{aligned} E\, e^{-\theta r(x_{1,n})} &= e^{-\theta b\gamma}(1 + \gamma)^{-n} \\ &= e^{-\theta b\gamma} p^{\frac{n}{n+a}}. \end{aligned}$$

If $a = b = 0$, then this quantity is p for all θ as it should be for the frequentist limit. Also this function of θ approaches the constant function p as n becomes infinite. The expectation of this with respect to $f'(\theta)$ is $p^{\frac{n}{n+a}} m_\theta(-b\gamma)$ which is equal to $p^{\frac{n}{n+a}}(1 + \gamma)^{-a} = p$.

The accuracy criterion is

$$q = \Pr\{\theta : |r(x_{1,n}) - d_{1-p}(\theta)| \le \Delta | x_{1,n}\}$$

$$= \Pr\{\theta : -\Delta \le (b + z)\gamma + \frac{\ln p}{\theta} \le \Delta | x_{1,n}\}$$

$$= \Pr\{\theta : \frac{-\ln p}{(b + z)\gamma + \Delta} \le \theta \le \frac{-\ln p}{(b + z)\gamma - \Delta} \mid x_{1,n}\}$$

$$= \Pr\{\theta : \frac{-2(b + z)\ln p}{(b + z)\gamma + \Delta} \le 2\theta(b + z) \le \frac{-2(b + z)\ln p}{(b + z)\gamma - \Delta} \mid x_{1,n}\} \quad (3.5)$$

$$= C(u_2; 2a + 2n) - C(u_1; 2a + 2n),$$

where u_2 and u_1 are the right- and left-hand terms, respectively, in the inequality of (3.5). Since u_2 and u_1 are functions of $x_{1,n}$ we need to find a function $k(x_{1,n})$ such that setting Δ equal to $m \cdot k(x_{1,n})$ will make (3.5) independent of $x_{1,n}$ and an increasing function of n. By letting $k(x_{1,n}) = (b + z)/(a + n)$, which will be approximately equal to the sample mean, equation (3.5) becomes

$$q = C(u_2; 2a + 2n) - C(u_1; 2a + 2n), \quad (3.6)$$

where

$$u_2 = \frac{-2 \ln p}{\gamma - \frac{m}{a+n}}; \quad u_1 = \frac{-2 \ln p}{\gamma + \frac{m}{a+n}},$$

and thus q will not be a function of the sample. Table IV tabulates (3.6) for p equal to .90 and .95.

To illustrate the concepts presented above consider the example of this section in Chapter II where the parameters of the prior were chosen to be $a = 3$ and $b = 2$ and the sum of 10 observations was 6. Then for $p = .90$,

$$r(x_{1,n}) = (2 + 6)[.90^{-\frac{1}{13}} - 1]$$

TABLE IV

ACCURACY CRITERION FOR BAYESIAN LOWER TOLERANCE LIMITS FOR

p-EXPECTED COVERAGE FOR THE EXPONENTIAL DENSITY

p = .90

m / a+n	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
5	.164	.323	.473	.603	.705	.775	.816	.843	.867	.885
10	.233	.450	.631	.763	.844	.889	.917	.937	.952	.964
15	.285	.537	.728	.846	.906	.938	.958	.972	.981	.987
20	.327	.604	.792	.893	.940	.964	.979	.987	.992	.995
25	.364	.657	.839	.924	.961	.979	.989	.994	.997	.998
30	.396	.709	.873	.945	.974	.987	.994	.997	.999	.999
35	.425	.737	.899	.959	.982	.992	.997	.999	.999	
40	.451	.769	.918	.970	.988	.995	.998	.999		
45	.475	.795	.934	.977	.992	.997	.999			
50	.497	.818	.946	.983	.994	.998	.999			

p = .95

m / a+n	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
5	.333	.617	.782	.847	.887	.915	.936	.951	.963	.970
10	.461	.774	.893	.940	.966	.980	.988	.993	.996	.997
15	.550	.854	.942	.974	.988	.995	.998	.999	.999	
20	.616	.900	.967	.989	.996	.999				
25	.670	.929	.981	.995	.999					
30	.714	.949	.989	.998	.999					
35	.750	.963	.993	.999						
40	.780	.972	.996	.999						
45	.807	.979	.998							
50	.829	.984	.999							

$$= .067$$

The coverage for this interval, $[.067, \infty)$, is $e^{-.067\theta}$, which is plotted in Figure 3, as is $f''(\theta|x_{1,n})$. To determine the accuracy of the limit obtained, we will evaluate (3.5) for $\Delta = .05$. Thus

$$u_2 = \frac{-2(2 + 6) \ln .90}{.067 - .05} = 43.07, \quad u_1 = \frac{-2(2 + 6) \ln .90}{.067 + .05} = 6.26,$$

and

$$q = C(43.07; 26) - C(6.26; 26)$$
$$= .980.$$

Dividing u_1 and u_2 by $2(b + z)$ in order to obtain the limits on θ in the line above (3.5) yields the results that for θ between .39 and 2.75, r is within .05 of $d_{1-p}(\theta)$ and this interval contains 98 per cent of $f''(\theta|x_{1,n})$. If the sample sum had been 10, instead of 6, then $r(x_{1,n})$ would have been .101 and the coverage of the interval obtained would be $e^{-.101\theta}$. For this case $u_2 = 21.54$ and

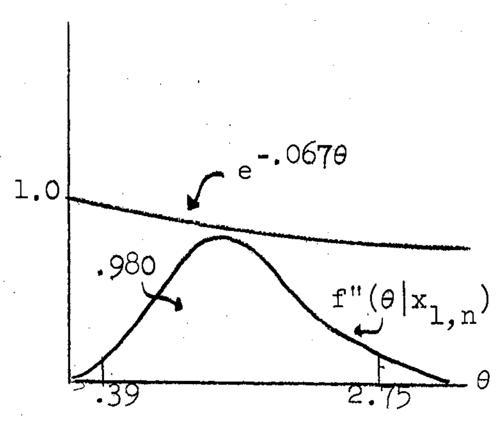

Figure 3

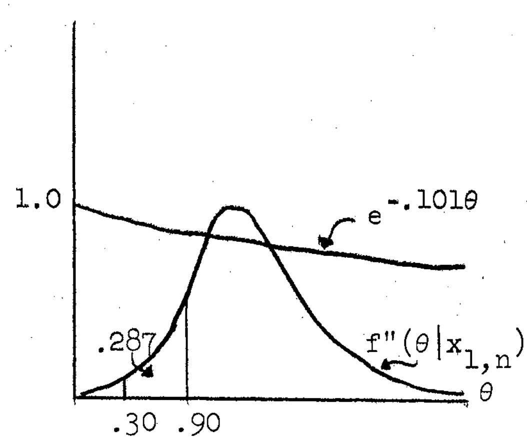

Figure 4

$u_1 = 7.27$ and thus the accuracy of the limit obtained is

$$q = C(21.54;\ 26) - C(7.27;\ 26)$$
$$= .287.$$

Thus for θ between .30 and .90, $r = .101$ is within .05 of $d_{1-p}(\theta)$ and this interval contains 28.7 per cent of $f''(\theta|x_{1,n})$. This latter situation is shown in Figure 4.

In order to investigate the average sample size for the multi-stage sampling procedure described above, a Monte Carlo study was done for some selected values of p, q, Δ, and for a and b, the parameters of $f'(\theta)$. As a choice for θ, we choose the case where θ takes on its expected value which is a/b. This may appear to be a utopian choice, but it is our intention to show the behavior of our sampling procedure in the expected situation and not the possible extreme situations. For the distributions under consideration, the procedure given on page 27 becomes:

1. Let $m_1 = \Delta/k(0) = a\Delta/b$ and determine n_1 such that (3.6) holds for the specified p, q, and m_1.

2. Take a sample of size n_1 from $f(x|\theta = a/b)$, calculate $m_1 \cdot k(x_{1,n}) = m_1(b + z)/(a + n)$, and compare this with the chosen Δ.

3. If $m_1 \cdot k(x_{1,n_1})$ is less than or equal to Δ, then (3.6) is satisfied. If it is greater than Δ, we adjust m downward, obtaining $m_2 = \Delta/k(x_{1,n_1}) = (a + n_1)\Delta/(b + z)$, determine the number of additional observations required, and take the additional sample. Then $m_2 \cdot k(x_{1,n_1+n_2})$ is compared with Δ and the decision made as to whether or not an additional sample is required.

To illustrate this, let

$$p = .90;\ q = .95;\ \Delta = .06;\ a = 3;\ b = 2.$$

Thus, assuming θ has a prior distribution which is a gamma with parameters $a = 3$ and $b = 2$, we wish to obtain a lower tolerance limit for expected coverage of .90 and in addition we want to have 95 per cent confidence in being within .06 of the actual lower .90 probability point. As an initial choice of m in equation (3.6) we take $m_1 = \Delta/k(0) = a\Delta/b = .09$, and from Table IV we see that $a + n$ should be ten and thus the initial sample size is seven. Suppose the sum of these seven observations is $z = 8$. Then $m \cdot k(x_{1,n_1})$ $= .09(2 + 8)/(3 + 7) = .09$, which is larger than $\Delta = .06$, and an additional sample is required. The next choice of m is $m_2 = (a + n_1)\Delta/(b + z_1) = 10(.06)/10 = .06$. Interpolating in Table IV we see that for $q = .95$, $a + n$ should be 17, and so an additional seven observations are required. If the sum of these seven observations is six, then $m_2 \cdot k(x_{1,n_1+n_2}) = m_2(b + z)/(a + n)$ is equal to $.06(2 + 14)/(3 + 14)$ which is less than the chosen Δ. Thus in two steps we have obtained a lower tolerance limit for .90 expected coverage and with the specified accuracy. Table V presents the average sample size for this procedure for some values of p, q, Δ, a, and b. Each entry is the average of 100 repetitions of the sampling procedure. Three entries are blank since the program was limited to $a + n = 50$.

As an alternative to letting $\Delta = m \cdot k(x_{1,n})$ we may let $\Delta = m[\mathrm{var}(y|\theta)]^{\frac{1}{2}}$ and (3.5) will then be the posterior probability that $r(x_{1,n})$ is within m standard deviations of $d_{1-p}(\theta)$. Since the variance of the exponential distribution is $1/\theta^2$, (3.5) becomes

TABLE V

AVERAGE SAMPLE SIZE FOR BAYESIAN LOWER TOLERANCE LIMITS

FOR p-EXPECTED COVERAGE FOR THE EXPONENTIAL DENSITY

p	.90					
q	.90			.95		
Δ	.06	.08	.10	.06	.08	.10
a,b						
1,3	36.37	25.91	21.95	---	---	31.28
1,2	23.06	21.16	15.85	---	31.73	25.29
2,3	19.74	14.94	9.34	30.65	23.77	15.59
1,1	12.30	8.52	6.19	19.19	13.04	10.23
2,2	11.34	7.42	4.98	18.15	12.48	8.34
3,3	9.79	6.52	4.31	17.80	10.59	7.43
3,2	5.69	3.22	1.59	9.02	5.28	3.70
2,1	4.20	2.61	1.84	6.75	4.79	3.93
3,1	1.47	1.46	1.50	2.68	1.59	1.49

p	.95					
q	.90			.95		
Δ	.06	.08	.10	.06	.08	.10
a,b						
1,3	11.91	8.77	7.91	17.82	13.39	12.16
1,2	8.43	7.78	6.07	12.70	11.84	9.47
2,3	6.73	4.71	2.91	10.84	8.02	5.79
1,1	5.16	3.86	2.84	7.55	5.14	4.12
2,2	3.89	2.67	1.68	6.11	4.14	2.86
3,3	2.93	1.58	1.61	5.37	2.97	1.73
3,2	1.64	1.48	1.54	2.83	1.58	1.40
2,1	1.45	1.43	1.44	2.53	1.63	1.39
3,1	1.49	1.43	1.53	1.53	1.51	1.57

$$q = \Pr\{\theta : |(b + z)\gamma + \frac{\ln p}{\theta}| \le \frac{m}{\theta} \mid x_{1,n}\}$$

$$= \Pr\{\theta : 2(\frac{-m-\ln p}{\gamma}) \le 2(b + z)\theta \le 2(\frac{m-\ln p}{\gamma}) \mid x_{1,n}\}.$$

Since $2(b + z)\theta$ has the $\chi^2(2a + 2n)$ density and since q is not a function of $x_{1,n}$, this function can be used to determine the sample size necessary to obtain a p-expected coverage tolerance interval with a given confidence q that the limit obtained is within m standard deviations of $d_{1-p}(\theta)$.

The Normal Distribution

We shall consider the same three cases as in Chapter II.

Case 1. σ known.

For this case

$$f(y|\mu) \propto e^{-(y-\mu)^2/2\sigma^2},$$

$$f'(\mu) \propto e^{-b(\mu-a)^2/2\sigma^2},$$

$$f''(\mu|x_{1,n}) \propto e^{-(b+n)(\mu-A)^2/2\sigma^2},$$

and

$$h''(y|x_{1,n}) \propto e^{-B(y-A)^2/2\sigma^2},$$

where

$$A = \frac{ba + n\bar{x}}{b + n}\ ; \quad B = \frac{b + n}{b + n + 1}\ .$$

Thus, $r(x_{1,n})$, the lower p probability point of $h''(y|x_{1,n})$, is

$$r(x_{1,n}) = A - z_p\sigma/B^{\frac{1}{2}}.$$

Taking $b = 0$ gives the usual frequentist limit. The accuracy is

$$q = \Pr\{\mu : |A - z_p\sigma/B^{\frac{1}{2}} - (\mu - z_p\sigma)| \le \Delta \mid x_{1,n}\}$$

$$= \Pr\{\mu : (b+n)^{\frac{1}{2}}[z_p(B^{-\frac{1}{2}}-1) - \Delta/\sigma] \le z \le (b+n)^{\frac{1}{2}}[z_p(B^{-\frac{1}{2}}-1) + \Delta/\sigma]\},$$

where z has a $N(0,1)$ distribution. Note that q is an increasing function of n, and for fixed n, a decreasing function of σ, both of which results we would expect intuitively. Also q is not a function of $x_{1,n}$ so that we can predetermine the sample size necessary for a given accuracy. Table VI gives values of q for some values of p, $b + n$, and Δ/σ.

TABLE VI

ACCURACY CRITERION FOR BAYESIAN LOWER TOLERANCE LIMITS FOR p-EXPECTED COVERAGE FOR THE NORMAL DENSITY WITH KNOWN VARIANCE

p	.90				.95			
Δ/σ	.2	.3	.4	.5	.2	.3	.4	.5
b+n								
5	.333	.482	.611	.719	.326	.472	.600	.708
10	.465	.648	.785	.879	.460	.642	.780	.875
15	.555	.748	.874	.944	.552	.745	.871	.942
20	.624	.816	.923	.973	.621	.813	.922	.972
25	.679	.863	.953	.987	.676	.861	.952	.986
30	.724	.897	.971	.994	.721	.896	.970	.993
35	.761	.922	.981	.997	.759	.921	.981	.997
40	.792	.941	.988	.998	.790	.940	.988	.998
45	.818	.955	.993	.999	.817	.954	.992	.999
50	.841	.965	.995		.840	.965	.995	

Case 2. μ known.

For this case

$$f(y|\sigma) \propto \frac{1}{\sigma} e^{-y^2/2\sigma^2},$$

$$f'(\sigma) \propto (\frac{1}{\sigma})^{w+1} e^{-wv/2\sigma^2},$$

$$f''(\sigma|x_{1,n}) \propto (\frac{1}{\sigma})^{W+1} e^{-WV/2\sigma^2},$$

and

$$h''(y|x_{1,n}) \propto [WV + y^2]^{-\frac{W+1}{2}}, \qquad (3.7)$$

where $W = w + n$ and $V = (wv + \Sigma x_i^2)/W$. In order to obtain $r(x_{1,n})$, we first make the change of variable in (3.7) of

$$y = u \cdot V^{\frac{1}{2}}.$$

Then

$$h''(\mu|x_{1,n}) \propto (1 + u^2/W)^{-\frac{W+1}{2}}$$

and so $u = y/V^{\frac{1}{2}}$ has Students t distribution with W degrees of freedom. Hence

$$r(x_{1,n}) = t_{1-p}(w + n)V^{\frac{1}{2}}$$

$$= t_{1-p}(w + n)(wv + \Sigma x_i^2)^{\frac{1}{2}}/(w + n)^{\frac{1}{2}}.$$

The accuracy of the limit in this case is

$$q = \Pr\{\sigma : |t_p(W)V^{\frac{1}{2}} - z_p\sigma| \le \Delta \mid x_{1,n}\}$$

$$= \Pr\{\sigma : \frac{t_p(W)V^{\frac{1}{2}}}{z_p} - \Delta \le \sigma \le \frac{t_p(W)V^{\frac{1}{2}}}{z_p} + \Delta \mid x_{1,n}\} \qquad (3.8)$$

for $p > .50$. In this case q is a function of $x_{1,n}$ and hence in order to apply the algorithm for obtaining a limit with a given degree of accuracy we need to find a function $k(x_{1,n})$ such that by letting Δ equal $m \cdot k(x_{1,n})$, (3.8) will be independent of $x_{1,n}$. Letting $k(x_{1,n}) = \left[\frac{wv + \Sigma x_i^2}{w + n}\right]^{\frac{1}{2}}$, which is approximately the standard deviation of x, will accomplish this. Equation (3.8) then becomes

$$q = \Pr\{\sigma : \frac{t_p(W) - m}{z_p W^{\frac{1}{2}}} \le \frac{\sigma}{(WV)^{\frac{1}{2}}} \le \frac{t_p(W) + m}{z_p W^{\frac{1}{2}}} \mid x_{1,n}\}$$

$$= \Pr\{\sigma : \frac{z_p^2 W}{(t_p(W) + m)^2} \le \frac{WV}{\sigma^2} \le \frac{z_p^2 W}{(t_p(W) - m)^2} \mid x_{1,n}\} \qquad (3.9)$$

$$= C(u_2; W) - C(u_1; W),$$

where u_2 and u_1 are the right and left side of (3.9) since as noted on page 17, WV/σ^2 has a $\chi^2(W)$ distribution. Table VII gives values of q for some values of p, $w + n$, and m.

As an alternative we might choose Δ equal to $m\sigma$. Then (3.8) becomes

$$q = \Pr\{\sigma : |t_p(W)V^{\frac{1}{2}} - z_p\sigma| \le m\sigma \mid x_{1,n}\}$$

$$= \Pr\{\sigma : \frac{W(z_p - m)^2}{t_p^2(W)} \le \frac{WV}{\sigma^2} \le \frac{W(z_p + m)^2}{t_p^2(W)} \mid x_{1,n}\}$$

$$= C(u_2; W) - C(u_1; W)$$

where u_2 and u_1 are the right and left sides, respectively, of the above inequality. Thus we can determine the sample size necessary to have q confidence that $r(x_{1,n})$ is within m standard deviations of $d_{1-p}(\sigma)$.

TABLE VII

ACCURACY CRITERION FOR BAYESIAN LOWER TOLERANCE LIMITS FOR p-EXPECTED COVERAGE FOR THE NORMAL DENSITY WITH KNOWN MEAN

p	.90		.95	
m	.25	.50	.25	.50
w+n				
5	.375	.709	.254	.513
10	.573	.893	.433	.779
15	.689	.947	.548	.884
20	.788	.970	.630	.933
25	.821	.982	.693	.958
30	.860	.989	.743	.972

Case 3. μ and σ unknown.

For this case

$$f(y|\mu,\sigma) \propto \frac{1}{\sigma} e^{-(y-\mu)^2/2\sigma^2},$$

and as in Chapter II

$$f'(\mu,\ \sigma) \propto (\frac{1}{\sigma})e^{-\frac{b}{2}(\frac{\mu-a}{\sigma})^2} (\frac{1}{\sigma})^{w+1} e^{-wv/2\sigma^2},$$

and

$$f''(\mu, \sigma|x_{1,n}) \propto (\frac{1}{\sigma})e^{-\frac{B}{2}(\frac{\mu-A}{\sigma})^2}(\frac{1}{\sigma})^{W+1}e^{-WV/2\sigma^2}.$$

Thus

$$h''(y|x_{1,n}) \propto [WV + (y - A)^2B/(B + 1)]^{-\frac{W+1}{2}},$$

where A, B, V, and W are defined on page 19. Thus the variable $u = (y - A)[B/V(1 + B)]^{\frac{1}{2}}$ has a Students t distribution with W degrees of freedom, and so

$$r(x_{1,n}) = A + t_{1-p}(W)[V(1 + B)/B]^{\frac{1}{2}}$$

$$= \frac{ba+n\bar{x}}{b+n} + t_{1-p}(w+n)[(wv+ba^2+\Sigma x_i^2 - \frac{(ba+n\bar{x})^2}{b+n})(b+n+1)/(w+n)(b+n)]^{\frac{1}{2}} \quad (3.10)$$

This is the result for $b > 0$. If $b = 0$, then $w + n$ is replaced by $w + n - 1$. Note that if $b = w = 0$,

$$r(x_{1,n}) = \bar{x} + t_{1-p}(n - 1)[(\Sigma(x_i - \bar{x})^2/(n - 1))(n + 1)/n]^{\frac{1}{2}}$$

$$= \bar{x} + t_{1-p}(n - 1)s[\frac{n + 1}{n}]^{\frac{1}{2}},$$

which is the usual frequentist result. To determine the accuracy of the tolerance limit (3.10) we need to evaluate

$$q = \Pr\{(\mu, \sigma): |A + t_{1-p}(W)[V(1 + B)/B]^{\frac{1}{2}} - (\mu - z_p\sigma)| \le \Delta \mid x_{1,n}\}$$

$$= \Pr\{(\mu, \sigma): t_{1-p}(W)[V(1 + B)/B]^{\frac{1}{2}} - \Delta \le \mu - A - z_p\sigma \le t_{1-p}(W)[V(1 + B)/B]^{\frac{1}{2}} + \Delta \mid x_{1,n}\}.$$

Making the change of variables $\eta = B^{\frac{1}{2}}(\mu - A)/\sigma$ and $\nu = V/\sigma^2$, as in Chapter II, page 20, the accuracy is

$$q = \Pr\{(\eta, \nu) : t_{1-p}(W)(1 + B)^{\frac{1}{2}} - \Delta B^{\frac{1}{2}}/V^{\frac{1}{2}} \le (\eta - z_p B^{\frac{1}{2}})/\nu \le$$

$$t_{1-p}(W)(1 + B)^{\frac{1}{2}} + \Delta B^{\frac{1}{2}}/V^{\frac{1}{2}}\}. \tag{3.11}$$

$$= \Pr(u_1 \le u \le u_2),$$

where u_1 and u_2 are the left and right sides, respectively, of the inequality in (3.11) and u has $t'(W, -z_p B^{\frac{1}{2}})$ distribution. Since (3.11) is a function of $x_{1,n}$, we need to replace Δ by $m \cdot k(x_{1,n})$ in order that (3.11) will not be a function of $x_{1,n}$. Letting $k(x_{1,n}) = V^{\frac{1}{2}}$ will accomplish this and our algorithm for multi-stage sampling can be applied to obtain a lower tolerance limit for p-expected coverage and q accuracy.

Concluding Remarks

In this chapter we have considered a Bayesian approach to p-expected coverage tolerance intervals and have proposed a criterion for measuring the accuracy of the tolerance limits obtained. The criterion is the probability, conditional on the sample, $x_{1,n}$, that the value of the θ we drew was one for which the tolerance limit, calculated for the sample, was within some amount, Δ, of the true p probability point of $f(y|\theta)$. If this probability was independent of $x_{1,n}$ then the sample size could be determined so that a tolerance limit with a given accuracy could be obtained and if not, a multi-stage sampling procedure was developed so that a specific accuracy could be obtained.

CHAPTER IV

SENSITIVITY OF BAYESIAN TOLERANCE INTERVALS TO AN INCORRECT CHOICE OF THE PRIOR DISTRIBUTION

As we have observed in the two previous chapters, if the experimenter is willing to assume that his choice of a prior distribution for the parameter of interest actually describes the true situation, then the Bayesian approach will provide more accurate tolerance limits for a fixed sample size. To speak of the prior actually describing the true situation we must limit ourselves to the situation where the parameter is a random variable. If we consider the parameter fixed but unknown and choose a prior distribution to describe our state of knowledge, then it is illogical to talk about the inaccuracy of our "state of knowledge" distribution, since we presumably chose it to describe our state of knowledge as well as possible. Thus if θ is actually a random variable and we are faced with the task of selecting a function which describes its distribution, we need to be aware of the risk of making a wrong selection.

As an illustration, and for tractability, we will consider p-expected coverage tolerance intervals for the exponential distribution. As we showed on page 29, the expected coverage at each θ is $e^{-b\theta\gamma}p^{\frac{n}{n+a}}$, where $\gamma = p^{-\frac{1}{n+a}} - 1$, and the expected coverage in the long run is p when the prior distribution of θ is actually a gamma distribution with parameters a and b. But suppose the actual distribution of θ is still one of the gamma family, but with parameters a' and b'. Then the expected coverage in the long run is

$$E\ e^{-\theta\gamma} p^{\frac{n}{n+a}} = m_\theta(-b\gamma) p^{\frac{n}{n+a}}$$

$$= (1 + b\gamma/b')^{-a'} p^{\frac{n}{n+a}}, \tag{4.1}$$

where $m_u(t)$ is the moment generating function of the random variable u. This expectation is equal to p when $a = a'$ and $b = b'$. For fixed a, a', b, and b' the expected coverage approaches p as n increases. In fact as n becomes infinite it makes no difference what the prior distribution is, or was thought to be.

Consider now the mean square error (MSE) of the coverage. We need to evaluate

$$\text{MSE (coverage)} = E(e^{-\theta r(x_{1,n})} - p)^2$$

$$= \operatorname{var}[e^{-\theta r(x_{1,n})}] + (Ee^{-\theta r(x_{1,n})} - p)^2,$$

where $r(x_{1,n}) = (b + z)(p^{-\frac{1}{n+a}} - 1) = (b + z)\gamma$. We first evaluate $\operatorname{var}[e^{-\theta r(x_{1,n})}]$ by the familiar relationship

$$\operatorname{var}(u) = \operatorname{var}(Eu|v) + E\operatorname{var}(u|v), \tag{4.2}$$

where for our case $u = e^{-\theta r(x_{1,n})}$ and $v = \theta$. Now $E(e^{-\theta r(x_{1,n})}|\theta) = e^{-\theta b\gamma} p^{\frac{n}{n+a}}$ and the variance of this is $p^{\frac{2n}{n+a}} \operatorname{var}(e^{-\theta b\gamma})$.

We next obtain

$$\operatorname{var}(e^{-\theta b\gamma}) = E\, e^{-2\theta b\gamma} - (E\, e^{-\theta b\gamma})^2$$

$$= (1 + 2b\gamma/b')^{-a'} - (1 + b\gamma/b')^{-2a'}.$$

Thus the first term on the right side of (4.2) is

$$p^{\frac{2n}{n+a}} \left[(1 + 2b\gamma/b')^{-a'} - (1 + b\gamma/b')^{-2a'} \right].$$

Evaluating the second term, we first obtain

$$\operatorname{var}\ (e^{-\theta r(x_{1,n})} | \theta) = \operatorname{var}\ (e^{-\theta(b+z)\gamma} | \theta)$$

$$= e^{-2\theta b\gamma} \operatorname{var}\ (e^{-\theta z\gamma} | \theta)$$

$$= e^{-2\theta b\gamma} \left[E(e^{-2\theta z\gamma} | \theta) - (E\, e^{-\theta z\gamma} | \theta)^2 \right]$$

$$= e^{-2\theta b\gamma} \left[(1+2\gamma)^{-n} - (1+\gamma)^{-2n} \right]. \quad (4.3)$$

Taking the expectation of this with respect to θ gives

$$E\left[\operatorname{var}\ (e^{-\theta r(x_{1,n})} | \theta) \right] = (1 + 2b\gamma/b')^{-a'} \left[(1 + 2\gamma)^{-n} - (1 + \gamma)^{-2n} \right]$$

and thus (4.2) becomes

$$\operatorname{var}\left[e^{-\theta r(x_{1,n})} \right] = (1+2b\gamma/b')^{-a'} (1+2\gamma)^{-n} - (1+\gamma)^{-2n} (1+b\gamma/b')^{-2a'}, \quad (4.4)$$

where we have replaced $p^{-\frac{1}{n+a}}$ by $(1 + \gamma)$. If $a = a'$ and $b = b'$ then this reduces to

$$(1 + 2\gamma)^{-(a+n)} - (1 + \gamma)^{-2(a+n)}. \quad (4.5)$$

Note that if we let $a = b = 0$ in (4.3) we will obtain the variance of the coverage for the frequentist interval which is just (4.5) with $a = 0$. Since this is a decreasing function of $a + n$, the variance of the coverage for the Bayesian limit is less than that for the frequentist for a fixed sample size if the correct prior is assumed. Combining (4.4) with the bias squared we obtain

$$\text{MSE (coverage)} = (1 + 2b\gamma/b')^{-a'}(1 + 2\gamma)^{-n} - (1 + \gamma)^{-2n}(1 + b\gamma/b')^{-2'a} + [(1 + b\gamma/b')^{-a'}(1 + \gamma)^{-n} - p]^2$$

$$= (1 + 2b\gamma/b')^{-a'}(1 + 2\gamma)^{-n} - 2p(1 + b\gamma/b')^{-a'}(1 + \gamma)^{-n} + p^2.$$

Table VIII gives the expected coverage and MSE (coverage) for some assumed and actual prior distributions and sample sizes. To illustrate Table VIII suppose that a Bayesian lower tolerance limit was obtained for $p = .90$ and $n = 10$. If the prior density was assumed to have parameters $a = 1$ and $b = 1$, and the actual parameters were $a = 3$ and $b = 1$, then in the long run the expected coverage would be .883 and the mean square error of the coverage would be .001213. For the same situation a sample size of thirty would lead to an expected coverage of .894 and mean square error of .000341. The values of a and b are listed in order of increasing a/b, that is, in increasing order of the mean of the prior distribution. Note that the expected coverage decreases as a/b increases. We would expect this intuitively since for θ's which are larger than those expected, the $1 - p$ probability point will be smaller than that expected under the assumed prior, and thus we will be less likely to actually cover the proportion, p. Conversely, for θ's actually less than those expected we will be more likely to cover more than the proportion, p. The terms on the diagonal of Table VIII are the mean and variance of the coverage for each of the priors. These quantities are the same as those for the frequentist tolerance limit based on a sample of $a + n$. Thus if it can be determined that the prior distribution of θ is gamma with $a = 2$, $b = 3$, then, over the long run, the variance of the coverage for the Bayesian tolerance limit for sample of size 10 will be the same as that of the frequentist limit for sample size 12. Another fact to note is that for fixed a, b, and n, the MSE (coverage) is a decreasing function of p. Thus the coverage of intervals of the form $[0, r]$ will vary more than the coverage of interval of the form $[r, \infty]$, where in both cases p-expected coverage tolerance limits are desired for $p > .5$.

TABLE VIII

EXPECTED COVERAGE AND MEAN SQUARE ERROR FOR

BAYESIAN LOWER TOLERANCE LIMITS FOR

THE EXPONENTIAL DISTRIBUTION*

p = .90

n = 10

Assumed a,b	Actual a,b: 1,3	1,2	2,3	1,1	2,2	3,3	3,2	2,1	3,1
1,3	.900 810	.896 910	.891 941	.883 1608	.883 1315	.883 1213	.870 2025	.858 3565	.834 6599
1,2	.903 783	.900 810	.897 805	.892 1079	.891 941	.891 893	.883 1213	.875 1884	.858 3210
2,3	.908 756	.904 780	.900 743	.892 1197	.892 941	.892 853	.881 1368	.869 2524	.847 4797
1,1	.906 787	.904 781	.903 766	.900 810	.900 774	.900 761	.896 803	.891 941	.883 1213
2,2	.911 774	.908 756	.905 710	.900 863	.900 743	.900 702	.892 853	.884 1310	.869 2212
3,3	.915 816	.911 783	.907 698	.900 989	.900 764	.900 686	.889 977	.879 1834	.858 3518
3,2	.917 868	.915 816	.912 740	.907 803	.907 698	.907 662	.900 686	.893 980	.879 1558
2,1	.913 821	.912 794	.911 760	.908 756	.908 725	.908 714	.904 687	.900 743	.892 853
3,1	.920 944	.918 903	.917 855	.915 816	.915 789	.915 779	.911 703	.907 698	.900 686

*MSE (coverage) = 10^{-6} times quantity in the table.

TABLE VIII (Continued)

p = .90

n = 30

Actual a,b

Assumed a,b	1,3	1,2	2,3	1,1	2,2	3,3	3,2	2,1	3,1
1,3	.900 289	.898 302	.897 306	.894 395	.894 354	.894 341	.889 447	.885 659	.876 1079
1,2	.901 286	.900 289	.899 288	.897 324	.897 306	.897 300	.894 341	.891 430	.885 607
2,3	.903 282	.901 285	.900 280	.897 347	.897 309	.897 296	.893 371	.888 545	.880 891
1,1	.902 286	.902 285	.901 284	.900 289	.900 284	.900 283	.898 288	.897 306	.894 341
2,2	.904 285	.903 282	.902 275	.900 297	.900 280	.900 274	.897 296	.894 363	.888 495
3,3	.906 292	.904 287	.903 273	.900 320	.900 284	.900 272	.896 318	.891 459	.883 738
3,2	.907 300	.906 292	.905 280	.903 290	.903 273	.903 268	.900 272	.897 319	.891 412
2,1	.905 291	.904 287	.904 283	.903 282	.903 278	.903 276	.901 272	.900 280	.897 296
3,1	.908 311	.907 305	.907 298	.906 292	.906 288	.906 286	.904 274	.903 273	.900 272

TABLE VIII (Continued)

p = .95

n = 10

Assumed a,b	Actual a,b 1,3	1,2	2,3	1,1	2,2	3,3	3,2	2,1	3,1
1,3	.950 215	.948 243	.946 252	.941 438	.941 355	.941 327	.935 552	.928 990	.915 1860
1,2	.951 207	.950 215	.949 214	.946 290	.946 252	.946 239	.941 327	.937 514	.928 885
2,3	.954 199	.952 206	.950 197	.946 323	.946 252	.946 228	.940 370	.934 696	.922 1341
1,1	.953 207	.952 206	.951 202	.950 215	.950 205	.950 202	.948 214	.946 252	.941 327
2,2	.955 203	.954 199	.953 187	.950 229	.950 197	.950 186	.946 228	.942 354	.934 605
3,3	.958 213	.956 205	.954 183	.950 264	.950 203	.950 182	.944 263	.939 502	.928 977
3,2	.959 226	.958 213	.956 194	.954 211	.954 183	.954 174	.950 182	.946 263	.939 423
2,1	.957 215	.956 208	.955 199	.954 199	.954 190	.954 188	.952 181	.950 197	.946 228
3,1	.960 245	.959 235	.959 223	.958 213	.958 206	.958 203	.956 184	.954 183	.950 182

TABLE VIII (Continued)

p = .95

n = 30

Assumed a,b	Actual a,b 1,3	1,2	2,3	1,1	2,2	3,3	3,2	2,1	3,1
1,3	.950 76	.949 80	.948 81	.947 105	.947 94	.947 91	.945 120	.942 177	.938 292
1,2	.951 75	.950 76	.949 76	.948 86	.948 81	.948 79	.947 91	.945 115	.942 163
2,3	.952 74	.951 75	.950 74	.948 92	.948 82	.948 79	.946 99	.944 146	.939 241
1,1	.951 76	.951 75	.951 75	.950 76	.950 75	.950 75	.949 76	.948 81	.947 91
2,2	.952 75	.952 74	.951 73	.950 79	.950 74	.950 73	.948 78	.947 97	.944 133
3,3	.953 77	.952 75	.951 72	.950 85	.950 75	.950 72	.948 84	.946 123	.941 199
3,2	.953 79	.953 77	.952 74	.951 76	.951 72	.951 71	.950 72	.949 85	.946 110
2,1	.953 77	.952 76	.952 74	.952 74	.952 73	.952 73	.951 72	.950 74	.948 78
3,1	.954 82	.954 80	.953 78	.953 77	.953 76	.953 75	.952 72	.951 72	.950 72

CHAPTER V

SUMMARY AND EXTENSIONS

In this thesis we have investigated a Bayesian approach to tolerance intervals and have proposed criteria which can be used to measure the accuracy of the tolerance interval obtained or to determine the sample size necessary for a tolerance interval to have a specified accuracy. In Chapter II we considered q tolerance intervals for p coverage, that is, intervals in which we have q confidence that the coverage will be at least p. In Chapter III we considered p-expected coverage tolerance intervals and in Chapter IV we investigated the sensitivity of Bayesian p-expected coverage tolerance intervals for the exponential density to the assumption of a prior density on the parameter θ.

In Chapter II, we measured accuracy by considering the probability, given the sample, that the θ we drew was one for which the coverage of the interval obtained was at least some proportion p', greater than the desired coverage p. Denoting this probability by q', we saw that for some common densities and priors, q' was a decreasing function of the sample size and was not a function of the actual sample. One question to be answered would be whether these properties were due to choosing natural conjugate priors, good fortune, or both. In other words, for what class of prior densities is q' a decreasing function of sample size, but not a function of the actual sample.

For a specified prior and any statistic, $t(x_{1,n})$, it is possible, although perhaps untractable, to find $f''(\theta|t(x_{1,n}))$, and from this to determine a Bayesian tolerance limit, and also the accuracy, say $q'(t)$. The question then arises as to whether there exists a statistic $t^*(x_{1,n})$ such that $q'(t^*) \leq q'(t)$ for all t. Thus it would be of interest to determine what priors and what statistics, if any, lead to most accurate Bayesian tolerance intervals.

The accuracy criterion we considered for Bayesian p-expected coverage tolerance intervals was the probability, given the sample, that the limit obtained was within Δ of the lower p probability point of $f(y|\theta)$. For the case when this probability is a function of the sample, we proposed a multi-stage sampling procedure and did a Monte Carlo investigation of the average sample size required to obtain a specified accuracy. Again in this case it would be of interest to explore the possibility of the existence of a statistic $t^*(x_{1,n})$ which would lead to most accurate tolerance limits.

Other possible areas of investigation include an empirical Bayes approach to tolerance intervals, discrete prior densities, and multi-parameter densities such as the Weibull and generalized gamma. Hence it appears that there are several aspects of a Bayesian approach to tolerance limits to be investigated in addition to the sample size determination aspect with which this thesis has been concerned.

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