THE MULTIPLE RESPONSE PROBLEM

By

HOWARD LELAND DOCKERY

Bachelor of Science
University of Missouri at Rolla
Rolla, Missouri
1963

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1965

Submitted to the Faculty of the Graduate School
of the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
May, 1967

THE MULTIPLE RESPONSE PROBLEM

Thesis Approved:

J. Leroy Folks
Thesis Adviser

James E. Shamblin

David L. Weeks

Robert D. Morrison

D. D. Durham
Dean of the Graduate College

ACKNOWLEDGMENTS

The author is indebted to Dr. Leroy Folks for suggesting the problem and directing the research throughout the preparation of this thesis.

The author is especially indebted to Professor E. K. McLachlan for his careful review which resulted in a greatly improved thesis. Thanks is also extended to Professor Robert D. Morrison for his helpful comments.

Gratitude is extended to Professors Robert D. Morrison, E. K. McLachlan, James E. Shamblin, and Robert A. Hultquist for serving on the author's advisory committee.

Thanks is also extended to Dr. Charles E. Antle who kindled the author's interest in statistics.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
II.	ADMISSIBILITY AND COMPLETENESS	5
III.	ADMISSIBLE POINTS FOR SOME RESPONSE FUNCTIONS WITH SPECIAL TYPES OF CONTOURS: N = 2, P = 2	24
	Admissible Points for Y_1 Having Elliptic Contours and a Maximum Response, Y_2 Having Elliptic Contours and a Maximum Response	25
	Admissible Points for Y_1 Having Elliptic Contours and a Minimum Response, Y_2 Having Elliptic Contours and a Maximum Response	30
	Admissible Points for Y_1 Having Elliptic Contours and a Maximum Response, Y_2 Having Hyperbolic Contours	32
	Admissible Points for Y_1 Having Elliptic Contours and a Maximum Response, Y_2 Having Parabolic Contours	37
	Admissible Points for Y_1 Having Hyperbolic Contours and Y_2 Having Hyperbolic Contours	43
	Admissible Points for Y_1 Having Parabolic Contours and Y_2 Having Hyperbolic Contours	52
	Admissible Points for Y_1 Having Parabolic Contours and Y_2 Having Parabolic Contours	59
	Summary	63
IV.	ADMISSIBLE POINTS WHEN THERE IS A LINEAR CONSTRAINT ON THE CONTROLLED VARIABLES: N = 2, P = 2	64
V.	TANGENT POINTS FOR RESPONSE FUNCTIONS WITH SPECIAL CONTOURS: N = 2, P = 3	73
VI.	ELLIPTIC CONTOURS WITH MOVABLE CENTER	80
VII.	SUMMARY	85
SELECTED BIBLIOGRAPHY	. . . .	87
APPENDIX A	. . . .	88
APPENDIX B	. . . .	93

LIST OF TABLES

Table		Page
I.	Responses with Discrete Controlled Variables	4
II.	Direction of Gradient	41
III.	Admissible Points for Hyperbolic Contours	52

LIST OF FIGURES

Figure		Page
1.	Two Responses of Interest, $P = 1$	9
2.	Contours for Two Response Functions of Interest for $N = 2$, $P = 2$	12
3.	A Graph of the Two Response Functions of Interest	15
4.	Tangent Points for Two Responses with Elliptic Contours, $a<1$	27
5.	Tangent Points for Two Responses with Elliptic Contours, $a>1$	28
6.	Tangent Points for Elliptic and Hyperbolic Contours	33
7.	Tangent Points for Elliptic and Parabolic Contours, $b>0$	39
8.	Tangent Points for Elliptic and Parabolic Contours, $b<0$	40
9.	Region of a Parabolic Contour	41
10.	Asymptotes of Hyperbolic Contours, $a>0$, $b>0$	48
11.	Asymptotes of Hyperbolic Contours, $a>0$, $b<0$	49
12.	Asymptotes of Hyperbolic Contours, $a<0$, $b>0$	49
13.	Asumptotes of Hyperbolic Contours, $a<0$, $b<0$	50
14.	Tangent Points for Hyperbolic and Parabolic Contours, $-b/4>h$	57
15.	Tangent Points for Hyperbolic and Parabolic Contours, $-b/4<h$	58
16.	Linear Constraints on Elliptic Contours	67
17.	Linear Constraints on Elliptic and Hyperbolic Contours	70
18.	Tangent Surface I	76

LIST OF FIGURES (CONT.)

Figure | Page

19. Tangent Surface II 77

20. Tangent Points for Contours with a Movable Center 84

CHAPTER I

INTRODUCTION

The concept of a P + 1 dimensional surface is an algebraic concept. However, statisticians call upon their intuition developed from observing surfaces and objects in the world around us to describe the relationship between controlled variables $X_1, X_2, \ldots, X_p$ and a response variable Y. The possible values of Y graphed against the values of the controlled variable are thought of as tracing out a surface in the P + 1 dimensional space. Actually we are interested in the surface generated by the true or basal response, rather than the observed response Y.

If this true response is a function f of the controlled variables, then Y is given by

$$Y = f(X_1, X_2, \ldots, X_p) + e$$

where e is an experimental error. One problem of interest is to determine the level, or levels, at which each X_i should be set in order to maximize (or minimize) f. If the function f is known, then the problem is a standard optimization problem which may be solved by some optimization technique such as linear programming (7), dynamic programming (2), nonlinear programming (8), or one of the many other optimization techniques. However, in most response surface problems, the function f is not known; and in these cases a different formulation is required. In order to determine the level, or levels, at which each X_i should be set in order to maximize f, one must employ some search technique. When the function f is not known, the difficulty of the problem is compounded further, by the fact that, while performing our experiments, we are measuring the Y responses and not the f responses.

Some of the more popular search techniques at this time are the single-factor method (5), the method of steepest ascent (4), the method of random search (12), and Kempthorne's method of parallel tangents (10).

Unlike the problem where a single response is of interest, the problem where there are multiple responses of interest has received very little attention in mathematical literature. The responses may be written:

$$
\begin{aligned}
Y_1 &= f_1(X_1, X_2, \ldots, X_p) + e_1, \\
Y_2 &= f_2(X_1, X_2, \ldots, X_p) + e_2, \\
&\ldots\ldots\ldots\ldots\ldots, \\
Y_N &= f_N(X_1, X_2, \ldots, X_p) + e_N.
\end{aligned}
$$

The problem of selecting the settings, $(X_1^o, X_2^o, \ldots, X_p^o)$ of the controlled variables to simultaneously optimize the, say N, responses of interest is the main subject of the following chapters. Note that above we have said optimize rather than maximize (or minimize) because, in general, it is not possible to find a set of values $X_1^o, X_2^o, \ldots, X_p^o$, which will maximize all N responses simultaneously. Therefore, we search for some "best" points. In Chapter II, we will consider only maximization problems because if one of the responses of interest, say Y_i, is to be minimized, we may consider a new variable, say $Z_i = -Y_i$, and then maximize the Z_i.

The subsequent chapters are concerned with methods for solving problems similar to the following examples: Suppose one wishes to develop a fertilizer from nitrogen, phosphorus, and potash which will simultaneously produce a maximum yield for wheat, alfalfa, and corn. We would be very surprised if there exists some level of nitrogen, X_1, phosphorus, X_2, and potash, X_3, which would maximize the yield of wheat, Y_1, alfalfa, Y_2, and corn, Y_3, simultaneously. Thus we are interested in the settings of (X_1,X_2,X_3) which have associated with them some optimum property. In this example we note the number of controlled variables is equal to three. Also the number of responses of interest is equal to three.

It is of interest to mention that the units in which each response is measured need not be the same for the methods of the following chapters to apply. That is, no matter what units are used to measure the responses, whether they be the same for each response or different for each response, the set of optimum settings, X^o's, obtained by applying the methods developed in the following chapters will be the same. Since the units of measure of different responses may be different, a linear combination of the responses may or may not have much meaning. For example, suppose one wishes to develop a coolant which has maximum density, Y_1, and maximum boiling point, Y_2, as two of its responses of interest. A linear combination of Y_1 and Y_2 has very little meaning.

As a second example, suppose one is interested in building a boat of specified size and shape. It is desired to have a boat with maximum strength, Y_1, and minimum weight, Y_2. One may choose any level (amount) of wood, X_1, fiber-glass, X_2, or steel, X_3, to construct the boat. Again it is obvious that no combination of the three materials, the controlled variables, will simultaneously maximize the strength and minimize the weight of the boat. Therefore, we are interested, as before, in the combination of the controlled variables which has associated with it some optimum property.

To introduce a concept which will be developed in the next chapter, we present a third example. Suppose there are two controlled variables. Suppose the first controlled variable has three possible values and the second controlled variable has two possible values. Suppose there are three responses of interest and we wish to maximize each. It is possible to construct the following table where each entry is a vector representing (Y_1, Y_2, Y_3).

TABLE I

RESPONSES WITH DISCRETE CONTROLLED VARIABLES

VARIABLE 2

Variable 1	1	2
1	(1, 2, 3)	(2, 4, 5)
2	(1, 5, 5)	(1, 3, 3)
3	(1, 0, 1)	(2, 4, 4)

It is obvious that the variable combination (1, 2) is better than (1, 1), (2, 2), (3, 1) and (3, 2), but we are unable to say whether (1, 2) or (2, 1) is better. We shall develop in the following chapters methods for handling such problems when certain conditions are met.

CHAPTER II

ADMISSIBILITY AND COMPLETENESS

The purpose of this chapter is to develop some basic theorems which will prove to be valuable tools in the later chapters. The following notation and definitions will facilitate this development.

1. The symbol X_i will represent the ith controlled variable. Unless otherwise stated, $i = 1, 2, \ldots, P$.
2. The letter X will denote a P-dimensional vector of controlled variables; that is, $X = (X_1, X_2, \ldots, X_P)$. Each point X is a point in the domain of the response functions of interest. Unless otherwise stated, in the development that follows, the domain of the response functions of interest will be the points in E_P.
3. The jth response function of interest is denoted by Y_j. Unless otherwise stated, $j = 1, 2, \ldots, N$. The point $Y_j(X)$, $X \in E_P$, is the image of X under Y_j.
4. The letter Y will denote the N-dimensional vector of the response functions of interest; that is, $Y = (Y_1, Y_2, \ldots, Y_N)$.
5. The symbol ∇Y_j will represent a P-dimensional vector of derivatives; that is, $\nabla Y_j = (\frac{\partial Y_j}{\partial X_1}, \frac{\partial Y_j}{\partial X_2}, \ldots, \frac{\partial Y_j}{\partial X_P})$. This vector is called the gradient vector.

It was explained in Chapter I that any minimization problem can be changed to a maximization problem. Therefore, each problem we will consider can be put into the following context. There are N response functions of interest $Y_1, Y_2, \ldots, Y_N$, and one wishes to choose the set of values for the P controlled variables, $(X_1, X_2, \ldots, X_p)$, which will simultaneously give the highest possible values for all the response functions of interest.

In Chapter I we mentioned associating some optimum property with a point. In order to be able to determine if a point has such a property, one must first be able to compare different points. This motivates the following definition.

Definition 1: A point $X^o \in E_p$ is better than a point $X^1 \in E_p$ for the response functions of interest, $(Y_1, Y_2, \ldots, Y_N)$ if

(1) $Y_j(X^o) \geq Y_j(X^1)$ for $1 \leq j \leq N$ and

(2) $Y_k(X^o) > Y_k(X^1)$ for at least one k, $1 \leq k \leq N$.

If (1) of Definition 1 holds, then X^o is at least as good as X^1.

Definition 2: The point $X^o \in E_p$ is an admissible point for the response functions of interest, $(Y_1, Y_2, \ldots, Y_N)$, if there exists no point X^1 in E_p better than X^o.

We are obviously interested in the admissible points and, in case there exist more than one admissible point, in the set of all admissible points. On the other hand suppose we wish to find a better point than a given point. Where should we search? This leads us to the question of whether there is a set of points such that we are sure of finding a point in the set which is better than the given point. Such a set is now defined.

Definition 3: A complete set of points is a set S of points such that, given any point X^o in E_p not in the complete set, there exists a point $X^1 \epsilon$ S that is better than X^o.

It may also be of interest to speak of an essentially complete set.

<u>Definition 4</u>: An essentially complete set of points is a set of points such that, given any point $X^o \epsilon E_p$ not in the essentially complete set, there exists a point $X^1 \epsilon E_p$ in the set which is at least as good.

It is seen from the preceding definitions, that a complete set is an essentially complete set; whereas an essentially complete set need not be a complete set.

Another, even more important, set of points which we will make use of is a minimal complete set of points.

<u>Definition 5</u>: A minimal complete set of points, if it exists, is a complete set of points such that no proper subset is a complete set of points.

<u>Definition 6</u>: A contour Γ_c of Y_j is the $\{x \mid Y_j(X) = C\}$ where C is an arbitrary real constant.

A minimal essentially complete set of points could also be defined, but we make no use of such a set. Likewise, one could define an inadmissible point X^o as a point such that there exists a point X^1 that is better than X^o.

From Definition 2 and Definition 5, one realizes that an admissible point and a minimal complete set are closely related. Theorems I and II, that are stated below, serve to express some of the properties of an admissible point and the minimal complete set. The proofs of Theorems I and II are essentially the same as the proofs for similar theorems concerning decision rules given in reference (13).

<u>Theorem I</u>: If a minimal complete set of points exists, it is equal to the set of admissible points.

Theorem II: A necessary and sufficient condition for the existence of a minimal complete set of points is that the set of admissible points be a complete set.

We note that if a minimal complete set, A, exists, then for any point X^o not in A there is a point in A which is better than X^o. Therefore, if A exists, it is of special interest. It is clear from the above definitions that a minimal complete set is unique. However, there may be any number of complete sets. For example, let P = 1 and N = 2. If $Y_1(X) = 2X$ and $Y_2(X) = X^3$, then the set

$$\left\{X \mid X \in (a, \infty)\right\}$$

is a complete set for any real number a, but the minimal complete set does not exist. As an example of when the minimal complete set does exist, let P = 1, N = 2, $Y_1(X) = 4 - X^2$ and $Y_2(X) = -(x - 2)^2$. This situation is shown in Figure 1. It is seen from Figure 1 that the minimal complete set is

$$\left\{X \mid X \in [0, 2]\right\}.$$

Since we are interested in the minimal complete set, if we had some way of analyzing the response functions of interest and determining if the minimal complete set exists or not, it would be a powerful tool. As of now we have no such tool, but Antle (1) has proved a theorem stating sufficient conditions for the existence of a minimal complete set. Use will be made of this theorem in the following chapters.

Theorem III: If Y_j is everywhere continuous for all j and at least one of the sets $S_j(a) = \left\{X \mid Y_j(X) \geq a\right\}$ is bounded for all a, then the minimal complete set of points for the response functions

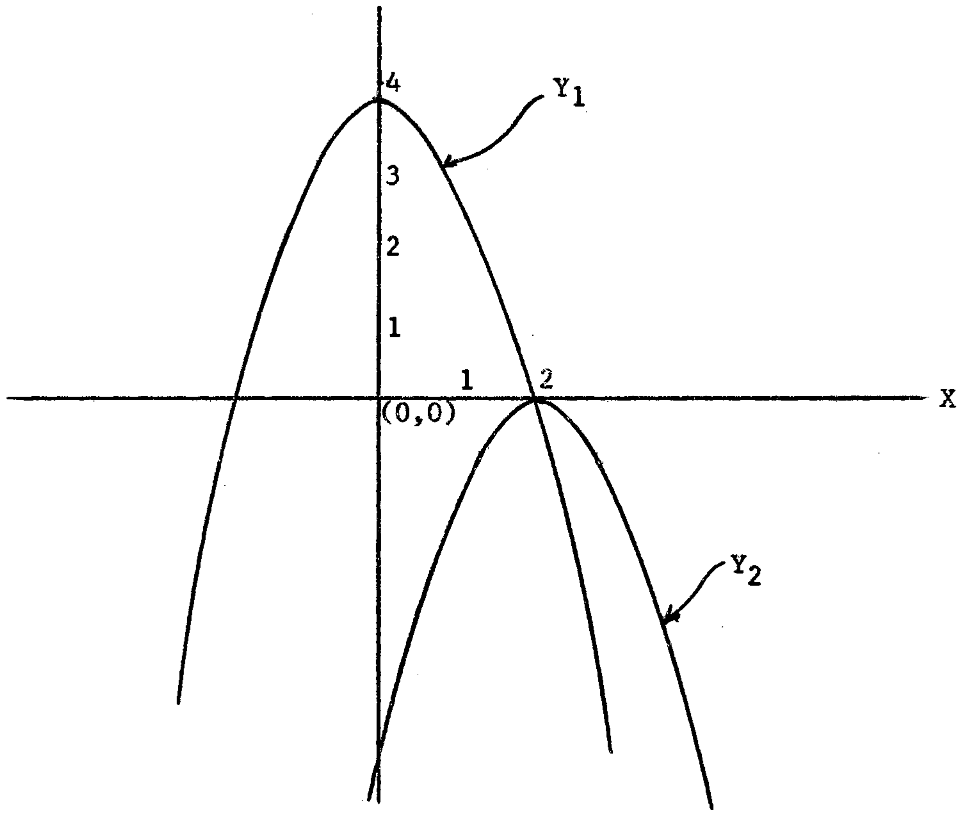

Figure 1. Two Responses of Interest, P = 1.

Y_1, Y_2, ..., Y_N exists.

Proof: See Antle (1), page 6.

Let us next consider the following example with P = 1, N = 2, $Y_1(X) = -X^2$ and $Y_2(X) = X + 2$. Then, since both $Y_1(X)$ and $Y_2(X)$ are continuous everywhere, and the set $S_1(a) = \left\{X \mid Y_1(X) \geq a\right\}$ is bounded for all a, we see that the minimal complete set for $Y_1(X) = -X^2$ and $Y_2(X) = X + 2$ exists. The minimal complete set is seen to be

$$\left\{X \mid X \in [0, \infty)\right\}.$$

Perhaps even more important than being able to determine if the minimal complete set exists, is being able to identify the admissible points. If we are able to identify the admissible points and if the minimal complete set of points exists, then we can identify the minimal complete set as the set of admissible points.

In the preceding examples, the sets of admissible points were easily identified; however, when considering problems with $P \geq 2$, the admissible points may be difficult to identify. The problem of finding a necessary and sufficient condition that a point be an admissible point was studied by Antle (1) and a necessary condition was obtained. Antle stated that if ∇Y_1, ∇Y_2, ..., ∇Y_N exist at a point X^o, then a necessary condition for X^o to be an admissible point is that there exists a vector α such that

$$\sum \alpha_i \ \nabla Y_i(X^o) = \phi, \ \alpha_i \geq 0 \text{ for all } i,$$

and

$$\sum \alpha_i = 1 \ .$$

This theorem is not true as stated because the proof assumes that each response increases as we move in the direction of the gradient. Although this is categorically stated to be the case in many calculus textbooks, it is easy to construct examples where the gradient does not lead us to higher responses but in fact leads us to lower responses.

Of course the utility of Antle's theorem is not decreased by such examples in that from an applied point of view, we would expect the theorem to hold on response surfaces actually encountered. However, it is desirable to determine what restrictions must be placed upon the functions Y_1, Y_2, ..., Y_N in order that Antle's theorem hold. In the theorem which follows we simply formulate restrictions which agree with the intuitive concept that the response increases in the direction of the gradient.

<u>Theorem IV</u>: If $\nabla Y_1(X)$, $\nabla Y_2(X)$, ..., $\nabla Y_N(X)$ exist at a point X^o and for every i with $\nabla Y_i(X^o) \neq \phi$ and every U with a positive component in the direction of $\nabla Y_i(X^o)$ there exists a $\delta(i, U) > 0$ such that $Y_i(X^o + tU) > Y_i(X^o)$ when $0<t<\delta(i, U)$, then a necessary condition for X^o to be an admissible point is that there exists a vector α such that

$$\sum \alpha_i \quad \nabla Y_i(X^o) = \phi, \ \alpha_i \geq 0 \text{ for all i, and } \sum \alpha_i = 1.$$

Proof: Assume no such vector α exists. Then none of the $\nabla Y_i(X^o)$ are equal to the null vector, and the convex hull generated by the vectors $X^o + \nabla Y_i(X^o)$ does not contain X^o. Call this hull D. Since X^o and D are convex and disjoint, there exists a hyperplane that strictly separates them. Call this hyperplane H. Thus H divides E_p into two half spaces: the half space H^+ which contains D and the half space H^- which contains X^o. Let the normal to H that is directed toward D be V. Therefore $V \cdot X^o < 0$ since X^o is in H^-. Also $V \cdot \sum B_i [X^o + \nabla Y_i(X^o)] > 0$ for all $B_i \geq 0$, $\sum B_i = 1$ since $\sum B_i [X^o + \nabla Y_i(X^o)]$ are the points in D and D is in H^+. But $V \cdot \sum B_i [X^o + \nabla Y_i(X^o)] > 0$ implies $\sum B_i \ V \cdot \nabla Y_i(X^o) > 0$ for all $B_i \geq 0$, $\sum B_i = 1$. This implies that $V \cdot \nabla Y_i(X^o) > 0$ for all i. Therefore, each $\nabla Y_i(X^o)$ has a positive component in the direction of V. By hypothesis there exists a δ_i for each i such that

$$Y_i(X^o + tV) > Y_i(X^o), \ 0<t<\delta_i.$$

Let $\delta = \min \delta_i$. Then

$$Y_i(X^o + tV) > Y_i(X^o), \quad 0<t<\delta$$

for all i. Therefore $X^o + tV$ $(0<t<\delta)$ is better than X^o; hence X^o is not an admissible point. This completes the proof of Theorem IV.

Theorem IV would be a much more powerful tool if it specified both a necessary and sufficient condition for a point to be an admissible point. However, it is easily seen from the next example that the conditions given in Theorem IV are not sufficient conditions.

Let P = 2, N = 2, $Y_1(X) = X_1 + X_2$ and Y_2 have contours as shown in Figure 2.

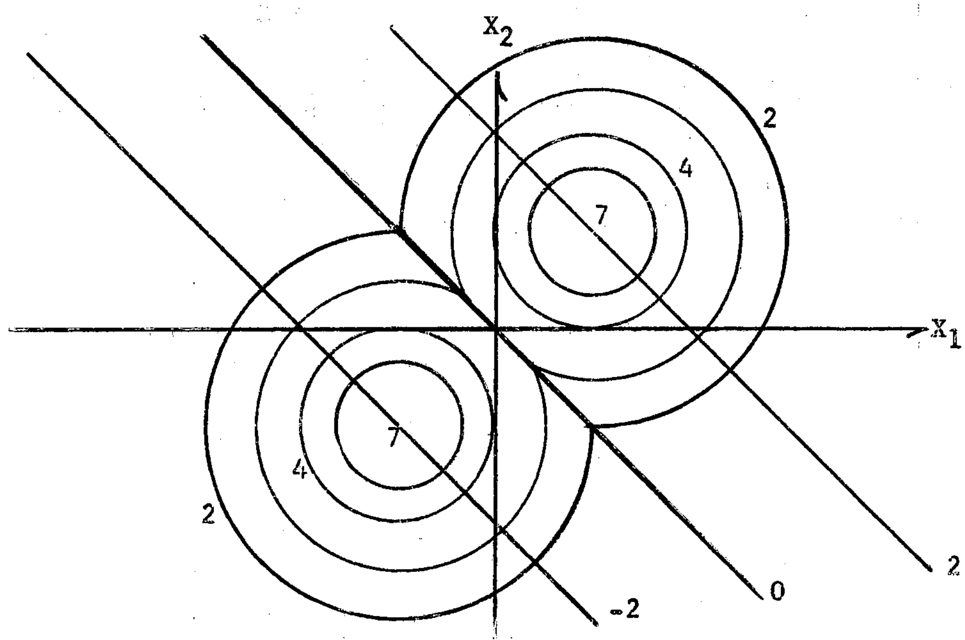

Figure 2. Contours for Two Response Functions of Interest for N = 2, P = 2.

It is easily seen that the set

$$\{X \mid X_1 = X_2,\ -1 \leq X_1 < 0\} \cup \{X \mid X_1 = X_2,\ 1 \leq X_1\}$$

satisfied Theorem IV. However, the set

$$\{X \mid X_1 = X_2,\ 1 \leq X_1\}$$

is the set of admissible points. Hence, the theorem does not provide sufficient conditions for a point to be an admissible point. It is of interest to note that the conditions of Theorem III are satisfied, so the set of admissible points forms the minimal complete set of points for the given response functions.

Although the restrictions imposed by the hypothesis of Theorem IV are not strict and although they are suggested by intuitive notions about the gradient, it would be difficult, in general, to verify that the hypothesis of Theorem IV is satisfied. Therefore, it is desirable to have conditions which can be more easily examined, even if the class of surfaces satisfying the conditions is further restricted. Thus we state as a corollary:

<u>Corollary 1</u>: If each $Y_1, Y_2, \ldots, Y_N$ has a differential at X^o, then a necessary condition for X^o to be an admissible point is that there exist a vector α such that

$$\sum_{i=1}^{N} \alpha_i \nabla Y_i(X^o) = \phi,\ \alpha_i \geq 0 \text{ for all } i,$$

and

$$\sum \alpha_i = 1.$$

Proof: It will suffice to show that if each Y_j has a differential at X^o, then the conditions of Theorem IV are satisfied. Let $\nabla Y_j(X^o) \neq \phi$; then we need to show that for every vector U such that $U \cdot \nabla Y_j(X^o) > 0$, there exists a $\delta(U)$ such that

$$Y_j(X^o + tU) > Y_j(X^o) \text{ when } 0<t<\delta(U).$$

Let U be any vector such that $\nabla Y_j(X^o) \cdot U = b$, $b>0$. Since Y_j has a differential at X^o, $\nabla Y_j(X^o) \cdot U = D_U Y_j(X^o)$. But $D_U Y_j(X^o) = \lim_{t \to 0} \dfrac{Y_j(X^o + tU) - Y_j(X^o)}{t}$. Therefore $\lim_{t \to 0} \dfrac{Y_j(X^o + tU) - Y_j(X^o)}{t} = b$.

From the definition of a limit we know that for every number $\varepsilon>0$, there is another number $\delta>0$ such that whenever $0<t<\delta$, then

$$\left| \frac{Y_j(X^o - tU) - Y_j(X^o)}{t} - b \right| < \varepsilon$$

Let $\varepsilon = \frac{b}{2}$. Then there is another number $\delta>0$ such that whenever $0<t<\delta$, then

$$\left| \frac{Y_j(X^o - tU) - Y_j(X^o)}{t} - b \right| < \frac{b}{2} \quad \text{or}$$

$$-\frac{b}{2} < \frac{Y_j(X^o - tU) - Y_j(X^o)}{t} - b < \frac{b}{2} \text{ . This implies}$$

that $\frac{bt}{2} < Y_j(X^o - tU) - Y_j(X^o)$. But $\frac{bt}{2} > 0$. Therefore $Y_j(X^o - tU) > Y_j(X^o)$ whenever $0<t<\delta$. This completes the proof of Corollary 1.

With a first reading of the preceding text, one may be led to the false conclusion that if the set of admissible points can be found, the problem is solved. However, this is the case only if the minimal complete set exists. The following example illustrates that there are cases where points other than the set of admissible points need to be considered. Suppose $P = 1$, $N = 2$, $Y_1(X) = |X|$, and

$$Y_2(X) = 2, \text{ if } X = 0$$
$$= 1, \text{ if } X \neq 0.$$

This situation is depicted in Figure 3. Note that the only admissible point is $X = 0$. However, if one is interested in large values of $Y_1(X)$, he would never choose X close to 0. Thus, one would be interested in points other than the admissible point. Clearly there exists no minimal complete set of points for these two response functions. If one did exist, then we would need to consider only the admissible points. It should be mentioned that many of the theorems developed in this chapter do not apply for response functions Y_1 and Y_2 of Figure 3.

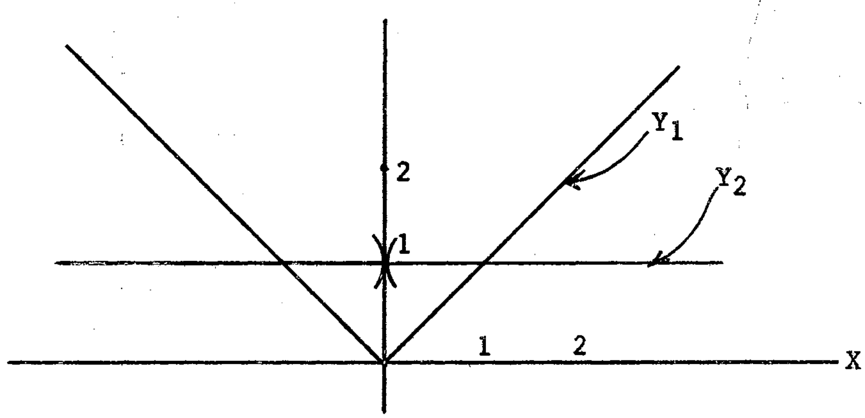

Figure 3. A Graph of the Two Response Functions of Interest.

In applying Theorem IV, it is generally more convenient to use the result stated in Corollary 2.

Corollary 2: If $N = 2$, Y_1 and Y_2 satisfy the conditions of Theorem IV, $\nabla Y_1(X^o) \neq \phi$, $\nabla Y_2(X^o) \neq \phi$, then a necessary condition for X^o to be an admissible point is that there exist a negative number $-c$ such that

$$\nabla Y_1(X^o) = -c \quad \nabla Y_2(X^o).$$

In studying response surfaces, the device of sketching contours is very helpful. Although one must be careful not to be misled by "special cases" we shall rely heavily upon such sketches throughout this thesis.

By inspecting the contours of two response surfaces, it seems intuitively obvious that points satisfying Theorem IV are, in fact, points at which a contour of one surface is "tangent" to a contour of the other surface. The concept of tangent points is therefore explored.

<u>Definition 7</u>: Let X^o be a point in F_p where $\nabla Y_1(X^o)$ and $\nabla Y_2(X^o)$ exist. The point X^o is a tangent point for the response functions Y_1 and Y_2 if there exist numbers b_1 and b_2, both not zero, such that

$$b_1 \ \nabla Y_1(X^o) = b_2 \ \nabla Y_2(X^o).$$

Of course, the set of tangent points is, in general, a larger set than the set of points satisfying the necessary condition of Theorem IV, so it might be argued that we are complicating our task of finding the admissible points by considering a larger set. However, the tangent points are easily obtained and we shall make use of them in this thesis. Considering Theorem IV and Definition 7, we are led to the following theorem:

Theorem V: If the hypothesis of Theorem IV holds, then a necessary condition for X^o to be an admissible point is that it be a tangent point.

It is obvious, but it should be stressed, that the condition stated in Theorem V is not a sufficient condition.

When one is searching for admissible points, it is very desirable to be able to eliminate some of the points in the P-dimensional space of the controlled variables as being not possible for admissible points. Theorem V sometimes aids us in performing such an operation. From Theorem V we are able to eliminate all points except the tangent points. Another such tool is the subject of the next theorem.

Theorem VI: If $N = 2$, Y_1 has a maximum response at some point, say X^1, Y_2 has a maximum response at some point, say X^2, then the set of admissible points is contained in the intersection of sets A and B, where

$$A = \left\{X \mid Y_1(X) \geq Y_1(X^2)\right\}$$

and

$$B = \left\{X \mid Y_2(X) \geq Y_2(X^1)\right\}.$$

Proof: Suppose X^o is an admissible point and not contained in A. Then $Y_1(X^o) < Y_1(X^2)$ and $Y_2(X^o) \leq Y_2(X^2)$; that is, X^2 is better than X^o. This contradicts the assumption that X^o is admissible; therefore X^o is contained in A. Similarly we can show $X^o \in B$. Therefore $X^o \in A \cap B$.

Theorem VI will prove to be a very powerful tool when $N = 2$ and both Y_1 and Y_2 have maximum responses. One may then wonder if there exists such a tool for the equally important problem when Y_1 has a maximum response and Y_2 has a minimum response. The answer is in the affirmative as is shown in Theorem VII.

Theorem VII: If $N = 2$, Y_1 has a maximum response at a point, say X^1, Y_2 has a minimum response at a point, say X^2, then the set of admissible points is contained in the set A, where

$$A = \left\{X \mid Y_2(X) \geq Y_2(X^1)\right\}.$$

Proof: To show all admissible points are contained in A, we show that for any point, say X^o, not in A, there is a point in A which is better than X^o. Hence, any point not in A would not be admissible. First, note that X^1 is in A, because

$$Y_2(X^1) = Y_2(X^1).$$

Let X^o be any point not in A. Since X^o is not in A, this implies that $Y_2(X^o) < Y_2(X^1)$. From the statement of Theorem VII,

$$Y_1(X^o) \leq Y_1(X^1);$$

because Y_1 has a maximum at X^1. Therefore, X^1 is better than X^o. However, X^o was any point not in A, and X^o is not admissible; hence, all admissible points are in the set A.

In attempting to characterize the set of tangent points in the case of two families of ellipses, it was noticed that when the tangent point lay on two contours which intersected (other than at another tangent point) that the given tangent point could not be admissible. This led to the formulation of Theorem VIII.

Before stating Theorem VIII, let us first prove the following lemma.

<u>Lemma</u>: If X^o is a point which is not an admissible point and X^1 is a point such that

$$Y(X^1) \leq Y(X^o),$$

then X^1 is not an admissible point.

Proof: $Y(X^1) \leq Y(X^o)$ implies $Y_i(X^1) \leq Y_i(X^o)$ for all i, $1 \leq i \leq N$. Since X^o is not an admissible point, then there exists some point, say X^2, that is better than X^o. That is,

$$Y_i(X^2) \geq Y_i(X^o) \text{ for all i, } 1 \leq i \leq N$$

and

$$Y_k(X^2) > Y_k(X^o) \text{ for at least one k, } 1 \leq k \leq N.$$

But

$$Y_i(X^o) \geq Y_i(X^1) \text{ for all i; therefore,}$$

$$Y_i(X^2) \geq Y_i(X^1) \text{ for all i, and}$$

$$Y_k(X^2) > Y_k(X^1) \text{ for at least one k.}$$

Hence, X^2 is better than X^1 so X^1 is not an admissible point. This completes the proof of the lemma.

Now we are prepared to state and prove Theorem VIII.

<u>Theorem VIII</u>: If $N = 2$, Y_1 and Y_2 satisfy the conditions of Theorem IV for all X, $\nabla Y_1(X)$ is not equal to ϕ for any X such that $Y_1(X) = C_1$, $\nabla Y_2(X)$ is not equal to ϕ for any X such that $Y_2(X) = C_2$, then if the contours with values C_1 and C_2 intersect other than at a tangent point, there exists no admissible point, say X^1, such that $Y_1(X^1) = C_1$ and $Y_2(X^1) = C_2$.

Proof: Let X^o be a point where the contours with values C_1 and C_2 intersect, and not be a tangent point. From the definition of C_1 and C_2

$$Y_1(X^o) = C_1 \quad \text{and} \quad Y_2(X^o) = C_2.$$

Suppose there exists a point X^1 such that

$$Y_1(X^1) = C_1 \quad \text{and} \quad Y_2(X^1) = C_2.$$

Then

$$Y_2(X^1) = Y_2(X^o) \text{ and } Y_1(X^1) = Y_1(X^o).$$

Then from the lemma, since X^o is not an admissible point, X^1 is not an admissible point. This completes the proof of Theorem VIII.

Many times in practice, the set of admissible points, hence the minimal complete set if it exists, will be much easier to determine if one knows where one of the admissible points is located. For a simple illustration of this, suppose $N = 2$, Y_1 has circular contours and Y_2 has circular contours. Theorem V implies that if there exists an admissible point, the point must lie on the line through the center of both sets of circular contours. Therefore, if one admissible point, say X^o, can be found, it is known that the set of admissible points is on a line through the point X^o. Of course, if two admissible points are found, then the line is completely determined. To assume circular contours may seem unrealistic but the point we wish to make by this example is the importance of being able to determine at least one admissible point.

The following theorem will be of much use in helping us determine an admissible point.

<u>Theorem IX</u>: If Y_1, Y_2, ..., Y_N are the response functions of interest and Y_k has a unique maximum at X^k, then X^k is an admissible point.

Proof: Suppose X^k is not an admissible point. Then there must exist some point, say X^o, which is better than X^k. That is,

$$Y(X^o) > Y(X^k)$$

or

$$Y_i(X^o) \geq Y_i(X^k) \text{ for all } i, \ 1 \leq i \leq N$$

and

$$Y_j(X^o) > Y_j(X^k) \text{ for at least one } j, \ 1 \leq j \leq N.$$

Suppose $i = k$. Then since X^o is better than X^k

$$Y_k(X^o) \geq Y_k(X^k).$$

But this contradicts the statement that Y_k has a unique maximum at the point X^k so the assumption that X^k is not an admissible point is false. This completes the proof of Theorem IX.

The importance of Theorem IX is further emphasized by noting that there exist simple search techniques, steepest ascent, one factor at a time, parallel tangent, etc., to determine the X^k mentioned in the theorem when there is only one response function of interest. When we find the maximum response of Y_k, we have found X_k, hence an admissible point.

It should be noted that Theorem IX is also true if there are n response functions of interest (Y_1, Y_2, ..., Y_n) each of which has a unique maximum. That is, if there exist n points, each of which is a unique maximum for one of the Y_i, then each of the n points is an admissible point.

As an illustration, consider the example

$$P = 2,\ N = 4$$

$$Y_1(X) = 3 - X_1^2 - X_2^2$$

$$Y_2(X) = 5 + e^{-X_1^2 - 3X_2^2}$$

$$Y_3(X) = 8 + (X_1 - 3)^2 + (X_2 - 4)^2$$

and

$$Y_4(X) = 13 + e^{-(X_1 + 2)^2 - (X_2 - 1)^2}.$$

All the $Y_i(X)$ have unique extrema so we know that for each $Y_i(X)$ that has a unique maximum, there is an admissible point (namely the point where it attains its maximum) associated with it. One notes that $Y_1(X)$ has a maximum response at (0,0), therefore (0,0) is an admissible point. Likewise, $Y_2(X)$ has a maximum response at (0,0) so again (0,0) is an admissible point. The response function, $Y_4(X)$, has a maximum response at (-2,1) so (-2,1) is an admissible point. Since $Y_3(X)$ does not have a maximum response, Theorem IX does not apply.

It is regrettable that efforts to find a sufficient condition for a point to be an admissible point have failed. It was possible to state a theorem for $N = 2$, $P = 1$ which gives a sufficient condition for a local property of admissibility but the obvious generalization of the theorem to $P > 1$ is not true. The theorem for $P = 1$ is now stated.

Theorem X: If Y_1 and Y_2 have derivatives at each point of E_1 and if at some point X^o, the derivatives are of opposite sign, then there is a neighborhood of X^o, $N(X^o,\delta)$ such that there is no point in the neighborhood which is better than X^o.

Proof: Without loss of generality suppose $Y_1'(X^o) > 0$ and $Y_2'(X^o) < 0$. From the definition of a derivative there exist $N(X^o,\delta_1)$ such that for every $X \in N'(X^o,\delta_1)$, $Y_1(X) < Y_1(X^o)$ if $X < X^o$ and $Y_1(X) > Y_1(X^o)$ if $X > X^o$. Also there exist $N(X^o,\delta_2)$ such that for every $X \in N'(X^o,\delta_2)$, $Y_2(X) > Y_2(X^o)$ if $X < X^o$ and $Y_2(X) < Y_2(X^o)$ if $X > X^o$. Let $\delta = \min(\delta_1,\delta_2)$. Then there is no point $X \in N(X^o,\delta)$ which is better than X^o. This completes the proof of Theorem X.

In the remaining paragraphs of this chapter we wish to discuss the problem of scale. It is well known, for instance, that the steepest ascent method for finding an optimum of a response surface is not scale invariant. Other techniques have been shown to be invariant under scale transformations. Naturally we should ask whether the set of admissible points is scale invariant. Fortunately, the answer is affirmative.

We have already stated that we shall make heavy use of sketches of the contours in the characterization of the sets of admissible points. It is important to note that, under changes of scale, elliptical contours are transformed into elliptical contours, parabolic into parabolic, and hyperbolic into hyperbolic.

CHAPTER III

ADMISSIBLE POINTS FOR SOME RESPONSE FUNCTIONS WITH SPECIAL TYPES OF CONTOURS: N = 2, P = 2

Many times, while performing the exploration of a response surface with $P = 2$, it has been found that the contours of the response surface are sufficiently close to some family of quadratic curves so that a function which has this family of quadratic curves as contours is employed as the basic model. For this reason, one sees that it is important to be able to find the set of admissible points for response functions having families of quadratic curves as their contours.

Models such as the following are but a few of those which have families of quadratic curves as their contours.

$$Y_i(X) = K_1 + K_2X_1^2 + K_3X_2^2 \tag{4}$$

$$Y_i(X) = K_1 + K_2 \exp(K_3X_1^2 + K_4X_2^2) \tag{5}$$

$$Y_i(X) = K_1 + (K_2)^{K_3X_1^2 + K_4X_2^2 + K_5X_1 + K_6X_2 + K_7}, \quad K_2 \quad 0 \tag{6}$$

In the development of this chapter, we will translate axes, rotate axes, and use the change of scale technique whenever necessary to make the problems as simple as possible. It is easily seen that a rotation or translation does not change the set of admissible points, also it can easily be shown that the change of scale technique likewise leaves the set of admissible points unchanged. If one wishes to determine the equation of the admissible points in the original coordinate system, he can go through the inverse transformations to get that result. However, here we are interested only in finding the admissible points for the simple problem, because , as was stated in Chapter II the set of admissible points for the simple problem will be on the same type of curve as the set of admissible points for the original problem.

Admissible Point for Y_1 Having Elliptic Contours and a Maximum Response, Y_2 Having Elliptic Contours and a Maximum Response

As our first problem, let us suppose that the contours of Y_1 form a family of ellipses with a maximum response at the center (h_1, k_1). Let the contours of Y_2 form a family of ellipses with a maximum response at the center (h_2, k_2).

Then, without loss of generality, one may translate the axis and have the Y_1 contours centered at (0,0) and the Y_2 contours centered at (h_2', k_2'). Now applying the change of scale technique, one may treat the problem as though Y_1 has circular contours centered at (0,0) and Y_2 has elliptic contours centered at (h_2', k_2'). After applying a rotation the new situation is: Y_1 has circular contours centered at (0,0) and Y_2 has elliptic contours centered at (h, k), $h \geq 0$, $k \geq 0$. The major axis of the Y_2 contours is parallel to one of the coordinate axes. Therefore, we may write the equations of the contours as follows.

The equation of the Y_1 contours is

$$X_1^2 + X_2^2 = K_1 \tag{7}$$

and the equation of the Y_2 contours is

$$a(X_1 - h)^2 + (X_2 - k)^2 = K_2 \quad (a > 0) \ . \tag{8}$$

We will first determine the set of tangent points, knowing the set of admissible points is contained in the set of tangent points from Theorem V, by setting $\partial X_2/\partial X_1 = X_2'$ from Equation 7 equal to $\partial X_2/\partial X_1 = X_2'$ from Equation 8.

Taking the derivatives of the functions in Equations 7 and 8 we have

$$2X_1 + 2X_2X_2' = 0 \tag{9}$$

and

$$2a(X_1 - h) + 2(X_2 - k)X_2' = 0 \ . \tag{10}$$

Solving for X_2' in Equations 9 and 10, we obtain

$$X_2' = -\frac{X_1}{X_2} \tag{11}$$

and

$$X_2' = -\frac{a(X_1 - h)}{X_2 - k} \ . \tag{12}$$

Setting X_2' of Equation 11 equal to X_2' of Equation 12, we have

$$\frac{a(X_1 - h)}{X_2 - k} = \frac{X_1}{X_2} \ , \tag{13}$$

or

$$(a - 1)X_1X_2 - ahX_2 + kX_1 = 0 \ . \tag{14}$$

If $a \neq 1$, that is, if the Y_2 contours are not circular, Equation 14 is the equation of a hyperbola.

Rewriting Equation 14, one has

$$X_2 = \frac{kX_1}{(1 - a)X_1 + ah} \quad . \tag{15}$$

Taking the limit of the right side of Equation 15 as $X_1 \to \infty$ to determine the horizontal asymptote of the hyperbola, we find the limit to be k/1-a. Therefore, $X_2 = k/1-a$ is the horizontal asymptote of the hyperbola. Rewriting Equation 14 again, we have

$$X_1 = \frac{ahX_2}{(a - 1)X_2 + k} \quad . \tag{16}$$

Taking the limit of the right side of Equation 16 as $X_2 \to \infty$ to determine the vertical asymptote of the hyperbola, we find the limit to be ah/a-1. Therefore, $X_1 = ah/a-1$ is the vertical asymptote of the hyperbola.

Noting there are two cases ($a < 1$ or $a > 1$), we may now draw the graph of the asymptotes in each case.

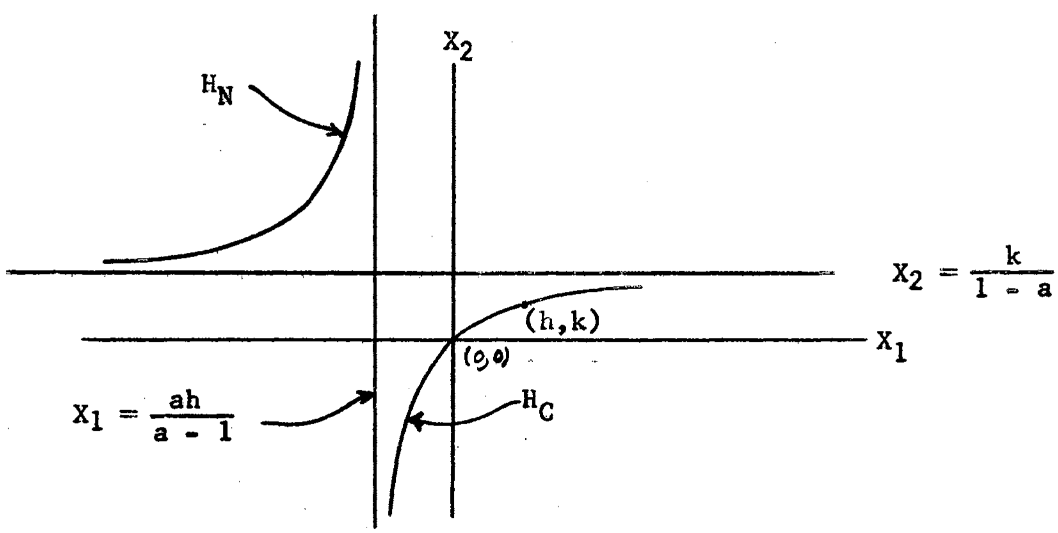

Figure 4. Tangent Points for Two Responses with Elliptic Contours, $a < 1$

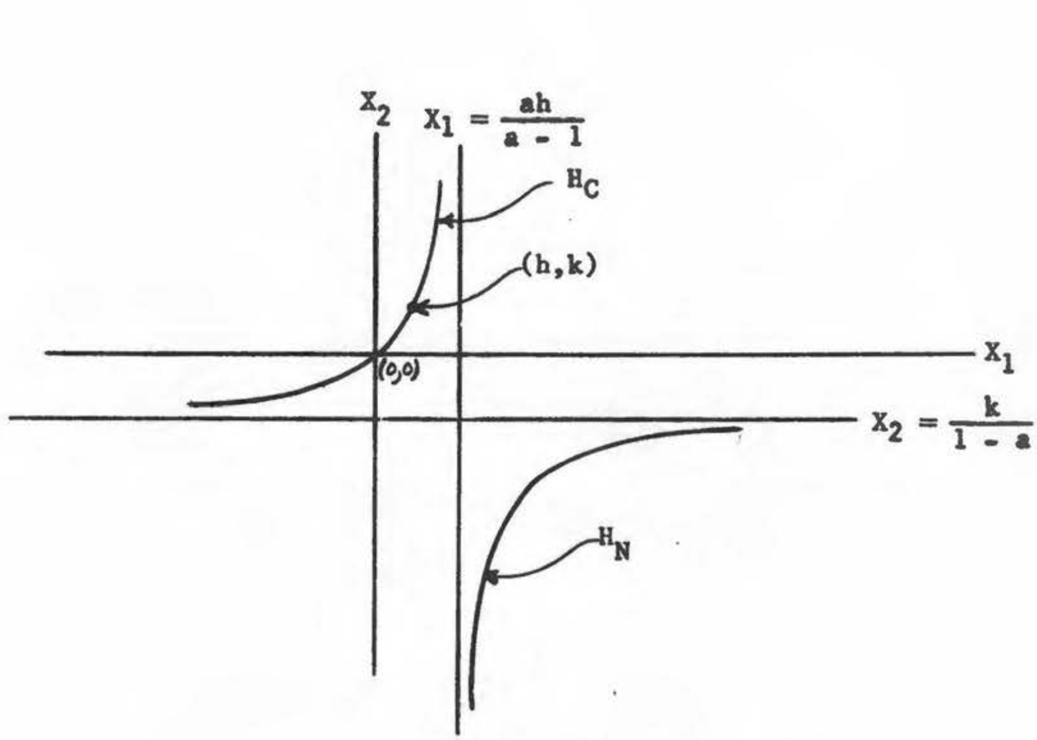

Figure 5. Tangent Points for Two Responses with Elliptic Contours, a > 1

From the asymptotes, and noting that the hyperbola passes through the points (0,0) and (h,k), one can now draw the graph of the hyperbola. For convenience in referring to the different branches of the hyperbola, we shall call the branch which passes through the origin, H_C, and the branch which does not pass through the origin, H_N.

We show in Appendix A that for each tangent point on H_N corresponding to a Y_1 and a Y_2 contour, with values, say C_1 and C_2, respectively, the contours with values C_1 and C_2 also intersect. We have already shown in Chapter II (Theorem VIII) that there exists no admissible point, say X^o, having $Y_1(X^o) = C_1$ and $Y_2(X^o) = C_2$ since contours with values

C_1 and C_2 intersect; hence there are no admissible points on H_N.

Consider Figure 4, since there are no admissible points on H_N, no admissible point has its X_1 coordinate less than ah/a-1. Let us look at the points whose X_1 coordinates satisfy ah/a-1 < X_1 < 0. Consider any point X^o on H_c and ah/a-1 < X_1^o < 0. If we draw a line from X^o through (h,k) and we move δ units, some infinitesimal amount, along that line, toward (h,k), to a point, say X^1, we increase both the Y_1 response and the Y_2 response. Thus, X^1 is a better point than X^o; therefore X^o is not an admissible point. From this, since X^o was any point on H_c with X_1 coordinate satisfying ah/a-1 < X_1 < 0, one knows there are no admissible points with X_1 coordinates less than zero.

Likewise, considering a point, say X^o, on H_c with $X_1^o > h$ and drawing the line from X^o through (h,k), we find that after moving δ units along this line, toward (h,k), one increases both the Y_1 response and the Y_2 response. Thus, there exist no admissible points corresponding to $X_1 > h$.

From this, it is seen that all admissible points must lie on H_c and the X_1 coordinate of all admissible points is bounded by zero and h. It is also easily seen that each point on H_c with X_1 coordinate satisfying $0 \leq X_1 \leq h$ is, in fact, an admissible point because as one moves (along H_c) from the origin to the point (h,k), the Y_1 response decreases while the Y_2 response increases with each move. Then, the set of admissible points for the case where a < 1 is given by

$$\{(X_1, X_2) | (a - 1)X_1X_2 - ahX_2 + kX_1 = 0, \ 0 \leq X_1 \leq h\} . \tag{17}$$

In a like manner, one can consider Figure 5 and find that the set of admissible points for the case where a > 1 is

$$\{(X_1, X_2) | (a - 1)X_1X_2 - ahX_2 + kX_1 = 0, \ 0 \leq X_1 \leq h\} . \qquad (18)$$

Clearly, from the preceding work one knows that if Y_1 has elliptic contours with a maximum response at the center of the ellipses and Y_2 has elliptic contours with a maximum response at the center of the ellipses, then the set of admissible points lie on a section of a hyperbola connecting the two maximum responses. The points on the section of the hyperbola are, of course, points in the set of tangent points.

It should be noted that the conditions for both $S_1(a)$ and $S_2(a)$ of Theorem III are satisfied. Hence, the set of admissible points is in fact the minimal complete set.

Admissible Points for Y_1 Having Elliptic Contours and a Minimum Response and Y_2 Having Elliptic Contours and a Maximum Response

Let us suppose, for our next problem, that the contours of Y_1 form a family of ellipses with a minimum response at their center (h_1, k_1). Let the contours of Y_2 form a family of ellipses with a maximum response at their center (h_2, k_2). Once again, we may go through translations, rotations, and change of scale techniques to obtain the following situation. The contours of Y_1 form a family of circles with a minimum response at their center $(0,0)$. The contours of Y_2 form a family of ellipses with a maximum response at their center (h,k), $h \geq 0$, $k \geq 0$. The major axes of the ellipses are parallel to one of the coordinate axes.

One should note that the contours of this problem and the contours of the first problem are the same, hence the set of tangent points are the same. Now, knowing what the set of tangent points is, we are ready to determine the set of admissible points. As before, there are no admissible points on H_N (Appendix A); so we need only determine which points on H_C are admissible points.

First, referring to Figure 4, let X^o be any point on H_C with $X_1^o < h$. Draw a line passing through X^o and through (h,k). Move along the line to (h,k), say the distance travelled is d, then move d units further along the line to a point, say X^1. Since (h,k) is the center of the elliptic contours, the $Y_2(X^1)$ response is equal to the $Y_2(X^o)$ response. It is seen that the distance from X^o to (0,0) is less than the distance from X^1 to (0,0) (radius of $Y_1(X^o)$ and $Y_1(X^1)$ contours, respectively). Since the Y_1 contours increase with increasing radii, $Y_1(X^1) > Y_1(X^o)$. Therefore, X^1 is a better point than X^o; so X^o is not an admissible point. However, X^o was an arbitrary point on H_C with $X_1^o < h$. Therefore, there are no admissible points with X_1 coordinates satisfying $X_1 < h$.

Clearly, each point on H_C with X_1 coordinate satisfying $X_1 \geq h$ is an admissible point. If we start at (h,k) and move along H_C in an increasing X_1 direction, each move increases the Y_1 response while decreasing the Y_2 response. Furthermore, since the sets $S_2(a)$ satisfy Theorem III, this set of admissible points is, in fact, the minimal complete set.

One can go through a similar argument for Figure 5, and the results will be similar. The set of admissible points when $a > 1$ is given by

$$\left\{(X_1, X_2) \mid (a - 1)X_1X_2 - ahX_2 + kX_1 = 0, \frac{ah}{a - 1} > X_1 > h\right\} . \qquad (19)$$

One sees from the preceding work, if Y_1 has elliptic contours with a minimum response at the center of the ellipses and Y_2 has elliptic contours with a maximum response at the center of the ellipses, then the set of admissible points, in fact, the minimal complete set, lies on a section of a hyperbola passing through the Y_2 center and directed away from the Y_1 center. The points on the hyperbola are points in the set of tangent points.

It may, at this time, seem natural to consider the problem where Y_1 has elliptic contours and a minimum response at its center and Y_2 has elliptic contours and a minimum response at its center. However, after close observation, it is seen that for this problem there exist no admissible points. By choosing X_1 or X_2 larger and larger, one may simultaneously make Y_1 and Y_2 as large as one wishes. Hence, this problem is not of interest. It could, of course, be of interest if there were boundary conditions placed on X_1 and X_2.

Admissible Points for Y_1 Having Elliptic Contours and a Maximum Response and Y_2 Having Hyperbolic Contours

As our third problem, let us determine the set of admissible points for the following responses. The Y_1 contours form a family of ellipses with the maximum response at their center, (h_1,k_1). The Y_2 contours form a family of hyperbolas with center (h_2,k_2). By rotating, translating, and applying the change of scale technique (much the same as was done in problem one of this chapter), we may reduce all problems of this type to one of the following type. The contours of the Y_1 response form a family of circles with a maximum response at their center $(0,0)$. The contours of the Y_2 response form a family of hyperbolas with center (h,k), $h \geq 0$, $k \geq 0$. One may then write the equations of the contours as follows.

The equation of the Y_1 contours is

$$X_1^2 + X_2^2 = K_1 \qquad (20)$$

and the equation of the Y_2 contours is

$$a(X_1 - h)^2 + (X_2 - k)^2 = K_2 \quad (a < 0) . \qquad (21)$$

From the framework of this problem, one is able to use much of what was done in problem one to finally obtain the equation of the tangent points (Equation 14). As before, the equation of the horizontal asymptote of the hyperbola is $X_2 = k/1-a$ and the equation of the vertical asymptote of the hyperbola is $X_1 = ah/a-1$. However, since in this case $(a < 0)$, we can draw a single figure showing the asymptotes of the hyperbola given by Equation 14 (see Figure 6).

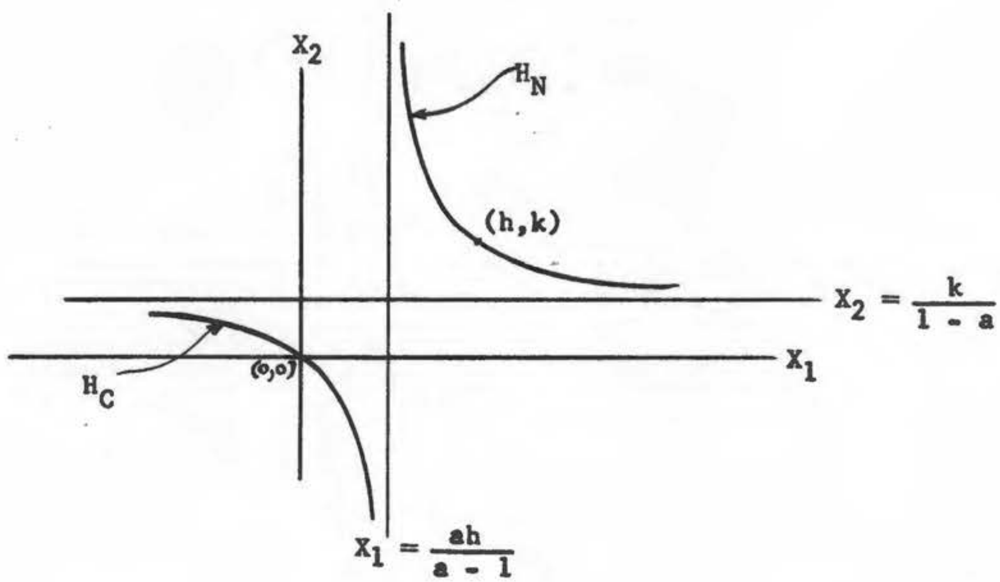

Figure 6. Tangent Points for Elliptic and Hyperbolic Contours

Since we know the hyperbola passes through (0,0) and (h,k), we can now draw the hyperbola corresponding to the set of tangent points. Let X^o be any point on H_N with $h > X_1^o > ah/a-1$. Draw a line through X^o and parallel to the X_2 axis. Follow the line through the point $X_2 = k$ to $X_2^o - k$ units below the line $X_2 = k$, call this point X^1.

Therefore, $Y_2(X^1) = Y_2(X^o)$, but the distance from X^1 to (0,0) is less than the distance from X^o to (0,0). Thus, the Y_1 contour passing through X^1 has smaller radius than the Y_1 contour passing through X^o. Since smaller radii correspond to larger Y_1 responses, $Y_1(X^1) > Y_1(X^o)$. Therefore, X^1 is a better point than X^o. Since X^o was an arbitrary point on H_N with $ah/a-1 < X_1^o < h$, there is no admissible point, X^o, on H_N with $X_1^o < h$.

Next, let X^o be any point on H_N with $X_1^o > h$. Consider the line passing through X^o, parallel to the X_1 axis. The point on this line, $X_1^o - h$ units to the left of the line $X_1 = h$ (call this point X^1) is better than X^o because $Y_1(X^1) > Y_1(X^o)$ and $Y_2(X^1) = Y_2(X^o)$. Therefore, there is no admissible point on H_N with X_1 coordinate greater than h.

It is seen that (h,k) is not an admissible point from the fact that one can follow one of the asymptotes of the family of hyperbolas, hence, keeping the Y_2 constant, equal to say b, to a point closer to (0,0) than is (h,k) and this point will be a better point than (h,k).

From the three preceding arguments, $ah/a-1 < X_1 < h$, $X_1 = h$, $X_1 > h$; we now conclude that there are no admissible points on H_N.

Since all admissible points are again on H_c and each Y_1 contour crosses H_c twice, once on the part of H_c with $0 < X_1 < ah/a-1$ and once on the part of H_c with $-\infty < X_1 < 0$, one may expect that in some cases the admissible points will be on one part of H_c $(0 < X_1 < ah/a-1)$

while at other times the admissible points will be on the other part of $H_c(-\infty < X_1 < 0)$. We now show that this is in fact the case. From the fact that the equation of the contours of Y_2 can be written as Equation 8 we know the axis of the hyperbolas (the axis of the Y_2 contours corresponding to values of the response function greater than zero) is either the line $X_1 = h$ or $X_2 = k$. First, suppose the axis of Y_2 contours, for contour values greater than zero, is $X_1 = h$. That is, if one starts at the point (h,k) and moves along the line $X_1 = h$ to values of $X_2 > k$ (or $X_2 < k$), then the Y_2 values increase. And if one starts at (h,k) and moves along the line $X_2 = k$ to values of $X_1 > h$ (or $X_1 < h$), then the value of the Y_2 contours decrease. Now, we will show that the set of admissible points is the part of H_c with $0 \leq X_1 < ah/a-1$. First, one observes that each point on H_c with $0 \leq X_1 < ah/a-1$ is, in fact, an admissible point because as we move from (0,0), where Y_1 has a maximum response, along H_c with increasing values of X_1 the Y_1 response decreases while the Y_2 response increases.

We need to show that no point on H_c with $X_1 < 0$ is an admissible point. To do this, let X^o be any point on H_c with $X_1^o < 0$. If we consider a point, X^1, an infinitesimal distance to the right of X^o along the line $X_2 = X_2^o$, then we note that in moving to the right we have increased both the Y_1 response and the Y_2 response. Hence, X^1 is a better point than X^o. But X^o was an arbitrary point on H_c with its X_1 coordinate less than 0. We then have the set of admissible points which can be expressed as

$$\{(X_1, X_2) | (a-1)X_1X_2 - ahX_2 + kX_1 = 0, \quad 0 \leq X_1 < \frac{ah}{a-1}\} . \qquad (22)$$

We can go through a similar argument for the case when the axis of the Y_2 contours is $X_2 = k$. In this case we would find that the part of H_c with $0 < X_1 < ah/a-1$ does not correspond to any admissible point and that the set of admissible points would be given by

$$\{(X_1, X_2) | (a - 1)X_1X_2 - ahX_2 + kX_1 = 0, \ -\infty < X_1 \leq 0\} . \quad (23)$$

Again the sets $S_1(a)$ satisfy the conditions of Theorem III, so that the above sets of admissible points are in fact the minimal complete sets.

From this, one sees that the minimal complete set with the Y_1 contours elliptic, with a maximum response at the center and the Y_2 contours hyperbolic is the set of points described as follows. This set is a section of the branch of the hyperbola, corresponding to a subset of the set of tangent points, which begins at the point where Y_1 has its maximum response and continues through the points where the contours of Y_2 increase. This part of H_c is the minimal complete set.

An algorithm for determining the minimal complete set for a problem of this type is given as follows:

(1) Determine the equation of the tangent points; this will be the equation of some hyperbola.

(2) Draw the branch which goes through the point where Y_1 has its maximum response (the branch that goes through (h_1, k_1) in the previous problem).

(3) Determine which end of this branch corresponds to large Y_2 responses.

(4) The section of this curve from the point where Y_1 has its maximum response toward the end which corresponds to large values for the Y_2 response is the minimal complete set.

One can easily observe from the set of tangent points that if the

Y_1 response had elliptic contours and a minimum response at the center, then there would be no admissible point since both Y_1 and Y_2 could be simultaneously increased without bound. This problem would only be of interest if there were boundary conditions on X_1 and X_2.

Admissible Points for Y_1 Having Elliptic Contours and a Maximum Response and Y_2 Having Parabolic Contours

As our fourth problem, we will consider the case where the Y_1 contours form a family of ellipses with maximum response at the center (h_1, k_1) and the Y_2 contours form a family of parabolas with axis $X_2 = mX_1 + b$ and their vertices at different points on $X_2 = mX_1 + b$. One may perform rotations, translations, and change of scale techniques to obtain a new situation. The new situation is stated as follows. The Y_1 contours form a family of circles with maximum response at their center $(0,0)$ and the Y_2 contours form a family of parabolas with axis $X_1 = h > 0$ and their vertices at different points along the axis. We may then write the equations of the contours for this new situation as follows.

The equation of the Y_1 contours is

$$X_1^2 = X_2^2 = K_1 \qquad (24)$$

and the equation of the Y_2 contours is

$$X_2 - a = b[X_1 - h]^2, \; -\infty < a < \infty, \; -\infty < b < \infty . \qquad (25)$$

Depending upon the problem, the contours of Y_2 will either increase as a increases or they will decrease as a increases. The sign of b tells us if the parabolas are concave up or concave down (if $b > 0$, then concave up, if $b < 0$, then concave down).

Again we go through the procedure of determining the set of tangent points by determing X_2' for each set of contours and setting these X_2''s equal.

From Equation 24, we have

$$X_2' = \frac{-X_1}{X_2} \quad , \tag{26}$$

and from Equation 25, we have

$$X_2' = 2b(X_1 - h) \quad . \tag{27}$$

Therefore, the set of tangent points is given by the equation

$$2bX_1X_2 + X_1 - 2bhX_2 = 0 \quad , \tag{28}$$

which is the equation of a hyperbola.

Rewriting Equation 28, we have

$$X_2 = \frac{-X_1}{2b(X_1 - h)} \quad . \tag{29}$$

Hence, the horizontal asymptote is

$$X_2 = -\frac{1}{2b} \quad .$$

Again rewriting Equation 28, we have

$$X_1 = \frac{2bhX_2}{2bX_2 + 1} \quad . \tag{30}$$

Therefore, the vertical asymptote is $X_1 = h$.

From Equation 28, we see (0,0) is a point on the hyperbola so that we may now draw the hyperbola.

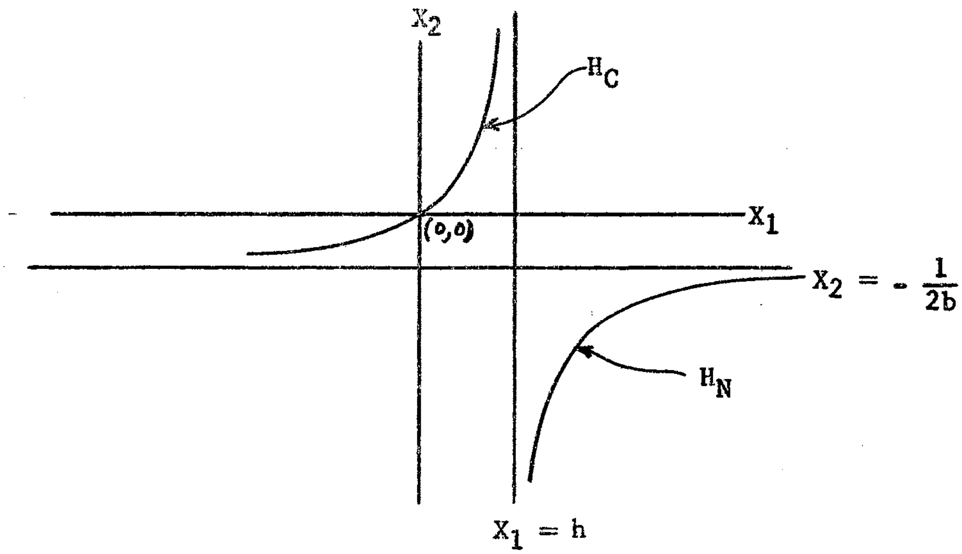

Figure 7. Tangent Points for Elliptic and Parabolic Contours, b > 0

Consider Figure 7 (b > 0). Let X^o be any point on H_N. Now, the point with coordinates $(2h - X_1^o, X_2^o)$ has the same Y_2 response as the point X^o and a larger Y_1 response; therefore the point $(2h - X_1^o, X_2^o)$ is a better point than X^o. Thus, X^o is not an admissible point. Since X^o was an arbitrary point on H_N, we know there are no admissible points on H_N; hence all admissible points are on H_C. Likewise, we could argue for Figure 8 (b < 0) that there are no admissible points on H_N; hence all admissible points are on H_C.

In order to facilitate the study of the problem, we designate the shaded area, A, in Figure 9 as the inside of the parabola. The region $\bar{A}$ will be referred to as the outside of the parabola. Therefore, when we say the gradient is directed toward the inside of the parabola, this means the gradient is in the direction indicated by the arrows in

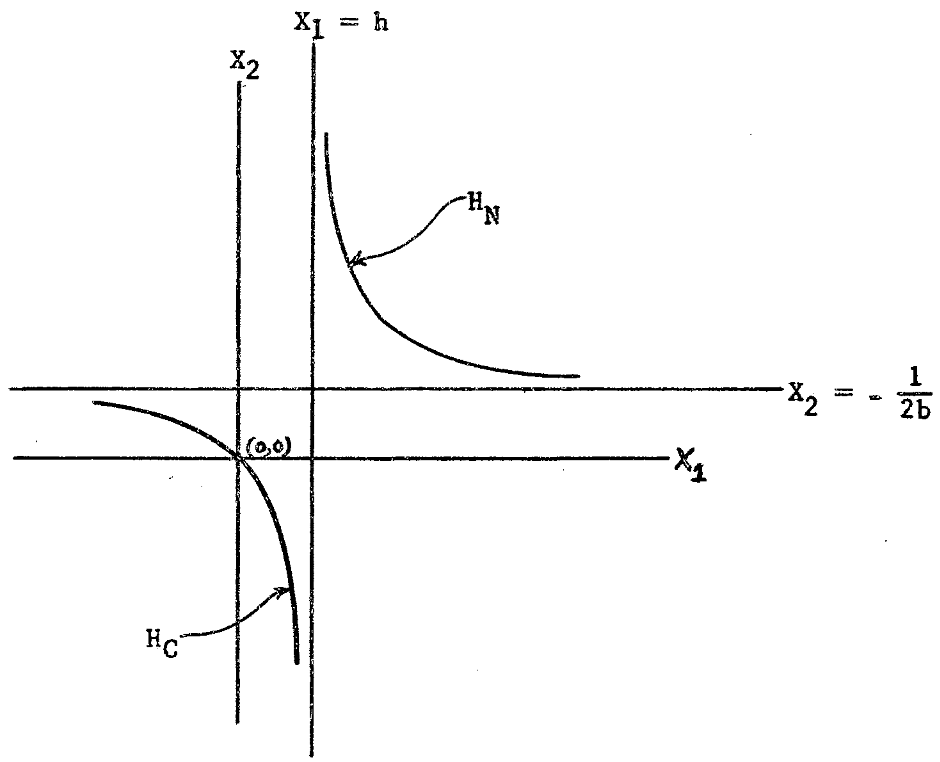

Figure 8. Tangent Points for Elliptic and Parabolic Contours, $b < 0$

Figure 9. Also, if we say the gradient is directed toward the outside of the parabola, the vectors will be in the direction opposite to that indicated in Figure 9.

Table II will help us to see the relationship between a, b, (of Equation 25) and the direction of the gradient of Y_2. The listings in the table indicate which values of a correspond to large values of the Y_2 contours.

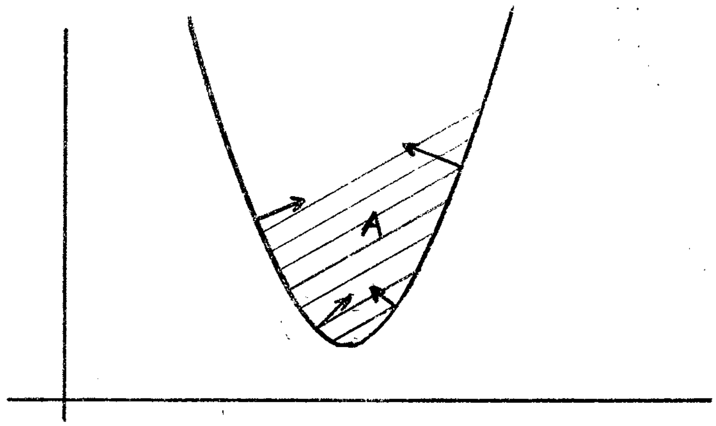

Figure 9. Region of a Parabolic Contour

TABLE II

DIRECTION OF GRADIENT

	INSIDE	OUTSIDE
$b > 0$	large a	small a
$b < 0$	small a	large a

One can see that there are four cases which must be considered in order to solve the preceding problem: one must specify whether the gradient is directed toward the inside or outside, then he must say if b is greater or less than zero.

Let us first consider the case where $b > 0$ and the gradient of Y_2 is directed toward the inside of the parabolic contours (large Y_2

responses correspond to large values of a). We now wish to show that the points H_c, with their X_1 coordinate less than zero, can not be admissible points. To show this, we will show that for any of these points, say X^o, we can find a point better than X^o.

Let X^o be any point on H_c with $X_1^o < 0$. Consider the point $(-X_1^o, X_2^o)$. This point has the same Y_1 response as X^o but it has a higher Y_2 response since it is on a Y_2 contour which has a larger value of a associated with it than did the $Y_2(X^o)$ contour. Therefore, $(-X_1^o, X_2^o)$ is a better point than X^o. Since X^o was an arbitrary point on H_c with $X_1^o < 0$ and X^o is not an admissible point, then there is no admissible point (X^o) on H_c with $X_1^o < 0$. However, one can see that the points on H_c with their X_1 coordinates greater than zero are each admissible points because if one starts at (0,0) and moves along H_c (as X_1 gets larger), each time he moves he will increase the Y_2 response while decreasing the Y_1 response.

It should be noted that the sets $S_1(a)$ satisfy the conditions of Theorem III; hence the minimal complete set exists. Therefore, if $b > 0$ and the gradient of Y_2 is directed toward the inside of the parabolic contours, then the minimal complete set is

$$\{(X_1, X_2) | 2bX_1X_2 + X_1 - 2bhX_2 = 0, \ \ 0 \leq X_1 < h\} . \qquad (31)$$

We can go through similar arguments for the other three cases to obtain the following results:

If $b > 0$ and ∇Y_2 is directed toward the outside of the parabolic contours, then the minimal complete set is

$$\{(X_1, X_2) | 2bX_1X_2 + X_1 - 2bhX_2 = 0, \ \ X_1 \leq 0\} . \qquad (32)$$

If $b < 0$ and ∇Y_2 is directed toward the inside of the parabolic contours, then the minimal complete set is

$$\{(X_1, X_2) \mid 2bX_1X_2 + X_2 - 2bhX_2 = 0, \;\; 0 \le X_1 < h\} \;. \tag{33}$$

If $b < 0$ and ∇Y_2 is directed toward the outside of the parabolic contours, then the minimal complete set is

$$\{(X_1, X_2) \mid 2bX_1X_2 + X_1 - 2bhX_2 = 0, \;\; X_1 \le 0\} \;. \tag{34}$$

From the preceding results, one is able to construct the following algorithm for determining the minimal complete set for the original problem.

(1) Determine the equation of the set of tangent points; this will be the equation of a hyperbola.

(2) Draw the branch of this curve which passed through the maximum of the Y_1 response.

(3) Determine which end of the drawn curve corresponds to large Y_2 responses.

(4) The section of the curve starting with the point where Y_1 has its maximum response and moving toward the end which corresponds to large Y_2 responses is the minimal complete set.

As one can easily see, if the Y_1 response had a minimum response at its center instead of a maximum response, then the set of admissible points would not exist. In fact, there would be no admissible point since both Y_1 and Y_2 could simultaneously be increased without bound.

Admissible Points for Y_1 Having Hyperbolic Contours and Y_2 Having Hyperbolic Contours

The fifth problem we wish to consider is of the type where the Y_1

contours form a family of hyperbolas with center (h_1, k_1) and the Y_2 contours form a family of hyperbolas with center (h_2, k_2). One should note that in many cases the set of admissible points does not exist for these type contours. After translating, rotating, and applying the change of scale technique, we may reduce all problems of this type to a situation which is as follows. The contours of the Y_1 response form a family of hyperbolas with $X_1 = X_2$ and $X_1 = -X_2$ as their asymptotes and $X_2 = 0$ as their axis. The contours of the Y_2 response form a family of hyperbolas with center (h, k), $h \geq 0$, $k \geq 0$.

Let us now determine the set of admissible points for this problem. The equation of the Y_1 contours is

$$X_1^2 - X_2^2 = K_1 \quad , \tag{35}$$

while the equation of the Y_2 contours is

$$a(X_1 - h)^2 + b(X_1 - h)(X_2 - k) + (X_2 - k)^2 = K_2, \; b^2 - 4a > 0 \; . \tag{36}$$

It should be noted that the cases where the Y_2 contours are of the form

$$(X_1 - h)(X_2 - k) = K_2$$

are not considered here. For these cases there exist no admissible points since both Y_1 and Y_2 responses can simultaneously be increased without bound.

We will go through the procedure which was performed in problem one. First, determine the set of tangent points by setting X_2' from Equation 35 equal to X_2' from Equation 36; then from this set of tangent points we eliminate those points which are not admissible points which leaves the set of admissible points.

From Equation 35, we have

$$X_2' = \frac{X_1}{X_2} \quad . \tag{37}$$

From Equation 36, we have

$$X_2' = -\frac{2a(X_1 - h) + b(X_2 - k)}{2(X_2 - k) + b(X_1 - h)} \quad . \tag{38}$$

Setting X_2' of Equation 37 equal to X_2' of Equation 38, we have the equation of the set of tangent points:

$$\frac{X_1}{X_2} = -\frac{2a(X_1 - h) + b(X_2 - k)}{2(X_2 - k) + b(X_1 - h)} \quad . \tag{39}$$

Simplifying Equation 39, we obtain

$$bX_1^2 + 2(a + 1)X_1X_2 + bX_2^2 - (2k + bh)X_1 - (2ah + bk)X_2 = 0 \quad . \tag{40}$$

After inspecting Equation 40, one sees that $(a + 1)^2 - b^2$ may be less than, equal to or greater than zero. If $(a + 1)^2 - b^2$ is less than zero, this indicates that the tangent points lie on an ellipse. However, from observing the Y_1 contours, it is obvious that the set of admissible points can not lie on an ellipse. Hence, for this case, the set of admissible points does not exist.

If $(a + 1)^2 - b^2$ is equal to zero, then Equation 40 is the equation of a parabola. There are two possible cases when the tangent points lie on a parabola. Case I is when one section of the parabola corresponds to large Y_1 and Y_2 responses, while the other section corresponds to small Y_1 and Y_2 responses. Case II is when one section of the parabola corresponds to large Y_1 responses and small Y_2 responses, while the other section corresponds to small Y_1 responses and large Y_2 responses.

In Case I there are no admissible points since by the correct choice of (X_1, X_2) one can simultaneously increase both Y_1 and Y_2 without bound. However, for Case II, the set of admissible points will be the complete parabola.

Now if $(a + 1)^2 - b^2$ is greater than zero, then Equation 40 is the equation of a hyperbola and in order to determine the set of admissible points, when they exist, we must break the problem into different cases. That is, we must consider subsets of the problem in which one is given more information about a and b. In each case, the set of tangent points will lie on either a branch of the hyperbola going through (0,0) and (h,k) referred to as H_c or on the other branch of the hyperbola referred to as H_N. If the set of admissible points exists, it will be the branch of the hyperbola referred to as H_N. Therefore, we need to determine conditions on a and b for the admissible points to exist.

To help us determine when the admissible points exist, we note the following:

$$\begin{bmatrix} \dfrac{\partial Y_2(X)}{\partial X_1} \\[2ex] \dfrac{\partial Y_2(X)}{\partial X_2} \end{bmatrix} = \pm \begin{bmatrix} 2a(X_1 - h) + b(X_2 - k) \\[2ex] 2(X_2 - k) + b(X_1 - h) \end{bmatrix} . \qquad \begin{matrix} (41) \\[2ex] (42) \end{matrix}$$

The plus or minus sign is determined from the problem but in the following work we will only consider problems in which the plus sign is appropriate. However, it should be noted that if we determine that the set of admissible points exists for certain values of a and b by using the plus sign, then for these values of a and b the admissible set does not exist with the minus sign. Thus, if the problem implies the use of a minus sign, then there exists no admissible set for this problem.

Another tool which we will find very useful is the ability to write the equation of the asymptotes of a hyperbola from the equation of the hyperbola (see (11) page 151).

From reference (11), we see that the slope, m, of the asymptotes of Equation 36 is given by

$$m = \frac{-b \pm \sqrt{b^2 - 4a}}{2} \quad . \tag{43}$$

Now, using m_1 to represent the slope of the asymptote with maximum slope, we have

$$m_1 = \frac{-b + \sqrt{b^2 - 4a}}{2} \quad . \tag{44}$$

Using m_2 to represent the slope of the asymptote with minimum slope, we have

$$m_2 = \frac{-b - \sqrt{b^2 - 4a}}{2} \quad . \tag{45}$$

One should note that $m_1 \neq m_2$ since $b^2 - 4a$ is greater than zero from the fact that Equation 36 is the equation of a hyperbola.

As a first case, let us consider $a > 0$, $b > 0$. Then Figure 10 will help us determine the conditions for the set of admissible points to exist. The arrows in Figure 10 indicate the direction of the Y_2 gradients. These directions were obtained from Equations 41 and 42. Equations 41 and 42 are easily applied along $X_1 = h$ and $X_2 = K$. One can see from the direction of the Y_2 gradients and knowing the direction of the Y_1 gradients, that the set of admissible points does not exist. If a point $(X_1, 0)$ is chosen with X_1 very large, we see that both Y_1 and Y_2 responses may be increased without bound.

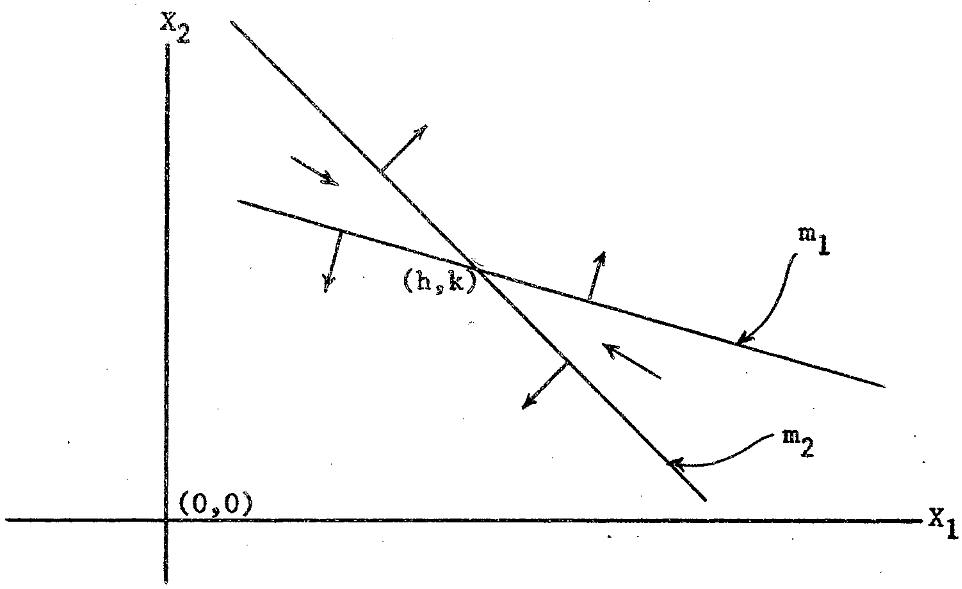

Figure 10. Asymptotes of Hyperbolic Contours, $a > 0$, $b > 0$

Next, let us consider the case when $a > 0$, $b < 0$. Then the asymptotes of Equation 36 can be drawn as in Figure 11.

From Figure 11 one can easily see that there exist no admissible points for this case. One can take points on the line $X_2 = 0$ with X_1 very large and we can see that both Y_1 and Y_2 responses may be increased without bound.

The third case we will consider is $a < 0$, $b > 0$. The asymptotes of Equation 36 for this particular case are drawn in Figure 12. From Figure 12 (noting the direction of the gradients of Y_2 and knowing the direction of the gradients of Y_1), we see the set of admissible points exists if and only if $m_1 > 1$ and $m_2 < -1$.

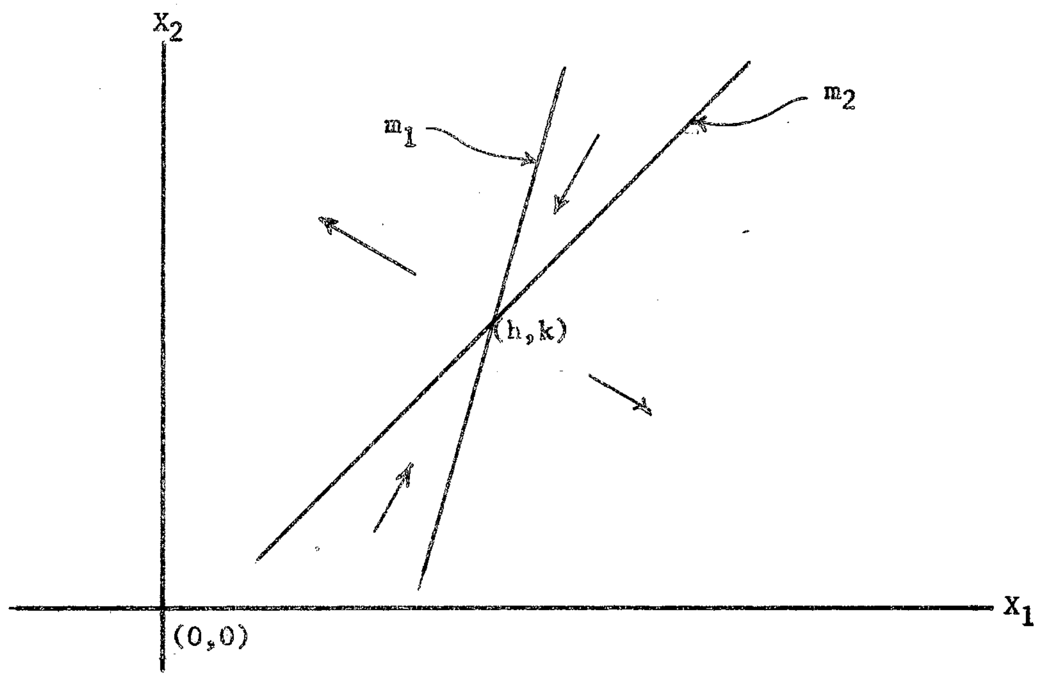

Figure 11. Asymptotes of Hyperbolic Contours, a > 0, b < 0

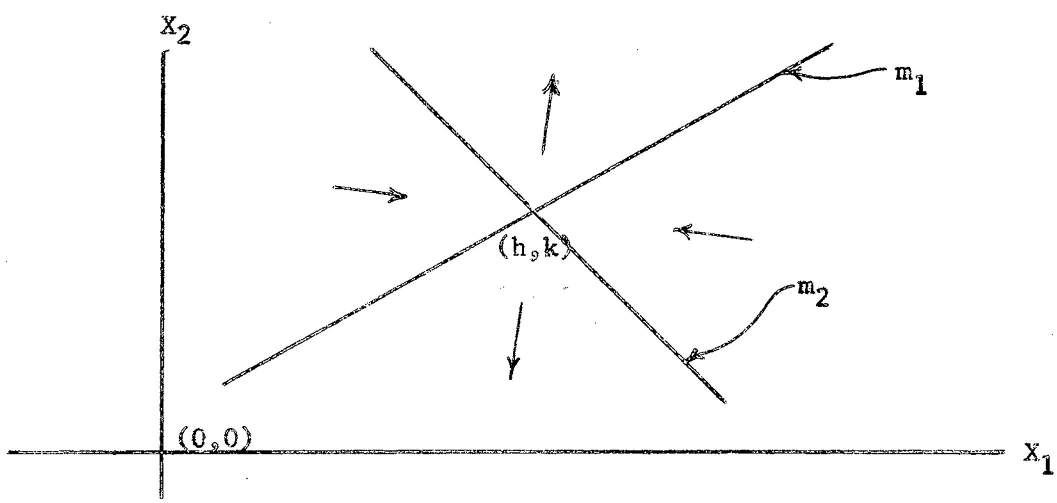

Figure 12. Asymptotes of Hyperbolic Contours, a < 0, b > 0

For the last case, $a < 0$, $b < 0$, we may draw the asymptotes of Equation 36 as is done in Figure 13. Clearly, from Figure 13, one sees that the set of admissible points exists if and only if $m_1 > 1$ and $m_2 < -1$.

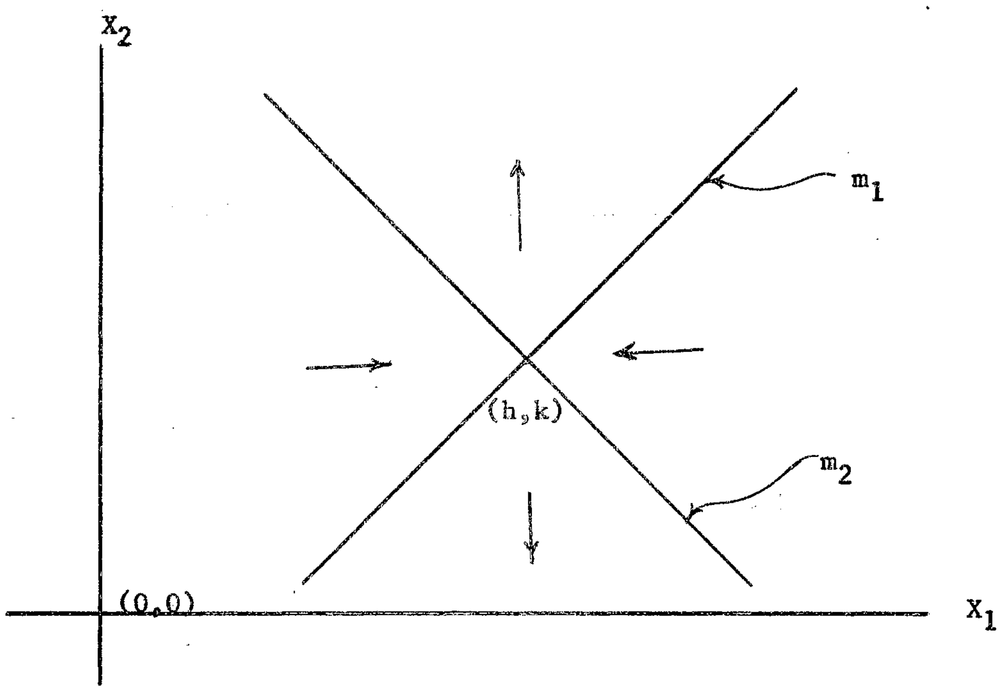

Figure 13. Asymptotes of Hyperbolic Contours, $a < 0$, $b < 0$

It may be convenient to show the results of the preceding situations in a table (Table III). It is of interest to note that if a given problem requires that we use the minus sign with Equations 41 and 42 instead of the plus sign, then the new table would be the image, through the b axis, of Table III.

If Equation 35 is the equation of a hyperbola and Equation 36 is

TABLE III

ADMISSIBLE POINTS FOR HYPERBOLIC CONTOURS

	b	
The set of admissible points exist if and only if $m_1 > 1$ and $m_2 < -1$.		There exist no admissible points.
	(0,0)	a
The set of admissible points exist if and only if $m_1 > 1$ and $m_2 < -1$.		There exist no admissible points.

the equation of a hyperbola, we will use the preceding work to construct an algorithm which determines: (1) if the set of admissible points exists, and (2) if the set of admissible points does exist, then the set of admissible points.

(1) Put the problem into a form having contours with Equations 35 and 36. This may require rotations, translations, and change of scale.

(2) By looking at a and b and referring to Table III (or the image thereof, if one has the minus sign associated with Equations 41 and 42), determine if the admissible set exists. One may have to calculate m_1 and m_2 by using Equations 44 and 45.

(3) If the set of admissible points exist, then calculate the set of tangent points (Equation 40).

(4) From this set of tangent points, pick the branch of the hyperbola which does not pass through the points (0,0) or (h,k) (that is, H_N). This is our set of admissible points for Y_1 and Y_2.

Admissible Points for Y_1 Having Parabolic Contours and Y_2 Having Hyperbolic Contours

As the sixth problem of this chapter, we will consider the problem of determining the set of admissible points, if it exists, when the contours of the Y_1 response function form a family of parabolas with vertices on the line $X_2 = mX_1 + b$ ($X_2 = mX_1 + b$ is also the axis of each parabola), and the contours of the Y_2 response function form a family of hyperbolas with center (h_2, k_2).

Without loss of generality, since we have at our disposal the tools of rotation, translation, and change of scale, we need to consider only problems of the following type. The contours of the Y_1 response function form a family of parabolas which has $X_1 = 0$ as its axis. The contours of the Y_2 response function form a family of hyperbolas which has its center at the point (h, k), $h \geq 0$, $k \geq 0$. After determining the set of admissible points (say A) for this new situation, if the set of admissible points exist, one may obtain the set of admissible points for the original problem by performing, on A, the inverse of the operations that were performed to get the original problem into the new problem format.

From the preceding discussion, we see that the equations of the contours for the response functions of the new problem may be as follows.

The equation of the contours for the Y_1 response may be written as

$$X_2 - c = dX_1^2, \quad d = \pm 1, \; c \in (-\infty, \infty) \; . \tag{46}$$

The equation of the contours for the Y_2 response may be written

$$a(X_1 - h)^2 + b(X_1 - h)(X_2 - k) + (X_2 - k)^2 = K_2$$

$$b^2 - 4a > 0, \; a \in (-\infty, \infty), \; b \in (-\infty, \infty) \; . \qquad (47)$$

The following discussion will be for the plus sign in Equations 41 and 42. It is obvious that the gradient of the Y_1 response must be directed toward the inside of the parabolic contours in order that the set of admissible points exist. Therefore, we will consider only problems with ∇Y_1 directed toward the inside of the parabolic contours. Applying Theorem V, we know that if we determine the set of tangent points for the contours, given by Equations 46 and 47, then the set of admissible points, if they exist, is a subset of this set of tangent points.

To determine the set of tangent points, we determine X_2' from Equation 46 and X_2' from Equation 47. Setting these two X_2''s equal, we have the equation of the set of tangent points.

From Equation 46, we have

$$X_2' = \frac{X_2}{X_1} = 2dX_1 \; , \qquad (48)$$

and from Equation 47, we have

$$X_2' = -\frac{2a(X_1 - h) + b(X_2 - k)}{2(X_2 - k) + b(X_1 - h)} \; . \qquad (49)$$

Setting these two X_2''s equal, we find that the equation of the set of tangent points is

$$-2dX_1 = \frac{2a(X_1 - h) + b(X_2 - k)}{2(X_2 - k) + b(X_1 - h)} \; . \qquad (50)$$

Now, simplifying Equation 50, we have

$$2bdX_1(X_1 - h) + 4dX_1(X_2 - k) + 2a\,(X_1 - h) + b(X_2 - k) = 0\ . \qquad (51)$$

We recognize that (51) is the equation of a hyperbola. Therefore, the set of tangent points lies on a hyperbola; hence if the set of admissible points exist, it will lie on a section of a hyperbola.

Solving Equation 51 for X_2 we have

$$-X_2 = \frac{2bdX_1(X_1 - h) + 2a(X_1 - h) - bk - 4dkX_1}{4dX_1 + b}\ . \qquad (52)$$

From Equation 52 one notes that the hyperbola has vertical asymptote $X_1 = -b/4d$. Also, noting that the hyperbola of tangent points passes through (h,k), we have an idea of what the graph of Equation 51 looks like. For the remainder of this discussion, we will choose $d = 1$ (that is, the equations of the Y_1 contours will be $X_2 - c = X_1^2$) in order to limit the number of cases it is necessary to consider. (One might just as well choose $d = -1$ and go through the following discussion.)

Putting $d = 1$ in Equation 52, we have

$$X_2 = -\frac{2bX_1(X_1 - h) + 2a(X_1 - h) - bk - 4kX_1}{4X_1 + b}\ . \qquad (53)$$

Dividing the numberator of Equation 53 by the denominator, we have

$$X_2 = -\frac{b}{2}X_1 + \frac{2bh - 2a + 4k + \frac{b^2}{2}}{4} + \frac{\text{constant}}{4X_1 + b}\ . \qquad (54)$$

Noting that as $X_1 \to \infty$ the last term of Equation 54 approaches zero, we have the equation of the other asymptote of the hyperbola.

$$X_2 = -\frac{b}{2}X_1 + \frac{2bh - 2a + 4k + \frac{b^2}{2}}{4}\ . \qquad (55)$$

Note that the slope of this asymptote is $-b/2$ which is the same as the slope of the axis of the hyperbola.

To better understand the hyperbola around the vertical asymptote ($X_1 = -b/4$), let $X_1 = -b/4$ in the numerator of Equation 53. The numerator of equation 53 reduces to

$$-\frac{1}{2}(b^2 - 4a)(h + \frac{b}{4}) . \tag{56}$$

We know $b^2 - 4a$ is greater than zero, since Equation 47 is the equation of a hyperbola, so the sign of the numerator of Equation 53 depends upon $(h + b/4)$.

If $(h + b/4)$ is greater than zero, then the hyperbola goes to $-\infty$ as $X_1 \to (-b/4)$ from the right$^+$ and the hyperbola goes to $+\infty$ as $X_1 \to (-b/4)$ from the left$^-$.

If $(h + b/4)$ is less than zero, then the hyperbola goes to $+\infty$ as $X_1 \to -b/4$ from the right$^+$ while the hyperbola goes to $-\infty$ as $X_1 \to -b/4$ from the left$^-$.

For convenience, we will refer to the branch of the hyperbola which passes through (h,k) as H_1 and the other branch will be referred to as H_2.

As our first case we choose $a > 0$, $b > 0$. However, as one can see from Figure 10, if $a > 0$, $b > 0$, then $m_1 < 0$, $m_2 < 0$, and from Equation 42

$$\left.\frac{\partial Y_2(X)}{\partial X_2}\right|_{X_1 = h} > 0 .$$

Hence, we see that we can choose points on the line $X_1 = h$ and X_2 very large which will simultaneously increase Y_1 and Y_2 without bound. Therefore, if $a > 0$, $b > 0$, there is no admissible point.

As Case II let us consider $a > 0$, $b < 0$. Again from Figure 11, or from the fact that

$$\left.\frac{\partial Y_2(X)}{\partial X_2}\right|_{X_1 = h} > 0 \, ,$$

we see that both Y_1 and Y_2 can be increased without bound as X_2 is increased. Therefore, no admissible point exists. From the two preceding examples, and referring to Figures 12 and 13, one sees that if Equations 41 and 42 have the plus sign associated with them, then there are no admissible points. However, if the problem should imply that Equations 41 and 42 should have the minus sign associated with them, then the set of admissible points does exist.

Suppose the minus sign is associated with Equations 41 and 42. Then the gradients of the Y_2 response (as shown by the arrows in Figures 10, 11, 12, and 13) will be opposite the direction shown in Figures 10, 11, 12, and 13. Now let us suppose the problem implies that the minus sign should be associated with Equations 41 and 42. Let us consider what the set of admissible points is when $a > 0$ and $b < 0$ (Figure 11 with gradient vectors of Y_2 in opposite direction). There are two situations we should consider. First, let $-b/4$ be greater than h. Figure 14 will help us determine the admissible points.

Clearly, the points on H_1 are not admissible points (For the points on H_1 (say X^o) with $h < X_1^o$, the point (h,k) is better than X^o). For the points on H_1 (say X^o) with $X_1^o \leq h$, we can find a point better than X^o by moving along the line going through X^o with slope $-b/2$ until we get to a point (say X^1) where $Y_2(X^1) = Y_2(X^o)$, $Y_1(X^1)$ will be greater than $Y_1(X^o)$, or we can refer to Appendix B for a proof that there are no admissible points on H_1.). Also, each point, X^o, on H_2 is the point with highest $Y_2(X)$ value such that $Y_1(X) = Y_1(X^o)$. If we move from one point on H_2 (say X^1) so as to increase Y_1, then we will decrease Y_2. That is, if $Y_1(X^1) > Y_1(X^o)$, then $Y_2(X^1) < Y_2(X^o)$.

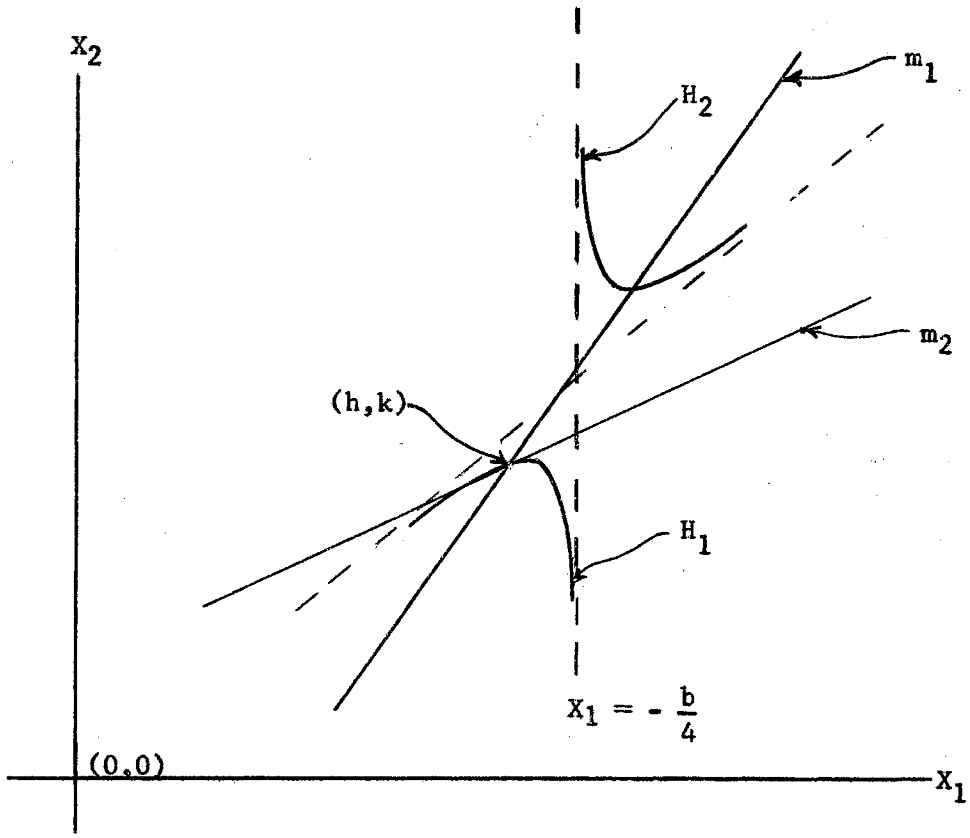

Figure 14. Tangent Points for Hyperbolic and Parabolic Contours, $-b/4 > h$

Next, suppose that $-b/4$ is less than h; a typical figure for this case is Figure 15. Clearly, as is shown in Appendix B, we see that there are no admissible points on H_1. Therefore, we are interested in only the points on H_2. If we let a and b take on different values, we will always find that the set of admissible points, if they exist, will be on H_2.

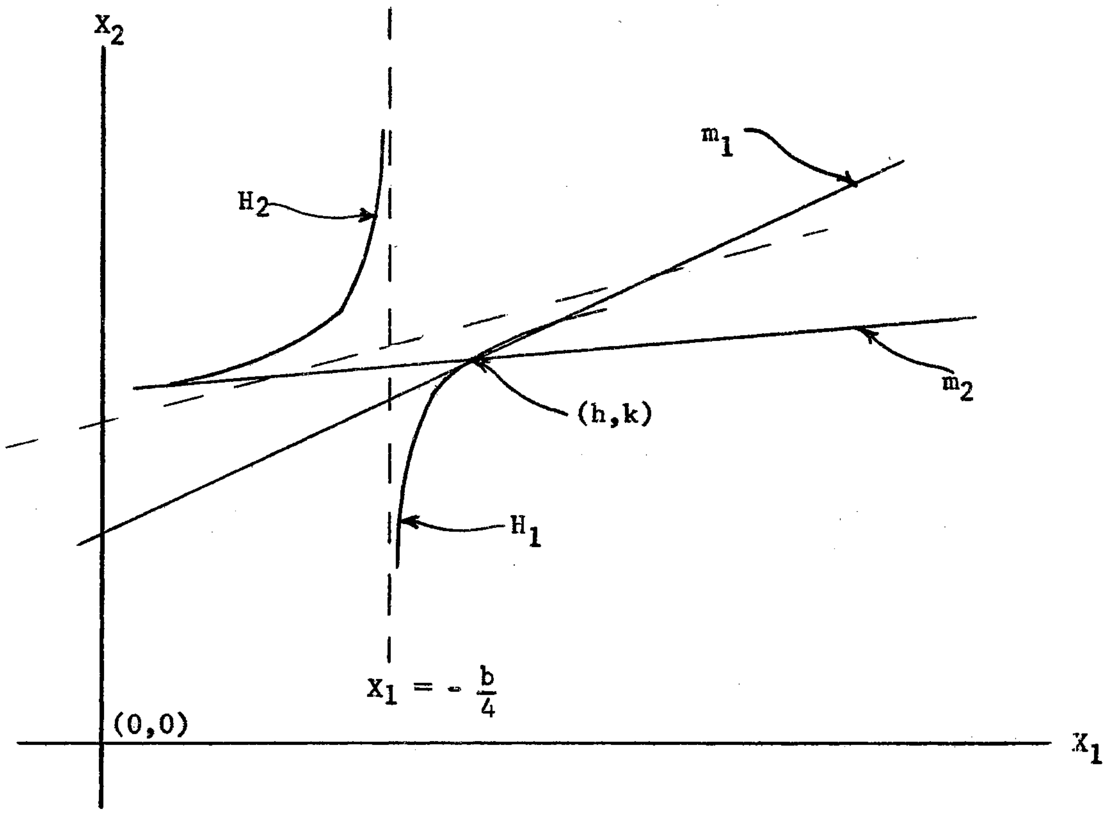

Figure 15. Tangent Points for Hyperbolic and Parabolic Contours, $-b/4 < h$

If Y_1 has parabolic contours and Y_2 has hyperbolic contours, then from the preceding work we can now state an algorithm which determines (1) if the set of admissible points exists and (2) the set of admissible points when it exists.

(1) Check the direction of the gradient of the parabolic contours (Y_1 contours). If the gradient is directed toward the inside of the parabola, then proceed. If the gradient is directed toward the outside of the parabola, then there exist no admissible points.

(2) Use a rotation, translation, and a change of scale to transform the problem into a new situation having contours with equations given by Equations 46 and 47.

(3) Check to see if the sign of Equations 41 and 42 should be plus or minus. If the problem implies the sign should be plus, then there exist no admissible points. If the problem implies the sign should be minus, then proceed.

(4) Determine the set of tangent points (H_1 and H_2, that is, the hyperbola given by Equation 51).

(5) From the set of tangent points determine the branch (H_2) which does not go through the center of the hyperbolic contours (h,k). This branch (H_2) is our set of admissible points for Y_1 and Y_2.

Admissible Points For Y_1 Having Parabolic Contours and Y_2 Having Parabolic Contours

As the final problem in this chapter, we need to consider the following situation. Suppose the Y_1 contours form a family of parabolas with axis $X_2 = m_1X_1 + b_1$ and the Y_2 contours form a family of parabolas with axis $X_2 = m_2X_2 + b_2$.

After performing a rotation, translation, and a change of scale, we may state any of the preceding situations as the following new situation. Now Y_1 has parabolic contours with $X_1 = 0$ as their axis and Y_2 has parabolic contours with axis $X_2 = m_3X_1 + b_3$. We may write the equations of the contours for response functions Y_1 and Y_2 as follows.

For the response function Y_1, the equation of the contours is

$$X_2 - c = dX_1^2, \quad d = \pm 1 \quad . \tag{57}$$

For the response function Y_2, the equation of the contours is

$$aX_1^2 + bX_1X_2 + X_2^2 + eX_1 + fX_2 + g = 0 ,$$
$$\text{where } b^2 - 4a = 0 . \tag{58}$$

Clearly, if the gradient of Y_1 or Y_2 or both Y_1 and Y_2 is directed toward the outside of the parabolic contours there is no admissible point since we will be able to simultaneously increase Y_1 and Y_2 without bound. Therefore, let us suppose, in the following discussion, that ∇Y_1 and ∇Y_2 are directed toward the inside of their respective parabolic contours. In order to limit the number of cases we need to consider, let $d = 1$. We could just as well use $d = -1$.

Before going any further we should mention that Equation 58 does not cover the case where the parabolas have parallel axes. If the parabolas have parallel axes and the gradients of Y_1 and Y_2 are in different directions (both being directed toward the inside of their respective contours), then the set of admissible points will be the line of tangent points. If the gradients of Y_1 and Y_2 are not in different directions, then there exists no admissible point.

To determine the set of admissible points, we will first determine the set of tangent points. To determine the set of tangent points, we set X_2' from Equation 57 equal to X_2' from Equation 58.

Calculating X_2' from Equation 57, with d = 1, we have

$$X_2' = 2X_1 \quad . \tag{59}$$

Calculating X_2' from Equation 58, we have

$$2aX_1 + bX_1X_2' + bX_2 + 2X_2X_2' + e + fX_2' = 0 \quad , \tag{60}$$

or

$$X_2' = -\frac{2aX_1 + bX_2 + e}{bX_1 + 2X_2 + f} \quad . \tag{61}$$

Setting X_2' of Equation 59 equal to X_2' of Equation 61, we find that the equation of the set of tangent points is

$$2X_1 = \frac{-(2aX_1 + bX_2 + e)}{bX_1 + 2X_2 + f} \quad . \tag{62}$$

Rewriting Equation 62, we have

$$2bX_1^2 + 4X_1X_2 + 2(a + f)X_1 + bX_2 + e = 0 \quad . \tag{63}$$

We note, by observing the discriminant of Equation 63, that Equation 63 is the equation of a hyperbola.

Solving Equation 63 for X_2, we have

$$X_2 = -\frac{2bX_1^2 + 2(a + f)X_1 + e}{4X_1 + b} \quad . \tag{64}$$

From Equation 64 we see that the hyperbola, which is the set of tangent points, has vertical asymptote $X_1 = -b/4$.

For convenience, let us call the branch of the hyperbola, which approaches $+\infty$ as $X_1 \to -b/4$, H_1. The branch of the hyperbola, which approaches $-\infty$ as $X_1 \to -b/4$, we will call H_2.

It is easily seen, from an argument similar to the argument in Appendix B, that there are no admissible points on H_2. Furthermore, if the section of H_1 which corresponds to the large values of Y_1, that is, the section of H_1 for X_1 values close to $-b/4$, also corresponds to small values of Y_2, then H_1 is the set of admissible points. However, if the section of H_1 which corresponds to large Y_1 values also corresponds to large Y_2 values then there exist no admissible points.

From the preceding work, we may state an algorithm for determining: (1) if the set of admissible points exists and (2) the set of admissible points (given the set of admissible points does exist) for response functions Y_1 and Y_2, when Y_1 and Y_2 have parabolic contours.

(1) Check the direction of the gradients of Y_1 and Y_2. If both gradients are directed toward the inside of their respective contours, then proceed. If one or both gradients are directed toward the outside of their respective contours, then there exists no admissible point.

(2) Determine the equation of the set of tangent points. This will be the equation of a hyperbola.

(3) From the set of tangent points determine which branch of the hyperbola corresponds to large values of Y_1, that is, determine H_1.

(4) Check the section of H_1 which corresponds to large Y_1 values. If this section corresponds to small Y_2 values, then H_1 is the set of admissible points. If the section of H_1 which corresponds to large Y_1 values also corresponds to large Y_2 values, then there exist no admissible points.

Summary

In this chapter there is introduced a procedure for determining a set, the set of tangent points, which contains the set of admissible points, if the set of admissible points exists (when $N = 2$). Using theorems of Chapter II, Appendix A, Appendix B, and various similar tools we were able to determine (1) if the set of admissible points exists and (2) (given that the set of admissible points exists) the set of admissible points for $N = 2$, $P = 2$, Y_1 having any family of quadratic curves as its contours, and Y_2 having any family of quadratic curves as its contours.

CHAPTER IV

ADMISSIBLE POINTS WHEN THERE IS A LINEAR CONSTRAINT ON THE CONTROLLED VARIABLES: N = 2, P = 2

Many times it is of interest to consider problems when the controlled variables are in some way constrained. In this chapter we will consider what effect a linear constraint on the controlled variables has on the set of admissible points for response functions having the contours considered in Chapter III. We will use the letter L to represent the line which is the linear constraint. L divides the plane of the controlled variables into two sets: the set A which is the set of points (X_1, X_2) that are permissible points (that is, the points which satisfy the constraint) and the set $\bar{A}$ which is the set of points (X_1, X_2) that are not permissible. The following definitions will facilitate the study of these linear constraints on the controlled variables.

Definition 8: The point $X = (X_1, X_2)$ is a feasible point if it is contained in the set A.

Definition 9: The point X^o is a feasible admissible point if X^o is feasible, that is, if X^o is in A, and considering only the points in A, X^o is an admissible point.

Definition 10: A complete set of feasible points is a set of points in A such that, given any point X^o in A which is not in the complete set of feasible points, there exists a point X^1 in the complete set of feasible points that is better than X^o.

Definition 11: A minimal complete set of feasible points, if it exists, is a set of points in A which is a complete set of feasible points such that no proper subset is a complete set of feasible points.

Definition 12: The point X^O is a feasible tangent point if X^O is feasible (that is, if X^O is in A) and X^O is a tangent point.

We are interested in obtaining, if it exists, the minimal complete set of feasible points. One should also note from the definitions that if an admissible point, for the problem without the linear constraint L, is feasible, then it is a feasible admissible point. Furthermore, if the minimal complete set, for the problem without the linear constraint L, is feasible, then the minimal complete set is in fact the minimal complete set of feasible points.

At first one may not know where to begin in his search for feasible admissible points. However, from the proof of Theorem IV (See page 11) one sees that in order for a point to be a feasible admissible point it must either be a point satisfying Theorem IV (or Theorem V since we are considering only cases where N = 2) or it must be a point on L, the linear constraint.

Again, since the set of tangent points contains all admissible points, for the problem without the linear constraint, in order to determine the set of feasible admissible points we will determine the set, say T, of feasible tangent points. We know the set of feasible admissible points is contained in the union of T and L.

In Chapter II it was shown that the admissible points are invariant under the change of scale technique. That is, if X^O is an admissible point in the original problem, then X^O, transformed by change of scale, is an admissible point for the new problem; and if Z^O is an admissible point for the new problem, then X^o is an admissible point for the original problem. Likewise, one can show that a feasible admissible point is invariant under the change of scale technique. If X^o is a feasible admissible point for the original problem, then Z^o is a feasible admissible point for the new problem; and if Z^o is a feasible admissible point for the new problem, then X^o is a feasible admissible point for the original problem.

It is seen that a rotation or translation does not change the set of feasible admissible points. Therefore, we see that we may translate, rotate, and use the change of scale technique to change the original problem into a new situation which has contours that are easier to study. After determining the set of feasible admissible points (if it exists) for this new situation, we may find the set of feasible admissible points for the original problem by applying the inverse change of scale, inverse translation, and the inverse rotation (that is, the inverse of those applied to obtain the new situation from the original problem) to the set of feasible admissible points of the new situation. Hence, we need to consider only contours of the types considered in the seven problems of Chapter III.

As our first problem, let us consider Y_1 contours which form a family of ellipses with a maximum at their center (h_1,k_1) and Y_2 contours which form a family of ellipses with a maximum at their center (h_2,k_2). As was done in problem one of Chapter III, the new situation can be stated as follows: The Y_1 contours form a family of circles with a maximum at their center $(0,0)$. The Y_2 contours form a family of ellipses with a maximum at their center (h,k), $h \geq 0$, $k \geq 0$ and the major axes of the ellipses are parallel to one of the coordinate axes.

The equation of the Y_1 contours is given by

$$X_1^2 + X_2^2 = K_1 \quad . \tag{65}$$

The equation of the Y_2 contours is given by

$$a(X_1 - h)^2 + (X_2 - k)^2 = k_2, \quad a > 0 \quad . \tag{66}$$

The equation of the linear constraint L is

$$X_2 \leq mX_1 + b, \quad -\infty < b < \infty, \quad -\infty < m < \infty \; . \tag{67}$$

For convenience let m be greater than zero and a be less than one. Figure 16 will help us to determine the set of feasible admissible points in the different cases. There are five different cases we will consider.

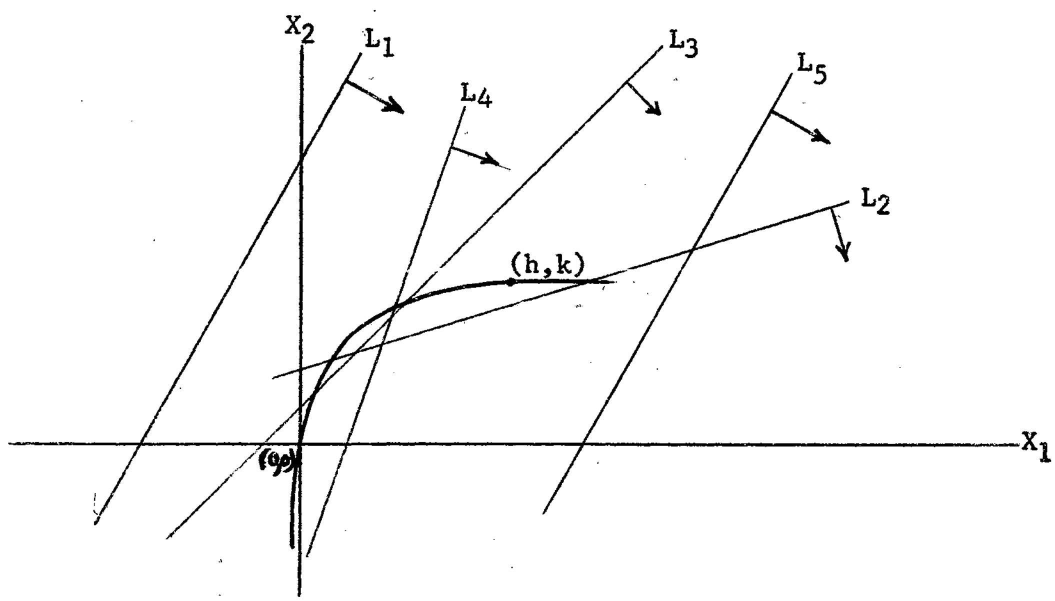

Figure 16. Linear Constraints on Elliptic Contours

Case I is the simplest of the five cases. It is the case when all admissible points are feasible; hence the set of feasible admissible points is just the set of admissible points. Also the minimal complete set of feasible points is just the minimal complete set.

Case II is the case where the admissible points which correspond to large values of the response function Y_1 are permissible but the admissible points which correspond to large values of the response function Y_2 are not permissible. Let P_2 be the point where L_2 intersects the curve of admissible points (that is, P_2 is the point where L_2 intersects the hyperbola and the X_1 coordinate of P_2 is between zero and h). Let P_2' be the point where L_2 is tangent to one of the elliptic contours of the Y_2 response function. Then the set of feasible admissible points for Case II is: (1) the points on the curve of admissible points from the point (0,0) to the point P_2; (2) the points on the line L_2 from P_2 to P_2'.

Case III occurs when the admissible points corresponding to large values of Y_1 are permissible and the admissible points corresponding to large values of Y_2 are permissible, but the admissible points corresponding to mid-range values of both Y_1 and Y_2 are not permissible.

Let P_3 and P_3' be the points where L_3 intersect the curve of admissible points. Also let the X_1 coordinate of P_3 be less than the X_1 coordinate of P_3'.

Then the set of feasible admissible points for Case III is as follows:

(1) The points on the curve of admissible points from (0,0) to P_3.

(2) The points on L_3 from the point P_3 to the point P_3'.

(3) The points on the curve of admissible points from P_3' to (h,k).

Case IV occurs when the admissible points associated with large values of Y_1 are not permissible and the admissible points associated with large values of Y_2 are permissible. Let P_4' be the point where L_4 is tangent to one of the circular contours of Y_1. Let P_4 be the point where L_4 intersects the curve of admissible points.

Then the set of feasible admissible points for Case IV is given as follows:

(1) The points on L_4 from P_4' to P_4.

(2) The points on the curve of admissible points from P_4 to (h,k).

Case V is the case when none of the admissible points are permissible. Let P_5 be the point where L_5 is tangent to one of the circular contours of Y_1. Also let P_5' be the point where L_5 is tangent to one of the elliptic contours of Y_2. The set of feasible admissible points for Case V is the set of points on L_5 from the point P_5 to the point P_5'.

As one can see from the first problem, which is possibly the simplest problem we can consider, the number of cases one must consider in order to solve the general problem is large; and we have not solved the general problem.

As our second problem let us consider Y_1 contours which form a family of circles with a maximum response at their center (0,0). Let the Y_2 contours form a family of hyperbolas with center (h,k), $h \geq 0$, $k \geq 0$ and axis $X_2 = k$. Figure 17 will help us in our search for the set of feasible admissible points.

From Chapter III we know the set of admissible points for this problem is the set

$$\{(X_1, X_2) | (a - 1)X_1X_2 - ahX_2 + kX_1 = 0, \ -\infty < X_1 \leq 0\}$$

where a is given by Equation 21.

Let us consider the problem of determining the set of feasible admissible points when the linear constraint is given by L, Figure 17.
Let P be the point where L intersects the curve of admissible points.
Let P' be the point on L such that Y(P') is equal to Y(P") where P" is a point on H_N.

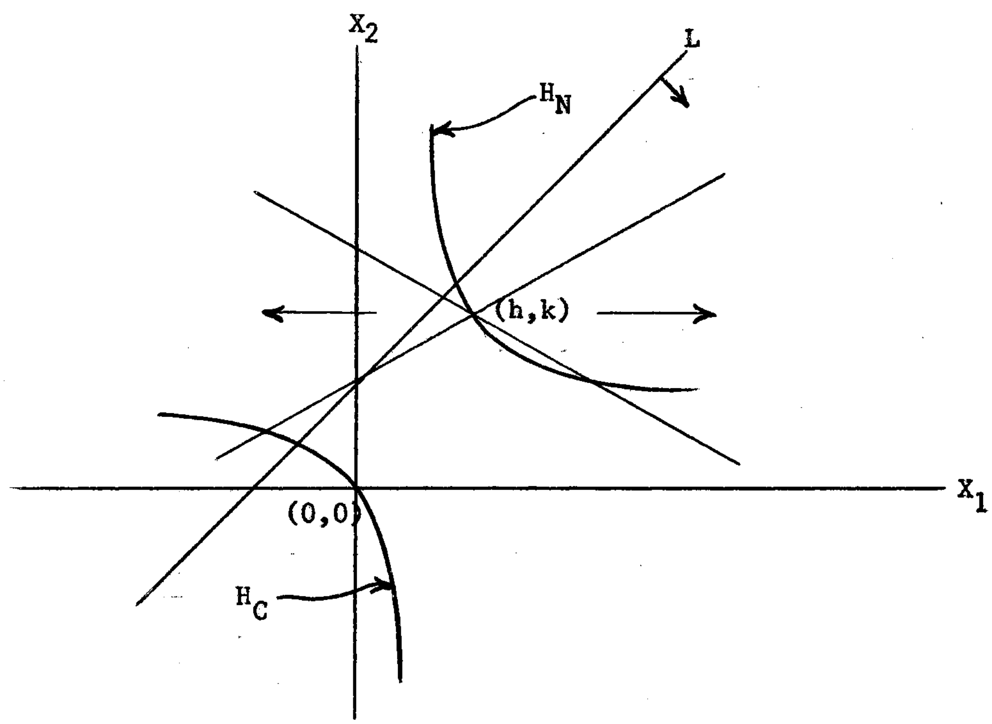

Figure 17. Linear Constraint on Elliptic and Hyperbolic Contours

The set of feasible admissible points for this problem is then the set described as follows:

(1) The set of points on the curve of admissible points from (0,0) to P.

(2) The points on L from P to P'.

(3) The points on H_N from P" toward the section of H_N which corresponds to large values of Y_2.

In Chapter III there were some cases when the set of admissible points did not exist because, by choosing X_1, X_2 or X_1 and X_2 in a certain manner, one was able to increase both Y_1 and Y_2 without bound. Many times, even though the set of admissible points does not exist, the set of feasible admissible points does exist. We illustrate this by the last example in this section.

Suppose the contours of the response function Y_1 are parabolic and have the following equation.

$$X_2 = X_1^2 + c, \quad -\infty < c < \infty \tag{68}$$

Let the contours of the response function Y_2 be parabolic and have Equation 69.

$$X_2 = (X_1 - 3)^2 + d, \quad -\infty < d < \infty \tag{69}$$

Furthermore, let the gradients of Y_1 and Y_2 be directed toward the inside of their respective contours. It is obvious then that the set of admissible points does not exist. Now let us impose the linear constraint

$$X_2 \leq 4 \quad . \tag{70}$$

Then one sees that the set of feasible admissible points is the set of points on $X_2 = 4$ with X_1 coordinates between 0 and 3, which written in set notation is

$$\{(X_1, X_2) | X_2 = 4, \ 0 \leq X_1 \leq 3\} \ .$$

Summary

In this chapter we considered what effect a linear constraint on the controlled variables would have on the set of admissible points. It should be mentioned that the results we obtained in the cases treated are in agreement with the conclusions reached by Antle (1). An example was also considered in which the set of admissible points did not exist, but the set of feasible admissible points did exist. The results of applying linear constraints to the controlled variables, of the seven problems considered in Chapter III, should be catalogued. Also the problems where the sets of admissible points did not exist should be considered.

CHAPTER V

TANGENT POINTS FOR RESPONSE FUNCTIONS WITH SPECIAL CONTOURS: N = 2, P = 3

In Chapter III and Chapter IV, the procedure used to obtain the set of admissible points consisted of first determining the set of tangent points, and then from this set of tangent points determining the set of admissible points. Clearly, one may use the same procedure when P = 3.

Let the response function Y_1 have contours which form a family of ellipsoids with center (h_1,k_1,P_1). Let the response function Y_2 have contours which form a family of any quadratic surfaces with center (h_2,k_2,P_2). After applying the techniques used in Chapters III and IV, for changing the original problem into a simpler problem, we have the following situation. The Y_1 contours form a family of spheres with center $(h,k,P) \geq (0,0,0)$. The Y_2 contours form a family of quadratic surfaces with center, if there is a center, at (0,0,0). The equation of the Y_2 contours will be in standard form. From the preceding discussion we see that the contours of the response functions for this new situation have the following equations.

The equation of the contours of the Y_1 response function is

$$(X_1 - h)^2 + (X_2 - k)^2 + (X_3 - P)^2 = k_2 \quad . \tag{71}$$

The equation of the contours of the Y_2 response function can be of the form:

$$aX_1^2 + bX_2^2 + X_3^2 = k_2 \quad . \tag{72}$$

Equation 72 is the form of all quadratic surfaces except the paraboloids. To represent the families of paraboloids we need an equation of the form:

$$X_1^2 + aX_2^2 + 2bX_3 = k_2 \ . \tag{73}$$

We will first consider the cases where the Y_2 contours do not form families of paraboloids. That is, we will find the equation of the set of tangent points for Equation 71 and Equation 72.

From Corollary 2, since N = 2 and the conditions of Theorem IV are met, we know

$$\nabla Y_1 = - c \ \nabla Y_2 \quad . \tag{74}$$

Now using Equation 74, we have

$$\begin{bmatrix} 2(X_1 - h) \\ 2(X_2 - k) \\ 2(X_3 - P) \end{bmatrix} = -c \begin{bmatrix} 2aX_1 \\ 2bX_2 \\ 2X_3 \end{bmatrix} \quad . \qquad \begin{matrix} (76) \\ (77) \\ (78) \end{matrix}$$

Dividing the terms of Equation 76 by $2aX_1$, dividing the terms of Equation 77 by $2bX_2$, and dividing the terms of Equation 78 by $2X_3$, we have

$$\frac{X_1 - h}{aX_1} = - c \quad , \tag{79}$$

$$\frac{X_2 - k}{bX_2} = - c \quad , \tag{80}$$

$$\frac{X_3 - P}{X_3} = -c \quad . \tag{81}$$

Setting -c of Equation 79 equal to -c of Equation 80, one obtains

$$\frac{X_1 - h}{aX_1} = \frac{X_2 - k}{bX_2} \tag{82}$$

or rewriting Equation 82, we have

$$(a - b)X_1X_2 - akX_1 + bhX_2 = 0 \quad . \tag{83}$$

We note that Equation 83 is the equation of a hyperbolic cylinder unless a = b when Equation 83 is the equation of a plane.

Next setting -c of Equation 80 equal to -c of Equation 81, we obtain

$$\frac{X_2 - k}{bX_2} = \frac{X_3 - P}{X_2} \tag{84}$$

or rewriting this, we have

$$(b - 1)X_2X_3 - bPX_2 + kX_3 = 0 \quad . \tag{85}$$

Equation 85 is the equation of a hyperbolic cylinder unless b = 1 when Equation 85 is the equation of a plane.

If one wants to analyze Equation 83 or Equation 85 more closely, he can write them in matrix notation and use the results (page 230 of reference (6)) to determine what each surface looks like.

That is, if we write Equation 83 as

$$(X_1, X_2, X_3, 1) \begin{bmatrix} 0 & a-b & 0 & -ak \\ a-b & 0 & 0 & bh \\ 0 & 0 & 0 & 0 \\ -ak & bh & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \end{bmatrix} = 0$$

and if we write Equation 85 as

$$(X_1, X_2, X_3, 1)\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b-1 & -bP \\ 0 & b-1 & 0 & k \\ 0 & -bP & k & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \end{bmatrix} = 0$$

then from (6) page 230 we are able to tell exactly what type of surface Equations 83 and 85 describe.

The results of an analysis of the preceding type may be best displayed in the following figure. Figure 18 shows, for the given values of a and b, the type of surface described by Equation 83.

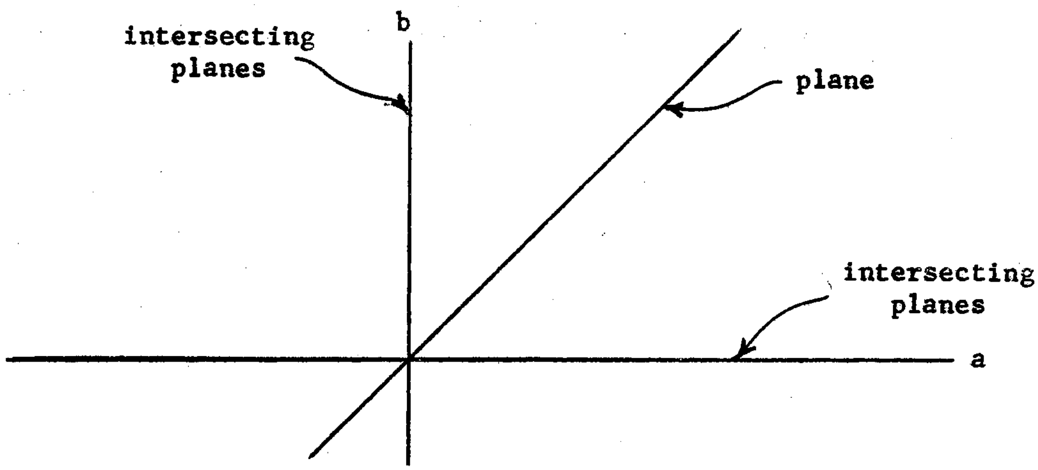

Figure 18. Tangent Surface I

For all points (a,b) not otherwise marked, Equation 83 describes a hyperbolic cylinder.

We can also display similar results for Equation 85. Figure 19 shows, for the given values of a and b, the type of surface which satisfies Equation 85.

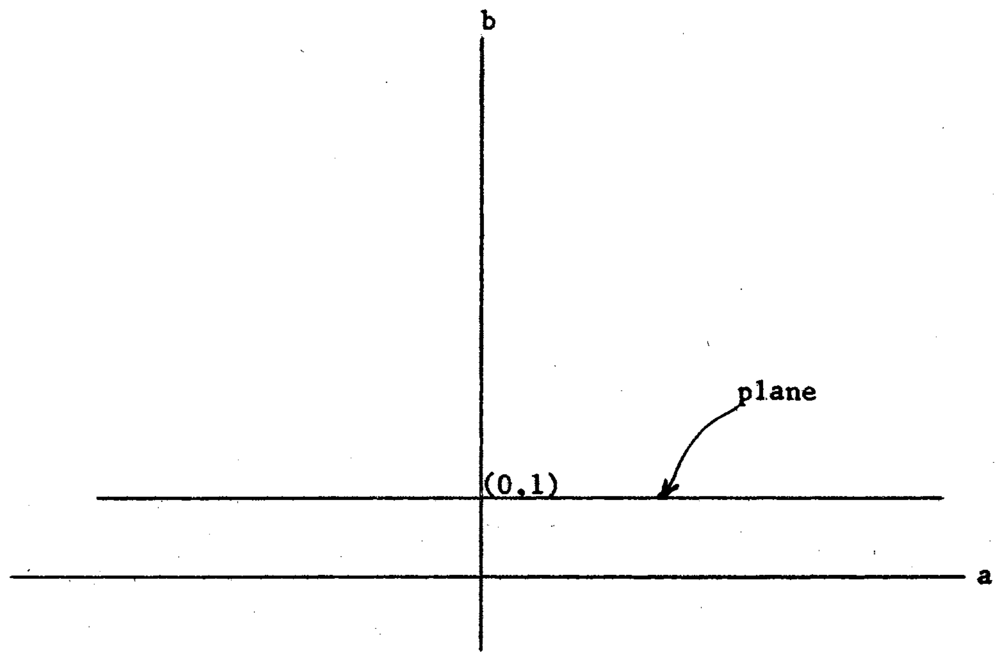

Figure 19. Tangent Surface II

For all points not otherwise marked, Equation 85 describes a hyperbolic cylinder. We could also have a figure showing the type of surface described by Equation 72 for different values of a and b.

From Figure 18 and Figure 19 and knowing a and b, we know what surfaces Equations 83 and 85 describe. The intersection of the surfaces described by Equations 83 and 85 is the set of tangent points for the response functions having contours described by Equations 71 and 72.

Let us next consider response functions whose contours are given by Equations 71 and 73.

Again applying Corollary 2, that is, Equation 74, we have

$$\begin{bmatrix} 2(X_1 - h) \\ 2(X_2 - k) \\ 2(X_3 - P) \end{bmatrix} = -c \begin{bmatrix} 2X_1 \\ 2aX_2 \\ 2b \end{bmatrix} . \quad \begin{matrix} (86) \\ (87) \\ (88) \end{matrix}$$

Dividing the terms of Equation 86 by $2X_1$, dividing the terms of Equation 87 by $2aX_2$, and dividing the terms of Equation 88 by 2b, we obtain the following equations.

$$\frac{X_1 - h}{X_1} = -c \qquad (89)$$

$$\frac{X_2 - k}{aX_2} = -c \qquad (90)$$

$$\frac{X_3 - P}{b} = -c \qquad (91)$$

Setting −c of Equation 89 equal to −c of Equation 91 and setting −c of Equation 90 equal to −c of Equation 91, we have

$$X_1X_3 - (b + P)X_1 + bh = 0 \qquad (92)$$

and

$$aX_2X_3 - (aP + b)X_2 + bk = 0 . \qquad (93)$$

Clearly Equations 92 and 93 are again hyperbolic cylinders or planes.

Therefore, the set of tangent points is again the set of points which is the intersection of some combination of hyperbolic cylinders and planes.

In the same manner that we have found the set of tangent points for one response function with a family of ellipsoids forming its contours and another response function with any family of quadratic surfaces forming its contours, we can find the set of tangent points for a response function with, say, a family of hyperboloids of one sheet forming its contours, and another response function with any family of quadratic surfaces forming its contours. However, it should be noted that the mathematics for some of these cases will be much more difficult.

After one obtains the set of tangent points, he still has the problem of picking, from this set of tangent points, the set of admissible points if the set of admissible points exist. At this point we are able to say what the set of admissible points is only when the contours of both response functions are ellipsoids.

CHAPTER VI

ELLIPTIC CONTOURS WITH MOVABLE CENTERS

When multiple responses are of interest, two things we must consider when choosing a model to fit a given response surface are: (1) how well the model fits and (2) if we use a certain model, then can we determine the set of admissible points. We mention this here because, as will be seen, the set of admissible points is sometimes difficult to determine.

As one may expect there are many cases when the quadratic curves of Chapter III do not approximate the contours of a response function sufficiently close to merit their use in a model. In an effort to obtain a model which will approximate the contours of a given response function sufficiently close, one may be led to consider a family of quadratic curves (as in Chapter III) with the center of the family moving in a given path. That is, each contour has a center which is on some given path.

Let us suppose one came to the conclusion that a family of non-intersecting ellipses with centers along a line, parallel to the major or minor axis, was an appropriate model for the contours of given response function Y_1. Furthermore, let us suppose the model which seemed appropriate for the response function Y_2 was a model with a family of circular contours with a common center.

Suppose we are confronted with the preceding problem, we first need to determine the set of tangent points. Let us suppose the line along which the elliptic contours of Y_1 have their center, is the non-negative part of the X_1 axis.

The equation of the contours of the response function Y_1 may be written

$$(X_1 - d)^2 + aX_2^2 = (cd)^2, \quad c > 1 \tag{94}$$

where d is the X_1 coordinate of the center of the ellipse.

The equation of the contours of the response function Y_2 may be written

$$(X_1 - h)^2 + (X_2 - k)^2 = k_2^2 \quad . \tag{95}$$

In order to determine the set of tangent points for response functions Y_1 and Y_2, we first obtain X_2' from Equations 94 and 95.

Taking the derivative of the functions in Equation 95 and solving for X_2', we have

$$X_2' = -\frac{(X_1 - h)}{(X_2 - k)} \quad . \tag{96}$$

Taking the derivative of the functions in Equation 94 and solving for X_2', we have

$$X_2' = -\frac{(X_1 - d)}{aX_2} \quad . \tag{97}$$

Setting the X_2' of Equation 96 equal to the X_2' of Equation 97, we have

$$\frac{(X_1 - h)}{(X_2 - k)} = \frac{X_1 - d}{aX_2} \quad . \tag{98}$$

Equation 98 determines the set of tangent points; but with the variable d in Equation 98, we are unable to tell much about what the graph of the curve looks like.

Solving Equation 98 for d, we find

$$d = \frac{X_1(X_2 - k) - aX_2(X_1 - h)}{X_2 - k} \quad . \tag{99}$$

Now to eliminate d from Equation 99 and obtain the equation of the set of tangent points in a form we can analyze, we solve Equation 94 for d. From Equation 94, we have

$$d = \frac{-X_1 \pm \sqrt{X_1^2 + (c^2 - 1)(X_1^2 + aX_2^2)}}{c^2 - 1} \quad . \tag{100}$$

Setting d from Equation 99 equal to d from Equation 100, we have

$$\frac{X_1(X_2 - k) - aX_2(X_1 - h)}{X_2 - k} = \frac{-X_1 \pm \sqrt{X_1^2 + (c^2 - 1)(X_1^2 + aX_2^2)}}{c^2 - 1} \quad . \tag{101}$$

Multiplying Equation 101 by $(c^2 - 1)(X_2 - k)$, we have

$$(c^2 - 1)[X_1(X_2 - k) - aX_2(X_1 - h)] + X_1(X_2 - k) = \pm (X_2 - k) \sqrt{X_1^2 + (c^2 - 1)(X_1^2 + aX_2^2)} \quad . \tag{102}$$

Squaring both sides of Equation 102, we have

$$(c^2 - 1)^2[X_1(X_2 - k) - aX_2(X_1 - h)]^2 + X_1^2(X_2 - k)^2 + 2(c^2 - 1)[X_1(X_2 - k) - aX_2(X_1 - h)]X_1(X_2 - k) = (X_2 - k)^2[X_1^2 + (c^2 - 1)(X_1^2 + aX_2^2)] \quad . \tag{103}$$

Dividing both sides of Equation 103 by $c^2 - 1$, we obtain

$$(X_2 - k)^2(X_1^2 + aX_2^2) = (c^2 - 1)[X_1(X_2 - k) - aX_2(X_1 - h)]^2$$

$$+ 2X_1(X_2 - k)[X_1(X_2 - k) - aX_2(X_1 - h)] \quad . \tag{104}$$

Rewriting Equation 104, we have

$$(X_2 - k)^2(aX_2^2) = c^2[X_1(X_2 - k) - aX_2(X_1 - h)]^2 - a^2X_2^2(X_1 - h)^2 . \tag{105}$$

Writing Equation 105 in descending powers of X_2, we have

$$aX_2^4 - 2akX_2^3 + [a^2(X_1 - h)^2 + ak^2 - c^2(a^2\{X_1 - h\}^2 + X_1^2$$

$$-2aX_1\{X_1 - h\}X_2^2 + [c^2\{2kX_1^2 - 2akX_1(X_1 - h)\}]X_2 - c^2k^2X_1 = 0 \quad . \tag{106}$$

Now, if we know the values of c, h, k, and a, we should be able to determine the type of curve given by Equation 106.

We may find it easier to determine what the set of tangent points looks like if we write Equation 105 in descending powers of X_1.

Writing Equation 105 in descending powers of X_1, we have

$$[a^2X_2^2 - c^2\{X_2 - k\}^2 + a^2X_2^2 - 2aX_2\{X_2 - k\}]X_1^2 +$$

$$[-2a^2X_2^2h - c^2\{2a^2hX_2^2 + 2ahX_2(X_2 - k)\}]X_1 +$$

$$aX_2^2[ah^2 + (X_2 - k)^2 - c^2ah^2] = 0 \; . \tag{107}$$

Again, if we know a, c, h, and k, we will be able to determine the type of curve given by Equation 107. As an example, let a = 1, c = 2, h = k = 1. Then the set of tangent points are illustrated in Figure 20.

From the preceding example one can readily see that problems of this type can become very difficult to treat. However, problems of this type are very important; hence, there should be further study in this area.

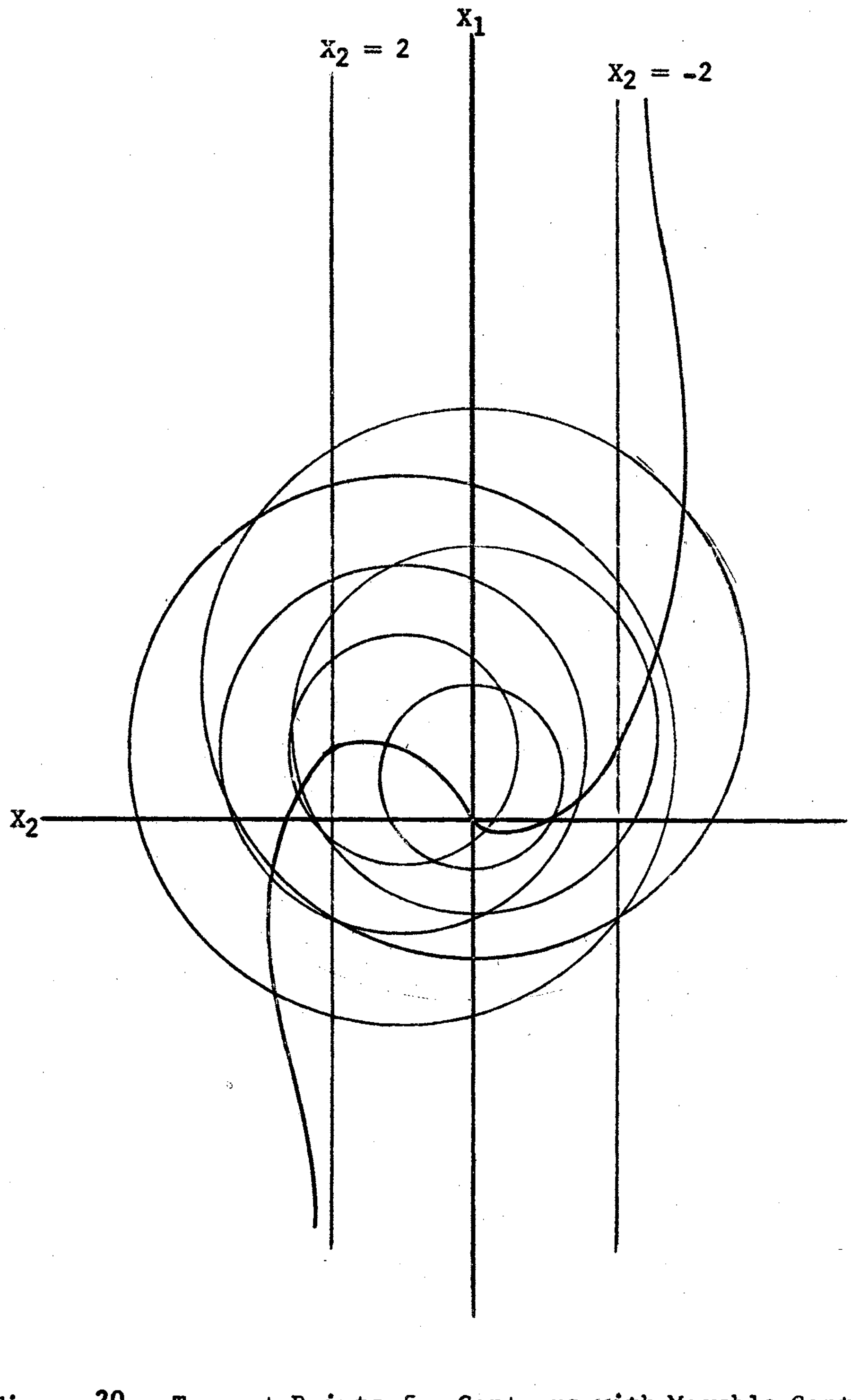

Figure 20. Tangent Points for Contours with Movable Center

CHAPTER VII

SUMMARY

When considering a problem which has multiple responses of interest, one must realize that, in general, we are unable to maximize all responses simultaneously. This led us to define admissible points. The definition of an admissible point led to the definition of a complete set, and then to the definition of a minimal complete set.

In Chapter II some very useful tools, which allow us to determine the set of admissible points for given response functions, and in some cases the minimal complete set, were introduced. We also showed that some of the theorems given in Chapter II could be very useful when one is searching a response surface for admissible points.

Chapter III dealt with cases where N = 2, P = 2, and the response functions of interest had contours which formed families of quadratic curves. Results for all different combinations of the quadratic curves were given. The results showed if the set of admissible points existed, then, if the set of admissible points did exist, described this set of admissible points.

Realizing that many times one is constrained to a certain region of the controlled variables, we considered, in Chapter IV, the effect on the set of admissible points of a linear constraint. Only a few problems were considered. All problems of Chapter III should be considered and their feasible admissible sets tabulated (if they exist). The idea of feasibility naturally came about with the introduction of constraints. Some less familiar concepts were introduced, such as the idea of a feasible tangent point, feasible admissible points and minimal complete set of feasible points. Besides the case of one linear constraint, one is also interested in the case where there is more than one linear constraint. Moreover, one is interested in all types of constraints; and this is an area where some future research should be done.

A procedure, for the cases when N = 2, was introduced in Chapters III and IV, in which one first obtains the set of tangent points, and then from this set of tangent points determines the set of admissible points. In Chapter V we determined the set of tangent points for some response functions which have special types of contours. The problem was considered for the cases N = 2 and P = 3. Only a limited number of cases was given in Chapter V and much more work could be done here.

In Chapter VI a method of building a model for a response surface by using families of quadratic curves with movable centers to approximate the contours was introduced. In Chapter III we considered the center of all families of quadratic curves as being fixed. If one allows the center of the quadratic curves to move along different paths, a very good model can be built for many problems. However, as was seen by an example, the set of admissible points are usually difficult to determine. Only the case with the centers of a family of ellipses moving along a line was considered; however, there are many other cases which should be considered. Thus, this is an area where much further study may be done.

SELECTED BIBLIOGRAPHY

(1) Antle, Charles E. "The N Response Problem." (unpub. Ph.D. thesis, Oklahoma State University, 1962).

(2) Bellman, R. E. and S. E. Drefus. Applied Dynamic Programming. Princeton: Princeton University Press, 1962.

(3) Blackwell, David, and M. Girshick. Theory of Games and Statistical Decisions. New York: John Wiley and Sons, 1954.

(4) Box, G. E. and K. B. Wilson. "On the Experimental Attainment of Optimum Conditions." Journal of the Royal Statistical Society. Vol. 13. (1951) 1-45.

(5) Cochran, W. G. and G. M. Cox. Experimental Designs. New York: John Wiley and Sons, Inc., 1962.

(6) Dresden, A. Solid Analytical Geometry and Determinants. New York: John Wiley and Sons, Inc., 1942.

(7) Garvin, W. W. Introduction to Linear Programming. New York: McGraw-Hill Book Co., 1960.

(8) Hadley, G. Nonlinear Programming. Reading: Addison-Wesley Publishing Co., 1964.

(9) Karlin, Samuel. Mathematical Methods and Theory in Games, Programming, and Economics. Reading: Addison-Wesley Publishing Co., 1959.

(10) Kempthorne, O., B. V. Shah, and R. J. Buehler. "The Method of Parallel Tangents (PARTAN) For Finding an Optimum." Annals of Mathematical Statistics. Vol. 32. (1961) 632.

(11) Kuhn, H. W. and A. W. Tucker. "Nonlinear Programming." Proceedings of The Second Berkeley Symposium. Univ. of California Press, 1951.

(12) Wilde, D. J. Optimum Seeking Methods. New York: Prentice-Hall Inc., 1964.

(13) Wilks, S. S. Mathematical Statistics. New York: John Wiley and Sons, 1962.

(14) Yates, Robert C. Analytic Geometry With Calculus. New York: Prentice-Hall, Inc., 1961.

APPENDIX A

The following problem will illustrate that if one has two families of elliptic contours, one family with center at the origin, then only those tangent points on H_c, the branch of the hyperbola through the origin, are possible admissible points. Without loss of generality we can let a^2 of Equation 108 be greater than b^2 of Equation 109. Therefore, to show this, we will show that all tangent points whose X_1 coordinates have values less than the value of the X_1 coordinate of the vertical asymptote are tangent points for contours which are tangent at one point and intersect at others.

Let the equation of the contours of one response be

$$a^2X_1^2 + X_2^2 = k_1 \tag{108}$$

and the equation of the contours of the other response be

$$b^2(X_1 - h)^2 + (X_2 - k)^2 = k_2 \quad . \tag{109}$$

Solving Equation 108 for X_2^2, we have

$$X_2^2 = k_1 - a^2X_1^2 \quad . \tag{110}$$

Expanding Equation 109 and eliminating X_2^2 by using Equation 110, we have

$$2kX_2 = k_1 - a^2X_1^2 + b^2(X_1 - h)^2 - k_2 + k^2 \quad . \tag{111}$$

Squaring both sides of Equation 111 and using Equation 110 to eliminate the X_2^2 term, we arrive at the following fourth degree equation:

$$(a^2 - b^2)^2X_1^4 + 4b^2h(a^2 - b^2)X_1^3 + [4k^2a^2 + 4b^4h^2 - 2a^2b^2h^2 + 2b^4h^2 - 2a^2k^2 + 2b^2k^2 - 2a^2k_1 + 2b^2k_1 + 2a^2k_2 - 2b^2k_2]X_1^2 + [-4b^4h^3 - 4b^2hk^2 - 4b^2hk_1 + 4b^2hk_2]X_1 + [b^4h^4 + k^4 + k_1^2 + k_2^2 - 2k^2k_1 + 2b^2k^2h^2 + 2b^2h^2k_1 - 2b^2h^2k_2 - 2k^2k_2 - 2k_1k_2] = 0 \quad . \tag{112}$$

If we let a^2 be greater than b^2 and solve for the tangent points of Equations 108 and 109, we find that the set of tangent points is a hyperbola (H_c) corresponds to points with X_1 coordinates greater than $-b^2h/a^2-b^2$. We wish to show that the contours which are tangent on H_N also intersect at other points.

Let c be the X_1 coordinate of any point where two elliptic contours are tangent, that is, any point in the set of tangent points. Therefore $(X_1 - c)^2$ must divide the Equation 112 evenly.

Dividing the terms of Equation 112 by $(X_1 - c)^2$ and using the fact that $(X_1 - c)^2$ must divide the terms of Equation 112 evenly, we have the following:

$$(a^2 - b^2)^2X_1^2 + [4b^2h + 2c(a^2 - b^2)][a^2 - b^2]X_1 + [(a^2 - b^2)(-c^2\{a^2 - b^2\} + 2\{k_2 - k_1 - k^2 - b^2h^2 + 4b^2hc + 2c^2\{a^2 - b^2\}\}) + 4b^4h^2 + 4k^2a^2] = 0 \quad ; \tag{113}$$

$$-2c[(a^2 - b^2)(-c^2\{a^2 - b^2\} + 2\{k_2 - k_1 - k^2 - b^2h^2 + 4b^2hc +$$

$$2c^2(a^2 - b^2)\}) + 4b^4h^2 + 4k^2a^2] =$$

$$(a^2 - b^2)(-c^2)(4b^2h + 2c\{a^2 - b^2\}) + 4b^2h[k_2 - k_1 - k^2 - b^2h^2] \quad (114)$$

$$c^2[(a^2 - b^2)(-c^2\{a^2 - b^2\} + 2\{k_2 - k_1 - k^2 - b^2h^2 + 4b^2hc +$$

$$2c^2(a^2 - b^2)\}) + 4b^4h^2 + 4k^2a^2] =$$

$$b^4h^4 + k^4 + k_1^2 + k_2^2 - 2k^2k_1 + 2b^2k^2h^2 + 2b^2h^2k_1 -$$

$$2b^2h^2k_2 - 2k^2k_2 - 2k_1k_2 \quad . \quad (115)$$

Equation 113 is the quotient of Equation 112 and $(X_1 - c)^2$. Equations 114 and 115 are from the X_1 coefficient and constant term respectively. (We use the fact that $(X_1 - c)^2$ must divide Equation 112 evenly). One should note the right side of Equation 115 is also equal to

$$[k_2 - k_1 - k^2 - b^2h^2]^2 \quad .$$

Solving Equation 114 for $k_2 - k_1$ we have

$$k_2 - k_1 = \frac{(a^2-b^2)^2c^3+(a^2-b^2)(3b^2h)c^2+(a^2k^2+3b^4h^2+b^2k^2-a^2b^2h^2)c-b^2h(k^2+b^2h^2)}{-[(a^2 - b^2)c + b^2h]} \quad (116)$$

The discriminant of Equation 113 is

$$[(a^2 - b^2)(4b^2h + 2c\{a^2 - b^2\})]^2$$

$$-4(a^2 - b^2)^2[(a^2 - b^2)(\{a^2 - b^2\}\{-c^2\} + 2\{k_2 - k_1 - k^2 - b^2h^2$$

$$+4b^2hc + 2c^2(a^2 - b^2)\}) + 4b^4h^2 + 4k^2a^2] \quad . \quad (117)$$

Since we are only interested in the sign of the discriminant and $4(a^2 - b^2)^2$ is positive, we will not change our conclusions if we divide the discriminant by $4(a^2 - b^2)^2$. Dividing the terms of 117 by $4(a^2 - b^2)^2$, expanding, and simplifying, one has

$$-(a^2 - b^2)^2 2c^2 - 4b^2h(a^2 - b^2)c - 4k^2a^2 + 2b^2h^2(a^2 - b^2) + 2k^2(a^2 - b^2) - 2(a^2 - b^2)(k_2 - k_1) \quad . \tag{118}$$

Substituting for $k_2 - k_1$ from Equation 116, factoring

$$\frac{-2}{(a^2 - b^2)c + b^2h} \ ,$$

and simplifying, one sees the sign of the discriminant will be given by the sign of the following expression:

$$\frac{-2}{(a^2 - b^2)c + b^2h}(2a^2b^2hk^2) \quad . \tag{119}$$

But $2a^2b^2hk^2$ is greater than zero, so the sign of the discriminant will be given by the sign of

$$\frac{-2}{(a^2 - b^2)c + b^2h} \quad . \tag{120}$$

We want to determine which values of c give rise to only tangent points, that is, for what values of c is the sign of the discriminant less than zero. Thus, we must determine for what values of c Equation 121 is greater than zero.

$$(a^2 - b^2)c + b^2h \tag{121}$$

We note that Equation 121 is greater than zero if and only if Equation 122 is satisfied.

$$c > \frac{-b^2h}{a^2 - b^2} \tag{122}$$

But, $X_1 = -b^2h/a^2-b^2$ is the vertical asymptote of the hyperbola which makes up the set of tangent points. Hence, tangent points with their X_1 coordinates greater than the X_1 coordinate of the vertical asymptote (of the hyperbola which is the set of tangent points) are only tangent points. Further, tangent points with their X_1 coordinates less than the X_1 coordinate of the vertical asymptote are not only tangent points but also give rise to contours which are both tangent and intersect at other points. Therefore, they can not give rise to admissible points (Theorem VIII).

APPENDIX B

Consider the response functions given by Equations 46 and 47. It is obvious from the geometry of the set of tangent points (Figures 16, 17, etc.) that any admissible point with a large Y_1 value must occur on H_2. Also we note that as we move on H_2 so as to increase the Y_2 response, then the Y_1 response is decreased. However, as we search for admissible points which have large Y_2 responses, it is not obvious that the admissible points are on H_2. Thus, we need to show that all the admissible points are in fact on H_2.

First, since there are admissible points on H_2 corresponding to large Y_1 responses and since Y_1 and Y_2 are continuous functions, then in order for there to be an admissible point on H_1 there must first be points, one point on H_1 and one point on H_2, which have the same Y_1 response and the same Y_2 response. That is, there must exist a point X^1 on H_1 and X^2 on H_2 such that $Y(X^1) = Y(X^2)$.

In the following we will show that there exist no points X^1 on H_1 and X^2 on H_2 such that $Y(X^1) = Y(X^2)$. This shows that all admissible points are on H_2. To show that X^1 and X^2 do not exist we will assume they do exist and show that we reach a condition that is true only in a special degenerate case.

Suppose there exist X^1 on H_1 and X^2 on H_2 such that $Y(X^1) = Y(X^2)$. This implies that $Y_1(X^1) = Y_1(X^2)$, or that both (X^1) and (X^2) are points on the same Y_1 contour. Let

$$X_2 = X_1^2 + c \tag{123}$$

be the equation of this Y_1 contour.

Now $Y(X^1) = Y(X^2)$ also implies that $Y_2(X^1) = Y_2(X^2)$, therefore that X^1 and X^2 are on the same Y_2 contour. Let

$$a(X_1 - h)^2 + b(X_1 - h)(X_2 - k) + (X_2 - k)^2 = k_2 \tag{124}$$

where $b^2 - 4a > 0$ be the equation of this Y_2 contour.

If we solve Equations 123 and 124 simultaneously, we will obtain two values for X_1 (also X_2). One of these values will correspond to a point on H_1 and the other to a point on H_2. Since these points are in fact tangent points they will each give rise to double roots.

Let us use Equation 123 to eliminate X_2 in Equation 124. Then

$$a(X_1 - h)^2 + b(X_1 - h)(X_1^2 + c - k) + (X_1^2 + c - k)^2 = k_2 \ . \tag{125}$$

Equation 125 is a fourth degree equation in X_1; but since X^1 and X^2 are tangent points which give rise to double roots, Equation 125 can be written as

$$(X_1 - e)^2(X_1 - f)^2 = 0 \quad , \tag{126}$$

where e and f are X_1^1 and X_1^2 (the X_1 coordinate of point X^1 and the X_1 coordinate of point X^2) respectively.

Expanding Equation 125, one obtains the following:

$$X_1^4 + bX_1^3 + (a - bh + 2c - 2k)X_1^2 + (cb - kb - 2ah)X_1$$
$$+ ah^2 + bhk - bhc + c^2 - 2ck + k^2 - k_2 = 0 \ .$$

Expanding Equation 126, one obtains

$$X_1^4 - 2(f + e)X_1^3 + [(f + e)^2 + 2ef]X_1^2 - 2ef(f + e)X_1 + e^2f^2 = 0. \quad (128)$$

Considering the coefficients of like powers of X_1, we obtain the following equations.

From the coefficient of X_1^3, we have

$$b = -2(f + e) \quad . \quad (129)$$

From the coefficient of X_1^2, we have

$$a - bh + 2c - 2k = (f + e)^2 + 2ef \quad . \quad (130)$$

From the coefficient of X_1, we have

$$cb - kb - 2ah = -2ef(f + e) \quad . \quad (131)$$

Eliminating f + e in Equation 130 and 131 by using its value from Equation 129, we have

$$a - bh + 2c - 2k = \frac{b^2}{4} + 2ef \quad (132)$$

and

$$cb - kb - 2ah = bef \quad . \quad (133)$$

Multiplying the terms of Equation 132 by b and the terms of Equation 133 by 2, we have

$$ab - b^2h + 2bc - 2bk = \frac{b^3}{4} + 2bef \quad (134)$$

and

$$2bc - 2bk - 4ah = 2bef \quad . \quad (135)$$

Subtracting the terms of Equation 135 from the terms of Equation 134, we obtain

$$ab - b^2h + 4ah = \frac{b^3}{4} \quad . \tag{136}$$

Now rewriting Equation 136 and multiplying each term by four, we have

$$b^3 - 4ab + 4b^2h - 16ah = 0 \quad . \tag{137}$$

Factoring b out of the first two terms and 4h out of the last two terms, we obtain

$$b(b^2 - 4a) + 4h(b^2 - 4a) = 0 \tag{138}$$

or factoring $b^2 - 4a$ from both terms, we have

$$(b + 4h)(b^2 - 4a) = 0 \quad . \tag{139}$$

Since $b^2 - 4a$ is greater than zero, this implies that

$$b + 4h = 0 \quad \text{or} \quad h = -\frac{b}{4} \quad . \tag{140}$$

But $-b/4$ is the vertical asymptote of the hyperbola which makes up the set of tangent points. If $h = -b/4$, then the hyperbola, which is the set of tangent points, degenerates into two straight lines.

From the preceding, we see that the only time that points X^1 and X^2 exist is when $h = -b/4$. In this case the set of admissible points is the union of sets A and B where

$$A = \{(X_1, X_2) | X_1 = h, \ X_2 \geq k\}$$

$$B = \{(X_1, X_2) | X_2 = - \frac{(-4h)X_1(X_1 - h) + a(X_1 - h) + 2hk - 2kX_1}{2X_1 - 2h}, X_1 \neq h\} .$$

If $-b/4 \neq h$, then the set of admissible points is the set of points on H_2.

VITA

Howard Leland Dockery

Candidate for the Degree of

Doctor of Philosophy

Thesis: THE MULTIPLE RESPONSE PROBLEM

Major Field: Mathematics and Statistics

Biographical:

Personal Data: Born in Sheldon, Missouri, August 3, 1942, the son of Howard C. and Darlene Joan Dockery.

Education: Attended grade school in Lamar, Missouri; graduated from the Lamar High School in 1960; received the Bachelor of Science degree from the Missouri School of Mines, with a major in physics, in May, 1963; received the Master of Science degree at the Oklahoma State University in May, 1965; completed the requirements for the Doctor of Philosophy degree in May, 1967.

Scholarly Information: Received a National Merit Scholarship and a National Science Foundation Scholarship at the Missouri School of Mines; Receiving a National Defense Education Act Fellowship at the Oklahoma State University; member of Phi Kappa Phi, Kappa Mu Epsilon, Sigma Pi Sigma, and Tau Beta Pi honorary societies.

Professional Experience: Junior Engineer, Outboard Marine Corp., Summers, 1960, 1961; Statistician, Texas Instruments Inc., Summer, 1964; graduate teaching assistant, Department of Mathematics and Statistics, Oklahoma State University, 1965, 1966.