A COMPARISON OF TWO SEQUENCES FOR INTRODUCING POSITIONAL NUMERATION SYSTEMS WITH A BASE TO PRE-SERVICE ELEMENTARY TEACHERS

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PREFACE

For some time I have been interested in the training of elementary teachers. This study afforded me the opportunity to construct programmed materials and to use these materials to study two sequences for introducing positional numeration systems with a base to pre-service elementary teachers. I appreciate the opportunity to conduct this experiment and express special thanks to all who participated in or cooperated with this study.

In particular, I would like to thank Dr. Vernon Troxel for the guidance and assistance that he provided as my dissertation adviser and committee chairman. Special thanks go to Drs. Milton Berg and Gerald Goff, through whose permission and direction the programmed materials were constructed. I also with to thank Drs. J. W. Blankenship and Norman Wilson for serving on my advisory committee.

Thanks go to Dr. Robert Morrison and Mr. Gary Lance, who gave advice and assistance with the statistical analysis.

To my wife, Kaye, and my children, Kathy and Kevin, I express gratitude for the sacrifices they made while assisting in the completion of my program of study. Finally, thanks to my parents for the assistance and confidence they have given me.

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CHAPTER I

INTRODUCTION

One of the contributions of the current revolution in school mathematics is the influx of new topics into the elementary school mathematics curriculum. Positional numeration systems with a base, often referred to as basimal numeration systems, is one such topic. Many of the recent mathematics textbooks for the elementary school contain introductions to these numeration systems, thus creating the need for a corresponding introduction to be presented to pre-service elementary teachers.

Purpose of the Study

The purpose of this study was to give a partial answer to the question, "What method of instruction is best for presenting positional numeration systems with a base to students in Mathematics 253 at Oklahoma State University?" Specifically, this report concentrated on the problem of selecting an optimum organization of the pertinent content in this course.

One method of introducing basimal numeration systems could be formed by using as a foundation the student's familiarity with base ten. The fundamental understanding of the numeration systems would stem from generalizing base ten to other bases. The procedures for adding and multiplying would rely on the students' ability to add and multiply in base ten.

Another introduction, which would not be established by relying on the students' understanding of base ten, would have a foundation similar to the one the elementary student should have for base ten. The understanding of the numerals and basic properties of the numeration system would be founded on sets and grouping of sets. A numeral would be a shorthand notation for describing the results of a particular grouping of a set. The understanding of addition and multiplication would be based on union of disjoint sets and cross product of sets, respectively, with regrouping. Finally, the student would be taught to change a numeral in one base to a numeral in another base, both of which represent the same number.

The natural question is, "Which of these two methods will produce better student understanding?" The writer felt that this problem was too broad for investigation in this report so a restricted problem derived from the preceding one was studied.

Statement of the Problem

This research was designed to compare two sequences for presenting basimal numeration systems to pre-service elementary teachers. The two sequences are described as follows. Both sequences contained the same initial topic, an introduction to basimal numeration systems. The foundation for this topic was grouping of sets. Sequence I then introduced the procedures for changing a numeral in one base to a numeral in another relying upon the students' ability to add and multiply in base ten. Then the operations and their algorisms in various bases were introduced. Sequence II contained the same two sections, but the order in which they were presented was altered. Finally, both sequences had the same concluding segment, which was changing a numeral in one base to a numeral in another base without using base ten. The groups of students following Sequence I and Sequence II are denoted E_1 and E_2 respectively.

The writer constructed programmed materials introducing basimal numeration systems. These materials are a revision of the programmed materials used in the course, which were written by Berg and Goff (2). The programmed materials constructed by the writer were composed of four parts, one for each of the subtopics to be introduced. They were constructed in such a manner that two sections could be interchanged in order to create the two sequences. The major purpose of the programmed materials was to delimit the material presented in the two sequences.

The relative merit of the two sequences was determined on the basis of student achievement as demonstrated by scores on a test constructed by the writer. The test was divided into three parts, which evaluate three aspects of numeration systems: (1) understanding of the operations of addition, subtraction, multiplication, and division together with the algorisms for those operations, (2) changing a numeral in one base to a numeral in another base representing the same number, and (3) understanding of positional numeration systems with a base and the properties of those numeration systems. These three parts of the test are referred to as parts C, B, and A, respectively.

Review of the Literature

The writer could find no research that related directly to sequences for introduction of numeration systems. The following studies are indirectly related to this research in that they present the rationale for introducing numeration systems other than base ten.

Sister Constantine in a historical review of some writings of DeMorgan states:

> . . . If the child has developed the habit of mechanical manipulation of numbers, the change to bases other than ten is perhaps the best approach to use in giving him proper concepts.
>
> It is interesting to note, however, that as early as 1836 Augustus DeMorgan (1806-1871) had published a paper in which he also recommended using different bases to aid in forming correct concepts in the fundamental operations.
>
> Concerning change of bases, DeMorgan states, "The student should accustom himself to work questions in different bases of numeration, which will give him a clearer insight into the nature of mathematical processes than he could obtain by any other method." (6:261)

Stringfellow makes the following comment:

> Educators are beginning to realize that this mechanical use of the decimal number system has resulted in a generation of teachers and pupils who are hampered by their lack of basic understanding of the decimal system. Some work in the use of other number systems has been found to be helpful in bringing about a better understanding of the decimal system. Especially is this a profitable activity among rapid-learning pupils. (28:557)

To examine the question, "Why teach numeration?" Hollis conducted a study to determine if there is an increased understanding of base ten resulting from instruction in a non-decimal numeration system. The experiment was conducted under a pre-test post-test design with no control group. From the study the investigator made the following observations:

1. There was an increase in the median and mean test scores.

2. The teacher observed an increased interest in and understanding of the number system.

3. The other bases did not seem to confuse any of the students. (15:95)

Finally, he concluded, "The evidence, although not conclusive, did seem to indicate that instruction in a non-decimal base will increase the child's understanding of base ten." (15:95)

Scott introduced numeration systems other than ten to five kindergarten and six first-grade classes. Responses from the childrens' teachers indicated that this instruction could help children to understand addition and subtraction. Four of the teachers mentioned that they had learned something of the number system which would benefit their teaching. The investigator concluded:

1. Many children in kindergarten and first grade can gain an understanding of the role of the radix in our number system. They are able to understand the concepts of positional notation for two-digit numbers where radices other than ten are utilized.

2. The experiences of using radices other than ten may contribute to an understanding of decimal notations, particularly with younger children who have not yet been encumbered by an habitual use of ten as a radix.

3. Teachers believe that the concepts included in a change of radix as presented in this investigation are not too difficult for children in kindergarten and first grade. (26:10)

By introducing the base five numeration system to fourth-grade pupils, Lerch examined the following hypothesis:

1. Fourth-grade arithmetic pupils can effectively learn about a number system with base other than ten.

2. The study of a number system with base other than ten will increase fourth-grade arithmetic pupils' understandings of our Hindu-Arabic decimal number system.

3. The study of a number system with base other than ten will not have a detrimental effect upon pupils' other arithmetical concepts and abilities.

4. Pupils who study a number system in this way will have favorable opinions toward the nature and content of the study. (19:59)

All hypotheses were accepted.

Ingham and Payne introduced numeration systems other than ten to eighth graders. They made the following conclusions and observations:

1. We felt that this unit on other number bases helped pupils gain additional insight into our own number system. Every day we drew analogies from our own Base-10 system as we faced problems with the new system. Frequently students showed real surprise at a new insight into our own system.

2. There was little doubt that pupils became fully aware of the arbitrariness of our Base-10 system and that they had a richer appreciation for the pattern, structure, and "tenness" of our own system.

3. The less able pupils seemed to profit less from the unit than the more able pupils, although they appeared to be interested.

4. The students were proud of the fact that they knew something that others did not know. They seemed pleased to take problems home that their parents couldn't work.

5. Other students in the school were interested in this "strange" mathematics class. As a result, there was a brief article in the school newspaper about the unit.

6. For the first day or two after the unit was completed, pupils sometimes stopped and asked themselves questions such as, "What is 4x3--is it 12 or22?" This reaction did not last long, and we felt that no damage had been done to their skills in arithmetic. (16:395)

The following paragraph reviews an experimental study comparing two methods for teaching numeration systems in a general mathematics course for college students.

Hamilton (13) developed some experimental materials entitled "Make Believe Arithmetic," which used notation distinctly different from the usual symbols for base ten. These materials were compared with a conventional introduction to numeration systems. A test was constructed that evaluated the students' understanding of such mathematical bases of arithmetic as the number system, place value, and notation for fractions. Using an analysis of variance on gain scores, the experimental materials proved more effective than the conventional materials.

In summary, this review indicated that the study of non-decimal numeration systems by elementary pupils can foster an understanding of the Hindu-Arabic numeration system. The review also indicated that these studies did not have any detrimental effects upon either the pupils' other arithmetical concepts and abilities or upon their attitudes toward arithmetic.

The writer concluded that the presentation of non-decimal numeration systems to elementary teachers can be justified in three ways:

1. It is background material necessary for elementary teachers.

2. It creates better understanding of the Hindu-Arabic numeration system.

3. It places the teacher in a situation similar to the one the elementary pupil experiences when first introduced to base ten.

Theoretical Background

A comprehensive theory of instruction would explain and predict the interrelationships among the following classifications of factors.

Class I. Societal structure within which the instruction occurs. (democratic, communistic, European, and sub-structures such as educational institutions, industrial situations, or spontaneous learning situations.)

Class II. Instructional objectives. (assemblage of facts, improvement of problem-solving ability, production of motivation, increase of conceptual understanding, competence in skills, and other changes in the learner's behavior.)

Class III. Teacher variables. (personality, education, experience, and abilities; also includes characteristics of the program.)

Class IV. Student variables. (motivation, subject experience, personal experience, maturity, ability, and age.)

Class V. Theory of learning. (gestalt-field or conditioning-association.)

Class VI. Instructional variables. (a. type of presentation, such as lecture, discussion, discovery, and recitation, and b. type of materials used such as textbook, programmed materials, overhead projector, movies or television.

A theory explaining the relationship among all these factors is not expected by educational researchers in the foreseeable future any more than a medical scientist expects to discover a pill that will be the universal cure for all illnesses. Instead, micro-theories must be examined. The micro-theory for this research is derived by a restriction of these six classes of factors to the problem as previously stated.

The societal structure was held constant since each experimental group was composed of two Mathematics 253 classes at Oklahoma State University.

The instructional objectives, in general, were held constant in that instruction was directed toward an understanding of numeration systems to bases other than ten and the operations and their algorisms. The objectives stated more specifically were:

1. To understand the properties of basimal numeration systems. (e.g. base, digits, place value, and expanded notation.)

2. To perform the algorisms for addition, subtraction, multiplication, and division in numeration systems other than base ten, and to understand the processes involved.

3. To change a numeral in one base to a numeral in another base, and to understand the processes involved.

The writer developed the programmed materials and also taught all four classes in the experiment. Thus, the teachers' characteristics were held constant. However, any teacher bias with respect to either sequence was not controlled. It is assumed that such bias did not effect the results of the experiment.

A pre-test was given to test for equivalency of the groups with respect to previous knowledge of the subject matter. Factors such as motivation, maturity, and age were assumed to be equivalent. The students' mathematics subscores on the American College Test (ACT) were checked for equivalency of the groups with respect to mathematical ability.

There is no one learning theory that is universally accepted by mathematics educators, but Lankford (18) gives several points of agreement among those learning theories that are accepted by mathematicians. The following are some of those points that particularly pertain to this research. Effective learning in mathematics results when:

1. Meaning is emphasized in contrast to manipulation.

2. Teaching is adapted to the variations among individuals.

3. The learner is appropriately motivated.

4. Practice is provided as needed by individuals but is not relied upon to develop meaning.

5. The learner is helped to discover ideas of mathematics through developmental, reflective activities.

6. The learner is an active participant rather than merely a passive listener.

In general, this research subscribed to the Gestalt or field theory of learning.

Instructional activities were controlled by defining the format of each class meeting to consist of lecture followed by discussion with a homework assignment consisting of specified frames from the programmed materials. The lectures introduced the topics presented in the programmed materials that would be assigned during that class session. The discussion was student initiated. The programmed materials were the same in both groups except that the order of presentation was changed to establish the two sequences.

Thus, the micro-theory for this research predicted the relationship between the two sequences when the other factors were controlled.

Micro-Theory

Before the revolution in mathematics, mathematical learning theory subscribed to the associationist principles for learning. The objectives of mathematics instruction were problem-solving skills. The methods used to obtain those skills basically involved drill. But within the last two decades, Gestalt or field theories of learning have been adopted. These theories of learning rely more on understanding the structure within the subject than on manipulative skills. After the understanding has been acquired, drill might be used in order to perfect any desired skill; but, the basis for the skill should be understanding instead of memorization.

This learning theory implies that an introduction to basimal numeration systems should emphasize the basic structure of those numeration systems. The procedures used in the algorisms should be guided by the students' understanding of the algorisms rather than a memorized procedure for computing.

Since both sequences introduce these numeration systems through grouping of sets, the micro-theory indicated there would be no significant difference between groups on Part A of the test, which is checking for basic understandings of positional numeration systems with a base.

Part B of the test checks the students' understanding of changing from one base to another. The writer felt there would be no significant difference between groups on Part B of the test. Sequence I could produce a little more proficiency, since the procedures are introduced earlier, affording the student the opportunity to use those procedures in the subsequent sections. But, the test checks for understanding, not proficiency.

Part C of the test checks for understanding of the operations and their algorisms. Sequence I, as previously defined, presented the opportunity for the student to rely upon base ten in his computational procedures and, thereby, possibly to develop a memorized procedure instead of a procedure guided by understanding. In sequence II the ability to change a numeral in one base to a numeral in another base was developed after the operations and their algorisms were introduced. Thus, the student would be encouraged to study the algorisms as they relate to the various bases, which should develop an understanding of the basic structure.

The writer also felt there would be a relationship between the students' mathematical ability and the effect of the sequence upon his understandings. For this reason, the students in each sequence were divided into high, middle, and low ability groups on the basis of ACT mathematics subscores. These subgroups of sequence I are denoted H_1, M_1, and L_1, respectively. ACT mathematics subscores were not available on all students; thus, the group of all students in sequence I for whom the scores were available is denoted W_1. The symbols H_2, M_2, L_2, and W_2 represent corresponding subgroups of sequence II.

The writer contended that students with high ACT mathematics subscores should understand the structure of the algorisms no matter which sequence of presentation they have followed. Students in the middle groups should react in accord with the previous theoretical position in that students in sequence II will develop a better understanding of the operations and the algorisms than the students in sequence I. The students in the low group would probably not develop an understanding in either sequence.

Hypotheses

Thus, the micro-theory supports the following hypotheses:

1. There will be no significant difference in post-test means on Part A of the test when comparing groups:

a. H_1 and H_2

b. M_1 and M_2

c. L_1 and L_2

d. W_1 and W_2

e. E_1 and E_2

2. There will be no significant difference in post-test means on Part B of the test when comparing groups:

a. H_1 and H_2

b. M_1 and M_2

c. L_1 and L_2

d. W_1 and W_2

e. E_1 and E_2

3. There will be a significant difference in post-test means on Part C of the test when comparing groups:

a. M_1 and M_2

b. W_1 and W_2

c. E_1 and E_2

Each difference should favor subgroups of sequence II.

4. There will be no significant difference in post-test means on Part C of the test when comparing groups:

a. H_1 and H_2

b. L_1 and L_2

5. There will be a significant difference in post-test means on the total test when comparing groups:

a. M_1 and M_2

b. W_1 and W_2

c. E_1 and E_2

Each difference should favor subgroups of sequence II.

6. There will be no significant difference in post-test means on the total test when comparing groups:

a. H_1 and H_2

b. L_1 and L_2

CHAPTER II

PROCEDURE

This was an experimental study comparing two sequences for introducing basimal numeration systems in Mathematics 253 at Oklahoma State University. During January and February of 1967, the programmed materials for use in the experiment were prepared. Also during this period, a test evaluating understanding of the concepts presented in the programmed materials was constructed.

Four of the six sections of Mathematics 253 offered the spring semester of 1967 were selected as subjects for the experiment. Each treatment was then randomly assigned to a pair of these sections.

The pre-test was given on March 1, 1967, and the post-test was given on March 17, 1967. Between those two dates, the two treatments were administered. Subsequent to the post-test, the mathematics subscores on the ACT test for subjects in the experiment were collected.

Mathematics 253 is the first course in a two-semester sequence for elementary teachers presented at Oklahoma State University. It is a three-hour course meeting three fifty-minute periods per week for one semester. The course is entitled "Arithmetic for Elementary Teachers," and is designed to acquaint the student with number systems beginning with the natural numbers and ending with the rational numbers. Lecture with discussion is the usual method of instruction.

The text for the sequence is a set of programmed materials written by Berg and Goff (2). The first seven chapters of the materials are covered in the first course. The first chapter is an intuitive development of sets emphasizing union, intersection, cross product, and their properties. Four chapters involve rigorous developments of the whole numbers, fractions, integers, and the rational numbers (including decimals). In these chapters, the properties of the operations are derived using properties of operations from systems developed in the preceding chapters. The remaining two chapters are on numeration systems for the whole numbers and the number line.

Selection of Subjects

This experiment was conducted during the second semester, 1966-67, at which time Oklahoma State University had an enrollment of 14,676. Six sections of Mathematics 253 were offered that semester. In order to match the classes with respect to days of the week and hour of the day as nearly as possible, four classes, which met at 7:30, 9:30, 2:30, and 3:30 on Monday, Wednesday, and Friday, were chosen for the experiment. These were sections 1, 2, 5, and 6, respectively. This selection was made with the understanding that each sequence would be taught to one morning class and one afternoon class.

The students in these classes were primarily freshman, sophomores, and juniors. Almost all of these students were female, and the majority of the students were education majors. The minority was composed of home economics majors. A very small number of graduate students were enrolled. Thus, the subjects in this experiment could be described as pre-service elementary teachers.

Programmed Materials and Programming Techniques

The writer constructed a set of programmed materials that were used in this experiment. The concepts presented in these materials were determined by the usual content of the course. The programmed materials were constructed by revising the original programmed materials for the course into four sections.

The first section was an introduction to the system of notation used in basimal numeration systems. The basic understandings were developed through grouping of sets.

The second section explained the procedure for changing from one base to another using base ten. These procedures were dependent upon the first section, expanded notation, and the students' familiarity with the operations in base ten.

The third section introduced the four operations: addition, subtraction, multiplication, and division. Algorisms for computing in various bases were stressed. The fact that the algorisms are a numerical procedure for computing was emphasized. The algorisms depended upon the students' ability to add or multiply digits in the various bases and positional notation for the numerals. Addition of digits was developed through grouping of sets, and multiplication of digits was developed as repeated addition. These foundations were selected so that the algorisms could be performed without using addition and multiplication in base ten. Often, multiplication of digits is performed by multiplying in base ten, then changing that numeral to the required base.

The last section was not essential to the experiment but was included in the programmed materials because the topic was normally introduced in the course. This topic was changing from base to base without using base ten. The procedures in this section were dependent upon expanded notation for the numerals and the students' ability to multiply and add in the second base. It was necessary, therefore, for section four to be presented after section three had been presented.

The writer used both Skinner- and Crowder-type frames in the construction of the programmed materials. The Skinner frames were used when leading the student step by step through the development of some concept, while the Crowder frames were used for review and reinforcement. These materials do not contain the usual amount of repetition found in most self-contained programs. The program was designed to be used in conjunction with lectures or discussion or both.

A copy of the programmed materials is in Appendix B.

Test and Test Construction

The test used for both a pre-test and a post-test was a forty-two question multiple choice test, constructed by the writer. Each question had four possible responses. Each question had one and only one correct response. The first nine questions checked for basic understanding of basimal numeration systems. Question ten through eighteen checked for understanding of the procedures for changing a numeral in one base to a numeral in another base. The last twenty-four questions were concerned with the operations and their algorisms in various bases.

Before being used in the experiment, the test was administered to two Education 4M2 classes at Oklahoma State University. Education 4M2 is a course concerned with methods in mathematics for the elementary teacher. Most of the students in these classes had previously taken Mathematics 253, but their scores on the test indicated something short of mastery of number bases. Therefore, the reliability of the test could not be established before the experiment.

The first administration of the test indicated that the test was too long for completion in forty-five minutes and that the format of the test needed to be changed in order to permit the examinee to read the test more efficiently and effectively. The items on the test were analyzed with respect to discrimination between high and low students on the total test. Many of the test items with low discrimination were revised or eliminated. An item with low discrimination was sometimes kept when the item tested a concept or procedure peculiar to the programmed materials. This was judged to be the case when almost all examinees missed the test item. The revised form of the test had forty-two test items whereas the original form had fifty.

A copy of the test is in Appendix A.

Design of the Experiment

The basic design of the experiment is one comparing two experimental treatments under a random sample post-test situation. A pretest was given in order to produce assurance of random sampling. The sample for the experiment was a handy sample, consisting of four sections of Mathematics 253 at Oklahoma State University. The population from which this sample was drawn was a hypothetical population having characteristics determined by the sample.

The writer elected to teach all four sections in the experiment in order to assure the presentation of the two sequences. The writer was the regularly assigned instructor in two of the four sections, section 1 and section 6. In order to attempt to balance any intervening effect due to the original instructor, sections 1 and 5 were selected for one treatment and sections 2 and 6 for the other. Assignment of the treatments to these pairs of sections was made by random assignment. Sequence I was assigned to sections 1 and 5 and sequence II was assigned to sections 2 and 6.

The independent variable was the sequence of presentation and the dependent variable was the set of post-test scores.

There were several possible intervening variables. One such variable was the transfer of knowledge from one sequence to the other. This communication could not be controlled, but hopefully it did not contaminate the data. Another intervening variable could have been the reaction of the students to the experiment or the investigator due to a previously established rapport with the original instructor. The investigator did not observe any difference in the reactions to the experiment among the four sections.

Instructional Procedures

The programmed materials were the focal point for the instruction. These materials were separated into six homework assignments. The first assignment was allotted to the basic numeration system which was Part I of the programmed materials. Three assignments were on the operations and the algorisms. There were two assignments on changing from one base to another, one for each of the two different methods.

Lectures introducing the concepts in each of these six assignments were prepared. Each lecture was approximately thirty minutes long except for the two lectures on changing from one base to another base. Those two lectures were approximately fifteen minutes long. A lecture was given at the first of every instructional period.

The remainder of the period, except with the last lecture, was spent in student initiated discussion followed by work on the programmed materials if time allowed. The purpose of a discussion period was to clarify concepts from previous lectures or frames. On the day preceding the post-test, the students responded to the assigned frames following the lecture and the discussion was deferred until the final ten minutes of the period so that any misconceptions in the last topic could be discussed.

An attempt was made to use essentially the same lectures in each of the sequences, except for the order in which they were presented. Also, an attempt was made to keep the discussion sessions equivalent. This was done by using the same examples and illustrations whenever appropriate. The writer sometimes asked questions during the discussion period in order to focus the attention on concepts emphasized in discussion in other classes.

To guarantee that the students followed the sequences of presentation, the four parts of the programmed materials were distributed to the students one part at a time. Each part was given to the students when it would accompany the appropriate lecture.

Directions to Students

The students were informed that the purpose of the pre-test was to measure the previous knowledge of numeration systems possessed by the class as a whole. Emphasis was placed upon the insignificance of individual test scores. Also, the students were told that the test scores would not be used in any way to evaluate them with respect to the course.

The class period following the pre-test the mean score for each class was reported to the students. A comment was made to the effect that since the mean score was only a point above guessing, it would probably be profitable to procede with the instruction of that chapter.

The students were not told specifically that they were subjects in an experiment, but it was apparent that they knew some sort of experiment was being conducted.

The post-test was announced in all classes two class periods before it was to be given. The announcement implied that the student's score on the test might be used to evaluate him in the course, but that the regular instructor would make the final decision in this matter. In the sections in which the writer was the original instructor, the students were aware that the test scores would be averaged with their other test scores when a final grade for the course was computed.

CHAPTER III

RESULTS OF THE STUDY

The objective of this study was to investigate the achievement test score differences of groups of students using two sequences of instructional materials written for use in introducing basimal numeration systems. Additional considerations were made relating the sequences and mathematical ability as indicated by mathematics subscores on the ACT test. These comparisons were made on the basis of post-test scores. Supplementary data were statistically analyzed to establish the equivalence of groups.

The purposes of this chapter are indicated by the chapter subheadings: "Summary of the Data," "Analysis of the Test," "Statistical Design," "Tests of Significance," "Equivalence of Groups," "Testing of the Hypotheses," and "Summary."

Summary of the Data

Originally, there were 160 students enrolled in the four sections of Mathematics 253 used in this experiment. Students that failed to take both the pre-test and the post-test were eliminated from consideration in the statistical analysis. There were 13 students that were eliminated, 8 in sequence I and 5 in sequence II, leaving 147 subjects for the experiment.

The data collected were of three types: mathematics subscores on the ACT test, pre-test scores, and post-test scores. Each subject has four pre-test scores, one for each of the three parts of the test and the fourth for the total test. A tabular summary of the data is contained in Appendix C.

The mathematics subscores on the ACT were available only for the students that entered Oklahoma State University as freshmen. Thus, these data were not available for transfer students. These subscores were used to separate the subjects with ACT scores into High, Middle, and Low ability groups. The Middle group was defined to be the set of students whose subscores were within one standard deviation of the total mean of the set of subscores. The mean was 18.88 and the standard deviation was 4.59. Therefore, the range of subscores for the Middle group was 15 to 23 inclusive. The High and Low groups were then defined to be the set of students whose subscores were, respectively, above and below the range for the Middle group.

Table I gives a breakdown of the 147 subjects in the experiment indicating the number of students in each of the categories determined by sequence and subgroup.

TABLE I

NUMBER OF SUBJECTS IN THE VARIOUS SUBCLASSES

Sequence	High	Middle	Low	With ACT Scores	Without ACT Scores	Total in Sequence	Students Eliminated
I	10	32	8	50	21	71	8
II	6	39	6	51	25	76	5

Analysis of the Test

Almost all students completed the forty-two item test in the allotted time of 45 minutes. Since the test is considered a power test, an internal consistency reliability was computed for the test. Because there was a wide dispersion in the item difficulties, Horst's modification of the Kuder-Richardson formula 20 (12:461) was used to compute the reliability of the total test and of each of the parts. This formula for estimating the reliability of a test uses the maximum variance, σ_m^2 , that a test could achieve with the distribution of item difficulties indicated by the test scores. The maximum variance is given by:

$$\sigma_m^2 = 2\sum R_i p_i - M_t \left(1 + M_t \right)$$

where R_i is the rank position of an item in the test, the easiest item being ranked 1. p_i is the item mean and M_t is the total-test mean. The modified formula reads

$$r_{tt} = \frac{\sigma_t^2 - \sum p_i q_i}{\sigma_m^2 - \sum p_i q_i} \cdot \frac{\sigma_m^2}{\sigma_t^2}$$

where σ_m^2 and p_i were defined previously, $q_i = 1 - p_i$, and σ_t^2 is the total-test variance.

The reliabilities were established on the post-test scores, and are summarized in Table II.

TABLE II

RELIABILITY OF THE TEST AND SUBTESTS

Test	Reliability
Part A	.43
Part B	.59
Part C	.77
Total	.82

Statistical Design

The analysis of the data for this experiment relied upon multiple use of a two-way factorial analysis of variance with a disproportionate number of observations in the subclasses. An excellent discussion of this analysis is given in Steel and Torre (27:252).

They make the following comment concerning analysis of variance:

> In the analysis of variance where tests of significance are made, the basic assumptions are:
>
> 1. Treatment and environmental effects are additive.
>
> 2. Experimental errors are random, independently and normally distributed about mean zero and with a common variance.
>
> The assumption of normality is not required for estimating components of variance. In practice, we are never certain that all these assumptions hold; often there is good reason to believe some are false (27:128).

The nine analyses of this type that were used to assure the randomization and to examine the hypotheses fall into two categories.

The first design (Type A) is represented in the following diagram:

		Sequence I	Sequence II
Ability Groups	High		
	Middle		
	Low		

Type A designs were used in the first five analyses summarized in this report. This design allowed three comparisons. The first was a comparison between columns; therefore, it was used to test hypotheses relating total students with ACT subscores in each sequence (W_1 and W_2). The second allowed comparisons among rows, which were the total students in both sequences with high, middle, and low ACT subscores. This opportunity was not pursued since the hypotheses to be examined were concerned with sequence. The final one made possible a statistical test for interaction. This allowed the writer to compare three pairs: the two High groups, the two Middle groups, and the two Low groups.

The second design (Type B) is shown in the diagram below:

		Sequence I	Sequence II
With or Without ACT Subscores	With		
	Without		

This type of design was used to test hypotheses concerning the total groups in each sequence (E_1 and E_2) by using the comparison between columns. The other information resulting from this design was not used to test any hypotheses.

Tests of Significance

The raw data in Appendix C were prepared for the Oklahoma State University Computing Center so calculation could be performed on the computer. Two-way analysis of variance was computed using a program entitled, "Hierarchical Analysis of Variance," by E. L. Butler. This program was adapted to this experiment by Gary Lance. The program calculated means, sum of squares, and mean squares for the various subgroups.

$\underline{F}$ ratios for the two main effects and the interaction were calculated by dividing the mean square of the effect under consideration by the mean square of the error. These $\underline{F}$ ratios were used to establish the equivalence between groups and to test the hypotheses.

It was planned to follow any significant $\underline{F}$ values with Tukey's w-procedure (27:114). However, this analysis was not necessary in order to test the hypotheses.

The .05 level of significance was arbitrarily selected for this research since it is the common level accepted for research in the behavioral sciences.

Equivalence of Groups

The equivalence of the subgroups denoted H_1 and H_2, M_1 and M_2, L_1 and L_2, and W_1 and W_2 were established by two-way analysis of variance with Type A design based on the ACT subscores (See Table III). The $\underline{F}$ value for main effects of sequence is .01, which is not significant at the .05 level. This indicates that groups W_1 and W_2 are equivalent with respect to ACT subscores. The $\underline{F}$ value for interaction

TABLE III

ANALYSIS OF VARIANCE FOR ACT LEVELS AND SEQUENCE BASED ON ACT SCORES

Source of Variation	df	Sum of Squares	Mean Squares	F
Total	100	2114.5747		
Sequence	1	.0352	.0352	.01
ACT Level	2	1505.4238	752.7119	118.51*
Interaction	2	5.7217	2.8609	.45
Error	95	603.3940	6.3515	

*Significant difference at the .005 level

was .45, which is not significant at the .05 level.

According to Steel and Torre:

> If interaction is nonsignificant, it is concluded that the factors under consideration act independently of each other; the simple effects of a factor are the same for all levels of the other factors, within chance variations as measured by experimental error. . . . the simple effects are equal to the corresponding main effects and a main effect, in a factorial experiment, is estimated as accurately as if the entire experiment had been devoted to that factor. (27:199)

Therefore, the pairs of groups, H_1 and H_2, M_1 and M_2, and L_1 and L_2, have the same relationship to sequence as W_1 and W_2. Thus, these three pairs were equivalent with respect to ACT subscores.

Eight analyses of variance were computed on pre-test scores. They were used to provide additional substance to the assumption of equivalent groups resulting from random assignments of classes to treatments. These analyses are summarized in Appendix D. The reader should note that significant F values for sequence were found on Parts A and B of the pre-test. These significances were discounted since the total means of the four pre-test scores were 2.83, 2.97, 6.36, and 12.16 (See Appendix C, Table XIII). Guessing scores are 2.25, 2.25, 6.00, and 10.50; so it may safely be concluded that the scores on the pre-test were largely due to chance. The writer assumed the random assignment was sufficient to assure equivalent groups.

Testing the Hypotheses

To test the twenty hypotheses stated in Chapter I, eight two-way analyses of variance were necessary. There were two analyses for each of the three parts of the test and two for the total test. The hypotheses involving groups W_1 and W_2 (subgroups of the two sequences for whom ACT scores were available) and the High, Middle, and Low subgroups were examined using a factorial analysis with Type A design on each part of the test and the total test (See Tables IV, V, VI, and VII).

The $\underline{F}$ ratios for sequence indicate the significance or non-significance of the difference between the means for W_1 and W_2. These ratios were .00, 1.50, .02, and .19 for Parts A, B, C, and the Total Test, respectively. The .05 level of significance for $\underline{F}$ with 1 and 95 degrees of freedom is 3.94, thus indicating no significant difference between means of groups W_1 and W_2 with respect to scores on the three parts of the test and the total test. The $\underline{F}$ ratios for interaction were 1.20, 1.31, .67, and 1.13. The interaction had 2 and 95 degrees of freedom. The .05 level of significance required an $\underline{F}$ ratio of 3.09. That the computed values were considerably less than 3.09 indicated no significant difference for interaction in any of the four analyses.

TABLE IV

ANALYSIS OF VARIANCE FOR ACT LEVELS
AND SEQUENCE BASED ON POST-TEST PART A

Source of Variation	df	Sum of Squares	Mean Square	F
Total	100	204.9901		
Sequence	1	0.0097	0.0097	.00
ACT Level	2	3.9655	1.9827	.96
Interaction	2	4.9436	2.4718	1.20
Error	95	196.0714	2.0639	

TABLE V

ANALYSIS OF VARIANCE FOR ACT LEVELS
AND SEQUENCE BASED ON POST-TEST PART B

Source of Variation	df	Sum of Squares	Mean Square	F
Total	100	288.6733		
Sequence	1	3.4674	3.4674	1.50
ACT Level	2	58.8494	29.4247	12.69*
Interaction	2	6.0584	3.0292	1.31
Error	95	220.2981	2.3189	

*Significant difference at the .005 level

Since the main effect of sequence was nonsignificant, no interaction implied no significant differences between the two High, the two Middle, and the two Low groups in the two sequences with respect to any of the three subtests or to the total test.

TABLE VI

ANALYSIS OF VARIANCE FOR ACT LEVELS
AND SEQUENCE BASED ON POST-TEST PART C

Source of Variation	df	Sum of Squares	Mean Square	F
Total	100	1451.1292		
Sequence	1	0.2036	0.2036	.02
ACT Level	2	319.4319	159.7159	13.60*
Interaction	2	15.6831	7.8416	.67
Error	95	1115.8105	11.7454	

*Significant difference at the .005 level

TABLE VII

ANALYSIS OF VARIANCE FOR ACT LEVELS
AND SEQUENCE BASED ON TOTAL POST-TEST

Source of Variation	df	Sum of Squares	Mean Square	F
Total	100	3251.3076		
Sequence	1	4.9053	4.9053	.19
ACT Level	2	749.8643	374.9321	14.61*
Interaction	2	58.2578	29.1289	1.13
Error	95	2438.2803	25.6661	

*Significant difference at the .005 level

To compare E_1 and E_2 (total subjects in the respective sequences), a factorial analysis with Type B design was used for each of the four sets of post-test scores (Tables VIII, IX, X, XI). The $\underline{F}$ ratios for

TABLE VIII

ANALYSIS OF VARIANCE FOR WITH OR WITHOUT ACT SCORES AND SEQUENCE BASED ON POST-TEST PART A

Source of Variation	df	Sum of Squares	Mean Square	F
Total	146	287.7960		
Sequence	1	0.5052	0.5052	.26
With or Without ACT	1	5.6537	5.6537	2.88*
Interaction	1	1.1443	1.1443	.58
Error	143	280.4929	1.9615	

*Significant difference at the .10 level

TABLE IX

ANALYSIS OF VARIANCE FOR WITH OR WITHOUT ACT SCORES AND SEQUENCE BASED ON POST-TEST PART B

Source of Variation	df	Sum of Squares	Mean Square	F
Total	146	469.5647		
Sequence	1	0.1315	0.1315	.04
With or Without ACT	1	0.0218	0.0218	.01
Interaction	1	14.9673	14.9673	4.71*
Error	143	454.4441	3.1779	

*Significant difference at the .05 level

TABLE X

ANALYSIS OF VARIANCE FOR WITH OR WITHOUT ACT SCORES AND SEQUENCE BASED ON POST-TEST PART C

Source of Variation	df	Sum of Squares	Mean Square	F
Total	146	2418.2588		
Sequence	1	8.4434	8.4434	.51
With or Without ACT	1	0.1729	0.1729	.01
Interaction	1	26.0498	26.0498	1.56
Error	143	2383.5923	16.6685	

TABLE XI

ANALYSIS OF VARIANCE FOR WITH OR WITHOUT ACT SCORES AND SEQUENCE BASED ON TOTAL POST-TEST

Source of Variation	df	Sum of Squares	Mean Square	F
Total	146	5097.4023		
Sequence	1	15.8340	15.8340	.45
With or Without ACT	1	3.2900	3.2900	.09
Interaction	1	100.3057	100.3057	2.88*
Error	143	4977.9727	34.8110	

*Significant difference at the .10 level

the main effect of sequence were .26, .04, .51, and .45. The .05 level of significance for F with 1 and 143 degrees of freedom is 3.91. Thus, there is insufficient evidence to reject the null hypotheses.

In summary, for a hypothesis to be accepted, it should have been stated in the null form. Therefore, the six directed hypotheses, 3a, 3b, 3c, 5a, 5b, and 5c, were rejected. All the other hypotheses were accepted.

Summary

Since there were no significant differences between any compared subgroup means on the four post-test scores, the analysis indicated that neither sequence is statistically preferred to the other. They proved to be equivalent with respect to (1) basic understanding of basimal numeration systems, (2) the procedures for changing a numeral in one base to a numeral in another, (3) the understanding of the operations and the algorisms, and (4) the combination of the previous three topics. Also, the analysis relating mathematical ability and sequence showed there were no significant differences between the corresponding ability groups in the two sequences with respect to those same four areas.

In four of the five Type A analyses of post-test scores significant differences were indicated among the ability levels. These differences were not pursued since they revealed no information concerning the hypotheses. For the same reason, the significant $\underline{F}$ value for interaction on the Type B analysis of variance of post-test scores on Part B of the test was not examined.

CHAPTER IV

IMPLICATIONS

Review of the Study

As stated previously, this experiment was designed to investigate the relative effect of two sequences for introducing positional numeration systems with a base to students in Mathematics 253 at Oklahoma State University. The two sequences differ in that the procedure for changing a numeral in one base to a numeral in another base is introduced earlier in one sequence than it is in the other.

Before the instruction of the experiment, the writer constructed a set of programmed materials and a test over the content in those materials. The order in which the frames in the programmed materials were presented was changed to establish the two sequences. The relative merit of the two sequences was evaluated in terms of student understanding as demonstrated by scores on the post-test.

A pre-test was given to assure the equivalence of groups. ACT scores were used for the same purpose. The ACT mathematics subscores were also used to facilitate an analysis relating mathematics ability and sequence. The statistical analysis consisted of analysis of variance based on the post-test scores.

Limitations

This study was institutional research conducted in the Mathematics 253 classes at Oklahoma State University. Two limitations follow from the fact that it was institutional research. First, the sample was a handy sample. Thus, any generalizations from the study must be restricted to the hypothetical population determined by the sample. Second, some topics were included in the instruction because they were normal course content, rather than because they were important to the sequences.

Another limitation is the reliability of Parts A and B of the test. Any conclusion concerning these two parts should be drawn with reservations. It is obvious that they do not measure unitary concepts.

Also, in a classroom situation, one cannot be completely certain the students followed the sequence of presentation. The tendency for the student to attempt to understand the material could cause the student to deviate from the sequence by using the more familiar base ten.

Finally, the programmed materials were just one from a universe of potential programs. Possibly, another program could produce entirely different conclusions.

Relationship Between the Conclusions and the Micro-Theory

The directed hypotheses were not accepted in this research. The prediction of significant differences was made on the assumption that one sequence would develop a procedure dependent upon base ten and memorization while the other would develop a procedure based upon an understanding of the algorisms as they relate to the various bases. The acceptance of no significant difference implies that this basis for prediction was in error.

The writer observed, when returning the post-test, that some students in each of the two sequences were using procedures utilizing base ten. This could indicate the familiarity with base ten was so strong that students in both sequences used procedures relying on base ten.

The other possible explanation suggests the students used the two different procedures, but there is actually no difference in the understandings produced.

Unfortunately, this research does not indicate which of these two possibilities was actually the case.

Implications for Further Research

The writer suggests five areas for further research related to this report.

1. A study should be made that is restricted to the operations and their algorisms. This study would compare two methods for performing the algorisms. One method would rely on the students' knowledge of base ten while the other would use the procedures developed in the programmed materials for this research.

2. A study should be made that investigates sequences for introducing basimal numeration systems to elementary pupils.

3. A study should be made that compares two methods of introducing basimal numeration systems where one method relies on generalizations from base ten and the other upon grouping.

4. An observation made when returning the tests after the experiment indicated the possibility that students in both sequences relied upon base ten when performing the algorisms. Thus, a study should be made that investigates the procedures that the students used in each of these sequences.

5. The two sequences compared in this research should be re-examined using a different post-test which would be more sensitive and restricted to the differences proposed in the micro-theory.

Practical Implications

This study showed that there was no significant difference in student understanding resulting from these two sequences. Thus, either sequence can be used to introduce basimal numeration systems in Mathematics 253.

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APPENDIX A

Test Booklet Number

Positional Numeration Systems

With a Base

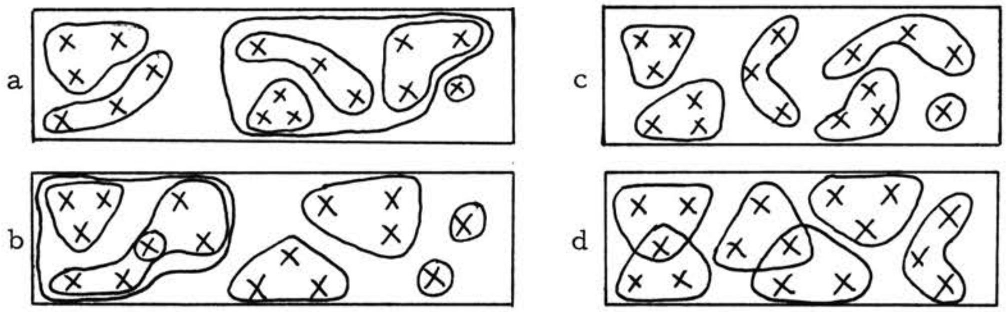
Directions: This test consists of 42 multiple choice questions. For each question, there is one and only one correct answer. Record all answers on the answer sheet provided. There is no penalty for guessing, so respond to each item.

Do not write in the test booklet. Though few of the problems require computation, use the scratch paper provided if necessary.

Time will be noted at the end of 20 minutes, 40 minutes, and at the end of the period. To complete the test you should plan to average one item per minute.

Multiple Choice Questions:

1. Which one of the following sets has been grouped by threes?



2. Which one of the following is the largest that the sum of two one-digit numerals in base seven can be?

   a. 14_{seven}
   b. 15_{seven}
   c. 16_{seven}
   d. 20_{seven}

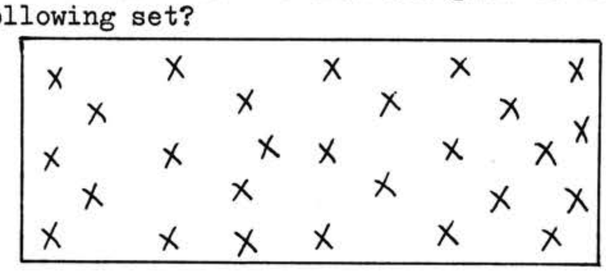
3. 300_{five} is how many times larger than 30_{five}?

   a. five
   b. ten
   c. thirty
   d. one hundred

4. What does the digit 3 in the numeral $3e0_{twelve}$ represent?

   a. three
   b. three twelves
   c. three twelve twelves
   d. 300

5. What is the base four numeral assigned to the number property of the following set?



   a. 27
   b. 63_{four}
   c. 123
   d. 123_{four}

6. What is the base four numeral that is one larger than 33_{four}?

   a. 34

   b. 34_{four}

   c. 40_{four}

   d. 100_{four}

7. What is the maximum number of nine nines that could be left over when grouping a set by nines?

   a. seven

   b. eight

   c. nine

   d. ten

8. Why is VII = 7?

   a. They are numbers representing the same numeral.

   b. They are numerals representing the same number.

   c. They are the same number.

   d. They are the same numeral.

9. The number of digits in a base four numeral is always __________ the number of digits in an equal numeral in base eight.

   a. greater than or equal to

   b. less than or equal to

   c. equal to

   d. less than

10. Which one of the following represents the largest number?

   a. 8_{twelve}

   b. 13_{five}

   c. 21_{four}

   d. 1000_{two}

11. Which one of the following is expanded notation for 356_{eight}?

    a. $[3 \cdot 10 \cdot 10 + 5 \cdot 10 + 6 \cdot 1]_{eight}$

    b. $[300 + 50 + 6]_{eight}$

    c. $3 \cdot 8 \cdot 8 + 5 \cdot 8 + 6 \cdot 1$

    d. $3 \cdot 10 \cdot 10 + 5 \cdot 10 + 6 \cdot 1$

12. Which one of the following would be the correct second step when 134 is changed to an equal numeral in base six using the procedure dependent upon the operations in base ten?

    $134 = \underline{\quad ?? \quad}$

    a. $[3 \cdot 10 \cdot 10 + 4 \cdot 10 + 2 \cdot 1]_{six}$

    b. $100 + 30 + 4$

    c. $1 \cdot 10 \cdot 10 + 3 \cdot 10 + 4 \cdot 1$

    d. $3 \cdot 6 \cdot 6 + 4 \cdot 6 + 2 \cdot 1$

13. What is the base seven numeral equal to the expression $3 \cdot 7 \cdot 7 \cdot 7 + 6 \cdot 7 \cdot 7 + 0 \cdot 7 + 4 \cdot 1$?

    a. 3604_{seven}

    b. 364_{seven}

    c. 3604

    d. 364

14. Which one of the following would be the correct next step when changing 234_{five} to an equal numeral in base three directly?

    $234_{five} = [2 \cdot 10 \cdot 10 + 3 \cdot 10 + 4 \cdot 1]_{five} = \underline{\quad ?? \quad}$

    a. $2 \cdot 5 \cdot 5 + 3 \cdot 5 + 4 \cdot 1$

    b. $[2 \cdot 5 \cdot 5 + 3 \cdot 5 + 4 \cdot 1]_{three}$

    c. $[2 \cdot 12 \cdot 12 + 10 \cdot 12 + 11 \cdot 1]_{three}$

    d. $[2 \cdot 10 \cdot 10 + 3 \cdot 10 + 4 \cdot 1]_{three}$

15. Which one of the following numerals is equal to the base five numeral 23_{five}?

a. 30_{four}

b. 31_{four}

c. 32_{four}

d. 33_{four}

16. Which one of the following restrictions on the base indicates those bases and only those bases for which this equation is true?

$$[2 \cdot 10 \cdot 10 + 3 \cdot 10 + 1 \cdot 1]_{four} = [2 \cdot 4 \cdot 4 + 3 \cdot 4 + 1 \cdot 1]_{base}$$

a. the base is less than four

b. the base is greater than four

c. the base is less than five

d. the base is greater than five

17. What is the base ten numeral equal to 222_{six}?

a. 50

b. 62

c. 74

d. 86

18. Which one of the following numerals is equal to the base ten numeral 18?

a. 26_{six}

b. 33_{five}

c. 101_{four}

d. 123_{three}

19. When adding three two-digit numerals in base four using positional notation, how many partial sums are computed?

a. one

b. two

c. three

d. four

20. What is the base for the computation on the right?

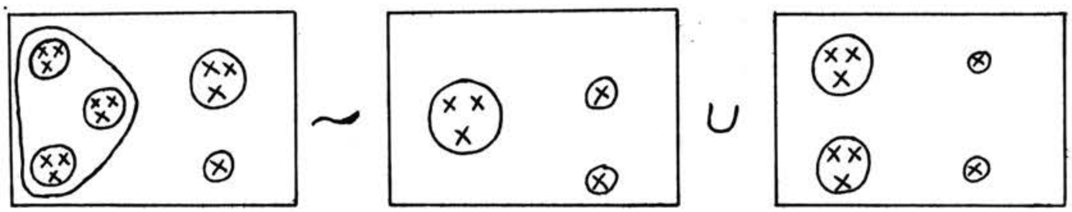
$$\left[\begin{array}{r} 324 \\ +\ 246 \\ \hline 603 \end{array}\right]_{\text{base}}$$

a. five

b. six

c. seven

d. eight

21. 

The preceding diagram illustrates which of the following sentences?

I. $[111 = 12 + 22]_{\text{three}}$ II. $[111 - 12 = 22]_{\text{three}}$

a. both I and II

b. I only

c. II only

d. neither I nor II

22. In the addition algorism to the right, where does the 1000_{four} come from?

$$\left[\begin{array}{r} 2222 \\ +\ 2222 \\ \hline 10 \\ 100 \\ 1000 \\ 10000 \\ \hline 11110 \end{array}\right]_{\text{four}}$$

a. $[2 + 2]_{\text{four}}$

b. $[20 + 20]_{\text{four}}$

c. $[200 + 200]_{\text{four}}$

d. $[2000 + 2000]_{\text{four}}$

23. What is the sum of $t5_{\text{twelve}}$ and $e6_{\text{twelve}}$?

a. $19e_{\text{twelve}}$

b. $1t1_{\text{twelve}}$

c. $21e_{\text{twelve}}$

d. 221_{twelve}

24. In the addition algorism to the right, $\square$ and $\triangle$ represent digits. Which one of the following is true?

$$\left[\begin{array}{r} 26 \\ +\underline{5\square} \\ \triangle 0 \end{array}\right]_{\text{nine}}$$

a. $\square$ is 3 and $\triangle$ is 7

b. $\square$ is 3 and $\triangle$ is 8

c. $\square$ is 4 and $\triangle$ is 7

d. $\square$ is 4 and $\triangle$ is 8

25. In the addition algorism to the right, $\square$ represents a digit. Which one of the following is $\square$?

$$\left[\begin{array}{r} 2\,\square \\ 3\,\square \\ +\underline{4\,\square} \\ 143 \end{array}\right]_{\text{six}}$$

a. 1

b. 3

c. 5

d. 7

26. In the substraction algorism to the right, $\triangle$ represents a digit. Which one of the following is $\triangle$?

$$\left[\begin{array}{r} 2\triangle 3 \\ -\underline{42} \\ 121 \end{array}\right]_{\text{five}}$$

a. 1

b. 2

c. 6

d. 11

27. In the subtraction algorism to the right, where does the 22_{six} come from?

$$\left[\begin{array}{r} 423 \\ -\underline{234} \\ 22 \\ +\underline{123} \\ 145 \end{array}\right]_{\text{six}}$$

a. $[400 = 22 + 334]_{\text{six}}$

b. $[423 = 300 + 123]_{\text{six}}$

c. $[145 = 22 + 123]_{\text{six}}$

d. $[300 = 22 + 234]_{\text{six}}$

28. What is 78_{nine} subtracted from 235_{nine}?

a. 146_{nine}

b. 156_{nine}

c. 157_{nine}

d. 246_{nine}

29. What is the answer to the subtraction problem to the right?

$$\left[\begin{array}{r} 100110 \\ -\ 10111 \\ \hline \\ \end{array}\right]_{two}$$

a. 1001_{two}

b. 1111_{two}

c. 10001_{two}

d. 11001_{two}

30. What is the base for the subtraction algorism to the right?

$$\left[\begin{array}{r} 1121 \\ -\ 657 \\ \hline 575 \end{array}\right]_{base}$$

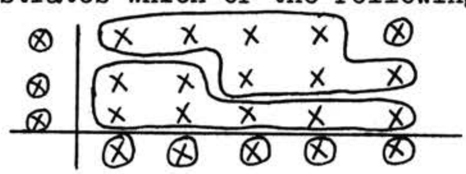
a. nine

b. ten

c. eleven

d. twelve

31. The diagram to the right illustrates which of the following sentences?



I. $[5 \cdot 3 = 21]_{seven}$ II. $[3 \cdot 5 = 21]_{seven}$

III. $[21 \div 3 = 5]_{seven}$ IV. $[21 \div 5 = 3]_{seven}$

a. I and III

b. I and IV

c. II and III

d. II and IV

32. How many partial products are computed when multiplying 346_{five} and 24_{five}?

a. one

b. two

c. three

d. six

33. What is the product of e_{twelve} and e_{twelve}?

a. $t1_{twelve}$

b. 1_{twelve}

c. 101_{twelve}

d. 121_{twelve}

34. In the multiplication algorism to the right, where does the 100_{four} come from?

$$\left[\begin{array}{r} 32 \\ \underline{\times\ 21} \\ 1200 \\ 100 \\ 30 \\ \underline{2} \\ 1332 \end{array}\right]_{four}$$

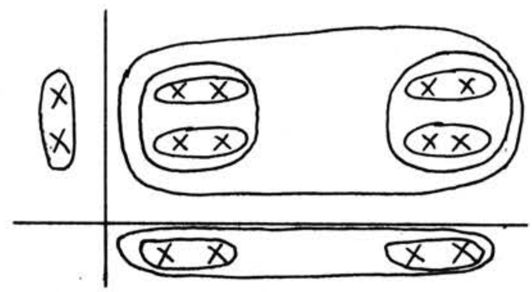
a. $[2 \cdot 2]_{four}$

b. $2 \cdot 20]_{four}$

c. $[1 \cdot 100]_{four}$

d. $[10 \cdot 10]_{four}$

35. Which one of the following numerical sentences is illustrated by the diagram on the right?



a. $100 \cdot 10 = 1000$

b. $10 \cdot 100 = 1000$

c. $[100 \cdot 10 = 1000]_{two}$

d. $[10 \cdot 100 = 1000]_{two}$

36. In the multiplication algorism to the right, Δ denotes a digit and $\square$ a numeral. What must Δ be for the product to be correct?

$$\left[\begin{array}{r} 23 \\ \underline{\times \Delta 3} \\ 124 \\ \underline{\square} \\ 1414 \end{array}\right]_{five}$$

a. 1

b. 2

c. 3

d. 4

37. What is the base for the multiplication algorism to the right?

$$\left[\begin{array}{r} 23 \\ \underline{\times\ 14} \\ 140 \\ \underline{230} \\ 410 \end{array}\right]_{base}$$

a. five

b. six

c. seven

d. eight

38. In the division algorism to the right, $\triangle$ and $\square$ represent digits. What is $\triangle$ for the algorism to be correct?

$$\left[\begin{array}{r} 16 \\ 5\overline{)\triangle\ 3} \\ \underline{\square\ 0} \\ 33 \\ \underline{33} \end{array}\right]_{\text{nine}}$$

a. 5

b. 6

c. 7

d. 8

39. What is the base for the division algorism to the right?

$$\left[\begin{array}{r} 13 \\ 23\overline{)343} \\ \underline{230} \\ 113 \\ \underline{113} \end{array}\right]_{\text{base}}$$

a. four

b. five

c. six

d. seven

40. What is 402_{eight} divided by 6_{eight}?

a. 53_{eight}

b. 54_{eight}

c. 56_{eight}

d. 63_{eight}

41. What are the correct four numbers omitted in the division algorism to the right?

$$\left[\begin{array}{rl} & \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ \\ 3 & \overline{421} \\ & \underline{220} \\ & 201 \\ & \underline{110} \\ & 41 \\ & \underline{30} \\ & 11 \\ & \underline{11} \end{array}\right]_{\text{five}}$$

a. $[30 + 20 + 10 + 2]_{\text{five}}$

b. $[40 + 20 + 10 + 2]_{\text{five}}$

c. $[40 + 30 + 10 + 2]_{\text{five}}$

d. $[40 + 30 + 20 + 2]_{\text{five}}$

42. What is the largest two-digit numeral in base six that 4_{six} divides evenly?

a. 52_{six}

b. 54_{six}

c. 56_{six}

d. 60_{six}

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Cross cut the letter corresponding to the correct response.

1.	a	b	c	d	22.	a	b	c	d
2.	a	b	c	d	23.	a	b	c	d
3.	a	b	c	d	24.	a	b	c	d
4.	a	b	c	d	25.	a	b	c	d
5.	a	b	c	d	26.	a	b	c	d
6.	a	b	c	d	27.	a	b	c	d
7.	a	b	c	d	28.	a	b	c	d
8.	a	b	c	d	29.	a	b	c	d
9.	a	b	c	d	30.	a	b	c	d
10.	a	b	c	d	31.	a	b	c	d
11.	a	b	c	d	32.	a	b	c	d
12.	a	b	c	d	33.	a	b	c	d
13.	a	b	c	d	34.	a	b	c	d
14.	a	b	c	d	35.	a	b	c	d
15.	a	b	c	d	36.	a	b	c	d
16.	a	b	c	d	37.	a	b	c	d
17.	a	b	c	d	38.	a	b	c	d
18.	a	b	c	d	39.	a	b	c	d
19.	a	b	c	d	40.	a	b	c	d
20.	a	b	c	d	41.	a	b	c	d
21.	a	b	c	d	42.	a	b	c	d

APPENDIX B

POSITIONAL NUMERATION SYSTEMS FOR THE WHOLE NUMBERS IN VARIOUS BASES

1. A number is that abstract property common to a given set and all sets

   ____________________ to that set.

---

equivalent

2. A numeral is a symbol used to denote the number property of a particular

   ________________.

---

set

3. Which of the following are numerals?

   _____ (a) 5   _____ (b) ε   _____ (c) V   _____ (d) 11111

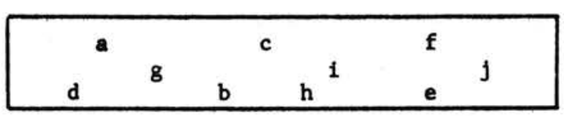
All of these are correct. They represent the number property of the set $\{a, b, c, d, e\}$ and all sets equivalent to that set. They are numerals from the Hindu-Arabic, Ionic Greek, Roman, and Egyptian systems of numeration, in that order.

Definition:

A system of numeration is a set of symbols that are used according to some scheme to assign a unique numeral to each number.

4. What is the Roman numeral assigned to the set U?

U



X

5. The Roman numerals ________________ a numeration system for the whole numbers.
   (are, are not)

Are not. There is no symbol for zero. If there were such a symbol, then the other response would be correct.

6. Which of the following could be systems of numeration for the whole numbers?

_____ (a) $\{0, 1, 2, 3, \ldots\}$

_____ (b) {0, 1, 11, 111, 1111, ~~1111~~, . . .}

_____ (c) $\{0, 1, 2, 3, \ldots, 9\}$

_____ (d) {0, I, 2, III, 4, V, 6, VII, . . .}

(a), (b), and (d) could be since a unique numeral can be assigned to each whole number. (c) could not be used since there are only a finite number of symbols.

7. Which of the following are true?

_____ (a) 2 + 4 = 6

_____ (b) II + IV = VII

_____ (c) 11 + 1111 = ~~1111~~ 1

_____ (d) If L = n{a,b}, □ = n{w,x,y,z} and ⊠ = n{a,b,w,x,y,z} then L + □ = ⊠

Responses (a), (c) and (d) are correct. Each sentence is representing a number fact for addition of whole numbers using different systems of numeration. Response (b) is incorrect since II + IV = VI.

8. Which of the following are true number sentences?

_____ (a) 6 + 4 = 4 + 6

_____ (b) VI + X = X + VI

_____ (c) ~~1111~~ 111 + 1111 = 1111 + ~~1111~~ 111

_____ (d) If n{a,b} = L and n{x} = △ then L + △ = △ + L

All responses are correct. Each sentence is representing an example of the commutative property of addition of whole numbers using different systems of numeration.

9. Which of the following are true?

_____ (a) II · IV = IX

_____ (b) 2 · 3 = 6

_____ (c) 111 · ~~1111~~ 1 = ~~1111~~ ~~1111~~ ~~1111~~ 111

_____ (d) II · III = VI

Response (a) is incorrect since II · IV = VIII. All other responses are correct, (b), (c), and (d) are examples of number facts for multiplication of whole numbers.

10. Which of the following are true?

_____ (a) II · (III · IV) = (II · III) · IV

_____ (b) VI - IV = IV - VI

_____ (c) X · I = I · X = X

_____ (d) XII ÷ (VI ÷ II) = (XII ÷ VI) ÷ II

_____ (e) XII · XVI = CXCII

(a) correct. The associative law for multiplication is true for whole numbers.
(b) incorrect. IV - VI is not a whole number. Also, this is a counter example for the commutative law for subtraction of whole numbers.
(c) correct. I is the identity for multiplication of whole numbers.
(d) incorrect. Division of whole numbers is not associative.
(e) correct. This is a multiplication fact even though there is no apparent method for performing the multiplication.

11. Generalizing from frames 7-10, the properties of the whole numbers under addition and multiplication ______________ (do, do not) depend upon the system of numeration used to denote the numbers.

---

do not

Definition:

A number system is comprised of: a set of numbers, the operations defined on that set and the properties of those operations.

12. Which of the following are number systems:

_____ (a) $\{0, 1, 2, 3, \ldots\}$

_____ (b) The whole numbers.

_____ (c) The whole numbers with addition and multiplication.

_____ (d) The whole numbers with addition and multiplication including the properties of those operations.

(a) incorrect. This response is a system of numeration for the whole numbers.
(b) incorrect. This response does not specify the operations nor their properties.
(c) incorrect. This response does not include the properties of the operations.
(d) correct. This response states the set of numbers, the operations on that set and includes the properties of the operations.

13. In a number system, the operations and the properties of the operations

_____________ depend upon the system of numeration used.
(do, do not)

---

do not

Note: We adopt the convention that the English name or number word is used to mean number, not numeral. A numeral is an abstract symbol used in a numeration system to represent the number word or number.

14. The Roman numeral V represents the number _________.

---

five

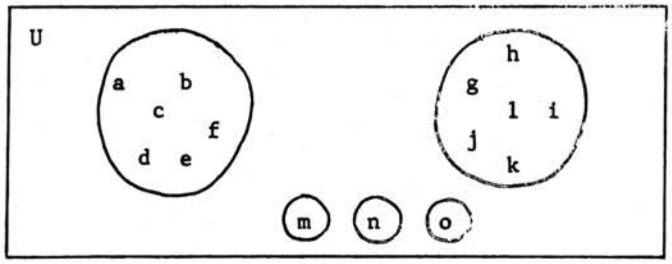
15. The number of elements in $\{a, b, c, d, e, f, g\}$ is:

_____ (a) 7 _____ (b) seven _____ (c) VII

(a) Incorrect. 7 is a numeral which represents the number of elements in the set. The number is seven.

(b) Correct. The number of elements in the set is seven. The other responses are numerals representing seven.

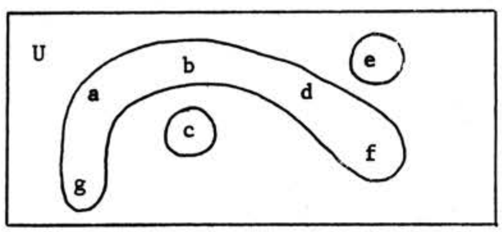
(c) Incorrect. VII is a Roman numeral which represents the number of elements in the set. The number is seven.

16. 

The universe above has been partitioned into disjoint subsets such that ___________ (how many) of these subsets have six elements each and ___________ (how many) of these subsets have one element each.

---

two; three

17. 

The set U has been partitioned into disjoint subsets such that ___________ subset has five elements and ___________ subsets have one element each.

---

one; two

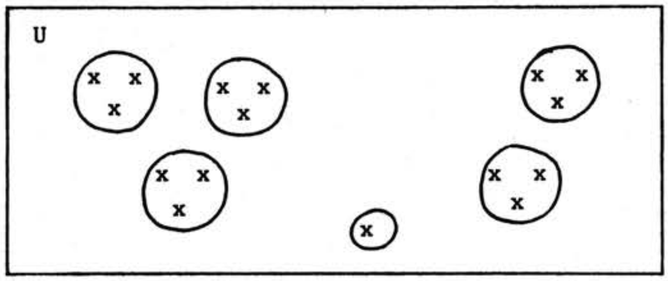
<u>Note</u>: We have used different letters to denote different elements in the Venn diagrams in frames 16 and 17. In subsequent frames the same symbol will be used to denote the elements with the interpretation that the symbol denotes distinct elements when located in distinct positions.

18. The number of elements in the set $\{x, x, x, x, x\}$ is ___________.

---

five

19.

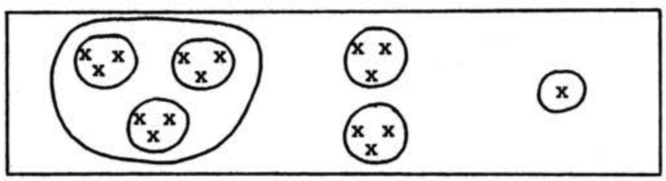


The set U has been partitioned into disjoint subsets such that five of these subsets have _______ elements each and one of these subsets has _____ element.

---

three; one

20. The set U of frame 19 can be partitioned into disjoint subsets again producing the following diagram:



The number of subsets containing three disjoint sets having three elements each is ___________, the number of disjoint subsets (disjoint from the set containing three sets haveing three elements) containing three elements each is ___________, and the number of subsets having one element each is __________.

---

one; two; one

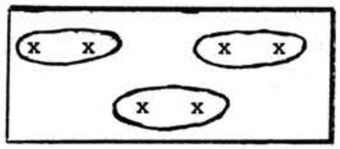
We will adopt the following terminology: two threes will mean two disjoint subsets having three elements each; three five fives will mean three disjoint subsets containing five disjoint subsets having five elements each; similar interpretations will be given to two three three threes, etc.

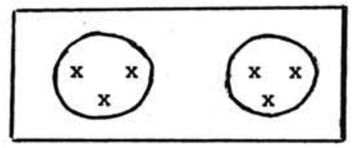
21. Four fives means ________ disjoint subsets having _________ elements each.

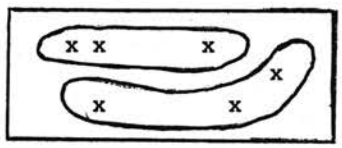
---

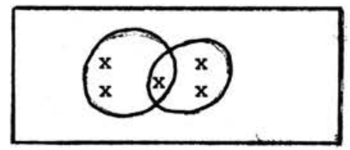
four; five

22. Which of the following sets have been partitioned into two threes?

____ (a) 

____ (b) 

____ (c) 

____ (d) 

(a) Incorrect. This set has been partitioned into three twos.

(b) Correct. The set has been partitioned into two disjoint sets having three elements each. The partitioning is not unique since response (c) is also correct.

(c) Correct. The set has been partitioned into two disjoint subsets having three elements each. The response (b) is also correct, thus the partitioning can be done in more than one way.

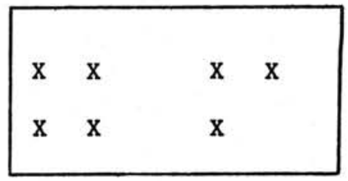
(d) Incorrect. Two threes means two disjoint subsets having three elements each. These subsets are not disjoint.

The phrase "group by (any number)" refers to a repeated partitioning of a finite set into disjoint subsets. In any partition after the first, disjoint subsets are formed whose elements are the subsets from the preceding partition. To "group by fives" we procede as follows.

Partition the universe into the maximum number of disjoint sets of fives. There may be some disjoint sets of ones left over which are disregarded for the time being. Then, partition the sets of fives into the maximum number of disjoint sets of five fives. Again, there may be some sets left over. However, they will be sets of fives rather than sets of ones. Disregard the sets of fives and the sets of ones. Continue partitioning in this manner until you cannot establish a subset containing five disjoint subsets from the previous partition. This repeated partitioning will terminate since the universe was a finite set.

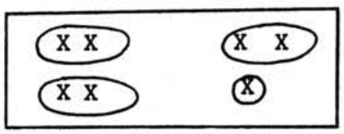
Similar meanings are assigned to the phrases "group by twos," "group by threes," etc. Frames 23 and 24 form an example of grouping by twos.

23. Consider the set U



If the set U were partitioned into the maximum number of sets of twos, the following diagram could result.

U



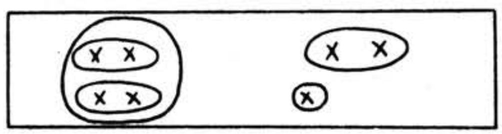
There are ________ twos and _________ ones.

---

three; one

24. If the three sets of twos were partitioned into the maximum number of sets of two sets of twos, the following diagram results.

U



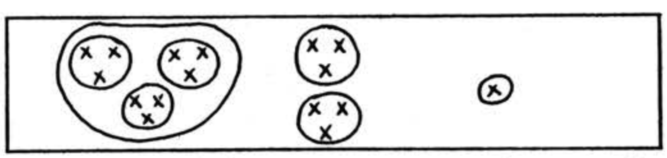
There are _________ two twos, _________ twos, and ________ ones.

Note the set cannot be partitioned again so the process terminates, so the set U has been grouped by twos.

---

one; one; one

25. The following set has been grouped by threes.

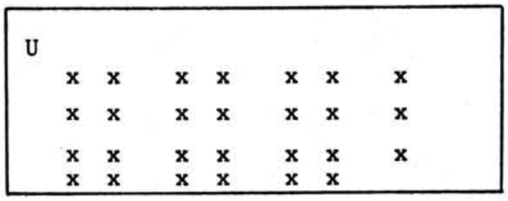


There are _________ three threes, __________ threes, and ________ ones.

---

one; two; one

26.

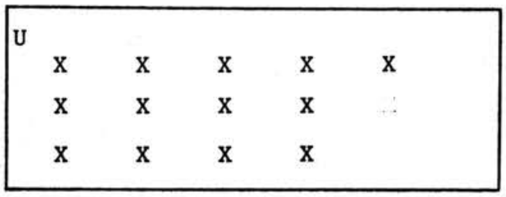


If the set U were grouped by fours, there would be _____ four fours, _____ fours and _____ ones.

---

one; two; three

27.



If the set U were grouped by twos, there would be _____ two two twos, _____ two twos, _____ twos and _____ ones.

---

one; one; zero; one

A positional numeration system with a base has three fundamental ingredients:

1) an arbitrarily selected base

2) a finite set of digits determined by the base

3) a principle of place-value

A basic interpretation for the numerals is in terms of grouping. The base is the number used in that grouping.

28. The base when grouping by fives is ________ and when grouping by ________ the base is eight.

---

five; eights

The choice of a number as a base is arbitrary; and each different number chosen gives rise to a different system of numeration. To indicate the numeration system to which a specific numeral belongs, the base will be indicated by writing the English name of the base as a subscript. Thus 12_{five} is a numeral in the base five numeration system. A numeral with no subscript will be a Hindu-Arabic or base ten numeral.

29. The numeral 43_{six} is a numeral in the base _______ numeration system.

---

six

30. In the set of numerals $\{11_{two}, 12_{three}, 41_{nine}, 101\}$, the base three numeral is:

_____ (a) 11_{two} _____ (b) 12_{three} _____ (c) 41_{nine} _____ (d) 101

(a) Incorrect. 11_{two} is a base two numeral.

(b) Correct. 12_{three} is a base three numeral as indicated by the subscript "three."

(c) Incorrect. 41_{nine} is a base nine numeral.

(d) Incorrect. 101 is a base ten numeral since no subscript appears.

The set of digits is a set of one character symbols associated with a particular base. A unique digit is assigned to each of the possible number of sets of ones that could be left over when grouping by the base. The digits are used with the written name for the base to form the numerals in that system of numeration.

31. The minimum number of sets of ones that could be left over when grouping by sevens is ___________.

---

zero

32. The maximum number of sets of ones that could be left over when grouping by sevens is ________.

---

six

33. Frames 31 and 32 indicate that in base seven we need digits assigned to each of the numbers zero, _____, _____, _____, _____, _____, six.

---

one, two, three, four, five

34. In base five, the maximum number of sets of ones left over is ________.

---

four

35. Frame 34 indicates that in base five we need digits associated with each of the numbers zero through ________ inclusive.

---

four

The selection of the symbols for the digits is arbitrary. We will utilize the students familiarity with base ten by selecting the following set of symbols: {0, 1, 2, 3, . . . , 9, t, e} to denote the corresponding numbers in the set - {zero, one, two, three, . . . , nine, ten, eleven}. So, the set of digits for a base between two and twelve inclusively will be a subset of {0, 1, 2, 3, . . . , 9, t, e}.

36. In base seven, we need digits to represent numbers zero, one, two, . . . six; therefore, the set of digits would be {0, 1, 2, . . . , ___}.

6

37. The finite set of digits for base three would be {____, ____, ____}.

0; 1; 2

38. The set of digits for base five is {______________________}.

0, 1, 2, 3, 4.

39. The set of digits {0, 1, 2, 3} would be used in a base ________ system.

---

four

40. The set of digits for a base two system would be {_____}.

0, 1

41. Which of the following sets of digits could be used in base twelve?

____ (a) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

____ (b) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t\}$

____ (c) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e\}$

---

41 (a) Incorrect. In base twelve, digits need to be assigned to the numbers zero, one, two, . . . , nine, ten, eleven. This response has no symbols for ten or eleven.

41 (b) Incorrect. In base twelve, digits need to be assigned to the numbers zero, one, two, . . . , nine, ten eleven. This response has no symbol for eleven.

41 (c) Correct. A unique symbol is assigned to each of the numbers zero, one, two, . . . , nine, ten, eleven.

42. In general, a numeral system base b would employ the finite set of digits $\{0, 1, 2, . . ., \_\_\_\_\}$.

---

b - 1

The principle of place-value is the unifying notational scheme that enables us to assign a unique numeral to every whole number. After a set has been grouped, a numeral is constructed (composed of digits and the English name of the base) with the position of the digit in the numeral denoting the type of subset represented by the digit. For example: 132_{four} represents the number property of a set that can be grouped by fours into one four four, three fours, and two ones. Thus, considering a numeral in an arbitrary base, the first digit from the right represents the number of ones, the second digit from the right represents the number of bases, the third digit from the right represents the number of base bases, etc.

43. The numeral 32 represents the number of a set which can be grouped by tens into ____ tens and ____ ones.

---

three; two

44. The numeral 32_{five} represents the number of a set that can be grouped by fives into ____ fives and ____ ones.

---

three; two

45. Thus, 56_{nine} represents _____ nines and _____ ones.

---

five; six

46. The base three numeral which represents two threes and one one would be:

_____ (a) 21 _____ (b) 31_{three} _____ (c) 21_{three}

---

46 (a) Incorrect. 21 is a base ten numeral and represents two tens and one one.

46 (b) Incorrect. The digit 3 is not used in base three.

46 (c) Correct. 21_{three} represents two threes and one one.

47. In the numeral 234_{nine} the 2 represents two _____ _____ and the 3 represents three _____.

---

nine nines; nines

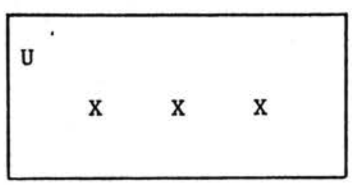
48. In the numeral $t24_{\text{twelve}}$ the t represents _______________.

---

ten twelve twelves

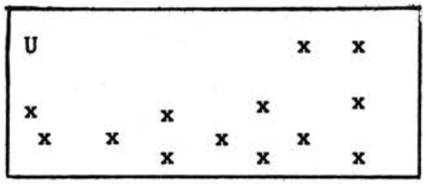
To find the numeral assigned to the number of a set the following procedure is used. First a base is arbitrarily selected. Next the set is grouped by bases. Finally the numeral is assigned that indicates the base and the result of that grouping.

49.



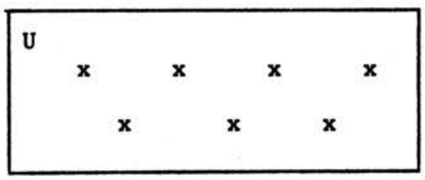
The base five numeral for the number of set U is __________.

3_{five}

50. 

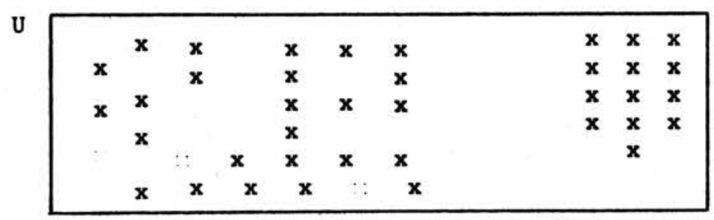
The base five numeral for the number of set U is ____________.

23_{five}

51. 

The base two numeral for the number of set U is ____________.

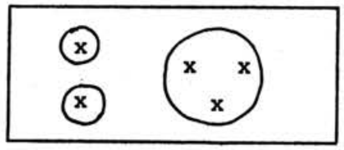
111_{two}

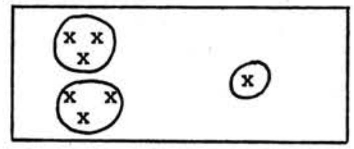
52. 

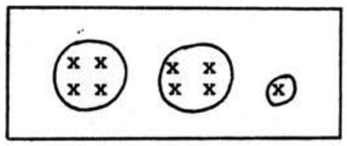
The base five numeral for the number of the set U is ____________.

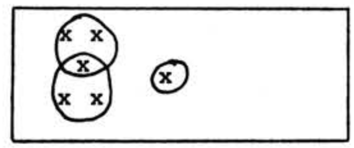
123_{five}

53. The numeral 21_{three} denotes the number of which of these sets:

_____ (a) 

_____ (b) 

_____ (c) 

_____ (d) 

---

53 (a) Incorrect. This set has been partitioned into one three and two ones and its number would be represented by the numeral 12_{three}.

53 (b) Correct. This set has been partitioned into two threes and one one and its number would be represented by 21_{three}.

53 (c) Incorrect. This set has been grouped by fours into two fours and one one and its number would be represented by 21_{four}.

53 (d) Incorrect. The subsets are not disjoint so this set has not been grouped by threes. It can be grouped by threes into two threes and zero ones and its number would be represented by 20_{three}.

Definition:

Two numerals are equal if and only if they represent the same number.

54. Which of the following are true?

_____ (a) VII = 7

_____ (b) $2 + 3 = 5$

_____ (c) $2 \cdot 3 = 5$

_____ (d) $2 \cdot 6 = 3 \cdot 4$

---

54 (a) Correct. VII and 7 each represent the number seven and by the definition of equal numerals, they are equal.

54 (b) Correct. $2 + 3$ represents the number five as does the numeral 5. Therefore, they are equal by the definition of equal numerals,

54 (c) $2 \cdot 3$ represents the number six while 5 represents the number five. Therefore, they are not equal by the definition of equal numerals.

54 (d) Correct. $2 \cdot 6$ represents the number twelve. $3 \cdot 4$ represents the number twelve. Therefore, they are equal by the definition of equal numerals.

CHANGING A NUMERAL IN ONE BASE TO AN EQUAL NUMERAL IN ANOTHER BASE USING BASE TEN

1. Which of the following are true?

    _____ (a) $2_{three} = 2_{five}$

    _____ (b) $10_{four} = 4_{five}$

    _____ (c) $12_{six} = 7_{twelve}$

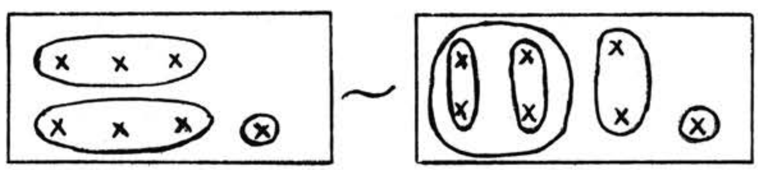
---

1 (a) Correct. 2_{three} represents two ones and 2_{five} represents two ones. So, they both represent the number two and are equal.

1 (b) Correct. 10_{four} represents one four and zero ones while 4_{five} represents four ones. So, they both represent the number four and are equal.

1 (c) Incorrect. 12_{six} represents one six and two ones or the number eight while 7_{twelve} represents seven ones or the number seven. They do not represent the same number and are not equal.

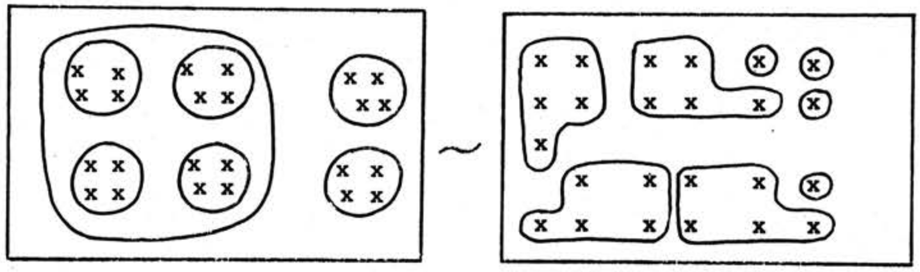
2.



The diagram above shows two equivalent sets one grouped by threes, the other by twos. The diagram illustrates the numerical equation:

$$\underline{\qquad\qquad}_{three} = \underline{\qquad\qquad}_{two}$$

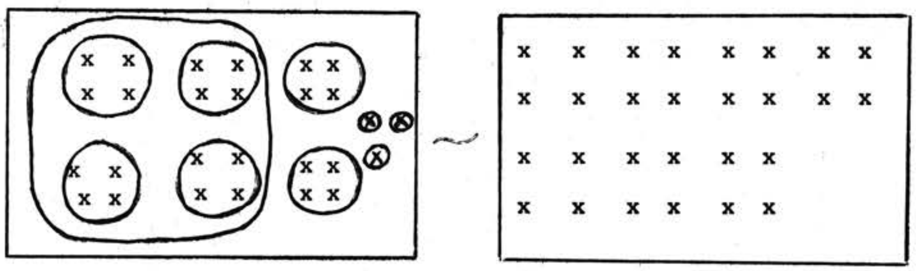
$21_{three} = 111_{two}$

3. 

The numerical equation illustrated by the diagram above is ______________.

$120_{four} = 44_{five}$

Frames 2 and 3 indicate that a numeral in one base could be changed to an equal numeral in another by using Venn diagrams and regrouping.

4. 

If the set on the right were grouped by sixes, it would show that $123_{four} =$

$\_\_\_\_\_\__{six}$.

43

The objective of this section is to develop a numerical procedure for changing a numeral in one base to an equal numeral in another base utilizing base ten. This procedure depends on:

1) place-value interpretation of numerals

2) the operations in base ten

The place-value interpretation is developed in frames 5-14 but the necessary skills with regard to the operations in base ten are assumed.

5. The numeral 21_{three} represents the number of elements in a set that can be grouped by threes into _____ threes and _____ one.

---

two; one

6. In base ten, the expression $2 \cdot 3 + 1 \cdot 1$ represents the number of elements in a set that could be grouped by threes into _____ threes and _____ one.

---

two; one   Note: $2 \cdot 3 + 1 \cdot 1$ is a base ten numeral since no base is indicated by a subscript number.

7. Frames 5 and 6 indicate that $21_{three} = 2 \cdot 3 + 1 \cdot 1$ since two numerals are equal if and only if they represent the same _______________.

---

number

8. 342_{five} represents the number of elements in a set that could be grouped by threes into _____ five fives, _____ fives and _____ ones.

---

three; four; two

9. In base ten, the expression $3 \cdot 5 \cdot 5 + 4 \cdot 5 + 2 \cdot 1$ represents the number of elements in a set that could be grouped by fives into _____ five fives, _____ fives and _____ ones.

---

three; four; two

10. Frames 8 and 9 indicate

$$342_{five} = \_\_\_\_ \cdot 5 \cdot 5 + \_\_\_\_ \cdot 5 + \_\_\_\_ \cdot 1$$

3; 4; 2

The expression $3 \cdot 5 \cdot 5 + 4 \cdot 5 + 2 \cdot 1$ is the place-value interpretation for the numeral 342_{five}.

11. By the place-value interpretation

$$283_{nine} = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_.$$

$2 \cdot 9 \cdot 9 + 8 \cdot 9 + 3 \cdot 1$

12. By the place-value interpretation

$t27_{\text{eleven}}$ = ______________________.

$10 \cdot 11 \cdot 11 + 2 \cdot 11 + 7 \cdot 1$

13. What numeral in base six has the following place-value interpretation?

$3 \cdot 6 \cdot 6 + 2 \cdot 6 + 5 \cdot 1$ = ______________________.

325_{six}

14. $3 \cdot 11 \cdot 11 \cdot 11 + 8 \cdot 11 \cdot 11 + 10 \cdot 11 + 2 \cdot 1$ = ____________$_{\text{eleven}}$.

38t2

By using place-value interpretation a numeral in any base can be changed to an equal numeral in base ten and any numeral in base ten can be changed to an equal numeral in any base. Using a combination of those two procedures, a numeral in one base can be changed to an equal numeral in another base. The following frames illustrate this procedure.

15. 122_{three} = ____ $\cdot 3 \cdot 3$ + ____ $\cdot 3$ + ____ $\cdot 1$ = ________

1; 2; 2; 17

16. 17 = ____ $\cdot 5$ + ____ $\cdot 1$ = ____$_{\text{five}}$

3; 2; 32

17. From frames 15 and 16

122_{three} = ____$_{\text{five}}$

32

18. 35_{seven} = $3 \cdot$ ____ + $5 \cdot$ ____ = ____

7; 1; 26

19. $26 = \_\_\_\_ \cdot 4 \cdot 4 + \_\_\_\_ \cdot 4 + \_\_\_\_ \cdot 1 = \_\_\_\__{four}$

1; 2; 2; 122

20. Frames 18 and 19 indicate $35_{seven} = \_\_\_\__{four}$

122

21. $321_{six} = \underset{\text{(base ten)}}{\_\_\_\_\_\_\_\_}$

---

121 since $321_{six} = 3 \cdot 6 \cdot 6 + 2 \cdot 6 + 1 \cdot 1$

22. $121 = \_\_\_\_\_\__{nine}$

---

144 since $121 = 1 \cdot 9 \cdot 9 + 4 \cdot 9 + 4 \cdot 1$

23. Frames 20 and 21 indicate $321_{six} = \_\_\_\_$ nine.

144

24. 221_{five} written as a base four numeral is: ________.

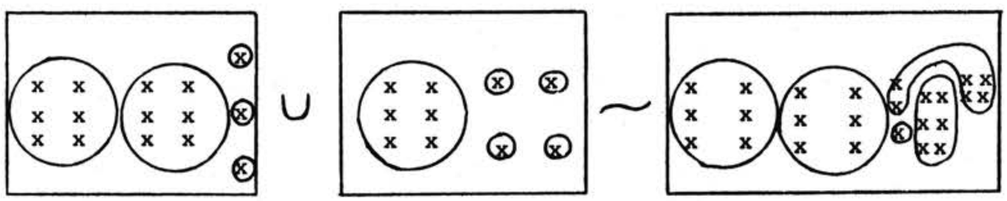
331_{four}

OPERATIONS AND ALGORISMS FOR THE OPERATIONS IN VARIOUS BASES

1. Recall the definition of addition of whole numbers in terms of disjoint sets. If $A \cap B = \emptyset$, then $nA + nB = n(\_\_\_\_\_)$ where nA, nB and $n(A \cup B)$ are the number properties of the sets A, B, and $A \cup B$ respectively.

$A \cup B$

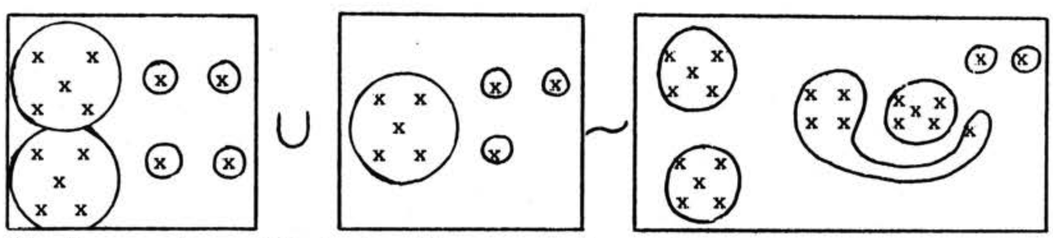
In frames 2-5 the diagrammatical interpretation of addition of whole numbers is demonstrated.

2. 

The diagram above shows the union of two disjoint sets that is equivalent to a third set. Each of the sets has been grouped by sixes. The numerical equation illustrated by the diagram is:

$$23_{six} + 14_{six} = \_\_\_\_\_\__{six}$$

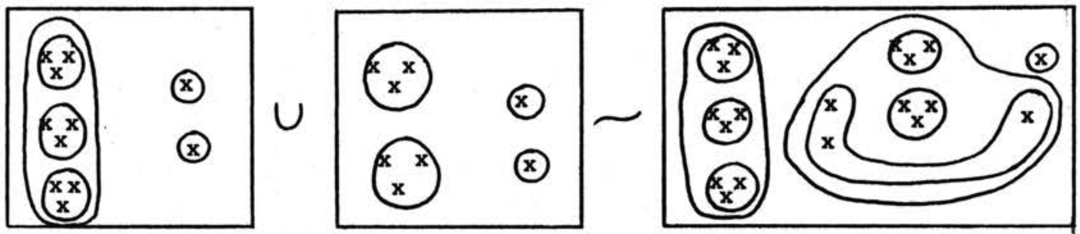
41

3. 

The diagram above illustrates the numerical equation:

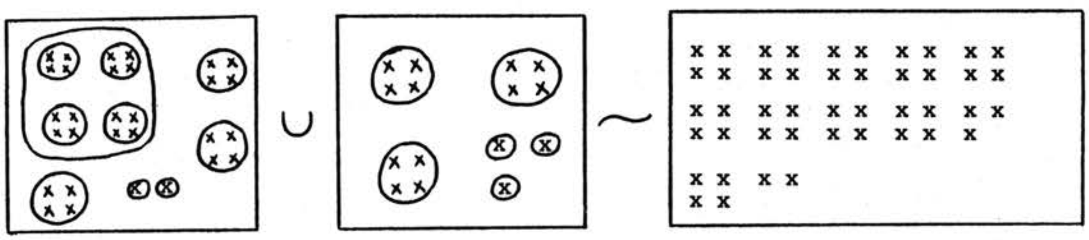
$$\_\_\_\_\_\__{five} + \_\_\_\_\_\__{five} = \_\_\_\_\_\__{five}$$

24; 13; 42

4. 

The diagram above illustrates the numerical equation ____________________.

$102_{three} + 22_{three} = 201_{three}$

5. 

If the set on the right were grouped by fours, it would show that

$132_{four} + 33_{four} =$ ________.

231_{four}

In a system of numeration, the numerical methods for performing the operations have varying degrees of difficulty. These numerical methods are called algorisms. Positional numeration systems with a base have algorisms for the operations of addition, multiplication, subtraction, and division that are relatively simple. The basic objective of this program is to introduce those algorisms.

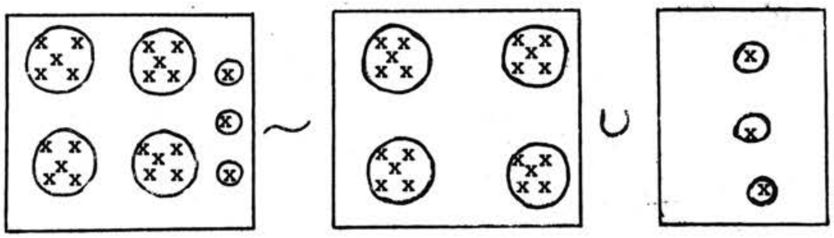
The algorism for addition is highly dependent upon:

1) positional notation

2) addition of the digits in that base.

Positional notation is introduced in frames 6-14 and addition of the digits is explained in frames 15-33.

6.

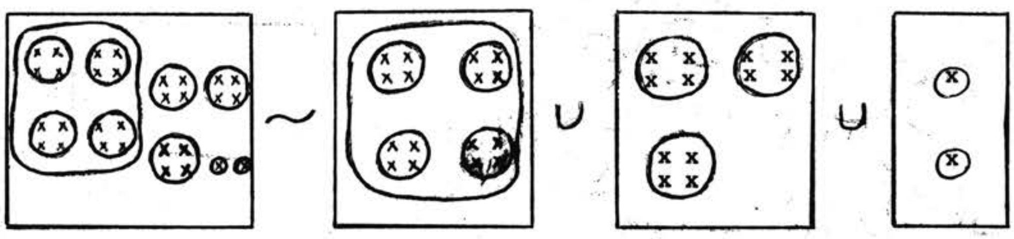


The diagram above shows a set equivalent to the union of two disjoint sets. Each of the sets has been grouped by fives. The numerical equation illustrated by the diagram is:

$$43_{five} = \_\_\_\__{five} + \_\_\_\__{five}$$

40; 3

7.



The diagram above shows a set equivalent to the union of three disjoint sets. Each set has been grouped by fours. The numerical equation illustrated by the diagram is:

$$132_{four} = \_\_\_\__{four} + \_\_\_\__{four} + \_\_\_\__{four}$$

100; 30; 2

The expression $100_{four} + 30_{four} + 2_{four}$ is the positional notation for the numeral 132_{four}

8. 269_{eleven} in positional notation would be:

$$\_\_\_\__{eleven} + \_\_\_\__{eleven} + \_\_\_\__{eleven}$$

200; 60; 9

To save time and effort the notation $[3 + 5 + 6]_{seven}$ will be used to denote the expression $3_{seven} + 5_{seven} + 6_{seven}$.

That is, the brackets with the base as a subscript indicate that each numeral within the brackets is a numeral in that base.

9. Which of the following are true:

a. $[(2 + 3) + 5]_{six} = (2_{six} + 3_{six}) + 5_{six}$

b. $[200 + 30 + 4]_{six} = 200_{six} + 30_{six} + 4_{six}$

c. $\left[\begin{array}{r} 23 \\ \underline{+14} \end{array}\right]_{six}$ means $\begin{array}{r} 23_{six} \\ \underline{+14_{six}} \end{array}$

d. $[35 \div 14]_{six}$ means $35 \div 14_{six}$

---

9 (a), 9 (b), and 9 (c) are all correct responses.

According to the notational convention, the brackets with the base as a subscript indicates that each numeral within the brackets is a numeral in that base.

9 (d) Incorrect. The correct statement should be:

$$[35 \div 14]_{six} \text{ means } 35_{six} \div 14_{six}$$

10. 436_{seven} in positional notation would be:

$$[\_\_\_ + \_\_\_ + \_\_\_]_{seven}$$

400; 30; 6

11. 110_{two} in positional notation would be:

$$[\_\_\_ + \_\_\_ + \_\_\_]_{two}$$

100; 10; 0

12. 302 in positional notation would be ______________________________.

---

300 + 0 + 2 or 300 + 2

13. 212_{three} in positional notation would be ______________________________.

$[200 + 10 + 2]_{three}$ or $200_{three} + 10_{three} + 2_{three}$

14. $[2000 + 30 + 1]_{five}$ is positional notation for the numeral ________________.

3031_{five}

15. $8 + 4 = 8 + (2 + 2) = (8 + 2) + 2 = 10 + 2 =$ ____.

12

16. $[3 + 4]_{five} = [3 + (2 + 2)]_{five} = [(3 + 2) + 2]_{five} = [10 + 2]_{five} =$ $\_\_\_\__{five}$.

12

17. $[5 + 5]_{seven} = [5 + (2 + 3)]_{seven} = [(5 + 2) + 3]_{seven} = [10 + 3]_{seven} =$ $\_\_\_\__{seven}$.

13

18. $[6 + 3]_{eight} = [(6 + \_\_\_\_) + 1]_{eight} = [10 + 1]_{eight} = 11_{eight}$.

2

19. $[2 + 3]_{four} = [(2 + \_\_\_\_) + \_\_\_\_]_{four} = 11_{four}$.

2; 1

20. $[7 + 5]_{nine} = [10 + \_\_\_]_{nine} = 13_{nine}$.

3

21. $[5 + 6]_{eight} = [10 + \_\_\_]_{eight} = \_\_\__{eight}$.

3; 13

22. $[9 + 5]_{twelve} = [10 + \_\_\_]_{twelve} = \_\_\__{twelve}$.

2; 12

23. $[6 + 3]_{nine} = [10 + \_\_\_]_{nine} = \_\_\__{nine}$.

0; 10

24. $[t + e]_{twelve} = \_\_\__{twelve}$.

19

25. $[4 + 3]_{five} = \_\_\__{five}$.

12

26. $[20 + 20]_{three} = [(20 + 10) + 10]_{three} = [\_\_\_ + 10]_{three} = \_\_\__{three}$.

100; 110

27. $[20 + 40]_{five} = [(20 + \_\_\_) + 10]_{five} = [\_\_\_ + 10]_{five} = \_\_\__{five}$.

30; 100; 110

28. $[50 + 70]_{nine} = [100 + \_\_\_]_{nine} = \_\_\__{nine}$

30; 130

29. $[300 + 400]_{six} = [300 + \_\_\_\_ + 100]_{six} = [\_\_\_\_ + 100]_{six} = \_\_\_\__{six}$.

300; 1000; 1100

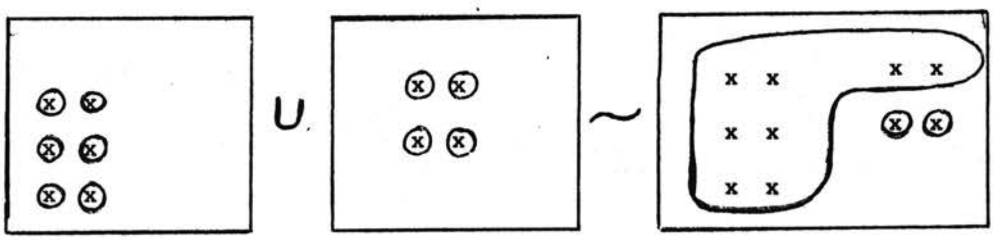
30. $[800 + 900]_{eleven} = [1000 + \_\_\_\_]_{eleven} = \_\_\_\__{eleven}$.

600; 1600

Notice that in frames 26-30 the answers could have been computed by using addition of the digits and assigning the appropriate place value to that sum.

Frames 31-33 give a diagrammatical interpretation for those addition problems.

31.

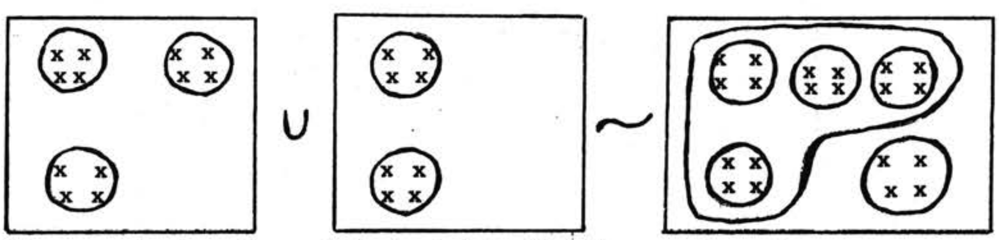


The numerical sentence illustrated by the diagram above where the sets were grouped by eights is:

$$[\_\_\_\_ + \_\_\_\_]_{eight} = \_\_\_\__{eight}$$

6; 4; 12

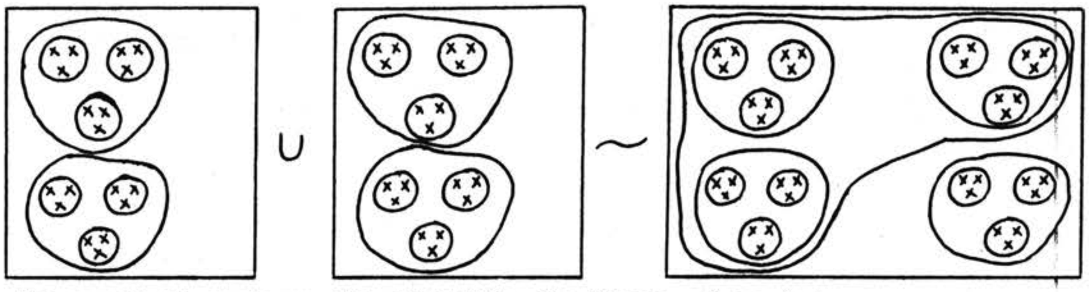
32.



The numerical sentence illustrated by the diagram above is:

$$[\_\_\_\_ + \_\_\_\_]_{four} = \_\_\_\__{four}$$

30; 20; 110

33. 

The numerical sentence illustrated by the diagram above is:

$[\_\_\_\_ + \_\_\_\_]_{three} = \_\_\_\__{three}$

200; 200; 1100

Using positional notation and the basic properties, the computations involved in finding 13 + 16 as a single numeral is illustrated in the ensuing seven frames.

34. 13 + 16 = (10 + 3) + (10 + _____).

6

35. (10 + 3) + (10 + 6) = ((10 + 3) + _____) + 6.

10

36. ((10 + 3) + 10) + 6 = (10 + (10 + 3)) + _____.

6

37. (10 + (10 + 3)) + 6 = ((10 + _____) + 3) + 6.

10

38. ((10 + 10) + 3) + 6 = (10 + 10) + (_____ + 6).

3

39. (10 + 10) + (3 + 6) = _____ + 9.

20

40. 20 + 9 = _____.

29

For the sake of brevity the computations involved in addition will be shortened by eliminating the steps involving the commutative and associative properties and retaining those which point out place-value. The following four frames illustrate the procedure to be followed.

41. 13 + 16 = (_____ + 3) + (_____ + 6).

10; 10

42. (10 + 3) + (10 + 6) = (10 + _____) + (3 + 6).

10

43. (10 + 10) + (3 + 6) = 20 + _____.

9

44. 20 + 9 = _____.

29

45. $[12 + 13]_{seven} = [(10 + 10) + (2 + \_\_\_\_\_)]_{seven} = [\_\_\_\_\_ + 5]_{seven} = \_\_\_\_\__{seven}$.

3; 20; 25

46. $[21 + 14]_{eight} = [(20 + 1) + (\_\_\_\_\_ + 4)]_{eight} = [(\_\_\_\_\_ + 10) + (1 + 4)]_{eight} = \_\_\_\_\__{eight}$.

10; 20; 35

47. $[45 + 31]_{seven} = [(40 + 5) + (30 + 1)]_{seven} = [(40 + 30) + (5 + 1)]_{seven}$ is equal to:

_____ (a) 76_{seven} _____ (b) 106_{seven}

---

47 (a) Incorrect. Four and three is seven, but in base seven $[4 + 3]_{seven} = 10_{seven}$. 7 is not a member of the set of digits base seven.

47 (b) Correct. 10_{seven} sevens in positional notation would be 100_{seven} and thus 106_{seven} is correct.

48. $[24 + 13]_{five} = [20 + 10 + 4 + 3]_{five} = [30 + 12]_{five} = [30 + \_\_\_\_ + 2]_{five} = 42_{five}$.

10

49. $[54 + 32]_{eight} = [50 + 30 + 4 + 2]_{eight} = [\_\_\_\_ + 6]_{eight} = 106_{eight}$.

100

50. $[102 + 23]_{five} = [100 + 20 + 2 + 3]_{five} = [100 + 20 + 10]_{five} = [100 + \_\_\_\_]_{five} = 130_{five}$.

30

51. $[32 + 41]_{seven} = [100 + 3]_{seven} = \_\_\_\__{seven}$.

103

52. $[123 + 24]_{six} = [100 + 20 + 20 + 3 + 4]_{six} = [100 + 40 + \_\_\_\_]_{six} = [100 + 40 + 10 + 1]_{six} = [100 + 50 + 1]_{six} = 151_{six}$.

11

53. $\begin{bmatrix} 235 \\ +152 \end{bmatrix}_{\text{eight}}$ may be written as $\begin{bmatrix} 200 + 30 + 5 \\ +100 + 50 + 2 \end{bmatrix}_{\text{eight}}$ and by adding

columns $\begin{bmatrix} 200 & + & 30 & + & 5 \\ 100 & + & 50 & + & 2 \\ \hline 300 & + & \_\_\_ & + & 7 \end{bmatrix}_{\text{eight}} = 407_{\text{eight}}.$

100

54. $\begin{bmatrix} 235 \\ +162 \end{bmatrix}_{\text{eight}} = \begin{bmatrix} 200 & + & 30 & + & 5 \\ +100 & + & 60 & + & 2 \end{bmatrix}_{\text{eight}}$ and adding gives

$[300 + \_\_\_\_ + 7]_{\text{eight}} = 417_{\text{eight}}.$

110

55. $\begin{bmatrix} 234 \\ +103 \end{bmatrix}_{\text{five}} = \begin{bmatrix} 200 & + & 30 & + & 4 \\ +100 & & & + & 3 \end{bmatrix}_{\text{five}}$ and adding gives

$[300 + 30 + \_\_\_\_]_{\text{five}} = 342_{\text{five}}.$

12

56. $\begin{bmatrix} 430 \\ +326 \end{bmatrix}_{\text{seven}} = \begin{bmatrix} 400 & + & 30 & & \\ +300 & + & 20 & + & 6 \end{bmatrix}_{\text{seven}}$ and adding gives

$[\_\_\_\_ + 50 + 6]_{\text{seven}} = 1056_{\text{seven}}.$

1000

In computing sums it is not necessary to write each numeral in positional notation. For, if one thinks in terms of positional notation, the partial sums can be obtained directly from the given numerals.

57. Thus: $\left[\begin{array}{r} 372 \\ +\ 456 \\ \hline 700 + 130 + 8 = \underline{\qquad} \end{array}\right]_{nine}$

838

58. $\left[\begin{array}{r} 212 \\ +\ 212 \\ \hline \underline{\qquad} + 20 + 10 \end{array}\right]_{four} = 1030_{four}$.

1000

59. $\left[\begin{array}{r} 537 \\ +\ 543 \\ \hline 1200 + 70 + 12 \end{array}\right]_{eight}$ equals

____ (a) 1282_{eight} ____ (b) 1302_{eight} ____ (c) 1202_{eight}

---

59 (a) Incorrect. $[70 + 10]_{eight} = 100_{eight}$ not 80_{eight} since 8 is not a member of the finite set of digits base eight.

59 (b) Correct. $[70 + 10]_{eight} = 100_{eight}$.

59 (c) Incorrect. $[70 + 10]_{eight} = 100_{eight}$ but

$[1200 + 100]_{eight} = 1300_{eight}$ not 1200_{eight}.

In addition, the partial sums may be written vertically one below another without loss of generality.

60. Thus:

$$\left[\begin{array}{r} 72 \\ +\ 56 \\ \hline 130 \\ +\ 8 \\ \hline \\ \underline{\quad\quad} \end{array}\right]_{nine}$$

138_{nine}

61.

$$\left[\begin{array}{r} 223 \\ +\ 403 \\ \hline 1100 \\ +\ 20 \\ +\ \underline{\quad} \\ \hline 1131 \end{array}\right]_{five}$$

11

62.

$$\begin{array}{r} 7459 \\ +\ 836 \\ \hline 7000 \\ +\ 1200 \\ +\ 80 \\ +\ 15 \\ \hline \\ \underline{\quad\quad} \end{array}$$

8295

Since addition is commutative and associative, the vertical order of listing the partial sums is reversible, thus allowing addition from right to left instead of from left to right.

63.

$$\left[\begin{array}{r} 1010 \\ +\ 101 \\ \hline 1 \\ +\ 10 \\ +\ 100 \\ +\ 1000 \\ \hline \_\_\_\_ \end{array}\right]_{two}$$

1111_{two}

64.

$$\left[\begin{array}{r} 525 \\ +\ 2041 \\ \hline 10 \\ +\ 100 \\ +\ 500 \\ +\ 2000 \\ \hline \_\_\_\_ \end{array}\right]_{six}$$

3010

65.

$$\begin{array}{r} 789 \\ +\ 9416 \\ \hline 15 \\ +\ 90 \\ +\ 1100 \\ +\ 9000 \\ \hline \_\_\_\_ \end{array}$$

10205

The standard algorism or computational procedure for finding the sum of two numerals does not require the recording the partial sums. The sum may be expressed directly provided the place-value of each partial sum is carefully treated.

66. $$\left[\begin{array}{r} 359 \\ +\ 136 \\ \hline \underline{\qquad} \end{array}\right]_{\text{six}}$$

494

67. $$\left[\begin{array}{r} 1023 \\ +\ 665 \\ \hline \underline{\qquad} \end{array}\right]_{\text{seven}}$$

2021

68. $$\left[\begin{array}{r} 1101 \\ +\ 111 \\ \hline \underline{\qquad} \end{array}\right]_{\text{two}}$$

10100

69. **Recall the definition of subtraction in the set of whole numbers. Let a, b, c denote whole numbers. Then a - b = c if and only if a = ______.**

---

b + c

This definition indicates the answer to a subtraction problem is found from a previously known addition fact.

70. Since $[5 = 2 + 3]_{seven}$, $[5 - 2 = \_\_\_\_]_{seven}$

3

71. Since $[10 = 3 + 5]_{eight}$, $[10 - 3 = \_\_\_\_]_{eight}$

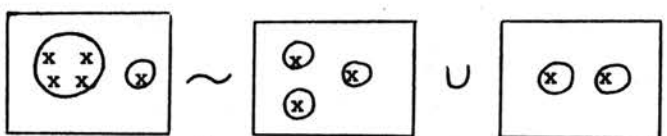
5

72. Since $[645 = 256 + 378]_{nine}$, $[645 - 256 = \_\_\_\_]_{nine}$

378

The necessary addition fact can be found by using a diagram as illustrated in the next two frames.

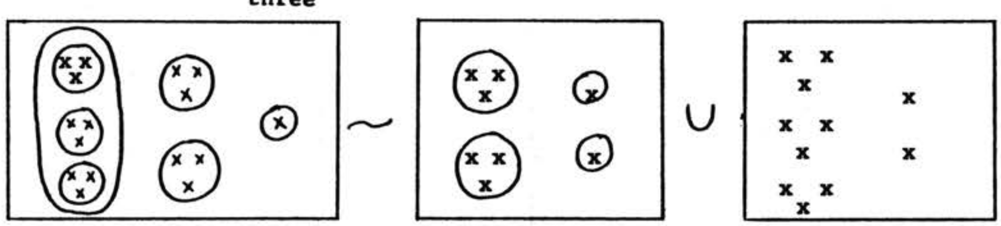
73. To find $[11 - 3]_{four}$:



So $[11 = 3 + \_\_\_\_]_{four}$ or $[11 - 3 = \_\_\_\_]_{four}$

2; 2

74. To find $[121 - 22]_{three}$:



If the set on the right were grouped by threes

$[121 = 12 + \_\_\_\_]_{three}$ or $[121 - 12 = \_\_\_\_]_{three}$

102; 102

75. Since $[10 = 4 + \_\_\_\_]_{twelve}$, $[10 - 4 = \_\_\_\_]_{twelve}$

8; 8

76. Since $[13 = 10 + 3 = (5 + 4) + 3 = 5 + (4 + 3)]_{nine}$ Thus,

$[13 - 5 = \_\_\_\_]_{nine}$

---

7 or 4 + 3

77. To find $[13 - 4]_{seven}$:

$[13 = 10 + 3 = (4 + 3) + 3 = 4 + (3 + 3) = 4 + \_\_\_\_]_{seven}$

Thus, $[13 - 4 = \_\_\_\_]_{seven}$

6; 6

78. To find $[20 - 3]_{five}$:

$[20 = 10 + 10 = (3 + 2) + 10 = 3 + (2 + 10) = 3 + \_\_\_\_]_{five}$

Thus, $[20 - 3 = \_\_\_\_]_{five}$

12; 12

79. To find $[32 - 3]_{five}$:

$[32 = (3 + 2) + 22 = 3 + (2 + 22) = 3 + \_\_\_\_]_{five}$

Thus, $[32 - 3 = \_\_\_\_]_{five}$

24; 24

80. To find $[32 - 15]_{eight}$: $[32 = 20 + 12 = 15 + 3 + 12]_{eight}$.

Hence, $[32 - 15 = \_\_\_\_\_]_{eight}$.

15

81. To find $[121 - 12]_{three}$: $[121 = 20 + 101 = 12 + 1 + 101]_{three}$.

Thus, $[121 - 12 = \_\_\_\_\_]_{three}$.

102

82. To find 325 - 76 : 325 = 100 + 225

= 76 + 24 + 225

Hence, 325 - 76 = _____.

249

It is possible to perform subtraction in this way by writing the numerals in traditional vertical pattern. The following frame will illustrate this procedure using the same problem as in the preceding frame.

83.
$$\begin{array}{rl} 325 & \\ -\ 76 & \\ \hline 24 & \text{since } 76 + 24 = 100 \\ +\ 225 & \text{Since } 100 + 225 = 325 \\ \hline 249 & \end{array}$$

Thus, 325 = 76 + 249 and 325 - 76 = _____.

249

84.

$$\left[\begin{array}{r} 432 \\ -\ 144 \\ \hline 23 \\ +\ 232 \\ \hline \underline{\hspace{2em}} \end{array}\right]_{\text{seven}}$$

since $[144 + 23 = 200]_{\text{seven}}$

since $[200 + 232 = 432]_{\text{seven}}$

Thus, $[432 - 144 = \underline{\hspace{3em}}]_{\text{seven}}$

255; 255

85.

$$\left[\begin{array}{r} 432 \\ -\ 144 \\ \hline 45 \\ +\ \underline{\hspace{2em}} \\ \hline \underline{\hspace{2em}} \end{array}\right]_{\text{nine}}$$

since $[144 + 45 = 200]_{\text{nine}}$

$$\begin{array}{r} +\ 232 \\ \hline 277 \end{array}$$

86.

$$\left[\begin{array}{r} 232 \\ -\ 144 \\ \hline \underline{\hspace{2em}} \\ +\ \underline{\hspace{2em}} \\ \hline \underline{\hspace{2em}} \end{array}\right]_{\text{six}}$$

$$\begin{array}{r} 12 \\ +\ 32 \\ \hline 44 \end{array}$$

87.

$$\left[\begin{array}{r} 632 \\ -\ 458 \\ \hline \_\_\_ \\ +\ \_\_\_ \\ \hline \_\_\_ \end{array}\right]_{\text{twelve}}$$

$$\begin{array}{r} 64 \\ +\ 132 \\ \hline 196 \end{array} \quad \text{or} \quad \begin{array}{r} 164 \\ +\ 32 \\ \hline 196 \end{array}$$

88. Using the vertical pattern of procedure developed in the preceding frames, which of the following would be correct for 847 - 586?

___ (a) $\begin{array}{r} 847 \\ -\ 586 \\ \hline 14 \\ +\ 247 \\ \hline 261 \end{array}$

___ (b) $\begin{array}{r} 847 \\ -\ 586 \\ \hline 114 \\ +\ 147 \\ \hline 261 \end{array}$

___ (c) $\begin{array}{r} 847 \\ -\ 586 \\ \hline 214 \\ +\ 47 \\ \hline 261 \end{array}$

___ (d) $\begin{array}{r} 847 \\ -\ 586 \\ \hline 164 \\ +\ 97 \\ \hline 261 \end{array}$

All responses are correct. However, in (a), (b), and (c), the first partial sums, namely 14, 114 and 214, were determined by selecting 600, 700, and 800 respectively for the intermediate values. In the (d) response, the first partial sum, namely 164, was determined by choosing 750 for the intermediate value. Any numeral between 586 and 847 may be selected for the intermediate value in determining the first partial sum. But, numerals involving the maximum number of zeros are probably more desirable.

The traditional way of doing subtraction may be illustrated by using a modified form of positional notation so that subtraction in the several positions is always possible.

89.

$$\begin{array}{rcrcr} 32 & = & 20 & + & 12 \\ -\ 17 & = & 10 & + & 7 \\ \hline \_\_\_ & = & 10 & + & 5 \end{array}$$

15

90.

$$\left[\begin{array}{rcrcr} 42 & = & 30 & + & 12 \\ -\ 15 & = & 10 & + & 5 \\ \hline \_\_\_ & = & 20 & + & 4 \end{array}\right]_{\text{seven}}$$

24

91.

$$\left[\begin{array}{rcrcr} 132 & = & 130 & + & 2 \\ -\ 52 & = & 50 & + & 2 \\ \hline \_\_\_ & = & \_\_\_ & + & 0 \end{array}\right]_{\text{nine}}$$

70; 70

92.

$$\left[\begin{array}{rcrcrcr} 304 & = & 200 & + & 70 & + & 14 \\ -\ 126 & = & 100 & + & 20 & + & 6 \\ \hline \_\_\_ & = & 100 & + & 50 & + & \_\_\_ \end{array}\right]_{\text{eight}}$$

156; 6

93.

$$\left[\begin{array}{rcrcrcr} 1312 & = & 1200 & + & 100 & + & 12 \\ -\ 343 & = & 300 & + & 40 & + & 3 \\ \hline \_\_\_ & = & \_\_\_ & + & \_\_\_ & + & \_\_\_ \\ \hline \end{array}\right]_{\text{five}}$$

---

414 = 400 + 10 + 4

94.

$$\begin{array}{rcrcrcrcr} & 3025 & = & 2000 & + & 900 & + & 110 & + & 15 \\ - & 1579 & = & 1000 & + & 500 & + & 70 & + & 9 \\ \hline & \_\_\_ & = & \_\_\_ & + & \_\_\_ & + & \_\_\_ & + & \_\_\_ \\ \hline \end{array}$$

1446 = 1000 + 400 + 40 + 6

95. **Recall the definition of multiplication of whole numbers in terms of sets. Let A and B denote sets. Then**

$$nA \cdot nB = n(\_\_\_\_\_)$$

A x B

96. Complete the following diagram representing A x B where A = $\{a, b, c\}$ and B = $\{x, y\}$

y	$\underset{(1)}{(\_\_,\_\_)}$	$\underset{(2)}{(\_\_,\_\_)}$	(c, y)
x	(a, x)	(b, x)	$\underset{(3)}{(\_\_,\_\_)}$
	a	b	c

---

(1) (a, y) (2) (b, y) (3) (c, x)

Since we are interested in the number properties of the sets instead of the specific elements, we will use the following diagram to represent A x B where the number property of A is three and the number property of B is two.

$$\begin{array}{c|ccc} 2\left\{\begin{array}{c} X \\ X \end{array}\right. & \begin{array}{c} X \\ X \end{array} & \begin{array}{c} X \\ X \end{array} & \begin{array}{c} X \\ X \end{array} \\ \hline & \multicolumn{3}{c}{\underbrace{X \quad X \quad X}_{3}} \end{array}$$

97.

$$\begin{array}{r|ccccc} 2\left\{\begin{matrix} X \\ X \end{matrix}\right. & \begin{matrix} X \\ X \end{matrix} & \begin{matrix} X \\ X \end{matrix} & \begin{matrix} X \\ X \end{matrix} & \begin{matrix} X \\ X \end{matrix} & \begin{matrix} X \\ X \end{matrix} \\ \hline & \multicolumn{5}{c}{\underbrace{X \quad X \quad X \quad X \quad X}_{5}} \end{array}$$

The numerical sentence in base ten illustrated by the diagram above is:

$$5 \cdot 2 = \_\_\_\_\_$$

10

98.

$$\begin{array}{r|cccc} \left\{\begin{matrix} X \\ X \\ X \\ X \\ X \end{matrix}\right. & \begin{matrix} X \\ X \\ X \\ X \\ X \end{matrix} & \begin{matrix} X \\ X \\ X \\ X \\ X \end{matrix} & \begin{matrix} X \\ X \\ X \\ X \\ X \end{matrix} & \begin{matrix} X \\ X \\ X \\ X \\ X \end{matrix} \\ \hline & \multicolumn{4}{c}{\underbrace{X \quad X \quad X \quad X}} \end{array}$$

The base ten numerical sentence illustrated by the diagram above is:

$$\_\_\_\_\_ \cdot \_\_\_\_\_ = 20$$

$4 \cdot 5$

99. The base ten sentence $3 \cdot 4 = 12$ is represented by which of the following diagrams:

___ (a)

$$\begin{array}{c|cccc} X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \\ \hline & X & X & X & X \end{array}$$

___ (b)

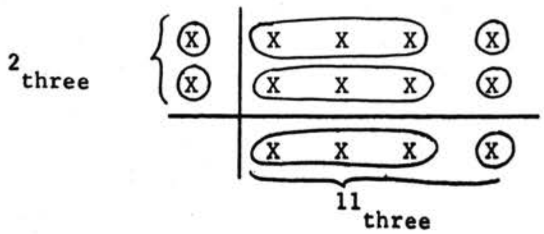
$$\begin{array}{c|ccc} X & X & X & X \\ X & X & X & X \\ X & X & X & X \\ X & X & X & X \\ \hline & X & X & X \end{array}$$

---

99 (a) Incorrect. This diagram represents the sentence $4 \cdot 3 = 12$

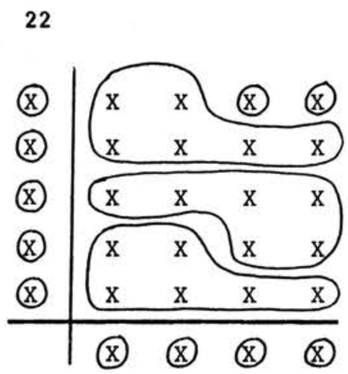
99 (b) Correct. This diagram represents the sentence $3 \cdot 4 = 12$

100. Multiplication in a number base can be illustrated by simply grouping the diagram for the cross product in that base.



The diagram above illustrates the multiplication sentence $[11 \cdot 2 = \_\_\_\_]_{three}$

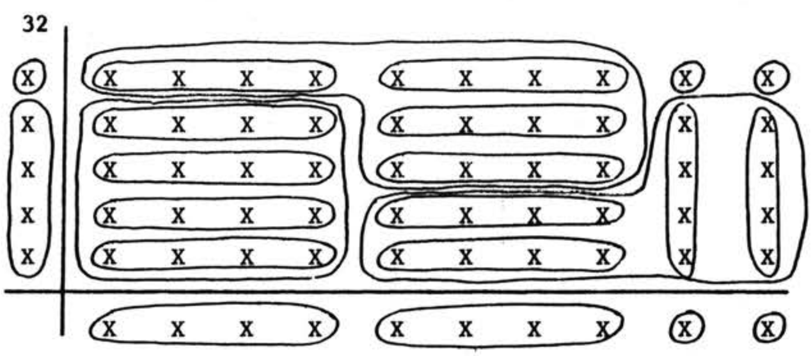
101.



The diagram represents the sentence:

$[4 \cdot 5 = \_\_\_\_\_]_{six}$

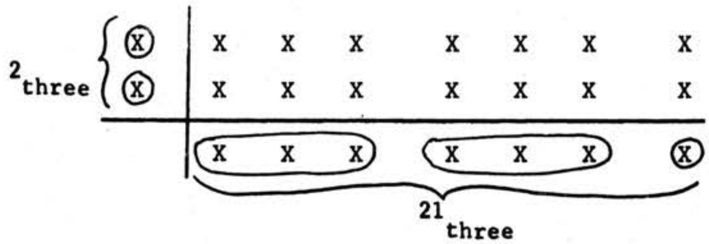
102.



This diagram represents the sentence:

$[\_\_\_\_ \cdot \_\_\_\_ = \_\_\_\_]_{four}$

$[22 \cdot 11 = 302]_{four}$

103. 

If the cross product were grouped by threes, it would show that:

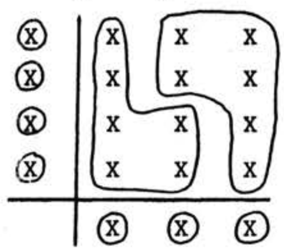
$$[21 \cdot 2 = \_\_\_\_\_]_{three}$$

112

The numerical algorism for multiplication is dependent upon:

1) Multiplication of digits and
2) Positional notation.

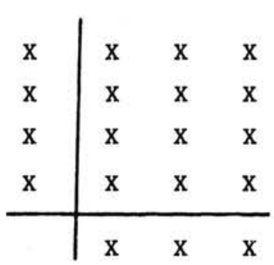
Positional notation was introduced in conjunction with addition, and frames 104-112 develop the procedure for the multiplication of digits.

104. 

The diagram above illustrates the numerical sentence:

$$[3 \cdot 4 = \_\_\_\_\_]_{six}$$

20

105. 

If the cross product above were grouped by fives, it would indicate

$$[3 \cdot 4 = \_\_\_\_\_]_{five}$$

22

Frames 104 and 105 indicate that the product of two digits could be found by using a diagram and grouping. A numerical method of multiplying digits can be based on the repeated addition interpretation of multiplication which is explained in frames 106-109.

106. $3 \cdot 4 = 4 \cdot 3 = 4 \cdot (1 + 1 + 1) = 4 \cdot 1 + 4 \cdot 1 + 4 \cdot 1 =$

$\_\_\_\_ + \_\_\_\_ + \_\_\_\_$

$4 + 4 + 4$

107. $[2 \cdot 5 = 5 \cdot 2 = 5 \cdot (1 + 1) = 5 \cdot 1 + 5 \cdot 1 = \_\_\_\_ + \_\_\_\_]_{six}$

5; 5

Multiplication interpreted as repeated addition implies $2 \cdot 5 = 5 + 5$

108. So, interpreted as repeated addition

$[4 \cdot 7 = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_]_{nine}$

$7 + 7 + 7 + 7$

109. Interpreted as repeated addition

$[3 \cdot 2 = \_\_\_\_\_\_\_\_]_{six}$

$2 + 2 + 2$

The numerical method for multiplying digits is illustrated in the following frame.

110. $[3 \cdot 2 = 2 + 2 + 2 = (2 + 2 + 1) + 1 = \_\_\_\_ + 1 = \_\_\_\_]_{five}$

10; 11

111. $[4 \cdot 3 = 3 + 3 + 3 + 3 = (3 + 3 + 2) + (1 + 3) = \_\_\_\_ + 4 =$

$\_\_\_\_]_{eight}$

10; 14

112. $$\left[\begin{array}{ll} 5 \cdot 7 & = 7 + 7 + 7 + 7 + 7 \\ & = 7 + 7 + 7 + 7 + 2 + 2 + 2 + 1 \\ & = (7 + 2) + (7 + 2) + (7 + 2) + (7 + 1) \\ & = \_\_\_\_ + \_\_\_\_ + \_\_\_\_ + \_\_\_\_ \\ & = \_\_\_\_\_\_ \end{array}\right]_{\text{nine}}$$

---

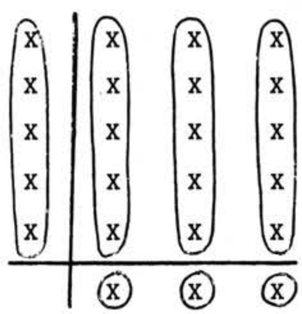
10 + 10 + 10 + 8; 38

113. $[5 \cdot 6 = \_\_\_\_]_{\text{seven}}$

42

114. $[4 \cdot 4 = \_\_\_\_]_{\text{six}}$

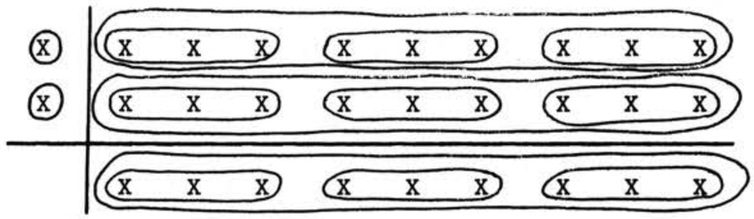
24

115. 

The numerical sentence illustrated by the diagram is:

$$[3 \cdot 10 = \_\_\_\_]_{\text{five}}$$

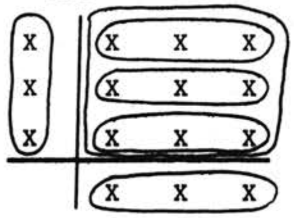
30

116. 

The numerical sentence illustrated by the diagram is

$[100 \cdot 2 = \_\_\_\_]_{three}$

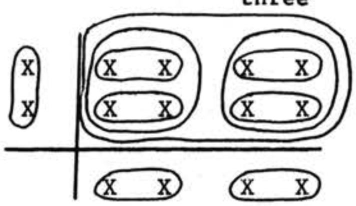
200

117. 

The diagram illustrates the sentence

$[\_\_\_\_ \cdot \_\_\_\_ = \_\_\_\_]_{three}$

$10 \cdot 10 = 100_{three}$

118. 

The diagram illustrates the sentence

$[\_\_\_\_ \cdot \_\_\_\_ = \_\_\_\_]_{two}$

$100 \cdot 10 = 1000$

119. Generalizing from frames 115-118, which of the following are true in any base greater than five?

_____ (a) $[10 \cdot 10 = 100]_{base}$

_____ (b) $[20 = 2 \cdot 10]_{base}$

_____ (c) $[100 \cdot 10 = 1000]_{base}$

_____ (d) $[60 = 6 \cdot 10]_{base}$

---

119 (a) Correct. This is a generalization of frame 117.

119 (b) Correct. This is a generalization of frame 115.

119 (c) Correct. This is a generalization of frame 118.

119 (d) Incorrect. In bases five or six the digit 6 is not used. So this sentence would not exist in those two bases. This response would be correct if the base were restricted to being greater than six.

120. $[20 \cdot 30 = (2 \cdot 10) \cdot (3 \cdot 10) = (2 \cdot 3) \cdot (10 \cdot 10) =$

$\underline{\quad\quad} \cdot \underline{100} = \underline{\quad\quad}]_{five}$

11; 1100

121. $[40 \cdot 3 = (4 \cdot 10) \cdot 3 = (4 \cdot 3) \cdot 10 = \underline{\quad\quad} \cdot 10 = \underline{\quad\quad}]_{six}$

20; 200

The work in frames 120 and 121 can be simplified by multiplying the digits mentally and writing down the answer using the appropriate place value.

122. $[30 \cdot 50 = \underline{\quad\quad}]_{seven}$

2100

123. $[20 \cdot 30 = \_\_\_\_]_{six}$

1000

124. $[40 \cdot 50 = \_\_\_\_]_{eight}$

2400

125. $[12 \cdot 2 = (10 + 2) \cdot 2 = (10 \cdot 2) + (2 \cdot 2) = 20 + 4 =$

$\_\_\_\_]_{six}$

24

126. $[32 \cdot 3 = (30 + 2) \cdot 3 = (30 \cdot 3) + (2 \cdot 3) = 120 + 6 =$

$\_\_\_\_]_{seven}$

126

Note that in Frames 125 and 126 the product of a two-digit numeral and a one-digit numeral produces two partial products whose sum is the product. The partial products can be written in the familiar vertical form.

127.
$$\left[\begin{array}{rrl} & 34 & \\ \cdot & 5 & \\ \hline & 24 & = 4 \cdot 5 \\ + & 170 & = 30 \cdot 5 \\ \hline & \_\_\_\_ & = (30 + 4) \cdot 5 \end{array}\right]_{eight}$$

214

128.

$$\left[\begin{array}{rl} & 32 \\ \cdot & 4 \\ \hline & 11 = 2 \cdot 4 \\ + & 150 = 30 \cdot 4 \\ \hline & \underline{\qquad} = (30 + 2) \cdot 4 \end{array}\right]_{seven}$$

161

129.

$$\left[\begin{array}{rl} & 54 \\ \cdot & 5 \\ \hline & 22 = 4 \cdot 5 \\ + & 270 = 50 \cdot 5 \\ \hline & \underline{\qquad} \end{array}\right]_{nine}$$

302

130.

$$\left[\begin{array}{rl} & 23 \\ \cdot & 3 \\ \hline & 21 \\ + & 120 \\ \hline & \underline{\qquad} \end{array}\right]_{four}$$

201

131.
$$\left[\begin{array}{rcl} 53 & & \\ \cdot\ \ 4 & & \\ \hline 180 & = & 50 \cdot 4 \\ +\ \ 10 & = & 3 \cdot 4 \\ \hline 190 & = & \underline{(\_\_\_\_ + \_\_\_\_)} \cdot 4 \end{array}\right]_{\text{twelve}}$$

(50 + 3) or (3+50) since addition is commutative in whole numbers

132. $[32 \cdot 45 = 32 \cdot (\_\_\_\_ + 5)]_{\text{seven}}$ **by positional notation**

40

133. $[32 \cdot (40 + 5) = (\_\_\_\_ \cdot 40) + (\_\_\_\_ \cdot 5)]_{\text{seven}}$ **by the distributive property.**

32; 32

134. $[(32 \cdot 40) + (32 \cdot 5) = ((30 + 2) \cdot 40) + ((30 + 2) \cdot 5)]_{\text{seven}}$ by ________________ notation

---

positional

135. $[(30 + 2) \cdot 40) + ((30 + 2) \cdot 5) = (30 \cdot 40) + (2 \cdot 40) + (30 \cdot 5) + (2 \cdot 5)]_{\text{seven}}$ **by applying the** ______________________ **property twice.**

---

distributive

136. $[(30 \cdot 40) + (2 \cdot 40) + (30 \cdot 5) + (2 \cdot 5) = 1500 + \_\_\_\_ + \_\_\_\_ + 13]_{\text{seven}}$

110; 210

137. Thus, from Frames 132 through 136, $[32 \cdot 45 = \underline{\qquad}]_{seven}$

2133

For the sake of brevity we utilize the vertical form to illustrate multiplication of a pair of two-digit numerals.

138.

$$\left[\begin{array}{rrcl}
 & 23 & = & 20 + 3 \\
\cdot & 42 & = & 40 + 2 \\
\hline
 & 1200 & = & 20 \cdot 40 \\
+ & 200 & = & 3 \cdot 40 \\
+ & 40 & = & 20 \cdot 2 \\
+ & 10 & = & 3 \cdot 2 \\
\hline
 & \underline{\qquad} & = & (20 \cdot 40) + (3 \cdot 40) + (20 \cdot 2) + (3 \cdot 2)
\end{array}\right]_{six}$$

1450

139.

$$\left[\begin{array}{rrcl}
 & 54 & & \\
\cdot & 36 & & \\
\hline
 & 1600 & = & 50 \cdot 30 \\
+ & 130 & = & 4 \cdot 30 \\
+ & 330 & = & 50 \cdot 6 \\
+ & 26 & = & 4 \cdot 6 \\
\hline
 & \underline{\qquad} & &
\end{array}\right]_{nine}$$

2186

140.

$$\left[\begin{array}{rcr} & 36 & \\ \cdot & 25 & \\ \hline & 600 & = \; 30 \cdot 20 \\ + & 140 & = \; 6 \cdot 20 \\ + & \underline{\hspace{1cm}} & = \; 30 \cdot 5 \\ + & \underline{\hspace{1cm}} & = \; 6 \cdot 5 \\ \hline & \underline{\hspace{1cm}} & \end{array}\right]_{\text{eight}}$$

170; 36; 1166

141. Since addition and multiplication are both commutative and associative, the order of listing the partial products can be rearranged.

$$\left[\begin{array}{rcr} & 23 & \\ \cdot & 42 & \\ \hline & 11 & = \; 2 \cdot 3 \\ + & 40 & = \; 2 \cdot 20 \\ + & 220 & = \; 40 \cdot 3 \\ + & \underline{\hspace{1cm}} & = \; 40 \cdot 20 \\ \hline & \underline{\hspace{1cm}} & \end{array}\right]_{\text{five}}$$

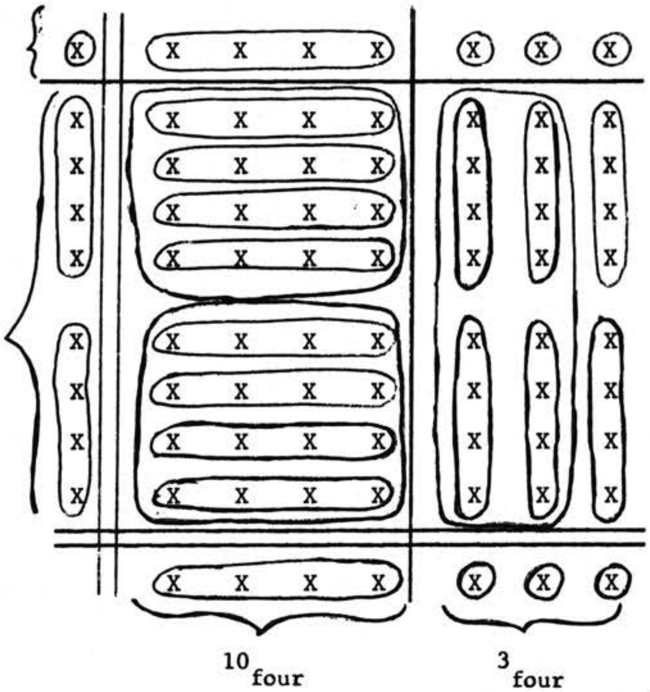
1300; 2121

142.

$$\left[\begin{array}{rcr} & 11 & \\ \cdot & 11 & \\ \hline & 1 & = \; 1 \cdot 1 \\ + & 10 & = \; 1 \cdot 10 \\ + & \underline{\hspace{1cm}} & = \; 10 \cdot 1 \\ + & \underline{\hspace{1cm}} & = \; 10 \cdot 10 \\ \hline & \underline{\hspace{1cm}} & \end{array}\right]_{\text{two}}$$

10; 100; 1001

143. 1_{four}, 20_{four}



The preceding diagram illustrates the following multiplication algorism. Notice the cross product has been partitioned into four parts each representing one of the partial products in the algorism.

$$\left[\begin{array}{rcl} 21 & = & 20 + 1 \\ 13 & = & 10 + 3 \\ \hline \underline{\quad\quad} & = & 20 \cdot 10 \\ + \underline{\quad\quad} & = & 1 \cdot 10 \\ + \underline{\quad\quad} & = & 20 \cdot 3 \\ + \underline{\quad\quad} & = & 1 \cdot 3 \\ \hline \underline{\quad\quad} & & \end{array}\right]_{four}$$

200; 10; 120; 3; 333

144. The familiar algorithm for multiplication of two-digit numerals can be derived by finding two partial sums instead of four.

$$\left[\begin{array}{rcl} & 34 & \\ \cdot & 25 & \\ \hline & 236 & = 5 \cdot 34 \\ + & \_\_\_ & = 20 \cdot 34 \\ \hline & \_\_\_ & \end{array}\right]_{\text{seven}}$$

1010; 1246

145.

$$\left[\begin{array}{rcl} & 21 & \\ \cdot & 32 & \\ \hline & \_\_\_ & = 2 \cdot 21 \\ + & \_\_\_ & = 30 \cdot 21 \\ \hline & \_\_\_ & \end{array}\right]_{\text{four}}$$

102; 1230; 1332

146. By extrapolation

$$\left[\begin{array}{rcl} & 234 & \\ \cdot & 35 & \\ \hline & \_\_\_ & = 5 \cdot 4 \\ + & \_\_\_ & = 5 \cdot 30 \\ + & \_\_\_ & = 5 \cdot 200 \\ + & \_\_\_ & = 30 \cdot 4 \\ + & \_\_\_ & = 30 \cdot 30 \\ + & \_\_\_ & = 30 \cdot 200 \\ \hline & \_\_\_ & \end{array}\right]_{\text{six}}$$

32; 230; 1400; 200; 1300; 10000; 14002

Division of whole numbers is defined in terms of multiplication. Thus:

$$a \div b = c \quad \text{if and only if} \quad a = b \cdot c$$

147. Since $12 = 3 \cdot 4$, $12 \div 4 =$ ______.

3

148. Since $[11 = 2 \cdot 4]_{seven}$, $[11 \div 4 = \_\_\_\_\_]_{seven}$.

2

149. Since $[22 = 4 \cdot 3]_{five}$, $[22 \div 3 = \_\_\_\_\_]_{five}$.

4

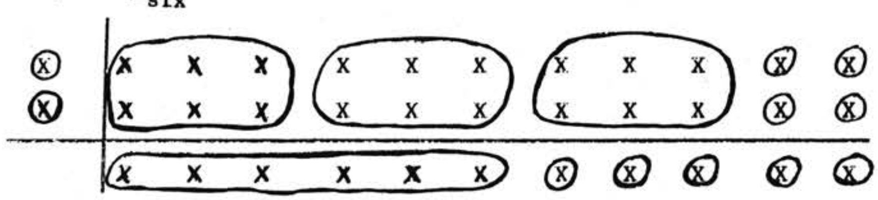
Thus, the solution to a division problem is found in a previously known multiplication fact. A diagram can be used to discover the multiplication fact. The following procedure is used. A rectangular array representing the cross product of two sets is constructed under two conditions:

1) the number of rows is the same as the number represented by the divisor

2) the number of elements in the cross product is equal to the number represented by the dividend

The number of elements in the cross product is the product of the number of rows and the number of columns. So by the definition of division, the quotient would be the number of columns. The numeral representing this number could be found by grouping. This procedure is illustrated in the following diagram.

150. To find $[34 \div 2]_{six}$

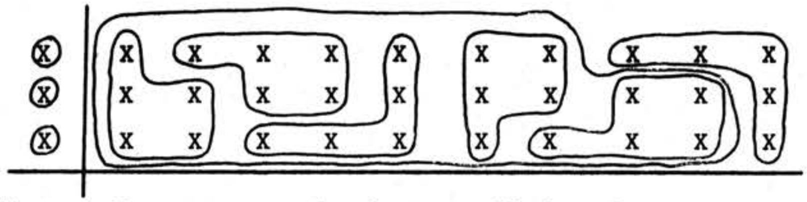


The multiplication sentence illustrated by the cross product above is:

$[34 = \_\_\_\_\_ \cdot 2]_{six}$ therefore, $[34 \div 2 = \_\_\_\_\_]_{six}$

15; 15

151. To find $[110 \div 3]_{five}$



If the diagram above were completed, it would show that:

$[110 = \_\_\_\_ \cdot 3]_{five}$ therefore, $[110 \div 3 = \_\_\_\_]_{five}$

20; 20

152. To find $[63 \div 5]_{seven}$: $[63 \div 5 = (50 + 13) \div 5 =$

$(50 \div 5) + (13 \div 5) = 10 + 2]_{seven}$. Thus, $[63 \div 5 = \_\_\_\_]_{seven}$

12

153. To find $[2400 \div 2]_{five}$:

$[2400 \div 2 = (2000 + 400) \div 2 = 1000 + 200]_{five}$

Thus, $[2400 \div 2 = \_\_\_\_]_{five}$.

1200

154. To find $[440 \div 4]_{seven}$: $[440 \div 4 = (400 + 40) \div 4 = 100 +$

$10]_{seven}$

Hence, $[440 \div 4 =] \_\_\_\_ {}_{seven}$.

110

155. To find $[24 \div 3]_{seven}$: $[24 \div 3 = (12 + \_\_\_\_) \div 3 = 3 + 3]_{seven}$

Thus, $[24 \div 3 = \_\_\_\_]_{seven}$.

12; 6

156. To find $[214 \div 5]_{eight}$:

$[214 \div 5 = (120 + 74) \div 5 = (120 + 50 + 24) \div 5]_{eight}$

$[(120 + 50 + 24) \div 5 = 20 + 10 + \_\_\_\_]_{eight}$

Thus, $[214 \div 5 = \_\_\_\_]_{eight}$.

4; 34

157. To find $[211 \div 2]_{three}$:

$[211 \div 2 = (200 + 11) \div 2 = 100 + \_\_\_\_]_{three}$

Thus, $[211 \div 2 = \_\_\_\_]_{three}$

2; 102

Changing the notation to a form closer to the familiar form of division, the process can be carried out as follows:

158. $$\left[4 \overline{)54} = 4 \overline{)40 + 14}^{\,10 \;+\; 3} \right]_{eight}$$

Thus, $[54 \div 4 = \_\_\_\_]_{eight}$.

13

159. $$4 \overline{)124} = 4 \overline{)120 + 4}^{\,30 \;+\; 1}$$

Thus, $124 \div 4 = \_\_\_\_$.

31

160.
$$\left[\begin{array}{lcl} & & \qquad\quad 10 \quad + \quad 1 \\ 101\overline{)1111} & = & 101\overline{)1010 \quad + \quad 101}\end{array}\right]_{two}$$

Thus, $[1111 \div 101 = \underline{\qquad}]_{two}$.

11

161.
$$\left[\begin{array}{lcl} & & \quad \underline{\qquad} + \underline{\qquad} + \underline{\qquad} + \underline{\qquad} \\ 4\overline{)301} & = & 4\overline{)110 \quad + \quad 110 \quad + \quad 40 \quad + \quad 11}\end{array}\right]_{seven}$$

Thus, $[301 \div 4 = \underline{\qquad}]_{seven}$.

20; 20; 10; 2; 52

162.
$$\left[\begin{array}{lcl} & & \quad \underline{\qquad} + \underline{\qquad} + \underline{\qquad} \\ 3\overline{)203} & = & 3\overline{)100 \quad + \quad 100 \quad + \quad 3}\end{array}\right]_{nine}$$

Thus, $[203 \div 3 = \underline{\qquad}]_{nine}$.

30; 30; 1; 61

The following three frames exhibit a correspondence between the steps in the method of division up to this point and the steps in the more familiar form.

163. To find $228 \div 12$:

$$\begin{array}{l} \qquad 10 \quad + \\ 12\overline{)120 \quad + \quad 108} \end{array}$$

$$\begin{array}{r} \underline{\qquad} \\ 12\overline{)228} \\ \underline{120} \\ 108 \end{array}$$

10

164.
$$\begin{array}{l} \qquad 10 \quad + \quad 5 \\ 12\overline{)120 \quad + \quad 60 \quad + \quad 48} \end{array}$$

$$\begin{array}{r} 10 \quad + \quad \underline{\qquad} \\ 12\overline{)228} \\ \underline{120} \\ 108 \\ \underline{60} \\ 48 \end{array}$$

5

165.

$$\begin{array}{r|l} & 10 \;+\; 5 \;+\; 4 \;=\; 19 \\ \hline 12 & 120 \;+\; 60 \;+\; 48 \end{array}$$

$$\begin{array}{r|l} & 10 \;+\; 5 \;+\; \_\_\_\_ \;=\; \_\_\_\_ \\ \hline 12 & 228 \\ & \underline{120} \\ & 108 \\ & \underline{\;\;60} \\ & \;\;48 \\ & \underline{\underline{\;\;48}} \end{array}$$

4; 19

166.

$$\left[\begin{array}{r|l} & 1000 \;+\; \_\_\_\_ \;=\; \_\_\_\_ \\ \hline 11 & 11011 \\ & \underline{11000} \\ & \;\;\;\;11 \\ & \underline{\underline{\;\;\;\;11}} \end{array}\right]_{\text{two}}$$

1; 1001

167.

$$\left[\begin{array}{r|l} & 20 \;+\; 10 \;+\; \_\_\_\_ \;=\; \_\_\_\_ \\ \hline 6 & 226 \\ & \underline{130} \\ & \;\;86 \\ & \underline{\;\;60} \\ & \;\;26 \\ & \underline{\underline{\;\;26}} \end{array}\right]_{\text{nine}}$$

4; 34

168.

$$\left[\begin{array}{r|l} & 20 \;+\; 20 \;+\; \_\_\_\_ \;=\; \_\_\_\_ \\ \hline 5 & 303 \\ & \underline{130} \\ & 143 \\ & \underline{130} \\ & \;\;13 \\ & \underline{\underline{\;\;13}} \end{array}\right]_{\text{seven}}$$

2; 42

169. $\left[\begin{array}{l} \underline{\quad} + \underline{\quad} = \underline{\quad} \\ 13 \overline{)43} \\ \quad \underline{30} \\ \quad 13 \\ \quad \underline{13} \\ \quad\ \ 0 \end{array}\right]_{six}$

2; 1; 3 reml.

170. $\left[\begin{array}{l} \underline{\quad} + \underline{\quad} = \underline{\quad} \\ 31 \overline{)204} \\ \quad \underline{102} \\ \quad 102 \\ \quad \underline{\underline{102}} \end{array}\right]_{six}$

2; 2; 4

171. $\left[\begin{array}{l} \underline{\quad} + \underline{\quad} + \underline{\quad} = \underline{\quad} \\ 18 \overline{)56} \\ \quad \underline{18} \\ \quad 37 \\ \quad \underline{18} \\ \quad 18 \\ \quad \underline{\underline{18}} \end{array}\right]_{nine}$

1; 1; 1; 3

172. $\left[\begin{array}{l} \underline{\quad} + \underline{\quad} = \underline{\quad} \\ 12 \overline{)2412} \\ \quad \underline{2400} \\ \quad\quad 12 \\ \quad \underline{\underline{\quad 12}} \end{array}\right]_{twelve}$

200; 1; 201

173.

$$\left[\begin{array}{rl} & \underline{\qquad} + \underline{\qquad} = \underline{\qquad} \\ 111 & \big)\overline{10101} \\ & \underline{\ \ 1110} \\ & \ \ \ 111 \\ & \underline{\ \ \ 111} \\ & \end{array}\right]_{two}$$

10; 1; 11

In the preceding frames, the two numerals for division were chosen so that one is a factor of the other. In general, this rarely happens since an arbitrary choice of the numerals in a division will usually produce a remainder different from zero. The division algorism for whole numbers is:

$n = (q \cdot d) + r \quad \text{where} \quad 0 \leq r < d.$

174.

$$\begin{array}{rl} & 20 + 3 = 23 \\ 32 & \big)\overline{738} \\ & \underline{640} \\ & \ \ 98 \\ & \ \ \underline{96} \\ & \ \ \ 2 \end{array}$$

Thus, $738 = (\underline{\qquad} \cdot 32) + \underline{\qquad}$.

23; 2

175.

$$\left[\begin{array}{rl} & \underline{\qquad} + \underline{\qquad} = \underline{\qquad} \\ 13 & \big)\overline{432} \\ & \underline{310} \\ & 122 \\ & \underline{112} \\ & \ \ 10 \end{array}\right]_{five}$$

Thus, $[432 = (\underline{\qquad} \cdot 13) + \underline{\qquad}]_{five}$.

20; 4; 24; 24; 10

176.

$$\left[\begin{array}{rl} & \underline{\qquad} \\ 13 & \big)\overline{432} \\ & \underline{420} \\ & \underline{\qquad} \end{array}\right]_{seven}$$

Thus, $[432 = (\underline{\qquad} \cdot 13) + \underline{\qquad}]_{seven}$

30; 12; 30; 12

CHANGING A NUMERAL IN ONE BASE TO AN EQUAL NUMERAL IN ANOTHER BASE WITHOUT USING BASE TEN

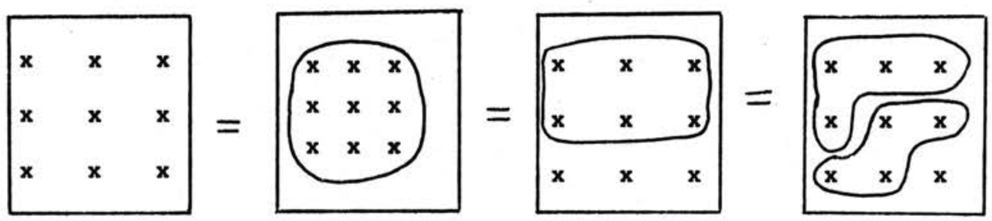
This section will develop the procedure for changing a numeral in one base to an equal numeral in another base without using the operations in base ten. This procedure depends on:

1) the ability to change digits in the first base to equal numerals in the second.

2) expanded notation.

3) the ability to add and multiply in the second base.

The ability to change digits and the first base to equal numerals in the second is founded on mentally grouping sets, which is developed in frames 1-4.

1. 

Each of the above sets have nine elements. They have been grouped by twelves, nines, sixes, and fours. Therefore, $9_{\text{twelve}} = \underline{\quad\quad}_{\text{nine}} = \underline{\quad\quad}_{\text{six}} = \underline{\quad\quad}_{\text{four}}$

10; 13; 21

2. $4_{\text{five}} = \underline{\quad\quad}_{\text{eight}} = \underline{\quad\quad}_{\text{six}} = \underline{\quad\quad}_{\text{four}} = \underline{\quad\quad}_{\text{two}}$

4; 4; 10; 100

3. When changing from base three to base five: $0_{\text{three}} = \underline{\quad\quad}_{\text{five}}$, $1_{\text{three}} = \underline{\quad\quad}_{\text{five}}$, $2_{\text{three}} = \underline{\quad\quad}_{\text{five}}$ and $10_{\text{three}} = \underline{\quad\quad}_{\text{five}}$.

0; 1; 2; 3

4. When changing from base five to base three

0_{five} = ________, 1_{five} = ________, 2_{five} = ________

3_{five} = ________, 4_{five} = ________, 10_{five} = ________

0_{three}; 1_{three}; 2_{three}; 10_{three}; 11_{three}; 12_{three}

5. $342_{five} = [300 + 40 + 2]_{five} = [\_\_ \cdot 10 \cdot 10 + \_\_ \cdot 10 + \_\_ \cdot 1]_{five}$

3; 4; 2

6. $246_{seven} = [200 + 40 + 6]_{seven} = [2 \cdot \_\_ \cdot \_\_ + 4 \cdot \_\_ + 6 \cdot \_\_]_{seven}$

$10 \cdot 10$; 10; 1

The expression $[2 \cdot 10 \cdot 10 + 4 \cdot 10 + 6 \cdot 1]_{seven}$ is called expanded notation for the numeral 246_{seven}.

7. In expanded notation

$523_{eight} = [\_\_\_\_ + \_\_\_\_ + \_\_\_\_]_{eight}$

$5 \cdot 10 \cdot 10 + 2 \cdot 10 + 3 \cdot 1$

8. In expanded notation $4te2_{twelve}$ = ______________________________.

$[4 \cdot 10 \cdot 10 \cdot 10 + t \cdot 10 \cdot 10 + e \cdot 10 + 2 \cdot 1]_{twelve}$

9. Which of the following are true sentences:

____ (a) $[2 \cdot 10 + 3 \cdot 1]_{four} = [2 \cdot 4 + 3 \cdot 1]_{six}$

____ (b) $[5 \cdot 10 + 2 \cdot 1]_{six} = [5 \cdot 11 + 2 \cdot 1]_{five}$

____ (c) $[2 \cdot 10 \cdot 10 + 3 \cdot 10 + 4 \cdot 1]_{five} = [2 \cdot 11 \cdot 11 + 3 \cdot 11 + 10 \cdot 1]_{four}$

____ (d) $[5 \cdot 10 + 3 \cdot 1]_{six} = [12 \cdot 13 + 10 \cdot 1]_{three}$

---

9 (a) Correct. These are both numeral expressions for the number expression two · four + three · one.

9 (b) Incorrect. The digit 5 is not used in base five. The correct sentence should be $[5 \cdot 10 + 2 \cdot 1]_{six} = [10 \cdot 11 + 2 \cdot 1]_{five}$

9 (c) Correct. These are both numeral expressions for two · five · five + three · five + four · one.

9 (d) Incorrect. $10_{six} = 20_{three}$ instead of 13_{three}. The correct sentence should be $[5 \cdot 10 + 3 \cdot 1]_{six} = [12 \cdot 20 + 10 \cdot 1]_{three}$

The procedure to find the base three numeral equal to 32_{five} is shown in frames 10-13.

10. $32_{five} = [\_\_\_\_ \cdot \_\_\_\_ + \_\_\_\_ \cdot \_\_\_\_]_{five}$

$3 \cdot 10 + 2 \cdot 1$

11. $[3 \cdot 10 + 2 \cdot 1]_{five} = [\_\_\_\_ \cdot \_\_\_\_ + \_\_\_\_ \cdot \_\_\_\_]_{three}$

$10 \cdot 12 + 2 \cdot 1$

12. $[10 \cdot 12 + 2 \cdot 1]_{three} = [\_\_\_\_ + 2]_{three} = \_\_\_\__{three}$

120; 122

13. Frames 10-12 indicate the base three numeral equal to 32_{five} is __________.

122_{three}

14. To find the base four numeral equal to 21_{three}.

$21_{\text{three}} = [2 \cdot 10 + 1 \cdot 1]_{\text{three}} = [2 \cdot 3 + 1 \cdot 1]_{\text{four}} = [\underline{\quad} + 1]_{\text{four}}$

$= \underline{\qquad}_{\text{four}}$

12; 13

15. $45_{\text{eight}} = [4 \cdot 10 + 5 \cdot 1]_{\text{eight}} = [4 \cdot 8 + 5 \cdot 1]_{\text{twelve}} =$

$[28 + 5]_{\text{twelve}} = \underline{\qquad}_{\text{twelve}}$

31

16. $1101_{\text{two}} = [1 \cdot 10 \cdot 10 \cdot 10 + 1 \cdot 10 \cdot 10 + 0 \cdot 10 + 1 \cdot 1]_{\text{two}}$

$= [1 \cdot 2 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 2 + 0 \cdot 2 + 1 \cdot 1]_{\text{six}}$

$= [12 + 4 + 0 + 1]_{\text{six}} = \underline{\qquad\qquad}_{\text{six}}$

21

17. The base three numeral equal to 24_{five} is ____________________.

112_{three}

18. The base five numeral equal to 45_{eight} is ____________________.

122_{five}

19. The correct missing base for

$321_{seven} = 1122$ __________ would be:

_____ (a) six _____ (b) five _____ (c) four _____ (d) three

---

19 (a) Incorrect. $321_{seven} = 430_{six}$

19 (b) Correct. $321_{seven} = 1122_{five}$

19 (c) Incorrect. $321_{seven} = 2202_{four}$

19 (d) Incorrect. $321_{seven} = 20000_{three}$

APPENDIX C

TABLE XII

RAW DATA

Student	Sequence	ACT Subscore	Pre-Test A	Pre-Test B	Pre-Test C	Pre-Test Total	Post-Test A	Post-Test B	Post-Test C	Post-Test Total
1	1	2	1	4	4	9	3	3	7	13
2	1	14	1	3	8	12	4	3	9	16
3	1	10	1	1	6	8	6	4	9	19
4	1	8	3	2	4	9	7	3	12	22
5	1	13	2	2	6	10	6	3	13	22
6	1	13	3	3	8	14	6	5	12	23
7	1	13	4	2	4	10	4	6	16	26
8	1	14	1	2	6	9	6	5	16	27
9	2	12	2	2	2	6	3	6	8	17
10	2	11	3	4	4	11	8	3	10	21
11	2	14	5	1	9	15	5	2	17	24
12	2	12	0	2	3	5	7	7	13	27
13	2	14	2	4	5	11	5	7	18	30
14	2	10	5	5	10	20	7	7	17	31
15	1	24	3	4	5	12	5	6	14	25
16	1	24	1	2	6	9	6	6	17	29
17	1	24	2	6	7	15	7	6	17	30
18	1	24	3	4	9	16	4	8	19	31
19	1	26	3	2	5	10	4	7	21	32
20	1	30	3	3	12	18	7	8	20	35
21	1	24	4	3	10	17	8	9	19	36
22	1	27	2	5	6	13	8	8	20	36
23	1	29	2	3	2	7	9	8	20	37
24	1	24	3	3	10	16	7	9	24	40
25	2	25	5	2	6	13	5	6	16	27
26	2	25	5	4	9	18	5	7	15	27
27	2	24	0	6	5	11	5	6	19	30
28	2	25	3	4	10	17	6	8	20	34
29	2	28	7	5	9	21	5	8	22	35
30	2	27	4	3	6	13	7	7	23	37
31	1	17	2	1	9	12	5	2	7	14
32	1	17	3	1	2	6	5	2	11	18
33	1	15	2	0	9	11	4	3	14	21
34	1	20	4	0	8	12	4	3	17	24
35	1	17	2	4	8	14	6	4	15	25
36	1	19	3	2	8	13	4	6	16	26
37	1	16	3	1	3	7	7	6	13	26
38	1	22	1	2	9	12	4	7	16	27
39	1	15	2	2	8	12	6	7	15	28
40	1	17	2	4	5	11	5	7	16	28
41	1	23	3	3	6	12	5	7	16	28
42	1	23	1	2	5	8	5	5	18	28

TABLE XII
(Continued)

			Pre-Test				Post-Test			
Student	Sequence	ACT Subscores	A	B	C	Total	A	B	C	Total
43	1	22	2	4	8	14	5	6	17	28
44	1	17	3	4	4	11	8	8	13	29
45	1	21	2	3	9	14	7	4	18	29
46	1	19	5	3	4	12	6	8	15	29
47	1	21	3	2	5	10	7	8	14	29
48	1	20	2	3	7	12	5	7	18	30
49	1	19	2	2	5	9	6	6	19	31
50	1	19	3	3	5	11	6	6	19	31
51	1	15	0	2	7	9	8	5	18	31
52	1	18	2	2	5	9	6	7	18	31
53	1	18	2	2	7	11	5	8	18	31
54	1	18	4	5	3	12	6	7	19	32
55	1	18	3	1	5	9	5	6	21	32
56	1	22	4	5	6	15	6	8	19	33
57	1	21	7	5	10	22	7	8	18	33
58	1	18	3	2	7	12	7	7	20	34
59	1	16	0	0	7	7	9	8	17	34
60	1	18	2	3	2	7	9	7	18	34
61	1	18	4	5	7	16	6	7	22	35
62	1	23	4	4	12	20	9	8	21	38
63	2	17	1	3	7	11	4	5	7	16
64	2	15	3	3	4	10	4	4	11	19
65	2	20	4	5	5	14	5	6	8	19
66	2	18	1	2	8	11	4	2	14	20
67	2	16	2	0	2	4	3	5	14	22
68	2	16	4	5	4	13	4	7	12	23
69	2	21	3	5	5	13	7	7	12	26
70	2	19	2	4	7	13	7	6	13	26
71	2	22	1	2	4	7	5	7	14	26
72	2	23	3	2	5	10	5	8	13	26
73	2	20	4	1	5	10	6	5	16	27
74	2	20	3	2	9	14	6	7	14	27
75	2	20	2	5	8	15	5	5	17	27
76	2	15	1	0	1	2	6	7	14	27
77	2	18	2	3	7	12	5	8	15	28
78	2	16	1	2	2	5	6	7	16	29
79	2	22	1	4	8	13	6	5	18	29
80	2	20	2	2	5	9	7	6	16	29
81	2	16	2	5	6	13	6	6	17	29
82	2	22	4	2	5	11	8	7	14	29
83	2	18	3	1	6	10	6	5	18	29
84	2	17	3	2	6	11	5	7	17	29
85	2	15	2	3	5	10	6	7	17	30
86	2	21	3	4	5	12	6	7	17	30

TABLE XII
(Continued)

Student	Sequence	ACT Subscores	Pre-Test A	Pre-Test B	Pre-Test C	Pre-Test Total	Post-Test A	Post-Test B	Post-Test C	Post-Test Total
87	2	18	3	6	8	17	6	8	17	31
88	2	19	3	3	5	11	5	8	18	31
89	2	16	2	5	7	14	6	7	18	31
90	2	16	5	4	9	18	7	8	17	32
91	2	17	3	3	6	12	6	7	19	32
92	2	19	3	3	3	9	6	7	19	32
93	2	23	7	4	10	21	8	5	20	33
94	2	17	4	3	6	13	8	7	20	35
95	2	18	3	1	10	14	6	7	23	36
96	2	18	3	5	9	17	9	7	20	36
97	2	21	7	4	9	20	8	7	21	36
98	2	23	8	4	17	29	8	8	21	37
99	2	20	5	4	9	18	8	8	21	37
100	2	22	4	5	9	18	7	8	22	37
101	2	21	5	6	14	25	7	8	24	39
102	1		3	1	4	8	5	3	10	18
103	1		6	2	4	12	5	7	8	20
104	1		1	6	5	12	5	5	14	24
105	1		2	0	5	7	5	6	14	25
106	1		0	2	7	9	7	3	17	27
107	1		3	2	5	10	5	7	15	27
108	1		2	2	4	8	5	6	17	28
109	1		4	4	5	13	6	8	14	28
110	1		4	2	3	9	6	8	16	30
111	1		5	4	8	17	8	7	16	31
112	1		4	2	5	11	9	6	17	32
113	1		5	2	8	15	6	8	18	32
114	1		2	2	7	11	6	6	20	32
115	1		1	4	6	11	6	9	17	32
116	1		8	5	16	29	6	8	20	34
117	1		0	4	3	7	7	7	20	34
118	1		3	6	11	20	8	7	21	36
119	1		1	3	7	11	8	8	21	37
120	1		3	4	9	16	9	7	23	39
121	1		7	7	15	29	9	8	23	40
122	1		4	4	9	17	8	9	23	40
123	2		2	3	7	12	4	0	9	13
124	2		1	0	8	9	5	4	6	15
125	2		3	2	5	10	7	5	9	21
126	2		0	2	0	2	6	7	10	23
127	2		3	0	3	6	7	5	11	23
128	2		3	2	5	10	7	6	10	23
129	2		2	3	7	12	4	6	13	23

TABLE XII
(Continued)

Student	Sequence	ACT Subscores	Pre-Test A	Pre-Test B	Pre-Test C	Pre-Test Total	Post-Test A	Post-Test B	Post-Test C	Post-Test Total
130	2		2	5	1	8	8	4	13	25
131	2		3	2	4	9	7	8	10	25
132	2		1	2	3	6	6	3	16	25
133	2		1	2	2	5	6	6	13	25
134	2		3	2	1	6	6	2	19	27
135	2		2	3	3	8	5	6	17	28
136	2		2	5	6	13	6	4	18	28
137	2		4	2	6	12	6	6	17	29
138	2		0	4	3	7	5	5	19	29
139	2		4	2	7	13	6	8	16	30
140	2		2	0	7	9	7	8	17	32
141	2		3	3	10	16	6	9	18	33
142	2		4	3	3	10	6	6	22	34
143	2		3	2	10	15	6	6	22	34
144	2		3	3	9	15	9	9	16	34
145	2		0	3	7	10	8	6	22	36
146	2		4	5	6	15	6	7	24	37
147	2		3	4	5	12	7	9	23	39

TABLE XIII

ACT SUBSCORE MEANS, PRE-TEST MEANS, AND POST-TEST MEANS BY SUBGROUP

			Pre-Test				Post-Test			
Group	Sequence	ACT	Part A	Part B	Part C	Total	Part A	Part B	Part C	Total
High	I	25.60	2.60	3.50	7.20	13.30	6.50	7.50	19.10	33.10
	II	25.67	4.00	4.00	7.50	15.50	5.50	7.00	19.17	31.67
Middle	I	18.81	2.66	2.56	6.41	11.63	6.03	6.19	16.75	28.97
	II	18.85	3.13	3.26	6.67	13.05	6.08	6.56	16.51	29.15
Low	I	10.88	2.00	2.38	5.75	10.13	5.25	4.00	11.75	21.00
	II	12.17	2.83	3.00	5.50	11.33	5.83	5.33	13.83	25.00
With ACT Subscores	I	18.90	2.54	2.72	6.46	11.72	6.00	6.10	16.42	28.52
	II	18.86	3.20	3.31	6.63	13.14	5.98	6.47	16.51	28.96
Without ACT Subscores	I		3.24	3.24	6.95	13.43	6.62	6.81	17.33	30.76
	II		2.32	2.56	5.12	10.00	6.24	5.80	15.60	27.64
Total in Sequence	I		2.75	2.87	6.61	12.22	6.18	6.31	16.69	29.18
	II		2.91	3.07	6.13	12.11	6.07	6.25	16.21	28.53
Total - Both Sequences Combined			2.83	2.97	6.36	12.16	6.12	6.28	16.44	28.84

APPENDIX D

TABLE XIV

ANALYSIS OF VARIANCE FOR ACT LEVELS
AND SEQUENCE BASED ON PRE-TEST PART A

Source of Variation	df	Sum of Squares	Mean Square	F
Total	100	247.3267		
Sequence	1	10.8675	10.8675	4.51*
ACT Level	2	4.8695	2.4347	1.01
Interaction	2	2.7787	1.3894	.58
Error	95	228.8111	2.4085	

*Significant difference at the .05 level

TABLE XV

ANALYSIS OF VARIANCE FOR ACT LEVELS
AND SEQUENCE BASED ON PRE-TEST PART B

Source of Variation	df	Sum of Squares	Mean Square	F
Total	100	223.9604		
Sequence	1	8.9000	8.9000	4.15*
ACT Level	2	9.5340	4.7670	1.22
Interaction	2	1.8405	.9203	.43
Error	95	203.6859	2.1441	

*Significant difference at the .05 level

TABLE XVI

ANALYSIS OF VARIANCE FOR ACT LEVELS
AND SEQUENCE BASED ON PRE-TEST PART C

Source of Variation	df	Sum of Squares	Mean Square	F
Total	100	773.0496		
Sequence	1	0.7079	0.7079	.09
ACT Level	2	20.8203	10.4101	1.39
Interaction	2	1.0358	.5179	.07
Error	95	710.4855	7.4788	

TABLE XVII

ANALYSIS OF VARIANCE FOR ACT LEVELS
AND SEQUENCE BASED ON TOTAL PRE-TEST

Source of Variation	df	Sum of Squares	Mean Square	F
Total	100	1928.8318		
Sequence	1	50.7124	50.7124	2.71
ACT Level	2	90.7124	45.3562	2.42
Interaction	2	8.2010	4.1005	.22
Error	95	1779.2059	18.7285	

TABLE XVIII

ANALYSIS OF VARIANCE FOR WITH OR WITHOUT ACT SCORES AND SEQUENCE BASED ON PRE-TEST PART A

Source of Variation	df	Sum of Squares	Mean Square	F
Total	146	388.7483		
Sequence	1	0.9564	0.9564	.37
With or Without ACT	1	0.5520	0.5520	.21
Interaction	1	19.5312	19.5312	7.60*
Error	143	367.7088	2.5714	

*Significant difference at the .005 level

TABLE XIX

ANALYSIS OF VARIANCE FOR WITH OR WITHOUT ACT SCORES AND SEQUENCE BASED ON PRE-TEST PART B

Source of Variation	df	Sum of Squares	Mean Square	F
Total	146	339.8912		
Sequence	1	1.3609	1.3609	.60
With or Without ACT	1	0.7134	0.7134	.31
Interaction	1	12.7870	12.7870	5.63*
Error	143	325.0294	2.2729	

*Significant difference at the .025 level

TABLE XX

ANALYSIS OF VARIANCE FOR WITH OR WITHOUT ACT SCORES AND SEQUENCE BASED ON PRE-TEST PART C

Source of Variation	df	Sum of Squares	Mean Square	F
Total	146	1217.8912		
Sequence	1	8.2491	8.2491	1.01
With or Without ACT	1	10.9285	10.9285	1.34
Interaction	1	30.7795	30.7795	3.77
Error	143	1167.9340	8.1674	

*Significant difference at the .10 level

TABLE XXI

ANALYSIS OF VARIANCE FOR WITH OR WITHOUT ACT SCORES AND SEQUENCE BASED ON TOTAL PRE-TEST

Source of Variation	df	Sum of Squares	Mean Square	F
Total	146	3172.0818		
Sequence	1	0.5293	0.5293	.03
With or Without ACT	1	33.9456	23.9456	1.16
Interaction	1	184.3444	184.3444	8.90*
Error	143	2963.2625	20.7221	

*Significant difference at the .005 level

VITA

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