DERIVED FUNCTORS IN RELATIVE HOMOLOGICAL ALGEBRA

by

FRANKLIN S. BRENNEMAN

Bachelor of Arts
Goshen College
Goshen, Indiana
1960

Master of Arts
Pennsylvania State University
University Park, Pennsylvania
1965

Submitted to the faculty of the Graduate College
of the Oklahoma State University
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Thesis Approved:

Hiroshi Uehara
Thesis Adviser

E. K. McLachlan

F. A. McCollom

R B Deal

Jeanne Agnew

D D Durham
Dean of the Graduate College

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TABLE OF CONTENTS

INTRODUCTION

The techniques and results of homological algebra are currently being used in many areas of mathematics. In particular, the functors derived from a given functor are very useful in the investigation of algebraic and topological problems. One of the central activities in homological algebra is the investigation of derived functors in particular, the discovery and axiomatization of derived functors and the demonstration of the existence of a suitable product on the derived functors.

Milnor and Moore, [16], and Gugenheim, [11] discussed the functor, cotensor product of comodules over a coalgebra. The functor Cotor, derived from the cotensor product, was defined and used by Moore in Cartan Séminaire, [18] page 7-25, for calculating some properties of differential projective modules over a ring which are also comodules over a coalgebra. However, the Cotor functor has not been investigated in full detail. This paper gives such an investigation and presents a derived functor Coext which is new and of importance equal to that of Cotor.

This study is begun in Chapter I by developing the theory, relative to an injective class of sequences of derived functors. The theory includes an axiomatization. The author presents in this chapter two classical examples, one from the theory of R-modules and the other from the theory of sheaves. These examples involve the classical injective class of sequences, namely the class of all exact sequences; therefore, an example is presented in Chapter II where the injective class of sequences is not equal to the class of all exact sequences.

Since the cotensor product is shown in Chapter III to satisfy the conditions of Chapter I, Cotor is axiomatized. Using the same class of sequences as used for cotensor product the author shows that the functor Hom_Λ satisfies the conditions of Chapter I; consequently, there exists a derived functor for Hom_Λ, which he calls Coext. Immediately, Coext is axiomatized by Chapter I. Finally, in Chapter III, some relations between Ext, Tor, Coext and Cotor are established.

In Chapter IV it is shown that Cotor and Coext each have a product and that the product for Coext yields an algebra. A summary of the results and a presentation of some problems for further research are given in Chapter V.

The notation and techniques of Eilenberg and Moore, [6], are used extensively in this paper. Numbers in brackets refer to the Bibliography at the end of the paper. For example, [3] refers to Bibliography reference number three and [3-13] refers to Bibliography reference number three, page 13.

CHAPTER I

RELATIVE COHOMOLOGY THEORY

S. Eilenberg and J. C. Moore, [6-7], introduced the concept of a cohomology theory relative to a particular injective class of sequences, $\mathcal{E}$, and refer to an unpublished work. Since the details of this have not yet appeared, the theory is developed in this chapter as preparation for the author's work appearing in later chapters.

Considerable work has been done on the case where $\mathcal{E}$ is the class of all exact sequences, denoted by $\mathcal{E}_1$, in an abelian category or exact category; MacLane [14], Buchsbaum [5], Heller [12] and Uehara [20]. This case will be referred to as "Absolute" cohomology theory.

Definition of Relative Cohomology Theory

Definition 1.1: Let $\mathfrak{A}$ be an additive category with cokernels, $\mathfrak{B}$ an abelian category, $T:\mathfrak{A} \to \mathfrak{B}$ an additive functor, $\mathcal{E}$ an injective class in $\mathfrak{A}$ with $\mathcal{E} \overset{*}{\Rightarrow} \mathcal{J}$. A cohomology theory $H_{\mathcal{E}}$ relative to $\mathcal{E}$ over T is a sequence of functors $H^n:\mathfrak{A} \to \mathfrak{B}$; $n \geq 0$; such that:

Axiom I: For each sequence $E:0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ in $\mathcal{E}$ and for each $n \geq 0$ there exists a morphism $\delta_E^n \in \operatorname{Hom}(H^n(A''), H^{n+1}(A'))$ satisfying the "naturality condition"; i.e., for a commutative diagram

$$\begin{tikzcd} E_1 : 0 \arrow[r] & A' \arrow[r] \arrow[d, "\varphi'"] & A \arrow[r] \arrow[d, "\varphi"] & A'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\ E_2 : 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

of two sequences E_1, E_2 in $\mathcal{E}$ the diagram

$$\begin{tikzcd} H^n(A'') \arrow[r, "\delta^n_{E_1}"] \arrow[d, "H^n(\varphi'')"'] & H^{n+1}(A') \arrow[d, "H^{n+1}(\varphi')"] \\ H^n(B'') \arrow[r, "\delta^n_{E_2}"'] & H^{n+1}(B') \end{tikzcd}$$

is commutative.

<u>Axiom II</u>: For each sequence $E: 0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ in $\mathcal{E}$, the sequence

$$\cdots \xrightarrow{\delta^{n-1}_E} H^n(A') \xrightarrow{H^n(i)} H^n(A) \xrightarrow{H^n(j)} H^n(A'') \xrightarrow{\delta^n_E} H^{n+1}(A') \xrightarrow{H^{n+1}(i)} \cdots$$

is exact in $\mathfrak{B}$.

<u>Axiom III</u>. There exists a natural equivalence $\eta: T \to H^0$.

<u>Axiom IV</u>: For each $A \in \mathfrak{A}$ there exists $i: A \to I$; where $i \in \mathfrak{M}$ and $I \in \mathcal{J}$; such that $H^n(i) = 0$ for $n > 0$. $(\mathcal{E} \overset{*}{\Rightarrow} \mathfrak{M})$.

For clarity the definitions of $\mathcal{E} \overset{*}{\Rightarrow} \mathfrak{M}$ and $\mathfrak{M} \overset{*}{\Rightarrow} \mathcal{E}$ are included. They are dual to the definitions of paragraph 4 in Eilenberg and Moore [6].

<u>Definition 1.2</u>: $\mathcal{E} \overset{*}{\Rightarrow} \mathfrak{M}$ means $f \in \mathfrak{M}$; $A \xrightarrow{f} A$; if and only if $0 \to A \xrightarrow{f} A' \in \mathcal{E}$.

<u>Definition 1.3</u>: $\mathfrak{M} \overset{*}{\Rightarrow} \mathcal{E}$. The sequence

$$\begin{tikzcd} & & C \arrow[dr, "\ell"] & \\ E: A' \arrow[r, "i"] & A \arrow[ur, "c"] \arrow[rr, "j"] & & A'' \end{tikzcd}$$

where c is the cokernel of i, belongs to $\mathcal{E}$ if and only if $\ell \in \mathfrak{M}$.

Existence of the Relative Cohomology Theory

Let $\mathfrak{A}$ be an additive category with cokernels, $\mathfrak{B}$ an abelian category and $\mathcal{E}$ an injective class in $\mathfrak{A}$ with $\mathcal{E} \overset{*}{\Rightarrow} \mathcal{J}$; $\mathcal{E} \overset{*}{\Rightarrow} \mathfrak{M}$.

<u>Definition 1.4</u>: A functor $T:\mathfrak{A} \to \mathfrak{B}$ is said to be <u>$\mathcal{E}$-left exact</u> if and only if for any sequence $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ in $\mathcal{E}$ the sequences $0 \to T(A') \xrightarrow{T(i)} T(A)$ and $T(A') \xrightarrow{T(i)} T(A) \xrightarrow{T(j)} T(A'')$ are exact.

Let $T:\mathfrak{A} \to \mathfrak{B}$ be an additive, covariant, $\mathcal{E}$-left exact functor. Let A be any object in $\mathfrak{A}$ and X an $\mathcal{E}$-injective resolution of A, one such exists by the dual of Proposition 3.1, Eilenberg and Moore [6]. The following notation will be used:

$$\hat{X} : 0 \to A \xrightarrow{\epsilon} X_0 \xrightarrow{\partial^0} X_1 \xrightarrow{\partial^1} X_2 \xrightarrow{\partial^2} \cdots$$

and $A \xrightarrow{\epsilon} X$ denotes $\hat{X}$.

Then there is a complex

$$T(X) : 0 \to T(X_0) \underset{T(\partial^0)}{\overset{\delta^0}{\longrightarrow}} T(X_1) \xrightarrow{\delta^1} T(X_2) \to \cdots \to T(X_n) \xrightarrow{\delta^n} T(X_{n+1}) \to \cdots$$

in $\mathfrak{B}$. Since $\mathfrak{B}$ is an abelian category, for each n one has the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & T(X_{n-1}) \arrow[rr, "\delta^{n-1}"] \arrow[dr, "b_{n-1}"'] & & T(X_n) \arrow[r, "\delta^n"] & T(X_{n+1}) \arrow[r] & \cdots \\
& & Z_n \arrow[ur, "k_n"'] \arrow[d, "p_n"'] & & & \\
& & H^n(T(X)) & & &
\end{tikzcd}$$

Diagram 1.1.

where k_n is the kernel of δ^n, b_{n-1} is uniquely determined by the definition of kernel since $\delta^n\delta^{n-1} = 0$, and p_n is the cokernel of b_{n-1}. $H^n(T(X))$ can be shown to depend only on A, up to a natural equivalence. Hence, for each $n \geq 0$ one defines the derived functors $H^n:\mathfrak{A} \to \mathfrak{B}$ by (i) $H^n(A) = H^n(T(X))$ for each A in $\mathfrak{A}$ and (ii) $H^n(f):H^n(A) \to H^n(A')$, for each morphism $f:A \to A'$ in $\mathfrak{A}$, defined by Diagram 1.2; $A \xrightarrow{\epsilon} X$, $A' \xrightarrow{\epsilon'} Y$;

$$\begin{tikzcd}
\cdots \arrow[r] & T(X_{n-1}) \arrow[rrr, "\delta_X^{n-1}"] \arrow[dr, "b_{n-1}"] \arrow[ddd, "\bar{f}_{n-1}"'] & & & T(X_n) \arrow[r, "\delta_X^n"] \arrow[ddd, "\bar{f}_n"] & T(X_{n+1}) \arrow[r] \arrow[ddd, "\bar{f}_{n+1}"] & \cdots \\
& & Z_n \arrow[urr, "k_n"] \arrow[r, "p_n"] \arrow[d, "\bar{\bar{f}}_{n-1}"'] & H^n(A) \arrow[d, "H^n(f)"'] & & & \\
& & \bar{Z}_n \arrow[r, "\bar{p}_n"'] \arrow[drr, "\bar{k}_n"] & H^n(A') & & & \\
\cdots \arrow[r] & T(Y_{n-1}) \arrow[rrr, "\delta_Y^{n-1}"'] \arrow[ur, "\bar{b}_{n-1}"] & & & T(Y_n) \arrow[r, "\delta_Y^n"'] & T(Y_{n+1}) \arrow[r] & \cdots
\end{tikzcd}$$

Diagram 1.2.

where each square is commutative.

Remark 1.1: If $A \xrightarrow{f} B$ is any morphism in $\mathfrak{A}$, then $A \xrightarrow{f} B \xrightarrow{c} C$ is in $\mathcal{E}$ where c is the cokernel of f.

Proof: Let $I \in \mathcal{J}$. Then consider

$\mathrm{Hom}(C,I) \xrightarrow{c^*} \mathrm{Hom}(B,I) \xrightarrow{f^*} \mathrm{Hom}(A,I)$. Let $g \in \ker f^*$. From the definition of cokernel there exists a unique $\ell \in \mathrm{Hom}(C,I)$ such that $c^*(\ell) = g$ and $A \xrightarrow{f} B \xrightarrow{c} C \in \mathcal{E}$.

Remark 1.2: If A is any object in $\mathfrak{A}$ and $A \xrightarrow{g} B$ is an epic, then $A \xrightarrow{g} B \to 0 \in \mathcal{E}$.

Proof: (Immediate).

Lemma 1.1: If $0 \to Z_1 \xrightarrow{f_1} Z_2 \xrightarrow{f_2} Z_3$ is in $\mathcal{E}$, then $0 \to T(Z_1) \xrightarrow{T(f_1)} T(Z_2) \xrightarrow{T(f_2)} T(Z_3)$ is exact.

Proof: Let $c_i : Z_{i+1} \to C_i$ be the cokernel of f_i and consider the diagram

$$
\begin{tikzcd}
0 \arrow[r] & Z_1 \arrow[r, "f_1"] & Z_2 \arrow[r, "f_2"] \arrow[dr, "c_1"'] & Z_3 \arrow[dr, "c_2"] & & \\
& & 0 \arrow[r] & C_1 \arrow[u, "h"] \arrow[dr] & C_2 \arrow[dr] & \\
& & & & 0 & 0
\end{tikzcd}
$$

Since $f_2 f_1 = 0$ there exists a unique $h: C_1 \to Z_3$ such that $hc_1 = f_2$ because c_1 is a cokernel of f_1. By Remarks 1.1 and 1.2 the following sequences are all in $\mathcal{E}$.

$$Z_1 \xrightarrow{f_1} Z_2 \xrightarrow{c_1} C_1$$

$$Z_2 \xrightarrow{f_2} Z_3 \xrightarrow{c_2} C_2$$

$$Z_2 \xrightarrow{c_1} C_1 \longrightarrow 0$$

$$Z_3 \xrightarrow{c_2} C_2 \longrightarrow 0$$

It can now be shown that $0 \to C_1 \xrightarrow{h} Z_3 \xrightarrow{c_2} C_2 \to 0$ is in $\mathcal{E}$. Let $I \in \mathcal{J}$ and consider:

(i) $\mathrm{Hom}(Z_3, I) \xrightarrow{h^*} \mathrm{Hom}(C_1, I) \to 0.$

Let $\alpha \in \mathrm{Hom}(C_1, I)$. Then $\alpha c_1 \in \mathrm{Hom}(Z_2, I)$ and $f_1^*(\alpha c_1) = \alpha c_1 f_1 = 0.$

Since $Z_1 \xrightarrow{f_1} Z_2 \xrightarrow{f_2} Z_3$ is in $\mathcal{E}$, it is known that there exists a $\beta \in \mathrm{Hom}(Z_3, I)$ such that $\alpha c_1 = f_2^*(\beta) = \beta f_2 = \beta h c_1$. Hence $\alpha = \beta h$. Therefore, $0 \to C_1 \xrightarrow{h} Z_3 \in \mathcal{E}$.

(ii) $\mathrm{Hom}(C_2, I) \xrightarrow{c_2^*} \mathrm{Hom}(Z_3, I) \xrightarrow{h^*} \mathrm{Hom}(C_1, I)$.

Let $\delta \in \ker h^*$. Then $\delta f_2 = \delta h c_1 = 0$ and $\delta \in \ker f_2^*$. Since $Z_2 \xrightarrow{f_2} Z_3 \xrightarrow{c_2} C_2 \in \mathcal{E}$, there exists a $\gamma \in \mathrm{Hom}(C_2, I)$ such that $c_2^*(\gamma) = \delta$. This implies $\gamma \in \mathrm{im}\ c_2^*$ and $C_1 \xrightarrow{h} Z_3 \xrightarrow{c_2} C_2 \in \mathcal{E}$.

Since T is $\mathcal{E}$-left exact, in the diagram

$$\begin{tikzcd}
0 \arrow[r] & T(Z_1) \arrow[r, "T(f_1)"] & T(Z_2) \arrow[rr, "T(f_2)"] \arrow[dr, "T(c_1)"'] & & T(Z_3) \arrow[d, "T(c_2)"] \\
& & & T(C_1) \arrow[ur, "T(h)"] & T(C_2) \\
& & 0 \arrow[ur] & &
\end{tikzcd}$$

the sequences $0 \to T(Z_1) \xrightarrow{T(f_1)} T(Z_2) \xrightarrow{T(c_1)} T(C_1)$ and

$$0 \to T(C_1) \xrightarrow{T(h)} T(Z_3) \xrightarrow{T(c_2)} T(C_2) \text{ are exact.}$$

Therefore, $T(f_1)$ and $T(h)$ are monics and $T(f_1)$ is a kernel of $T(c_1)$. To show $T(f_1)$ is a kernel of $T(f_2)$, let $g: G \to T(Z_2)$ be any morphism such that $T(f_2)g = 0$. Then $T(h)T(c_1)g = 0$. Hence, $T(c_1)g = 0$ because $T(h)$ is a monic. Therefore, there exists a unique $\ell: G \to T(Z_1)$ such that $T(f_1)\ell = g$ and the proof is completed.

The theory of abelian categories is discussed in detail in Mitchell [17], Freyd [8] and Uehara [20].

Theorem 1.1: There exists a natural equivalence $\eta: T \to H^0$.

Proof: From the definition of $H^0(A)$, $H^0(A) = Z_0$. Moreover, T is $\mathcal{E}$-left exact. Hence $0 \to T(A) \xrightarrow[T(\epsilon)]{\epsilon_* \,=} T(X_0) \xrightarrow{\delta^0} T(X_1)$ is exact and ϵ_* is a monic. Therefore, $\epsilon_* = T(\epsilon)$ is a kernel of δ^0. But $H^0(A) \xrightarrow{k_0} T(X)_0$ is a kernel of δ^0. Hence, there exists a unique isomorphism $\beta_A : T(A) \to H^0(A)$ such that $h_0\beta_A = \epsilon_*$.

Now, define $\eta: T \to H^0$. For any A in $\mathfrak{A}$ let $\eta(A) = \beta_A$. Then η is a natural equivalence. Commutativity can be verified using Diagram 1.2.

Theorem 1.2: For each A in $\mathfrak{A}$ there exists $i: A \to I$ where $i \in \mathfrak{M}$, $I \in \mathcal{J}$ and $H^n(i) = 0$ for $n > 0$.

Proof: By the dual of Proposition 4.1 of Eilenberg and Moore [6], it is known that for each A in $\mathfrak{A}$ there exists $i \in \mathfrak{M}$ such that $i: A \to I$ where $I \in \mathcal{J}$. It can be verified that $0 \to I \xrightarrow{1} I \to 0 \to 0 \to \cdots$ is an $\mathcal{E}$-injective resolution of I. Therefore, $H^n(I) = \begin{cases} T(I) \text{ if } n = 0 \\ 0 \text{ if } n > 0. \end{cases}$

Lemma 1.2: If $\mathfrak{A}$ is an additive category, $\mathcal{E}$ an injective class in $\mathfrak{A}$ with $\mathcal{E} \overset{*}{\Rightarrow} \mathcal{J}$ and if $\{i_\sigma : A_\sigma \to A \mid \sigma = 1,2\}$ is a biproduct in $\mathfrak{A}$ then $A_1 \xrightarrow{i_1} A \xrightarrow{\pi_2} A_2$ and $A_2 \xrightarrow{i_2} A \xrightarrow{\pi_1} A_1$ are in $\mathcal{E}$. In fact $0 \to A_{\sigma_1} \xrightarrow{i_{\sigma_1}} A \xrightarrow{\pi_{\sigma_2}} A_{\sigma_2} \to 0$ is in $\mathcal{E}$ where $\sigma_1, \sigma_2 \in \{1,2\}$ and $\sigma_1 \neq \sigma_2$.

Proof: Let $I \in \mathcal{J}$ and let $f \in \ker i_1^*$, then $f: A \to I$ and $fi_1 = 0$. Consequently, $fi_2 : A_2 \to I$ and $fi_2\pi_2 = f$ because $1 = i_1\pi_1 + i_2\pi_2$. Hence, $\pi_2^*(fi_2) = f$ and $A_1 \xrightarrow{i_1} A \xrightarrow{\pi_2} A_2$ is in $\mathcal{E}$. Similarly $A_2 \xrightarrow{i_2} A \xrightarrow{\pi_1} A_1$

is in $\mathcal{E}$.

Consider $\mathrm{Hom}(A,I) \xrightarrow{i_1^*} \mathrm{Hom}(A_1,I) \to 0$. Let $f \in \mathrm{Hom}(A_1,I)$. Then one has the family $\{f:A_1 \to I;\ 0:A_2 \to I\}$. Since $\{i_1,i_2\}$ is a coproduct, there exists a unique morphism $k \in \mathrm{Hom}(A,I)$ such that $ki_1 = f$. Therefore, $0 \to A_1 \xrightarrow{i_1} A$ belongs to $\mathcal{E}$. To complete the proof one needs to show $0 \to \mathrm{Hom}(A_2,I) \xrightarrow{\pi_2^*} \mathrm{Hom}(A,I)$ is exact; i.e., show π_2^* is a monomorphism. Let $f \in \mathrm{Hom}(A_2,I)$ such that $f\pi_2 = 0$. Then $f = 0$ because π_2 is an epic.

<u>Remark 1.3</u>: In an additive category $\mathfrak{A}$ for any A_1, A_2, X in $\mathfrak{A}$ $\mathrm{Hom}(A_1 + A_2,X) \cong \mathrm{Hom}(A_1,X) + \mathrm{Hom}(A_2,X)$, MacLane [15 - 250].

<u>Proof</u>: Let $\{i_\sigma:A_\sigma \to A_1 + A_2 \mid \sigma = 1,2\}$ be the biproduct of A_1 and A_2. Let ψ: $\mathrm{Hom}(A_1 + A_2,X) \to \mathrm{Hom}(A_1,X) + \mathrm{Hom}(A_2,X)$ be defined as follows: for any $f \in \mathrm{Hom}(A_1 + A_2,X)$, $\psi(f) = (fi_1,fi_2)$. It is clear that ψ is a homomorphism. Let f, g be any morphisms in $\mathrm{Hom}(A_1 + A_2,X)$ such that $\psi(f) = \psi(g)$. Then $fi_1 = gi_1$ and $fi_2 = gi_2$. Since $\{i_1,i_2\}$ is a coproduct, $f = g$.

Let $(g,h) \in \mathrm{Hom}(A_1,X) + \mathrm{Hom}(A_2,X)$. By the definition of coproduct there exists a unique morphism $k:A_1 + A_2 \to X$ such that $ki_1 = g$ and $ki_2 = h$, hence ψ is surjective.

<u>Lemma 1.3</u>: Let $\mathfrak{C}$ be a pointed category and $\mathcal{E}$ an injective class in $\mathfrak{C}$ with $\mathcal{E} \overset{*}{\Rightarrow} \mathcal{J}$. If $0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ is in $\mathcal{E}$ and $0 \to A'' \xrightarrow{\epsilon} X$ is an $\mathcal{E}$-injective resolution of A'', then $A' \xrightarrow{i} A \xrightarrow{\epsilon j} X_0$ and $A \xrightarrow{\epsilon j} X_0 \xrightarrow{\partial^0} X_1$ are in $\mathcal{E}$.

<u>Proof</u>: Note that $A' \xrightarrow{i} A \xrightarrow{\epsilon j} X_0$ and $A \xrightarrow{\epsilon j} X_0 \xrightarrow{\partial^0} X_1$ are sequences.

Let $I \in \mathcal{J}$ and consider the following diagrams:

$$\text{(a)}\quad \begin{tikzcd} \mathrm{Hom}(X_0,I) \arrow[rr, "(\epsilon j)^*"] \arrow[dr, "\epsilon^*"] & & \mathrm{Hom}(A,I) \arrow[r, "i^*"] & \mathrm{Hom}(A',I) \\ & \mathrm{Hom}(A'',I) \arrow[ur, "j^*"] & & \\ & 0 \arrow[u] & & \end{tikzcd}$$

$$\text{(b)}\quad \begin{tikzcd} \mathrm{Hom}(X_1,I) \arrow[r, "(\partial^0)^*"] & \mathrm{Hom}(X_0,I) \arrow[rr, "(\epsilon j)^*"] \arrow[dr, "\epsilon^*"] & & \mathrm{Hom}(A,I) \\ & & \mathrm{Hom}(A'',I) \arrow[ur, "j^*"] & \\ & & 0 \arrow[u] & \end{tikzcd}$$

In (a) let $f \in \ker i^*$. From the hypothesis, $\operatorname{im} j^* = \ker i^*$ and ϵ^* is surjective. Hence, there exists $h \in \mathrm{Hom}(X_0,I)$ such that $(\epsilon j)^*(h) = f$, and $A' \xrightarrow{i} A \xrightarrow{\epsilon j} X_0$ is in $\mathcal{E}$. Similarly, using (b) one can show that $A \xrightarrow{\epsilon j} X_0 \xrightarrow{\partial^0} X_1$ is in $\mathcal{E}$.

Definition 1.5: A sequence of complexes; $0 \to X \xrightarrow{F} Y \xrightarrow{G} Z \to 0$ where $F = \{f_n : X_n \to Y_n \mid \text{all } n\}$ and $G = \{g_n : Y_n \to Z_n \mid \text{all } n\}$; is called an <u>$\mathcal{E}$-sequence of complexes</u> if and only if X, Y, Z are in $\mathcal{E}$ and for each n the following conditions are satisfied:

(i) $0 \to X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n \to 0$ is in $\mathcal{E}$,

(ii) the diagram

$$\begin{tikzcd} 0 \arrow[r] & X_n \arrow[r, "f_n"] \arrow[d, "\partial_X^n"'] & Y_n \arrow[r, "g_n"] \arrow[d, "\partial_Y^n"'] & Z_n \arrow[r] \arrow[d, "\partial_Z^n"'] & 0 \\ 0 \arrow[r] & X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \arrow[r, "g_{n+1}"] & Z_{n+1} \arrow[r] & 0 \end{tikzcd}$$

is commutative.

<u>Theorem 1.3</u>: If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is in $\mathcal{E}$, then there exists an $\mathcal{E}$-sequence of complexes $0 \to X \xrightarrow{F} Y \xrightarrow{G} Z \to 0$ and augmentations ϵ_1, ϵ_2, ϵ_3 such that $A \xrightarrow{\epsilon_1} X$, $B \xrightarrow{\epsilon_2} Y$ and $C \xrightarrow{\epsilon_3} Z$ are $\mathcal{E}$-injective resolutions for A, B and C, respectively.

<u>Proof</u>: One constructs $A \xrightarrow{\epsilon_1} X$ and $C \xrightarrow{\epsilon_3} Z$ in the usual manner; Eilenberg and Moore [6].

Define Y by the following construction (see Diagram 1.3).

(1) Let $Y_n = X_n + Z_n$. By Proposition 2.2 of [6]; $Y_n \in \mathcal{J}$ for $n \geq 0$.

For notational purposes, let $f_n = i_n^X$ and $g_n = \pi_n^Z$. Then by Lemma 1.2, $0 \to X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n \to 0$ is in $\mathcal{E}$.

(2) <u>Augmentation</u>: $0 \to A \xrightarrow{f} B$ belongs to $\mathcal{E}$ and $X_0 \in \mathcal{J}$ hence there exists an $\alpha : B \to X_0$ such that $\alpha f = \epsilon_1$. Let $\epsilon_2 = f_0\alpha + i_0^Z\epsilon_3 g$. Then $\epsilon_2 f = (f_0\alpha + i_0^Z\epsilon_3 g)f = f_0\alpha f = f_0\epsilon_1$ and $g_0\epsilon_2 = g_0(f_0\alpha + i_0^Z\epsilon_3 g) = \epsilon_3 g$.

In order to show $0 \to B \xrightarrow{\epsilon_3} Y_0$ belongs to $\mathcal{E}$ it needs to be shown that given any $I \in \mathcal{J}$ the sequence $\mathrm{Hom}(Y_0, I) \xrightarrow{\epsilon_3^*} \mathrm{Hom}(B, I) \to 0$ is exact; i.e., show ϵ_3^* is surjective. By Lemma 1.2 and Remark 1.3 the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\epsilon_1"'] & B \arrow[r, "g"] \arrow[d, "\epsilon_2"] \arrow[dl, dashed, "\alpha"'] & C \arrow[r] \arrow[d, "\epsilon_3"] & 0 \\
0 \arrow[r] & X_0 \arrow[r, shift left, "f_0"] \arrow[d, "\partial_X^0"'] & Y_0 \arrow[l, dashed, shift left, "\pi_0^X"] \arrow[r, shift left, "g_0"] \arrow[d, "\partial_Y^0"] & Z_0 \arrow[l, dashed, shift left, "i_0^Z"] \arrow[r] \arrow[d, "\partial_Z^0"] \arrow[dll, dashed, "\beta_0"] & 0 \\
0 \arrow[r] & X_1 \arrow[r, shift left, "f_1"] \arrow[d, "\partial_X^1"'] & Y_1 \arrow[l, dashed, shift left, "\pi_1^X"] \arrow[r, shift left, "g_1"] \arrow[d, "\partial_Y^1"] & Z_1 \arrow[l, dashed, shift left, "i_1^Z"] \arrow[r] \arrow[d, "\partial_Z^1"] \arrow[dll, dashed, "\beta_1"] & 0 \\
0 \arrow[r] & X_2 \arrow[r, shift left, "f_2"] \arrow[d] & Y_2 \arrow[l, dashed, shift left, "\pi_2^X"] \arrow[r, shift left, "g_2"] \arrow[d] & Z_2 \arrow[l, dashed, shift left, "i_2^Z"] \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & X_n \arrow[r, shift left, "f_n"] \arrow[d, "\partial_X^n"'] & Y_n \arrow[l, dashed, shift left, "\pi_n^X"] \arrow[r, shift left, "g_n"] \arrow[d, "\partial_Y^n"] & Z_n \arrow[l, dashed, shift left, "i_n^Z"] \arrow[r] \arrow[d, "\partial_Z^n"] \arrow[dll, dashed, "\beta_n"] & 0 \\
0 \arrow[r] & X_{n+1} \arrow[r, shift left, "f_{n+1}"] \arrow[d, "\partial_X^{n+1}"'] & Y_{n+1} \arrow[l, dashed, shift left, "\pi_{n+1}^X"] \arrow[r, shift left, "g_{n+1}"] \arrow[d, "\partial_Y^{n+1}"] & Z_{n+1} \arrow[l, dashed, shift left, "i_{n+1}^Z"] \arrow[r] \arrow[d, "\partial_Z^{n+1}"] \arrow[dll, dashed, "\beta_{n+1}"] & 0 \\
0 \arrow[r] & X_{n+2} \arrow[r, shift left, "f_{n+2}"] \arrow[d] & Y_{n+2} \arrow[l, dashed, shift left, "\pi_{n+2}^X"] \arrow[r, shift left, "g_{n+2}"] \arrow[d] & Z_{n+2} \arrow[l, dashed, shift left, "i_{n+2}^Z"] \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

Diagram 1.3

$$\begin{matrix} \mathrm{Hom}(X_0,I) \\ + \\ \mathrm{Hom}(Z_0,I) \end{matrix} = \begin{tikzcd} 0 & 0 & \\ \mathrm{Hom}(X_0,I) \arrow[u] \arrow[r, "\epsilon_1^*"] & \mathrm{Hom}(A,I) \arrow[u] \arrow[r] & 0 \\ \mathrm{Hom}(Y_0,I) \arrow[u, "\rho"] \arrow[r, "\epsilon_2^*"] & \mathrm{Hom}(B,I) \arrow[u, "f^*"'] \arrow[r] & 0 \\ \mathrm{Hom}(Z_0,I) \arrow[u, "\iota"] \arrow[r, "\epsilon_3^*"] & \mathrm{Hom}(C,I) \arrow[u, "g^*"'] \arrow[r] & 0 \\ 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

is obtained with rows 1, 3 and columns 1, 2 exact. Note, $\rho(h) = hf_0$ for any $h \in \mathrm{Hom}(Y_0,I)$ and for any $k \in \mathrm{Hom}(Z_0,I)$, $\iota(k)$ is the unique morphism in $\mathrm{Hom}(Y_0,I)$ such that $\iota(k)i_0^Z = k$ and $\iota(k)f_0 = 0$. Then each square is commutative because given any $k \in \mathrm{Hom}(Z_0,I)$, $\epsilon_2^*(\iota(k)) =$ $= \iota(k)(f_0\alpha + i_0^Z\epsilon_3 g) = k\epsilon_3 g$ and $g^*(\epsilon_3^*(k)) = k\epsilon_3 g$. Similarly $\epsilon_1^*\rho = f^*\epsilon_2^*$. Therefore, by the Five Lemma, ϵ_2^* is surjective.

(3) <u>Define ∂_Y^0:</u> By Lemma 1.3, $A \xrightarrow{f} C \xrightarrow{\epsilon_3 g} Z_0$ belongs to $\mathcal{E}$. Moreover, $\partial_X^0\alpha f = 0$, hence there exists $\beta_0 : Z_0 \to X_1$ such that $\beta_0\epsilon_3 g = \partial_X^0\alpha$.

Define ∂_Y^0 by

$$\partial_Y^0 = f_1(\partial_X^0\pi_0^X - \beta_0 g_0) + i_1^Z\partial_Z^0 g_0.$$

One can immediately verify that each square is commutative. One still needs to show:

(a) $\partial_Y^0\epsilon_2 = 0$

(b) $B \xrightarrow{\epsilon_2} Y_0 \xrightarrow{\partial_Y^0} Y_1$ belongs to $\mathcal{E}$.

(a). $\partial_Y^0\epsilon_2 = [f_1(\partial_X^0\pi_0^X - \beta_0 g_0) + i_1^Z\partial_Z^0 g_0](f_0\alpha + i_0^Z\epsilon_3 g)$

$$= f_1(\partial_X^0 \pi_0^X - \beta_0 g_0)(f_0\alpha + i_0^Z \epsilon_3 g) + i_1^Z \partial_Z^0 g_0 (f_0 \alpha + i_0^Z \epsilon_3 g)$$

$$= f_1 \partial_X^0 \alpha - f_1 \beta_0 \epsilon_3 g = f_1 \partial_X^0 \alpha - f_1 \partial_X^0 \alpha = 0.$$

(b). Let I be any object in $\mathcal{I}$. Then the following diagram is obtained

$$\begin{tikzcd}
0 & 0 & 0 \\
\mathrm{Hom}(X_1,I) \arrow[u] \arrow[r, "\partial_X^{0*}"] \arrow[d, dashed, shift left] \arrow[ddr, dashed, "\beta_0^*"] & \mathrm{Hom}(X_0,I) \arrow[u] \arrow[r, "\epsilon_1^*"] \arrow[d, dashed, shift left] \arrow[dr, dashed, "\alpha^*"] & \mathrm{Hom}(A,I) \arrow[u] \\
\mathrm{Hom}(Y_1,I) \arrow[u, "f_1^*"] \arrow[r, "\partial_Y^{0*}"] \arrow[d, dashed, shift left] & \mathrm{Hom}(Y_0,I) \arrow[u, "f_0^*"] \arrow[r, "\epsilon_2^*"] \arrow[d, dashed, shift left] & \mathrm{Hom}(B,I) \arrow[u, "f^*"'] \\
\mathrm{Hom}(Z_1,I) \arrow[u, "g_1^*"] \arrow[r, "\partial_Z^{0*}"'] & \mathrm{Hom}(Z_0,I) \arrow[u, "g_0^*"] \arrow[r, "\epsilon_3^*"'] & \mathrm{Hom}(C,I) \arrow[u, "g^*"'] \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

where columns 1, 2, 3 are exact, rows 1 and 3 are exact and every square of solid arrows is commutative. Then row 2 is exact by the following remark from the category of abelian groups. Hence $B \xrightarrow{\epsilon_2} Y_0 \xrightarrow{\partial_Y^0} Y_1$ belongs to $\mathcal{E}$.

<u>Remark 1.4</u>: Consider the following diagram where the objects and morphisms are in the category of abelian groups,

$$\begin{tikzcd}
G_1' \arrow[r, "f_3"] \arrow[d, dashed, shift left, "i_1'"] \arrow[dr, dashed] & G_0' \arrow[r, "g_3"] \arrow[d, dashed, shift left, "i_0'"] \arrow[dr, dashed, "\xi"] & A'' \\
G_1 + G_1' \arrow[u, "\pi_1'"] \arrow[r, "f_2"] \arrow[d, dashed, shift left, "\pi_1"] \arrow[dr, dashed, "\xi'"] & G_0 + G_0' \arrow[u, "\pi_0'"] \arrow[r, "g_2"] \arrow[d, dashed, shift left, "\pi_0"] & A \arrow[u, "\gamma"'] \\
G_1 \arrow[u, "i_1"] \arrow[r, "f_1"'] & G_0 \arrow[u, "i_0"] \arrow[r, "g_1"'] & A' \arrow[u, "\beta"']
\end{tikzcd}$$

and the following conditions are satisfied on the morphisms:

(i) rows 1, 2, and column 3 are exact sequences (i.e., ker = im),

(ii) β is a monomorphism,

(iii) commutativity in each square of solid arrows,

(iv) $\beta g_1 \xi' = \xi f_3$ and $\gamma\xi = g_3$,

(v) $g_2 = \xi\pi'_0 + \beta g_1 \pi_0$,

(vi) $f_2 = (i'_0 f_3 - i_0 \xi')\pi'_1 + i_0 f_1 \pi_1$.

Then $G_1 + G'_1 \xrightarrow{f_2} G_0 + G'_0 \xrightarrow{g_2} A$ is an exact sequence.

Proof of the remark: $g_2 f_2 = 0$ by an argument similar to that used to show $\partial^0_Y \epsilon_2 = 0$, so im $f_2 \subset$ ker g_2. Now, let $(x,x') \in$ ker g_2. This implies that $0 = g_2(x,x') - \beta g_1(x) + \xi(x')$. By commutativity $g_3(x') = \gamma g_2(x,x') = 0$, hence $x' \in$ ker g_3 = im f_3. Therefore, there exists $y' \in G'_1$ such that $f_3(y') = x'$. Now, consider $z = x + \xi'(y') \in G_0$; $\beta g_1(z) = \beta g_1 \xi'(y') + \beta g_1(x) = \xi f_3(y') + \beta g_1(x)$ $= \xi(x') + \beta g_1(x) = 0$. β is a monomorphism, hence $z \in$ ker g_1 = im f_1 and there exists $y \in G_1$ such that $f_1(y) = z$. Then $f_2(y,y') =$

$= [(i'_0 f_3 - i_0\xi')\pi'_1 + i_0 f_1 \pi_1](y,y') = (i'_0 f_3 - i_0 \xi')(y') + i_0 f_1(y)$

$= i'_0 f_3(y') - i_0\xi'(y') + i_0(z) = i'_0(x') - i_0\xi'(y') + i_0\xi'(y') + i_0(x)$

$= (x,x')$.

(4) Define ∂^1_Y: From Lemma 1.3 the sequence $B \xrightarrow{\epsilon_3 g} Z_0 \xrightarrow{\partial^0_Z} Z_1$ belongs to $\mathcal{E}$. Also $\partial^1_X \beta_0 \epsilon_3 g = \partial^1_X \partial^0_X \alpha = 0$. Hence, there exists $\beta_1 : Z_1 \to X_2$ such that $\beta_1 \partial^0_Z = \partial^1_X \beta_0$. Now define ∂^1_Y by setting

$$\partial^1_Y = f_2(\partial^1_X \pi^X_1 + \beta_1 g_1) + i^Z_2 \partial^1_Z g_1.$$

One can readily verify that $\partial_Y^1 f_1 = f_2 \partial_X^1$ and $g_2 \partial_Y^1 = \partial_Z^1 g_1$. It remains to be shown that ∂_Y^1 has the following properties:

(a) $\partial_Y^1 \partial_Y^0 = 0$

(b) $Y_0 \xrightarrow{\partial_Y^0} Y_1 \xrightarrow{\partial_Y^1} Y_2$ belongs to $\mathcal{E}$.

A straightforward calculation establishes (a). To show (b) consider the following diagram for any $I \in \mathcal{J}$.

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathrm{Hom}(X_2,I) \arrow[r, "(\partial_X^1)^*"] \arrow[d, shift left, dashed, "(\pi_2^X)^*"] \arrow[ddr, dashed, "\beta_1^*"] & \mathrm{Hom}(X_1,I) \arrow[r, "(\partial_X^0)^*"] \arrow[d, shift left, dashed, "(\pi_1^X)^*"] \arrow[ddr, dashed, "\beta_0^*"] & \mathrm{Hom}(X_0,I) \arrow[d, shift left, dashed, "(\pi_0^X)^*"] \\
\mathrm{Hom}(Y_2,I) \arrow[r, "(\partial_Y^1)^*"] \arrow[u, shift left, "f_2^*"] \arrow[d, shift left, dashed, "(i_2^Z)^*"] & \mathrm{Hom}(Y_1,I) \arrow[r, "(\partial_Y^0)^*"] \arrow[u, shift left, "f_1^*"] \arrow[d, shift left, dashed, "(i_1^Z)^*"] & \mathrm{Hom}(Y_0,I) \arrow[u, shift left, "f_0^*"] \arrow[d, shift left, dashed, "(i_0^Z)^*"] \\
\mathrm{Hom}(Z_2,I) \arrow[r, "(\partial_Z^1)^*"] \arrow[u, shift left, "g_2^*"] & \mathrm{Hom}(Z_1,I) \arrow[r, "(\partial_Z^0)^*"] \arrow[u, shift left, "g_1^*"] & \mathrm{Hom}(Z_0,I) \arrow[u, shift left, "g_0^*"]
\end{tikzcd}$$

where rows 1 and 3 are exact and the columns are direct sum diagrams. By the remark which follows this proof, the middle row is exact and $Y_0 \xrightarrow{\partial_Y^0} Y_1 \xrightarrow{\partial_Y^1} Y_2$ is in $\mathcal{E}$.

(5) Assume, for $k \leq n$, there exists $\beta_k : Z_k \to X_{k+1}$ such that

$\beta_k \partial_Z^{k-1} = \partial_X^k \beta_{k-1}$ and that ∂_Y^k has been defined by

$$\partial_Y^k = f_{k+1}(\partial_X^k \pi_k^X + (-1)^{k+1} \beta_k g_k) + i_{k+1}^Z \partial_Z^k g_k$$

with the properties

(a) $\partial_Y^k \partial_Y^{k-1} = 0$,

(b) $Y_{k-1} \xrightarrow{\partial_Y^{k-1}} Y_k \xrightarrow{\partial_Y^k} Y_{k+1}$ belongs to $\mathcal{E}$,

(c) $\partial_Y^k f_k = f_{k+1}\partial_X^k$ and $g_{k+1}\partial_Y^k = \partial_Z^k g_k$.

(6) <u>Define</u> ∂_Y^{n+1}: Since $Z_{n-1} \xrightarrow{\partial_Z^{n-1}} Z_n \xrightarrow{\partial_Z^n} Z_{n+1}$ belongs to $\mathcal{E}$ and $\partial_X^{n+1}\beta_n\partial_Z^{n-1} = \partial_X^{n+1}\partial_X^n\beta_{n-1} = 0$, there exists $\beta_{n+1}:Z_{n+1} \to X_{n+2}$ such that $\beta_{n+1}\partial_Z^n = \partial_X^{n+1}\beta_n$. Now define ∂_Y^{n+1} by

$$\partial_Y^{n+1} = f_{n+2}(\partial_X^{n+1}\pi_{n+1}^X + (-1)^{n+2}\beta_{n+1}g_{n+1}) + i_{n+2}^Z\partial_Z^{n+1}g_{n+1}.$$

Commutativity can be verified without difficulty. The following two properties are also satisfied:

(a) $\partial_Y^{n+1}\partial_Y^n = 0$,

(b) $Y_n \xrightarrow{\partial_Y^n} Y_{n+1} \xrightarrow{\partial_Y^{n+1}} Y_{n+2}$ belongs to $\mathcal{E}$.

(a) $$\partial_Y^{n+1}\partial_Y^n = \left[f_{n+2}(\partial_X^{n+1}\pi_{n+1}^X + (-1)^{n+2}\beta_{n+1}g_{n+1}) + i_{n+2}^Z\partial_Z^{n+1}g_{n+1}\right]\circ$$

$$\circ\left[f_{n+1}(\partial_X^n\pi_n^X + (-1)^{n+1}\beta_n g_n) + i_{n+1}^Z\partial_Z^n g_n\right]$$

$$= (f_{n+2}\partial_X^{n+1}\pi_{n+1}^X + (-1)^{n+2}f_{n+2}\beta_{n+1}g_{n+1} + i_{n+2}^Z\partial_Z^{n+1}g_{n+1})\circ$$

$$\circ(f_{n+1}\partial_X^n\pi_n^X + (-1)^{n+1}f_{n+1}\beta_n g_n + i_{n+1}^Z\partial_Z^n g_n)$$

$$= f_{n+2}\partial_X^{n+1}\pi_{n+1}^X f_{n+1}\partial_X^n\pi_n^X + (-1)^{n+1}f_{n+2}\partial_X^{n+1}\pi_{n+1}^X f_{n+1}\beta_n g_n$$

$$+ f_{n+2}\partial_X^{n+1}\pi_{n+1}^X i_{n+1}^Z\partial_Z^n g_n + (-1)^{n+2}f_{n+2}\beta_{n+1}g_{n+1}f_{n+1}\partial_X^n\pi_n^X$$

$$+ (-1)^{2n+3}f_{n+2}\beta_{n+1}g_{n+1}f_{n+1}\beta_n g_n + (-1)^{n+2}f_{n+2}\beta_{n+1}g_{n+1}i_{n+1}^Z\partial_Z^n g_n$$

$$+ i^Z_{n+2}\partial^{n+1}_Z g_{n+1} f_{n+1}\partial^n_X \pi^X_n + (-1)^{n+1} i^Z_{n+2}\partial^{n+1}_Z g_{n+1} f_{n+1}\beta_n g_n$$

$$+ i^Z_{n+2}\partial^{n+1}_Z g_{n+1} i^Z_{n+1}\partial^n_Z g_n$$

$$= (-1)^{n+1} f_{n+2}\partial^{n+1}_X \beta_n g_n + (-1)^{n+2} f_{n+2}\beta_{n+1}\partial^n_Z g_n$$

$$= (-1)^{n+1} f_{n+2}\left[\partial^{n+1}_X \beta_n - \beta_{n+1}\partial^n_Z\right] g_n = 0.$$

(b) Let $I \in \mathcal{J}$ and consider the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Hom}(X_{n+2},I) \arrow[r, "(\partial^{n+1}_X)^*"] \arrow[d, dashed, shift left, "(\pi^X_{n+2})^*"] \arrow[ddr, dashed, "\beta^*_{n+1}" near end] & \mathrm{Hom}(X_{n+1},I) \arrow[r, "(\partial^n_X)^*"] \arrow[d, dashed, shift left, "(\pi^X_{n+1})^*"] \arrow[ddr, dashed, "\beta^*_n" near end] & \mathrm{Hom}(X_n,I) \arrow[d, dashed, shift left, "(\pi^X_n)^*"] \\
\mathrm{Hom}(Y_{n+2},I) \arrow[r, "(\partial^{n+1}_Y)^*"] \arrow[u, shift left, "f^*_{n+2}"] \arrow[d, dashed, shift left, "(i^Z_{n+2})^*"] & \mathrm{Hom}(Y_{n+1},I) \arrow[r, "(\partial^n_Y)^*"] \arrow[u, shift left, "f^*_{n+1}"] \arrow[d, dashed, shift left, "(i^Z_{n+1})^*"] & \mathrm{Hom}(Y_n,I) \arrow[u, shift left, "f^*_n"] \arrow[d, dashed, shift left, "(i^Z_n)^*"] \\
\mathrm{Hom}(Z_{n+2},I) \arrow[r, "(\partial^{n+1}_Z)^*"'] \arrow[u, shift left, "g^*_{n+2}"] & \mathrm{Hom}(Z_{n+1},I) \arrow[r, "(\partial^n_Z)^*"'] \arrow[u, shift left, "g^*_{n+1}"] & \mathrm{Hom}(Z_n,I) \arrow[u, shift left, "g^*_n"]
\end{tikzcd}$$

where rows 1 and 3 are exact and the columns are direct sum diagrams. By the remark that follows, the middle row is exact and

$Y_n \xrightarrow{\partial^n_Y} Y_{n+1} \xrightarrow{\partial^{n+1}_Y} Y_{n+2}$ belongs to $\mathcal{E}$. This completes the desired construction.

<u>Remark 1.5</u>: Consider the following diagram where the objects and morphisms are in the category of abelian groups:

$$\begin{tikzcd}[row sep=huge, column sep=huge]
G_2' \arrow[r, "g_3"] \arrow[d, dashed, shift left, "i_2'"] \arrow[ddr, dashed, "\xi'"] & G_1' \arrow[r, "f_3"] \arrow[d, dashed, shift left, "i_1'"] \arrow[ddr, dashed, "\xi"] & G_0' \arrow[d, dashed, shift left, "i_0'"] \\
G_2 + G_2' \arrow[r, "g_2"] \arrow[u, shift left, "\pi_2'"] \arrow[d, dashed, shift left, "\pi_2"] & G_1 + G_1' \arrow[r, "f_2"] \arrow[u, shift left, "\pi_1'"] \arrow[d, dashed, shift left, "\pi_1"] & G_0 + G_0' \arrow[u, shift left, "\pi_0'"] \arrow[d, dashed, shift left, "\pi_0"] \\
G_2 \arrow[r, "g_1"'] \arrow[u, shift left, "i_2"] & G_1 \arrow[r, "f_1"'] \arrow[u, shift left, "i_1"] & G_0 \arrow[u, shift left, "i_0"]
\end{tikzcd}$$

If the following conditions are satisfied on the morphisms:

(i) the rows 1 and 3 are exact sequences,

(ii) the columns are direct sum diagrams,

(iii) commutativity in each square with solid arrows,

(iv) $\xi g_3 = f_1 \xi'$,

(v) $f_2 = i_0' f_3 + (-1)^k i_0 \xi)\pi_1' + i_0 f_1 \pi_1$,

(vi) $g_2 = (i_1' g_3 + (-1)^{k+1} i_1 \xi')\pi_2' + i_1 g_1 \pi_2$,

then row 2 is an exact sequence in the sense that im g_2 = ker f_2.

Proof: By a direct computation, as done previously, it can be verified that $f_2 g_2 = 0$. Now, let $(x,x') \in G_1 + G_1'$ such that $f_2(x,x') = 0$. Then by commutativity, $f_3(x') = 0$. Hence $x' \in \ker f_3 = \operatorname{im} g_3$ and there exists $y' \in G_2'$ such that $g_3(y') = x'$.

Consider $z = x + (-1)^k \xi'(y') \in G_1$. Then $i_0 f_1(z) =$

$$= i_0 f_1(x) + (-1)^k i_0 f_1 \xi'(y') = i_0 f_1 \pi_1(x,x') + (-1)^k i_0 \xi g_3(y')$$

$$= i_0 f_1 \pi_1(x,x') + (-1)^k i_0 \xi \pi_1'(x,x') = f_2(x,x') - i_0' f_3 \pi_1'(x,x') = 0.$$

Hence $z \in \ker f_1 = \operatorname{im} g_1$ because i_0 is a monomorphism. Let $y \in G_2$ such that $g_1(y) = z$, then $g_2(y,y') = (i_1' g_3 + (-1)^{k+1} i_1 \xi')(y') + i_1 g_1(y)$

$$= i_1'(x') + (-1)^{k+1} i_1 \xi'(y') + i_1(z) = i_1'(x') + (-1)^{k+1} i_1 \xi'(y') + i_1(x) + (-1)^k i_1 \xi'(y')$$

$= i_1'(x') + i_1(x) = (x,x')$ and the proof is completed.

__Lemma 1.4__: If $X = \{X_n \xrightarrow{\partial_X^n} X_{n+1} \mid n \geq 0\}$ and $Y = \{Y_n \xrightarrow{\partial_Y^n} Y_{n+1} \mid n \geq 0\}$ are chain complexes in $\mathfrak{B}$ and if $\xi = \{X_n \xrightarrow{\xi_n} Y_{n+1} \mid n \geq 0\}$ is a sequence of morphisms with the property; for each $n \geq 0$, $\xi_{n+1}\partial_X^n = \partial_Y^{n+1}\xi_n$; then for each $n \geq 0$ there exists a morphism $\Delta^n : H^n(X) \to H^{n+1}(Y)$.

__Proof__: Consider the following diagram (recall Diagram 1.1):

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n-1} \arrow[rrr, "\partial_X^{n-1}"] \arrow[dr, "b_{n-1}"] \arrow[ddddd, "\xi_{n-1}"'] & & & X_n \arrow[r, "\partial_X^n"] \arrow[ddddd, "\xi^n"] & X_{n+1} \arrow[r] \arrow[ddddd, "\xi_{n+1}"] & \cdots \\
& & Z_n \arrow[urr, "k_n"'] \arrow[dr, "p_n"] \arrow[ddd, "\overline{\xi}_n"'] & & & & \\
& & & H^n(X) \arrow[d, "\Delta^n"] & & & \\
& & & H^{n+1}(Y) & & & \\
& & Z'_{n+1} \arrow[ur, "p'_{n+1}"] \arrow[drr, "k'_{n+1}"] & & & & \\
\cdots \arrow[r] & Y_n \arrow[ur, "b'_n"] \arrow[rrr, "\partial_Y^n"'] & & & Y_{n+1} \arrow[r, "\partial_Y^{n+1}"'] & Y_{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

Diagram 1.4

By commutativity $\partial_Y^n \xi_n k_n = \xi_{n+1} \partial_X^n k_n = 0$, hence there exists a unique $\bar{\xi}_n : Z_n \to Z'_{n+1}$ such that $k'_{n+1} \bar{\xi}_n = \xi_n k_n$ because k'_{n+1} is the kernel of ∂_Y^{n+1}. Also, by commutativity, $k'_{n+1} \bar{\xi}_n b_{n-1} = \xi_n k_n b_{n-1} = \xi_n \partial_X^{n-1} = \partial_Y^n \xi_{n-1}$ $= k'_{n+1} b'_n \xi_{n-1}$. k'_{n+1} is a monic, hence $\bar{\xi}_n b_{n-1} = b'_n \xi_{n-1}$, $p'_{n+1} \bar{\xi}_n b_{n-1} =$ $= p'_{n+1} b'_n \xi_{n-1}$ and $p'_{n+1} b'_n \xi_{n-1} = 0$ because p'_{n+1} is the cokernel of b'_n. Thus there exists a unique morphism $\Delta^n : H^n(X) \to H^{n+1}(X)$ such that $\Delta^n p_n = p'_{n+1} \bar{\xi}_n$.

Proposition 1.1: For each sequence $E: 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{E}$ and for each $n \geq 0$ there exists a morphism $\Delta_E^n : H^n(C) \to H^{n+1}(A)$ such that the following is a sequence.

$$0 \to T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \xrightarrow{\Delta_E^0} H^1(A) \xrightarrow{H^1(f)} H^1(B) \to \cdots$$

$$\cdots \xrightarrow{\Delta_E^{n-1}} H^n(A) \xrightarrow{H^n(f)} H^n(B) \xrightarrow{H^n(g)} H^n(C) \xrightarrow{\Delta_E^n} \cdots$$

Proof: First Δ_E^0 will be constructed and $\Delta_E^0 T(g) = 0 = H^1(f)\Delta_E^0$ will be verified. From Lemma 1.4, there exist morphisms $\Delta_E^n : H^n(C) \to H^{n+1}(A)$ for all $n \geq 1$. $\Delta_E^1 H^1(g) = 0 = H^2(f)\Delta_E^1$ will be shown. The proof that $\Delta_E^n H^n(g) = 0 = H^{n+1}(f)\Delta_E^n$ is exactly the same, taking into consideration the definition of ∂_Y^n for $n \geq 1$.

By commutativity, $\delta_X^1 T(\beta_0) T(\epsilon_3) = T(\beta_1)\delta_Z^0 T(\epsilon_3) = 0$. Therefore, there exists a unique morphism $\gamma : T(C) \to Z_1$ such that $k_1 \gamma = T(\beta_0)T(\epsilon_3)$. Define $\Delta_E^0 = p_1 \gamma$. $\gamma T(g) = b_0 T(\alpha)$ because k_1 is a monic and $k_1 \gamma T(g)$ $= T(\beta_0)T(\epsilon_3)T(g) = \delta_X^0 T(\alpha) = k_1 b_0 T(\alpha)$. Using this commutativity and the fact that p_1 is the cokernel of b_0, $\Delta_E^0 T(g) = p_1\gamma T(g) = p_1 b_0 T(\alpha) = 0$.

To prove $\Delta_E^1 H^1(g) = 0$, recall three facts:

(a) From Lemma 1.4, $p_2\,\overline{T(\beta_1)} = \Delta_E^1\bar{\bar{p}}_1$, $T(\beta_1)\bar{\bar{k}}_1 = k_2\overline{T(\beta_1)}$;

(b)
$$\begin{aligned}\delta_Y^1 = T(\partial_Y^1) &= T[f_2(\partial_X^1\pi_1^X + \beta_1 g_1) + i_2^Z\partial_Z^1 g_1)\\ &= T(f_2)[T(\partial_X^1)T(\pi_1^X) + T(\beta_1)T(g_1)]\\ &\qquad + T(i_2^Z)T(\partial_Z^1)T(g_1)\\ &= T(f_2)[\delta_X^1 T(\pi_1^X) + T(\beta_1)T(g_1)] + T(i_2^Z)\delta_Z^1\,T(g_1);\end{aligned}$$

(c) by commutativity

$$T(i_2^Z)\delta_Z^1 T(g_1)\bar{k}_1 = T(i_2^Z)T(g_2)\delta_Y^1\,\bar{k}_1 = 0$$

$$T(f_2)\delta_X^1 T(\pi_1^X)\bar{k}_1 = \delta_Y^1 T(f_1)T(\pi_1^X)\bar{k}_1 = \delta_Y^1\bar{k}_1 = 0.$$

Therefore, $T(f_2)T(\beta_1)T(g_1)\bar{k}_1 = 0$. T is additive, hence preserves biproducts. $T(f_2)$ is a monic, therefore $T(\beta_1)T(g_1)\bar{k}_1 = 0$. Now, $0 = T(\beta_1)T(g_1)\bar{k}_1 = T(\beta_1)\bar{\bar{k}}_1\bar{g}_0 = k_2\overline{T(\beta_1)}\,\bar{g}_0$. k_2 is a monic so $\overline{T(\beta_1)}\,\bar{g}_0 = 0$. Using commutativity, $\Delta_E^1 H^1(g)\bar{p}_1 = \Delta_E^1\bar{\bar{p}}_1\bar{g}_0 = p_2\overline{T(\beta_1)}\,\bar{g}_0 = 0$. $\bar{p}_1$ is an epic, hence $\Delta_E^1 H^1(g) = 0$.

Since $\partial_Y^0 = f_1\partial_X^0\pi_0^X - f_1\beta_0 g_0 + i_1^Z\partial_Z^0 g_0$; $\delta_Y^0 T(i_0^Z)T(\epsilon_3)$ $= -T(f_1)T(\beta_0)T(\epsilon_3) = -T(f_1)k_1\gamma = -\bar{k}_1\bar{f}_0\gamma$. Therefore, $\bar{k}_1\bar{b}_0 T(i_0^Z)T(\epsilon_3)$ $= -\bar{k}_1\bar{f}_0\gamma$ and $-\bar{b}_0 T(i_0^Z)T(\epsilon_3) = \bar{f}_0\gamma$. Now, $H^1(f)\Delta_E^0 = H^1(f)p_1\gamma$ $= \bar{p}_1\bar{f}_0\gamma = -\bar{p}_1\bar{b}_0 T(i_0^Z)T(\epsilon_3) = 0$ because $\bar{p}_1$ is a cokernel of $\bar{b}_0$. Similarly $H^2(f)\Delta_E^1 = 0$ because $\bar{k}_2\bar{b}_1 T(i_1^Z)\bar{\bar{k}}_1 = \delta_Y^1 T(i_1^Z)\bar{\bar{k}}_1 = T(f_2)T(\beta_1)\bar{\bar{k}}_1$ $= T(f_2)k_2\overline{T(\beta_1)} = \bar{k}_2\bar{f}_1\overline{T(\beta_1)}$ and, since $\bar{k}_2$ is a monic, $\bar{b}_1 T(i_1^Z)\bar{\bar{k}}_1 = \bar{f}_1\overline{T(\beta_1)}$.

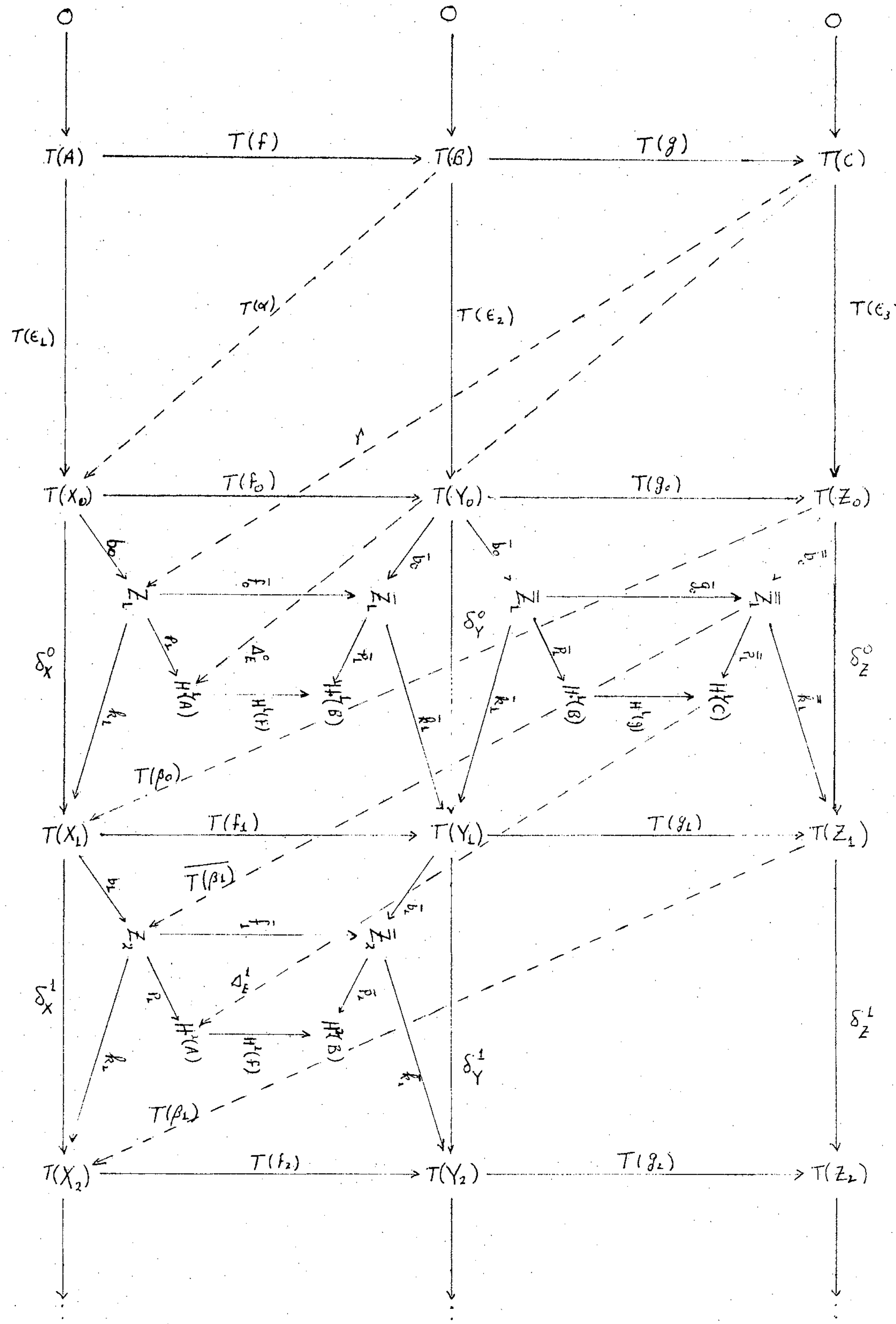

Diagram 1.5

Therefore $H^2(f)\Delta^1_E\bar{\bar{p}}_1 = H^2(f)p_2\overline{T(\beta_1)} = \bar{p}_2\bar{f}_1\overline{T(\beta_1)} = \bar{p}_2\bar{b}_1T(i^Z_1)\bar{\bar{k}}_1 = 0$ and $H^2(f)\Delta^1_E = 0$.

<u>Theorem 1.4</u>: (Naturality Condition) For a commutative diagram

$$\begin{tikzcd}
E_1 : 0 \arrow[r] & A' \arrow[r, "f"] \arrow[d, "\varphi'"] & A \arrow[r, "g"] \arrow[d, "\varphi"] & A'' \arrow[r] \arrow[d, "\varphi''"] & 0 \\
E_2 : 0 \arrow[r] & B' \arrow[r, "h"] & B \arrow[r, "k"] & B'' \arrow[r] & 0
\end{tikzcd}$$

of two sequences E_1, E_2 in $\mathcal{E}$ and for each $n \geq 0$ the diagram

$$\begin{tikzcd}
H^n(A'') \arrow[r, "\Delta^n_{E_1}"] \arrow[d, "H^n(\varphi'')"'] & H^{n+1}(A') \arrow[d, "H^{n+1}(\varphi')"] \\
H^n(B'') \arrow[r, "\Delta^n_{E_2}"'] & H^{n+1}(B')
\end{tikzcd}$$

is commutative.

<u>Proof</u>: The following notation will be used throughout this proof:

(i)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r, "f"] \arrow[d, "\epsilon'"'] & A \arrow[r, "g"] \arrow[d, "\epsilon"'] & A'' \arrow[r] \arrow[d, "\epsilon''"'] & 0 \\
0 \arrow[r] & X' \arrow[r, "F"] & X \arrow[r, "G"] & X'' \arrow[r] & 0
\end{tikzcd}$$

where $X' = \{X'_n \xrightarrow{\partial^n_{X'}} X'_{n+1} \mid n \geq 0\}$; similarly for X and X'' and $\beta_n : X''_n \to X'_{n+1}$ for $n \geq 0$.

(ii)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B' \arrow[r, "h"] \arrow[d, "\eta'"'] & B \arrow[r, "k"] \arrow[d, "\eta"'] & B'' \arrow[r] \arrow[d, "\eta''"'] & 0 \\
0 \arrow[r] & Y' \arrow[r, "H"] & Y \arrow[r, "K"] & Y'' \arrow[r] & 0
\end{tikzcd}$$

where $Y' = \{Y'_n \xrightarrow{\partial^n_{Y'}} Y'_{n+1} \mid n \geq 0\}$, similarly for Y and Y'' and $\xi_n : Y''_n \to Y'_{n+1}$ for $n \geq 0$.

(iii) Let $W'_1 \xrightarrow{k_1^{W'}} T(Y'_1)$ denote the kernel of $\delta^1_{Y'}$ and $Z'_1 \xrightarrow{k_1^{Z'}} T(X'_1)$ the kernel of $\delta^1_{X'}$.

Consider the diagram:

$$\begin{tikzcd}
T(A'') \arrow[rrr, "T(\epsilon'')"] \arrow[dr, "T(\varphi'')"] \arrow[ddd, "\gamma''_A"'] & & & T(X''_0) \arrow[dl, "T(\varphi''_0)"'] \arrow[ddd, "T(\beta_0)"] \\
 & T(B'') \arrow[r, "T(\eta'')"] \arrow[d, "\gamma_{B''}"] \arrow[ur, phantom, "(\mathrm{IV})"] \arrow[d, phantom, shift right=8ex, "(\mathrm{V})"] & T(Y''_0) \arrow[d, "T(\xi_0)"] \arrow[dl, phantom, "(\mathrm{I})"] \arrow[d, phantom, shift left=8ex, "(\mathrm{III})"] & \\
 & W'_1 \arrow[r, "k_1^{W'}"] & T(Y'_1) \arrow[dl, phantom, "(\mathrm{II})"] & \\
Z'_1 \arrow[ur, "\hat{\varphi}"] \arrow[rrr, "k_1^{Z'}"'] & & & T(X'_1) \arrow[ul, "T(\varphi'_1)"']
\end{tikzcd}$$

Diagram 1.6

where I, II and IV are known to be commutative.

Now $\delta^1_{Y'} T(\varphi'_1) k_1^{Z'} = T(\varphi'_2) \delta^1_X k_1^{Z'} = 0$. Therefore, there exists a unique $\hat{\varphi}: Z'_1 \to W'_1$ such that $k_1^{W'} \hat{\varphi} = T(\varphi'_1) k_1^{Z'}$. In the diagram

$$
\begin{tikzcd}[column sep=large, row sep=huge]
T(A'') \arrow[r, "T(\epsilon'')"] \arrow[d, shift right, "\gamma_{B''}T(\varphi'')"'] \arrow[d, shift left, "\hat{\varphi}\gamma_{A''}"] & T(X''_0) \arrow[r, "\delta^0_{X''}"] \arrow[dl, "\lambda_0"'] \arrow[d, shift right, "T(\xi_0)T(\varphi''_0)"'] \arrow[d, shift left, "T(\varphi'_1)T(\beta_0)"] & T(X''_1) \arrow[r] \arrow[dl, "\lambda_1"'] \arrow[d, shift right, "T(\xi_1)T(\varphi''_1)"'] \arrow[d, shift left, "T(\varphi'_2)T(\beta_1)"] & \cdots \arrow[r] & T(X''_n) \arrow[r, "\delta^n_{X''}"] \arrow[d, shift right, "T(\xi_n)T(\varphi''_n)"'] \arrow[d, shift left, "T(\varphi'_{n+1})T(\beta_n)"] & T(X''_{n+1}) \arrow[r] \arrow[dl, "\lambda_{n+1}"'] \arrow[d, shift right, "T(\xi_{n+1})T(\varphi''_{n+1})"'] \arrow[d, shift left, "T(\varphi'_{n+2})T(\beta_{n+1})"] & \cdots \\
W'_1 \arrow[r, "k^{W'}_1"'] & T(Y'_1) \arrow[r, "\delta^1_{Y'}"'] & T(Y'_2) \arrow[r] & \cdots \arrow[r] & T(Y'_{n+1}) \arrow[r, "\delta^{n+1}_{Y'}"'] & T(Y'_{n+2}) &
\end{tikzcd}
$$

each square is commutative, therefore, if there exist morphisms λ_n, for all $n \geq 0$,

$$\lambda_0 : T(X''_0) \to W'_1$$

$$\lambda_n : T(X''_n) \to T(Y'_n)$$

such that $$\gamma_{B''}\, T(\varphi'') - \hat{\varphi}\, \gamma_{A''} = \lambda_0 T(\epsilon'')$$

$$T(\xi_0)T(\varphi''_0) - T(\varphi'_1)T(\beta_0) = k^{W'}_1 \lambda_0 + \lambda_1 \delta^0_{X''}$$

and for $n \geq 0$

$$T(\xi_n)T(\varphi''_n) - T(\varphi'_{n+1})T(\beta_n) = \delta^n_{Y'}\lambda_n + \lambda_{n+1}\delta^n_{X''}$$

then the theorem will be proved because the homology morphisms induced by homotopic chain maps are equal. It can be verified that these are chain maps.

<u>1st Step</u>. Definition of λ_0 and λ_1.

Recall that $\partial^0_X = f_1(\partial^0_{X'}\pi^{X'}_0 - \beta_0 g_0) + i^{X''}_1 \partial^0_{X''} g_0$ and

$$\partial^0_Y = h_1(\partial^0_{Y'}\pi^{Y'}_0 - \xi_0 k_0) + i^{Y''}_1 \partial^0_{Y''} k_0.$$

Therefore, (1) $i^{X''}_1 \partial^0_{X''} g_0 = \partial^0_X - f_1(\partial^0_{X'}\pi^{X'}_0 - \beta_0 g_0)$,

(2) $i^{X''}_1 \partial^0_{X''} = \partial^0_X i^{X''}_0 + f_1 \beta_0$,

$$(3)\quad \pi_1^{Y'}\varphi_1 i_1^{X''}\partial_{X''}^{0} = \pi_1^{Y'}\varphi_1\partial_X^{0} i_0^{X''} + \pi_1^{Y'}\varphi_1 f_1\beta_0,$$

$$= \pi_1^{Y'}\partial_Y^{0}\varphi_0 i_0^{X''} + \pi_1^{Y'}h_1\varphi_1'\beta_0,$$

$$= \pi_1^{Y'}\partial_Y^{0}\varphi_0 i_0^{X''} + \varphi_1'\beta_0,$$

$$(4)\quad \pi_1^{Y'}\partial_Y^{0}\varphi_0 i_0^{X''} = (\partial_{Y'}^{0}\pi_0^{Y'} - \xi_0 k_0)\varphi_0 i_0^{X''} \text{ because } \pi_1^{Y'}i_1^{Y''} = 0 \text{ and}$$

$$\pi_1^{Y'}h_1 = 1,$$

$$= \partial_{Y'}^{0}\pi_0^{Y'}\varphi_0 i_0^{X''} - \xi_0 k_0\varphi_0 i_0^{X''}$$

$$= \partial_{Y'}^{0}\pi_0^{Y'}\varphi_0 i_0^{X''} - \xi_0\varphi_0'' g_0 i_0^{X''}$$

$$= \partial_{Y'}^{0}\pi_0^{Y'}\varphi_0 i_0^{X''} - \xi_0\varphi_0''.$$

Substituting (4) into (3) one obtains

$$\pi_1^{Y'}\varphi_1 i_1^{X''}\partial_{X''}^{0} = \partial_{Y'}^{0}\pi_0^{Y'}\varphi_0 i_0^{X''} - \xi_0\varphi_0'' + \varphi_1'\beta_0$$

and

$$\xi_0\varphi_0'' - \varphi_1'\beta_0 = \partial_{Y'}^{0}\pi_0^{Y'}\varphi_0 i_0^{X''} - \pi_1^{Y'}\varphi_1 i_1^{X''}\partial_{X''}^{0}.$$

Now let $\mu_1 = -\pi_1^{Y'}\varphi_1 i_1^{X''}$ and $\lambda_1 = T(\mu_1)$. Recall, $T(\partial_{Y'}^{0}) = \delta_{Y'}^{0} = k_1^{W'}b_0^{W'}$.

So let $\lambda_0 = b_0^{W'}T(\pi_0^{Y'}\varphi_0 i_0^{X''})$. Then $k_1^{W'}\lambda_0 + \lambda_1\delta_{X''}^{0} = T(\xi_0)T(\varphi_0'') -$

$- T(\varphi_1')T(\beta_0)$.

Moreover, $k_1^{W'}\lambda_0 T(\epsilon'') = T(\xi_0)T(\varphi_0'')T(\epsilon'') - T(\varphi_1')T(\beta_0)T(\epsilon'')$

$= k_1^{W'}\hat{\varphi}\gamma_{A''} - k_1^{W'}\gamma_{B''}T(\varphi'')$ from Diagram 1.6.

$k_1^{W'}$ is a monic hence $\lambda_0 T(\epsilon'') = \hat{\varphi}\,\gamma_{A''} - k_1^{W'}\gamma_{B''}$.

Now for each $k \geq 1$ define $\mu_k = (-1)^k\pi_k^{Y'}\varphi_k i_k^{X''}$ and let $\lambda_k = T(\mu_k)$.

2nd Step: Induction Hypothesis. Assume that for each $k < n$ the following condition is satisfied

$$T(\xi_k)T(\varphi''_k) - T(\varphi'_{k+1})T(\beta_k) = \delta^k_{Y'}\lambda_k + \lambda_{k+1}\delta^k_{X''}.$$

3rd Step: Show that

$$T(\xi_n)T(\varphi''_n) - T(\varphi'_{n+1})T(\beta_n) = \delta^n_{Y'}\lambda_n + \lambda_{n+1}\delta^n_{X''}.$$

If $\xi_n\varphi''_n - \varphi'_{n+1}\beta_n = \partial^n_{Y'}\mu_n + \mu_{n+1}\partial^n_{X''}$ then the proof will be complete. Recall that

(a) $\partial^n_X = f_{n+1}(\partial^n_{X'}\pi^{X'}_n + (-1)^{n+1}\beta_n g_n) + i^{X''}_{n+1}\partial^n_{X''}g_n;$

(b) $\partial^n_Y = h_{n+1}(\partial^n_{Y'}\pi^{Y'}_n + (-1)^{n+1}\xi_n k_n) + i^{Y''}_{n+1}\partial^n_{Y''}k_n.$

Now,

(1) $\mu_{n+1}\partial^n_{X''} = (-1)^{n+1}\pi^{Y'}_{n+1}\varphi_{n+1}i^{X''}_{n+1}\partial^n_{X''},$

(2) from (a)

$$i^{X''}_{n+1}\partial^n_{X''}g_n = \partial^n_X - f_{n+1}(\partial^n_{X'}\pi^{X'}_n + (-1)^{n+1}\beta_n g_n) \text{ and}$$

$$i^{X''}_{n+1}\partial^n_{X''} = \partial^n_X i^{X''}_n + (-1)^{n+2}f_{n+1}\beta_n,$$

(3) $$\pi^{Y'}_{n+1}\varphi_{n+1}i^{X''}_{n+1}\partial^n_{X''} = \pi^{Y'}_{n+1}\varphi_{n+1}\partial^n_X i^{X''}_n + (-1)^{n+2}\pi^{Y'}_{n+1}\varphi_{n+1}f_{n+1}\beta_n,$$

$$= \pi^{Y'}_{n+1}\partial^n_Y\varphi_n i^{X''}_n + (-1)^{n+2}\varphi'_{n+1}\beta_n,$$

and from (b)

(4) $$\pi^{Y'}_{n+1}\partial^n_Y\varphi_n i^{X''}_n = \partial^n_{Y'}\pi^{Y'}_n\varphi_n i^{X''}_n + (-1)^{n+1}\xi_n k_n\varphi_n i^{X''}_n,$$

$$= \partial^n_{Y'}\pi^{Y'}_n\varphi_n i^{X''}_n + (-1)^{n+1}\xi_n\varphi''_n g_n i^{X''}_n,$$

$$= \partial^n_{Y'}\pi^{Y'}_n\varphi_n i^{X''}_n + (-1)^{n+1}\xi_n\varphi''_n.$$

Substituting (4) into (3) one obtains

$$\pi^{Y'}_{n+1}\varphi_{n+1}i^{X''}_{n+1}\partial^n_{X''} = \partial^n_{Y'}\pi^{Y'}_n\varphi_n i^{X''}_n + (-1)^{n+1}[\xi_n\varphi''_n - \varphi'_{n+1}\beta_n].$$

Therefore,

$$\pi^{Y'}_{n+1}\varphi_{n+1}i^{X''}_{n+1}\partial^n_{X''} - \partial^n_{Y'}\pi^{Y'}_n\varphi_n i^{X''}_n = (-1)^{n+1}[\xi_n\varphi''_n - \varphi'_{n+1}\beta_n]$$

and

$$\xi_n\varphi''_n - \varphi'_{n+1}\beta_n = (-1)^{n+2}\partial^n_{Y'}\pi^{Y'}_n\varphi_n i^{X''}_n + (-1)^{n+1}\pi^{Y'}_{n+1}\varphi_{n+1}i^{X''}_{n+1}\partial^n_{X''}$$

$$= \partial^n_{Y'}\mu_n + \mu_{n+1}\partial^n_{X''}.$$

<u>Theorem 1.5</u>. If $\mathfrak{B}$ has a projective generator P, then for each sequence $E: 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{E}$ the sequence

$$0 \to T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \xrightarrow{\Delta^0_E} H^1(A) \xrightarrow{H^1(f)} H^1(B) \xrightarrow{H^1(g)} H^1(C) \to$$

$$\xrightarrow{\Delta^1_E} H^2(A) \to \cdots$$

is exact.

<u>Proof</u>: By 5.2 of [6] $\mathfrak{B}$ is projectively perfect. Moreover, $P \in \mathcal{P}_1$ where $\mathcal{E}_1 \Rightarrow \mathcal{P}_1$. Therefore the functor $\mathfrak{B}(P,-) = \mathrm{Hom}_{\mathfrak{B}}(P,-): \mathfrak{B} \to \mathcal{G}$ where $\mathcal{G}$ is the category of abelian groups, has the properties:

(i) preserves and reflects exactness because $P \in \mathcal{P}_1$; and $\mathcal{E}_1$ is closed;

(ii) is faithful; from the definition of projective generator;

(iii) preserves monics, kernels and products because is the adjoint of a functor (Proposition 5.1 in [6-19]);

(iv) preserves biproducts because is additive; Proposition 6.4 in [17];

(v) reflects epics and monics and exact sequences (Proposition 1.1 and 1.2 in [6]).

Now apply $\mathfrak{B}(P,-)$ to Diagram 1.5. Recall, for $n > 0$

$T(X_{n-1}) \xrightarrow{b_{n-1}} Z_n \xrightarrow{p_n} H^n(A) \to 0$ is exact because p_n is the cokernel of b_{n-1}. Similarly for Z and Y. Also, recall, $0 \to Z_n \xrightarrow{k_n} T(X_n) \xrightarrow{\delta^n_X}$ $\to T(X_{n+1})$ is exact.

(1) $\ker H^n(g)_* \subset \operatorname{im} H^n(f)_*$.

Let $x \in \ker H^n(g)_*$, then there exists $y \in \mathrm{Hom}(P,\overline{Z}_n)$ such that $(\overline{p}_n)_*(y) = x$. Then $(\bar{\bar{p}}_n)_*(\overline{g}_{n-1})(y) = H^n(g)_*(\overline{p}_n)_*(y) = 0$. By exactness there exists $z \in \mathrm{Hom}(P,T(Z_{n-1}))$ such that $(\bar{\bar{b}}_{n-1})_*(z) = (\overline{g}_{n-1})_*(y)$. $T(g_{n-1})_*$ is surjective hence there exists $w \in \mathrm{Hom}(P,T(Y_{n-1}))$ such that $T(g_{n-1})_*(w) = z$. Let $a = (\overline{k}_n)_*(y) - \hat{\delta}^{n-1}_Y(w) \in \mathrm{Hom}(P,T(Y_n))$.

Then $T(g_n)_*(a) = [T(g_n)_*(\overline{k}_n)_*](y) - [T(g_n)_*\hat{\delta}^{n-1}_Y](w)$

$= (\bar{\bar{k}}_n)_*(\overline{g}_{n-1})_*(y) - \hat{\delta}^{n-1}_Z T(g_{n-1})_*(w) = (\bar{\bar{k}}_n)_*(\bar{\bar{b}}_{n-1})_*(z) - \hat{\delta}^{n-1}_Z(Z) = 0.$

Therefore, by exactness, there exists $b \in \mathrm{Hom}(P,T(X_n))$ such that $T(f_n)_*(b) = a$. Moreover, $T(f_{n+1})_*\hat{\delta}^n_X(b) = \hat{\delta}^n_Y T(f_n)_*(b) = \hat{\delta}^n_Y(a)$ $= \hat{\delta}^n_Y(\overline{k}_n)_*(y) - \hat{\delta}^n_Y\, \hat{\delta}^{n-1}_Y(w) = 0$. Hence $\hat{\delta}^n_X(b) = 0$ and there exists $c \in \mathrm{Hom}(P,Z_n)$ such that $(k_n)_*(c) = b$. $T(f_n)_*(k_n)_*(c) = T(f_n)_*(b) = a$ $= (\overline{k}_n)_*(y) - \hat{\delta}^{n-1}_Y(w)$, hence $(\overline{f}_{n-1})_*(c) = y - (\overline{b}_{n-1})(w)$ because $(\overline{k}_n)_*$ is injective. Therefore, by commutativity, $H^n(f)_*(p_n)_*(c) =$ $= (\overline{p}_n)_*(\overline{f}_{n-1})_*(c) = (\overline{p}_n)_*(y - (\overline{b}_{n-1})_*(w)) = (\overline{p}_n)_*(y) = x$.

(2) $\ker \Delta^n_* \subset \operatorname{im} H^n(g)_*$.

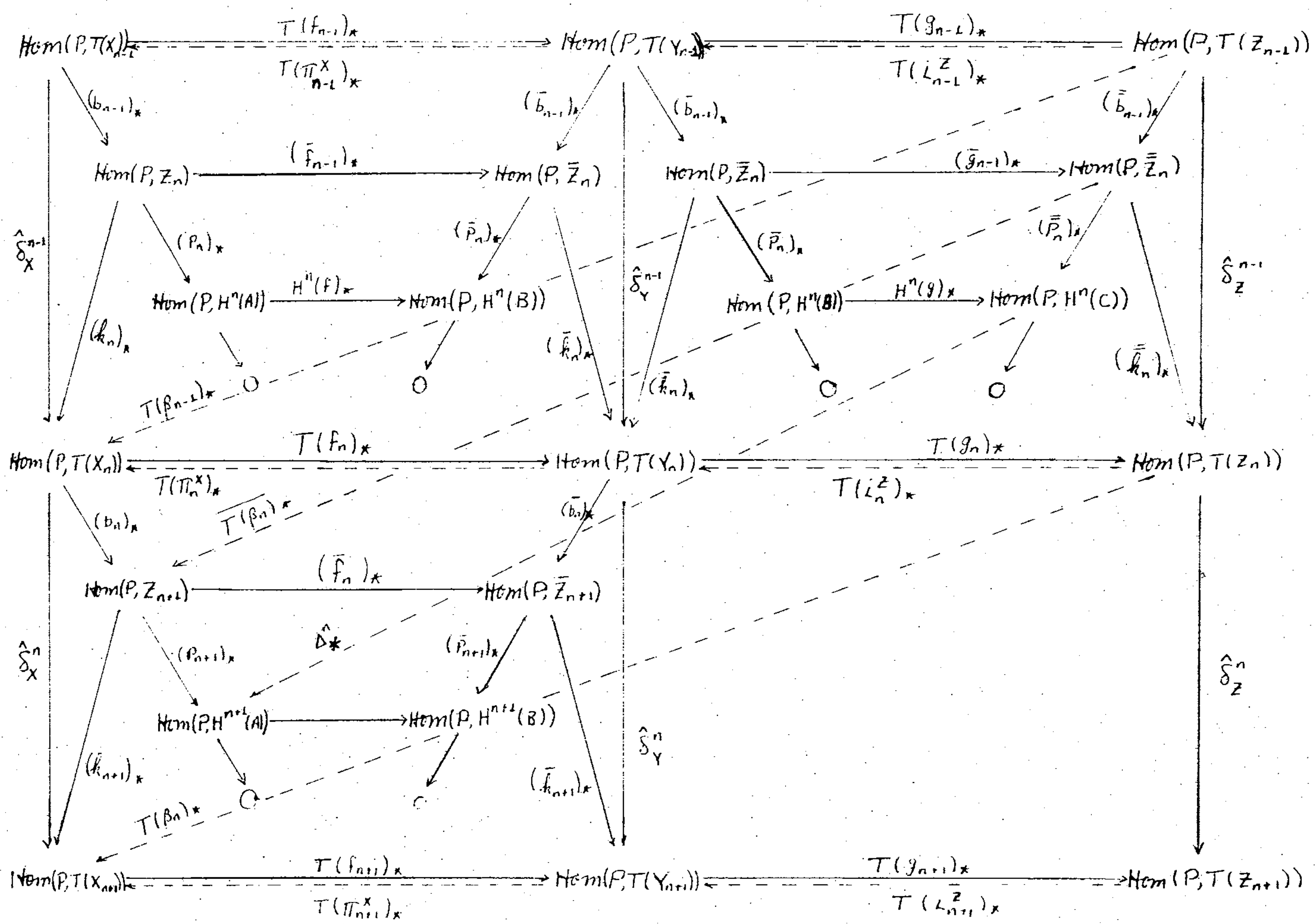
$Hom(P,T(X_{n-1}))$
$T(f_{n-1})_*$
$T(\pi^X_{n-1})_*$
$Hom(P,T(Y_{n-1}))$
$T(g_{n-1})_*$
$T(i^Z_{n-1})_*$
$Hom(P,T(Z_{n-1}))$
$(b_{n-1})_*$
$(\bar{b}_{n-1})_*$
$(\bar{\bar{b}}_{n-1})_*$
$Hom(P,Z_n)$
$(\bar{f}_{n-1})_*$
$Hom(P,\bar{Z}_n)$
$Hom(P,\bar{Z}_n)$
$(\bar{g}_{n-1})_*$
$Hom(P,\bar{\bar{Z}}_n)$
$\hat{\delta}^{n-1}_X$
$(p_n)_*$
$(\bar{p}_n)_*$
$\hat{\delta}^{n-1}_Y$
$(\bar{p}_n)_*$
$(\bar{\bar{p}}_n)_*$
$\hat{\delta}^{n-1}_Z$
$Hom(P,H^n(A))$
$H^n(f)_*$
$Hom(P,H^n(B))$
$Hom(P,H^n(B))$
$H^n(g)_*$
$Hom(P,H^n(C))$
$(k_n)_*$
$(\bar{k}_n)_*$
$(\bar{k}_n)_*$
$(\bar{\bar{k}}_n)_*$
$T(\beta_{n-1})_*$
0
0
0
0
$Hom(P,T(X_n))$
$T(f_n)_*$
$T(\pi^X_n)_*$
$Hom(P,T(Y_n))$
$T(g_n)_*$
$T(i^Z_n)_*$
$Hom(P,T(Z_n))$
$(b_n)_*$
$\overline{T(\beta_n)}_*$
$(\bar{b}_n)_*$
$Hom(P,Z_{n+1})$
$(\bar{f}_n)_*$
$Hom(P,\bar{Z}_{n+1})$
$\hat{\delta}^n_X$
$(p_{n+1})_*$
$\hat{\Delta}_*$
$(\bar{p}_{n+1})_*$
$\hat{\delta}^n_Z$
$Hom(P,H^{n+1}(A))$
$Hom(P,H^{n+1}(B))$
$\hat{\delta}^n_Y$
$(k_{n+1})_*$
$(\bar{k}_{n+1})_*$
0
0
$T(\beta_n)_*$
$Hom(P,T(X_{n+1}))$
$T(f_{n+1})_*$
$T(\pi^X_{n+1})_*$
$Hom(P,T(Y_{n+1}))$
$T(g_{n+1})_*$
$T(i^Z_{n+1})_*$
$Hom(P,T(Z_{n+1}))$

Diagram 1.7

Let $x \in \ker \Delta^n_*$. $(\bar{\bar{p}}_n)_*$ is surjective, hence there exists $y \in \mathrm{Hom}(P,\bar{\bar{Z}}_n)$ such that $(\bar{\bar{p}}_n)_*(y) = x$. By commutativity, $\overline{T(\beta_n)}_*(y) \in \ker\,(p_{n+1})_*$, and there exists $z \in \mathrm{Hom}(P,T(X_n))$ such that $(b_n)_*(z) = \overline{T(\beta_n)}_*(y)$. Let $a = T(f_n)_*(z) + (-1)^n T(i^Z_n)_*(\bar{\bar{k}}_n)_*(y) \in \mathrm{Hom}(P,T(Y_n))$. Then $\hat{\delta}^n_Y(a) = T(f_{n+1})_*\hat{\delta}^n_X(z) + (-1)^{n+1}(-1)^n T(f_{n+1})_* T(\beta_n)_*(\bar{\bar{k}}_n)_*(y) + 0$

$= T(f_{n+1})_*[\hat{\delta}^n_X(z) - T(\beta_n)_*(\bar{\bar{k}}_n)_*(y)]$

$= T(f_{n+1})_*[(k_{n+1})_*(b_n)_*(z) - T(\beta_n)_*(\bar{\bar{k}}_n)_*(y)]$

$= T(f_{n+1})_*[(k_{n+1})_*\,\overline{T(\beta_n)}_*(y) - T(\beta_n)_*(\bar{\bar{k}}_n)_*(y)] = 0$ because of commutativity. Hence, there exists $b \in \mathrm{Hom}(P,\bar{Z}_n)$ such that $(\bar{k}_n)_*(b) = a$. $(\bar{\bar{k}}_n)_*(\bar{g}_{n-1})_*(b) = T(g_n)_*(\bar{k}_n)_*(b) = T(g_n)_*(a) = (-1)^n(\bar{\bar{k}}_n)_*(y)$. Hence $(\bar{g}_{n-1})_*(b) = (-1)^n y$. Let $c = (-1)^n(\bar{p}_n)_*(b)$, then $H^n(g)_*(c) =$
$= (-1)^n H^n(g)_*(\bar{p}_n)_*(b) = (-1)^n(\bar{\bar{p}}_n)_*(\bar{g}_{n-1})(b) = (-1)^{2n}(\bar{\bar{p}}_n)_*(y) = x$.

(3) $\ker H^{n+1}(f)_* \subset \mathrm{im}\,\Delta^n_*$.

Let $x \in \ker H^{n+1}(f)_*$. $(p_{n+1})_*$ is surjective so there exists $y \in \mathrm{Hom}(P,Z_{n+1})$ such that $(p_{n+1})_*(y) = x$. $(\bar{p}_{n+1})_*(\bar{f}_n)_*(y) =$
$= H^{n+1}(f)_*(x) = 0$ and $(\bar{f}_n)_*(y)$ is in the $\ker\,(\bar{p}_{n+1})_* = \mathrm{im}\,(\bar{b}_n)_*$. Hence there exists $z \in \mathrm{Hom}(P,T(Y_n))$ such that $(\bar{b}_n)_*(z) = (\bar{f}_n)_*(y)$.
$\hat{\delta}^n_Z T(g_n)_*(z) = T(g_{n+1})_*\hat{\delta}^n_Y(z) = T(g_{n+1})_*(\bar{k}_{n+1})_*(\bar{b}_n)_*(z) =$
$= T(g_{n+1})_*(\bar{k}_{n+1})_*(\bar{f}_n)_*(y) = T(g_{n+1})_* T(f_{n+1})_*(k_{n+1})_*(y) = 0$, so there

exists $w \in \mathrm{Hom}(P,\bar{\bar{Z}}_n)$ such that $(\bar{\bar{k}}_n)_*(w) = T(g_n)_*(z)$. By commutativity $(k_{n+1})_*\overline{T(\beta_n)}_*(w) = T(\beta_{n+1})_*(\bar{\bar{k}}_n)_*(w)$.

Let $a = y - (b_n)_*T(\pi_n^X)_*(z)$ and show that $\overline{T(\beta_n)}_*((-1)^n w) = a$. If so, then by commutativity and because $(p_{n+1})_*(b_n)_* = 0$; $\Delta_*^n(\bar{\bar{p}}_n)_*((-1)^n w)$ $= x$ and $x \in \mathrm{im}\,\Delta_*^n$. $T(f_{n+1})_*(k_{n+1})_*(a + (-1)^n\overline{T(\beta_n)}_*(w)) =$

$$= T(f_{n+1})_*(k_{n+1})_*(a) + (-1)^n T(f_{n+1})_*(k_{n+1})_*\,\overline{T(\beta_n)}_*(w)$$

$$= T(f_{n+1})_*(k_{n+1})_*(y) - T(f_{n+1})_*(k_{n+1})_*(b_n)_*T(\pi_n^X)_*(z) +$$

$$+ (-1)^n T(f_{n+1})_*(k_{n+1})_*\overline{T(\beta_n)}_*(w)$$

$$= T(f_{n+1})_*(k_{n+1})_*(y) - T(f_{n+1})_*\hat{\delta}_X^n T(\pi_n^X)_*(z) +$$

$$+ (-1)^n T(f_{n+1})_*T(\beta_n)_*T(g_n)_*(z)$$

$$= (\bar{k}_{n+1})_*(\bar{f}_n)_*(y) - T(f_{n+1})_*[\hat{\delta}_X^n T(\pi_n^X)_*(z) + (-1)^{n+1}T(\beta_n)_*T(g_n)_*(z)]$$

$$= (\bar{k}_{n+1})_*(\bar{b}_n)_*(z) - T(f_{n+1})_*[\hat{\delta}_X^n T(\pi_n^X)_* + (-1)^{n+1}T(\beta_n)_*T(g_n)_*](z)$$

$$= \{\hat{\delta}_Y^n - T(f_{n+1})_*[\hat{\delta}_X^n T(\pi_n^X)_* + (-1)^{n+1}T(\beta_n)_*T(g_n)_*]\}(z)$$

$= T(g_{n+1})_*\hat{\delta}_Z^n T(g_n)_*(z) = 0$ by definition of $\hat{\delta}_Y^n$ and because

$\hat{\delta}_Z^n T(g_n)_*(z) = 0$.

Since $T(f_{n+1})_*$ and $(k_{n+1})_*$ are monics, $a + (-1)^n\overline{T(\beta_n)}_* = 0$.

(4) $\ker \Delta_*^0 \subset \mathrm{im}\, T(g)_*$ (Diagram 1.8).

Let $x \in \ker \Delta_*^0$. Then $(p_1)_*\gamma_*(x) = \Delta_*^0(x) = 0$ and $\gamma_*(x) \in \ker\,(p_1)_*$.

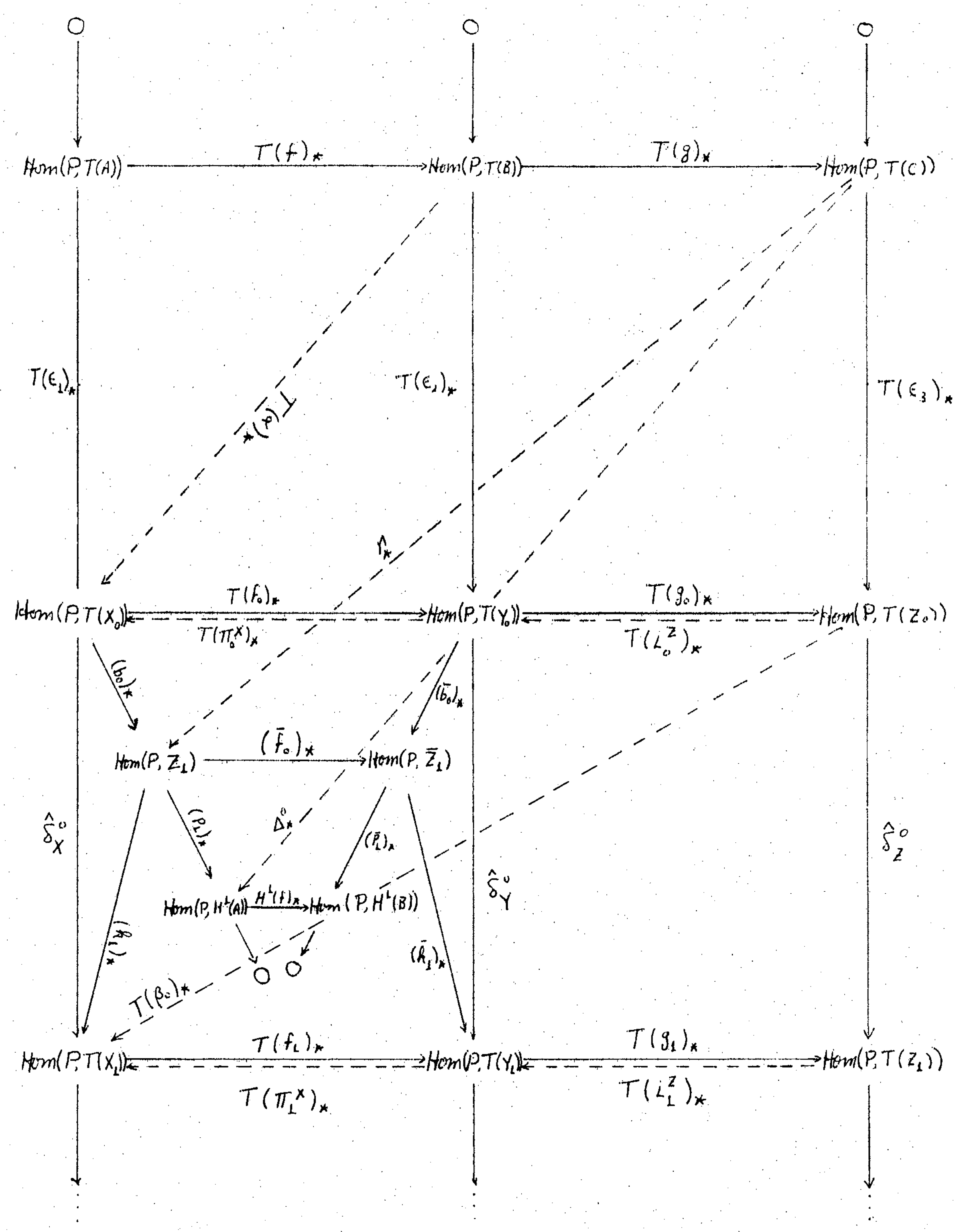
0
0
0
$Hom(P,T(A))$
$T(f)_*$
$Hom(P,T(B))$
$T(g)_*$
$Hom(P,T(C))$
$T(\epsilon_1)_*$
$T(\epsilon_2)_*$
$T(\epsilon_3)_*$
h_*
$Hom(P,T(X_0))$
$T(f_0)_*$
$T(\pi_0^X)_*$
$Hom(P,T(Y_0))$
$T(g_0)_*$
$T(i_0^Z)_*$
$Hom(P,T(Z_0))$
$(b_0)_*$
$(\bar{b}_0)_*$
$Hom(P,Z_1)$
$(\bar{f}_0)_*$
$Hom(P,\bar{Z}_1)$
$\hat{\delta}_X^0$
$(p_1)_*$
Δ_*^0
$(\bar{p}_1)_*$
$\hat{\delta}_Y^0$
$\hat{\delta}_Z^0$
$Hom(P,H^1(A))$
$H^1(f)_*$
$Hom(P,H^1(B))$
$(k_1)_*$
$(\bar{k}_1)_*$
0
0
$T(\beta_0)_*$
$Hom(P,T(X_1))$
$T(f_1)_*$
$T(\pi_1^X)_*$
$Hom(P,T(Y_1))$
$T(g_1)_*$
$T(i_1^Z)_*$
$Hom(P,T(Z_1))$

Diagram 1.8

Hence there exists $y \in \mathrm{Hom}(P,T(X_0))$ such that $(b_0)_*(y) = \gamma_*(x)$. Let $a = T(f_0)_*(y) + T(i_0^Z)_*T(\epsilon_3)_*(x) \in \mathrm{Hom}(P,T(Y_0))$. Then $\hat{\delta}_Y^0(a) =$
$= \hat{\delta}_Y^0 T(f_0)_*(y) + \hat{\delta}_Y^0 T(i_0^Z)_*T(\epsilon_3)_*(x) = T(f_1)_*\hat{\delta}_X^0(y) - T(f_1)_*T(\beta_0)_*T(\epsilon_3)_*(x)$
$= T(f_1)_*[\hat{\delta}_X^0(y) - (k_1)_*\gamma_*(x)] = T(f_1)_*[(k_1)_*(b_0)_*(y) - (k_1)_*\gamma_*(x)]$
$= T(f_1)_*[(k_1)_*\gamma_*(x) - (k_1)_*\gamma_*(x)] = 0$. Therefore, there exists $b \in \mathrm{Hom}(P,T(B))$ such that $T(\epsilon_2)_*(b) = a$. By commutativity, $T(\epsilon_3)_*T(g)_*(b) = T(g_0)_*T(\epsilon_2)_*(b) = T(g_0)_*(a) = T(\epsilon_3)_*(x)$ and $T(g)_*(b) = x$.

(5) $\ker H^1(f)_* \subset \mathrm{im}\, \Delta_*^0$.

Let $x \in \ker H^1(f)_*$. There exists $y \in \mathrm{Hom}(P,Z_1)$ such that $(p_1)_*(y) = x$. By commutativity $(\overline{p}_1)_*(\overline{f}_0)_*(y) = 0$ and from exactness there exists $z \in \mathrm{Hom}(P,T(Y_0))$ such that $(\overline{b}_0)_*(z) = (\overline{f}_0)_*(y)$. Now
$\hat{\delta}_Z^0 T(g_0)_*(z) = T(g_1)_*\hat{\delta}_Y^0(z) = T(g_1)_*(\overline{k}_1)_*(\overline{b}_0)_*(z) = T(g_1)_*(\overline{k}_1)_*(\overline{f}_0)_*(y)$
$= T(g_1)_*T(f_1)_*(k_1)_*(y) = 0$. Therefore, there exists $w \in \mathrm{Hom}(P,T(C))$ such that $T(\epsilon_3)_*(w) = T(g_0)_*(z)$. Let $a = -y + (b_0)_*T(\pi_0^X)_*(z)$ in $\mathrm{Hom}(P,Z_1)$. If $\gamma_*(w) = a$, then $\Delta_*^0(-w) = x$ because $(p_1)_*(b_0)_* = 0$. $\gamma_*(w) = a$ if $T(f_1)_*(k_1)_*[a - \gamma_*(w)] = 0$ because composition of two monics is a monic. $T(f_1)_*(k_1)_*[a - \gamma_*(w)] = T(f_1)_*(k_1)_*(a) -$
$- T(f_1)_*(k_1)_*\gamma_*(w) = - T(f_1)_*(k_1)_*(y) + T(f_1)_*(k_1)_*(b_0)_*\, T(\pi_0^X)_*(z) -$

$$- T(f_1)_*T(\beta_0)_*T(\epsilon_3)_*(w) = -(\overline{k}_1)_*(\overline{f}_0)_*(y) + T(f_1)_*[\hat{\delta}^0_X T(\pi^X_0)_*(z) -$$

$$- T(\beta_0)_*T(g_0)_*(z)] = -(\overline{k}_1)_*(\overline{b}_0)_*(z) + T(f_1)_*[\hat{\delta}^0_X T(\pi^X_0)_*(z) -$$

$$- T(\beta_0)_*T(g_0)_*(z)] = - \hat{\delta}^0_Y(z) + T(f_1)_*[\hat{\delta}^0_X T(\pi^X_0)_* - T(\beta_0)_*T(g_0)_*](z) = 0$$

because of the definition of $\hat{\delta}^0_Y(z)$ and because $\hat{\delta}^0_Z T(g_0)_*(z) = 0$.

The examples cited in Chapters II and III have projective generators. In fact, the usual examples that one is interested in do have projective generators. But this is not true for all abelian categories as the following example shows.

Example:

Definition [2-70]: An abelian group A is called a torsion group if for each $a \in A$ there exists a natural number $n_a \neq 0$ such that $n_a a = 0$.

Let $\mathcal{G}$ be the abelian category of all abelian groups. Define a full subcategory $\mathcal{T}$ of $\mathcal{G}$ by letting the objects of $\mathcal{T}$ be the torsion groups and $\mathrm{Hom}_{\mathcal{T}}(A,B) \equiv \mathrm{Hom}_{\mathcal{G}}(A,B)$ for any A,B in $\mathcal{T}$. It will first be shown that $\mathcal{T}$ is an abelian category, second that the only projective objects of $\mathcal{T}$ are the null objects and finally that a null object cannot be a generator.

That $\mathcal{T}$ is a pointed category and Hom(A,B) is an abelian group with the distributive laws satisfied are readily seen as inherited from $\mathcal{G}$. Also inherited from $\mathcal{G}$ is the fact that any morphism can be factored as the composition of an epic with a monic. So only three properties need to be shown:

(1) $\mathcal{J}$ has finite biproducts,

(2) every morphism has a kernel and cokernel,

(3) given a sequence $E: A' \xrightarrow{i} A \xrightarrow{j} A''$ with i a monic and j an epic, then i is a kernel of j if and only if j is a cokernel of i.

(1) <u>$\mathcal{J}$ has finite biproducts</u>:

For any objects A, B of $\mathcal{J}$ consider the abelian group $A + B = \{(a,b) \mid a \in A,\ b \in B\}$. Then for any element $(a,b) \in A + B$ consider the integer $n_a n_b \neq 0$, $n_a n_b(a,b) = n_a(n_b a,\ n_b b) = n_a(n_b a, 0) = (n_a n_b a, 0) = (n_b n_a a, 0) = (0,0)$. The usual properties on injections and projections hold.

(2) <u>Every morphism has a kernel and cokernel</u>:

Given any morphism $A \xrightarrow{f} B$, consider the abelian group $\ker f \subset A$. This is a torsion group and $k:\ker(f) \to A$ defined by $k(a) = a$, for any $a \in \ker f$, is the kernel morphism of f. Similarly consider $B \xrightarrow{\pi} B/\mathrm{im}\, f$ where $\pi(b) = b + \mathrm{im}\, f$ for any $b \in B$. $B/\mathrm{im}\, f$ is an abelian group and for $\overline{b} \in B/\mathrm{im}\, f$; $n_b(\overline{b}) = \overline{n_b b} = \overline{0}$, hence $B/\mathrm{im}\, f$ is a torsion group and π is a cokernel of f.

(3) Let $E: A' \xrightarrow{i} A \xrightarrow{j} A''$ be a sequence in $\mathcal{J}$ with a monic i and an epic j. Then i is a kernel of j if and only if j is a cokernel of i.

(i) If $A \xrightarrow{i} A'$ is a monic in $\mathcal{J}$ then i is an injective function (hence is a monic in $\mathcal{G}$).

<u>Proof of i)</u>: Assume there exist $x_1, x_2 \in A$ such that $i(x_1) = i(x_2)$ but $x_1 \neq x_2$. A is a torsion group hence there exists $n_1 \neq 0$, $n_2 \neq 0$ such that $n_1 x_1 = 0 = n_2 x_2$. Consider $Z_{n_1} + Z_{n_2}$. This is in $\mathcal{J}$. Define

$$f_1 : Z_{n_1} + Z_{n_2} \to A \text{ by } f_1(1,0) = x_1 \text{ and } f_1(0,1) = x_2$$

$$f_2 : Z_{n_1} + Z_{n_2} \to A \text{ by } f_2(1,0) = x_2 \text{ and } f_2(0,1) = x_1.$$

Then $f_1 \neq f_2$ but $if_1 = if_2$. This is a contradiction to the definition of monic.

(ii): If $A \xrightarrow{j} A'$ is an epic in $\mathcal{J}$ then j is a surjection (hence j is an epic in $\mathcal{G}$).

<u>Proof of ii)</u>: If j is not a surjection then im $j \neq A'$, hence $A \xrightarrow{j} A' \underset{0}{\xrightarrow{\pi}} A'/\text{im } j$ where $\pi j = 0j = 0$ but $\pi \neq 0$, contradiction.

<u>Proof of 3)</u>:

I) Assume i is a kernel of j.

$$\begin{tikzcd}
 & Z_{n_0} \arrow[dl, dashed, "k"'] \arrow[d, "g"] & \\
A' \arrow[r, "i"] & A \arrow[r, "j"] \arrow[d, "f"] & A'' \arrow[dl, dashed, "h"] \\
 & B &
\end{tikzcd}$$

Let $f : A \to B$ such that $fi = 0$. Define $h : A'' \to B$ by $h(a) = f(b)$ where $j(a) = b$. Let $b_1, b_2 \in A$ such that $j(b_1) = j(b_2)$. Then, if $f(b_1) = f(b_2)$, h will be well-defined. $b_1 - b_2 \in \ker j$ and there exists $n_0 \neq 0$ such that $n_0(b_1 - b_2) = 0$. Consider Z_{n_0} in $\mathcal{J}$ and define $g : Z_{n_0} \to A$ by $g(1) = b_1 - b_2$. Then $jg = 0$ and there exists a unique $k : Z_{n_0} \to A'$ such that $ik = g$. Therefore $ik(1) = g(1) = b_1 - b_2$. Hence

$f(b_1 - b_2) = 0$. It can be verified that h is a group homomorphism and is unique with the property that $hj = f$.

II) Assume j is a cokernel of i. Then j is equivalent to the morphism $c: A \to {}^{A}/\mathrm{im}\ i$ and without loss of generality, assume $j = c$.

$$\begin{tikzcd} & B \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ A' \arrow[r, "i"] & A \arrow[r, "j"] & {}^{A}/\mathrm{im}\ i \end{tikzcd}$$

Let $f:B \to A$ be such that $jf = 0$. Then $\mathrm{im}\ f \subset \ker j = \mathrm{im}\ i$. Now define $g:B \to A'$ by $g(b) = a$ where $i(a) = f(b)$. Since i is a monic, it can be readily verified that g is a function, a group homomorphism and is unique with the property $ig = f$. The proof of 3) is therefore completed.

It is shown in [6-23] that the only projective objects of $\mathcal{J}$ are the null objects and a null object cannot be a generator of $\mathcal{J}$ because consider $Z_2 \in \mathcal{J}$, $f_1:Z_2 \to Z_2$, defined by $f_1(1) = 1$ and $f_2:Z_2 \to Z_2$, defined by $f_2(1) = 0$. Then $f_1 \neq f_2$ but $T_N(f_1) = T_N(f_2)$ where T_N is the functor Hom(N,—). Hence T_N is not faithful. This completes the example.

In the case where $\mathfrak{B}$ does not have a projective generator, the First Embedding Theorem [8], can be used, which says; given any small abelian category $\mathfrak{M}$ there exists an exact covariant additive embedding $T:\mathfrak{M} \to \mathcal{G}$ where $\mathcal{G}$ is the category of abelian groups. Therefore, T has the following properties:

1) T is both right - and left-exact [8-65],

2) T preserves biproducts because is additive,

3) reflects exactness by Proposition 1.2 in [6],

4) for each pair of objects A, B in $\mathfrak{M}$ there exists a group monomorphism $\varphi_{AB}: \mathrm{Hom}_{\mathfrak{B}}(A,B) \to \mathrm{Hom}_{\mathcal{G}}(T(A),T(B))$.

Consider the full subcategory $\mathfrak{B}_0$ of $\mathfrak{B}$ defined in the following manner:

i) objects: all objects appearing in Diagram 1.5,

ii) morphisms, for any A, B in $\mathfrak{B}_0$, $\mathrm{Hom}_{\mathfrak{B}_0}(A,B) \equiv \mathrm{Hom}_{\mathfrak{B}}(A,B)$.

Since $\mathfrak{B}_0$ has only a countable number of objects, $\mathfrak{B}_0$ is a small subcategory of $\mathfrak{B}$ and full. Then, by Lemma 2.7 of [17-101], there exists a small, full, abelian subcategory $\mathfrak{B}_1$ of $\mathfrak{B}$ such that $\mathfrak{B}_0$ is a subcategory of $\mathfrak{B}_1$. Now by the aforementioned embedding theorem there exists an exact covariant additive embedding $T:\mathfrak{B}_1 \to \mathcal{G}$. By a direct diagram chasing argument, similar to the one used on the proof of Theorem 1.5, the long sequence of homologies is exact. This completes existence.

Uniqueness of the Cohomology Theory (Uehara - [20]).

<u>Definition 1.6</u>: Let $H_{\mathcal{E}}$, $K_{\mathcal{E}}$ be two cohomology theories relative to $\mathcal{E}$ over a functor $T:\mathfrak{A} \to \mathfrak{B}$. They are said to be <u>equivalent</u> if and only if there exists a sequence of natural equivalences $\varphi^n:H^n \to K^n$, for $n \geq 0$ such that:

1) for each sequence $E:0 \to A' \to A \to A'' \to 0$ in $\mathcal{E}$ and for each $n \geq 0$ the diagram

$$\begin{array}{ccc}
H^n(A'') & \xrightarrow{\Delta^n_E} & H^{n+1}(A') \\
\downarrow{\scriptstyle \varphi^n(A'')} & & \downarrow{\scriptstyle \varphi^{n+1}(A')} \\
K^n(A'') & \xrightarrow{\bar{\Delta}^n_E} & K^{n+1}(A')
\end{array}$$

is commutative,

2) $$\begin{tikzcd} T \arrow[r, "\eta"] \arrow[dr, "\bar{\eta}"'] & H^0 \arrow[d, "\varphi^0"] \\ & K^0 \end{tikzcd}$$ is commutative.

Theorem 1.6: Let H,K be cohomology theories relative to $\mathcal{E}$ over the functors $S,T:\mathfrak{A} \to \mathfrak{B}$, respectively, and let $\nu:S \to T$ be a natural transformation, then there exists a sequence of natural transformations $\varphi^n:H^n \to K^n$ such that:

1) For each sequence $E:0 \to A' \to A \to A'' \to 0$ in $\mathcal{E}$ and for each $n \geq 0$ the following diagram is commutative:

$$\begin{tikzcd} H^n(A'') \arrow[r, "\Delta_E^n"] \arrow[d, "\varphi^n(A'')"'] & H^{n+1}(A') \arrow[d, "\varphi^{n+1}(A')"] \\ K^n(A'') \arrow[r, "\bar{\Delta}_E^n"] & K^{n+1}(A') \end{tikzcd}$$

2) $$\begin{tikzcd} S \arrow[r, "\nu"] \arrow[d, "\eta"'] & T \arrow[d, "\bar{\eta}"] \\ H^0 \arrow[r, "\varphi^0"] & K^0 \end{tikzcd}$$ is a commutative diagram of functors and natural transformations.

Proof: For each A in $\mathfrak{A}$ define $\varphi^0(A):H^0(A) \to K^0(A)$ by $\varphi^0(A) = \bar{\eta}(A)\nu(A)\eta^{-1}(A)$. Then φ^0 is a natural transformation and 2) is satisfied because η and $\bar{\eta}$ are natural equivalences.

Now, assume that $\varphi^i:H^i \to K^i$ have been constructed for all $i < n$ so as to satisfy the commutativity conditions of the theorem.

By Axiom IV, for each A in $\mathfrak{A}$, there exists a morphism $i:A \to I$ where

$i \in \mathfrak{M}$ and $I \in \mathcal{J}$ such that $H^n(i) = 0$ for $n > 0$. (In fact by the proof of Theorem 1.2 for any $\alpha: A \to I'$ where $\alpha \in \mathfrak{M}$ and $I' \in \mathcal{J}$ one has $H^n(\alpha) = 0$ for $n > 0$.) By Remarks 1.1, 1.2, $E: 0 \to A \xrightarrow{i} I \xrightarrow{j} Q \to 0$ is in $\mathcal{E}$ where j is the cokernel of i. Then by Axiom II and the induction hypothesis there exists a commutative diagram with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n-1}(A) \arrow[r, "H^{n-1}(i)"] \arrow[d, "\varphi^{n-1}(A)"'] & H^{n-1}(I) \arrow[r, "H^{n-1}(j)"] \arrow[d, "\varphi^{n-1}(I)"'] & H^{n-1}(Q) \arrow[r, "\Delta_E^{n-1}"] \arrow[d, "\varphi^{n-1}(Q)"'] & H^n(A) \arrow[r, "H^n(i)"] \arrow[d, "\varphi^n(A)"] & H^n(I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K^{n-1}(A) \arrow[r, "K^{n-1}(i)"'] & K^{n-1}(I) \arrow[r, "K^{n-1}(j)"'] & K^{n-1}(Q) \arrow[r, "\overline{\Delta}_E^{n-1}"'] & K^n(A) \arrow[r, "K^n(i)"'] & K^n(I) \arrow[r] & \cdots
\end{tikzcd}$$

$H^n(i) = 0$ and $H^{n-1}(Q) \xrightarrow{\Delta_E^{n-1}} H^n(A) \xrightarrow{H^n(i)} H^n(I) = 0$ is exact, hence Δ_E^{n-1} is an epic. Consider

$$\begin{tikzcd}
H^{n-1}(I) \arrow[r, "H^{n-1}(j)"] \arrow[dr, "\ell"'] & H^{n-1}(Q) \arrow[r, "\Delta_E^{n-1}"] & H^n(A) \\
& B \arrow[u, "k"'] &
\end{tikzcd}$$

where k is a kernel of Δ_E^{n-1} and ℓ is an epic. Since Δ_E^{n-1} is an epic; Δ_E^{n-1} is a cokernel of k. Moreover, $\overline{\Delta}_E^{n-1}\varphi^{n-1}(Q)\, H^{n-1}(j) =$

$= \overline{\Delta}_E^{n-1}\, K^{n-1}(j)\varphi^{n-1}(I) = 0$. Hence there exists a unique morphism $\varphi^n(A): H^n(A) \to K^n(A)$ such that $\varphi^n(A)\Delta_E^{n-1} = \overline{\Delta}_E^{n-1}\, \varphi^{n-1}(Q)$. It must be shown that this definition is independent of E.

Let $E': 0 \to A \xrightarrow{i'} I' \xrightarrow{j'} Q' \to 0$ be another choice and obtain $\varphi'^n(A)$. Since $\mathfrak{A}$ has biproducts there exists a unique morphism $\alpha: A \to I + I'$ such that $p\alpha = i$ and $p'\alpha = i'$ where p, p' are the projections. Denote the injections by k, k'. Then $H^n(\alpha) =$ $= H^n((kp + k'p')\alpha) = H^n(ki + k'i') = 0$. Moreover, $0 \to A \xrightarrow{\alpha} I + I' \in \mathcal{E}$ because given any $J \in \mathcal{J}$ and any $f: A \to J$ there exists $g \in \mathrm{Hom}(I,J)$ such that $gi = f$. Consider $gp \in \mathrm{Hom}(I + I', J)$. Then $gp\alpha = gi = f$ and $0 \to A \xrightarrow{\alpha} I + I' \in \mathcal{E}$. Hence the sequence $E'': 0 \to A \xrightarrow{\alpha} I + I' \xrightarrow{\beta} Q'' \to 0$ is in $\mathcal{E}$ where β is the cokernel of α. Then E'' defines $\varphi''^n(A)$ in a similar manner as $\varphi^n(A)$ and $\varphi'^n(A)$ are defined by E and E', respectively. Now, consider the following commutative diagram:

$$
\begin{tikzcd}
E: 0 \arrow[r] & A \arrow[r, "i"] & I \arrow[r, "j"] & Q \arrow[r] & 0 \\
E'': 0 \arrow[r] & A \arrow[u, "1_A"] \arrow[r, "\alpha"] & I + I' \arrow[u, "p"] \arrow[r, "\beta"] & Q'' \arrow[u, dashed, "q"] \arrow[r] & 0
\end{tikzcd}
$$

Since $jp\alpha = ji = 0$, there exists a unique morphism $q: Q'' \to Q$ such that $q\beta = jp$. Therefore, the diagram

$$
\begin{tikzcd}
H^{n-1}(Q'') \arrow[rrr, "H^{n-1}(q)"] \arrow[dr, "\Delta^{n-1}_{E''}"] \arrow[ddd, "\varphi^{n-1}(Q'')"'] & & & H^{n-1}(Q) \arrow[dl, "\Delta^{n-1}_{E'}"'] \arrow[ddd, "\varphi^{n-1}(Q)"] \\
& H^n(A) \arrow[r, "H^n(1_A) = 1"] \arrow[d, "\varphi''^n(A)"] & H^n(A) \arrow[d, "\varphi^n(A)"'] & \\
& K^n(A) \arrow[r, "1 = K^n(1_A)"'] & K^n(A) & \\
K^{n-1}(Q'') \arrow[rrr, "K^{n-1}(q)"] \arrow[ur, "\overline{\Delta}^{n-1}_{E''}"'] & & & K^{n-1}(Q) \arrow[ul, "\overline{\Delta}^{n-1}_{E'}"]
\end{tikzcd}
$$

is known to have every square commutative except the inside one. Since $\Delta_{E''}^{n-1}$ is an epic the inside square commutes and $\varphi''^{n}(A) = \varphi^{n}(A)$. Similarly, $\varphi'^{n}(A) = \varphi''^{n}(A)$. Therefore $\varphi^{n}(A)$ is well-defined.

To complete the proof it must be shown that φ^{n} satisfies the commutativity condition and that φ_{n} is a natural transformation.

Let $E: 0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0$ be in $\mathcal{E}$. For A, there exists $k: A \to I$ with $k \in \mathfrak{M}$ and $I \in \mathcal{J}$ such that $H^{n}(k) = 0$ for $n > 0$. Now, consider $0 \to A' \xrightarrow{ki} I$ and show that $ki \in \mathfrak{M}$. This will be true if $0 \to A' \xrightarrow{ki} I$ belongs to $\mathcal{E}$. Let $J \in \mathcal{J}$ and consider

$$\begin{tikzcd}
\mathrm{Hom}(I,J) \arrow[rr, "(ki)^*"] \arrow[dr, "k^*"'] & & \mathrm{Hom}(A',J) \arrow[r] & 0 \\
& \mathrm{Hom}(A,J) \arrow[ur, "i^*"'] \arrow[dr] & & \\
& & 0 &
\end{tikzcd}$$

Since $k,i \in \mathfrak{M}$; i^* and k^* are surjective hence $(ki)^* = i^*k^*$ is surjective and $0 \to A' \xrightarrow{ki} I \in \mathcal{E}$. Then $E': 0 \to A' \xrightarrow{ki} I \xrightarrow{q} Q \to 0$ is in $\mathcal{E}$ where q is the cokernel of ki and E' defines $\varphi^{n}(A')$. Consequently, the diagram

$$\begin{tikzcd}
H^{n-1}(Q) \arrow[r, "\Delta_{E'}^{n-1}"] \arrow[d, "\varphi^{n-1}(Q)"'] & H^{n}(A') \arrow[d, "\varphi^{n}(A')"] \\
K^{n-1}(Q) \arrow[r, "\overline{\Delta}_{E'}^{n-1}"] & K^{n}(A')
\end{tikzcd}$$

is commutative.

Now, consider the commutative diagram:

$$\begin{tikzcd}
E:0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, "1_{A'}"'] & A \arrow[r, "j"] \arrow[d, "k"'] & A'' \arrow[r] \arrow[d, dashed, "\ell"] & 0 \\
E':0 \arrow[r] & A' \arrow[r, "ki"] & I \arrow[r, "q"] & Q \arrow[r] & 0
\end{tikzcd}$$

Then there exists a unique $\ell:A'' \to Q$ such that $\ell j = qk$ because q is the cokernel of ki. Hence, one has the diagram

$$\begin{tikzcd}
H^{n-1}(A'') \arrow[rrr, "\Delta_E^{n-1}"] \arrow[dr, "H^{n-1}(\ell)"] \arrow[ddd, "\varphi^{n-1}(A'')"'] & & & H^n(A') \arrow[dl, "H^n(1_{A'})"'] \arrow[ddd, "\varphi^n(A')"] \\
& H^{n-1}(Q) \arrow[r, "\Delta_{E'}^{n-1}"] \arrow[d, "\varphi^{n-1}(Q)"'] & H^n(A') \arrow[d, "\varphi^n(A')"] & \\
& K^{n-1}(Q) \arrow[r, "\overline{\Delta}_{E'}^{n-1}"] & K^n(A') & \\
K^{n-1}(A'') \arrow[rrr, "\overline{\Delta}_E^{n-1}"'] \arrow[ur, "K^{n-1}(\ell)"] & & & K^n(A') \arrow[ul, "K^n(1_{A'})"']
\end{tikzcd}$$

where all squares, except the outside one, are known to be commutative. Since $K^n(1_{A'})$ is a monic, the outside square commutes and the desired commutativity holds for φ^n.

The remaining task is the verification of the naturality of φ^n. Let $f:A \to B$ be any morphism in $\mathfrak{A}$. Let $E:0 \to A \xrightarrow{i} I \xrightarrow{j} Q \to 0$ and $E':0 \to B \xrightarrow{i'} I' \xrightarrow{j'} Q' \to 0$ be sequences in $\mathcal{E}$ defining $\varphi^n(A)$ and $\varphi^n(B)$

respectively. Then by the definition of biproduct there exists a unique morphism $\alpha: A \to I + I'$ such that $p\alpha = i$ and $p'\alpha = i'f$ where p, p' are the projections of the biproduct. To show $0 \to A \xrightarrow{\alpha} I + I'$ is in $\mathcal{E}$, let $J \in \mathcal{J}$ and $g: A \to J$. Then there exists $h: I \to J$ such that $hi = g$. Consider $hp: I + I' \to J$, then $hp\alpha = hi = g$ and $0 \to A \xrightarrow{\alpha} I + I'$ is in $\mathcal{E}$. Then $E'': 0 \to A \xrightarrow{\alpha} I + I' \xrightarrow{\beta} Q'' \to 0$ is in $\mathcal{E}$ where β is the cokernel of α. Hence E'' defines $\varphi^n(A)$.

The diagram

$$\begin{tikzcd}
E'': 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & I + I' \arrow[r, "\beta"] \arrow[d, "p'"] & Q'' \arrow[r] \arrow[d, dashed, "q"] & 0 \\
E': 0 \arrow[r] & B \arrow[r, "i'"] & I' \arrow[r, "j'"] & Q' \arrow[r] & 0
\end{tikzcd}$$

is commutative, therefore, since β is a cokernel of α, there exists a unique morphism $q: Q'' \to Q'$ such that $q\beta = j'p'$. Consider the diagram:

$$\begin{tikzcd}
H^{n-1}(Q'') \arrow[rrr, "H^{n-1}(q)"] \arrow[dr, "\Delta^{n-1}_{E''}"] \arrow[ddd, "\varphi^{n-1}(Q'')"'] & & & H^{n-1}(Q') \arrow[dl, "\Delta^{n-1}_{E'}"'] \arrow[ddd, "\varphi^{n-1}(Q')"] \\
\arrow[r, phantom, "(1)"] & H^n(A) \arrow[r, "H^n(f)"] \arrow[d, "\varphi^n(A)"] \arrow[u, phantom, "(4)"] & H^n(B) \arrow[d, "\varphi^n(B)"'] & \arrow[l, phantom, "(3)"] \\
& K^n(A) \arrow[r, "K^n(f)"] \arrow[d, phantom, "(2)"] & K^n(B) & \\
K^{n-1}(Q'') \arrow[ur, "\overline{\Delta}^{n-1}_{E''}"] \arrow[rrr, "K^{n-1}(q)"'] & & & K^{n-1}(Q') \arrow[ul, "\overline{\Delta}^{n-1}_{E'}"']
\end{tikzcd}$$

There is commutativity in

(1) because φ^n satisfies the commutativity condition of the theorem,

(3) similar to (1)

(2) and (4) by Axiom I.

The outside square commutes because φ^{n-1} is assumed to be a natural transformation. Since $\Delta^{n-1}_{E''}$ is an epic, the inside square commutes and φ^n is a natural transformation.

Theorem 1.7: Two cohomology theories over the same $\mathcal{E}$-left exact functor $T:\mathfrak{A} \to \mathfrak{B}$ are equivalent.

Proof: Let H and K be two cohomology theories over T, and let $1_T:T \to T$ be the identity transformation. Then, from the previous theorem, there exist maps $\varphi:H \to K$ and $\psi:K \to H$ such that $\psi\varphi:H \to H$ and $\varphi\psi:K \to K$ are identity maps. Using induction, one can prove that φ^n, ψ^n are natural equivalences for each $n \geq 0$, so that the proof is completed.

The techniques of this section are similar to those used by Uehara [20] for the absolute case.

Examples of Classical Cohomology Theory

If $\mathfrak{A}$ is the abelian category of R-modules where R is a commutative ring with identity and $\mathfrak{B}$ is the abelian category of abelian groups, then it is well known that the class $\mathcal{E}$ of all exact sequences is an injective class in $\mathfrak{A}$. The functor $T = \mathrm{Hom}_{\mathfrak{A}}(A,-):\mathfrak{A} \to \mathfrak{B}$, for an R-module A, is a covariant additive functor. Moreover T is $\mathcal{E}$-left exact, MacLane [15]. Therefore, there exists a unique cohomology theory $H_{\mathcal{E}}$ relative to $\mathcal{E}$ over T. One defines $\mathrm{Ext}^n(A,B)$ to be $H^n(B)$ for $B \in \mathfrak{A}$.

A second example is from the theory of sheaves, Swan [19]. Let $\mathfrak{A}$ be the abelian category of sheaves over a fixed space X and commutative ring R with identity. The morphisms are sheaf homomorphisms $f:(S,\pi,X) \to (S',\pi',X)$, Uehara [20] and Swan [19]. Let $\mathfrak{B}$ be the abelian category of R-modules and $T:\mathfrak{A} \to \mathfrak{B}$ be the functor which associates with a sheaf $\mathcal{G} = (S,\pi,X)$ the R-module $\mathrm{Hom}_{\mathfrak{A}}(\Lambda,\mathcal{G})$ where $\Lambda = (X\times R,\pi,X)$. Then T is a covariant additive functor.

Consider the class $\mathcal{E}$ of all coexact sequences. Then $\mathcal{E}$ is an injective class, the proof of this involves establishing a pair of adjoint functors between $\mathfrak{A}$ and the category of protosheaves. This has been done by Professor H. Uehara in [20; 5.12-5.18]. Moreover, T is an $\mathcal{E}$-left exact functor. Therefore, by the general theory developed above, there exists a unique cohomology theory $H_{\mathcal{E}}$ relative to $\mathcal{E}$ over T.

CHAPTER II

THE DERIVED FUNCTOR EXT FOR MODULES OVER AN ALGEBRA

Let (Λ, μ, η) be a graded R-algebra over a commutative ring R with unity where $\mu:\Lambda\otimes\Lambda \to \Lambda$ is the multiplication and $\eta:R \to \Lambda$ is the unit. ${}_\Lambda\mathfrak{M}$ denotes the category of graded left Λ-modules; [16] and [15]; where the morphisms are the Λ-module homomorphisms of degree zero and $\mathfrak{M}$ denotes the category of graded R-modules with R-homomorphisms of degree zero. ${}_\Lambda\mathfrak{M}$ and $\mathfrak{M}$ are abelian categories. $\mathfrak{M}_\Lambda$ denotes the category of graded right Λ-modules.

Properties of the Category ${}_\Lambda\mathfrak{M}$ ($\mathfrak{M}_\Lambda$)

Let $T:{}_\Lambda\mathfrak{M} \to \mathfrak{M}$ be the forgetful functor and let $S:\mathfrak{M} \to {}_\Lambda\mathfrak{M}$ be defined by $S(A) = \Lambda \otimes A$ for any object A in $\mathfrak{M}$ where the Λ-module structure of $\Lambda \otimes A$ is given by ${}_{\Lambda\otimes A}\varphi = \mu \otimes 1$. Then it can be shown that $S \dashv T:({}_\Lambda\mathfrak{M}, \mathfrak{M})$. It follows from [6], that T preserves monics, products and kernels. Moreover, since T is a faithful kernel preserving functor, T reflects epics and exact sequences. It can also be shown that a morphism is an epic in ${}_\Lambda\mathfrak{M}$ if and only if it is a surjective function. T, therefore, preserves epics.

Similarly, a morphism in ${}_\Lambda\mathfrak{M}$ is a monic in ${}_\Lambda\mathfrak{M}$ if and only if it is an injective function. Therefore, by the corollaries to the Kan Adjoint Theorem; [6-15,16]; ${}_\Lambda\mathfrak{M}$ is projectively perfect.

The following notation will be used, for any objects A, A' in $\mathfrak{M}$

$\mathrm{Hom}_R^d(A,A')$ denotes the set of all R-homomorphisms of degree $d \geq 0$ from A to A' and $\mathrm{Hom}_R(A,A') = \{\mathrm{Hom}_R^d(A,A') \mid d \geq 0\}$. Then $\mathrm{Hom}_R(A,A')$ is a graded R-module.

Proposition 2.1: Given any A in $\mathfrak{M}$ there exists an R-homomorphism of degree zero,

$$\varphi:\Lambda \otimes \mathrm{Hom}_R(\Lambda,A) \to \mathrm{Hom}_R(\Lambda,A),$$

such that $(\mathrm{Hom}_R(\Lambda,A), \varphi)$ is a left Λ-module.

Proof: For any $f \in \mathrm{Hom}_R(\Lambda,A)$ and λ, $\lambda' \in \Lambda$ define $\varphi(\lambda \otimes f)(\lambda') = = (-1)^{|\lambda||f|+|\lambda||\lambda'|}f(\lambda\lambda')$. Then it needs to be shown that $\varphi(\lambda \otimes f) \equiv \lambda f \in \mathrm{Hom}_R(\Lambda,A)$, $|\lambda f| = |\lambda| + |f|$ and $(\mathrm{Hom}_R(\Lambda,A), \varphi)$ is a left Λ-module.

The verification that λf is an R-homomorphism of degree $|\lambda| + |f|$ is straightforward, so omitted here.

Let λ_1, $\lambda_2 \in \Lambda$, then $(\lambda_1\lambda_2)f = \lambda_1(\lambda_2 f)$; because given any $\lambda \in \Lambda$, $[(\lambda_1\lambda_2)f](\lambda) = (-1)^{\epsilon}f(\lambda\lambda_1\lambda_2)$, $\epsilon = |\lambda_1||f| + |\lambda_2||f| + |\lambda_1||\lambda| + |\lambda_2||\lambda|$, and $[\lambda_1(\lambda_2 f)](\lambda) = (-1)^{\rho_1}(\lambda_2 f)(\lambda\lambda_1)$, $\rho_1 = |\lambda_1||\lambda_2| + |\lambda_1||f| + |\lambda_1||\lambda|$, $= (-1)^{\rho_2}f(\lambda\lambda_1\lambda_2)$, $\rho_2 = \rho_1 + |\lambda_2||f| + |\lambda_2||\lambda| + |\lambda_2||\lambda_1|$. Similarly, one can show $\eta(1)f = f$ and the proof is completed.

For the following theorem one needs to refer to paragraph 4 of [6] and verify the dual statements for categories with cokernels.

Theorem 2.1: ${}_\Lambda\mathfrak{M}$ is injectively perfect.

Proof: The following characterization of injectively perfect will be verified for the category ${}_\Lambda\mathfrak{M}$:

> A category $\mathfrak{A}$ with cokernels is injectively perfect if and only if given any object A in $\mathfrak{A}$ there exists a monic $i:A \to I$, where I is an injective object.

Let Q be an injective module in $\mathfrak{M}$. Then $\mathrm{Hom}_R(\Lambda,Q)$ is an injective object in ${}_\Lambda\mathfrak{M}$ because given any monic $i:M \to M'$ and any Λ-morphism $\alpha:M \to \mathrm{Hom}_R(\Lambda,Q)$ it can be shown there exists a Λ-morphism $\beta:M' \to \mathrm{Hom}_R(\Lambda,Q)$ such that $\beta i = \alpha$.

$$\begin{tikzcd}
M \arrow[rr, "i"] \arrow[dr, "\alpha"] \arrow[ddr, "\alpha'"'] & & M' \arrow[dl, "\beta"'] \arrow[ddl, "\beta'"] \\
& \mathrm{Hom}_R(\Lambda,Q) & \\
& Q &
\end{tikzcd}$$

In $\mathfrak{M}$, consider $\alpha':M \to Q$ defined by $\alpha'(a) = \alpha(a)(e)$ for any $a \in M$, where $e = \eta(1) \in \Lambda$. Then α' is an R-homomorphism and $|\alpha'| = 0$. $\mathfrak{M}$ is injectively perfect hence there exists $\beta':M' \to Q$ such that $\beta' i = \alpha'$.

Define $\beta:M' \to \mathrm{Hom}_R(\Lambda,Q)$ setting $\beta(m)(\lambda) = (-1)^{|\lambda||m|}\beta'(\lambda m)$ for any $m \in M'$ and $\lambda \in \Lambda$.

1. $\beta(m) \in \mathrm{Hom}_R^{|m|}(\Lambda,Q)$ for any $m \in M'$. If $m \in M'_s$, then for any $\lambda \in \Lambda_n$, $\lambda m \in M'_{s+n}$ and, since $|\beta'| = 0$, $\beta'(\lambda m) \in Q_{s+n}$. Therefore, $|\beta(m)| = |m|$ and $|\beta| = 0$.

Furthermore, for any $\lambda,\ \lambda' \in \Lambda_s$, $\beta(m)(\lambda + \lambda') =$
$= (-1)^{|\lambda+\lambda'|m|}\beta'((\lambda + \lambda')m) = (-1)^{|\lambda||m|}\beta'(\lambda m) + (-1)^{|\lambda'||m|}\beta'(\lambda' m)$
$= \beta(m)(\lambda) + \beta'(m)(\lambda')$. Similarly, $\beta(m)(r\lambda) = r\beta(m)(\lambda)$ for $\lambda \in \Lambda$, $r \in R$.

2. β is a Λ-module homomorphism, because given any $m \in M'$ and

$\lambda, \lambda' \in \Lambda$, $\beta(\lambda m)(\lambda') = (-1)^{|\lambda'||\lambda m|}\beta'(\lambda'\lambda m) = (-1)^{|\lambda'||\lambda|+|\lambda'||m|}\beta'(\lambda'\lambda m)$ and $(\lambda\beta(m))(\lambda') = (-1)^{|\lambda||\beta(m)|+|\lambda||\lambda'|}\beta(m)(\lambda'\lambda) =$

$(-1)^{|\lambda||m|+|\lambda||\lambda'|}\beta(m)(\lambda'\lambda) = (-1)^{|\lambda||m|+|\lambda||\lambda'|+|\lambda'\lambda||m|}\beta'(\lambda'\lambda m)$

$= (-1)^{|\lambda||\lambda'|+|\lambda'||m|}\beta'(\lambda'\lambda m)$.

3. $\beta i = \alpha$. For any $m \in M'$ and $\lambda \in \Lambda$, $\beta(i(m))(\lambda) =$

$= (-1)^{|\lambda||i(m)|}\beta'(\lambda i(m)) = (-1)^{|\lambda||m|}\beta'(i(\lambda m)) = (-1)^{|\lambda||m|}\alpha'(\lambda m)$

$= (-1)^{|\lambda||m|}\alpha(\lambda m)(e) = (-1)^{|\lambda||m|}[\lambda\alpha(m)](e)$

$= (-1)^{|\lambda||\alpha(m)|+|\lambda||e|}[\lambda\alpha(m)](e) = \alpha(m)(\lambda)$. Hence $\beta i = \alpha$ and $\mathrm{Hom}_R(\Lambda, Q)$ is an injective object.

Now, given any M in ${}_{\Lambda}\mathfrak{M}$ consider M in $\mathfrak{M}$. Then, there exists an injective object Q in $\mathfrak{M}$ and a monic $i': M \to Q$. From the above $\mathrm{Hom}_R(\Lambda, Q)$ is an injective object in ${}_{\Lambda}\mathfrak{M}$. Define $i: M \to \mathrm{Hom}_R(\Lambda, Q)$ by $i(m)(\lambda) = (-1)^{|\lambda||m|}i'(\lambda m)$ for $m \in M$ and $\lambda \in \Lambda$. Then $i(m) \in \mathrm{Hom}_R(\Lambda, Q)$ and $|i(m)| = |m|$. Moreover, i is a zero-degree Λ-homomorphism. If it can be shown that i is an injective set function, then the proof will be completed.

Let $i(m) = i(m')$ for $m, m' \in M$. Then $i(m)(e) = i(m')(e)$ and $i'(m) = i'(m')$. Hence $m = m'$.

Construction of Adjoint Functors $T \dashv S'(\mathfrak{M}, {}_{\Lambda}\mathfrak{M})$

The procedure of the above paragraph implies there exists a functor $S': \mathfrak{M} \to {}_{\Lambda}\mathfrak{M}$ such that S' is an adjoint of T where T is the forgetful functor. The following theorem states this. (Subsequent to the completion of this dissertation the author has noticed that S. Eilenberg and J. C. Moore have also obtained this result, [7-397].)

Therefore, if $\mathcal{E}$ is an injective class in $\mathfrak{M}$, $T^{-1}(\mathcal{E})$ is an injective class in ${}_\Lambda\mathfrak{M}$, Kan Adjoint Theorem for injective classes.

<u>Theorem 2.2</u>: There exists a functor $S':\mathfrak{M} \to {}_\Lambda\mathfrak{M}$ such that $T \dashv S'$ where $T:{}_\Lambda\mathfrak{M} \to \mathfrak{M}$ is the forgetful functor.

<u>Proof</u>: Given any A in $\mathfrak{M}$ let $S'(A) = \mathrm{Hom}_R(\Lambda, A)$ with multiplication defined by $(\lambda f)(\lambda') = (-1)^{|\lambda||f|+|\lambda||\lambda'|}f(\lambda'\lambda)$. If $g:A \to B$ is a morphism in $\mathfrak{M}$, then define $S'(g) \equiv g_*:\mathrm{Hom}_R(\Lambda,A) \to \mathrm{Hom}_R(\Lambda,B)$. It can be verified that g_* is an R-homomorphism of degree zero where $g_*(\alpha) = g\alpha$ for any $\alpha \in \mathrm{Hom}_R(\Lambda,A)$. Moreover, g_* is a Λ-homomorphism because given any $\lambda,\lambda' \in \Lambda$ and $\alpha \in \mathrm{Hom}_R(\Lambda,A)$; $g_*(\lambda\alpha)(\lambda') = [g(\lambda\alpha)](\lambda') = g((\lambda\alpha)(\lambda'))$ $= (-1)^{|\lambda||\alpha|+|\lambda||\lambda'|}g(\alpha(\lambda'\lambda)) = (-1)^{|\lambda||g_*(\alpha)|+|\lambda||\lambda'|}[g_*(\alpha)](\lambda'\lambda)$ $= [\lambda g_*(\alpha)](\lambda')$; i.e., $g_*(\lambda\alpha) = \lambda g_*(\alpha)$.

To complete the proof it must be shown that there exist set functions (for each pair (A,M) with A in $\mathfrak{M}$ and M in ${}_\Lambda\mathfrak{M}$)

$$b:\mathrm{Hom}_R^0(T(M),A) \to \mathrm{Hom}_\Lambda^0(M,S'(A)) \text{ and}$$

$$a:\mathrm{Hom}_\Lambda^0(M,S'(A)) \to \mathrm{Hom}_R^0(T(M),A)$$

such that $ab = 1$ and $ba = 1$.

For any $f \in \mathrm{Hom}_R^0(T(M),A)$ define $b(f)$ by, for any $m \in M$ and $\lambda \in \Lambda$, $b(f)(m)(\lambda) = (-1)^{|\lambda||m|}f(\lambda m)$. Then $|b(f)(m)(\lambda)| = |f(\lambda m)| = |\lambda m|$ $= |\lambda| + |m|$. Consequently, $|b(f)| = 0$. Moreover, $b(f)(m)$ is an R-homomorphism for each $m \in M$ and $b(f)$ is an R-homomorphism. Therefore, it need only be shown that $b(f)$ is a Λ-homomorphism.

For any $\lambda,\lambda' \in \Lambda$ and $m \in M$, $b(f)(\lambda m)(\lambda') = (-1)^{|\lambda'||\lambda m|}f(\lambda'\lambda m)$ $= (-1)^{|\lambda'||\lambda|+|\lambda'||m|}f(\lambda'\lambda m)$. Also, $[\lambda b(f)(m)](\lambda') =$

$= (-1)^{|\lambda||b(f)(m)|+|\lambda||\lambda'|}b(f)(m)(\lambda'\lambda) = (-1)^{|\lambda||m|+|\lambda||\lambda'|+|\lambda'\lambda||m|}f(\lambda'\lambda m)$

$= (-1)^{|\lambda||\lambda'|+|\lambda'||m|}f(\lambda'\lambda m)$ and $b(f)(\lambda m) = \lambda b(f)(m)$.

To define a, consider an arbitrary $g: M \to \mathrm{Hom}_R(\Lambda, A)$ and set $a(g)(m) = g(m)(e)$ for any $m \in M$. THen $|a(g)(m)| = |g(m)(e)| = |g(m)| = |m|$. Therefore, $|a(g)| = 0$ and one can verify that $a(g)$ is an R-homomorphism.

Let $f \in \mathrm{Hom}_R^0(T(M), A)$. Then $(ab)(f) = a(b(f)): T(M) \to A$ is defined by $a(b(f))(m) = b(f(m))(e) = f(em) = f(m)$ for any $m \in M$. Hence $ab = 1$.

Also, if $\alpha \in \mathrm{Hom}_\Lambda^0(M, S'(A))$, then for any $m \in M$ and $\lambda \in \Lambda$;

$[(ba)(\alpha)(m)](\lambda) = b(a(\alpha))(m)(\lambda) = (-1)^{|\lambda||m|}\alpha(\lambda m)(e)$

$= (-1)^{|\lambda||m|}[\lambda\alpha(m)](e) = (-1)^{|\lambda||\alpha(m)|+|\lambda||e|}\lambda\alpha(m)(e) = \alpha(m)(\lambda)$.

So, for any $m \in M$, $(ba)(\alpha)(m) = \alpha(m)$ and $(ba)(\alpha) = \alpha$.

Definition of $\mathrm{Ext}_{\Lambda,\tilde{\mathcal{E}}}^1$ and $\mathrm{Ext}_{\Lambda,\tilde{\mathcal{E}}}^0$

Given left Λ-modules M, M'; $\mathrm{Hom}_\Lambda(M, M') = \{\mathrm{Hom}_\Lambda^d(M, M') \mid d \geq 0\}$ is an R-module and the functor $\mathrm{Hom}_\Lambda(M, -): {}_\Lambda\mathfrak{M} \to \mathfrak{M}$ is a covariant additive functor.

<u>Theorem 2.3</u>: Given an exact sequence

$$0 \to M^1 \xrightarrow{i} M^2 \xrightarrow{j} M^3 \to 0$$

in ${}_\Lambda\mathfrak{M}$ and given any object M in ${}_\Lambda\mathfrak{M}$, the sequence

$$0 \to \mathrm{Hom}_\Lambda(M, M^1) \xrightarrow{i_*} \mathrm{Hom}_\Lambda(M, M^2) \xrightarrow{j_*} \mathrm{Hom}_\Lambda(M, M^3)$$

is exact in $\mathfrak{M}$.

<u>Proof</u>: To prove this it is sufficient to show that given any $d \geq 0$

the sequence

$$0 \to \mathrm{Hom}^d_\Lambda(M,M^1) \xrightarrow{(i_*)_d} \mathrm{Hom}^d_\Lambda(M,M^2) \xrightarrow{(j_*)_d} \mathrm{Hom}^d_\Lambda(M,M^3)$$

is exact as a sequence of (ungraded) R-modules.

Consider $f,g \in \mathrm{Hom}^d_\Lambda(M,M^1)$ such that $(i_*)_d(f) = (i_*)_d(g)$. This implies $if = ig$. Assume $f \neq g$. Then there exists $n \geq 0$ such that $f_n \neq g_n$ where $f_n, g_n : M_n \to M^1_{n+d}$. Therefore, there exists $x \in M_n$ such that $f_n(x) \neq g_n(x)$ and $i_{n+d}(f_n(x)) \neq i_{n+d}(g_n(x))$. This is a contradiction, hence $(i_*)_d$ is injective.

$$\begin{tikzcd} & M \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ M^1 \arrow[r, "i"] & M^2 \arrow[r, "j"] & M^3 \end{tikzcd}$$

Now, let $f \in \mathrm{Hom}^d_\Lambda(M,M^2)$ such that $(j_*)_d(f) = 0$. Then for each n; $f_n : M_n \to M^2_{n+d}$ and $\operatorname{im} f_n \subset \ker j_{n+2} = \operatorname{im} i_{n+d}$.

Define $g_n : M_n \to M^1_{n+d}$ by $g_n(m) = m'$ where m' is the unique element of M^1_{n+d} such that $i_{n+d}(m') = f_n(m)$. Then, for each n, g_n is an R-homomorphism of degree d and $i_{n+d}g_n = f_n$. To complete the proof it must be shown that $g = \{g_n \mid n \geq 0\}$ is a Λ-homomorphism. For any $m \in M$ and $\lambda \in \Lambda$, $f(\lambda m) = (-1)^{|\lambda||f|}\lambda f(m)$. Hence there exists a unique $m' \in M^1$ such that $i(m') = f(m)$ and since i is a zero degree homomorphism, $i((-1)^{|\lambda||f|}\lambda m') = (-1)^{|\lambda||f|}\lambda i(m') = f(\lambda m)$. By the definition of g and since i is an injection, $g(\lambda m) = (-1)^{|\lambda||f|}\lambda m' =$ $= (-1)^{|\lambda||g|}\lambda g(m)$ and the proof is completed.

If $\mathcal{E}^1$ denotes the class of all exact (exact = coexact in $\mathfrak{M}$) sequences in $\mathfrak{M}$ and $\mathcal{E}^0$ denotes the class of all split exact sequences in $\mathfrak{M}$, then $\mathcal{E}^1$ and $\mathcal{E}^0$ are injective classes in $\mathfrak{M}$. Hence by the Kan Adjoint Theorem, $\tilde{\mathcal{E}}^1 = T^{-1}(\mathcal{E}^1)$ and $\tilde{\mathcal{E}}^0 = T^{-1}(\mathcal{E}^0)$ are injective classes in ${}_\Lambda\mathfrak{M}$.

Moreover, by properties of T, it can be shown that $\mathcal{E}^1$ is the class of all exact sequences in ${}_\Lambda\mathfrak{M}$ and $\mathcal{E}^0$ is an exact class of sequences not equal to $\mathcal{E}^1$. By Theorem 2.3, given any M in ${}_\Lambda\mathfrak{M}$ the functor $\mathrm{Hom}_\Lambda(M,-)$ is both $\mathcal{E}^1$-left exact and $\mathcal{E}^0$-left exact.

From Chapter I there exist unique cohomology theories over $\mathrm{Hom}_\Lambda(M,-)$ relative to $\mathcal{E}^1$ and $\mathcal{E}^0$, respectively. These derived functors are denoted by $\mathrm{Ext}^n_{\Lambda,\mathcal{E}^1}(M,-)$ and $\mathrm{Ext}^n_{\Lambda,\mathcal{E}^0}(M,-)$, respectively, and are obtained in the following manner. Given M' in ${}_\Lambda\mathfrak{M}$ there exists an $\mathcal{E}^1$-injective resolution of M'. Let

$$X: 0 \to M' \xrightarrow{\epsilon} X_1 \xrightarrow{\partial^1} X_2 \to \cdots \to X_n \xrightarrow{\partial^n} X_{n+1} \to \cdots$$

denote this resolution. Then one obtains

$$\mathrm{Hom}^d_\Lambda(M,X): 0 \to \mathrm{Hom}^d_\Lambda(M,M') \xrightarrow{(\epsilon_*)_d} \mathrm{Hom}^d_\Lambda(M,X_1) \xrightarrow{\delta^1_d} \mathrm{Hom}^d_\Lambda(M,X^2) \to \cdots$$

for each $d \geq 0$ and $\mathrm{Ext}^{n,d}_{\Lambda,\mathcal{E}^1}(M,M') \equiv H^n(\mathrm{Hom}^d_\Lambda(M,X))$; $\mathrm{Ext}^n_{\Lambda,\mathcal{E}^1}(M,M') \equiv \{\mathrm{Ext}^{n,d}_{\Lambda,\mathcal{E}^1}(M,M') \mid d \geq 0\}$. Similarly, define $\mathrm{Ext}^n_{\Lambda,\mathcal{E}^0}(M,-)$.

The Canonical $\mathcal{E}^0$-injective Resolution

<u>Proposition 2.2</u>: For any A,B in $\mathfrak{M}$, $\mathrm{Hom}_R(\Lambda \otimes A, B)$ is a left Λ-module.

Proof: For $f: \Lambda \otimes A \to B$ define $\lambda f: \Lambda \otimes A \to B$, for any $\lambda \in \Lambda$, by the following:

$$\lambda f(\lambda' \otimes a) = (-1)^{|\lambda||f| + |\lambda||\lambda'|} f(\lambda'\lambda \otimes a) \text{ for any } \lambda' \in \Lambda,\ a \in A.$$

By extension λf is an R-homomorphism and $|\lambda f| = |\lambda| + |f|$. Moreover, for any $\lambda, \lambda' \in \Lambda$ and any $f \in \mathrm{Hom}_R(\Lambda \otimes A, B)$, $(\lambda\lambda')f = \lambda(\lambda' f)$ because

$$[(\lambda\lambda')f](\lambda'' \otimes a) = (-1)^{|\lambda\lambda'||f| + |\lambda\lambda'||\lambda''|} f(\lambda''\lambda\lambda' \otimes a)$$

$= (-1)^{\rho_1} f(\lambda''\lambda\lambda' \otimes a)$, where $\rho_1 = |\lambda||f| + |\lambda'||f| + |\lambda||\lambda''| + |\lambda'||\lambda''|$,

and $[\lambda(\lambda' f)](\lambda'' \otimes a) = (-1)^{|\lambda||\lambda' f| + |\lambda||\lambda''|}(\lambda' f)(\lambda''\lambda \otimes a)$

$= (-1)^{\rho_2} f(\lambda''\lambda\lambda' \otimes a)$, where $\rho_2 = |\lambda||\lambda'| + |\lambda||f| + |\lambda||\lambda''| + |\lambda'||f| +$

$+ |\lambda'||\lambda''| + |\lambda'||\lambda|$.

One can verify that $(-1)^{\rho_1} = (-1)^{\rho_2}$.

Similarly, $\eta(1)f = f$. Therefore, $\mathrm{Hom}_R(\Lambda \otimes A, B)$ is a left Λ-module with respect to the above multiplication.

Proposition 2.3: For any A,B in $\mathfrak{M}$, $\mathrm{Hom}_R(\Lambda, \mathrm{Hom}_R(A,B))$ is isomorphic to $\mathrm{Hom}_R(\Lambda \otimes A,B)$ as left Λ-modules.

Proof: Define $\psi:\mathrm{Hom}_R(\Lambda,\mathrm{Hom}_R(A,B)) \to \mathrm{Hom}_R(\Lambda \otimes A,B)$ setting $\psi(f)(\lambda \otimes a) = f(\lambda)(a)$ for any $f:\Lambda \to \mathrm{Hom}_R(A,B)$ and $\lambda \in \Lambda$, $a \in A$. By extension $\psi(f)$ is an R-homomorphism and $|\psi(f)| = |f|$. It can be shown that ψ is an R-homomorphism of degree zero. Moreover, ψ is a Λ-module homomorphism because for any $f \in \mathrm{Hom}_R(\Lambda, \mathrm{Hom}_R(A,B))$, $\lambda,\lambda' \in \Lambda$ and $a \in A$, $[\lambda\psi(f)](\lambda' \otimes a) = (-1)^{|\lambda||\psi(f)| + |\lambda||\lambda'|}\psi(f)(\lambda'\lambda \otimes a)$

$= (-1)^{|\lambda||f| + |\lambda||\lambda'|} f(\lambda'\lambda)(a)$ and $\psi(\lambda f)(\lambda' \otimes a) = [(\lambda f)(\lambda')](a)$

$= (-1)^{|\lambda||f| + |\lambda||\lambda'|} f(\lambda'\lambda)(a)$. It is straightforward to show ψ is surjective. To complete the proof, consider $f,g \in \mathrm{Hom}_R(\Lambda, \mathrm{Hom}_R(A,B))$ such that $\psi(f) = \psi(g)$. For any $a \in A$, $\psi(f)(\eta(1) \otimes a) = \psi(g)(\eta(1) \otimes a)$ and $f(\eta(1)(a) = g(\eta(1)(a)$. Then, for any $\lambda \in \Lambda$, $f(\lambda) =$

$= (-1)^{|\lambda||f|}\lambda f(\eta(1)) = (-1)^{|\lambda||g|}\lambda g(\eta(1)) = g(\lambda)$. $\psi(f) = \psi(g)$ implies that $|f| = |g|$; consequently, $f = g$ and the proof is completed.

From [6-10-17], there exists a canonical $\mathcal{E}^0$-injective resolution of the Λ-module M constructed in the following manner, where $(\Lambda, \mu, \eta, \epsilon)$ is a graded augmented R-algebra.

Remark: $(\Lambda, \mu, \eta, \epsilon)$ is an augmented graded R-algebra so the sequence $0 \to Q \underset{p}{\overset{i}{\rightleftarrows}} \Lambda \underset{\eta}{\overset{\epsilon}{\rightleftarrows}} R \to 0$ is split exact as a sequence of R-modules where $Q = \ker \epsilon$. Then $\Lambda \cong Q + \eta(R)$.

It has already been shown that there exists a functor $S' : \mathfrak{M} \to {}_{\Lambda}\mathfrak{M}$ defined by $S'(A) = \mathrm{Hom}_R(\Lambda, A)$ for any A in $\mathfrak{M}$, such that $T \dashv S'$ where $T : {}_{\Lambda}\mathfrak{M} \to \mathfrak{M}$ is the forgetful functor. The morphisms

$$\mathrm{Hom}^0_R(T(M), A) \underset{a}{\overset{b}{\rightleftarrows}} \mathrm{Hom}^0_{\Lambda}(M, S'(A))$$

are defined by:

$$b(f)(m)(\lambda) = (-1)^{|\lambda||m|} f(\lambda m)$$

$$a(g)(m) = g(m)(\eta(1)).$$

Let $c : \mathfrak{M}^2 \to \mathfrak{M}^2$ be the cokernel coresolvent of $\mathcal{E}^0$ in $\mathfrak{M}$, see paragraph 6 of [6]. By a corollary to the Kan Adjoint Theorem; [6]; there exists a coresolvent e for $\mathcal{E}^0$ in ${}_{\Lambda}\mathfrak{M}$. Given any $f : M \to M'$ in ${}_{\Lambda}\mathfrak{M}$ consider $f : M \to M'$ in $\mathfrak{M}$. Then $c(f) : M' \to \operatorname{coker} f$ and

$e(f) \equiv b(c(f)) : M' \to \mathrm{Hom}_R(\Lambda, \operatorname{coker} f)$; $e(f)(m)(\lambda) = (-1)^{|\lambda||m|} c(f)(\lambda m)$ for any $m \in M'$ and any $\lambda \in \Lambda$. Now, construct an $\mathcal{E}^0$-injective resolution for M using the coresolvent e of $\mathcal{E}^0$. This resolution is called the canonical resolution of M relative to the coresolvent e; [6-10].

Theorem 2.4: The canonical resolution of M relative to the coresolvent e is the cochain complex

$$0 \to M \underset{t^{-1}}{\overset{\alpha}{\rightleftarrows}} B_0 \underset{t^0}{\overset{\delta^0}{\rightleftarrows}} B_1 \underset{t^1}{\overset{\delta^1}{\rightleftarrows}} B_2 \rightleftarrows \cdots \rightleftarrows B_n \underset{t^n}{\overset{\delta^n}{\rightleftarrows}} B_{n+1} \rightleftarrows \cdots$$

where:

0) $\alpha(m)(\lambda) = (-1)^{|\lambda||m|}\lambda m$ for $m \in M$, $\lambda \in \Lambda$;

i) $B_k \equiv \mathrm{Hom}_R(\Lambda \otimes Q^k, M)$ for $k \geq 0$ and $Q^k \equiv Q \otimes \ldots \otimes Q$ (k factors);

ii) $t^{-1}(f) = f(\eta(1)$ for $f \in \mathrm{Hom}_R(\Lambda, M)$;

iii) $t^k(f)(\lambda \otimes q_1 \otimes \ldots \otimes q_k) = f(\eta(1) \otimes p(\lambda) \otimes q_1 \otimes \ldots \otimes q_k)$ for $f \in B_{k+1}$, $\lambda \in \Lambda$, $q_i \in Q$ and $k \geq 0$;

iv) $$\delta^k = [(\mu \otimes 1^k](\tau \otimes 1^k)]^* + \sum_{i=1}^{k}(-1)^i[(1 \otimes \ldots \otimes \mu \otimes \ldots \otimes 1)(1 \otimes \ldots \otimes \tau \otimes \ldots \otimes 1)]^* + (-1)^{k+1}({}_M\varphi\tau)_*, \text{ for } k \geq 0.$$

Notation: $\beta^*(g) = g\beta$ and $({}_M\varphi\tau)_*(g) = {}_M\varphi\tau(g \otimes 1)$. τ is the twisting morphism, [16-213].

Proof:

1. Conisder, in $\mathfrak{M}$, the sequence $0 \to M \xrightarrow{1_M} M$. 1_M is a cokernel of 0_M. Therefore $\alpha = e(0_M) = b(1_M)$ and from the definition of b $\alpha(m)(\lambda) = (-1)^{|\lambda||m|}\lambda m$ for any $\lambda \in \Lambda$, $m \in M$. By the construction of b, α is a Λ-homomorphism. This will always be true when b is used to define the morphism so will not be pointed out each time.

Define $t^{-1}: B_0 \to M$ by $t^{-1}(f) = f(\eta(1))$ for any $f: \Lambda \to M$. Then $t^{-1}(\alpha(m)) = \alpha(m)(\eta(1)) = \eta(1)m = m$ and $t^{-1}\alpha = 1_M$. Moreover, $B_0 = \mathrm{Re}(0_M) = \mathrm{Hom}_R(\Lambda, M)$.

2. Define $\alpha^0: \mathrm{Hom}_R(\Lambda, M) \to \mathrm{Hom}_R(\Lambda, M)$ by $\alpha^0 = 1 - \alpha t^{-1}$. It can be considered that $\alpha^0: \mathrm{Hom}_R(\Lambda, M) \to \mathrm{Hom}_R(Q, M)$ because the sequence

$$0 \to \mathrm{Hom}_R(Q,M) \xrightarrow{p^*} \mathrm{Hom}_R(\Lambda,M) \xrightarrow{\eta^*} \mathrm{Hom}_R(R,M) \to 0$$

is exact and $\eta^*\alpha^0 = 0$. $\eta^*\alpha^0 = 0$ because if $f \in \mathrm{Hom}_R(\Lambda,M)$ and $r \in R$, then $(\eta^*\alpha^0)(f)(r) = [\alpha^0(f)\eta](r) = \alpha^0(f)(\eta(r)) = (1 - \alpha t^{-1})(f)(\eta(r))$

$= f(\eta(r)) - \alpha(t^{-1}(f))(\eta(r)) = f(\eta(r)) - \alpha(f(\eta(1))(\eta(r))$

$= f(\eta(r)) - \eta(r)f(\eta(1)) = f(\eta(r)) - f(\eta(r)) = 0.$

Moreover, $\alpha^0\alpha = 0$ and $\ker \alpha^0 \subset \operatorname{im} \alpha$. Therefore, if α^0 is an epic, α^0 is a cokernel of α. Let $f \in \mathrm{Hom}_R(Q,M)$. Define $g:\Lambda \to M$ by setting $g(\lambda) = f(p(\lambda))$ for any $\lambda \in \Lambda$. Then $\alpha^0(g)(q) = g(q) - \alpha(t^{-1}(g))(q)$

$= f(q) - \alpha(g(\eta(1)))(q) = f(q) - qf(p(\eta(1))) = f(q)$ because $p\eta = 0$.

Therefore, $b(\alpha^0):\mathrm{Hom}_R(\Lambda,M) \to \mathrm{Hom}_R(\Lambda,\mathrm{Hom}_R(Q,M))$ and

$\mathrm{Hom}_R(\Lambda,\mathrm{Hom}_R(Q,M)) \cong \mathrm{Hom}_R(\Lambda \otimes Q,M)$ where; for any $f:\Lambda \to M$, $\lambda \in \Lambda$ and $q \in Q$; $b(\alpha^0)(f)(\lambda)(q) = (-1)^{|\lambda||f|}\alpha^0(\lambda f)(q) =$

$= (-1)^{|\lambda||f|}((\lambda f) - \alpha(t^{-1}(\lambda f)))(q) =$

$= (-1)^{|\lambda||f|}[(\lambda f)(q) - \alpha((\lambda f)(\eta(1)))(q)]$

$= (-1)^{|\lambda||f|}(\lambda f)(q) + (-1)^{|\lambda||f|+|q||\lambda f|+1}q(\lambda f)(\eta(1))$

$= (-1)^{|\lambda||q|}f(q\lambda) + (-1)^{|q||f(\lambda)|+1}qf(\lambda)$. Therefore,

$\delta^0:\mathrm{Hom}_R(\Lambda,M) \to \mathrm{Hom}_R(\Lambda \otimes Q,M) = B_1$ is given by $\delta^0 = (\mu\tau)^* - ({}_M\varphi\tau)_*$.

$\delta^0\alpha = 0$ because given any $m \in M$, $\lambda \in \Lambda$ and $q \in Q$, $\delta^0(\alpha(m))(\lambda \otimes q) =$

$= [\alpha(m)\mu\tau](\lambda \otimes q) - [{}_M\varphi\tau(\alpha(m) \otimes 1)](\lambda \otimes q)$

$= (-1)^{|\lambda||q|}\alpha(m)(q\lambda) - (-1)^{|q||\alpha(m)(\lambda)|}q\alpha(m)(\lambda)$

$= (-1)^{|\lambda||q|+|q\lambda||m|}(q\lambda)m + (-1)^{|q||m|+|q||\lambda|+|\lambda||m|+1}q(\lambda m)$

$= (-1)^{|\lambda||q|+|q||m|+|\lambda||m|}(q\lambda)m + (-1)^{|q||m|+|q||\lambda|+|\lambda||m|+1}q(\lambda m) = 0.$

Now, define $t^0:\mathrm{Hom}_R(\Lambda \otimes Q,M) \to \mathrm{Hom}_R(\Lambda,M)$, for any $f:\Lambda \otimes Q \to M$ and $\lambda \in \Lambda$, $t^0(f)(\lambda) = f(\eta(1) \otimes p(\lambda))$. t^0 is an R-homomorphism of degree zero and, for any $g:\Lambda \to M$, $\lambda \in \Lambda$, $(\alpha t^{-1} + t^0\delta^0)(g)(\lambda) =$

$= \alpha(t^{-1}(g))(\lambda) + t^0(\delta^0(g))(\lambda) = \alpha(g(\eta(1)))(\lambda) + \delta^0(g)(\eta(1) \otimes p(\lambda))$

$= (-1)^{|\lambda||g|}\lambda g(\eta(1)) + g(p(\lambda)) - (-1)^{|\lambda||g|}p(\lambda)g(\eta(1))$.

$\lambda \in \Lambda$, hence $\lambda = p(\lambda) + \eta(r)$ where $p(\lambda) \in Q$ and $r \in R$. If $|\lambda| = 0$, then $(\alpha t^{-1} + t^0\delta^0)(g)(\lambda) = (p(\lambda) + \eta(r))\, g(\eta(1)) + g(p(\lambda)) - p(\lambda)g(\eta(1))$ $= \eta(r)g(\eta(1)) + g(p(\lambda)) = g(\eta(r) + p(\lambda)) = g(\lambda)$. If $|\lambda| \neq 0$ then $\eta(r) = 0$ and $\lambda = p(\lambda)$. Therefore, $(\alpha t^{-1} + t^0\delta^0)(g)(\lambda) = g(\lambda)$ and $\alpha t^{-1} + t^0\delta^0 = 1_{B_0}$.

3. Consider the morphism $\alpha^1 = 1 - \delta^0 t^0$. In a similar manner as in the previous step it will be shown that $\operatorname{im} \alpha^1 \subset \operatorname{Hom}_R(Q \otimes Q, M)$ and the sequence

$$B_0 = \operatorname{Hom}_R(\Lambda, M) \underset{t^0}{\overset{\delta^0}{\rightleftarrows}} \operatorname{Hom}_R(\Lambda \otimes Q, M) \xrightarrow{\alpha^1} \operatorname{Hom}_R(Q \otimes Q, M) \text{ is}$$

well-defined.

The sequence

$$0 \to \operatorname{Hom}_R(Q \otimes Q, M) \xrightarrow{(p \otimes 1)^*} \operatorname{Hom}_R(\Lambda \otimes Q, M) \xrightarrow{(\eta \otimes 1)^*} \operatorname{Hom}_R(R \otimes Q, M) \to 0$$

is exact. So, if $(\eta \otimes 1)^*\alpha^1 = 0$, $\operatorname{im} \alpha^1 \subset \operatorname{Hom}_R(Q \otimes Q, M)$. Let $f:\Lambda \otimes Q \to M$, then, for any $r \in R$ and any $q \in Q$, $[(\eta\otimes 1)^*(\alpha^1(f))](r \otimes q)$

$= \alpha^1(f)(\eta(r)\otimes q) = [f - \delta^0(t^0(f))](\eta(r)\otimes q) = f(\eta(r)\otimes q) - \delta^0(t^0(f))(\eta(r)\otimes q)$

$= f(\eta(r) \otimes q) - [t^0(f)(q\eta(r)) - (-1)^{|q||t^0(f)|}qt^0(f)(\eta(r))]$

$= f(\eta(r) \otimes q) - f(\eta(1) \otimes rq) + (-1)^{|q||t^0(f)|}qf(\eta(1) \otimes p(\eta(r)))$

$= f(\eta(r) \otimes q) - f(\eta(r) \otimes q) = 0$.

One can readily verify that $\alpha^1\delta^0 = 0$ and $\ker \alpha^1 \subset \operatorname{im} \delta^0$. So, if α^1 is an epic, α^1 will be a cokernel of δ^0. Let $f:Q \otimes Q \to M$ and define

$g:\Lambda \otimes Q \to M$ by, for any $\lambda \in \Lambda$ and $q \in Q$, $g(\lambda \otimes q) = f(p(\lambda) \otimes q)$.

Consequently, for any $q,q' \in Q$, $\alpha'(g)(q \otimes q') = (1 - \delta^0 t^0)(g)(q \otimes q')$

$$= g(q \otimes q') - \delta^0(t^0(g))(q \otimes q') = f(q \otimes q') - \delta^0(t^0(g))(q \otimes q')$$

$$= f(q \otimes q') - (-1)^{|q||q'|} t^0(g)(q'q) + (-1)^{|q'||t^0(g)(q)|} q' t^0(g)(q)$$

$$= f(q \otimes q') - (-1)^{|q||q'|} g(\eta(1) \otimes q'q) + (-1)^{|q||t^0(g)(q)|} q' g(\eta(1) \otimes q)$$

$$= f(q \otimes q') - (-1)^{|q||q'|} f(p(\eta(1)) \otimes q'q) +$$
$$+ (-1)^{|q'||t^0(g)(q)|} q' f(p(\eta(1)) \otimes q)$$

$= f(q \otimes q')$ because $p\eta = 0$. Then

$$e(\delta^0) = b(\alpha^1): \mathrm{Hom}_R(\Lambda \otimes Q, M) \to \mathrm{Hom}_R(\Lambda, \mathrm{Hom}_R(Q \otimes Q, M))$$

and $\mathrm{Hom}_R(\Lambda, \mathrm{Hom}_R(Q \otimes Q, M)) \cong \mathrm{Hom}_R(\Lambda \otimes Q \otimes Q, M) \equiv B_2$. Now, for any

$f:\Lambda \otimes Q \to M$ and any $\lambda \in \Lambda$, $q,q' \in Q$, $b(\alpha^1)(f)(\lambda)(q \otimes q') =$

$$= (-1)^{|\lambda||f|} \alpha^1(\lambda f)(q \otimes q') = (-1)^{|\lambda||f|}(1 - \delta^0 t^0)(\lambda f)(q \otimes q')$$

$$= (-1)^{|\lambda||f|}[(\lambda f) - \delta^0(t^0(\lambda f))](q \otimes q')$$

$$= (-1)^{|\lambda||f|}(\lambda f)(q \otimes q') + (-1)^{|\lambda||f|+|q||q'|+1} t^0(\lambda f)(q'q)$$
$$+ (-1)^{|\lambda||f|+|q'||t^0(\lambda f)(q)|} q' t^0(\lambda f)(q)$$

$$= (-1)^{|q||\lambda|} f(q\lambda \otimes q') + (-1)^{|\lambda||f|+|q||q'|+1}(\lambda f)(\eta(1) \otimes q'q)$$
$$+ (-1)^{|\lambda||f|+|q'||t^0(\lambda f)(q)|} q'(\lambda f)(\eta(1) \otimes q)$$

$$= (-1)^{|q||\lambda|} f(q\lambda \otimes q') + (-1)^{|q||q'|+1} f(\lambda \otimes q'q)$$
$$+ (-1)^{|q'||f(\lambda \otimes q)|} q' f(\lambda \otimes q)$$

$$= [f(\mu \otimes 1)(\tau \otimes 1)](\lambda \otimes q \otimes q') - [f(1 \otimes \mu)(1 \otimes \tau)](\lambda \otimes q \otimes q')$$
$$+ [{}_M\varphi\tau(f \otimes 1)](\lambda \otimes q \otimes q').$$

Define $\delta^1 : \mathrm{Hom}_R(\Lambda \otimes Q, M) \to \mathrm{Hom}_R(\Lambda \otimes Q \otimes Q, M)$ by

$$\delta^1 = [(\mu \otimes 1)(\tau \otimes 1)]^* - [(1 \otimes \mu)(1 \otimes \tau)]^* + ({}_M\varphi\tau)_*$$

and define $t^1 : \mathrm{Hom}_R(\Lambda \otimes Q \otimes Q, M) \to \mathrm{Hom}_R(\Lambda \otimes Q, M)$ by $t^1(f)(\lambda \otimes q) =$
$= f(\eta(1) \otimes p(\lambda) \otimes q)$.

To complete this step of the construction it must be verified that $\delta^1\delta^0 = 0$ and $\delta^0 t^0 + t^1\delta^1 = 1_{B_1}$. From theoretical considerations $\delta^1\delta^0 = 0$ but it will be shown here by direct calculation. This will not be done in further steps. Let $f \in B_0$ and let $\lambda \in \Lambda, q, q' \in Q$. Then

$$\begin{aligned}
\delta^1(\delta^0(f))(\lambda \otimes q \otimes q') &= (-1)^{|\lambda||q|}\delta^0(f)(q\lambda \otimes q') + \\
&+ (-1)^{|q'||q|+1}\delta^0(f)(\lambda \otimes q'q) \\
&+ (-1)^{|q'||\delta^0(f)(\lambda \otimes q)|}q'\delta^0(f)(\lambda \otimes q) \\
= (-1)^{|\lambda||q|+|q\lambda||q'|}f(q'q\lambda) &- (-1)^{|\lambda||q|+|q'||f(q\lambda)|}q'f(q\lambda) \\
&+ (-1)^{|q'||q|+|q'q||\lambda|+1}f(q'q\lambda) \\
&+ (-1)^{|q'||q|+|q'q||f(\lambda)|+1}q'qf(\lambda) \\
&+ (-1)^{|q'||\delta^0(f)(\lambda \otimes q)|+|q||\lambda|}q'f(q\lambda) \\
&+ (-1)^{|q'||\delta^0(f)(\lambda \otimes q)|+|q||f(\lambda)|}q'qf(\lambda)
\end{aligned}$$

$= 0$ because

i) $|\lambda||q| + |q'||f(q\lambda)| = |\lambda||q| + |q'||f| + |q'||q| + |q'||\lambda|$

and $|q'||\delta^0(f)(\lambda \otimes q)| + |q||\lambda| =$

$= |q'||f| + |q'||\lambda| + |q'||q| + |q||\lambda|$;

ii) $|q'||\delta^0(f)(\lambda \otimes q)| + |q||f(\lambda)| =$

$= |q'||f| + |q'||\lambda| + |q'||q| + |q||f| + |q||\lambda|$ and

$|q'||q| + |q'q||f(\lambda)| =$

$= |q'||q| + |q'||f| + |q'||\lambda| + |q||f| + |q||\lambda|$.

To show $\delta^0 t^0 + t^1\delta^1 = 1_{B_1}$, let $f:\Lambda \otimes Q \to M$. Then

$$(\delta^0 t^0 + t^1\delta^1)(f)(\lambda \otimes q) = \delta^0(t^0(f))(\lambda \otimes q) + t^1(\delta^1(f))(\lambda \otimes q)$$

$$= (-1)^{|q||\lambda|} t^0(f)(q\lambda) - (-1)^{|q||t^0(f)(\lambda)|} qt^0(f)(\lambda) + \delta^1(f)(\eta(1)\otimes p(\lambda)\otimes q)$$

$$= (-1)^{|q||\lambda|} f(\eta(1) \otimes p(q\lambda)) - (-1)^{|q||t^0(f)(\lambda)|} qf(\eta(1) \otimes p(\lambda))$$

$$+ (-1)^{|\eta(1)||p(\lambda)|} f(p(\lambda) \otimes q) - (-1)^{|q||p(\lambda)|} f(\eta(1) \otimes qp(\lambda))$$

$$+ (-1)^{|q||f(\eta(1)\otimes p(\lambda)|} qf(\eta(1) \otimes p(\lambda))$$

$$= (-1)^{|q||\lambda|} f(\eta(1) \otimes p(q\lambda)) + f(p(\lambda) \otimes q) - (-1)^{|q||p(\lambda)|} f(\eta(1)\otimes qp(\lambda)).$$

The following two remarks complete this step of the construction:

i) $p(q\lambda) = p(q(p(\lambda) + \eta(r))) = p(qp(\lambda)) + p(q\eta(r)) = qp(\lambda) + rq$

because p is an R-homomorphism and $p_{|Q} = 1_Q$.

ii) $\Lambda = Q + \eta(R)$ means $\Lambda_0 = Q_0 + \eta(R)$ and $\Lambda_n = Q_n$ for $n > 0$.

Hence $\lambda = p(\lambda) + \eta(r)$. If $|\lambda| = 0$, then $(\delta^0 t^0 + t^1\delta^1)(f)(\lambda\otimes q)$

$= f(\eta(1) \otimes qp(\lambda)) + f(\eta(1) \otimes rq) + f(p(\lambda) \otimes q) - f(\eta(1)\otimes qp(\lambda))$

$= f(\eta(r)\otimes q) + f(p(\lambda) \otimes q) = f(\lambda \otimes q)$. On the other hand if

$|\lambda| \neq 0$ then $\eta(r) = 0$ and $(\delta^0 t^0 + t^1\delta^1)(f)(\lambda \otimes q) =$

$$= (-1)^{|q||\lambda|} f(\eta(1) \otimes qp(\lambda))$$

$$+ f(p(\lambda) \otimes q) - (-1)^{|q||\lambda|} f(\eta(1) \otimes qp(\lambda))$$

$= f(\lambda \otimes q)$. Hence $(\delta^0 t^0 + t^1\delta^1) = 1_{B_1}$ and this step of the

construction is completed.

4). Assume, for each $1 \leq k \leq n$,

0) $e(\delta^{k-1}) = \delta^k$ is Λ-module homomorphism of degree zero where

$$\delta^k = [(\mu \otimes 1^k)(\tau \otimes 1^k)]^*$$

$$+ \sum_{i=1}^{k} (-1)^i [1 \otimes \ldots \otimes \mu \otimes \ldots \otimes 1)(1 \otimes \ldots \otimes \tau \otimes \ldots \otimes 1)]^*$$

$+ (-1)^{k+1}({}_M\varphi\tau)_*$ and $\delta^k\delta^{k-1} = 0$;

i) $\mathrm{Re}(\delta^k) = B_{k+1} = \mathrm{Hom}_R(\Lambda \otimes Q^{k+1}, M)$;

ii) there exists an R-homomorphism of degree zero, $t^k : B_{k+1} \to B_k$, defined by $t^k(f)(\lambda \otimes q_1 \otimes \ldots \otimes q_k) =$
$= f(\eta(1) \otimes p(\lambda) \otimes q_1 \otimes \ldots \otimes q_n)$ such that
$\delta^{k-1}t^{k-1} + t^k\delta^k = 1_{B_k}$.

5). Consider the function $\alpha^{n+1} = 1 - \delta^n t^n$ where

$$\alpha^{n+1} : B_{n+1} \to \mathrm{Hom}_R(\Lambda \otimes Q^{n+1}, M).$$

Then it can be shown that the im $\alpha^{n+1} \subset \mathrm{Hom}_R(Q \otimes Q^{n+1}, M)$ because the following sequence is ecact and $(\eta \otimes 1^{n+1})^*\alpha^{n+1} = 0$,

$$0 \to \mathrm{Hom}_R(Q\otimes Q^{n+1}, M) \xrightarrow{(p\otimes 1^{n+1})^*} \mathrm{Hom}_R(\Lambda\otimes Q^{n+1}, M) \xrightarrow{(\eta\otimes 1^{n+1})^*} \mathrm{Hom}_R(R\otimes Q^{n+1}, M) \to 0.$$

To show $(\eta \otimes 1^{n+1})^*\alpha^{n+1} = 0$, let $f : \Lambda \otimes Q^{n+1} \to M$ and r, q_i be arbitrary in R and Q, respectively. Then $[(\eta \otimes 1^{n+1})^*\alpha^{n+1}](r \otimes q_1 \otimes \ldots \otimes q_{n+1}) =$

$$= \alpha^{n+1}(f)(\eta(r) \otimes q_1 \otimes \ldots \otimes q_{n+1})$$

$$= (f - \delta^n(t^n(f)))(\eta(r) \otimes q_1 \otimes \ldots \otimes q_{n+1})$$

$$= f(\eta(r) \otimes q_1 \otimes \ldots \otimes q_{n+1}) - \delta^n(t^n(f))(\eta(r) \otimes q_1 \otimes \ldots \otimes q_{n+1})$$

$$= f(\eta(r) \otimes q_1 \otimes \ldots \otimes q_{n+1}) - t^n(f)(rq_1 \otimes \ldots \otimes q_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^{i+1+|q_{i+1}||q_i|} t^n(f)(\eta(r)\otimes q_1 \otimes \ldots \otimes q_{i+1}q_i \otimes \ldots \otimes q_{n+1})$$
$$+ (-1)^{n+2+|q_{n+1}||t^n(f)(\eta(r)\otimes q_1\otimes\ldots\otimes q_n)|} q_{n+1} t^n(f)(\eta(r)\otimes q_1\otimes\ldots\otimes q_n)$$

$$= f(\eta(r) \otimes q_1 \otimes \ldots \otimes q_{n+1}) - f(\eta(1) \otimes rq_1 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{i=1}^{n}(-1)^{i+1+|q_{i+1}||q_i|}f(\eta(1)\otimes p(\eta(r)\otimes q_1\otimes\ldots\otimes q_{i+1}q_i\otimes\ldots\otimes q_{n+1})$$

$$+ (-1)^{n+2+|q_{n+1}||t^n(f)(\eta(r)\otimes q_1\otimes\ldots\otimes q_n)|}q_{n+1}f(\eta(1)\otimes p(\eta(r))\otimes q_1\otimes\ldots\otimes q_n)$$

$= f(\eta(r) \otimes q_1 \otimes \ldots \otimes q_{n+1}) - f(\eta(r) \otimes q_1 \otimes \ldots \otimes q_{n+1}) = 0$ because $p\eta = 0$. Therefore,

$$B_n = \mathrm{Hom}_R(\Lambda \otimes Q^n, M) \underset{t^n}{\overset{\delta^n}{\rightleftarrows}} \mathrm{Hom}_R(\Lambda \otimes Q^{n+1}, M) \xrightarrow{1-\delta^n t^n} \mathrm{Hom}_R(Q \otimes Q^{n+1}, M)$$

is a sequence, $(1 - \delta^n t^n)\delta^n = 0$ and $\ker(1 - \delta^n t^n) \subset \mathrm{im}\,\delta^n$. Now, let $f: Q \otimes Q^{n+1} \to M$ and define $g: \Lambda \otimes Q^{n+1} \to M$ by setting $g(\lambda \otimes q_1 \otimes \ldots \otimes q_{n+1}) = f(p(\lambda) \otimes q_1 \otimes \ldots \otimes q_{n+1})$. Then

$$(1 - \delta^n t^n)(g)(q_0 \otimes q_1 \otimes \ldots \otimes q_{n+1}) = g(q_0 \otimes \ldots \otimes q_{n+1}) -$$

$$- \delta^n(t^n(g))(q_0 \otimes \ldots \otimes q_{n+1})$$

$$= f(q_0 \otimes \ldots \otimes q_{n+1}) - (-1)^{|q_0||q_1|}t^n(g)(q_1 q_0 \otimes q_2 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{i=1}^{n}(-1)^{i+1+|q_{i+1}||q_i|}t^n(g)(q_0 \otimes q_1 \otimes \ldots \otimes q_{i+1}q_i \otimes \ldots \otimes q_{n+1})$$

$$+ (-1)^{n+2+|q_{n+1}||t^n(g)(q_0\otimes \ldots \otimes q_n|}q_{n+1}t^n(g)(q_0\otimes q_1\otimes \ldots \otimes q_n)$$

$$= f(q_0 \otimes \ldots \otimes q_{n+1}) + (-1)^{|q_0||q_1|+1}g(\eta(1) \otimes q_1 q_0 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{i=1}^{n}(-1)^{i+1+|q_{i+1}||q_i|}g(\eta(1) \otimes q_0\otimes q_1\otimes\ldots\otimes q_{i+1}q_i\otimes\ldots\otimes q_{n+1})$$

$$+ (-1)^{n+2+|q_{n+1}||t^n(g)(q_0\otimes\ldots\otimes q_n)|}q_{n+1}g(\eta(1) \otimes q_0\otimes\ldots\otimes q_n)$$

$= f(q_0 \otimes \ldots \otimes q_{n+1})$. Hence $(1 - \delta^n t^n)$ is an epic and $(1 - \delta^n t^n)$ is a cokernel of δ^n. Therefore, $\delta^{n+1} = e(\delta^n) = b(1 - \delta^n t^n)$;

$$\delta^{n+1}: \mathrm{Hom}_R(\Lambda \otimes Q^{n+1}, M) \to \mathrm{Hom}_R(\Lambda, \mathrm{Hom}_R(Q \otimes Q^{n+1}, M));$$

and $\mathrm{Hom}_R(\Lambda, \mathrm{Hom}_R(Q \otimes Q^{n+1}, M)) \cong \mathrm{Hom}_R(\Lambda \otimes Q^{n+2}, M)$. Now, let

$B_{n+2} \equiv \mathrm{Hom}_R(\Lambda \otimes Q^{n+2}, M)$. To calculate δ^{n+1}, let $f \in B_{n+1}$, $\lambda \in \Lambda$ and $q_i \in Q$. Then

$$b(1 - \delta^n t^n)(f)(\lambda)(q_0 \otimes \ldots \otimes q_{n+1}) = (-1)^{|\lambda||f|}(1 - \delta^n t^n)(\lambda f)(q_0 \otimes \ldots \otimes q_{n+1})$$

$$= (-1)^{|\lambda||f|}(\lambda f)(q_0 \otimes \ldots \otimes q_{n+1}) + (-1)^{|\lambda||f|+1}\delta^n(t^n(\lambda f))(q_0 \otimes \ldots \otimes q_{n+1})$$

$$= (-1)^{|q_0||\lambda|} f(q_0\lambda \otimes q_1 \otimes \ldots \otimes q_{n+1}) +$$

$$+ (-1)^{|\lambda||f|+1+|q_1||q_0|} t^n(\lambda f)(q_1 q_0 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{i=1}^{n} (-1)^{i+|\lambda||f|+1+|q_{i+1}||q_i|} t^n(\lambda f)(q_0 \otimes q_1 \otimes \ldots \otimes q_{i+1} q_i \otimes \ldots \otimes q_{n+1})$$

$$+ (-1)^{n+1+|\lambda||f|+1+|q_{n+1}||t^n(\lambda f)(q_0 \otimes \ldots \otimes q_n)|} q_{n+1} t^n(\lambda f)(q_0 \otimes \ldots \otimes q_n)$$

$$= (-1)^{|q_0||\lambda|} f(q_0\lambda \otimes q_1 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{j=0}^{n} (-1)^{j+1+|\lambda||f|+|q_{j+1}||q_j|} (\lambda f)(\eta(1) \otimes q_0 \otimes \ldots \otimes q_{j+1} q_j \otimes \ldots \otimes q_{n+1})$$

$$+ (-1)^{n+2+|\lambda||f|+|q_{n+1}||t^n(\lambda f)(q_0 \otimes \ldots \otimes q_n)|} q_{n+1} (\lambda f)(\eta(1) \otimes q_0 \otimes \ldots \otimes q_n)$$

$$= (-1)^{|q_0||\lambda|} f(q_0\lambda \otimes q_1 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{j=0}^{n} (-1)^{j+1+|q_{j+1}||q_j|} f(\lambda \otimes q_0 \otimes \ldots \otimes q_{j+1} q_j \otimes \ldots \otimes q_{n+1})$$

$$+ (-1)^{n+2+|q_{n+1}||f(\lambda \otimes q_0 \otimes \ldots \otimes q_n)|} q_{n+1} f(\lambda \otimes q_0 \otimes \ldots \otimes q_n)$$

$$= [f(\mu \otimes 1^{n+1})(\tau \otimes 1^{n+1})](\lambda \otimes q_0 \otimes q_1 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{j=0}^{n} (-1)^{j+1} [f(1 \otimes \ldots \otimes \mu \otimes \ldots \otimes 1)(1 \otimes \ldots \otimes \tau \otimes \ldots \otimes 1)](\lambda \otimes q_0 \otimes \ldots \otimes q_{n+1})$$

$$+ (-1)^{n+2} [{}_M\varphi\tau(f \otimes 1)](\lambda \otimes q_0 \otimes \ldots \otimes q_{n+1}).$$

Therefore, $\delta^{n+1} = [(\mu \otimes 1^{n+1})(\tau \otimes 1^{n+1})]^*$

$$+ \sum_{i=1}^{n+1}(-1)^i[(1 \otimes \ldots \otimes \mu \otimes \ldots \otimes 1)(1 \otimes \ldots \otimes \tau \otimes \ldots \otimes 1)]^*$$

$$+ (-1)^{n+2}({}_M\varphi\tau)_*.$$

Then, from theoretical considerations; i.e., the Kan Adjoint Theorem; $\delta^{n+1}\delta^n = 0$ because $(1 - \delta^n t^n)\delta^n = 0$.

Define $t^{n+1}:\mathrm{Hom}_R(\Lambda \otimes Q^{n+2}, M) \to \mathrm{Hom}_R(\Lambda \otimes Q^{n+1}, M)$ in the following manner, for any $f:\Lambda \otimes Q^{n+2} \to M$, $\lambda \in \Lambda$, $q_i \in Q$, set

$t^{n+1}(f)(\lambda \otimes q_1 \otimes q_2 \otimes \ldots \otimes q_{n+1}) = f(\eta(1) \otimes p(\lambda) \otimes q_1 \otimes \ldots \otimes q_{n+1})$.

Then by linear extension f is an R-homomorphism of degree zero.

To complete the proof of this theorem it must be shown that $\delta^n t^n + t^{n+1}\delta^{n+1} = 1_{B_n}$. Let $f:\Lambda \otimes Q^{n+1} \to M$, $\lambda \in \Lambda$ and $q_i \in Q$. Then

$(\delta^n t^n + t^{n+1}\delta^{n+1})(f)(\lambda \otimes q_1 \otimes \ldots \otimes q_{n+1}) =$

$$= \delta^n(t^n(f))(\lambda \otimes q_1 \otimes \ldots \otimes q_{n+1}) + t^{n+1}(\delta^{n+1}(f))(\lambda \otimes q_1 \otimes \ldots \otimes q_{n+1})$$

$$= (-1)^{|\lambda||q_1|}t^n(f)(q_1\lambda \otimes q_2 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{i=1}^{n}(-1)^{i+|q_{i+1}||q_i|}t^n(f)(\lambda \otimes q_1 \otimes \ldots \otimes q_{i+1}q_i \otimes \ldots \otimes q_{n+1})$$

$$+ (-1)^{n+1+|q_{n+1}||t^n(f)(\lambda \otimes q_1 \otimes \ldots \otimes q_n)|}q_{n+1}t^n(f)(\lambda \otimes q_1 \otimes \ldots \otimes q_n)$$

$$+ \delta^{n+1}(f)(\eta(1) \otimes p(\lambda) \otimes q_1 \otimes \ldots \otimes q_{n+1})$$

$$= [(-1)^{|\lambda||q_1|}f(\eta(1) \otimes p(q_1\lambda) \otimes q_2 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{i=1}^{n}(-1)^{i+|q_{i+1}||q_i|}f(\eta(1) \otimes p(\lambda) \otimes q_1 \otimes \ldots \otimes q_{i+1}q_i \otimes \ldots \otimes q_{n+1})$$

$$+ (-1)^{n+1+|q_{n+1}||f(\eta(1)\otimes p(\lambda)\otimes q_1\otimes\ldots\otimes q_n)|} q_{n+1} f(\eta(1)\otimes p(\lambda)\otimes q_1\otimes\ldots\otimes q_n)]$$

$$+ [f(p(\lambda) \otimes q_1 \otimes \ldots \otimes q_{n+1})$$

$$+ \sum_{\substack{j=0 \\ q_0=p(\lambda)}}^{n} (-1)^{j+1+|q_{j+1}||q_j|} f(\eta(1) \otimes p(\lambda) \otimes q_1\otimes\ldots\otimes q_{j+1}q_j\otimes\ldots\otimes q_{n+1})$$

$$+ (-1)^{n+2+|q_{n+1}||f(\eta(1)\otimes p(\lambda)\otimes q_1\otimes\ldots\otimes q_n|} q_{n+1} f(\eta(1)\otimes p(\lambda)\otimes q_1\otimes\ldots\otimes q_n)]$$

$$= (-1)^{|\lambda||q_1|} f(\eta(1) \otimes p(q_1\lambda) \otimes q_2 \otimes \ldots \otimes q_{n+1})$$

$$+ f(p(\lambda)\otimes q_1\otimes\ldots\otimes q_i\otimes\ldots\otimes q_{n+1}) -$$

$$- (-1)^{|q_1||p(\lambda)|} f(\eta(1) \otimes q_1 p(\lambda) \otimes q_2 \otimes\ldots\otimes q_{n+1})$$

$$= (-1)^{|\lambda||q_1|} f(\eta(1) \otimes q_1 p(\lambda) \otimes q_2 \otimes\ldots\otimes q_{n+1})$$

$$+ (-1)^{|\lambda||q_1|} f(\eta(1) \otimes rq_1 \otimes q_2 \otimes\ldots\otimes q_{n+1}) + f(p(\lambda) \otimes q_1\otimes\ldots\otimes q_{n+1})$$

$$- (-1)^{|q_1||p(\lambda)|} f(\eta(1) \otimes q_1 p(\lambda) \otimes q_2 \otimes\ldots\otimes q_{n+1})$$

$= (-1)^{|\lambda||q|} f(\eta(r) \otimes q_1\otimes q_2\otimes\ldots\otimes q_{n+1}) + f(p(\lambda) \otimes q_1 \otimes\ldots\otimes q_{n+1})$ where $\lambda = p(\lambda) + \eta(r)$. Then, by the remarks made in step 3, $(\delta^n t^n + t^{n+1}\delta^{n+1})(f) = f$ and by induction the proof is completed.

An $\mathcal{E}^1$-injective Resolution

Let M be any Λ-module. Then, by using the forgetful functor, M can be considered as an R-module. There exists an injective module Q_0 in $\mathfrak{M}$ such that $0 \to M \xrightarrow{\hat{\alpha}} Q_0$ is in $\mathcal{E}^1$. From the lemma to the Kan Adjoint theorem it is known that

$$0 \to M \xrightarrow{\alpha} \mathrm{Hom}_R(\Lambda,\ Q_0)$$

is in $\mathcal{E}^1$ where $\alpha = b(\hat{\alpha})$; recall the definition of b. Moreover, $\mathrm{Hom}_R(\Lambda, Q_0) \in \mathfrak{I}^1$ where $\mathcal{E}^1 \overset{*}{\Rightarrow} \mathfrak{I}^1$. ${}_\Lambda\mathfrak{M}$ is an abelian category hence there exists a morphism $\pi_0 : \mathrm{Hom}_R(\Lambda, Q_0) \to C_0$ such that π_0 is a cokernel of α. Consider $M \xrightarrow{\alpha} \mathrm{Hom}_R(\Lambda, Q_0) \xrightarrow{\pi_0} C_0$ in $\mathfrak{M}$. Then there exists an injective module Q_1 and a monic $i_0 : C_0 \to Q_1$ such that the sequence $M \xrightarrow{\alpha} \mathrm{Hom}_R(\Lambda, Q_0) \xrightarrow{\hat{\delta}^0} Q_1$ is in $\mathcal{E}^1$, where $\hat{\delta}^0 = i_0\pi_0$. Then $M \xrightarrow{\alpha} \mathrm{Hom}_R(\Lambda, Q_0) \xrightarrow{\delta^0} \mathrm{Hom}_R(\Lambda, Q_1)$ is in $\mathcal{E}^1$, $\delta^0 = b(\hat{\delta}^0)$, and $\mathrm{Hom}_R(\Lambda, Q) \in \mathfrak{I}^1$.

Assume that, for each $k \leq n$, the following sequence has been constructed

$$0 \to M \xrightarrow{\alpha} B_0 \xrightarrow{\delta^0} B_1 \xrightarrow{\delta^1} \cdots \to B_{k-2} \xrightarrow{\delta^{k-2}} B_{k-1} \xrightarrow{\delta^{k-1}} B_k$$

which is in $\mathcal{E}^1$ and such that $B_i \in \mathfrak{I}^1$ for each $0 \leq i \leq k$.

Now, consider $B_{k-1} \xrightarrow{\delta^{k-1}} B_k \xrightarrow{\pi_k} C_k$ in ${}_\Lambda\mathfrak{M}$ where π_k is the cokernel of δ^{k-1}. Then there exists an injective module Q_{k+1} and a monic, in $\mathfrak{M}$, $i_k : C_k \to Q_{k+1}$ such that $B_{k-1} \xrightarrow{\delta^{k-1}} B_k \xrightarrow{\hat{\delta}^k} Q_{k+1}$ is in $\mathcal{E}^1$; $\hat{\delta}^k = i_k\pi_k$. Hence by the lemma to the Kan Adjoint Theorem,

$B_{k-1} \xrightarrow{\delta^{k-1}} B_k \xrightarrow{\delta^k} \mathrm{Hom}_R(\Lambda, Q_{k+1})$ is in $\mathcal{E}^1$ and $B_{k+1} \equiv \mathrm{Hom}_R(\Lambda, Q_{k+1}) \in \mathfrak{I}^1$.

CHAPTER III

THE DERIVED FUNCTORS COTOR AND COEXT FOR COMODULES OVER A COALGEBRA

Let $(\Lambda, \Delta, \epsilon)$ be a graded coalgebra over a commutative ring R with unity where $\Delta:\Lambda \to \Lambda \otimes \Lambda$ is the comultiplication and $\epsilon:\Lambda \to R$ is the counit; see Milnor and Moore [16] and Gugenheim [11]. Let ${}^{\Lambda}\mathfrak{M}$ and $\mathfrak{M}^{\Lambda}$ denote the categories of left Λ-comodules and right Λ-comodules, respectively, where the morphisms are the Λ-homomorphisms of degree zero. Let $\mathfrak{M}$ denote the category of graded R-modules with R-homomorphisms of degree zero. Then ${}^{\Lambda}\mathfrak{M}$ and $\mathfrak{M}^{\Lambda}$ are additive categories with cokernels and $\mathfrak{M}$ is an abelian category. If Λ is a flat R-module then ${}^{\Lambda}\mathfrak{M}$ and $\mathfrak{M}^{\Lambda}$ are abelian categories; Milnor and Moore [16].

Construction of Adjoint Functors $T \dashv S(\mathfrak{M}, {}^{\Lambda}\mathfrak{M})$

Let T be the forgetful functor and let $S(A) = \Lambda \otimes A$ where the Λ-comodule structure is given by ${}_{\Lambda\otimes A}\varphi = \Delta \otimes 1:\Lambda \otimes A \to \Lambda \otimes \Lambda \otimes A$. Moreover, if $f:A \to B$ is in $\mathfrak{M}$ then define $S(f) = 1 \otimes f:\Lambda \otimes A \to \Lambda \otimes B$.

<u>Proposition 3.1</u>: S is an adjoint functor of T.

<u>Proof</u>: Define functions

$$b:\mathrm{Hom}_R^0(T(M), A) \to \mathrm{Hom}_\Lambda^0(M, S(A))$$

$$a:\mathrm{Hom}_\Lambda^0(M, S(A)) \to \mathrm{Hom}_R^0(T(M), A)$$

for any M in ${}^{\Lambda}\mathfrak{M}$ and any A in $\mathfrak{M}$ such that ab = 1 and ba = 1.

For any $f:M \to A$ in $\mathfrak{M}$ define $b(f):M \to \Lambda \otimes A$ by the following diagram:

$$\begin{tikzcd} M \arrow[r, "M^{\varphi}"] \arrow[dr, "b(f)"'] & \Lambda\otimes M \arrow[d, "1\otimes f"] \\ & \Lambda\otimes A \end{tikzcd}$$

Then b(f) is an R-homomorphism of degree zero. Moreover, the diagram

$$\begin{tikzcd} M \arrow[r, "b(f)"] \arrow[d, "M^{\varphi}"'] & \Lambda\otimes A \arrow[d, "\Lambda\otimes A^{\varphi}"] \\ \Lambda\otimes M \arrow[r, "1\otimes b(f)"] & \Lambda\otimes\Lambda\otimes A \end{tikzcd}$$

commutes because it can be written as

$$\begin{tikzcd} M \arrow[r, "M^{\varphi}"] \arrow[d, "M^{\varphi}"'] \arrow[dr, phantom, "\mathrm{(I)}"] & \Lambda\otimes M \arrow[r, "1\otimes f"] \arrow[d, "\Delta\otimes 1"] \arrow[dr, phantom, "\mathrm{(II)}"] & \Lambda\otimes A \arrow[d, "\Delta\otimes 1"] \\ \Lambda\otimes M \arrow[r, "1\otimes {}_M\varphi"'] & \Lambda\otimes\Lambda\otimes M \arrow[r, "1\otimes 1\otimes f"'] & \Lambda\otimes\Lambda\otimes A \end{tikzcd}$$

where (I) commutes by the definition of a Λ-comodule and (II) is an identity. Therefore, b(f) is a Λ-comodule homomorphism.

For any $g:M \to \Lambda \otimes A$ in ${}^{\Lambda}\mathfrak{M}$ define $a(g):M \to A$ by the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[ddr, "a(g)"'] & \Lambda\otimes A \arrow[d, "\epsilon\otimes 1"] \\ & R\otimes A \arrow[d, "\xi"] \\ & A \end{tikzcd}$$

where $\xi(r \otimes a) = ra$ is the natural isomormorphism.

Then a(g) is an R-homomorphism of degree zero.

Now, for any $f:T(M) \to A$; $(ab)(f) = a(b(f)) = a((1 \otimes f)_M\varphi)$

$= \xi(\epsilon \otimes 1)(1 \otimes f)_M\varphi = \xi(\epsilon \otimes f)_M\varphi = \xi(1 \otimes f)(\epsilon \otimes 1)_M\varphi = f$ because $\xi(1 \otimes f)(\epsilon \otimes 1)_M\varphi(m) = \xi(1 \otimes f)(1 \otimes m) = \xi(1 \otimes f(m)) = f(m)$. Hence $ab = 1$.

Also $ba = 1$ because given $g: M \to \Lambda \otimes A$ in ${}^{\Lambda}\mathfrak{M}$; $(ba)(g) = b(\xi(\epsilon \otimes 1)g)$ $= \{1 \otimes [\xi(\epsilon \otimes 1)g]\}_M\varphi = (1 \otimes \xi)(1 \otimes \epsilon \otimes 1)(1 \otimes g)_M\varphi$.

Consider the following commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "M^{\varphi}"'] & \Lambda\otimes A \arrow[r, "\Delta\otimes 1"] \arrow[d, "\Delta\otimes 1"] & \Lambda\otimes\Lambda\otimes A \arrow[d, "1\otimes\epsilon\otimes 1"] & \\
\Lambda\otimes M \arrow[r, "1\otimes g"'] & \Lambda\otimes\Lambda\otimes A \arrow[r, "1\otimes\epsilon\otimes 1"'] & \Lambda\otimes R\otimes A \arrow[r, "1\otimes\xi=\xi\otimes 1"'] & \Lambda\otimes A
\end{tikzcd}$$

Hence $(ba)(g) = (1 \otimes \xi)(1 \otimes \epsilon \otimes 1)(\Delta \otimes 1)g = g$ because $\xi(1 \otimes \epsilon)\Delta = 1_\Lambda$. Furthermore, one can verify that a and b are R-homomorphisms.

Therefore, T has the following properties:

i) faithful (by the definition of T),

ii) reflects epics and monics (because it is faithful),

iii) preserves epics, coproducts and cokernels (because it is a coadjoint functor of S),

iv) reflects coexact sequences (because it is faithful and cokernel preserving).

But T does not preserve monics as the following example shows. Moreover, from adjoint properties S preserves monics, kernels and products.

<u>Example 3.1</u>: T does not necessarily preserve monics.

To do this it will be shown that there exist monics in ${}^{\Lambda}\mathfrak{M}$ which not injective functions. Recall; S preserves monics, hence, if

$f:A \to A'$ is a monic in $\mathfrak{M}$ then $1 \otimes f:\Lambda \otimes A \to \Lambda \otimes A'$ is a monic in ${}^{\Lambda}\mathfrak{M}$.

Let $R = Z$, then $\mathfrak{M}$ is the category of all graded abelian groups. Let $\Lambda = (Z, Z_2, 0, 0, \ldots)$. Then

i) $(\Lambda \otimes \Lambda)_0 = Z \otimes Z$; $(\Lambda \otimes \Lambda)_1 = Z \otimes Z_2 + Z_2 \otimes Z$; $(\Lambda \otimes \Lambda)_2 = Z_2 \otimes Z_2$ and $(\Lambda \otimes \Lambda)_m = 0$ for $m > 2$;

ii) $(\Lambda \otimes \Lambda \otimes \Lambda)_0 = Z \otimes Z \otimes Z$; $(\Lambda \otimes \Lambda \otimes \Lambda)_1 = Z \otimes Z \otimes Z_2 +$
$+ Z \otimes Z_2 \otimes Z + Z_2 \otimes Z \otimes Z$;

$(\Lambda \otimes \Lambda \otimes \Lambda)_2 = Z_2 \otimes Z \otimes Z_2 + Z_2 \otimes Z_2 \otimes Z + Z \otimes Z_2 \otimes Z_2$;

$(\Lambda \otimes \Lambda \otimes \Lambda)_3 = Z_2 \otimes Z_2 \otimes Z_2$; $(\Lambda \otimes \Lambda \otimes \Lambda)_k = 0$ for $k > 3$.

Define $\Delta:\Lambda \to \Lambda \otimes \Lambda$ by:

$\Delta_0:Z \to Z \otimes Z$; $\Delta_0(1) = 1 \otimes 1$;

$\Delta_1:Z_2 \to Z \otimes Z_2 + Z_2 \otimes Z$; $\Delta_1(\bar{1}) = 1 \otimes \bar{1} + \bar{1} \otimes 1$

$\Delta_m = 0$ for $m > 1$.

Then Δ is a Z-homomorphism of degree zero. Moreover the following diagram commutes:

$$\begin{tikzcd}
\Lambda \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \Lambda\otimes\Lambda \arrow[d, "1\otimes\Delta"] \\
\Lambda\otimes\Lambda \arrow[r, "\Delta\otimes 1"'] & \Lambda\otimes\Lambda\otimes\Lambda
\end{tikzcd}$$

because;

i) <u>0th degree</u>

$(1 \otimes \Delta)_0\Delta_0(1) = (1 \otimes \Delta)_0(1 \otimes 1) = 1 \otimes \Delta_0(1) = 1 \otimes 1 \otimes 1,$

$(\Delta \otimes 1)_0\Delta_0(1) = (\Delta \otimes 1)_0(1 \otimes 1) = \Delta_0(1) \otimes 1 = 1 \otimes 1 \otimes 1;$

ii) <u>1st degree</u>:

$(1 \otimes \Delta)_1\Delta_1(\bar{1}) = (1 \otimes \Delta)_1(1 \otimes \bar{1} + \bar{1} \otimes 1) = 1 \otimes \Delta_1(\bar{1}) + \bar{1} \otimes \Delta_0(1)$

$$= 1 \otimes 1 \otimes \overline{1} + 1 \otimes \overline{1} \otimes 1 + \overline{1} \otimes 1 \otimes 1;$$

$$(\Delta \otimes 1)_1 \Delta_1(\overline{1}) = (\Delta \otimes 1)_1(1 \otimes \overline{1} + \overline{1} \otimes 1) = \Delta_0(1) \otimes \overline{1} +$$

$$+ \Delta_1(\overline{1}) \otimes 1 =$$

$$= 1 \otimes 1 \otimes \overline{1} + 1 \otimes \overline{1} \otimes 1 + \overline{1} \otimes 1 \otimes 1.$$

Now define $\epsilon:\Lambda \to Z$ by; $\epsilon_0 = 1_Z$ and $\epsilon_1 = 0$. Then ϵ is a Z-homomorphism of degree zero and the diagram

$$\begin{tikzcd}
Z\otimes\Lambda \arrow[r, equal] & \Lambda \arrow[r, equal] \arrow[dl, "\Delta"] \arrow[dr, "\Delta"] & \Lambda\otimes Z \\
\Lambda\otimes\Lambda \arrow[u, "\epsilon\otimes 1"] & & \Lambda\otimes\Lambda \arrow[u, "1\otimes\epsilon"']
\end{tikzcd}$$

is commutative because:

i) <u>0th degree</u>

$$(\epsilon \otimes 1)_0 \Delta_0(1) = (\epsilon \otimes 1)_0(1 \otimes 1) = 1 \otimes 1 \text{ and } (1 \otimes \epsilon)_0(1 \otimes 1) =$$

$$= 1 \otimes 1;$$

ii) <u>1st degree</u>

$$(\epsilon \otimes 1)_1 \Delta_1(\overline{1}) = (\epsilon \otimes 1)_1(1 \otimes \overline{1} + \overline{1} \otimes 1) = 1 \otimes \overline{1},$$

$$(1 \otimes \epsilon)_1 \Delta_1(\overline{1}) = (1 \otimes \epsilon)_1(1 \otimes \overline{1} + \overline{1} \otimes 1) = \overline{1} \otimes 1.$$

Therefore, $(\Lambda, \Delta, \epsilon)$ is a Z-coalgebra.

Now, $i:2Z \to Z$ defined by $i(2) = 2$ is a monic in $\mathfrak{M}$ hence $1 \otimes i:\Lambda \otimes 2Z \to \Lambda \otimes Z$ is a monic in ${}^{\Lambda}\mathfrak{M}$ but is not an injection because $(1 \otimes i)_1(\overline{1} \otimes 2) = 0$ where $(1 \otimes i)_1:Z_2 \otimes 2Z \to Z_2 \otimes Z$.

By the Kan Adjoint Theorem one knows that if $\mathcal{E}$ is an injective class in $\mathfrak{M}$, then $\tilde{\mathcal{E}} = T^{-1}(\mathcal{E})$ is an injective class in ${}^{\Lambda}\mathfrak{M}$. In particular, if we consider the class $\mathcal{E}^0$ of all split exact sequences in $\mathfrak{M}$, the class $\tilde{\mathcal{E}}^0 = T^{-1}(\mathcal{E}^0)$ is a coexact injective class in ${}^{\Lambda}\mathfrak{M}$. Note that a sequence $M^1 \xrightarrow{f} M^2 \xrightarrow{g} M^3$ is in $\tilde{\mathcal{E}}^0$ if and only if it is a split exact sequence when considered in $\mathfrak{M}$. Also $\tilde{\mathfrak{J}}^0$; where $\tilde{\mathcal{E}}^0 \overset{*}{\Rightarrow} \tilde{\mathfrak{J}}^0$;

consists of all retracts of objects $\Lambda \otimes A$ for any A in $\mathfrak{M}$.

Definition and Properties of the Cotensor Product

<u>Definition 3.1</u>, [16-219]: If N is a right Λ-comodule and M is a left Λ-comodule the <u>cotensor product of N and M over Λ</u>; denoted by $N \underset{\Lambda}{\square} M$; is defined to be the R-module such that the sequence

$$0 \to N \underset{\Lambda}{\square} M \xrightarrow{i} N \otimes M \xrightarrow{\varphi_N \otimes 1 - 1 \otimes {}_M\varphi} N \otimes \Lambda \otimes M$$

is exact as graded R-modules where φ_N and ${}_M\varphi$ are the multiplications of N and M respectively; i.e., $N \underset{\Lambda}{\square} M = \ker(\varphi_N \otimes 1 - 1 \otimes {}_M\varphi)$.

<u>Proposition 3.2</u>: Given a Λ-comodule homomorphism of degree zero, $f:M \to M'$, there exists a unique morphism $1 \square f : N \square M \to N \square M'$ in $\mathfrak{M}$ for each right Λ-comodule N.

<u>Proof</u>: Consider the diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
N \square M \arrow[r, "1 \square f"] \arrow[d, "i"'] & N \square M' \arrow[d, "i'"] \\
N \otimes M \arrow[r, "1 \otimes f"] \arrow[d, "\varphi_N \otimes 1 - 1 \otimes {}_M\varphi"'] & N \otimes M' \arrow[d, "\varphi_N \otimes 1 - 1 \otimes {}_{M'}\varphi"] \\
N \otimes \Lambda \otimes M \arrow[r, "1 \otimes 1 \otimes f"] & N \otimes \Lambda \otimes M'
\end{tikzcd}$$

Since f is a comodule homomorphism, ${}_{M'}\varphi f = (1 \otimes f)_M\varphi$. Hence $(1\otimes 1\otimes f)(\varphi_N \otimes 1 - 1 \otimes_M \varphi) = (\varphi_N \otimes 1 - 1 \otimes_{M'} \varphi)(1 \otimes f)$ and $(\varphi_N \otimes 1 - 1 \otimes_{M'} \varphi)(1 \otimes f)i = 0$. Hence there exists a unique morphism $1 \square f : N \square M \to N \square M'$ such that $(1 \otimes f)i = i'(1 \square f)$.

<u>Proposition 3.3</u>: If N is in $\mathfrak{M}^{\Lambda}$ and M, M', M'' are in ${}^{\Lambda}\mathfrak{M}$, then the following properties are satisfied:

i) $1_N \square 1_M = 1_{N \square M}$;

ii) if $f,g:M \to M'$ are morphisms in ${}^{\Lambda}\mathfrak{M}$ then $1 \square (f + g) = (1 \square f) + (1 \square g)$;

iii) if $f: M \to M'$ and $g: M' \to M''$ then $(1 \square gf) = (1 \square g)(1 \square f)$.

<u>Proof</u>: i) and iii) are immediate from the previous proposition and because of the uniqueness which the kernel guarantees. To prove ii) consider:

$$\begin{array}{ccc} N \square M & \overset{1 \square f}{\underset{1 \square g}{\longrightarrow}} & N \square M' \\ \downarrow i & & \downarrow i' \\ N \otimes M & \overset{1 \otimes f}{\underset{1 \otimes g}{\longrightarrow}} & N \otimes M' \\ \downarrow & & \downarrow \\ N \otimes \Lambda \otimes M & \overset{1 \otimes 1 \otimes f}{\underset{1 \otimes 1 \otimes g}{\longrightarrow}} & N \otimes \Lambda \otimes M' \end{array}$$

We know that $(1 \otimes (f + g)) = (1 \otimes f) + (1 \otimes g)$ and $1 \otimes 1 \otimes (f + g) = 1 \otimes 1 \otimes f + 1 \otimes 1 \otimes g$. Hence $i'(1 \square (f + g))$
$= (1 \otimes (f + g))i$
$= [(1 \otimes f) + (1 \otimes g)]i$
$= (1 \otimes f)i + (1 \otimes g)i$
$= i'(1 \square f) + i'(1 \square g)$ and the proof is completed.

From the above properties an additive covariant functor can be defined $N \square \_ : {}^{\Lambda}\mathfrak{M} \to \mathfrak{M}$ for each N in $\mathfrak{M}^{\Lambda}$. Moreover, this functor is $\mathcal{E}^{0}$-left exact as the following theorem shows.

<u>Theorem 3.1</u>: If $0 \to M_1 \underset{s}{\overset{f}{\rightleftarrows}} M_2 \underset{\sigma}{\overset{g}{\rightleftarrows}} M_3 \to 0$ is in $\mathcal{E}^{0}$, then for each N in $\mathfrak{M}^{\Lambda}$

$$0 \to N \square M_1 \xrightarrow{1 \square f} N \square M_2 \xrightarrow{1 \square g} N \square M_3$$

is an exact sequence in $\mathfrak{M}$.

<u>Proof</u>: Consider the following diagram (in $\mathfrak{M}$):

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & N \,\square\, M' \arrow[rr, "1 \,\square\, f"] \arrow[d, "i_1"'] & & N \,\square\, M_2 \arrow[rr, "1 \,\square\, g"] \arrow[d, "i_2"'] & & N \,\square\, M_3 \arrow[d, "i_3"'] & \\
0 \arrow[r] & N \otimes M_1 \arrow[rr, shift left, "1 \otimes f"] \arrow[d] & & N \otimes M_2 \arrow[ll, dashed, shift left, "1 \otimes s"] \arrow[rr, shift left, "1 \otimes g"] \arrow[d] & & N \otimes M_3 \arrow[ll, dashed, shift left, "1 \otimes \sigma"] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \otimes \Lambda \otimes M_1 \arrow[rr, shift left, "1 \otimes 1 \otimes f"] & & N \otimes \Lambda \otimes M_2 \arrow[ll, dashed, shift left, "1 \otimes 1 \otimes s"] \arrow[rr, shift left, "1 \otimes 1 \otimes g"] & & N \otimes \Lambda \otimes M_3 \arrow[ll, dashed, shift left, "1 \otimes 1 \otimes \sigma"] \arrow[r] & 0
\end{tikzcd}$$

where the three columns are exact and the bottom two rows are exact from a property of tensor product. $1 \,\square\, f$ is an injection because if $(1 \,\square\, f)(x) = (1 \,\square\, f)(y)$ then $i_2(1 \,\square\, f)(x) = i_2(1 \,\square\, f)(y)$ and $(1 \otimes f)(i_1(x)) = (1 \otimes f)(i_1(y))$. $1 \otimes f$ is an injection hence $i_1(x) = i_1(y)$ and $x = y$. By the previous theorem $(1 \,\square\, g)(1 \,\square\, f) = 0$ and the im $(1 \,\square\, f) \subset$ ker $(1 \,\square\, g)$. So to complete the proof one need only show; ker $(1 \,\square\, g) \subset$ im $(1 \,\square\, f)$. Let $x \in$ ker $(1 \,\square\, g)$. Then $i_2(x) \in$ ker $(1 \otimes g) =$ im $(1 \otimes f)$. Hence, there exists $y \in N \otimes M_1$ such that $(1 \otimes f)(y) = i_2(x)$. Thus $(1 \otimes 1 \otimes f)(\varphi_N \otimes 1 - 1 \otimes {}_{M_1}\varphi)(y)$ $= (\varphi_N \otimes 1 - 1 \otimes {}_{M_2}\varphi)(1 \otimes f)(y) = (\varphi_N \otimes 1 - 1 \otimes {}_{M_2}\varphi)(i_2(x)) = 0$, so there exists $z \in N \,\square\, M_1$ such that $i_1(z) = y$ because $1 \otimes 1 \otimes f$ is an injection. Therefore $(1 \,\square\, f)(z) = x$.

The following example shows that the condition of being split exact as a sequence of R-modules is necessary in the above theorem.

<u>Example 3.2</u>: Define $\Delta: R \to R \otimes R$ by $\Delta(r) = 1 \otimes r$ and $\epsilon = 1_R: R \to R$ then (R, Δ, ϵ) is an R-coalgebra. Given any R-module A define ${}_A\varphi: A \to R \otimes A$ by ${}_A\varphi(a) = 1 \otimes a$. Then $(A, {}_A\varphi)$ is a left R-comodule. Similarly define

right R-comodules. Then $A \underset{R}{\square} B = A \underset{R}{\otimes} B$ and it is well known that tensor product does not preserve monics.

Definition of the Derived Functor Cotor and the Cobar Construction

Let $(\Lambda,\Delta,\epsilon)$ be an R-coalgebra where R is a commutative ring with unity. ${}^{\Lambda}\mathfrak{M}$, $\mathfrak{M}^{\Lambda}$ and $\mathfrak{M}$ are the categories of left Λ-comodules, right Λ-comodules and R-modules, respectively. It has been shown that there exists a functor $S:\mathfrak{M} \to {}^{\Lambda}\mathfrak{M}$ such that S is an adjoint of the forgetful functor $T:{}^{\Lambda}\mathfrak{M} \to \mathfrak{M}$. Consider the injective class $\mathcal{E}^0$ of all split exact (exact = coexact in $\mathfrak{M}$) sequences in $\mathfrak{M}$, then $T^{-1}(\mathcal{E}^0) = \bar{\mathcal{E}}^0$ is a coexact injective class in ${}^{\Lambda}\mathfrak{M}$ and the $\bar{\mathcal{E}}^0$-injective objects are the retracts of $S(A)$ for any A in $\mathfrak{M}$.

If N is a right Λ-comodule the functor $N \underset{\Lambda}{\square} \_ : {}^{\Lambda}\mathfrak{M} \to \mathfrak{M}$ is an additive, covariant, $\bar{\mathcal{E}}^0$-left exact functor. So by Chapter I there exists a unique cohomology theory over $N \underset{\Lambda}{\square} \_$ relative to $\bar{\mathcal{E}}^0$. Define $\mathrm{Cotor}_{\Lambda,\bar{\mathcal{E}}^0}(N,\_)$ to be the derived functor of $N \underset{\Lambda}{\square} \_$. This means given any M in ${}^{\Lambda}\mathfrak{M}$ consider an $\bar{\mathcal{E}}^0$-injective resolution

$$0 \to M \underset{\longleftarrow}{\overset{\alpha}{\Longrightarrow}} X$$

of M. Then $\mathrm{Cotor}_{\Lambda,\bar{\mathcal{E}}^0}(N,M) \equiv H(N \underset{\Lambda}{\square} X)$, [18] page 7-25.

From now on in this paragraph it will be assumed that Λ is an augmented R-coalgebra with augmentation $\eta:R \to \Lambda$. Hence the following sequence

$$\begin{array}{c} 0 \to Q \underset{\longleftarrow}{\overset{i}{\Longrightarrow}} \Lambda \underset{\eta}{\overset{\epsilon}{\rightleftarrows}} R \to 0 \\ \| \\ \ker \epsilon \end{array}$$

is in $\mathcal{E}^0$ and $\Lambda \cong Q + R$.

The cobar resolution, Adams [1], for a left Λ-comodule is given by

$$0 \to M \underset{t^{-1}}{\overset{{}_M\varphi}{\rightleftarrows}} B_0(\Lambda,M) \underset{t^0}{\overset{\delta^0}{\rightleftarrows}} B_1(\Lambda,M) \underset{t^1}{\overset{\delta^1}{\rightleftarrows}} \cdots \underset{t^{n-1}}{\overset{\delta^{n-1}}{\rightleftarrows}} B_n(\Lambda,M) \underset{t^n}{\overset{\delta^n}{\rightleftarrows}} \cdots$$

where $\{t^n : n \geq -1\}$ is a contracting homotopy and

$$B_0(\Lambda,M) = \Lambda \otimes M;$$

$$B_n(\Lambda,M) = \Lambda \otimes \underbrace{Q \otimes \ldots \otimes Q}_{n \text{ factors}} \otimes M, \text{ for } n > 0;$$

$$\delta^n = \Delta \otimes 1_Q^n \otimes 1_M + 1 \otimes [\textstyle\sum_{i=1}^{n} (-1)^i 1_Q \otimes \ldots \otimes \Delta \otimes \ldots \otimes 1_Q \otimes 1_M]$$

$$+ (-1)^{n+1}\, 1_\Lambda \otimes 1_Q^n \otimes {}_M\varphi, \text{ for } n \geq 0;$$

$$t^n(\lambda \otimes q_1 \otimes \ldots \otimes q_n \otimes m) = \epsilon(\lambda) q_1 \otimes q_2 \otimes \ldots \otimes q_n \otimes m, \text{ for } n \geq 0,$$

and $t^{-1}(\lambda \otimes m) = \epsilon(\lambda) m$. Therefore, the cobar construction is an $\mathcal{E}^0$-injective resolution of M.

Let $e : \mathfrak{M}^2 \to \mathfrak{M}^2$ be the cokernel functor, then e is a coresolvent for $\mathcal{E}^0$ and $e' = b(e(T(f)))$ is a coresolvent for $\mathcal{E}^0$.

<u>Theorem 3.2</u>: The cobar resolution is the canonical [6-10] $\mathcal{E}^0$-injective resolution determined by the coresolvent e'.

<u>Proof</u>: Let $(M, {}_M\varphi)$ be a left Λ-comodule and consider the sequence

$0 \to M \underset{1_M}{\overset{1_M = e(\varphi_M)}{\rightleftarrows}} M$ in $\mathfrak{M}$. Then $b(e(0_M)) = (1_\Lambda \otimes 1_M)\,{}_M\varphi = {}_M\varphi$ and

$0 \to M \xrightarrow{{}_M\varphi} \Lambda \otimes M$ is precisely $0 \to M \xrightarrow{e'(0_M)} Re'(0_M)$.

Now, in the diagram,

$$\begin{tikzcd}
0 \arrow[r, no head] & M \arrow[r, shift left, "{}_M\varphi"] & \Lambda \otimes M \arrow[l, shift left, "t^{-1}"] \arrow[r, shift left, "\delta^0 = \Delta \otimes 1 - 1 \otimes {}_M\varphi"] \arrow[dr, "1 - {}_M\varphi t^{-1}"'] & Q \otimes M \arrow[l, shift left, "t^0"] \\
 & & & Q \otimes M
\end{tikzcd}$$

the sequence $M \xrightarrow{{}_M\varphi} M \xrightarrow{1 - {}_M\varphi t^{-1}} Q \otimes M$ is exact because

$(1 \otimes 1) - {}_M\varphi t^{-1})_M\varphi = {}_M\varphi - {}_M\varphi = 0$ and if $x \in \ker (1 - {}_M\varphi t^{-1})$, then

$x = {}_M\varphi t^{-1}(x)$. Moreover, $1 \otimes 1 - {}_M\varphi t^{-1}$ is an epic because given any

$q \otimes m \in Q \otimes M$; $q \otimes m + 0 \in \Lambda \otimes M$ and $(1 \otimes 1 - {}_M\varphi t^{-1})(q \otimes m) = q \otimes m$

because $q \in Q = \ker \epsilon$. Hence $1 \otimes 1 - {}_M\varphi t^{-1}$ is a cokernel of ${}_M\varphi$, $e({}_M\varphi) =$

$= 1 \otimes 1 - {}_M\varphi t^{-1}$ and $e'({}_M\varphi) = b(1 \otimes 1 - {}_M\varphi t^{-1}) = (1 \otimes 1 \otimes 1 - 1 \otimes {}_M\varphi t^{-1})\circ$

$\circ(\Delta \otimes 1) = \Delta \otimes 1 - (1 \otimes {}_M\varphi)(1 \otimes \epsilon \otimes 1)(\Delta \otimes 1) = \Delta \otimes 1 - 1 \otimes {}_M\varphi = \delta^0$.

So the only thing left to verify at this step is that the

$\operatorname{im} (1 \otimes 1 - {}_M\varphi t^{-1}) \subset Q \otimes M$. But this is true because

$(\epsilon \otimes 1)(1 \otimes 1 - {}_M\varphi t^{-1}) = 0$ and $0 \to Q \otimes M \xrightarrow{i \otimes 1} M \xrightarrow{\epsilon \otimes 1} R \otimes M$

is exact.

In the third step we consider the diagram:

$$\begin{tikzcd}
\Lambda \otimes M \arrow[r, shift left, "\delta^0"] & \Lambda \otimes Q \otimes M \arrow[l, shift left, dashed, "t^0"] \arrow[r, shift left, dashed, "\delta^1 = \Delta\otimes1\otimes1 - 1\otimes\Delta\otimes1 + 1\otimes1\otimes{}_M\varphi"] \arrow[dr, "1 - \delta^0 t^0"'] & \Lambda \otimes Q \otimes Q \otimes M \arrow[l, shift left, dashed, "t^1"] \\
 & & Q \otimes Q \otimes M
\end{tikzcd}$$

First it must be verified that the $\operatorname{im} (1 - \delta^0 t^0) \subset Q \otimes Q \otimes M$. But the sequence

$$0 \to Q \otimes Q \otimes M \xrightarrow{i \otimes 1 \otimes 1} \Lambda \otimes Q \otimes M \xrightarrow{\epsilon \otimes 1 \otimes 1} R \otimes Q \otimes M \to 0$$

is exact and $(\epsilon \otimes 1_Q \otimes 1_M)(1 - \delta^0 t^0) =$

$= (\epsilon \otimes 1_Q \otimes 1_M)(1_\Lambda \otimes 1_Q \otimes 1_M) - (\epsilon \otimes 1 \otimes 1)[\Delta \otimes 1_M - 1_\Lambda \otimes {}_M\varphi]t^0$

$= \epsilon \otimes 1 \otimes 1 - (\epsilon \otimes 1 \otimes 1)(\Delta \otimes 1)t^0 + (\epsilon \otimes 1 \otimes 1)(1 \otimes {}_M\varphi)t^0 = 0$

because $(\epsilon \otimes 1)\Delta = 1_\Lambda$ and, for any $\lambda \otimes q \otimes m \in \Lambda \otimes Q \otimes M$,

$(\epsilon \otimes 1 \otimes 1)(1 \otimes {}_M\varphi)(t^0)(\lambda \otimes q \otimes m) = (\epsilon \otimes 1 \otimes 1)(\epsilon(\lambda)q \otimes {}_M\varphi(m))$

$= \epsilon(\lambda)\epsilon(q) \otimes {}_M\varphi(m) = 0.$

Secondly, $\Lambda \otimes M \underset{t^0}{\overset{\delta^0}{\rightleftarrows}} Q \otimes M \xrightarrow{1 - \delta^0 t^0} Q \otimes Q \otimes M$ is exact

because $(1 - \delta^0 t^0)\delta^0 = \delta^0(1 - t^0\delta^0) = \delta^0(1 - (1 - {}_M\varphi t^{-1})) = \delta^0\,{}_M\varphi t^{-1} = 0$ and for any $x \in \ker(1 - \delta^0 t^0)$, $x = \delta^0(t^0(x))$. Moreover, $1 - \delta^0 t^0$ is an epimorphism, hence $1 - \delta^0 t^0$ is a cokernel of δ^0. Therefore, $\epsilon(\delta^0) =$

$= 1 - \delta^0 t^0$ and $\epsilon'(\delta^0) = b(1 - \delta^0 t^0) = (1 \otimes (1 - \delta^0 t^0))(\Delta \otimes 1 \otimes 1)$

$= (1_\Lambda \otimes 1_\Lambda \otimes 1_Q \otimes 1_M - 1 \otimes \delta^0 t^0)(\Delta \otimes 1 \otimes 1)$

$= \Delta \otimes 1 \otimes 1 - (1 \otimes \Delta \otimes 1)t^0(\Delta \otimes 1 \otimes 1) + (1 \otimes 1 \otimes {}_M\varphi)t^0(\Delta \otimes 1 \otimes 1)$

$= \Delta \otimes 1 \otimes 1 - 1 \otimes \Delta \otimes 1 + 1 \otimes 1 \otimes {}_M\varphi$ since $t^0(\Delta \otimes 1 \otimes 1) = 1$.

Similarly, one can verify that $\delta^n = e'(\delta^{n-1})$ and the theorem is proved.

It should be noted that in a similar manner one can verify that the bar construction of MacLane, [15-306ff.], is the canonical resolution of $\mathcal{E}_0$, based on the kernel functor as a resolvent for the projective class $\mathcal{E}_0$, of all split exact sequences in $\mathfrak{M}$ where $\mathcal{E}_0$ is a projective class in ${}_\Lambda\mathfrak{M}$, considering Λ as an augmented graded R-algebra.

Commutative Coalgebra

The following discussion yields a useful computational technique for working with comodules over a coalgebra Λ. An example of the technique will be given in this section. The technique will also be used extensively in Chapter IV.

Let R be a commutative ring with identity, $\mathfrak{M}$ the category of all graded R-modules where the morphisms are the R-homomorphisms of degree zero. For each object X in $\mathfrak{M}$ define a covariant functor $T_X : \mathfrak{M} \to \mathfrak{M}$ by

i) for each object Y in $\mathfrak{M}$, $T_X(Y) = Y \otimes X$;

ii) for each morphism $f : Y \to Y'$, $T_X(f) = f \otimes 1_X$.

Similarly, define a covariant functor $S_X : \mathfrak{M} \to \mathfrak{M}$ by $S_X(Y) = X \otimes Y$. All tensor products are over R.

<u>Definition 3.2</u>, [16-215]: For X, Y in $\mathfrak{M}$ the morphism $\tau : X \otimes Y \to Y \otimes X$ defined by

$$\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x \text{ where } x \in X_{|x|} \text{ and } y \in Y_{|y|}$$

is called the <u>twisting morphism on $X \otimes Y$</u>. (Note that τ is an R-homomorphism of degree zero.)

<u>Proposition 3.4</u>: For each object X in $\mathfrak{M}$ there exists a natural equivalence $\tau_X : T_X \to S_X$ $(\sigma_X : S_X \to T_X)$.

<u>Proof</u>: Let X be any object of $\mathfrak{M}$. For each Y let $\tau_X(Y)$ be the twisting morphism on $Y \otimes X$ ($\sigma_X(Y)$ the twisting morphism on $X \otimes Y$). For any $f : Y \to Y'$ in $\mathfrak{M}$ the diagram

$$\begin{tikzcd}
Y \otimes X \arrow[rr, "f \otimes 1"] \arrow[d, shift right, "\tau_X(Y)"'] & & Y' \otimes X \arrow[d, shift left, "\tau_X(Y')"] \\
X \otimes Y \arrow[rr, "1 \otimes f"'] \arrow[u, shift right, dashed, "\sigma_X(Y)"'] & & X \otimes Y' \arrow[u, shift left, dashed, "\sigma_X(Y')"]
\end{tikzcd}$$

is commutative because $|f| = 0$. Furthermore; $\sigma_X(Y)\tau_X(Y) = 1_{Y \otimes X}$ and $\tau_X(Y)\sigma_X(Y) = 1_{X \otimes Y}$.

Definition 3.3, [16-215]: The R-coalgebra $(\Lambda,\Delta,\epsilon)$ is said to be commutative if the diagram

$$\begin{tikzcd}
\Lambda\otimes\Lambda \arrow[r, "\Delta"] & \Lambda \arrow[dl, "\Delta"] \\
\Lambda\otimes\Lambda \arrow[u, "\tau"] &
\end{tikzcd}$$

is commutative where τ is the twisting morphism on $\Lambda\otimes\Lambda$.

Theorem 3.3, [11-355]: If $(\Lambda,\Delta,\epsilon)$ is a commutative R-coalgebra, then any left Λ-comodule can be considered as a right Λ-comodule.

Proof: Let $(M,{}_M\varphi)$ be any left Λ-comodule with multiplication ${}_M\varphi : M \to \Lambda\otimes M$. Define ${}_M\varphi' : M \to M\otimes\Lambda$ by the composition of the morphisms;
$M \xrightarrow{{}_M\varphi} \Lambda\otimes M \xrightarrow[\sigma_\Lambda(M)]{\tau_M(\Lambda)} M\otimes\Lambda$. Then ${}_M\varphi'$ is an R-homomorphism of degree zero. To complete the proof one needs to verify commutativity of the following diagrams:

$$\begin{tikzcd}
M \arrow[r, "{}_M\varphi'"] \arrow[d, "{}_M\varphi'"'] \arrow[dr, phantom, "(a)"] & M\otimes\Lambda \arrow[d, "1\otimes\Delta"] \\
M\otimes\Lambda \arrow[r, "{}_M\varphi'\otimes 1"'] & M\otimes\Lambda\otimes\Lambda
\end{tikzcd}
\qquad
\begin{tikzcd}
M\otimes R \arrow[r, equal] & R \arrow[dl, "{}_M\varphi'", "(b)"'] \\
M\otimes\Lambda \arrow[u, "1\otimes\epsilon"] &
\end{tikzcd}$$

(a) can be written as

$$\begin{tikzcd}
M \arrow[r, "{}_M\varphi"] \arrow[dd, "{}_M\varphi"'] \arrow[ddr, phantom, "(1)"] & \Lambda\otimes M \arrow[rr, "\tau_M(\Lambda)", "(5)"'] \arrow[dr, "\Delta\otimes 1"] \arrow[dd, "\Delta\otimes 1"'] & & M\otimes\Lambda \arrow[d, "1\otimes\Delta"] \\
& & \Lambda\otimes\Lambda\otimes M \arrow[r, "\tau_M(\Lambda\otimes\Lambda)", "(4)"'] \arrow[dl, "1"] & M\otimes\Lambda\otimes\Lambda \arrow[d, "\tau_\Lambda(M)\otimes 1"] \\
\Lambda\otimes M \arrow[r, "1\otimes{}_M\varphi"] \arrow[d, "\sigma_\Lambda(M)"'] \arrow[dr, phantom, "(2)"] & \Lambda\otimes\Lambda\otimes M \arrow[rr, "1\otimes\sigma_\Lambda(M)"] \arrow[d, "\sigma_\Lambda(\Lambda\otimes M)"] \arrow[drr, phantom, "(3)"] & & \Lambda\otimes M\otimes\Lambda \arrow[d, "\sigma_\Lambda(M\otimes\Lambda)"] \\
M\otimes\Lambda \arrow[r, "{}_M\varphi\otimes 1"'] & \Lambda\otimes M\otimes\Lambda \arrow[rr, "\sigma_\Lambda(M)\otimes 1"'] & & M\otimes\Lambda\otimes\Lambda
\end{tikzcd}$$

where (1) is commutative because $(M, {}_M\varphi)$ is a left Λ-comodule and (2), (3), and (5) are commutative by the proposition.

Let $\lambda \otimes \lambda' \otimes m \in \Lambda \otimes \Lambda \otimes M$. Then $(\tau_\Lambda(M) \otimes 1)(\tau_M(\Lambda \otimes \Lambda))(\lambda\otimes\lambda'\otimes m) =$ $= (\tau_\Lambda(M)\otimes 1)[(-1)^{|m||\lambda\otimes\lambda'|}(m\otimes\lambda\otimes\lambda')] = (-1)^{|m||\lambda|+|m||\lambda'|+|\lambda||m|}(\lambda\otimes m\otimes\lambda')$ $= (1 \otimes \sigma_\Lambda(M))(\lambda \otimes \lambda' \otimes m)$ and (4) is commutative. So, if $(\sigma_\Lambda(M \otimes \Lambda))(\tau_\Lambda(M) \otimes 1)(1 \otimes \Delta) = 1 \otimes \Delta$, the proof will be complete. Since Λ is commutative, one need only show $\sigma_\Lambda(M \otimes \Lambda)(\tau_\Lambda(M) \otimes 1) = 1 \otimes \tau$ where τ is the twisting morphism on $\Lambda \otimes \Lambda$.

Let $m \otimes \lambda \otimes \lambda' \in M \otimes \Lambda \otimes \Lambda$. Then $\sigma_\Lambda(M \otimes \Lambda)(\tau_\Lambda(M) \otimes 1)(m \otimes \lambda \otimes \lambda') =$ $= \sigma_\Lambda(M \otimes \Lambda)(-1)^{|m||\lambda|}(\lambda \otimes m \otimes \lambda') = (-1)^{|m||\lambda|+|\lambda||m\otimes\lambda'|}(m \otimes \lambda' \otimes \lambda)$ $= (-1)^{|\lambda||\lambda'|}(m \otimes \lambda' \otimes \lambda) = (1 \otimes \tau)(m \otimes \lambda \otimes \lambda')$.

Similarly one can show (b) is commutative by writing (b) as

$$\begin{tikzcd}
M \otimes R & R \otimes M \arrow[l, "\tau_M(R)"'] \arrow[r, equal] & M \arrow[dl, "{}_M\varphi"] \\
M \otimes \Lambda \arrow[u, "1\otimes\epsilon"] \arrow[ur, phantom, "(2)"] & \Lambda \otimes M \arrow[l, "\tau_M(\Lambda)"] \arrow[u, "\epsilon\otimes 1"'] \arrow[ur, phantom, "(1)"] &
\end{tikzcd}$$

where (1) is commutative because $(M, {}_M\varphi)$ is a left Λ-comodule and (2) is commutative by the proposition.

From the above theorem, if Λ is a commutative coalgebra, the cotensor product is a bifunctor on the category of all left Λ-comodules.

Definition of the Derived Functor Coext

R is a commutative ring with unity and $(\Lambda, \Delta, \epsilon)$ is a graded R-coalgebra. Let M be any left Λ-comodule. Recall that

$\mathrm{Hom}_\Lambda(M, M') \equiv \{ \mathrm{Hom}^d_\Lambda(M, M') \mid d \geq 0 \}$ for M' in ${}^\Lambda\mathfrak{M}$.

<u>Proposition 3.5</u>: $\mathrm{Hom}_\Lambda(M,M')$ is a graded R-module.

<u>Proof</u>: For each $d \geq 0$ it will be shown that $\mathrm{Hom}^d_\Lambda(M,M')$ is an R-module. For any $f \in \mathrm{Hom}^d_\Lambda(M,M')$ and any $r \in R$ let $(rf)(m) = rf(m)$. Then $|(rf)(m)| = |rf(m)| = |f| + |m|$ and $|rf| = d$. Moreover, $(1 \otimes rf)(\lambda \otimes m)$ $= \lambda \otimes (rf)(m) = \lambda \otimes rf(m) = r(\lambda \otimes f(m)) = r(1 \otimes f)(m)$.

<u>Proposition 3.6</u>: $\mathrm{Hom}_\Lambda(M,\_) : {}^\Lambda\mathfrak{M} \to \mathfrak{M}$ is a covariant additive functor for any M in ${}^\Lambda\mathfrak{M}$.

<u>Proof</u>: Given any M' in ${}^\Lambda\mathfrak{M}$ it has already been shown that $\mathrm{Hom}_\Lambda(M,M')$ is in $\mathfrak{M}$. Now let $f : M' \to M''$ be any morphism in ${}^\Lambda\mathfrak{M}$ and define $\mathrm{Hom}_\Lambda(M,f) \equiv f_* : \mathrm{Hom}_\Lambda(M,M') \to \mathrm{Hom}_\Lambda(M,M'')$ by $f_*(g) = fg$ for any $g \in \mathrm{Hom}_\Lambda(M,M')$. Then fg is an R-homomorphism of the same degree as g. Moreover, fg is a Λ-comodule homomorphism because the following diagram is commutative.

$$\begin{tikzcd}
M \arrow[r, "g"] \arrow[d, "M^\varphi"'] & M' \arrow[r, "f"] \arrow[d, "M'^\varphi"'] & M'' \arrow[d, "M''^\varphi"'] \\
\Lambda \otimes M \arrow[r, "1 \otimes g"'] & \Lambda \otimes M' \arrow[r, "1 \otimes f"'] & \Lambda \otimes M''
\end{tikzcd}$$

So f_* is an R-homomorphism of degree zero and $\mathrm{Hom}_\Lambda(M,\_)$ is a covariant functor.

Moreover $\mathrm{Hom}_\Lambda(M,\_)$ is an additive functor because given $f,g : M' \to M''$, $(f+g)_*(h) = (f+g)h = fh + gh = f_*(h) + g_*(h) =$ $= (f_* + g_*)(h)$ for any $h \in \mathrm{Hom}_\Lambda(M,M')$.

<u>Theorem 3.4</u>: If $E : 0 \to M^1 \underset{\alpha}{\overset{f}{\rightleftarrows}} M^2 \underset{\beta}{\overset{g}{\rightleftarrows}} M^3 \to 0$ is a sequence in $\mathcal{E}^0$, then for any M in ${}^\Lambda\mathfrak{M}$ the sequence

$$0 \to \mathrm{Hom}_\Lambda(M,M^1) \xrightarrow{f_*} \mathrm{Hom}_\Lambda(M,M^2) \xrightarrow{g_*} \mathrm{Hom}_\Lambda(M,M^3)$$

is exact in $\mathfrak{M}$.

<u>Proof</u>: Note that $|f| = |g| = 0$ and f, $1 \otimes f$ are injective set functions because E is split exact when considered in $\mathfrak{M}$.

For each $d \geq 0$ we need to show that

$$0 \to \mathrm{Hom}^d_\Lambda(M,M^1) \xrightarrow{(f_*)_d} \mathrm{Hom}^d_\Lambda(M,M^2) \xrightarrow{(g_*)_d} \mathrm{Hom}^d_\Lambda(M,M^3)$$

is exact; i.e., show $(f_*)_d$ is an injection and $\ker (g_*)_d = \mathrm{im}\ (f_*)_d$. Suppose $h,h':M \to M'$ such that $|h| = |h'| = d$ and $fh = fh'$. Then $h = h'$ because if $h \neq h'$, then there exists $n \geq 0$ and $x \in M_n$ such that $h_n(x) \neq h'_n(x)$. But $f_{n+d}(h_n(x)) = f_{n+d}(h'_n(x))$ and f_{n+d} is an injection. Contradiction, therefore $h = h'$ and $(f_*)_d$ is an injection.

We know $(g_*)_d(f_*)_d = 0$ hence need only show $\ker (g_*)_d \subset \mathrm{im}\ (f_*)_d$. Let $h:M \to M^2$ such that $|h| = d$ and $gh = 0$. Since $\ker (g)_k = \mathrm{im}\ (f)_k$, for all $k \geq 0$, one can define an R-homomorphism $k:M \to M^1$, of degree d, by setting $k(m) = m'$ where m' is the unique element of M^1 such that $f(m') = h(m)$. In order that k be a Λ-comodule homomorphism the following diagram must be commutative

$$\begin{tikzcd}
M \arrow[r, "{}_M\varphi"] \arrow[d, "k"'] & \Lambda \otimes M \arrow[d, "1 \otimes k"] \\
M^1 \arrow[r, "{}_{M^1}\varphi"] & \Lambda \otimes M^1
\end{tikzcd}$$

we know f and h are Λ-comodule homomorphisms, hence for any $m \in M$;
$[(1 \otimes f)\,{}_{M^1}\varphi k](m) = [{}_{M^2}\varphi f k](m) = [{}_{M^2}\varphi h](m) = [(1 \otimes h)_M\varphi](m)$
$= [(1 \otimes f)(1 \otimes k)_M\varphi](m)$. Since $(1 \otimes k)$ is injective, ${}_{M^1}\varphi k = (1 \otimes k)_M\varphi$

and k is a Λ-comodule homomorphism.

From the above propositions and theorem one sees that, for each M in ${}^{\Lambda}\mathfrak{M}$ where $(\Lambda,\Delta,\epsilon,\eta)$ is an augmented graded R-coalgebra, $\mathrm{Hom}_{\Lambda}(M\_)$ is a covariant, additive, $\mathcal{E}^0$-left exact functor from ${}^{\Lambda}\mathfrak{M}$ to $\mathfrak{M}$. Hence, from Chapter I, there exists a unique cohomology theory ${}_{M}H_{\mathcal{E}^0}$ relative to $\mathcal{E}^0$ over $\mathrm{Hom}_{\Lambda}(M,\_)$. We will call this derived functor $\mathrm{Coext}_{\Lambda,\mathcal{E}^0}(M,\_)$ and define $\mathrm{Coext}^n_{\Lambda,\mathcal{E}^0}(M,M')$ as ${}_{M}H^n_{\mathcal{E}^0}(M')$.

A natural question to ask is whether the conditions on E in Theorem 3.4 can be weakened and still have the desired result; i.e., if $E:0 \to M^1 \xrightarrow{f} M^2 \xrightarrow{g} M^3 \to 0$ is in $\mathcal{E}^1$ then is $0 \to \mathrm{Hom}_{\Lambda}(M,M^1) \xrightarrow{f_*} \mathrm{Hom}_{\Lambda}(M,M^2) \xrightarrow{g_*} \mathrm{Hom}_{\Lambda}(M,M^3)$ exact? The answer is no as the following example shows.

Example 3.3: Consider $R = Z$; $\Lambda = (Z, Z_2, 0, 0, \dots)$ and the Z-coalgebra $(\Lambda,\Delta,\epsilon)$ where Δ and ϵ are defined as in the example on p. 74. Let $M^1 = (2Z, Z_2, 0, 0, \dots)$ and $\varphi:M' \to \Lambda \otimes M^1$ be defined by;

$$\varphi_0:2Z \to Z \otimes 2Z = (\Lambda \otimes M^1)_0$$
$$\varphi_0(2) = 1 \otimes 2;$$
$$\varphi_1:Z_2 \to Z \otimes Z_2 + Z_2 \otimes 2Z = (\Lambda \otimes M^1)_1$$
$$\varphi_1(\overline{1}) = 1 \otimes \overline{1}$$

and $\varphi_k = 0$ for $k > 0$. Then the following diagrams are commutative

$$\begin{tikzcd} M^1 \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & \Lambda \otimes M^1 \arrow[d, "\Delta \otimes 1"] \\ \Lambda \otimes M^1 \arrow[r, "1 \otimes \varphi"] & \Lambda \otimes \Lambda \otimes M^1 \end{tikzcd} \qquad \begin{tikzcd} Z \otimes M^1 \arrow[r, equal] & M^1 \arrow[dl, "\varphi"] \\ \Lambda \otimes M^1 \arrow[u, "\epsilon \otimes 1"] & \end{tikzcd}$$

and (M^1,φ) is a left Λ-comodule because:

i) <u>0th degree</u>

$$(\Delta \otimes 1)_0\varphi_0(2) = (\Delta \otimes 1)_0(1 \otimes 2) = \Delta_0(1) \otimes 2 = 1 \otimes 1 \otimes 2,$$

$$(1 \otimes \varphi)_0\varphi_0(2) = (1 \otimes \varphi_0)(1 \otimes 2) = 1 \otimes \varphi_0(2) = 1 \otimes 1 \otimes 2,$$

$$(\epsilon \otimes 1)_0\varphi_0(2) = (\epsilon \otimes 1)_0(1 \otimes 2) = 1 \otimes 2;$$

ii) <u>1st degree</u>

$$(\Delta \otimes 1)_1\varphi_1(\bar{1}) = (\Delta \otimes 1)_1(1 \otimes \bar{1}) = \Delta_0(1) \otimes \bar{1} = 1 \otimes 1 \otimes \bar{1},$$

$$(1 \otimes \varphi)_1\varphi_1(\bar{1}) = (1 \otimes \varphi)_1(1 \otimes \bar{1}) = 1 \otimes \varphi_1(\bar{1}) = 1 \otimes 1 \otimes \bar{1},$$

$$(\epsilon \otimes 1)_1\varphi_1(\bar{1}) = (\epsilon \otimes 1)_1(1 \otimes \bar{1}) = 1 \otimes \bar{1}.$$

Let $M^2 = \Lambda$ and $\hat{\Delta}: M^2 \to \Lambda \otimes M^2$ be defined by

$$\hat{\Delta}_0 = \Delta_0 : Z \to Z \otimes Z$$

$$\hat{\Delta}_1 : Z_2 \to Z \otimes Z_2 + Z_2 \otimes Z \text{ where } \hat{\Delta}_1(\bar{1}) = 1 \otimes \bar{1}.$$

Then $(\Lambda, \hat{\Delta})$ is a left Λ-comodule because:

i) <u>0th degree</u>

$$(\Delta \otimes 1)_0\hat{\Delta}_0(1) = (\Delta \otimes 1)_0(1 \otimes 1) = \Delta_0(1) \otimes 1 = 1 \otimes 1 \otimes 1,$$

$$(1 \otimes \hat{\Delta}_0)\hat{\Delta}_0(1) = (1 \otimes \hat{\Delta})_0(1 \otimes 1) = 1 \otimes \hat{\Delta}_0(1) = 1 \otimes 1 \otimes 1,$$

$$(\epsilon \otimes 1)_0\hat{\Delta}_0(1) = (\epsilon \otimes 1)_0(1 \otimes 1) = 1 \otimes 1;$$

ii) <u>1st degree</u>

$$(\Delta \otimes 1)_1\hat{\Delta}_1(\bar{1}) = (\Delta \otimes 1)_1(1 \otimes \bar{1}) = \Delta_0(1) \otimes \bar{1} = 1 \otimes 1 \otimes \bar{1},$$

$$(1 \otimes \hat{\Delta})_1\hat{\Delta}_1(\bar{1}) = (1 \otimes \hat{\Delta})_1(1 \otimes \bar{1}) = 1 \otimes \hat{\Delta}_1(\bar{1}) = 1 \otimes 1 \otimes \bar{1},$$

$$(\epsilon \otimes 1)_1\hat{\Delta}_1(\bar{1}) = (\epsilon \otimes 1)_1(1 \otimes \bar{1}) = \epsilon_0(1) \otimes \bar{1} = 1 \otimes \bar{1}.$$

Define M^3 by; $M^3 = (Z_2, 0, 0, \ldots)$ and $\alpha: M^3 \to \Lambda \otimes M^3$ where $\alpha_0(\bar{1}) = 1 \otimes \bar{1}$ and $\alpha_k = 0$ for $k > 0$. Then the following diagrams are commutative and (M^3, α) is a left Λ-comodule;

$$\begin{tikzcd} M^3 \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \Lambda \otimes M^3 \arrow[d, "\Delta \otimes 1"] \\ \Lambda \otimes M^3 \arrow[r, "1 \otimes \alpha"'] & \Lambda \otimes \Lambda \otimes M^3 \end{tikzcd} \qquad \begin{tikzcd} Z \otimes M^3 \arrow[r, equal] & M^3 \arrow[dl, "\alpha"] \\ \Lambda \otimes M^3 \arrow[u, "\epsilon \otimes 1"] & \end{tikzcd}$$

because

$(\Delta \otimes 1)_0\alpha_0(\bar{1}) = (\Delta \otimes 1)_0(1 \otimes \bar{1}) = 1 \otimes 1 \otimes \bar{1}$ and $(1 \otimes \alpha)_0\alpha_0(\bar{1}) =$

$= (1 \otimes \alpha)_0(1 \otimes \bar{1}) = 1 \otimes \alpha_0(\bar{1}) = 1 \otimes 1 \otimes \bar{1}$. Similarly, $(\epsilon \otimes 1)\alpha = 1_{M^3}$.

Now consider the sequence

$$0 \to M^1 \xrightarrow{f} M^2 \xrightarrow{g} M^3 \to 0$$

where $f_0 = i{:}2Z \to Z$

$f_1 = 1_{Z_2}{:}Z_2 \to Z_2$

$g_0 = \pi{:}Z \to Z_2$

$g_1 = 0{:}Z_2 \to 0.$

Then we can show f,g are zero-degree Λ-comodule homomorphisms.

i)
$$\begin{tikzcd} M^1 \arrow[r, "\varphi"] \arrow[d, "f"'] & \Lambda \otimes M^1 \arrow[d, "1 \otimes f"] \\ \Lambda \arrow[r, "\hat{\Delta}"] & \Lambda \otimes \Lambda \end{tikzcd}$$

<u>0th degree</u>

$(1 \otimes f)_0\varphi_0(2) = (1 \otimes f)_0(1 \otimes 2) = 1 \otimes 2$

$\hat{\Delta}_0 f_0(2) = \hat{\Delta}_0(2) = \hat{\Delta}_0(1) + \hat{\Delta}_0(1)$

$= 1 \otimes 1 + 1 \otimes 1 = 1 \otimes 2.$

<u>1st degree</u>

$(1 \otimes f)_1\varphi_1(\bar{1}) = (1 \otimes f)_1(1 \otimes \bar{1}) = 1 \otimes \bar{1}$

$\hat{\Delta}_1 f_1(\bar{1}) = \hat{\Delta}_1(\bar{1}) = (1 \otimes \bar{1})$ and f is a Λ-comodule homomorphism.

ii)
$$\begin{tikzcd} \Lambda \arrow[r, "\Delta"] \arrow[d, "g"'] & \Lambda \otimes \Lambda \arrow[d, "1 \otimes g"] \\ M^3 \arrow[r, "\alpha"] & \Lambda \otimes M^3 \end{tikzcd}$$

<u>0th degree</u>

$(1 \otimes g)_0\hat{\Delta}_0(1) = (1 \otimes g)_0(1 \otimes 1)$

$= 1 \otimes \bar{1}$, $g_0(1) = \alpha_0(\bar{1}) = 1 \otimes \bar{1}$;

<u>1st degree</u>

$$(1 \otimes g)_1 \hat{\Delta}_1(\overline{1}) = (1 \otimes g)_1(1 \otimes \overline{1}) = 0$$

$\hat{\Delta}_1 g_1(\overline{1}) = \hat{\Delta}_1(0) = 0$ and g is a Λ-comodule homomorphism.

Now consider $M = \Lambda$, then (Λ, Δ) is a left Λ-comodule. Define $h{:}M \to M^2$ by: $h_0{:}Z \to Z$; $h_0(1) = 2$ and $h_1 = 1_{Z_2}{:}Z_2 \to Z_2$. Then $gh = 0$ and h is a Λ-comodule homomorphism of degree zero because

$$\begin{tikzcd} \Lambda \arrow[r, "\Delta"] \arrow[d, "h"'] & \Lambda \otimes \Lambda \arrow[d, "1 \otimes h"] \\ \Lambda \arrow[r, "\hat{\Delta}"] & \Lambda \otimes \Lambda \end{tikzcd}$$

<u>0th degree</u>

$(1 \otimes h)_0 \Delta_0(1) = (1 \otimes h_0)(1 \otimes 1)$

$= 1 \otimes 2$; $\hat{\Delta}_0 h_0(1) = \hat{\Delta}_0(2) = 1 \otimes 2$;

<u>1st degree</u>

$$(1 \otimes h)_1 \Delta_1(\overline{1}) = (1 \otimes h)_1(1 \otimes \overline{1} + \overline{1} \otimes 1) = 1 \otimes h_1(\overline{1}) + \overline{1} \otimes h_0(1)$$

$$= 1 \otimes \overline{1} + \overline{1} \otimes 2 = 1 \otimes \overline{1};\ \hat{\Delta}_1 h_1(\overline{1}) = \hat{\Delta}_1(\overline{1}) = 1 \otimes \overline{1}.$$

But there does not exist a Λ-comodule homomorphism $k{:}M \to M^1$ such that $fk = h$ because we know that $k{:}M \to M^1$ defined by;

$k_0{:}Z \to 2Z$; $k_0(1) = 2$

$k_1{:}Z_2 \to Z_2$; $k_1 = 1_{Z_2}$

is unique such that when considered as Z-homomorphisms, $fk = h$. But k is not a Λ-comodule homomorphism because in the 1st degree,

$(1 \otimes k)_1 \Delta_1(\overline{1}) = (1 \otimes k)_1(1 \otimes \overline{1} + \overline{1} \otimes 1) = 1 \otimes \overline{1} + \overline{1} \otimes 2$ and

$\varphi_1 k_1(\overline{1}) = \varphi_1(\overline{1}) = 1 \otimes \overline{1}$.

Some Relations Between Derived Functors

Let R be a commutative ring with identity and let $\mathfrak{M}$ be the category of graded R-modules. (If the ungraded case is to be specifically considered this will be noted in the particular theorems.) If M is a graded R-module define $M^* \equiv \{M_n^* = \mathrm{Hom}_R(M_n, R) \mid n \geq 0\}$ (for a discussion of dual module see [15-146-148]) where $\mathrm{Hom}_R(M_n, R)$ is all R-homomorphisms from the module M_n to R. Then M^* is a graded R-module and is in $\mathfrak{M}$. We will assume $(\Lambda, \mu, \eta, \epsilon)$ is an augmented graded R-algebra and Λ is projective of finite type; i.e., for each $n \geq 0$ Λ_n is a finitely generated projective R-module. Then one can verify that $(\Lambda^*, \mu^*, \epsilon^*, \eta^*)$ is an augmented R-coalgebra and $Q^* = \ker \eta^*$. It can also be shown, if $(M, {}_M\varphi)$ is a graded left (right) Λ-module, then $(M^*, {}_M\varphi^*)$ is a graded left (right) Λ^*-comodule.

<u>Lemma 3.1</u>: If $(M, {}_M\varphi)$ is a right Λ-module, projective of finite type, then $(M \otimes_\Lambda N)^* \cong M^* \underset{\Lambda^*}{\square} N^*$ for any left Λ-module N.

<u>Proof</u>: Consider the diagram:

$$\begin{array}{ccccc}
(M \otimes_\Lambda N)^* & \longrightarrow & (M \otimes N)^* & \xrightarrow{(\varphi_M \otimes 1 - 1 \otimes {}_N\varphi)^*} & (M \otimes \Lambda \otimes N)^* \\
 & & \Vert\wr & & \Vert\wr \\
M^* \underset{\Lambda^*}{\square} N^* & \longrightarrow & M^* \otimes N^* & \xrightarrow{\varphi_M^* \otimes 1 - 1 \otimes {}_N\varphi^*} & M^* \otimes \Lambda^* \otimes N^*
\end{array}$$

<u>Convention</u>: $\mathrm{Hom}_\Lambda(M,N) \equiv \{\mathrm{Hom}_\Lambda^p(M,N) \mid p \in Z\}$

$\mathrm{Hom}_R(M,N) \equiv \{\mathrm{Hom}_R^p(M,N) \mid p \in Z\}$.

<u>Remark 3.1</u>: $\mathrm{Hom}_\Lambda(\Lambda \otimes M, N) \cong \mathrm{Hom}_R(M,N)$ and $\mathrm{Hom}_{\Lambda^*}(N^*, \Lambda^* \otimes N^*) \cong \mathrm{Hom}_R(N^*, M^*)$ for left Λ-modules M, N.

<u>Proof</u>: To do this it needs to be shown that $\mathrm{Hom}_\Lambda^d(\Lambda \otimes M, N) \cong \mathrm{Hom}_R^d(M,N)$ for $d \in Z$. The technique is the same as that for $d = 0$ which was

proved on pp. 72 - 74. For $d \in Z$ define $b_d : \mathrm{Hom}^d_\Lambda(\Lambda \otimes M, N) \to \mathrm{Hom}^d_R(M,N)$ by the diagram

$$\begin{tikzcd} M \arrow[r, "b_d(f)"] \arrow[d] & N \\ R \otimes M \arrow[r, "\eta \otimes 1"] & \Lambda \otimes M \arrow[u, "f"'] \end{tikzcd}$$

for any $f \in \mathrm{Hom}^d_\Lambda(\Lambda \otimes M, N)$, and define $a_d : \mathrm{Hom}^d_R(M,N) \to \mathrm{Hom}^d_\Lambda(\Lambda \otimes M, N)$ by the diagram

$$\begin{tikzcd} \Lambda \otimes M \arrow[d, "M^\varphi"'] \arrow[dr, "a_d(g)"] & \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

for any $g \in \mathrm{Hom}^d_R(M,N)$. One can verify that b_d and a_d are R-homomorphisms and $a_d b_d = 1$, $b_d a_d = 1$.

Since the direct summand of a projective R-module of finite type is also projective of finite type, $Q = \ker \epsilon$ is a finitely generated projective R-module. Recall, from the Universal Coefficient Theorem, [15-77], if K is a chain complex of free abelian groups K_n and if C is any abelian group, then $H^n(K,C) \cong \mathrm{Hom}_Z(H_n(K),C) + \mathrm{Ext}^1_Z(H_{n-1}(K),C)$. Moreover, if K is a chain complex of vector spaces K_n over a field F and C is a vector space over F, then $H^n(K,C) \cong \mathrm{Hom}_F(H_n(K),C)$.

<u>Theorem 3.5</u>: If R is a field and M is a finite dimensional vector space over R, where (M,φ_M) is a right Λ-module, then for any left Λ-module $(N, {}_N\varphi)$

$$[\mathrm{Tor}_\Lambda(M,N)]^* \cong \mathrm{Cotor}_{\Lambda^*}(M^*,N^*).$$

<u>Proof</u>: Let B(N) denote the bar resolution for N where $B(N)_k = \Lambda \otimes Q^k \otimes N$ for $k \geq 0$. Then $\mathrm{Tor}^{n,p}_\Lambda(M,N) \equiv H_n([M \otimes_\Lambda B(N)]_p)$ where the

k^{th} term of the complex $[M \otimes_\Lambda B(N)]_p$ is $[M \otimes_\Lambda (\Lambda \otimes Q^k \otimes N)]_p$. Passing to the dual we have $[\mathrm{Tor}_\Lambda^{n,p}(M,N)]^* = \mathrm{Hom}_R(H_n([M \otimes_\Lambda B(N)]_p),R)$. But $B(N)^*$; where $B(N)_k^* = \Lambda^* \otimes Q^{*k} \otimes N^*$; is the cobar resolution for N^* and $\mathrm{Cotor}_{\Lambda^*}^{n,p}(M^*,N^*) \equiv H_n([M^* \underset{\Lambda^*}{\square} B(N)^*]_p)$ where the k^{th} term of the complex $[M^* \underset{\Lambda^*}{\square} B(N)^*]_p$ is $[M^* \underset{\Lambda^*}{\square} (\Lambda^* \otimes Q^{*k} \otimes N^*)]_p$. From Lemma 3.1,

$(M \otimes_\Lambda N)^* \cong M^* \underset{\Lambda^*}{\square} N^*$ and for $k \geq 0$; $(M \otimes_\Lambda (\Lambda \otimes Q^k \otimes N))^*$

$\cong M^* \underset{\Lambda^*}{\square} (\Lambda^* \otimes Q^{*k} \otimes N^*)$. Therefore, $\mathrm{Cotor}_{\Lambda^*}^{n,p}(M^*,N^*) \cong H^n([M \otimes_\Lambda B(N)]_p,R)$.

Hence, by the Universal coefficient theorem, for any $n \geq 0$ and for any $p \in Z$, $[\mathrm{Tor}_\Lambda^{n,p}(M,N)]^* \cong \mathrm{Cotor}_{\Lambda^*}^{n,p}(M^*,N^*)$ and $[\mathrm{Tor}_\Lambda(M,N)]^*$ $\cong \mathrm{Cotor}_{\Lambda^*}(M^*,N^*)$.

Lemma 3.2: $R \otimes_\Lambda (\Lambda \otimes Q^k \otimes N) \cong Q^k \otimes N$ for any left Λ-module N and $k \geq 0$.

Proof: Consider R as a right Λ-module with multiplication $\varphi_R : R \otimes \Lambda \to R$ defined by $\varphi_R(r \otimes \lambda) = r\epsilon(\lambda) = \epsilon(\lambda)r$ for any $r \in R$ and any $\lambda \in \Lambda$.

Then consider the following diagram, for any $k \geq 0$:

(Notation: $q^k = q_1 \otimes q_2 \otimes \ldots \otimes q_k$ where the q_i are arbitrary in Q.)

$$\begin{tikzcd}
R \otimes \Lambda \otimes \Lambda \otimes Q^k \otimes N \arrow[r, "\varphi_R \otimes 1 - 1 \otimes \mu \otimes 1"] \arrow[d, equal, "\wr"] & R \otimes \Lambda \otimes Q^k \otimes N \arrow[r, "\pi"] \arrow[d, equal, "\wr"] & R \otimes_\Lambda (\Lambda \otimes Q^k \otimes N) \\
\Lambda \otimes \Lambda \otimes Q^k \otimes N \arrow[r, "f \otimes 1"] & \Lambda \otimes Q^k \otimes N &
\end{tikzcd}$$

where $(f \otimes 1)(\lambda \otimes \lambda' \otimes q^k \otimes n) = \epsilon(\lambda)\lambda' \otimes q^k \otimes n - \lambda\lambda' \otimes q^k \otimes n$, which belongs to $Q \otimes Q^k \otimes N$ because ϵ is an algebra homomorphism and $\epsilon(\epsilon(\lambda)\lambda' - \lambda\lambda') = 0$. Now, $\Lambda \otimes Q^k \otimes N = R \otimes Q^k \otimes N + Q \otimes Q^k \otimes N$ hence if $Q \otimes Q^k \otimes N \subset \operatorname{im}(f \otimes 1)$, the proof is complete.

Consider $q \otimes q^k \otimes n \in Q \otimes Q^k \otimes N$. Then $x = -q \otimes \eta(1) \otimes q^k \otimes n$ is in $\Lambda \otimes \Lambda \otimes Q^k \otimes N$ and $(f \otimes 1)(x) = q \otimes q^k \otimes n$ because $q \in Q = \ker \epsilon$.

<u>Lemma 3.3</u>: If $(M, {}_M\varphi)$ is a left Λ-module and trivially graded, then $\operatorname{Hom}_R(R_\epsilon \otimes_\Lambda M, R) \cong \operatorname{Hom}_\Lambda(M, {}_\epsilon R)$.

<u>Proof</u>: ${}_\epsilon R$ is a left Λ-module with respect to ${}_R\varphi : \Lambda \otimes R \to R$ defined by ${}_R\varphi(\lambda \otimes r) = \epsilon(\lambda)r$. Let $f : M \to {}_\epsilon R$ be a left Λ-homomorphism. Then $f(\lambda m) = \epsilon(\lambda)f(m)$ for any $\lambda \in \Lambda$ and any $m \in M$. Now consider; where $\psi(r \otimes m) = rm$;

$$\begin{array}{ccccc} R_\epsilon \otimes \Lambda \otimes M & \xrightarrow{\varphi_R \otimes 1 - 1 \otimes {}_N\varphi} & R_\epsilon \otimes M & \xrightarrow{\pi} & R_\epsilon \otimes_\Lambda M \\ & & \big\downarrow \psi & & \big\downarrow g \\ & & M & \xrightarrow{f} & {}_\epsilon R \end{array}$$

If $f\psi(\varphi_R \otimes 1 - 1 \otimes {}_N\varphi) = 0$, then there exists a unique R-homomorphism $g : R_\epsilon \otimes_\Lambda N \to {}_\epsilon R$ such that $f\psi = g\pi$. $f\psi(\varphi_R \otimes 1 - 1 \otimes {}_N\varphi)(1 \otimes \lambda \otimes m) =$

$= f\psi(\epsilon(\lambda) \otimes m - 1 \otimes \lambda m) = f(\epsilon(\lambda)m - \lambda m) = \epsilon(\lambda)f(m) - \epsilon(\lambda)f(m) = 0.$

Now, define $\omega : \operatorname{Hom}_\Lambda(M, {}_\epsilon R) \to \operatorname{Hom}_R(R_\epsilon \otimes_\Lambda M, R)$ by $\omega(f) = g$. Then ω is an R-homomorphism and is injective because ψ is an epic.

Let $g : R_\epsilon \otimes_\Lambda M \to R$ and let $f = g\pi\psi^{-1}$. Then $\psi f = g\pi$ and f is an R-homomorphism. If f is a left Λ-homomorphism, then ω is an isomorphism; i.e., if $f_M\varphi = {}_R\varphi(1 \otimes f)$. Let $\lambda \otimes m \in \Lambda \otimes M$, then $f_M\varphi(\lambda \otimes m) =$

$= f(\lambda m) = g(1 \otimes_\Lambda \lambda m)$ and ${}_R\varphi(1 \otimes f)(\lambda \otimes m) = {}_R\varphi(\lambda \otimes f(m)) = \epsilon(\lambda)f(m)$

$= \epsilon(\lambda)g(1 \otimes_\Lambda m) = g(\epsilon(\lambda) \otimes_\Lambda m)$. Now, if $\epsilon(\lambda) \otimes m -$

$- 1 \otimes \lambda m \in \operatorname{im}(\varphi_R \otimes 1 - 1 \otimes {}_N\varphi)$ then the proof will be completed.

But $(\varphi_R \otimes 1 - 1 \otimes {}_N\varphi)(1 \otimes \lambda \otimes m) = \epsilon(\lambda) \otimes m - 1 \otimes \lambda m$.

For the following theorem we will use the subsequent notation because ${}_\epsilon R$ and R^{ϵ^*} are trivially graded.

${}_\epsilon R$ - denotes R as a left Λ-module and

R^{ϵ^*} - denotes R as a right Λ-comodule.

$\mathrm{Hom}^p_\Lambda(M, {}_\epsilon R) \equiv \{f: M_p \to {}_\epsilon R\}$ for any left Λ-module M.

$\mathrm{Hom}^p_R(M,R) \equiv \{f: M_p \to R\}$ for any R-module M.

<u>Theorem 3.6</u>: If $(N, {}_N\varphi)$ is a left Λ-module, then

$$\mathrm{Ext}_{\Lambda,\mathcal{E}_0}(N, {}_\epsilon R) \cong \mathrm{Cotor}_{\Lambda^*,\mathcal{E}^0}(R^\epsilon, N^*).$$

<u>Proof</u>: For any $n,p \geq 0$ (because of our convention above) it needs to be shown that $\mathrm{Ext}^{n,p}_{\Lambda,\mathcal{E}_0}(N, {}_\epsilon R) \cong \mathrm{Cotor}^{n,p}_{\Lambda^*,\mathcal{E}^0}(R^{\epsilon^*}, N^*)$. Let

$$B(N): \cdots \rightleftarrows B_n \underset{\sigma_n}{\overset{\delta_n}{\rightleftarrows}} B_{n-1} \rightleftarrows \cdots \rightleftarrows B_1 \underset{\sigma_1}{\overset{\delta_1}{\rightleftarrows}} B_0 \underset{\sigma_0}{\overset{{}_N\varphi}{\rightleftarrows}} N \to 0$$

be the bar resolution for N; $(B_k = \Lambda \otimes Q^k \otimes N)$; and consider

$\mathrm{Hom}_\Lambda(B(N), {}_\epsilon R): 0 \to \mathrm{Hom}_\Lambda(N, {}_\epsilon R) \to \mathrm{Hom}_\Lambda(B_0, {}_\epsilon R) \to \cdots \to \mathrm{Hom}_\Lambda(B_n, {}_\epsilon R) \to \cdots;$

see [Hml-185]; and

$\mathrm{Hom}^p_\Lambda(B(N), {}_\epsilon R): 0 \to \mathrm{Hom}^p_\Lambda(N, {}_\epsilon R) \to \cdots \to \mathrm{Hom}^p_\Lambda(B_n, {}_\epsilon R) \to \cdots.$

By definition, $\mathrm{Ext}^{n,p}_{\Lambda,\mathcal{E}_0}(N, {}_\epsilon R) \equiv H_n(\mathrm{Hom}^p_\Lambda(B(N), {}_\epsilon R))$.

Now, by considering Lemmas 3.1 and 3.2 and the remark of this section, keeping in mind the notation assumed just previous to this theorem, one obtains, for $k \geq 0$,

$$\mathrm{Hom}^p_\Lambda(B_k, {}_\epsilon R) = \mathrm{Hom}^p_\Lambda(\Lambda \otimes Q^k \otimes N, {}_\epsilon R) \cong \mathrm{Hom}^p_R(Q^k \otimes N, R)$$

$= \mathrm{Hom}_R((Q^k \otimes N)_p, R) = [(Q^k \otimes N)_p]^* = [(Q^k \otimes N)^*]_p \cong (R_\epsilon \otimes_\Lambda (\Lambda \otimes Q^k \otimes N))^*_p$

$\cong (R^{\epsilon^*} \underset{\Lambda^*}{\square} \Lambda^* \otimes Q^{*k} \otimes N^*)_p$ and $\mathrm{Hom}^p_\Lambda(N, {}_\epsilon R) \cong (R^{\epsilon^*} \underset{\Lambda^*}{\square} N^*)_p$ by Lemma 3.3.

Hence $\mathrm{Ext}^{n,p}_{\Lambda,\mathcal{E}_0}(N, {}_\epsilon R) \cong \mathrm{Cotor}^{n,p}_{\Lambda^*,\mathcal{E}^0}(R^{\epsilon^*}, N^*)$ because $\mathrm{Cotor}^{n,p}_{\Lambda^*,\mathcal{E}^0}(R^{\epsilon^*}, N^*)$

$\equiv H_n((R^{\epsilon^*} \underset{\Lambda^*}{\square} B(N)^*)_p)$.

<u>Theorem 3.7</u>: (ungraded case) Assume R is a field (Λ is a finite dimensional vector space over R). If $(M, {}_M\varphi)$ is a right Λ-module and M is a finite dimensional vector space over R, then $\mathrm{Ext}_\Lambda(R_\epsilon, M) \cong \mathrm{Cotor}_{\Lambda^*}(M^*, {}^{\epsilon^*}R)$.

<u>Proof</u>: Let

$$B(R): \cdots \rightleftarrows \Lambda \otimes Q^n \underset{\sigma_n}{\overset{\delta_n}{\rightleftarrows}} \cdots \rightleftarrows \Lambda \otimes Q \underset{\sigma_1}{\overset{\delta_1}{\rightleftarrows}} \Lambda \underset{\eta}{\overset{\epsilon}{\rightleftarrows}} R_\epsilon \to 0$$

be the bar resolution for R_ϵ and consider

$$\mathrm{Hom}_\Lambda(B(R), M): 0 \to \mathrm{Hom}_\Lambda(R_\epsilon, M) \to \mathrm{Hom}_\Lambda(\Lambda, M) \to \cdots.$$

By definition $\mathrm{Ext}^n_\Lambda(R_\epsilon, M) \equiv H_n(\mathrm{Hom}_\Lambda(B(R), M))$. Moreover,

$$B(R)^*: 0 \to {}^{\epsilon^*}R \rightleftarrows \Lambda^* \rightleftarrows \Lambda^* \otimes Q^* \rightleftarrows \cdots \rightleftarrows \Lambda^* \otimes Q^{*n} \rightleftarrows \cdots$$

is the cobar resolution for R^{ϵ^*} and $\mathrm{Cotor}^n_{\Lambda^*}(M^*, {}^{\epsilon^*}R) \equiv H_n(M^* \underset{\Lambda^*}{\square} B(R)^*)$.

For $k \geq 1$; $\mathrm{Hom}_\Lambda(\Lambda \otimes Q^k, M) \cong \mathrm{Hom}_R(Q^k, M) \cong \sum_s \mathrm{Hom}_R(Q^k, R)$ where $\dim M = s$.

Therefore $\mathrm{Hom}_\Lambda(\Lambda \otimes Q^k, M) \cong \sum_s Q^{*k}$. Now $M^* \underset{\Lambda^*}{\square} (\Lambda^* \otimes Q^{*k})$

$\cong (M \otimes_\Lambda (\Lambda \otimes Q^k))^* \cong \sum_s (R \otimes_\Lambda (\Lambda \otimes Q^k))^* \cong \sum_s Q^{*k}$.

To complete this proof it needs to be shown that

$\mathrm{Hom}_\Lambda(R_\epsilon, M) \cong M^* \underset{\Lambda^*}{\square} {}^{\epsilon^*}R$ and $\mathrm{Hom}_\Lambda(\Lambda, M) \cong (M^* \underset{\Lambda^*}{\square} \Lambda^*)$. But if $\mathrm{Hom}_\Lambda(\Lambda, M)$

$\cong (M^* \underset{\Lambda^*}{\square} \Lambda^*)$, then, by the Five Lemma, $\mathrm{Hom}_\Lambda(R_\epsilon, M) \cong M^* \underset{\Lambda^*}{\square} R^{\epsilon^*}$.

$\mathrm{Hom}_\Lambda(\Lambda,M) \cong \mathrm{Hom}_\Lambda(\Lambda \otimes R,M) \cong \mathrm{Hom}_R(R,M) \cong M$ and $(M^* \underset{\Lambda^*}{\square} \Lambda^*) \cong (M \otimes_\Lambda \Lambda)^*$

$\cong \sum_s (R \otimes_\Lambda \Lambda)^* \cong \sum_s (R \otimes_\Lambda (\Lambda \otimes R))^* \cong \sum_s R^* \cong M.$

In a similar manner by being careful with the grading one can verify this theorem for the graded case.

Recall the following remarks about ungraded R-modules.

Remark 3.2: If M is a free R-module then for any R-module N there exists a monomorphism $\psi:\mathrm{Hom}_R(N,M) \to \mathrm{Hom}(M^*,N^*)$ where, for any $f:N \to M$, $\psi(f)(\alpha) = \alpha f$ for $\alpha:M \to R$.

Remark 3.3: If M,N are finitely generated free R-modules then $\mathrm{Hom}_R(N,M) \cong \mathrm{Hom}_R(M^*,N^*)$.

Theorem 3.8: (ungraded case): If Λ is a free finitely generated R-module; M,N are left Λ-modules and are free finitely generated R-modules then $\mathrm{Ext}_{\Lambda,\mathcal{E}_0}(M,N) \cong \mathrm{Coext}_{\Lambda^*,\mathcal{E}^0}(N^*,M^*)$.

Proof: Let

$$B(M):\dots \rightleftarrows B_n \underset{\sigma_n}{\overset{\delta_n}{\rightleftarrows}} B_{n-1} \rightleftarrows \dots \rightleftarrows B_1 \underset{\sigma_1}{\overset{\delta_1}{\rightleftarrows}} B_0 \underset{\sigma_0}{\overset{M^\varphi}{\rightleftarrows}} M \to 0$$

be the bar resolution for M. Then $\mathrm{Ext}^n_{\Lambda,\mathcal{E}_0}(M,N) = H_n(\mathrm{Hom}_\Lambda(B(M),N))$, where $[\mathrm{Hom}_\Lambda(B(M),N)]_k = \mathrm{Hom}_\Lambda(\Lambda \otimes Q^k \otimes M,N)$. Then for any $n \geq 0$ $\mathrm{Hom}_\Lambda(\Lambda \otimes Q^k \otimes M,N) \cong \mathrm{Hom}_R(Q^k \otimes M,N)$ by adjoint properties and

$$C(M^*):0 \to M^* \underset{\sigma_0^*}{\overset{M^{\varphi *}}{\rightleftarrows}} B_0^* \rightleftarrows B_1^* \rightleftarrows \dots \rightleftarrows B_{n-1}^* \underset{\sigma_n^*}{\overset{\delta_n^*}{\rightleftarrows}} B_n^* \rightleftarrows \dots$$

is the cobar resolution for M^* where $B_k^* = \Lambda^* \otimes Q^{*k} \otimes M^* \cong (\Lambda \otimes Q^k \otimes M)^*$. By definition $\mathrm{Coext}^n_{\Lambda^*,\mathcal{E}_0}(N,M) = H_n(\mathrm{Hom}_{\Lambda^*}(N^*,C(M^*)))$ where

$[\mathrm{Hom}_{\Lambda^*}(N^*, C(M^*))]_k = \mathrm{Hom}_{\Lambda^*}(N^*, \Lambda^* \otimes Q^{*^k} \otimes M^*)$. Then for any $k \geq 0$, by properties of adjoint functors, $\mathrm{Hom}_{\Lambda^*}(N^*, \Lambda^* \otimes Q^{*^k} \otimes M^*)$ $\cong \mathrm{Hom}_R(N^*, Q^{*^k} \otimes M^*) \cong \mathrm{Hom}_R(N^*, (Q^k \otimes M)^*)$. By remarks 3.2 and 3.3 $\mathrm{Hom}_R(Q^k \otimes M, N) \cong \mathrm{Hom}_R(N^*, (Q^k \otimes N)^*)$. Therefore, for $k \geq 0$, $\mathrm{Hom}_\Lambda(\Lambda \otimes Q^k \otimes M, N) \cong \mathrm{Hom}_{\Lambda^*}(N^*, \Lambda^* \otimes Q^{*^k} \otimes M^*)$. By the Five Lemma, $\mathrm{Hom}_\Lambda(M,N) \cong \mathrm{Hom}_{\Lambda^*}(N^*, M^*)$. Therefore $\mathrm{Ext}^n_{\Lambda, \mathcal{E}_0}(M,N) \cong \mathrm{Coext}^n_{\Lambda^*, \mathcal{E}^0}(N^*, M^*)$.

The above theorem can also be proved for the graded case by an argument similar to that for the ungraded case.

CHAPTER IV

PRODUCTS FOR THE DERIVED FUNCTORS COTOR AND COEXT

The classical derived functors Tor and Ext each have, in addition to the axioms, a property called product. In this chapter it is shown that Cotor and Coext each have a product.

Properties of the Cotensor Product

Some of the properties presented here are stated by V. Gugenheim; [11]; or J. W. Milnor and J. C. Moore; [16]. They are included by the author for completeness.

Let $(\Lambda, \Delta, \in)$ and $(\Lambda', \Delta', \in')$ be graded coalgebras over a commutative ring with unity. Let $\bar{\in} = \in \otimes \in'$ and define $\bar{\Delta}$ by the diagram

$$\begin{tikzcd} \Lambda \otimes \Lambda' \arrow[r, "\Delta \otimes \Delta'"] \arrow[dr, "\bar{\Delta}"'] & \Lambda \otimes \Lambda \otimes \Lambda' \otimes \Lambda' \arrow[d, "1 \otimes \tau \otimes 1"] \\ & \Lambda \otimes \Lambda' \otimes \Lambda \otimes \Lambda'. \end{tikzcd}$$

<u>Proposition 4.1</u>: $(\Lambda \otimes \Lambda', \bar{\Delta}, \bar{\in})$ is an R-coalgebra, [16 - 218].

<u>Proof</u>: The diagram

$$\begin{tikzcd} \Lambda \otimes \Lambda' \arrow[r, "\bar{\Delta}"] \arrow[d, "\bar{\Delta}"'] & \Lambda \otimes \Lambda' \otimes \Lambda \otimes \Lambda' \arrow[d, "1 \otimes \bar{\Delta}"] \\ \Lambda \otimes \Lambda' \otimes \Lambda \otimes \Lambda' \arrow[r, "\bar{\Delta} \otimes 1"'] & \Lambda \otimes \Lambda' \otimes \Lambda \otimes \Lambda' \otimes \Lambda \otimes \Lambda' \end{tikzcd}$$

can be written as Diagram 4.1 where (1) is commutative because $(\Lambda,\Delta,\epsilon)$ and $(\Lambda',\Delta',\epsilon')$ are coalgebras, (2) and (3) are commutative by naturality and (4) is an identity. The commutativity of (5) can be computed directly by a set theoretic argument.

Similarly one can verify the commutativity of

$$\begin{tikzcd}
R \otimes (\Lambda \otimes \Lambda') \arrow[rr, equal] & & \Lambda \otimes \Lambda' \arrow[rr, equal] & & (\Lambda \otimes \Lambda') \otimes R \\
(\Lambda \otimes \Lambda') \otimes (\Lambda \otimes \Lambda') \arrow[u, "\bar{\epsilon} \otimes 1 \otimes 1"'] \arrow[urr, "\bar{\Delta}"'] & & & & (\Lambda \otimes \Lambda') \otimes (\Lambda \otimes \Lambda'). \arrow[u, "1 \otimes 1 \otimes \bar{\epsilon}"'] \arrow[ull, "\bar{\Delta}"]
\end{tikzcd}$$

Proposition 4.2: If (M,φ_M) is a right Λ-comodule and $(M',\varphi_{M'})$ is a right Λ'-comodule then $M \otimes M'$ is a right $\Lambda \otimes \Lambda'$-comodule. A similar theorem is true for left comodules, [11-355].

Proof: Define $\varphi : M \otimes M' \to M \otimes M' \otimes \Lambda \otimes \Lambda'$ by the composition

$$M \otimes M' \xrightarrow{\varphi_M \otimes \varphi_{M'}} M \otimes \Lambda \otimes M' \otimes \Lambda' \xrightarrow[1 \otimes \tau_{M'}(\Lambda \otimes 1)]{1 \otimes \sigma_\Lambda(M') \otimes 1} M \otimes M' \otimes \Lambda \otimes \Lambda'.$$

Consider Diagram 4.2 where (1) is commutative by the definition of comodules, (2) and (3) are commutative by naturality and the commutativity of (4) is readily verified by a set-theoretic computation. Similarly one can verify the commutativity of

$$\begin{tikzcd}
(M \otimes M') \otimes R \arrow[r, equal] & M \otimes M' \arrow[dl, "\varphi"] \\
(M \otimes M') \otimes \Lambda \otimes \Lambda' \arrow[u, "1 \otimes \bar{\epsilon}"'] &
\end{tikzcd}$$

Proposition 4.3: If (M,φ_M), $(M',\varphi_{M'})$ are right Λ-, Λ'-comodules, respectively, and $(N,{}_N\varphi)$, $(N',{}_{N'}\varphi)$ are left Λ-, Λ'-comodules, respectively, then there exists a unique R-homomorphism, of degree zero,

$$
\begin{tikzcd}
\Lambda\otimes\Lambda' \arrow[r, "\Delta\otimes\Delta'"] \arrow[dd, "\Delta\otimes\Delta'"'] & \Lambda\otimes\Lambda\otimes\Lambda'\otimes\Lambda' \arrow[rrr, "1\otimes\sigma_\Lambda(\Lambda')\otimes 1"] \arrow[dr, "1\otimes 1\otimes\Delta'\otimes 1"] \arrow[dd, "\Delta\otimes 1\otimes\Delta'\otimes 1"'] & & & \Lambda\otimes\Lambda'\otimes\Lambda\otimes\Lambda' \arrow[dl, "1\otimes\Delta'\otimes 1\otimes 1"] \\
& & \Lambda\otimes\Lambda\otimes\Lambda'\otimes\Lambda'\otimes\Lambda' \arrow[r, "1\otimes\sigma_\Lambda(\Lambda'\otimes\Lambda')\otimes 1"] \arrow[dl, "\Delta\otimes 1\otimes 1\otimes 1"] & \Lambda\otimes\Lambda'\otimes\Lambda'\otimes\Lambda\otimes\Lambda' \arrow[dr, "\Delta\otimes 1\otimes 1\otimes 1\otimes 1"] & \\
\Lambda\otimes\Lambda\otimes\Lambda'\otimes\Lambda' \arrow[r, "1\otimes\Delta\otimes 1\otimes\Delta'"] \arrow[d, "1\otimes\tau_{\Lambda'}(\Lambda)\otimes 1"'] & \Lambda\otimes\Lambda\otimes\Lambda\otimes\Lambda'\otimes\Lambda'\otimes\Lambda' \arrow[rrr, "1\otimes 1\otimes\sigma_\Lambda(\Lambda'\otimes\Lambda')\otimes 1"] \arrow[d, "1\otimes\tau_{\Lambda'}(\Lambda\otimes\Lambda)\otimes 1\otimes 1"] & & & \Lambda\otimes\Lambda\otimes\Lambda'\otimes\Lambda'\otimes\Lambda\otimes\Lambda' \arrow[d, "1\otimes\tau_{\Lambda'}(\Lambda)\otimes 1\otimes 1\otimes 1"'] \\
\Lambda\otimes\Lambda'\otimes\Lambda\otimes\Lambda' \arrow[r, "1\otimes 1\otimes\Delta\otimes\Delta'"] & \Lambda\otimes\Lambda'\otimes\Lambda\otimes\Lambda\otimes\Lambda'\otimes\Lambda' \arrow[rrr, "1\otimes 1\otimes 1\otimes\sigma_\Lambda(\Lambda')\otimes 1"] & & & \Lambda\otimes\Lambda'\otimes\Lambda\otimes\Lambda'\otimes\Lambda\otimes\Lambda'
\arrow[from=1-1, to=3-2, phantom, "(1)"]
\arrow[from=3-1, to=4-2, phantom, "(2)"]
\arrow[from=1-5, to=2-3, phantom, "(3)"]
\arrow[from=2-3, to=3-5, phantom, "(4)"]
\arrow[from=3-2, to=4-5, phantom, "(5)"]
\end{tikzcd}
$$

Diagram 4.1

$$
\begin{tikzcd}
M \otimes M' \arrow[r, "\varphi_M \otimes \varphi_{M'}"] \arrow[d, "\varphi_M \otimes \varphi_{M'}"] \arrow[dr, phantom, "(1)"] & M \otimes \Lambda \otimes M' \otimes \Lambda' \arrow[r, "1 \otimes \tau_{M'}(\Lambda) \otimes 1"] \arrow[d, "1 \otimes \Delta \otimes 1 \otimes \Delta'"] \arrow[dr, phantom, "(3)"] & M \otimes M' \otimes \Lambda \otimes \Lambda' \arrow[d, "1 \otimes 1 \otimes \Delta \otimes \Delta'"] \\
M \otimes \Lambda \otimes M' \otimes \Lambda' \arrow[r, "\varphi_M \otimes 1 \otimes \varphi_{M'} \otimes 1"] \arrow[d, "1 \otimes \nabla_\Lambda(M') \otimes 1"] \arrow[dr, phantom, "(2)"] & M \otimes \Lambda \otimes \Lambda \otimes M' \otimes \Lambda' \otimes \Lambda' \arrow[r, "1 \otimes \tau_{M'}(\Lambda \otimes \Lambda) \otimes 1"] \arrow[d, "1 \otimes 1 \otimes \nabla_\Lambda(M' \otimes \Lambda') \otimes 1"] \arrow[dr, phantom, "(4)"] & M \otimes M' \otimes \Lambda \otimes \Lambda \otimes \Lambda' \otimes \Lambda' \arrow[d, "1 \otimes 1 \otimes 1 \otimes \tau_{\Lambda'}(\Lambda) \otimes 1"] \\
M \otimes M' \otimes \Lambda \otimes \Lambda' \arrow[r, "\varphi_M \otimes \varphi_{M'} \otimes 1 \otimes 1"'] & M \otimes \Lambda \otimes M' \otimes \Lambda' \otimes \Lambda \otimes \Lambda' \arrow[r, "1 \otimes \tau_{M'}(\Lambda) \otimes 1 \otimes 1 \otimes 1"'] & M \otimes M' \otimes \Lambda \otimes \Lambda' \otimes \Lambda \otimes \Lambda'
\end{tikzcd}
$$

Diagram 4.2

$$\alpha:(M \mathop{\square}_{\Lambda} N) \otimes (M' \mathop{\square}_{\Lambda'} N') \to (M \otimes M') \mathop{\square}_{\Lambda\otimes\Lambda'} (N \otimes N').$$

(A theorem similar to this is stated by Gugenheim, [11-357], but with stronger conditions on the comodules.)

<u>Proof</u>: Consider the diagram:

$$\begin{tikzcd}
(M \mathop{\square}_{\Lambda} N) \otimes (M' \mathop{\square}_{\Lambda'} N') \arrow[rr, dashed] \arrow[d, "i"] & & (M \otimes M') \mathop{\square}_{\Lambda\otimes\Lambda'} (N \otimes N') \arrow[d, "i'"] \\
M\otimes N\otimes M'\otimes N' \arrow[rr, "1\otimes\tau_{M'}(N)\otimes 1"] \arrow[d, "\beta"] \arrow[drr, phantom, "(*)"] & & M\otimes M'\otimes N\otimes N' \arrow[d, "\varphi_{M\otimes M'}\otimes 1\otimes 1 - 1\otimes 1\otimes_{N\otimes N'}\varphi"] \\
M\otimes\Lambda\otimes N\otimes M'\otimes\Lambda'\otimes N' \arrow[rr, "\alpha"'] & & M\otimes M'\otimes\Lambda\otimes\Lambda'\otimes N\otimes N'
\end{tikzcd}$$

where i, i' are the injections, the right column is exact and

$$\beta = [\varphi_M \otimes 1 + 1 \otimes {}_N\varphi) \otimes \varphi_{M'} \otimes 1] - [1 \otimes {}_N\varphi \otimes (\varphi_{M'} \otimes 1 + 1 \otimes_{N'}\varphi)]$$

$$\alpha = (1 \otimes 1 \otimes 1 \otimes \tau_{\Lambda'}(N) \otimes 1)(1 \otimes \tau_{M'}(\Lambda \otimes N) \otimes 1 \otimes 1).$$

Since i' is the kernel of $\varphi_{M\otimes M'} \otimes 1 \otimes 1 - 1 \otimes 1 \otimes_{N\otimes N'}\varphi$, if $i\beta = 0$ and $(*)$ is commutative, there exists a unique R-homomorphism $\alpha:(M \mathop{\square}_{\Lambda} N) \otimes (M' \mathop{\square}_{\Lambda'} N') \to M \otimes M' \mathop{\square}_{\Lambda\otimes\Lambda'} N \otimes N'$ such that $i'\alpha = (1\otimes\tau_{M'}(N)\otimes 1)i$.

Notice that $(\varphi_M \otimes 1 + 1 \otimes_N\varphi) \otimes \varphi_{M'} \otimes 1 = \varphi_M \otimes 1 \otimes \varphi_{M'} \otimes 1 + 1 \otimes {}_N\varphi \otimes \varphi_{M'} \otimes 1$ and $1 \otimes {}_N\varphi \otimes (\varphi_{M'} \otimes 1 + 1 \otimes {}_{N'}\varphi) = 1 \otimes_N\varphi \otimes_{M'}\varphi \otimes 1 + 1 \otimes_N\varphi \otimes 1 \otimes_{N'}\varphi$ and $\beta = \varphi_M \otimes 1 \otimes \varphi_{M'} \otimes 1 - 1 \otimes {}_N\varphi \otimes 1 \otimes {}_{N'}\varphi$ or one can write

$$\beta = [\varphi_M \otimes 1 - 1 \otimes {}_N\varphi) \otimes {}_{M'}\varphi \otimes 1] + [1 \otimes \varphi_N \otimes (\varphi_{M'} \otimes 1 - 1 \otimes {}_{N'}\varphi)].$$

Therefore, $\beta i = 0$ because $M \mathop{\square}_{\Lambda} N = \ker\,(\varphi_M \otimes 1 - 1 \otimes {}_N\varphi)$ and $M' \mathop{\square}_{\Lambda'} N' = \ker\,(\varphi_{M'} \otimes 1 - 1 \otimes {}_{N'}\varphi)$.

The square $(*)$ can be written as Diagram 4.3. Because of

$$
\begin{tikzcd}
M\otimes N\otimes M'\otimes N' \arrow[rrr, "1\otimes\nabla_N(M')\otimes 1 = 1\otimes\tau_{M'}(N)\otimes 1"] \arrow[dr, "1\otimes{}_N\varphi\otimes 1\otimes{}_{N'}\varphi"] \arrow[dd, "\varphi_M\otimes 1\otimes\varphi_{M'}\otimes 1"'] & & & M\otimes M'\otimes N\otimes N' \arrow[dr, "1\otimes 1\otimes{}_N\varphi\otimes{}_{N'}\varphi"] \arrow[dd, "\varphi_M\otimes\varphi_{M'}\otimes 1\otimes 1"'] & \\
& M\otimes\Lambda\otimes N\otimes M'\otimes\Lambda'\otimes N' \arrow[rrr, dashed, "1\otimes\tau_{M'}(\Lambda\otimes N)\otimes 1\otimes 1"] \arrow[dl, equal] & & & M\otimes M'\otimes\Lambda\otimes N\otimes\Lambda'\otimes N' \arrow[dd, "1\otimes 1\otimes 1\otimes\tau_{\Lambda'}(N)\otimes 1"] \\
M\otimes\Lambda\otimes N\otimes M'\otimes\Lambda'\otimes N' \arrow[rrr, "1\otimes 1\otimes\nabla_N(M'\otimes\Lambda')\otimes 1"] \arrow[dr, "1\otimes\tau_{M'}(\Lambda\otimes N)\otimes 1\otimes 1"] & & (**) & M\otimes\Lambda\otimes M'\otimes\Lambda'\otimes N\otimes N' \arrow[dr, "1\otimes\tau_{M'}(\Lambda)\otimes 1\otimes 1\otimes 1"'] & \\
& M\otimes M'\otimes\Lambda\otimes N\otimes\Lambda'\otimes N' \arrow[rrr, "1\otimes 1\otimes 1\otimes\tau_{\Lambda'}(N)\otimes 1"'] & & & M\otimes M'\otimes\Lambda\otimes\Lambda'\otimes N\otimes N'
\end{tikzcd}
$$

Diagram 4.3

naturality conditions if one can show commutativity in (**), then (*) is commutative and the proof will be completed. (**) is commutative because $(\tau_{M'}(\Lambda) \otimes 1 \otimes 1)(1 \otimes \sigma_N(M' \otimes \Lambda'))(\lambda \otimes n \otimes m' \otimes \lambda') =$ $(-1)^{|n||m'|+|n||\lambda'|+|m'||\lambda|}(m' \otimes \lambda \otimes \lambda' \otimes n)$ and $(1 \otimes 1 \otimes \tau_{\Lambda'}(N)) \circ$

$$\circ (\tau_{M'}(\Lambda \otimes N) \otimes 1)(\lambda \otimes n \otimes m' \otimes \lambda') =$$

$$= (-1)^{|m'||\lambda \otimes n|}(1 \otimes 1 \otimes \tau_{\Lambda'}(N))(m' \otimes \lambda \otimes n \otimes \lambda') =$$

$$= (-1)^{|m'||\lambda|+|m'||n|+|n||\lambda'|}(m' \otimes \lambda \otimes \lambda' \otimes n).$$

An External Product on Cotor

In this section we will use A,B,C,D for designating R-coalgebras as well as Λ, Λ', where R is a commutative ring with unity.

Let $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ be R-coalgebras. An R homomorphism $\alpha: A \to B$ is called a <u>coalgebra homomorphism</u> if the diagrams

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[d, "\alpha"'] & A \otimes A \arrow[d, "\alpha \otimes \alpha"] \\ B \arrow[r, "\Delta_B"'] & B \otimes B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\epsilon_A"] \arrow[d, "\alpha"'] & R \\ B \arrow[ur, "\epsilon_B"'] & \end{tikzcd}$$

are commutative, Milnor and Moore [16]. Let (M, φ_M) be a right A-comodule; i.e., $A \in \mathfrak{M}^A$; and let $(M', \varphi_{M'})$ be a right B-comodule. We are always considering graded objects unless specifically stated otherwise.

<u>Definition 4.1</u> [11-353]. An R-homomorphism $f: M \to M'$ is called an <u>α-right comodule</u> homomorphism if and only if the diagram

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\varphi_M"'] & M' \arrow[d, "\varphi_{M'}"] \\ M \otimes A \arrow[r, "f \otimes \alpha"'] & M' \otimes B \end{tikzcd}$$

is commutative.

Similarly, define α-left comodule homomorphism.

Note that if $A = B$ then an A-comodule homomorphism is a 1_A-right comodule homomorphism.

Proposition 4.4: If $\alpha: A \to B$ is a coalgebra homomorphism, $f: M \to M'$ is an α-right comodule homomorphism, and $g: N \to N'$ is an α-left comodule homomorphism, then there exists a unique R-homomorphism $f \underset{\alpha}{\square} g: M \underset{\alpha}{\square} N \to M' \underset{B}{\square} N'$.

Proof: Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \underset{A}{\square} N \arrow[r, "i"] \arrow[d, dashed, "f \underset{\alpha}{\square} g"] & M \otimes N \arrow[r, "\varphi_M \otimes 1 - 1 \otimes {}_N\varphi"] \arrow[d, "f \otimes g"] \arrow[dr, phantom, "(1)"] & M \otimes A \otimes N \arrow[d, "f \otimes \alpha \otimes g"] \\ 0 \arrow[r] & M' \underset{B}{\square} N' \arrow[r, "i'"] & M' \otimes N' \arrow[r, "\varphi_{M'} \otimes 1 - 1 \otimes {}_{N'}\varphi"] & M' \otimes B \otimes N' \end{tikzcd}$$

where (1) is commutative because f is an α-right comodule homomorphism and because g is an α-left comodule homomorphism. Therefore, $(\varphi_{M'} \otimes 1 - 1 \otimes {}_{N'}\varphi)(f \otimes g)i = 0$ and there exists a unique R-homomorphism $f \underset{\alpha}{\square} g: M \underset{A}{\square} N \to M' \underset{B}{\square} N'$ such that $(f \otimes g)i = i'(f \underset{\alpha}{\square} g)$ (i' is the kernel morphism of $\varphi_{M'} \otimes 1 - 1 \otimes {}_{N'}\varphi$).

Proposition 4.5: Assume A, B, C are R-coalgebras, $\alpha: A \to B$ and $\beta: B \to C$ are coalgebra homomorphisms. If

$f: M \to N$ is an α-right comodule homomorphism,

$g: N \to L$ is a β-right comodule homomorphism,

$f':M' \to N'$ is an α-left comodule homomorphism,

$g':N' \to L'$ is a β-left comodule homomorphism,

then i) $gf:M \to L$ is a $\beta\alpha$-right comodule homomorphism,

ii) $g'f':M' \to L'$ is a $\beta\alpha$-left comodule homomorphism,

and iii) $gf \underset{\beta\alpha}{\square} g'f' = (g \underset{\beta}{\square} g')(f \underset{\alpha}{\square} f')$.

<u>Proof</u>: Consider the diagrams

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\varphi_M"'] & N \arrow[r, "g"] \arrow[d, "\varphi_N"] & L \arrow[d, "\varphi_L"] \\
M \otimes A \arrow[r, "f \otimes \alpha"] & N \otimes B \arrow[r, "g \otimes \beta"] & L \otimes C
\end{tikzcd}$$

$$\begin{tikzcd}
M' \arrow[r, "f'"] \arrow[d, "{}_{M'}\varphi"'] & N' \arrow[r, "g'"] \arrow[d, "{}_{N'}\varphi"'] & L' \arrow[d, "{}_{L'}\varphi"'] \\
A \otimes M' \arrow[r, "\alpha \otimes f'"] & B \otimes N' \arrow[r, "\beta \otimes g'"] & C \otimes L'
\end{tikzcd}$$

Since each subdiagram commutes, gf is a $\beta\alpha$-right comodule homomorphism and $g'f'$ is a $\beta\alpha$-left comodule homomorphism.

From the commutativity of the diagram

$$\begin{tikzcd}
M \underset{A}{\square} M' \arrow[r] \arrow[d, "f \underset{\alpha}{\square} f'"] & M \otimes M' \arrow[r, "\varphi_M \otimes 1 - 1 \otimes {}_{M'}\varphi"] \arrow[d, "f \otimes f'"] & M \otimes A \otimes M' \arrow[d, "f \otimes \alpha \otimes f'"] \\
N \underset{B}{\square} N' \arrow[r] \arrow[d, "g \underset{\beta}{\square} g'"] & N \otimes N' \arrow[r, "\varphi_N \otimes 1 - 1 \otimes {}_{N'}\varphi"] \arrow[d, "g \otimes g'"] & N \otimes B \otimes N' \arrow[d, "g \otimes \beta \otimes g'"] \\
L \underset{C}{\square} L' \arrow[r, "i"] & L \otimes L' \arrow[r, "\varphi_L \otimes 1 - 1 \otimes {}_{L'}\varphi"] & L \otimes C \otimes L'
\end{tikzcd}$$

and because of the uniqueness guaranteed by the kernel morphism i,

$(g \underset{\beta}{\square} g')(f \underset{\alpha}{\square} f') = gf \underset{\beta\alpha}{\square} g'f'$.

Now, consider the following situation. If A,B,C,D are R-coalgebras; $\alpha:A \to B$, $\beta:C \to D$, $\delta:B \to D$, and $\gamma:A \to C$ are coalgebra homomorphisms; f is an α-right, f' is an α-left, h is a γ-right, h' is a γ-left, k is a δ-right, k' is a δ-left, g is a β-right and g' is a β-left comodule homomorphism, then, if $\beta\gamma = \delta\alpha$, $kf = gh$ and $k'f' = g'h'$, the diagram

$$\begin{tikzcd} M \underset{A}{\square} M' \arrow[r, "f \underset{\alpha}{\square} f'"] \arrow[d, "h \underset{\gamma}{\square} h'"'] & N \underset{B}{\square} N' \arrow[d, "k \underset{\delta}{\square} k'"] \\ K \underset{C}{\square} K' \arrow[r, "g \underset{\beta}{\square} g'"'] & L \underset{D}{\square} L' \end{tikzcd}$$

is commutative.

<u>Theorem 4.1</u>: If A, B are R-coalgebras, M,M' are right A-,B-comodules, respectively, and X, Y are cochain complexes of left A-,B-comodules, respectively, then there exists a cochain map

$$\alpha:(M \underset{A}{\square} X) \otimes (M' \underset{B}{\square} Y) \to (M \otimes M') \underset{A\otimes B}{\square} (X \otimes Y).$$

<u>Proof</u>: Let $X: 0 \to X^0 \xrightarrow{\delta_X^0} X^1 \to \cdots \to X^n \xrightarrow{\delta_X^n} X^{n+1} \to \cdots$ and

$Y: 0 \to Y^0 \xrightarrow{\delta_Y^0} Y^1 \to \cdots \to Y^n \xrightarrow{\delta_Y^n} Y^{n+1} \to \cdots$, then

$$M \underset{A}{\square} X: 0 \to M \underset{A}{\square} X^0 \xrightarrow{1 \underset{A}{\square} \delta_X^0} M \underset{A}{\square} X^1 \to \cdots \to M \underset{A}{\square} X^n \xrightarrow{1 \underset{A}{\square} \delta_X^n} M \underset{A}{\square} X^{n+1} \to \cdots$$

and

$$M' \underset{B}{\square} Y: 0 \to M' \underset{B}{\square} Y^0 \xrightarrow{1 \underset{B}{\square} \delta_Y^0} M' \underset{B}{\square} Y^1 \to \cdots \to M' \underset{B}{\square} Y^n \xrightarrow{1 \underset{B}{\square} \delta_Y^n} M' \underset{B}{\square} Y^{n+1} \to \cdots$$

are complexes. The complex $((M \underset{A}{\square} X) \otimes (M' \underset{B}{\square} Y), \delta)$ is given by

$$[(M \underset{A}{\square} X) \otimes (M' \underset{B}{\square} Y)]^n = \sum_{p+q=n} (M \underset{A}{\square} X^p) \otimes (M' \underset{B}{\square} Y^q) \text{ and}$$

$\delta^n(m \otimes x \otimes m' \otimes y) = [(1 \otimes \delta_X)(m \otimes x)] \otimes m' \otimes y +$

$+ (-1)^p m \otimes x \otimes [(1 \otimes \delta_Y)(m' \otimes y) =$

$= m \otimes \delta_X^p(x) \otimes m' \otimes y + (-1)^p m \otimes x \otimes m' \otimes \delta_Y^q(y)$ for any

$m \otimes x \otimes m' \otimes y \in [(M \otimes X) \otimes (M' \otimes Y)]^n$ and extend by linearity on $[(M \underset{A}{\square} X) \otimes (M' \underset{B}{\square} Y)]^n$, where $x \in X^p$ and $y \in Y^q$.

By Proposition 4.3, for each pair (p,q) such that $p + q = n$, there exists a unique R-homomorphism $\alpha(p,q)$ such that

$$\begin{tikzcd}
(M \underset{A}{\square} X^p) \otimes (M' \underset{B}{\square} Y^q) \arrow[r, "\alpha(p,q)"] \arrow[d] & (M \otimes M') \underset{A\otimes B}{\square} (X^p \otimes Y^q) \arrow[d] \\
(M \otimes X^p) \otimes (M' \otimes Y^q) \arrow[r, "1 \otimes \tau_{M'}(X^p) \otimes 1"] & (M \otimes M') \otimes (X^p \otimes Y^q)
\end{tikzcd}$$

is commutative. Define

$$\alpha_n : [(M \underset{A}{\square} X) \otimes (M' \underset{B}{\square} Y)]^n \to \sum_{p+q=n} [(M \otimes M') \underset{A\otimes B}{\square} (X^p \otimes Y^q)] =$$

$$= (M \otimes M') \underset{A\otimes B}{\square} (X \otimes Y)^n$$

as, $\alpha_n = \sum_{p+q=n} \alpha(p,q)$. The proof will be completed if $\alpha = (\alpha_n)_{n \geq 0}$ commutes with the coboundary where the coboundary $\overline{\delta} = (\overline{\delta}^n)$ for the complex $(M \otimes M') \underset{A\otimes B}{\square} (X \otimes Y)$ is given by; $\overline{\delta}^n = 1 \underset{A\otimes B}{\square} \delta^n_{X\otimes Y}$ and

$\delta^n_{X\otimes Y}(x \otimes y) = \delta_X^p(x) \otimes y + (-1)^p x \otimes \delta_Y^q(y)$ for any $p,q \geq 0$ such that $p + q = n$ and $x \in X^p$, $y \in Y^q$. In other words, the proof will be completed if $\overline{\delta}^n \alpha_n = \alpha_{n+1} \delta^n$.

Since any element $z \in (M \underset{A}{\square} X^p) \otimes (M' \underset{B}{\square} Y^q)$ is a finite linear combination of elements of the form $m \otimes x \otimes m' \otimes y$ it is sufficient to show, $\overline{\delta}^n \alpha_n(m \otimes x \otimes m' \otimes y) = \alpha_{n+1}\delta^n(m \otimes x \otimes m' \otimes y)$ for any $m \in M$,

$m' \in M'$, $x \in X^p$ and $y \in Y^q$. $\overline{\delta}^n\alpha_n(m \otimes x \otimes m' \otimes y) = \overline{\delta}^n\alpha(p,q)(m\otimes x\otimes m'\otimes y)$

$= \overline{\delta}^n[(-1)^{|x||m'|}m \otimes m' \otimes x \otimes y] = (-1)^{|x||m'|}(1_{A\otimes B} \,\square\, \delta^n_{X\otimes Y})(m \otimes m' \otimes x \otimes y)$

$= (-1)^{|x||m'|}[m \otimes m' \otimes \delta^n_{X\otimes Y}(x \otimes y)] =$

$= (-1)^{|x||m'|}[m \otimes m' \otimes \delta^p_X(x) \otimes y + (-1)^p m \otimes m' \otimes x \otimes \delta^q_Y(y)]$

and $\alpha_{n+1}\delta^n(m \otimes x \otimes m' \otimes y) =$

$= \alpha_{n+1}[m \otimes \delta^p_X(x) \otimes m' \otimes y + (-1)^p m \otimes x \otimes m' \otimes \delta^q_Y(y)]$

$= \alpha(p+1,q)[m \otimes \delta^p_X(x) \otimes m' \otimes y] + \alpha(p,q+1)[(-1)^p m \otimes x \otimes m' \otimes \delta^q_Y(y)]$

$= (-1)^{|m'||x|}m \otimes m' \otimes \delta^p_X(x) \otimes y + (-1)^{p+|m'||x|}m \otimes m' \otimes x \otimes \delta^q_Y(y)$,

because $|\delta^p_x(x)| = |x|$, and the proof is completed.

Now, consider the following where K,L are cochain complexes of graded R-modules:

$$K: 0 \to K^0 \xrightarrow{\delta^0_K} K^1 \to \cdots \to K^{n-1} \xrightarrow{\delta^{n-1}_K} K^n \to \cdots$$

$$K_\ell: 0 \to K^0_\ell \xrightarrow{(\delta^0_K)_\ell} K^1_\ell \to \cdots \to K^{n-1}_\ell \xrightarrow{(\delta^{n-1}_K)_\ell} K^n_\ell \to \cdots$$

$$H^{n,\ell}(K) \overset{\text{def.}}{=} H^n(K_\ell).$$

$$L: 0 \to L^0 \xrightarrow{\delta^0_L} L^1 \to L^2 \to \cdots \to L^{n-1} \to L^n \to \cdots$$

$$L_r: 0 \to L^0_r \xrightarrow{(\delta^0_L)_r} L^1_r \to L^2_r \to \cdots \to L^{n-1}_r \to L^n_r \to \cdots$$

$$H^{n,r}(L) \equiv H^n(L_r).$$

Moreover $(K \otimes L)^n_\ell = (\sum_{p+q=n} K^p \otimes L^q)_\ell = \sum_{p+q=n}\sum_{a+b=\ell} K^p_a \otimes L^q_b$.

Therefore, considering MacLane [15-163-166] and Theorem 4.1, there exists an R-homomorphism

$$(1)\ {}_{(n,p)}\emptyset_{(m,q)}\ H^{n,p}(M \underset{A}{\square} X) \otimes H^{m,q}(M' \underset{B}{\square} Y) \to H^{n+m,p+q}((M\otimes M') \underset{A\otimes B}{\square} (X \otimes Y)),$$

for all $n,p,m,q \geq 0$.

Proposition 4.6: If A,B are R-coalgebras and $\alpha: A \to B$ is a coalgebra homomorphism then there exists a functor $T_\alpha: \mathfrak{M}^A \to \mathfrak{M}^B$ where $\mathfrak{M}^A(\mathfrak{M}^B)$ is the category of all right A-comodules (right B-comodules). A similar proposition is true for left comodules.

Proof: Let (M,φ_M^A) be a right A-comodule. Define $T_\alpha(M,\varphi_M^A) = (M,\varphi_M^B)$ where $\varphi_M^B: M \to M \otimes B$ is defined as the composition $M \xrightarrow{\varphi_M^A} M \otimes A \xrightarrow{1 \otimes \alpha} M \otimes B$. Then the appropriate diagrams commute because α is a coalgebra homomorphism and (M,φ_M^B) is a right B-comodule.

Suppose $f: M \to M'$ is an A-comodule homomorphism, then $T(f) = f: M \to M'$ is a B-comodule homomorphism because the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi_M^A"] \arrow[d, "f"'] & M \otimes A \arrow[r, "1 \otimes \alpha"] \arrow[d, "f \otimes 1"] & M \otimes B \arrow[d, "f \otimes 1"] \\ M' \arrow[r, "\varphi_{M'}^A"] & M' \otimes A \arrow[r, "1 \otimes \alpha"] & M' \otimes B \end{tikzcd}$$

is commutative.

Proposition 4.7: If $\alpha: A \to B$ is a coalgebra homomorphism and $M \in \mathfrak{M}^A$, then there exists a canonical R-homomorphism from M to $T_\alpha(M)$ which is an α-right comodule homomorphism.

Proof: Define $\aleph_M: M \to T_\alpha(M)$ by $\aleph_M = 1_M$ when considered as an R-homomorphism. Then the diagram

$$\begin{tikzcd} M \arrow[r, "\aleph_M"] \arrow[d, "\varphi_M^A"'] & T_\alpha(M) = M \arrow[d, "\varphi_M^B"] \\ M \otimes A \arrow[r, "\aleph_M \otimes \alpha"'] & M \otimes B \end{tikzcd}$$

is commutative because, by definition, $\varphi_M^B = (1_M \otimes \alpha)\varphi_M^A = (\aleph_M \otimes \alpha)\varphi_M^A$.

Proposition 4.8: $\alpha: A \to B$ is a coalgebra homomorphism and A,B are augmented R-coalgebras. If $N \in {}^A\mathfrak{M}$, then there exists an α-left cochain map $\rho: \mathfrak{B}(A,N) \to \mathfrak{B}(B, {}_\alpha T(N))$ where $\mathfrak{B}(A,N)$ is the cobar resolution for N in the coalgebra A, similarly for $\mathfrak{B}(B, {}_\alpha T(N))$, paragraph 3 of Chapter III.

Proof: A sequence of R-homomorphisms, $\rho = (\rho_n)$, must be defined such that ρ is a chain map and each ρ_n is an α-left comodule homomorphism. Recall, where $Q = \ker \epsilon$ and $Q' = \ker \epsilon'$,

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r, shift left, "i"] & A \arrow[l, shift left, "p"] \arrow[r, shift left, "\epsilon"] \arrow[d, "\alpha"] & R \arrow[l, shift left, "\eta"] \arrow[r] & 0 \\ 0 \arrow[r] & Q' \arrow[r, shift left, "i'"] & B \arrow[l, shift left, "p'"] \arrow[r, shift left, "\epsilon'"] & R \arrow[l, shift left, "\eta'"] \arrow[r] & 0. \end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd} \mathfrak{B}(A,N): 0 \arrow[r] & N \arrow[r, "N\varphi^A"] \arrow[d, "\aleph_N"'] & A \otimes N \arrow[r, "\delta^0"] \arrow[d, "\rho_0 = \alpha \otimes 1"] & A \otimes Q \otimes N \arrow[r, "\delta^1"] \arrow[d, "\rho_1"] & A \otimes Q^2 \otimes N \arrow[r] \arrow[d, "\rho_2"] & \cdots \\ \mathfrak{B}(B, {}_\alpha T(N)): 0 \arrow[r] & N = {}_\alpha T(N) \arrow[r, "N\varphi^B"] & B \otimes N \arrow[r, "\hat{\delta}^0"] & B \otimes Q' \otimes N \arrow[r, "\hat{\delta}^1"] & B \otimes (Q')^2 \otimes N \arrow[r] & \cdots \end{tikzcd}$$

Define $\rho_{-1} = \aleph_N$ and $\rho_k = \alpha \otimes (\alpha')^k \otimes 1$ for $k \geq 0$ where $\alpha' = \alpha_{|Q}: Q \to Q'$. The first thing that must be verified is $\mathrm{im}(\alpha_{|Q}) \subset Q'$. Then it must be shown that ρ is a cochain map and each

ρ_n, for $n \geq 0$, is an α-left comodule homomorphism.

Since $\epsilon'\alpha i = \epsilon i = 0$ and i is an injection, $\mathrm{im}(\alpha_{|Q}) \subset Q'$. Also, by the definition of ${}_N\varphi^B$, ${}_N\varphi^B \aleph_N = \rho_{0N}\varphi^A$.

For $k \geq 0$ the diagram

$$\begin{tikzcd} A \otimes Q^k \otimes N \arrow[r, "\Delta_A \otimes 1"] \arrow[d, "\rho_k = \alpha \otimes (\alpha')^k \otimes 1"'] & A \otimes A \otimes Q^k \otimes N \arrow[d, "\alpha \otimes \rho_k"] \\ B \otimes (Q')^k \otimes N \arrow[r, "\Delta_B \otimes 1"'] & B \otimes B \otimes (Q')^k \otimes N \end{tikzcd}$$

is commutative because α is a homomorphism of coalgebras; i.e., $(\alpha \otimes \alpha)\Delta_A = \Delta_B\alpha$. Hence ρ_k is an α-left comodule homomorphism for $k \geq 0$.

Recall

i) $\delta^k = \Delta_A \otimes 1_Q^k \otimes 1_N + 1_A \otimes [\sum_{i=1}^{k}(-1)^i 1_Q \otimes \ldots \otimes \Delta_A \otimes \ldots \otimes 1_Q \otimes 1_N] +$
$+ (-1)^{k+1} 1_A \otimes 1_Q^k \otimes {}_N\varphi^A$

and

ii) $\hat{\delta}^k = \Delta_B \otimes 1_{Q'}^k \otimes 1_N + 1_B \otimes [\sum_{i=1}^{k}(-1)^i 1_{Q'} \otimes \ldots \otimes \Delta_B \otimes \ldots \otimes 1_{Q'} \otimes 1_N]$
$+ (-1)^{k+1} 1_B \otimes 1_{Q'}^k \otimes {}_N\varphi^B$.

Since α is a coalgebra homomorphism and by the definition of ${}_N\varphi^B$, for $k \geq 0$, $\rho_{k+1}\delta^k = (\alpha \otimes (\alpha')^{k+1} \otimes 1_N)\delta^k =$

$= (\alpha \otimes \alpha')\Delta_A \otimes (\alpha')^k \otimes 1_N + \alpha \otimes [\sum_{i=1}^{k}(-1)^i \alpha' \otimes \ldots \otimes (\alpha' \otimes \alpha')\Delta_A \otimes \ldots \otimes \alpha' \otimes 1_N]$
$+ (-1)^{k+1}\alpha \otimes (\alpha')^k \otimes (\alpha' \otimes 1)_N\varphi^A$

$= \Delta_B\alpha \otimes (\alpha')^k \otimes 1_N + \alpha \otimes [\sum_{i=1}^{k}(-1)^i \alpha' \otimes \ldots \otimes \Delta_B\alpha' \otimes \ldots \otimes \alpha' \otimes 1_N] +$
$+ (-1)^{k+1}\alpha \otimes (\alpha')^k \otimes {}_N\varphi^B$

$= \hat{\delta}^k\rho_k$ and the proof is completed.

<u>Proposition 4.9</u>: A,B are augmented R-coalgebras and $\alpha: A \to B$ is a coalgebra homomorphism. If $M \in \mathfrak{M}^A$ and $N \in {}^A\mathfrak{M}$, then there exists a cochain map $\gamma: M \underset{A}{\square} \mathfrak{B}(A,N) \to T_\alpha(M) \underset{B}{\square} \mathfrak{B}(B, {}_\alpha T(N))$.

<u>Proof</u>: Let $\gamma = \aleph_M \underset{\alpha}{\square} \rho$.

Therefore, if A,B are augmented R-coalgebras, $\alpha: A \to B$ is a coalgebra homomorphism and if $M \in \mathfrak{M}^A$, $N \in {}^A\mathfrak{M}$, there exists, for each $n, p \geq 0$, an R-homomorphism

(2) $\gamma^*_{n,p}: \mathrm{Cotor}^{n,p}_{A,\widetilde{\mathcal{E}}^0}(M,N) \to \mathrm{Cotor}^{n,p}_{B,\widetilde{\mathcal{E}}^0}(M,N)$.

Let $(\Lambda, \Delta, \in, \eta)$ and $(\Lambda', \Delta', \in', \eta')$ be augmented R-coalgebras and let $N \in {}^\Lambda\mathfrak{M}$, $N' \in {}^{\Lambda'}\mathfrak{M}$ with cobar resolutions $N \xrightarrow{N^\varphi} \mathfrak{B}(\Lambda,N)$, $N' \xrightarrow{N'^\varphi} \mathfrak{B}(\Lambda',N')$, respectively. (All mosules and coalgebras are assumed graded unless specifically stated otherwise.) Then it is known that $N \otimes N' \in {}^{\Lambda\otimes\Lambda'}\mathfrak{M}$ and

(3) $N \otimes N' \xrightarrow{N^\varphi \otimes N'^\varphi} \mathfrak{B}(\Lambda,N) \otimes \mathfrak{B}(\Lambda',N')$,

is a cochain complex. If (3) is an $\widetilde{\mathcal{E}}^0$-injective resolution of $N \otimes N'$, then the homology groups calculated using (3) or by using the cobar resolution for $N \otimes N'$ will be the same up to a natural equivalence. The following theorem shows that (3) is an $\widetilde{\mathcal{E}}^0$-injective resolution for $N \otimes N'$.

<u>Theorem 4.2</u>: Under the assumptions of the above paragraph $0 \to N \otimes N' \xrightarrow{N^\varphi \otimes N'^\varphi} \mathfrak{B}(\Lambda,N) \otimes \mathfrak{B}(\Lambda',N')$ is an $\widetilde{\mathcal{E}}^0$-injective resolution for $N \otimes N'$.

Proof: Recall that the cobar resolution is

$$\mathfrak{B}(\Lambda,N):0 \to N \underset{s^{-1}}{\overset{N^{\varphi}}{\rightleftarrows}} \Lambda \otimes N \underset{s^{0}}{\overset{\delta^{0}}{\rightleftarrows}} \Lambda \otimes Q \otimes N \underset{s^{1}}{\overset{\delta^{1}}{\rightleftarrows}} \Lambda \otimes Q^2 \otimes N \rightleftarrows \cdots$$

where $\{s^i \mid i \geq -1\}$ is the contracting homotopy and $1_{\mathfrak{B}(A,N)} \sim 0_{\mathfrak{B}(A,N)}$. Similarly

$$\mathfrak{B}(\Lambda'N'):0 \to N' \underset{\sigma^{-1}}{\overset{N'^{\varphi}}{\rightleftarrows}} \Lambda' \otimes N' \underset{\sigma^{0}}{\overset{\delta^{0}}{\rightleftarrows}} \Lambda' \otimes Q' \otimes N' \underset{\sigma^{1}}{\overset{\delta^{1}}{\rightleftarrows}} \Lambda \otimes (Q')^2 \otimes N' \rightleftarrows \cdots$$

is the cobar resolution of N' and $1_{\mathfrak{B}(\Lambda',N')} \sim 0_{\mathfrak{B}(\Lambda',N')}$. Therefore, by Proposition 9.1, [15-164], there exists a contracting homotopy $t = \{t^k \mid k \geq -1\}$ of R-homomorphisms for the cochain complex (3). To complete the proof,

$[\mathfrak{B}(\Lambda,N) \otimes \mathfrak{B}(\Lambda',N')]^n = \sum_{p+q=n} (\Lambda \otimes Q^p \otimes N) \otimes (\Lambda' \otimes (Q')^q \otimes N')$, for $n \geq 0$,

must be shown to be an $\widetilde{\mathcal{E}}^0_{\Lambda\otimes\Lambda'}$-injective object.

To show an object $M \in {}^{\Lambda\otimes\Lambda'}\mathfrak{M}$ is in $\widetilde{\mathfrak{J}}^0_{\Lambda\otimes\Lambda'}$, one needs to show there exists an $A \in \mathfrak{M}$ and comodule homomorphisms c,r such that, $M \underset{r}{\overset{c}{\rightleftarrows}} (\Lambda \otimes \Lambda') \otimes A$ and $rc = 1_M$. Recall that direct sums of objects in $\widetilde{\mathfrak{J}}^0_{\Lambda\otimes\Lambda'}$ are in $\widetilde{\mathfrak{J}}^0_{\Lambda\otimes\Lambda'}$, [6], hence the proof will be completed if $(\Lambda \otimes Q^p \otimes N) \otimes (\Lambda' \otimes (Q')^q \otimes N')$ is in $\widetilde{\mathfrak{J}}^0_{\Lambda\otimes\Lambda'}$ for any $p,q \geq 0$.

Consider the diagram

$$\begin{array}{ccc}
(\Lambda\otimes N)\otimes(\Lambda'\otimes N') & \underset{1\otimes\tau_N(\Lambda')\otimes 1}{\overset{1\otimes\tau_{\Lambda'}(N)\otimes 1}{\rightleftarrows}} & \Lambda\otimes\Lambda'\otimes N\otimes N' \\
\downarrow \varphi_0 & & \downarrow \varphi = \bar{\Delta}\otimes 1 \\
(\Lambda\otimes\Lambda')\otimes[(\Lambda\otimes N)\otimes(\Lambda'\otimes N')] & \underset{1\otimes 1\otimes 1\otimes\tau_{\Lambda'}(N)\otimes 1}{\overset{1\otimes 1\otimes 1\otimes\tau_N(\Lambda')\otimes 1}{\rightleftarrows}} & (\Lambda\otimes\Lambda')\otimes(\Lambda\otimes\Lambda')\otimes(N\otimes N').
\end{array}$$

where $\varphi_0 = (1 \otimes 1 \otimes 1 \otimes \tau_N(\Lambda') \otimes 1)(\bar{\Delta} \otimes 1)(1 \otimes \tau_{\Lambda'}(N) \otimes 1)$. If

$(\Lambda \otimes N \otimes \Lambda' \otimes N', \varphi_0)$ is a left $\Lambda \otimes \Lambda'$-comodule, let $c = 1 \otimes \tau_{\Lambda'}(N) \otimes 1$ and $r = 1 \otimes \sigma_{\Lambda'}(N) \otimes 1 = 1 \otimes \tau_N(\Lambda') \otimes 1$. Then from the definition of φ_0, r, c are Λ-comodule homomorphisms and $rc = 1$.

The diagram

$$\begin{tikzcd}
\Lambda\otimes N\otimes\Lambda'\otimes N' \arrow[r, "\varphi_0"] \arrow[d, "\varphi_0"'] & (\Lambda\otimes\Lambda')\otimes(\Lambda\otimes N\otimes\Lambda'\otimes N') \arrow[d, "1\otimes\varphi_0"] \\
(\Lambda\otimes\Lambda')\otimes(\Lambda\otimes N\otimes\Lambda'\otimes N') \arrow[r, "\bar{\Delta}\otimes 1"'] & (\Lambda\otimes\Lambda')^2\otimes(\Lambda\otimes N\otimes\Lambda'\otimes N')
\end{tikzcd}$$

can be written as Diagram 4.4 where (i) is commutative by definition of φ_0, (ii) is commutative because φ is coscalar multiplication and (iii) is an identity. Similarly, one can verify the commutativity of

$$\begin{tikzcd}
R\otimes(\Lambda\otimes N\otimes\Lambda'\otimes N') \arrow[r, equal] & \Lambda\otimes N\otimes\Lambda'\otimes N' \arrow[dl, "\varphi_0"] \\
(\Lambda\otimes\Lambda')\otimes(\Lambda\otimes N\otimes\Lambda'\otimes N') \arrow[u, "\bar{\varepsilon}\otimes 1"] &
\end{tikzcd}$$

Define for each $p, q \geq 0$ with either $p \neq 0$ or $q \neq 0$,

$\varphi_{p,q}: \Lambda \otimes Q^p \otimes N \otimes \Lambda' \otimes (Q')^q \otimes N' \to (\Lambda \otimes \Lambda') \otimes (\Lambda \otimes Q^p \otimes N \otimes \Lambda' \otimes (Q')^q \otimes N')$

by the diagram

$$\begin{tikzcd}
\Lambda\otimes Q^p\otimes N\otimes\Lambda'\otimes(Q')^q\otimes N' \arrow[r, leftrightarrow, dashed, "1\otimes\tau_{\Lambda'}(Q^p\otimes N)\otimes 1^{q+1}"] \arrow[d, "\varphi_{p,q}"'] & \Lambda\otimes\Lambda'\otimes Q^p\otimes N\otimes(Q')^q\otimes N' \arrow[d, "\bar{\Delta}\otimes 1^{p+q+2}"] \\
(\Lambda\otimes\Lambda')\otimes(\Lambda\otimes Q^p\otimes N\otimes\Lambda'\otimes(Q')^q\otimes N') \arrow[r, leftrightarrow, dashed, "1\otimes 1\otimes 1\otimes\sigma_{\Lambda'}(Q^p\otimes N)\otimes 1^{q+1}"] & (\Lambda\otimes\Lambda')^2\otimes(Q^p\otimes N\otimes(Q')^q\otimes N')
\end{tikzcd}$$

and the reader can verify, as done for $p = 0 = q$, that $(\Lambda \otimes Q^p \otimes N \otimes \Lambda' \otimes (Q')^q \otimes N', \varphi_{p,q})$ is a left $\Lambda \otimes \Lambda'$-comodule.

Therefore, by (1) and Theorem 4.2, for n, m, p, $q \geq 0$, there exists an R-homomorphism

$$
\begin{tikzcd}
\Lambda\otimes N\otimes\Lambda'\otimes N' \arrow[r, "1\otimes\tau_{\Lambda'}(N)\otimes 1"] \arrow[d, "1\otimes\tau_{\Lambda'}(N)\otimes 1"'] & \Lambda\otimes\Lambda'\otimes N\otimes N' \arrow[r, "\bar{\Delta}\otimes 1^2"] \arrow[dr, phantom, "(i)"] & (\Lambda\otimes\Lambda')^2\otimes N\otimes N' \arrow[r, "1\otimes 1\otimes 1\otimes\tau_N(\Lambda')\otimes 1"] & (\Lambda\otimes\Lambda')\otimes(\Lambda\otimes N\otimes\Lambda'\otimes N') \arrow[d, "1\otimes 1\otimes 1\otimes\tau_{\Lambda'}(N)\otimes 1"] \\
\Lambda\otimes\Lambda'\otimes N\otimes N' \arrow[rrr, "\bar{\Delta}\otimes 1^2 = \varphi"] \arrow[d, "\varphi"'] & & \arrow[d, phantom, "(ii)"] & (\Lambda\otimes\Lambda')\otimes(\Lambda\otimes\Lambda'\otimes N\otimes N') \arrow[d, "1\otimes 1\otimes\varphi"] \\
(\Lambda\otimes\Lambda')^2\otimes(N\otimes N') \arrow[rrr, "\bar{\Delta}\otimes 1^2\otimes 1^2"] \arrow[d, "1\otimes 1\otimes 1\otimes\tau_N(\Lambda')\otimes 1"'] & & \arrow[d, phantom, "(iii)"] & (\Lambda\otimes\Lambda')^3\otimes(N\otimes N') \arrow[d, "1\otimes 1\otimes 1\otimes 1\otimes 1\otimes\tau_N(\Lambda')\otimes 1"] \\
(\Lambda\otimes\Lambda')\otimes(\Lambda\otimes N\otimes\Lambda'\otimes N') \arrow[rrr, "\bar{\Delta}\otimes 1\otimes 1\otimes 1\otimes 1"] & & & (\Lambda\otimes\Lambda')^2\otimes(\Lambda\otimes N\otimes\Lambda'\otimes N')
\end{tikzcd}
$$

Diagram 4.4

$$_{(n,p)}\phi_{(m,q)} \mathrm{Cotor}^{n,p}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,N) \otimes \mathrm{Cotor}^{m,q}_{\Lambda',\widetilde{\mathcal{E}}^0}(M',N') \to \mathrm{Cotor}^{n+m,p+q}_{\Lambda\otimes\Lambda',\widetilde{\mathcal{E}}^0}(M\otimes M',N\otimes N')$$

for any $M \in \mathfrak{M}^{\Lambda}$, $N \in {}^{\Lambda}\mathfrak{M}$, $M' \in \mathfrak{M}^{\Lambda'}$ and $N' \in {}^{\Lambda'}\mathfrak{M}$ where $(\Lambda, \Delta, \epsilon, \eta)$ and $(\Lambda', \Delta', \epsilon', \eta')$ are augmented R-coalgebras.

If $(\Lambda, \Delta, \mu, \epsilon, \eta)$ is a Hopf algebra where $\mu:\Lambda \otimes \Lambda \to \Lambda$ is the multiplication and $\eta:R\to\Lambda$ is the unit, then μ is a coalgebra homomorphism; Milnor and Moore [16-227]; thus, by (2), there exists, for $n,m,p,q \geq 0$,

$$_{(n,p)}\phi_{(m,q)}: \mathrm{Cotor}^{n,p}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,N) \otimes \mathrm{Cotor}^{m,q}_{\Lambda,\widetilde{\mathcal{E}}^0}(M',N') \to \mathrm{Cotor}^{n+m,p+q}_{\Lambda,\widetilde{\mathcal{E}}^0}(M\otimes M',N\otimes N')$$

for $M,M' \in \mathfrak{M}^{\Lambda}$ and $N,N' \in {}^{\Lambda}\mathfrak{M}$. Further, if we consider $N = {}^{\epsilon}R$ and $M' = R^{\epsilon}$, then

$$_{(n,p)}\phi_{(m,q)}: \mathrm{Cotor}^{n,p}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,R) \otimes \mathrm{Cotor}^{m,q}_{\Lambda,\widetilde{\mathcal{E}}^0}(R,N') \to \mathrm{Cotor}^{n+m,p+q}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,N')$$

An Internal Product for Coext

Let $(\Lambda, \Delta, \epsilon, \eta)$ be an augmented graded R-coalgebra where R is a commutative ring with identity. It will be shown that for each M,N,L in ${}^{\Lambda}\mathfrak{M}$ and for each $m,n,p,q \geq 0$ there exists an R-homomorphism.

$$\phi:\mathrm{Coext}^{n,p}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,N) \otimes \mathrm{Coext}^{m,q}_{\Lambda,\widetilde{\mathcal{E}}^0}(N,L) \to \mathrm{Coext}^{n+m,p+q}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,L).$$

A similar result can be obtained for right Λ-comodules.

Lemma 4.1: If M is a left Λ-comodule, A an R-module, then for any $d \geq 0$, $\mathrm{Hom}^d_R(M,A) \cong \mathrm{Hom}^d_{\Lambda}(M, \Lambda \otimes A)$ as R-modules.

Proof: For each $d \geq 0$ define $b_d:\mathrm{Hom}^d_R(M,A) \to \mathrm{Hom}^d_{\Lambda}(M, \Lambda \otimes A)$ by the diagram

$$\begin{tikzcd} M \arrow[r, "M^{\varphi}"] \arrow[dr, "b_d(f)"'] & \Lambda \otimes M \arrow[d, "1 \otimes f"] \\ & \Lambda \otimes A \end{tikzcd}$$

for any $f \in \mathrm{Hom}_R^d(M,A)$ and define $a_d:\mathrm{Hom}_\Lambda^d(M, \Lambda \otimes A) \to \mathrm{Hom}_R^d(M,A)$ by the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[ddr, "a_d(g)"'] & \Lambda \otimes A \arrow[d, "\epsilon \otimes 1"] \\ & R \otimes A \arrow[d, equal] \\ & A \end{tikzcd}$$

for any $g \in \mathrm{Hom}_\Lambda^d(M, \Lambda \otimes A)$. Then, one can verify that b_d and a_d are R-homomorphisms and $a_d b_d = 1$, $b_d a_d = 1$.

<u>Theorem 4.3</u>: If $E:M^1 \xrightarrow{f} M^2 \xrightarrow{g} M^3$ is in $\widetilde{\mathcal{E}}^0$ and $I \in \widetilde{\mathcal{J}}^0$, then for any $d \geq 0$, the sequence

$$\mathrm{Hom}_\Lambda^d(M^3,I) \xrightarrow{g^*} \mathrm{Hom}_\Lambda^d(M^2,I) \xrightarrow{f^*} \mathrm{Hom}_\Lambda^d(M^1,I)$$

is exact.

<u>Proof</u>: Without loss of generality, assume $I = \Lambda \otimes A$ where A is an R-module. Since $f^*g^* = 0$, it needs only be shown that $\ker f^* \subset \operatorname{im} g^*$. Because of Lemma 4.1, if the sequence

(1) $\mathrm{Hom}_R^d(M^3,A) \xrightarrow{g^*} \mathrm{Hom}_R^d(M^2,A) \xrightarrow{f^*} \mathrm{Hom}_R^d(M^1,A)$

is exact, then the proof will be completed.

The sequence E is R-split exact, so there exists an R-homomorphism $m:M^2/\operatorname{im} f \to M^3$ and $e:M^3 \to M^2/\operatorname{im} f$ such that $g = mc$, where c is the cokernel of f, m is a monomorphism, e is an epimorphism and em = 1. Since hf = 0, there exists a unique morphism $k: M^2/\text{im } f \to A$ such that kc = h. Then keg = kemc = kc = f and (1) is exact as a sequence of R-modules.

$$\begin{tikzcd} M' \arrow[r, "f"] & M^2 \arrow[r, "g"] \arrow[d, "h"'] \arrow[dr, "c"] & M^3 \arrow[d, shift left, "e"] \\ & A & M^2/\text{im } f \arrow[l, "k"] \arrow[u, shift left, "m"] \end{tikzcd}$$

Let M,N,L be left Λ-comodules. Let B(N) and B(L) denote the cobar resolutions of N and L; i.e., the canonical $\widetilde{\mathcal{E}}^0$-injective resolutions of N and L, respectively, see paragraph 3 of Chapter III. B(N) and B(L) are

$$B(N): 0 \to N \underset{s^{-1}}{\overset{N\varphi}{\rightleftarrows}} B_0 \underset{s^{1}}{\overset{\delta^0}{\rightleftarrows}} B_1 \rightleftarrows \cdots \rightleftarrows B_n \underset{s^{n}}{\overset{\delta^n}{\rightleftarrows}} B_{n+1} \rightleftarrows \cdots$$

$$B(L): 0 \to L \underset{\sigma^{-1}}{\overset{L\varphi}{\rightleftarrows}} \bar{B}_0 \underset{\sigma^{0}}{\overset{\hat{\delta}^0}{\rightleftarrows}} \bar{B}_1 \rightleftarrows \cdots \rightleftarrows \bar{B}_n \underset{\sigma^{n}}{\overset{\hat{\delta}^n}{\rightleftarrows}} \bar{B}_{n+1} \rightleftarrows \cdots .$$

The Hom_Λ sequences are

$$\text{Hom}_\Lambda(M,B(N)): 0 \to \text{Hom}_\Lambda(M,N) \xrightarrow{N\varphi_*} \text{Hom}_\Lambda(M,B_0) \xrightarrow{\delta^0_*} \cdots$$

$$\text{Hom}_\Lambda(N,B(L)): 0 \to \text{Hom}_\Lambda(N,L) \xrightarrow{L\varphi_*} \text{Hom}_\Lambda(N,\bar{B}_0) \xrightarrow{\hat{\delta}^0_*} \cdots$$

$$\text{Hom}_\Lambda(M,B(L)): 0 \to \text{Hom}_\Lambda(M,L) \xrightarrow{L\varphi_*} \text{Hom}_\Lambda(M,B_0) \xrightarrow{\hat{\delta}^0_*} \cdots$$

and, for each $d \geq 0$,

$$\text{Hom}^d_\Lambda(M,B(N)): 0 \to \text{Hom}^d_\Lambda(M,N) \xrightarrow{(N\varphi_*)_d} \text{Hom}^d_\Lambda(M,B_0) \xrightarrow{(\delta^0_*)_d} \cdots$$

$$\text{Hom}^d_\Lambda(N,B(L)): 0 \to \text{Hom}^d_\Lambda(N,L) \xrightarrow{(L\varphi_*)_d} \text{Hom}^d_\Lambda(N,\bar{B}_0) \xrightarrow{(\hat{\delta}^0_*)_d} \cdots$$

$$\text{Hom}^d_\Lambda(M,B(L)): 0 \to \text{Hom}^d_\Lambda(M,L) \xrightarrow{(L\varphi_*)_d} \text{Hom}^d_\Lambda(M,\bar{B}_0) \xrightarrow{(\hat{\delta}^0_*)_d} \cdots$$

Then, for each $n,p,m,q \geq 0$;

$$\operatorname{Coext}^{n,p}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,N) \equiv H^n(\operatorname{Hom}^p_\Lambda(M,B(N))) = \frac{\ker(\delta^n_*)_p}{\operatorname{im}\,(\delta^{n-1}_*)_p},$$

$$\operatorname{Coext}^{m,q}_{\Lambda,\widetilde{\mathcal{E}}^0}(N,L) \equiv H^m(\operatorname{Hom}^q_\Lambda(N,B(L))) = \frac{\ker(\hat{\delta}^m_*)_q}{\operatorname{im}(\hat{\delta}^{m-1}_*)_q}$$

and

$$\operatorname{Coext}^{n+m,p+q}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,L) \equiv H^{n+m}(\operatorname{Hom}^{p+q}_\Lambda(M,B(L))).$$

To define an R-homomorphism

$$\phi:\operatorname{Coext}^{n,p}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,N) \otimes \operatorname{Coext}^{m,q}_{\Lambda,\widetilde{\mathcal{E}}^0}(N,L) \to \operatorname{Coext}^{n+m,p+q}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,L)$$

consider the following diagram where $f \in \operatorname{Hom}^p_\Lambda(M,B_m)$ and $\delta^n f = 0$; $g \in \operatorname{Hom}^q_\Lambda(M,B_m)$ and $\delta^m g = 0$.

$$\begin{tikzcd}
 & & & & & & M \arrow[d, "f"] & & \\
B(N):0 \arrow[r] & N \arrow[r, "{}_N\varphi"] \arrow[d, "1_N"'] & B_0 \arrow[r, "\delta^0"] \arrow[d, dashed, "g_0"'] & B_1 \arrow[r] \arrow[d, dashed, "g_1"'] & \cdots \arrow[r] & B_{n-1} \arrow[r, "\delta^{n-1}"] \arrow[d, dashed, "g_{n-1}"'] & B \arrow[r, "\delta^n"] \arrow[d, dashed, "g_n"'] & B_{n+1} \arrow[r] \arrow[d, dashed, "g_{n+1}"'] & \cdots \\
K_g:0 \arrow[r] & N \arrow[r, "g"'] & \bar{B}_m \arrow[r, "\hat{\delta}^m"'] & \bar{B}_{m+1} \arrow[r] & \cdots \arrow[r] & \bar{B}_{m+n-1} \arrow[r, "\hat{\delta}^{m+n-1}"] & \bar{B}_{m+n} \arrow[r, "\hat{\delta}^{m+n}"] & \bar{B}_{m+n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Diagram 4.5

By Theorem 4.3 we can define a Λ-cochain map $G:B(N) \to K_g$, i.e., a sequence of Λ-comodule homomorphisms $g_k:B_k \to \bar{B}_{m+k}$, for $k \geq 0$, such that $g_{k+1}\delta^k = \hat{\delta}^{m+k}g_k$ and $|g_k| = |g|$. Since $0 \to N \to B_0$ is in $\widetilde{\mathcal{E}}^0$ and $\bar{B}_m \in \widetilde{\mathfrak{J}}^0$, there exists a Λ-comodule homomorphism $g_0:B_0 \to \bar{B}_m$ such that $g_0\,{}_N\varphi = g1_N$. Then $\hat{\delta}^m g_0\,{}_N\varphi = \hat{\delta}^m g = 0$ and there exists a Λ-comodule homomorphism $g_1:B_1 \to \bar{B}_{m+1}$ such that $g_1\delta^0 = \hat{\delta}^m g_0$. Assume there exists a Λ-comodule homomorphism $g_n:B_n \to \bar{B}_{m+n}$ such that $g_n\delta^{n-1} = \hat{\delta}^{m+n-1}g_{n-1}$.

Then $\hat{\delta}^{m+n} g_n \delta^{n-1} = \hat{\delta}^{m+n}\hat{\delta}^{m+n-1} g_{n-1} = 0$. $B_{n-1} \xrightarrow{\delta^{n-1}} B_n \xrightarrow{\delta^{n}} B_{n+1}$ is in $\widetilde{\mathcal{E}}^0$ and $\bar{B}_{m+n+1} \in \widetilde{\mathcal{J}}^0$, hence there exists a Λ-comodule homomorphism $g_{n+1}: B_{n+1} \to \bar{B}_{m+n+1}$ such that $g_{n+1}\delta^n = \hat{\delta}^{m+n} g_n$. Therefore, G is constructed by induction.

Let $\phi(\bar{f} \otimes \bar{g}) = \overline{g_n f}$ where $\bar{f}$ denotes $f + \text{im}\,(\delta_*^{n-1})_p$, similarly for $\bar{g}$ and $\overline{g_n f}$. To show ϕ is well-defined it must be verified that the definition is independent of the choice of the representative of the cosets, independent of the $\widetilde{\mathcal{E}}^0$-injective resolutions of N and L and independent of the choice of the Λ-cochain map G. Since the homology groups are independent of the particular $\widetilde{\mathcal{E}}^0$-injective resolution the definition of ϕ is independent of the choice of resolution.

Let $G' = \{g_k' : B_k \to \bar{B}_{m+k} \mid k \geq 0\}$ be another Λ-cochain map derived from 1_N. Then using Theorem 4.3, one can show G is homotopic to G', i.e., there exists a sequence of Λ-homomorphisms $\{t_k : B_k \to \bar{B}_{m+k-1} \mid k \geq 1\}$ such that $g_0 - g_1 = t_1\delta^0$ and $g_k - g_k' = t_{k+1}\delta^k + \hat{\delta}^{m+k-1} t_k$ for $k \geq 1$. Then $(g_n - g_n')f \in \text{im}\,\hat{\delta}^{m+n-1}$, because $(g_n - g_n')f = \hat{\delta}^{m+n-1} t_n f + t_{n+1}\delta^n f$ $= \hat{\delta}^{m+n-1} t_n f$. Therefore, the definition of ϕ is independent of the cochain map.

Now suppose $\bar{g} = 0$. Then $g \in \text{im}\,(\hat{\delta}_*^{m-1})_q$ and there exists a Λ-comodule homomorphism of degree q, $h: N \to \bar{B}_{m-1}$ such that $\hat{\delta}^{m-1} h = g$. From h, one obtains a Λ-cochain map $H = \{h_i : B_i \to \bar{B}_{m-1+i} \mid i \geq 0\}$ where $H: B(N) \to K_h$;

$$\begin{tikzcd}
B(N):0 \arrow[r] & N \arrow[r, "N^{\varphi}"] \arrow[d, "1"] & B_0 \arrow[r, "\delta^0"] \arrow[d, "h_0"] & B_1 \arrow[r] \arrow[d, "h_1"] & \cdots \arrow[r] & B_n \arrow[r] \arrow[d, "h_n"] & \cdots \\
K_h:0 \arrow[r] & N \arrow[r, "h"] & \bar{B}_{m-1} \arrow[r, "\hat{\delta}^{m-1}"] & \bar{B}_m \arrow[r] & \cdots \arrow[r] & B_{m+n-1} \arrow[r] & \cdots;
\end{tikzcd}$$

consequently,

$$\begin{array}{ccccccccc}
B(N):0 \to N & \longrightarrow & B_0 & \longrightarrow & B_1 & \to \cdots \to & B_n & \to \cdots \\
\downarrow 1 & & g_0 \downarrow\; h'_0 & & g_1 \downarrow\; h'_1 & & g_n \downarrow\; h'_n & \\
K_g = K_{\hat{\delta}^{m-1}h}:0 \to N & \xrightarrow[\hat{\delta}^{m-1}h]{g} & \bar{B}_m & \longrightarrow & \bar{B}_{m+1} & \to \cdots \to & \bar{B}_{m+n} & \to \cdots
\end{array}$$

where $h'_k = \hat{\delta}^{m+k-1}h_k$ for $k \geq 0$. Therefore $H' \sim G$ and there exists a sequence $\{\rho_n:B_n \to \bar{B}_{m+n-1} \mid n \geq 0\}$ of Λ-comodule homomorphisms such that $g_n - h'_n = \hat{\delta}^{m+n-1}\rho_n + \rho_{n+1}\,\delta^n$. Then $(g_n - h'_n)f = \hat{\delta}^{m+n-1}\rho_n f$ and $g_n f = \hat{\delta}^{m+n-1}\rho_n f + \hat{\delta}^{m+n-1}h_n f = \hat{\delta}^{m+n-1}(\rho_n f + h_n f)$ and $g_n f \in \text{im}\,(\hat{\delta}_*^{m+n-1})_{p+q}$.

Finally, suppose $\bar{f} = 0$, then there exists a Λ-comodule homomorphism $\ell:M \to B_{n-1}$ of degree p such that $\delta^{n-1}\ell = f$. Thus $\hat{\delta}^{m+n-1}g_{n-1}\ell = g_n\,\delta^{n-1}\ell = g_n f$ and $g_n f \in \text{im}\,(\delta_*^{m+n-1})_{p+q}$. Therefore ϕ is a function. It can readily be verified that ϕ is an R-homomorphism and the proof of the following theorem is complete.

<u>Theorem 4.4:</u> If M, N, L are left Λ-comodules, then there exists for each $n,m,p,q \geq 0$ an R-homomorphism

$$\phi:\text{Coext}^{n,p}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,N) \otimes \text{Coext}^{m,q}_{\Lambda,\widetilde{\mathcal{E}}^0}(N,L) \to \text{Coext}^{n+m,p+q}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,L).$$

<u>Theorem 4.5:</u> If M is a left Λ-comodule, then $C = \{\text{Coext}^{n,p}_{\Lambda,\widetilde{\mathcal{E}}^0}(M,M) \mid n,p \geq 0\}$ is a bigraded R-algebra.

<u>Proof:</u> One can readily verify that C is a bigraded R-module. Let

$C_{n,p}$ denote $\mathrm{Coext}^{n,p}_{\Lambda,\tilde{\mathcal{E}}^0}(M,M)$ for $n,p \geq 0$. Define

$$\mu : C \otimes C \to C$$

by letting ${}_{(n,p)}\mu_{(m,q)} = \phi$, where ϕ is defined in the proof of Theorem 4.4. ${}_{(n,p)}\mu_{(m,q)} : C_{n,p} \otimes C_{m,q} \to C_{n+m,p+q}$ for $n,p,m,q \geq 0$. Define $\eta : R \to C$ by $\eta(1) = \varphi_M \in \mathrm{Hom}_R(M, \Lambda \otimes M)$ i.e., $\eta(1) = \bar{\varphi}_M \in C_{0,0}$.

Then commutativity must be verified in the diagrams

$$\begin{tikzcd} C \otimes C \otimes C \arrow[r, "\mu \otimes 1"] \arrow[d, "1 \otimes \mu"'] \arrow[dr, phantom, "\mathrm{(I)}"] & C \otimes C \arrow[d, "\mu"] \\ C \otimes C \arrow[r, "\mu"'] & C \end{tikzcd} \qquad \begin{tikzcd} R \otimes C \arrow[r, equal] \arrow[d, "\eta \otimes 1"'] \arrow[dr, phantom, "\mathrm{(II)}"] & C \arrow[dr, phantom, "\mathrm{(III)}"] & C \otimes R \arrow[l, equal] \arrow[d, "1 \otimes \eta"] \\ C \otimes C \arrow[ur, "\mu"] & & C \otimes C \arrow[ul, "\mu"'] \end{tikzcd}$$

1) <u>Commutativity in (I):</u>

Let $\bar{f} \in C_{n,p}$, $\bar{g} \in C_{m,q}$ and $\bar{h} \in C_{r,s}$. Then $\mu(\mu \otimes 1)(\bar{f} \otimes \bar{g} \otimes \bar{h}) =$

$= \mu(\overline{g_n f} \otimes \bar{h}) = \overline{h_{n+m}(g_n f)}$ and $\mu(1 \otimes \mu)(\bar{f} \otimes \bar{g} \otimes \bar{h}) = \mu(\bar{f} \otimes \overline{h_m g}) = \overline{(h_m g)_n f}$.

By considering the following diagrams and because of chain homotopy, $\overline{h_{m+n} g_n f} = \overline{(h_m g)_n f}$.

$$\begin{tikzcd} & & & & M \arrow[d, "f"] & \\ 0 \arrow[r] & M \arrow[r, "M^{\varphi}"] \arrow[d, "1"] & B_0 \arrow[r] \arrow[d, "g_0"] & \cdots \arrow[r] & B_n \arrow[r] \arrow[d, "g_n"] & \cdots \\ 0 \arrow[r] & M \arrow[r, "g"'] \arrow[d, "1"] & B_m \arrow[r] \arrow[d, "h_m"] & \cdots \arrow[r] & B_{m+n} \arrow[r] \arrow[d, "h_{m+n}"] & \cdots \\ 0 \arrow[r] & M \arrow[r, "h_m g"'] & B_{m+r} \arrow[r] & \cdots \arrow[r] & B_{m+n+r} \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "M^{\varphi}"] \arrow[d, "1"'] & B_0 \arrow[r] \arrow[d, "(h_m g)_0"] & \cdots \arrow[r] & B_n \arrow[r] \arrow[d, "(h_m g)_n"] & \cdots \\
0 \arrow[r] & M \arrow[r, "h_m g"'] & B_{m+r} \arrow[r] & \cdots \arrow[r] & B_{m+n+r} \arrow[r] & \cdots .
\end{tikzcd}$$

2) <u>Commutativity in III (Similarly in II)</u>:

Let $\bar{f} \in C_{n,p}$ then $(1 \otimes \eta)(\bar{f} \otimes 1) = \bar{f} \otimes 1$ and $\mu(\bar{f} \otimes {}_M\bar{\varphi}) = \bar{f}$

by construction of the chain maps from K_f.

CHAPTER V

SUMMARY AND CONCLUSIONS

This paper is concerned with two objectives, an investigation of the properties of the cotorsion functor and a presentation of the functor coextension. Relative homological algebra is the principal tool used in this research.

An exposition of derived functors relative to an injective class of sequences is given and then in Chapter II an example, Ext, is stated where the injective class considered, $\mathcal{E}^0$, is not equal to the class of all exact sequences. The writer also shows that the category, ${}_\Lambda\mathfrak{M}$, of left modules over a given algebra Λ is injectively perfect. It is also shown that the functor $\mathrm{Hom}_R(\Lambda, --)$ from $\mathfrak{M}$ to ${}_\Lambda\mathfrak{M}$ is an adjoint functor of the forgetful functor. The canonical $\mathcal{E}^0$-injective resolution is constructed.

Using the theory of Chapter I, it is shown that the Cotor functor can be derived, relative to the injective class $\mathcal{E}^0$, from the cotensor product. Furthermore, the writer shows that Hom_Λ, relative to $\mathcal{E}^0$, satisfies the conditions of Chapter I. Hence, a derived functor exists which is called Coext.

Finally, in Chapter IV, products are obtained for Cotor and Coext. It is also shown that the product for Coext yields an algebra.

In relation to this investigation and subsequent to its completion Professor N. Shimada, Professor H. Uehara and the author have found that triple cohomology (M. Barr and J. Beck [4], M. Barr [3], S. Eilenberg and J. C. Moore [7]) can be discussed as a derived functor in the relative homological algebra of [6]; in particular the standard complex used in [4] is a resolution with respect to a suitable projective class in a category of functors. Hence, the Acyclic Model Theorem (Theorem 3.1, [4]) is exactly the comparison theorem (Proposition 3.2, [6]). (The author has noticed that S. MacLane reported a similar result in the April, 1967 issue of the Notices of the American Mathematical Society.) This discovery unifies all known cohomology theories of algebras including Lie algebras, from the standpoint of relative homological algebra.

By consideration of Grothendieck's fibred category (Grothendieck [10] and Gray [9]) it is proposed that the product of Chapter IV can be added to the axioms of a derived functor, discussed in Chapter I. This proposal has the effect of unifying cohomology and homology theory in relative homological algebra. Preliminary investigation indicates that this can be done.

Another proposal for further research is to apply the results of this paper, to the calculation of the Ext functor of modules over the Steenrod algebra. It is proposed that this application can then be used to study not only the usual multiplicative structure but also some characteristic features of the cohomology of Hopf algebras--for example, the algebraic Steenrod operations defined in the cohomology (A. Liulevicius [13]).

BIBLIOGRAPHY

1. Adams, J. F. "On the Cobar Construction," Proc. NAS USA, 42 (1956), 409-412.

2. Barnes, W. E. Introduction to Abstract Algebra, D. C. Heath and Company, Boston, 1963.

3. Barr, M. "Shukla Cohomology and Triples," J. of Alg., 5 (1967), 222-231.

4. Barr, M. and Beck, J. "Acyclic Models and Triples," Proceedings of the Conference on Categorical Algebra, La Jolla (1966), 336-343.

5. Buchsbaum, D. A. "A Note on Homology in Categories," Ann. of Math., 69 (1959), 66-74.

6. Eilenberg, S. and Moore, J. C. "Foundations of Relative Homological Algebra," Memoirs of the Amer. Math. Soc., No. 55 (1965).

7. Eilenberg, S. and Moore, J. C. "Adjoint Functors and Triples," Ill. J. of Math., 9 (1965), 381-398.

8. Freyd, P. Abelian Categories, Harper and Row, New York, 1964.

9. Gray, J. W. "Fibred and Cofibred Categories," Proceedings of the Conference on Categorical Algebra, La Jolla (1966), 21-83.

10. Grothendieck, A. "Catégories Fibrées et Descente," Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques, Paris, 1961.

11. Gugenheim, V.K.A.M. "On Extensions of Algebras, Coalgebras and Hopf Algebras, I," Am. J. Math., 84 (1962), 349-382.

12. Heller, A. "Homological Algebra in Abelian Categories," Ann. of Math., 68 (1958), 484-525.

13. Liulevicius, A. "The Factorization of Cyclic Reduced Powers by Secondary Cohomology Operations," Memoirs of the Amer. Math. Soc., 42 (1962).

14. MacLane, S. "Categorical Algebra," Bull. Amer. Math. Soc., 71 (1965), 40-106.

15. MacLane, S. Homology, Springer-Verlag, Berlin, Academic Press, New York, 1963.

16. Milnor, J. W. and Moore, J. C. "On the Structure of Hopf Algebras," Ann. of Math., 81 (1965), 211-264.

17. Mitchell, B. Theory of Categories, Academic Press, New York, 1965.

18. Moore, J. C. "Algèbre Homologique et Homologie des Espaces Classifiants," Séminaire H. Cartan, 12e année, (1959), 7.01-7.37.

19. Swan, R. G. The Theory of Sheaves, U. of Chicago Press, Chicago, 1964.

20. Uehara, H. Mimeographed Seminar Notes on the Theory of Sheaves, Oklahoma State University, Stillwater, 1966.

VITA

Franklin S. Brenneman

Candidate for the Degree of

Doctor of Philosophy

Thesis: DERIVED FUNCTORS IN RELATIVE HOMOLOGICAL ALGEBRA

Major Field: Mathematics

Biographical:

Personal Data: Born in Dhamptri, C.P., India, February 17, 1938, the son of Frederick and Millie Brenneman.

Education: Attended elementary schools in Hesston and Moundridge, Kansas; received a high school diploma from Hesston College, Hesston, Kansas, in 1956; received the Bachelor of Arts degree from Goshen College, Goshen, Indiana, with a major in Chemistry, in June, 1960; attended The Pennsylvania State University, University Park, Pennsylvania, 1960-63; completed requirements for the Master of Arts degree in mathematics at The Pennsylvania State University, in August, 1965; was a National Aeronautics and Space Administration Trainee at Oklahoma State University, 1965-67; completed requirements for the Doctor of Philosophy degree at Oklahoma State University, Stillwater, Oklahoma, in July, 1967.

Professional Experience: Taught science and mathematics in Belleville Mennonite High School, Belleville, Pennsylvania, 1957-58; Teaching assistant in the Division of Natural Sciences, Goshen College, 1959-60; Graduate assistant in the Department of Mathematics, The Pennsylvania State University, 1960-63; Instructor in the Department of Mathematics, Susquehanna University, Selinsgrove, Pennsylvania, 1963-64; Graduate Assistant in the Department of Mathematics, Oklahoma State University, fall of 1964, 1966-67.

Organizations: Member of the Mathematical Association of America; institutional member of the American Mathematical Society; member of the American Scientific Affiliation; member of the American Association of University Professors.