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OPTIMAL FIELD PLANNING VIA DYNAMIC

PROGRAMMING TECHNIQUE

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ABSTRACT

Optimal planning or the determination of an optimal strategy for developing an oil field is considered. A mathematical model based on system theory notation is formulated to approximate the actual reservoir behavior and to simulate the development operations. This model and an optimization procedure based on dynamic programming are used to find a chronological schedule for drilling and equipping the field. The schedule obtained for the study period maximizes the profit made in the operation and satisfies the desired production requirements.

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CHAPTER I

INTRODUCTION

A specific engineering problem rarely has a unique solution. In most cases, rather, it has a multitude of solutions each of which in a practical sense is satisfactory. A measure of value can be associated with each solution by evaluating some economic indicator of its performance (such as the cost index, index of performance, etc.). The value of the economic indicator will probably be different for each solution, and the criterion for determining which solution is best is based on maximizing or minimizing it. The solution corresponding to the best value of the economic indicator is called the optimum solution. The process of searching for the optimum solution out of all possible ones is referred to as optimization.[21] This process is taking on an extremely important role in all the areas of engineering.

Some of the reasons behind searching for an optimized solution of operation includes man's desire to perform in the best possible fashion and a changing world in which he is faced with problems of overpopulation, pollution of all types and an ever-shrinking supply of natural resources. Furthermore, in the presence of severe competition, only those engineering solutions which are optimal in some defined sense can be accepted if a company is to survive.

Oil reservoir development, being an engineering problem and requiring a tremendous capital investment needs an optimal program for its development and operation. The optimum program here, defines a drilling and production rate schedule that results in the best economic criterion when practical constraints are in effect. One such economic criterion is the overall profit made in the operation within a given study period. The question in this case concerns the drilling and production policy that must be adopted to maximize the profit made in a field.

This apparently simple question, being associated with a great deal of uncertainty, poses a very complex problem. The problem in fact is one of strategy determination under uncertainty. Presently no definite solution to the problem exists partly because of its complexity. However, the main reason for the lack of a satisfactory solution is that so little work has been done on a problem of this type.[54]

There appear to be two ways of attacking this problem. One is to treat it in a probabilistic manner in which the uncertainty is recognized explicitly. A second way is the simulation method in which the physical problem is represented by a suitable model.

Conventional mechanistic models have been developed in the literature to describe the behavior of the petroleum reservoirs. A model of this type approximates the internal mechanics of the pool by a combination of such tools as a material balance equation and Darcy's law. The usefulness of such models are normally marred by the presence of parameters whose values are not accurately known. Examples of these uncertain values are the amount of original oil and gas in place, rate of water invasion into the reservoir and properties of the formation rock.

An alternative approach in establishing a model is the systems theory. In this technique the reservoir is considered to be a dynamic system whose behavior is described by a set of differential equations. The coefficients of the equations are to be obtained from the information on production and pressure. These coefficients can be updated as time elapses. In the absence of accurate data the coefficients may be obtained by using the mechanistic equations. In either event this unconventional modeling approach does not require specification of uncertain parameters. One has to compute the coefficients of the state differential equations, instead. Since identification of the coefficients is more convenient and a great deal of optimization literature is closely associated with modern systems theory, the latter approach will be followed in this research.

The object of the report would then be the determination of an optimal development plan for an oil field which is operated as a unit under the control of a single operator.

When using the systems theory terminology, the field is the system whose development operation is to be optimized. A set of algebraic and ordinary differential equations is chosen to define the state of the field in its various stages of development and depletion

The equations chosen to represent the reservoir behavior are either algebraic equations or linear differential equations whose constant coefficients may be updated as time elapses. The economic criterion function selected to judge the various possible solutions is the total profit made in the operation within a given study period.

Having the development model, the criterion function and the practical constraints, the dynamic programming technique of optimization is applied to determine the best development schedule for the field.

Previous Research

The oil industry became aware of optimization procedures through the work of Charnes, Cooper and Mellon.[13]

Their achievement was applied to the optimization of refining processes. They applied linear programming to obtain an optimum solution to the problem of blending aviation gasoline. A complete refinery operation was also optimized and presented by Conway.[16]

Many segments of the oil industry were reported to be using optimization techniques to determine "best" operating plans for their operations. Garvin, Crandall, John and Spellman surveyed a number of areas of the oil industry in which linear programming was being applied to obtain optimal policies.[24] In the area of production they gave an example from the Arabian American Oil Company involving production from several oil fields. The authors demonstrated a method for obtaining a production schedule for each field so that their composite behavior satisfied a commitment on the total production from all reservoirs (such as keeping a pipeline full or a refinery supplied). The objective was to determine the production schedule for each field so that the profit over a given number of years became a maximum. The model, the constraints and the objective function were all assumed to be linear. The linear programming technique, therefore, was applicable for the solution of the problem.

A great number of linear programming applications noted by these authors were in the area of petroleum refining optimization. An example, contributed by the Atlantic

Refining Company, was related to a refinery producing gasoline, furnace oil and other products. A large number of crude oils with different properties and yielding different volumes of finished products, could supply the refinery. Specific volumes of some of the crudes had to be refined to satisfy some of the requirements of the special products. From the remaining crudes, which were available, volumes were to be selected which could supply the required products most economically. Marketing and distribution were also reported as other segments of the oil industry to which the linear programming technique was being applied. A great deal of work had been done in this area, particularly by oil companies.

The authors concluded that linear programming had made a place for itself in the oil industry, particularly in the manufacturing phase. They also mentioned that not everything in this world is linear and therefore a great deal more basic research on optimization methods in the universities and industrial laboratories was needed.

Later work related to the optimal scheduling done by Aronofsky and Lee pertained to crude oil production.[7] As an example the authors started with the scheduling of shipments of a commodity from a number of sources to a number of destinations. According to the authors, if shipments were to be made from, say, ten sources to several hundred destinations, even a competent and experienced scheduler might find it difficult to obtain a reasonable solution. Such a problem requires choosing from a very large number of combinations, all of them being possible solutions. Without access to modern digital computers little could be done with such problems. The answer to the problem, however, was easily obtained by the use of linear programming technique and utilization of high speed digital computers.

The authors then proceeded to apply the optimization techniques to a production scheduling problem. Admitting the fact that even the simplest reservoir behavior is nonlinear, they nevertheless elected to describe the system and the constraints by linear expressions. The problem was to determine the schedule of crude oil production from five reservoirs which, over a given period and subject to certain restrictions, results in maximum profit. They assumed that all of the reservoir parameters remained constant throughout the study period. Further assumptions included the presence of an infinite water drive and applicability of the radial flow equations. The authors divided the study period into equal time increments and then used the linear programming technique to obtain a schedule for the rate of production for all periods from each reservoir. The plan was shown to maximize the profit for the given study period.

The desirability of the extension of linear programming techniques, to the area of reservoir planning was noticed by Aronofsky and Williams.[8] The authors confirmed that the method of linear programming had been used extensively in the oil industry. They, none the less, declared that very little work had been done in extending this technique to the area of underground oil production.

It is true that even the simplest reservoir problem is nonlinear and does not lend itself readily to linear programming models. However, they contended that this same objection had been made to all the other areas to which linear programming had been applied. The research dealt with the development of a model for a group of crude oil resources, such as inland reservoirs, whose produced fluids were delivered to a facility of a known capacity. A possible interpretation of this facility was a trunk pipeline "consuming" the crude oil. The behavior of the underground reservoir was described by

$$P_o - P_w(t) = q \cdot f(t).$$

It was assumed that the function f(t) had been precalculated for discrete values of time. In the equation P_o was the initial reservoir pressure, $P_w(t)$ the flowing well pressure at any time t and q was the constant flow rate of crude oil. This expression, a relation between flowing bottom hole pressure and flow rate, was the linear model selected to simulate the reservoir behavior. A set of relations describing the practical constraints, all in linear form, were also chosen. The purpose of the optimization was to compute that production schedule for these sources over a series of time periods for which resulting profit became a maximum.

The authors then turned to solution of a somewhat different problem. In this part it was desired to obtain the optimum drilling schedule of wells in a given reservoir. The production rate of any well was assumed to decline in a specified manner. To solve the problem, the reservoir was divided into small cells and one well was associated with each cell. A decline function, $h_n(t)$ was used to define the production rate of the nth well at time t. The purpose was to determine that schedule of drilling that resulted in an optimum rate of return.

By taking a close look at the investigations surveyed so far the following observations can be made.

1. Very little effort has been spent to extend the optimization techniques to the area of reservoir planning and underground oil production.

2. The only optimization procedure which has been utilized is that of linear programming which incorporates the assumption of linearity of reservoir behavior.

Recent research in the area of reservoir optimal planning was carried out by Rowan and Warren.[54] The extension of systems notations and nonlinear models to the reservoir behavior was introduced by these authors for the first time.

The reservoir was considered as a system whose behavior was characterized by the pressure-production history or by the reservoir and fluid properties.[40] The mathematical model selected to describe the behavior of the system was the following set of equations.

1. The pressure-production relation was the first order linear differential equation

$$\frac{d\Delta P}{dt} + \gamma\Delta P = R(q - q_i)\cdot$$

In this equation γ and R were the constant coefficients.

2. The expression relating the rate of production with the number of wells was the linear relation

$$q = n_t \cdot \frac{\beta}{R} (\Delta_o - \Delta P)$$

where β was a numberical constant.

Identification of the system which in fact is equivalent to computing the numerical values of the coefficients was also carried out. The economic criterion function was the present worth of net cash flow from the operation. The purpose of the research was to find a drilling and production policy for the development operations such that the criterion function became a maximum.

Formulation of the Problem

An oil field, newly discovered or partially developed, which is operated as a unit under a single company management is considered for further development. The entire assemblage, as shown in Figure 1, is the production system whose best development plan is to be determined. To schedule such an operation a planning horizon must first be set. In this study a planning horizon of 5 years has been adopted. A crude oil market demand with a constant percentage annual increase in demand has been assumed. Within the framework of these requirements and the production system characteristics there is an optimum plan for producing and equipping the field over the study period. The selected optimality for judging various possible solutions is the total profit made in the operation. The profit function is nonlinear and reservoir properties vary with time. Thus it is necessary to use nonlinear programming procedures to solve the problem.[65,28] The planning horizon, the characteristics of the production system, the crude oil market demand function and the profit function all affect the results obtained from the optimization. However, it is not the objective of this work to systematically study the effects of these parameters on a particular system. Rather, it is intended to develop an optimization technique which can be applied to this problem. Consequently, the determination of the optimum development schedule of an oil field using the dynamic programming optimization technique is the subject of this report.

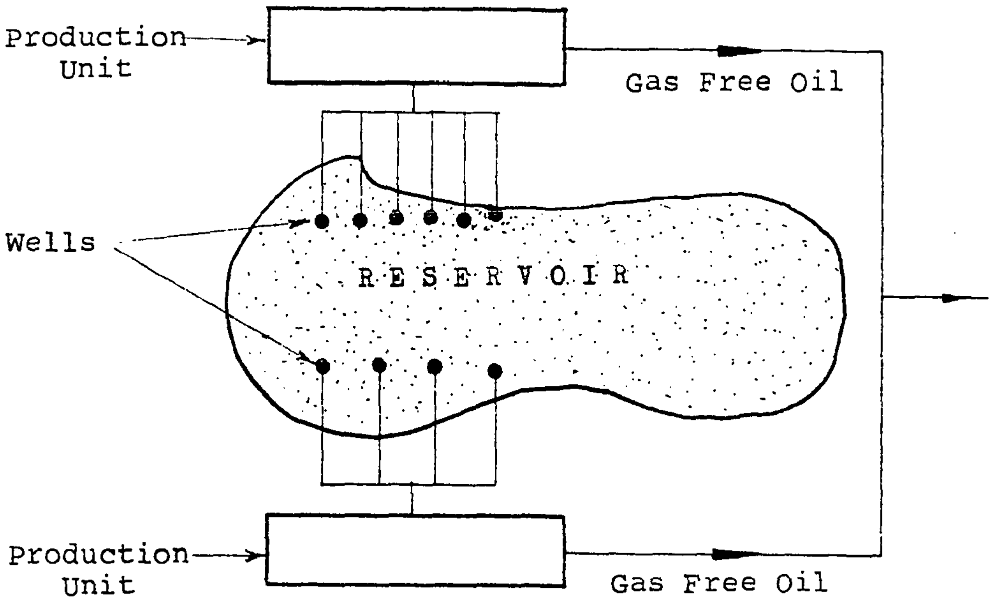

Figure 1. Schematic of the System.

CHAPTER II

THE MATHEMATICAL MODEL

To study a practical optimization problem it is necessary to represent the physical system by means of a suitable mathematical model. Furthermore, an appropriate performance criterion for the evaluation of the various possible solutions is also required. The model must describe correctly the features of the practical system in the range of the possible operating conditions, and the criterion function must be a valid representation of a practical optimality. The model, however, has to be simple enough for the manipulation and handling if it is to be of any engineering value.[27]

Representation of the practical systems by the mathematical models is, by no means, an easy task. Physical systems are, as a rule, very complex, time variant, nonlinear, and they contain sources of uncertainty. Observations and measurements being always clouded by the possibility of experimental error leave the systems undeterministic, and the physical laws used to establish the models are approximate.

Since the incorporation of all the above mentioned details is not convenient, the models can not be a complete representation of the real system. Thus in establishing a mathematical model for a system one is faced with a compromise between accuracy and complexity on one hand and approximation and simplicity on the other.

For the particular problem in hand we are faced with an oil field which we want to develop. Not enough information is available on the internal characteristics of the reservoir and the future behavior of the market. One can state, however, that any change in production will be reflected in a change in reservoir pressure and the actual magnitude of pressure change depends upon the internal mechanism of the reservoir. Furthermore, the previous behavior of the market may be extrapolated into the future. Thus the behavior of the field is a dynamic, time-varying phenomenon and presumably could be described by manipulating the following set of relations and differential equations.[40]

State Equation of Pressure

This equation as developed in appendices A and B is

$$\frac{d\Delta P}{dt} + \gamma \cdot \Delta P = R \cdot (q + q_{wp} - q_i) \qquad \text{(II-1)}$$

where

$\Delta P \quad = P_o - P_t$

$P_o \quad$ = the original reservoir pressure

$P_t \quad$ = the average reservoir pressure at time t

$\gamma \quad = \frac{K_t}{N \cdot C}$

$K \quad$ = the water influx constant as in $q_e = K_t \cdot \Delta P$

$q_{wp} \quad$ = the rate of water production

$q_e \quad$ = the rate of water influx

$q \quad$ = the rate of oil production, stock tank volume per day

$q_i \quad$ = the rate of fluid injection into the reservoir

$R \quad = 1/N \cdot C$

$N \quad$ = the volume of the original oil in place

$C \quad$ = the compressibility factor of oil

The solution of this state equation as obtained in Appendix E is

$$\frac{\Delta P}{\Delta P_o} = e^{-\int_{t_o}^{t} \gamma \cdot dt} \circ \left\{ 1 + \frac{1}{\Delta P_o} \int_{t_o}^{t} R \circ (q + q_{wp} - q_i) \cdot e^{\int_{t_o}^{\tau} \gamma \cdot dt} \circ d\tau \right\} \tag{II-2}$$

<u>State Equation of the Production Rate</u>

This equation as developed in Appendix C is

$$\frac{dq}{dt} + \xi_t \, q = \eta_t$$

where

q = the rate of oil production in stock tank volume per day

$\xi_t = \gamma + \lambda_t - \frac{1}{\lambda t} \cdot \frac{d\lambda_t}{dt}$

$\lambda t = n_t \cdot \beta_t$

n_t = the number of oil wells in production at time t

$\beta_t = \frac{A_t}{N \cdot C}$

A_t = a constant as in $q = n_t \cdot A_t \cdot [\Delta_o - \Delta P]$

$\Delta_o = P_o - P_w$

P_o = the original reservoir pressure

P_w = the flowing bottom hole pressure

$\Delta P = P_o - P_t$

P_t = the average reservoir pressure at time t

$\eta_t = (q_i - q_{wp} + \frac{\gamma \Delta_o}{R}) \cdot \lambda_t$

The solution of this equation as obtained in Appendix E, is

$$\frac{q}{q_o} = \frac{n_t \beta_t}{n_o \beta_o} e^{-\int_{t_o}^{t} (\gamma + n_t \beta_t) \cdot dt}$$

$$\cdot \left\{ 1 + \frac{n_o \beta_o}{q_o} \int_{t_o}^{t} (q_i - q_{wp} + \frac{\gamma \Delta_o}{R}) e^{\int_{t_o}^{\tau} (\gamma + n_t \beta_t) dt} d\tau \right\} \cdot$$

Productivity Equation

The following relation between the number of wells, rate of oil production, and the field pressure drop is also assumed to exist.

$$q_t = n_t \cdot A_t \cdot [\Delta_o - \Delta P]_t \qquad (II-4)$$

where

q = the rate of oil production

A_t = proportionality constant

Other variables were previously defined.

From this last relation the number of wells in production, as a function of the rate of production, could be obtained as follows:

$$n_t = \frac{q_t}{A_t \cdot [\Delta_o - \Delta P]_t} \, .$$

Number of Separation Stations

Separation stations may be designed to handle various amounts of crude oil per day. This value, denoted by Cp_t, is the stock tank volume of crude handled by the production unit per day. The necessary number of separation stations at any time then becomes

$$SU_t = \frac{q_t}{Cp_t}$$

where

SU_t = number of separation stations

q_t = the production rate of the field in terms of stock tank volume per day

CHAPTER III

COST ANALYSIS

Considering a given study period during field development, expressions are formulated to calculate the cost pertaining to the entire study period. This value is referred to as the total cost of development.

Dividing the study period into small time increments, called stages, there will be a cost corresponding to each individual stage. This latter value is denoted as the development cost of an individual stage.

Both of these two costs are composed of the cost of drilling and the cost of oil and gas separation. Expressions are developed in the following for computation of the total and the individual stage costs.

Total Cost of Development

This cost, as mentioned above, equals the cost of wells plus the cost of separation. In equation form this becomes

$$T_c = C_w + C_s \tag{III-1}$$

where

T_c = total cost

C_w = cost of wells

C_s = cost of separation.

The cost of the wells is made up of drilling and operating costs. Or

$$C_w = C_d + C_o \qquad \text{(III-2)}$$

where

C_d = drilling cost of wells

C_o = operating cost of wells.

The drilling cost is the product of the total number of wells drilled in the study period by the drilling cost of a well,

or

$$C_d = [n_t - n_{to}] \cdot C_{dw} \qquad \text{(III-3)}$$

where

n_t = total number of wells drilled to the end of the study period

n_{to} = number of wells existing at the beginning of the study period

C_{dw} = average cost of drilling a well.

The operating cost may be obtained from

$$C_o = C_{ow} \int_{\tau=t_o}^{\tau=t} \frac{q_\tau}{A_\tau [\Delta_o - \Delta P]_\tau} \cdot d\tau \qquad \text{(III-4)}$$

where

C_{ow} = operating cost of one well in a unit of time

$\frac{q_\tau}{A_\tau[\Delta_o - \Delta P]_\tau}$ is the number of producing wells at time τ.

t_o = the time at which the study period started

t = the time at which the study period ended

By summing these two costs, C_w becomes

$$C_w = C_{dw}[n_t - n_{to}] + C_{ow} \int_{\tau=t_o}^{\tau=t} \frac{q_\tau}{A_\tau[\Delta_o - \Delta P]_\tau} \cdot d\tau \qquad \text{(III-5)}$$

In order to obtain an expression for the cost of separation it is assumed that all of the production units are of the same capacity Cp. The necessary number of production units built within the study period then becomes

$$NU = \frac{q_{max} - q_{to}}{Cp} \qquad \text{(III-6)}$$

where

q_{max} = the highest production rate from the field in the study period

q_{to} = the production rate from the field at the beginning of the study period.

The cost of separation is equal to the initial cost plus the operating cost of the production units:

$$C_s = C_{is}\left[\frac{q_{max} - q_{to}}{Cp}\right] + O_{cs}\int_{\tau=to}^{\tau=t} \frac{q_\tau}{Cp}\, d\tau \qquad \text{(III-7)}$$

where

C_{is} = the initial cost of the production unit

O_{cs} = the operating cost of the production unit in a unit time.

Having C_w and C_s the expression for the total cost of development becomes

$$T_c = C_{dw}[n_t - n_{to}] + C_{ow}\int_{\tau=to}^{\tau=t} \frac{q_\tau}{A_\tau[\Delta_o - \Delta P]_\tau}\, d\tau +$$

$$C_{is}\left[\frac{q_{max} - q_{to}}{Cp}\right] + O_{cs}\int_{\tau=to}^{\tau=t} \frac{q_\tau}{Cp}\, d\tau. \qquad \text{(III-8)}$$

Development Cost of an Individual Stage

An oil field which is being developed assumes a multitude of conditions or states during a given study period. The variables which typify these states are the number of wells and separation stations present in the field. Using the previous terminology one may say that the system is transformed from one state to the next as the development operations proceed. Assuming that these development operations are carried out at discrete time points separated from each other by the constant time increment, Δ, the transformation will have the schematic presentation of Figure 2. The study period begins with the system in the Ith stage. The field contain n_I number of producing wells and SU_I number of separation stations at this time. It will be assumed that wells are drilled and separation stations are installed at the beginning of each individual stage.

The summation of the initial cost of a stage and the operating cost of the lapse time between that stage and the next one is the total operating cost of any stage. This value is termed the development cost of an individual stage. To obtain expressions for computation of the individual stage cost fųrther simplifying assumptions are made as follows:

1. The number of wells in production at any time point can be obtained from

$$n_t = \frac{q_t}{A_t[\Delta_o - \Delta P]}, \qquad \text{(III-9)}$$

where

q_t = production rate from the field in stock tank volume per day

$[\Delta_o - \Delta P]_t$ = pressure drawdown

n_t = number of wells in production

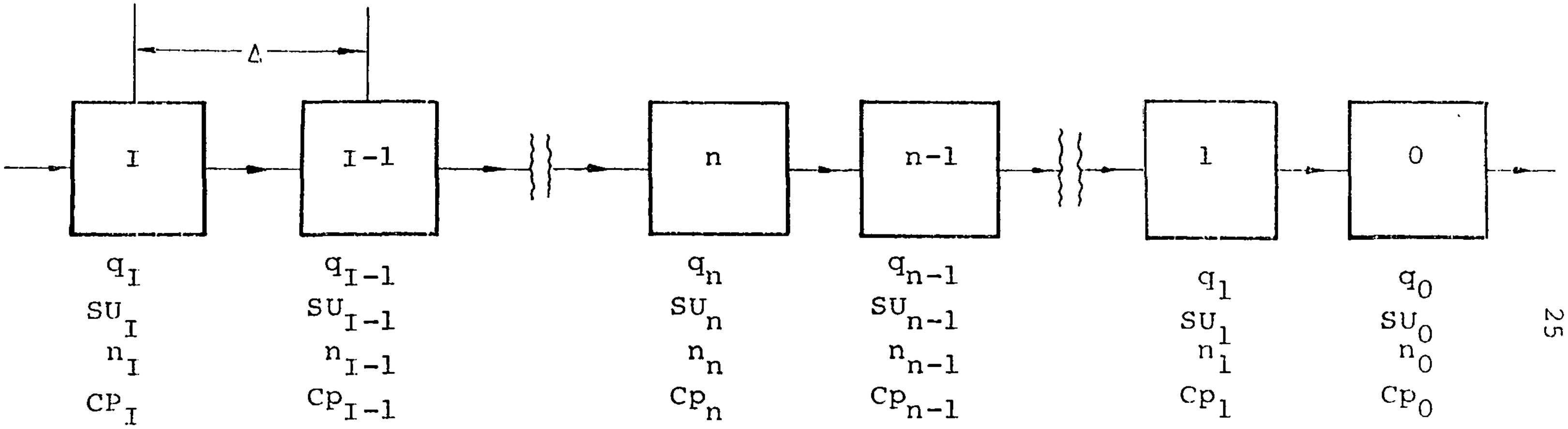

Figure 2. System's Transformation between Stages.

2. Number of production units in operation at any time t equals

$$SU_t = \frac{q_t}{Cp_t} , \qquad \text{(III-10)}$$

where

SU_t = the number of production units in operation at time t.

q_t = the rate of production from the field at time t,

Cp_t = the capacity of the production unit that began operation at time t.

Dividing the length of the study period into I equal time increments, the system goes through I different stages during the study period. Numbering these stages backward, the system starts at stage I and passes through the stages I-1, ... n, n-1, ... 3, 2, 1 and finally reaches the stage 0. Figure 3 shows the stages the system passes through during the development period. The number of time increments, I, is obtained from

$$I = \frac{\text{length of the study period}}{\text{length of the time increment}}$$

3. There will not be any drilling and installation operation in the final stage of the study period (i.e. stage 0) and, therefore, there is no corresponding cost attributable to that stage.

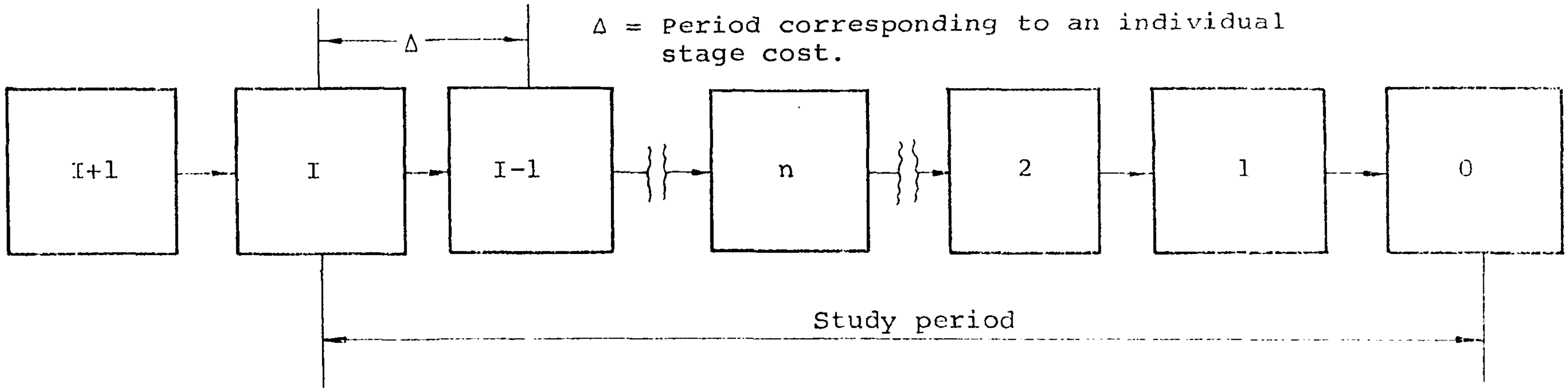

Figure 3. Study Period and the Time Increment between Stages.

Based on the above comments and assumptions various components of the development cost of an individual stage are computed in the following.

Drilling Cost of Wells

The cost of drilling at any stage of the system is

$$C_d(n) = \left[\frac{q_n}{A_n[\Delta_o - \Delta P]_n} - \frac{q_{n+1}}{A_{n+1}[\Delta_o - \Delta P]_{n+1}} \right] C_{dw},$$

$$n = I,\ I-1, \ldots, 3, 2, 1. \qquad \text{(III-11)}$$

where

q_n = the rate of production from the field

$[\Delta_o - \Delta P]_n$ = the pressure drawdown

A_n = the productivity constant.

All of the three terms mentioned above pertain to the time point, n.

C_{dw} = the drilling cost of a well

$C_d(n)$ = the drilling cost incurred when the system is at the stage, n.

When the term inside the bracket has a positive sign, drilling of new wells may be necessary.

Operating Cost of Wells

Wells in production at the nth stage have to be operated for a Δ lapse of time. After which the system enters (n-1)th stage. Thus the operating cost of the wells can be formulated as

$$C_o(n) = \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot C_{ow},$$

$$n = I,\ I-1,\ \ldots.,\ 3,\ 2,\ 1. \qquad \text{(III-12)}$$

where

$\frac{q_n+q_{n+1}}{2}$ = the average production rate of the field within the time increment Δ.

Δ = the time increment between the stages.

C_{ow} = the operating cost of a well for one unit of production.

$C_o(n)$ = the operating cost of wells while the system passes from stage n to stage n-1.

Total cost pertaining to the wells is the summation of the two equations (III-11) and (III-12).

$$C_w(n) = \left[\frac{q_n}{A_n[\Delta_o - \Delta P]_n} - \frac{q_{n+1}}{A_{n+1}[\Delta_o - \Delta P]_{n+1}}\right] \cdot C_{dw} +$$

$$\frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot C_{ow},\ n = I,\ I-1,\ \ldots.,\ 3,\ 2,\ 1.$$

$$\text{(III-13)}$$

Initial Cost of a Separation Station

As mentioned before, the number of production units in operation at any stage is q_t/Cp_t. Additional number of production units, required at the nth stage would then be

$$\frac{q_n}{Cp_n} - \frac{q_{n+1}}{Cp_{n+1}} .$$

Thus the initial cost of separation units at the nth stage becomes

$$C_i(n) = \left[\frac{q_n}{Cp_n} - \frac{q_{n+1}}{Cp_{n+1}}\right] \cdot C_{is}, \quad n = I, I-1, \ldots, 3, 2, 1. \tag{III-14}$$

where

C_{is} = the installation cost of one production unit

$C_i(n)$ = the initial cost of separation units at the nth stage

Operating Cost of a Separation Station

The operating cost of separation stations when the system is in the nth stage of development would be

$$C_{os}(n) = \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot O_{cs}, \quad n = I, I-1, \ldots, 3, 2, 1. \tag{III-15}$$

where

O_{cs} = the operating cost of a production station for one unit of production

Δ = the time interval between any stage and the subsequent one

$C_{os}(n)$ = the operating cost of production unit incurred in the nth stage of development

$\frac{q_n + q_{n+1}}{2}$ = the average rate of production from the field within the time increment

Total cost of oil and gas separation is the summation of the two equations (III-14) and (III-15).

$$C_s(n) = \left[\frac{q_n}{Cp_n} - \frac{q_{n+1}}{Cp_{n+1}}\right] \cdot C_{is} + \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot O_{cs},$$

$$n = I, I-1, \ldots . 2, 1. \qquad \text{(III-16)}$$

Overproduction Costs

Any production exceeding that of the market demand is not desirable. Excessive production requires storage facilities and brings about handling expenses. Furthermore, it may waste the reservoir energy and the potential reserves. In order to eliminate the additional costs and discourage excessive production, an overproduction cost is considered. This term is assumed to be directly proportional to the square of excessive production, $(q_n - MD_n)$. The expression,

$$C_{op}(n) = O_c \cdot (q_n - MD_n)^2 \cdot \Delta, \; n = I, I-1, \ldots, 3, 2, 1 \qquad \text{(III-17)}$$

is chosen for computation of this cost. O_c in this expression is the constant of proportionality, Δ is the time interval between stages, and MD_n is the market demand.

$C_{op}(n)$ is the cost of overproduction in the nth stage of development.

Addition of all the terms so far computed composes the total expense incurred at any stage of the field development. The sum is

$$T_c(n) = \left[\frac{q_n}{A_n[\Delta_o - \Delta P]_n} - \frac{q_{n+1}}{A_{n+1}[\Delta_o - \Delta P]_{n+1}}\right] \cdot C_{dw} +$$

$$\frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot C_{ow} + \left[\frac{q_n}{Cp_n} - \frac{q_{n+1}}{Cp_{n+1}}\right] \cdot C_{is} +$$

$$\frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot O_{cs} + O_c(q_n - MD_n)^2 \cdot \Delta ,$$

$$n = I, I\text{-}1, \ldots 2, 1. \qquad \text{(III-18)}$$

$T_c(n)$ in this relation is the development cost of an individual stage.

Total development cost of the entire study period is the summation of $T_c(n)$ terms of all the stages from $n = I$ to $n = 1$. A concise notation would be

$$T_c = \sum_{n=I}^{n=1} T_c(n).$$

Profit Function

Assuming that the only revenue in the operation comes from the sale of crude oil, the amount of income at at the nth stage would be

$$I(n) = \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot P_{pc}$$

P_{Pc} in this relation is the price of one barrel of crude oil paid to the company at the port.

The nth stage profit $T_p(n)$ is computed from

$$T_p(n) = I(n) - T_c(n)$$

$$T_p(n) = \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot P_{pc} - \Bigg\{ \left[\frac{q_n}{A_n[\Delta_o - \Delta P]_n} - \frac{q_{n+1}}{A_{n+1}[\Delta_o - \Delta P]_{n+1}} \right] \cdot C_{dw} + \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot C_{ow} + \left[\frac{q_n}{Cp_n} - \frac{q_{n+1}}{Cp_{n+1}} \right] \cdot C_{is} + \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot O_{cs} + O_c \cdot (q_n - MD_n)^2 \cdot \Delta \Bigg\}. \qquad \text{(III-19)}$$

Total profit pertaining to the entire study period then becomes

$$T_p = \sum_{n=I}^{n=1} T_p(n). \qquad \text{(III-20)}$$

Variables of this function are subject to practical constraints such as mentioned below.

The production rate should not exceed the current market demand.

In many cases it is necessary to maintain a lower limit of production from the field. Examples of this are the situations when we have to keep a pipeline full or a refinery supplied.

Production units require large investments and can not be economically designed for capacities lower than a minimum value.

Cumulative pressure drop of the field is a positive value for all times.

The restrictions on the variables are further explained in Chapter V of this report.

The purpose of optimization is to maximize the total profit T_p subject to the practical constraints of the field.

CHAPTER IV

DESCRIPTION OF PARAMETERS

The expression obtained for the individual stage profit is

$$T_p(n) = \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot P_{pc} - \left\{ \left[\frac{q_n}{A_n[\Delta_o - \Delta P]_n} - \frac{q_{n+1}}{A_{n+1}[\Delta_o - \Delta P]_{n+1}} \right] \cdot C_{dw} + \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot C_{ow} + \left[\frac{q_n}{Cp_n} - \frac{q_{n+1}}{Cp_{n+1}} \right] \cdot C_{is} + \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot O_{cs} + O_c \cdot (q_n - MD_n)^2 \cdot \Delta \right\}. \qquad \text{(IV-1)}$$

Total profit of the operation pertaining to the entire study period is the summation of the profits made in all of the stages, i.e.

$$T_p = \sum_{n=I}^{n=1} T_p(n). \qquad \text{(IV-2)}$$

Prior to the commencement of actual maximization of the profit function, one has to find numerical values or suitable functions for the parameters in the expression for the total profit. Description and/or evaluation of these parameters are carried out in the present chapter.

1. q_n

q_n is the rate of oil production in terms of stock tank volume at the time point n. It is one of the variables based on which the maximization is to be carried out.

2. Δ_o

Δ_o is the difference between the original reservoir pressure P_o and the flowing bottom hole pressure P_w. Since P_w is assumed to be constant throughout the study period Δ_o would also be a constant value. In the form of an equation we will have

$$\Delta_o = P_o - P_w.$$

3. ΔP_n

ΔP_n is defined by the following expression

$$\Delta P_n = P_o - P_n.$$

Here P_o is the original reservoir pressure and P_n the average reservoir pressure at time n.

4. A_n

A_n is a parameter that possibly varies with time and was defined through the expression

$$q_t = n_t \cdot A_t[\Delta_o - \Delta P]_t \qquad \text{(IV-3)}$$

n_t in this relation is the number of wells in production at time t. The equation could also be written as

$$q_t = n_t \cdot A_t [P_o - P_w - (P_o - P)]_t$$

$$q_t = n_t \cdot A_t [P - P_w]_t \qquad \text{(IV-4)}$$

Solving for A_t from this expression yields

$$A_t = \frac{q_t}{n_t \cdot [P - P_w]_t}$$

or

$$A_t = \frac{1}{n_t} \cdot \frac{q_t}{[P - P_w]_t} .$$

The definition of productivity index, on the other hand, is [17,52]

$$J = \frac{q}{P - P_w} \qquad \text{(IV-5)}$$

Dividing both sides of this relation by n_t, there will result

$$\frac{J}{n_t} = \frac{1}{n_t} \cdot \frac{q}{P - P_w} .$$

Thus one concludes that

$$A_t = \frac{J}{n_t} . \qquad \text{(IV-6)}$$

In this relation J is the productivity index, and n_t is the number of wells in production at time t.

5. C_{dw}

This item denotes the drilling cost of one well. The best source from which this information may be obtained is past experience. Data collected from the wells previously drilled in the same or a similar field could be of great assistance. Figures for the cost of drilling are also gathered in the literature.[34] This information is available for different areas of the country. Tables1 and 2 show several values for the drilling cost of a well.[32, 35] The change in the cost of drilling is normally surveyed and reported annually in the industry's publications.[4,1,5] Extensive data on the cost of equipment, labor, and utilities abroad as compared to the United States costs are gathered in the literature. Various articles by Nelson in the Oil and Gas Journal and its reprints can be helpful in estimating the cost of overseas operations.[46, 47]

6. Δ

This item is the time increment selected to divide the study period into equal intervals. For the purpose of optimal planning this time increment is taken to be six months. Since most of the modifications and preparations in an oil field need about a few months to be carried out, a six month interval seems appropriate. It is possible, however, to shorten the time step size and detect any change in the results obtained. Emergence of drastic changes in the results is an indication of too large a time increment

TABLE 1

DEEP WELL DRILLING COST

Area	Number of Wells	Total Footage (feet)	Total Drilling Cost (dollars)
Louisiana onshore	488	8,257,633	329,238,989.00
Louisiana offshore	186	2,998,695	184,058,766.00
West Texas	121	2,231,312	133,475,168.00
Texas (remainder)	130	2,301,370	98,756,991.00
Mississippi	53	879,442	24,724,387.00
Oklahoma	39	652,791	21,286,723.00
New Mexico	7	124,783	6,924,562.00
Total	1,024	17,446,029	798,465,588.00

Average drilling cost/ft = $45.76

TABLE 2

AVERAGE DRILLING COST/FT (IN DOLLARS)

Area	Average Drilling Cost/ft (dollars)
Mississippi	28.11
Oklahoma	32.61
Central Texas Gulf Coast	38.89
Louisiana Intermediate	39.27
Central Louisiana	40.46
South Texas Gulf Coast	40.53
Upper Texas Gulf Coast East Texas	44.63
Louisiana Above Intermediate	45.16
New Mexico	55.49
West Texas	57.25
West Louisiana offshore	61.25
East Louisiana offshore	60.68
Texas Panhandle	88.49

in the previous run. It is preferred, then, to repeat the computations with the shorter time increment.

7. C_{ow}

C_{ow} is the operating cost of a well for one unit of production. The operating cost in this report refers to the overall cost which is composed of the costs of running, testing, maintenance and possibly reworking of the wells. Furthermore, salaries paid to the employees of the related sections, such as geological and petroleum engineering sections, are also considered to be parts of this operating cost. To obtain realistic values for the operating cost one has to make use of previous experience in the same field or a similar one. Based on the past data, an average figure for the operating cost can be obtained to be used for the entire study period. Furthermore, one can project these data into the future and compute values for any particular time interval. In the case of the absence of all the previous information one may use the estimates available in the literature. An expense of $1 per barrel or 25% of the net yearly income of the well are the operating costs estimated by some of the cost analysts.[33]

8. Cp_n

The variable Cp_n is the capacity of the production unit at the nth time interval of the study period. It is the number of barrels of stock tank crude oil processed by the separation station daily.

The capacity of the oil and gas separation facilities can vary within a wide range of values at the operator's option. To obtain the best processing capacity it was decided to consider it as a second variable of the profit function. The optimization is then carried out with respect to both the production rate from the field and the capacity of the production unit. The numerical value obtained for the capacity has to be a positive one.

The answer obtained from the optimization calculations would be the optimal capacity of the production unit which is to be installed in the nth time interval of the study period.

9. C_{is}

Initial cost of the separation station C_{is}, could be obtained from the past experiences. Quotations pertaining to the units build previously may be utilized to estimate the cost of the unit at hand. These values are updated and modified for changes with time and differences in capacity, as described below:

1. Previous data on a similar production unit are based on the economic conditions at some time in the past. Some method must be used to convert the costs applicable to a past date to equivalent values at the present time.[61] Several cost indexes are published regularly in the literature for this purpose.[20, 46, 58] A cost index for a given

year is a number which shows the cost at that time relative to the cost at a base year. If the cost at some time in the past is known, the present cost can be obtained from

$$\text{Present cost} = \text{previous cost} \cdot \frac{\text{index at the present time}}{\text{previous index}}$$

2. The cost of a unit with a different capacity can be estimated when the cost of a given unit is known.[50] The concept is known as the six-tenths factor rule according to which[14, 63]

$$\text{cost of unit B} = \text{cost of unit A} \cdot \left(\frac{\text{Capacity of Unit B}}{\text{Capacity of Unit A}}\right)^{.6} .$$

Using these ideas and utilizing some of the quotations given in manufacturer's catalogs, the initial cost of a production unit with a capacity of 150,000 BPD is estimated at \$500,000.00. The cost of a similar plant with a capacity of Cp_n then becomes

$$C_{is} = 500{,}000 \left(\frac{Cp_n}{150{,}000}\right)^{.6}$$

$$C_{is} = \frac{500{,}000}{1270} (Cp_n)^{.6}$$

$$C_{is} = 394 \; (Cp_n)^{.6}$$

Cp_n in this expression is in terms of barrels per day and C_{is} in dollars.

10. O_{cs}

Using the idea of operating cost per barrel of oil as discussed by Nelson, one can obtain a value for the operating cost of the production unit.[47] Owing to the absence of more specific data, an operating cost of 20 cents per barrel is assumed for the numerical computations. More accurate data can be obtained from past experience.

CHAPTER V

THEORY

As discussed in the previous chapters, total profit made in the operation can be expressed by

$$T_p = \sum_{n=I}^{n=1} T_p(n). \qquad \text{(V-1)}$$

In this expression n is a decreasing integer index running between I and 1, and $T_p(n)$ is the total profit made in the nth stage of development. The value of $T_p(n)$ is computed by using the relation given below.

$$T_p(n) = \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot P_{pc} - \left\{ \left[\frac{q_n}{A_n[\Delta_o - \Delta P]_n} - \frac{q_{n+1}}{A_{n+1}[\Delta_o - \Delta P]_{n+1}} \right] \cdot C_{dw} + \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot C_{ow} + \left[\frac{q_n}{Cp_n} - \frac{q_{n+1}}{Cp_{n+1}} \right] \cdot C_{is} + \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot O_{cs} + O_c \cdot (q_n - MD_n)^2 \cdot \Delta \right\}. \qquad \text{(V-2)}$$

It is desired to obtain a particular combination of Cp_n and q_n for all stages, so that the total profit, T_p, becomes a maximum. Presence of the practical constraints limit the range of values that can be assumed by the variables Cp_n and q_n. The restrictions on the variables are discussed below.

1. q_n and Cp_n are positive values for all times.

2. The production rate of the field should never exceed that of the current market demand. Market demand values can be obtained by using forecasting techniques which will not be covered here. A simple case of the market trend is when a constant percentage yearly increase in demand prevails.

If the percentage increase is denoted by d, we will have

$$\frac{dMD_n}{MD_n \cdot dt_n} = d$$

or

$$\ln MD_n = d \cdot t_n + \ln C$$

in which $\ln C$ is the constant of integration and MD_n is the market demand pertaining to the nth stage of development. The expression could be simplified as

$$\ln (MD_n/C) = d \cdot t_n$$

Assuming that the market demand at the initial time t_I, is MD_I, the constant C becomes

$$C = \frac{MD_I}{e^{d \cdot t_I}}$$

Substituting for C in the previous expression it becomes

$$\ell n\left(\frac{MD_n}{MD_I/e^{d\cdot t_I}}\right) = d\cdot t_n$$

or

$$\frac{MD_n\cdot e^{d\cdot t_I}}{MD_I} = e^{d\cdot t_n}$$

and finally

$$MD_n = MD_I\cdot e^{d\cdot(t_n-t_I)}$$

The value of t_n-t_I according to Figure 4 is $(I-n)\cdot\Delta$ and

$$MD_n = MD_I\cdot e^{d\cdot[(I-n)\cdot\Delta]}. \qquad (V\text{-}3)$$

The rate of production of the field should be less than or equal to the demand value, MD_n.

Thus

$$q_n \le MD_I\cdot e^{d[(I-n)\cdot\Delta]}.$$

3. In general there will be more than one production unit in the field and the capacity of the production unit is less than or equal to the total production rate of the field, i.e.

$$cp_n \le q_n.$$

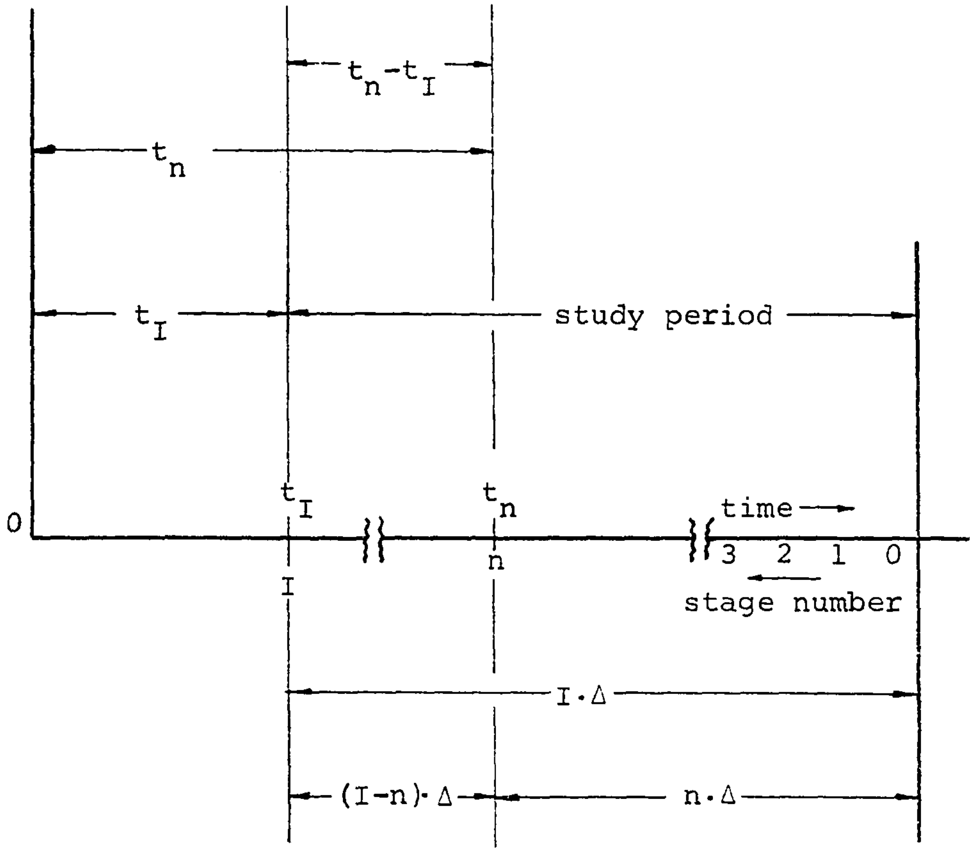

Figure 4. Time Interval between the Ith and the nth Stage.

Production units requiring fairly large investment can not be designed for capacities lower than a minimum value, Cp_{min}. Assuming that this minimum capacity is 50,000 barrels per day we will have

$$Cp_n > 50,000.$$

Similarly the lower limit for the rate of production of the field is assumed to be equal to 20 per cent of its upper limit. Thus

$$q_n > .20 \cdot MD_n$$

In this study the development process of the field is assumed to be of discrete and deterministic nature. The decisions are made at the beginning of each individual stage. The behavior of the system during each stage depends upon the condition of the system at the time of decision and upon the decision itself. Therefore each decision affects the next immediate stage and all the other subsequent stages of development.

Maximization of T_p under the conditions imposed by the above constraints and assumptions is a multistage decision making process which will be solved by using the dynamic programming optimization technique. The obtained solution is a sequence of values for the variables q_n and

Cp_n for all the stages of development which maximizes the total profit made in the operation.

The Recursion Equation of Dynamic Programming

The application of dynamic programming to a problem of this nature is carried out by the use of the principle of optimality. According to this principle any optimal policy has the property that the remaining decisions must constitute an optimal policy with regard to the state resulting from the previous decision.[10, 48]

By use of this principle a multistage decision process can be translated into a number of one-stage subproblems. Other concepts necessary for this transformation are

1. The resulting profit at any stage does not depend on the decisions made afterwards. This simply means that the decisions made in a later date do not affect the previously decided developments.

2. The maximum value of the function $h_1(u_1) + h_2(u_1, u_2)$ with respect to u_1 and u_2 could be obtained by

$$\max_{u_1} [h_1(u_1) + \max_{u_2} h_2(u_1, u_2)].$$

In the form of an equation

$$\max_{u_1,u_2} [h_1(u_1)+h_2(u_1,u_2)] = \max_{u_1}[h_1(u_1)+\max_{u_2} h_2(u_1,u_2)] \tag{V-4}$$

Bringing the maximization with respect to u_2 inside the outside bracket is a crucial step and results from the fact that h_1 is a function of u_1 only.

3. There are I stages in the study period each one of them having two variables. Maximization of T_p starting with the present condition of the field is equivalent to finding 2 I unknown variables. However, if we start from the final condition there would be only two unknown variables which could be obtained by maximizing the final stage profit function. Knowing the optimum values of the variables of the final stage we can proceed step by step to the initial state of the system and find all the other variables. This procedure is referred to as the backward multistage problem solving and will be followed in this research. If we want to have the subscripts of all the stages to agree with the order in which the variables are determined, it is necessary to number the stages in reverse order. Therefore the backward indexing is being followed throughout this work.

Utilizing these ideas to the profit function, T_p, we will have [11, 19]

$$\max T_{p,I} = \max_{q_I, \dots q_1, Cp_I \dots Cp_1} [T_p]$$

$$\max T_{p,I} = \max_{q_I, \dots q_1, Cp_I \dots Cp_1} \left[\sum_{n=I}^{n=1} T_p(n) \right]$$

$$\max T_{p,I} = \max_{q_I, Cp_I} \left\{ T_p(I) + \max_{q_{I-1}, \ldots q_1, Cp_{I-1}, \ldots Cp_1} \left[\sum_{n=I-1}^{1} T_p(n) \right] \right\},$$

and finally,

$$\max T_{p,I} = \max_{q_I, Cp_I} \left\{ T_p(I) + \max_{q_{I-1}, \ldots q_1, Cp_{I-1}, \ldots Cp_1} [T_p(I-1) + \ldots + T_p(1)] \right\}.$$

$\max T_{p,I}$ in this relation is the maximum profit made in the development operations of the field from the beginning of the Ith stage to the end of the study period. It could be noticed from the last relation that the original problem is transformed into two smaller optimization problems as shown below.

The first one is an (I-1)-stage optimization problem, namely

$$\max_{q_{I-1}, \ldots q_1, Cp_{I-1}, \ldots Cp_1} [T_p(I-1) + T_p(I-2) + \ldots + T_p(1)].$$

Using the maximum profit notation this item could be denoted by $\max T_{p,I-1}$. The second problem is a one-stage optimization which is

$$\max T_{p,I} = \max_{q_I, Cp_I} \left[T_p(I) + \max_{q_{I-1}, \ldots q_1, Cp_{I-1}, \ldots Cp_1} [T_p(I-1) + \ldots T_p(1)] \right]$$

or

$$\max T_{p,I} = \max_{q_I, Cp_I} [T_p(I) + \max T_{p,I-1}]. \qquad \text{(V-5)}$$

This final form is the recursion equation of dynamic programming which will be used for computing the optimal development policy. The solution to this equation could be obtained by a single stage optimization operation only when the value of $\max T_{p,I-1}$ is already defined. The value of $\max T_{p,I-1}$ could be obtained by the repeated use of the same recursion equation as given below.

$$\max T_{p,I-1} = \max_{q_{I-1}, Cp_{I-1}} [T_p(I-1) + \max T_{p,I-2}]$$

$$\max T_{p,I-2} = \max_{q_{I-2}, Cp_{I-2}} [T_p(I-2) + \max T_{p,I-3}]$$

$$\vdots$$

$$\max T_{p,3} = \max_{q_3, Cp_3} [T_p(3) + \max T_{p,2}]$$

and finally

$$\max T_{p,2} = \max_{q_2, Cp_2} [T_p(2) + \max T_{p,1}].$$

since it had been assumed that the stage number 1 was the final time increment in which any development operation took place, the value of max $T_{p,1}$ can be computed without using the recursion equations. Furthermore, since Tp,1 pertains to the final stage of development it is equal to $T_p(1)$. In other words

$$T_{p,1} = T_p(1) = \frac{q_1 + q_2}{2} \cdot \Delta \cdot P_{pc} - \left\{ \left[\frac{q_1}{A_1[\Delta_c - \Delta P]_1} - \frac{q_2}{A_2[\Delta_o - \Delta P]_2} \right] \cdot \right.$$

$$C_{dw} + \frac{q_1 + q_2}{2} \cdot \Delta \cdot C_{ow} + \left[\frac{q_1}{Cp_1} - \frac{q_2}{Cp_2} \right] \cdot C_{is} + \frac{q_1 + q_2}{2} \cdot$$

$$\left. \Delta \cdot O_{cs} + O_c \cdot (q_1 - MD_1)^2 \cdot \Delta \right\} . \qquad \text{(V-6)}$$

q_1, q_2, ΔP_1, ΔP_2, Cp_1 and Cp_2 are unknown values in this equation. By maximizing the function one can obtain numerical values for two of these variables. Therefore, it is necessary to have realistic estimates for q_1 and ΔP_1. Knowing ΔP_1 and using the state equation of pressure, ΔP_2 can be calculated. Furthermore, we may assume that $Cp_1 = Cp_2$. Under these conditions, numerical optimization techniques can be followed to compute the optimum values for the variables. Values thus obtained increase the profit of the operation to its highest feasible level while meeting the conditions imposed by the constraints. A discussion related to these numerical techniques will follow in the next section.

Maximization of the Profit

The recursion equation of the previous section involves the maximization of the individual profit functions of all the stages. The variables in these functions are the rate of oil production from the field and the capacity of the production units. General form of the profit function and a set of pertaining constraints are given below.

$$T_p(n) = \frac{q_n + q_{n+1}}{2} \cdot \Delta \cdot P_{pc} - \left\{ \left[\frac{q_n}{A_n [\Delta_o - \Delta P]_n} - \frac{q_{n+1}}{A_{n+1} [\Delta_o - \Delta P]_{n+1}} \right] \cdot C_{dw} + \left[\frac{q_n}{Cp_n} - \frac{q_{n+1}}{Cp_{n+1}} \right] \cdot C_{is} + \Delta \cdot \frac{q_n + q_{n+1}}{2} (C_{ow} + O_{cs}) + O_c \cdot (q_n - MD_n)^2 \cdot \Delta \right\}$$

with

$$q_n \leq q_I \cdot e^{d[(I-n) \cdot \Delta]}$$

$$q_n > 0$$

$$Cp_n > 0$$

$$Cp_n \le q_n$$

$$q_n > \cdot 20 * q_I \cdot e^{[(I-n)\cdot\Delta]}$$

$$Cp_n > 50{,}000 \text{ B.P.D.} \qquad \text{(V-7)}$$

The profit function in its concise mathematical notation form is expressed as

$$T_p(n) = T_p(q_n, Cp_n). \qquad \text{(V-8)}$$

For a differentiable $T_p(n)$ the Lagrange multiplier optimization technique can be applied.[23, 18] An alternative approach which is more conveniently adaptable to numerical computations and nondifferentiable functions is the multidimensional geometry technique. The method is a trial and error procedure for computation of a set of operating conditions yielding a value of the criterion function which is close to the best possible. Every effort is made to utilize the basic concepts of multidimensional algebra to design an experimental plan in which the search will be completed with a minimum number of trials. Ordinarily a search plan of this type is comprised of three phases. At the beginning when nothing at all is known about the function one has to explore the situation by running randomly chosen experiments. In the middle of the search, an effort is made to get close to the optimum point with as few experiments as possible. Finally when close to the optimum, further exploration becomes necessary to attain any improvement in the situation.

Beginning Experiments

Looking at the profit function, $T_p(n) = T_p(q_n, Cp_n)$ it could be noticed that for each pair of values on the q_n and Cp_n axis, we can compute a value for $T_p(n)$. A series of these pairs sketches a surface in the three dimensional space of q_n, Cp_n and $T_p(n)$. This surface will be called the profit surface or the criterion function. Since $T_p(n)$ is a complicated function we have no advance information about the shape of the surface. A group of opening trials, therefore, are necessary to show the corresponding elevations on the criterion surface. Based on the information obtained from these initial experiments it becomes possible to find the way to the maximum point of this surface. To begin the experiments we start with an arbitrarily chosen point in the $q_n - Cp_n$ plane. Caution must be exercised, however, to select this point in the region where all the constraints are satisfied. Denoting this point by (q_{no}, Cp_{no}) and substituting these values in the profit function the corresponding profit, $T_{po}(n)$, (shown in Figure 5) is computed. In order to locate the position of the following trials it is necessary to know the general slope of the criterion

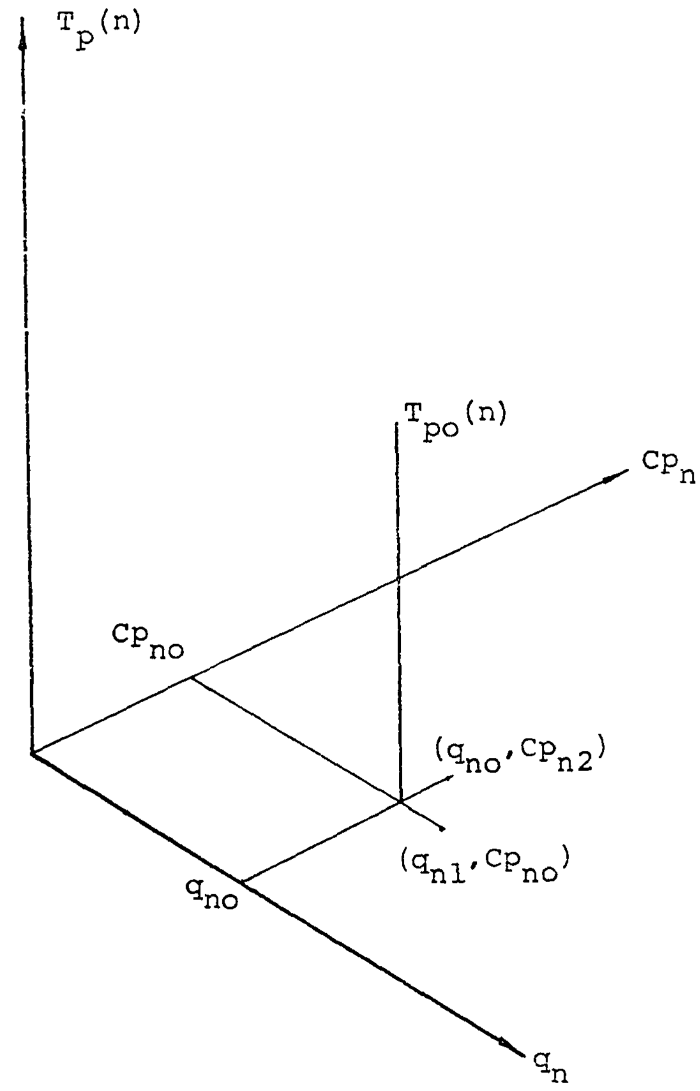

Figure 5. Three Dimensional Space of q_n, Cp_n and $T_p(n)$. "Beginning Experiments"

function in the neighborhood of $T_{po}(n)$. To find the slope of the profit surface in the q_n direction, an experiment with $Cp_1 = Cp_o$ is run. The value of q_n should, however, be slightly different from q_{no}. The new trial would then be at the point (q_{n1}, Cp_{no}). The slope of the criterion function in the direction parallel to q_n becomes:

$$\left(\frac{\partial T_p(n)}{\partial q_n}\right)_o = \frac{T_{p1}(n) - T_{po}(n)}{q_{n1} - q_{no}} \qquad \text{(V-9)}$$

In this relation $T_{p1}(n)$ is the elevation on the profit surface corresponding to the point (q_{n1}, Cp_{no}). A similar experiment with the previous q_n and slightly different Cp_n furnishes the slope of the surface in the direction parallel to the Cp_n axis. This third experiment then has the property of

$$q_{n2} = q_{no}$$

$$Cp_{n2} \neq Cp_{no}$$

and is positioned at the point (q_{no}, Cp_{n2}). Using the information obtained from this last trial the slope of the surface in the Cp_n direction becomes

$$\left(\frac{\partial\ T_p(n)}{\partial Cp_n}\right)_o = \frac{T_{p2}(n) - T_{po}(n)}{Cp_{n2} - Cp_{no}}\ . \qquad \text{(V-10)}$$

The slope thus obtained pertains to the locations in the neighborhood of the initial point (q_{no}, Cp_{no}). $T_{p2}(n)$ is the value of the profit function calculated in the third experiment. Having these data, it becomes possible to determine the plane approximately tangent to the profit surface. The equation of such a tangent plane could be found with the aid of deviations of q_n, Cp_n and $T_p(n)$ from the initial point, (q_{no}, Cp_{no}). It could be proved that this equation is in the form of [49]

$$\Delta T_p(n) = m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n, \qquad \text{(V-11)}$$

where

$$\Delta T_p(n) = T_p(n) - T_{po}(n)$$

$$\Delta q_n = q_{n1} - q_{no}$$

$$\Delta Cp_n = Cp_{n2} - Cp_{no}.$$

m_{q_n} and m_{Cp_n} are the calculated slopes in the q_n and Cp_n directions respectively. The tangent plane so obtained shall be used as an approximate representation of the criterion function $T_p(n)$ in the vicinity of the original point. This approximate representation will be used as a guide for locating the future trials. Taylor's series expansion of the profit function proves that for sufficiently small deviations the tangent plane approximates very closely the behavior of the criterion function.

To raise the value of the profit function one must find a combination of Δq_n and ΔCp_n for which

$$\Delta T_p(n) > 0.$$

It is desired, therefore, to obtain a solution to the inequality

$$m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n > 0. \qquad \text{(V-12)}$$

Graphically speaking the expression

$$m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n = 0$$

depicts a straight line on the q_n - Cp_n plane. This line divides the plane into favorable and unfavorable regions. Based on the information gained from the beginning experiments, a large section of q_n - Cp_n plane will become unfavorable for further exploration, and one gets a rough idea where to conduct the next experiments.

Middle Strategies

After the elementary exploration of the beginning experiments one must decide where to look for further improvement in the value of the profit function. Starting in the favorable section of the experimental region which was isolated by the beginning work, one guides the new search in the direction which gives the greatest rate of change of the criterion function. This simply suggests following a path in which the slope of the function is the greatest.[30, 53] Thus the idea of ascending the steepest path is the basis for this middle search strategy which is called the gradient method. Even though the direction of the movement may change from point to point, it should always be in the direction perpendicular to the local contour of the profit surface.

Determination of the local contour to the profit surface should be carried out by following the steps given below.

1. By running three experiments in the favorable section of q_n - Cp_n plane the equation of the tangent plane to the profit surface becomes available. This equation is in the form of

$$\Delta T_p(n) = m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n \qquad \text{(V-13)}$$

where

$$\Delta T_p(n) = T_p(n) - T_{po}(n)$$

m_{q_n} = slope of the profit surface in the direction parallel to the q_n axis

$$\Delta q_n = q_{n1} - q_{no}$$

m_{Cp_n} = slope of the profit surface in the direction parallel to the Cp_n axis

$$\Delta Cp_n = Cp_{n2} - Cp_{no}$$

2. The contour at any point of the profit surface has the property of $\Delta T_p(n) = 0$. This same relation holds for the contour line located at the point of tangency of the tangent plane to the profit surface. Thus the line of intersection of the tangent plane with this contour has the equation of

$$m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n = 0$$

This equation could also be written as

$$\Delta Cp_n = -(m_{q_n}/m_{Cp_n}) \cdot \Delta q_n$$

3. The projection of this line on the q_n - Cp_n plane has the very same equation

$$\Delta Cp_n = -\ (m_{q_n}/m_{Cp_n}) \cdot \Delta q_n \qquad \text{(V-14)}$$

This line is called the contour tangent. The slope of this line on q_n - Cp_n plane is $-(m_{q_n}/m_{Cp_n})$.

4. The direction in which the value of the profit function increases most rapidly is the one perpendicular to

the contour tangent. This direction has a slope of (m_{Cp_n}/m_{q_n}). This is the direction of the gradient of the criterion function and is the suggested path to follow for the middle stage trials.

An index for the effectiveness of this operation is the value of the profit improvement divided by the profit at the previous point. The movement in the gradient direction should be continued as long as this index shows any significant improvement.

Final Search

As a result of the beginning and middle stage experiments we are close to the optimum point and reaching the near-stationary region of the profit surface. An extension of the previous techniques, as suggested by Box and Wilson, calls for the utilization of the derivatives of higher order for the next stage of exploration.[12] The reason behind their suggestion for the use of higher derivatives is discussed below.

All of the approximation techniques applied so far were of the linear type. The tangent to the profit surface at the vicinity of the maximum point, however, is close to a horizontal plane. Thus the values of the slopes parallel to q_n and Cp_n axis in this zone are negligible. The coefficients of the linear terms of the approximating equation in that neighborhood are very small and other nonlinear effects become overwhelming there. Therefore we are led to fit the unknown criterion function by a nonlinear (quadratic or higher degree) expression.

The nonlinear exploration is carried out through the use of the Taylor expansion of the profit function in the region of interest. The form of this series neglecting the terms of higher than the second degree is

$$\Delta T_p(n) = m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n + \frac{1}{2}\left[m_{q_n^2} \cdot (\Delta q_n)^2 + 2m_{q_n Cp_n}(\Delta q_n \cdot \Delta Cp_n) + m_{Cp_n^2}(\Delta Cp_n)^2\right]. \quad \text{(V-15)}$$

where

$m_{q_n^2}$ = second derivative of the profit function with respect to q_n

$m_{q_n Cp_n}$ = second derivative of the profit function with respect to q_n and Cp_n in that order

$m_{Cp_n^2}$ = second derivative of the profit function with respect to Cp_n.

The rest of the terms were defined previously.

A simpler expansion would result if the interaction term is neglected. This simplified form is

$$\Delta T_p(n) = m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n + \frac{1}{2}\left[m_{q_n^2} \cdot (\Delta q_n)^2 + m_{Cp_n^2} \cdot (\Delta Cp_n)^2 \right]. \qquad \text{(V-16)}$$

A check measurement which will be described later serves to indicate whether this simplified form or the one with the interaction term should be used.

1. The Noninteracting Taylor Series Form. In practice one begins with fitting this simplified form of the Taylor expansion to the profit function. This approximation is then used to estimate the location of the maximum profit. In the case that the check measurement calls for the inclusion of the interaction term, the necessary adjustment is made after the calculations with the noninteracting form have been made.

In order to fit the series to the criterion function and find its maximum point, one may follow the steps given below.

1. Beginning with the last experiment of the middle stage exploration, one selects four new points in the experimental region. The points are positioned on a cross-like arrangement with the last experiment being located in the center of the cross. The distance of all the points from the center one is selected to be the same. Denoting

the center point by (q_{no}, Cp_{no}), the other four points (as shown in Figure 6) must have the coordinates of (q_{n1}, Cp_{no}), (q_{no}, Cp_{n3}), (q_{n2}, Cp_{no}) and (q_{no}, Cp_{n4}). The value of the criterion function at these points are designated by $T_{p1}(n)$, $T_{p3}(n)$, $T_{p2}(n)$ and $T_{p4}(n)$ respectively.

2. Applying the simplified expansion, the value of the criterion function at the point (q_{n1}, Cp_{no}) is computed from

$$\Delta T_{p1}(n) = m_{q_n} \cdot \Delta q_{n1} + \frac{1}{2} m_{q_n^2} \cdot (\Delta q_{n1})^2.$$

In this expression $\Delta T_{p1}(n) = T_{p1}(n) - T_{po}(n)$ and $T_{po}(n)$ is the value of profit at the center point. Similarly the value of the profit function at the point (q_{n2}, Cp_{no}) may be obtained from

$$\Delta T_{p2}(n) = m_{q_n} \cdot \Delta q_{n2} + \frac{1}{2} m_{q_n^2} \cdot (\Delta q_{n2})^2$$

where

$$\Delta T_{p2}(n) = T_{p2}(n) - T_{po}(n).$$

Since $\Delta q_{n2} = - \Delta q_{n1}$, then

$$\Delta T_{p2}(n) = - m_{q_n} \cdot \Delta q_{n1} + \frac{1}{2} m_{q_n^2} \cdot (\Delta q_{n1})^2.$$

Addition of the last equation and the one pertaining to the

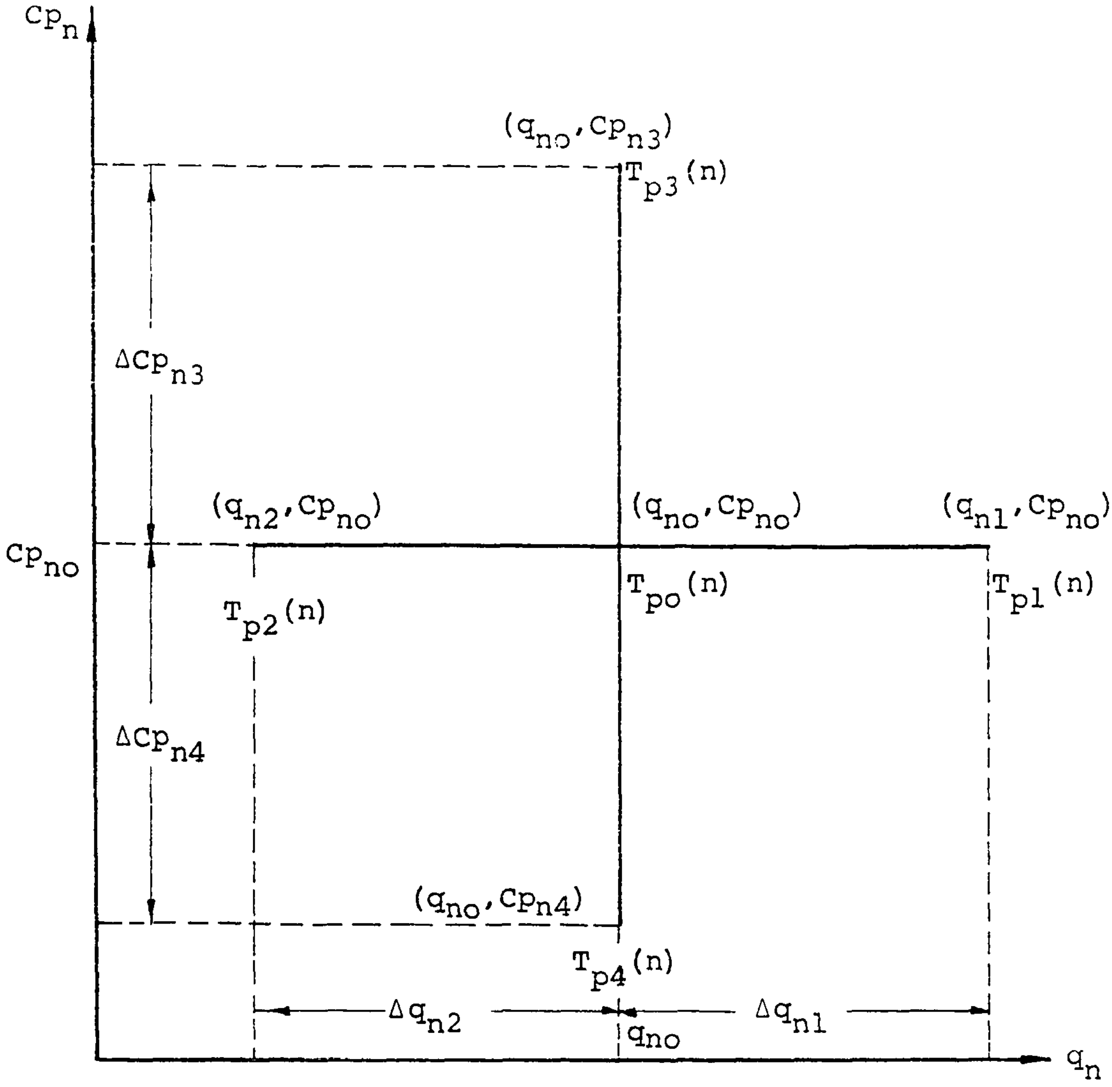

Figure 6. Crosslike Arrangement of the Final Search Experiments when Neglecting the Interaction of the Variables.

(q_{n1}, Cp_{no}) gives

$$m_{q_n^2} = \frac{\Delta T_{p1}(n) + \Delta T_{p2}(n)}{(\Delta q_{n1})^2}. \qquad \text{(V-17)}$$

Subtraction of the two equations gives

$$m_{q_n} = \frac{\Delta T_{p1}(n) - \Delta T_{p2}(n)}{2 \cdot \Delta q_{n1}} . \qquad \text{(V-18)}$$

3. Repeating the step number 2 above on the points (q_{no}, Cp_{n3}) and (q_{no}, Cp_{n4}), similar relations for the calculation of $m_{Cp_n^2}$ and m_{Cp_n} are obtained.

$$\Delta T_{p3}(n) = m_{Cp_n} \cdot \Delta Cp_{n3} + \frac{1}{2} m_{Cp_n^2} \cdot (\Delta Cp_{n3})^2$$

$$\Delta T_{p4}(n) = m_{Cp_n} \cdot \Delta Cp_{n4} + \frac{1}{2} m_{Cp_n^2} \cdot (\Delta Cp_{n4})^2$$

where

$$\Delta T_{p3}(n) = T_{p3}(n) - T_{po}(n)$$

$$\Delta T_{p4}(n) = T_{p4}(n) - T_{po}(n) .$$

Since $\Delta Cp_{n4} = - \Delta Cp_{n3}$, then

$$\Delta T_{p4}(n) = - m_{Cp_n} \cdot \Delta Cp_{n3} + \frac{1}{2} m_{Cp_n^2} \cdot (\Delta Cp_{n3})^2 .$$

Similar operations as in step number 2 result in

$$m_{Cp_n^2} = \frac{\Delta T_{p3}(n) + \Delta T_{p4}(n)}{(\Delta Cp_{n3})^2} \qquad \text{(V-19)}$$

and

$$m_{Cp_n} = \frac{\Delta T_{p3}(n) - \Delta T_{p4}(n)}{2\ \Delta Cp_{n3}}. \qquad \text{(V-20)}$$

Now that all the coefficients of the simplified Taylor expansion is computed, the noninteracting approximation of the criterion function is known.

4. Differentiating the $\Delta T_p(n)$ expression partially with respect to Δq_n and ΔCp_n we will have

$$\frac{\partial \Delta T_p(n)}{\partial \Delta q_n} = m_{q_n} + m_{q_n^2} \cdot \Delta q_n \qquad \text{(V-21)}$$

$$\frac{\partial \Delta T_p(n)}{\partial \Delta Cp_n} = m_{Cp_n} + m_{Cp_n^2} \cdot \Delta Cp_n. \qquad \text{(V 22)}$$

Setting these partial derivatives equal to zero, values for Δq_n and ΔCp_n are computed as given below.

$$\Delta q_n = - \frac{m_{q_n}}{m_{q_n^2}} \qquad \text{(V-23)}$$

and

$$\Delta Cp_n = \frac{-m_{Cp_n}}{m_{Cp_n^2}}\ . \qquad \text{(V-24)}$$

These values are the corrections computed from the final search which should be considered when shifting the previously calculated maximum point to its more accurate location.

The maximum point of the profit function thus obtained is the maximum of the fitting of noninteracting form of the Taylor series to the criterion function. This procedure will be satisfactory when the contour lines of the profit surface are close to circles or ellipses. However, if the contours are irregular curves, it becomes necessary to include the interaction term, $\Delta q_n \cdot \Delta Cp_n$ in the series.

2. The Interacting Form of the Taylor Series. In order to determine the necessity of the inclusion of the interaction term in the Taylor series a new experiment should be run. This trial, as suggested by Wild, should be located at the point (q_{n2}, Cp_{n3}).[62] This point, as shown in Figure 7, has a profit of $T_{p5}(n)$.

Ordinarily a different value of profit will be obtained by substituting the (q_{n2}, Cp_{n3}) into the simplified Taylor series expansion. Denoting this second value by $T_{p5T}(n)$, the decision as to whether the discrepancy

$$T_{p5}(n) - T_{p5T}(n)$$

is tolerable or not is based on the value of the ratio

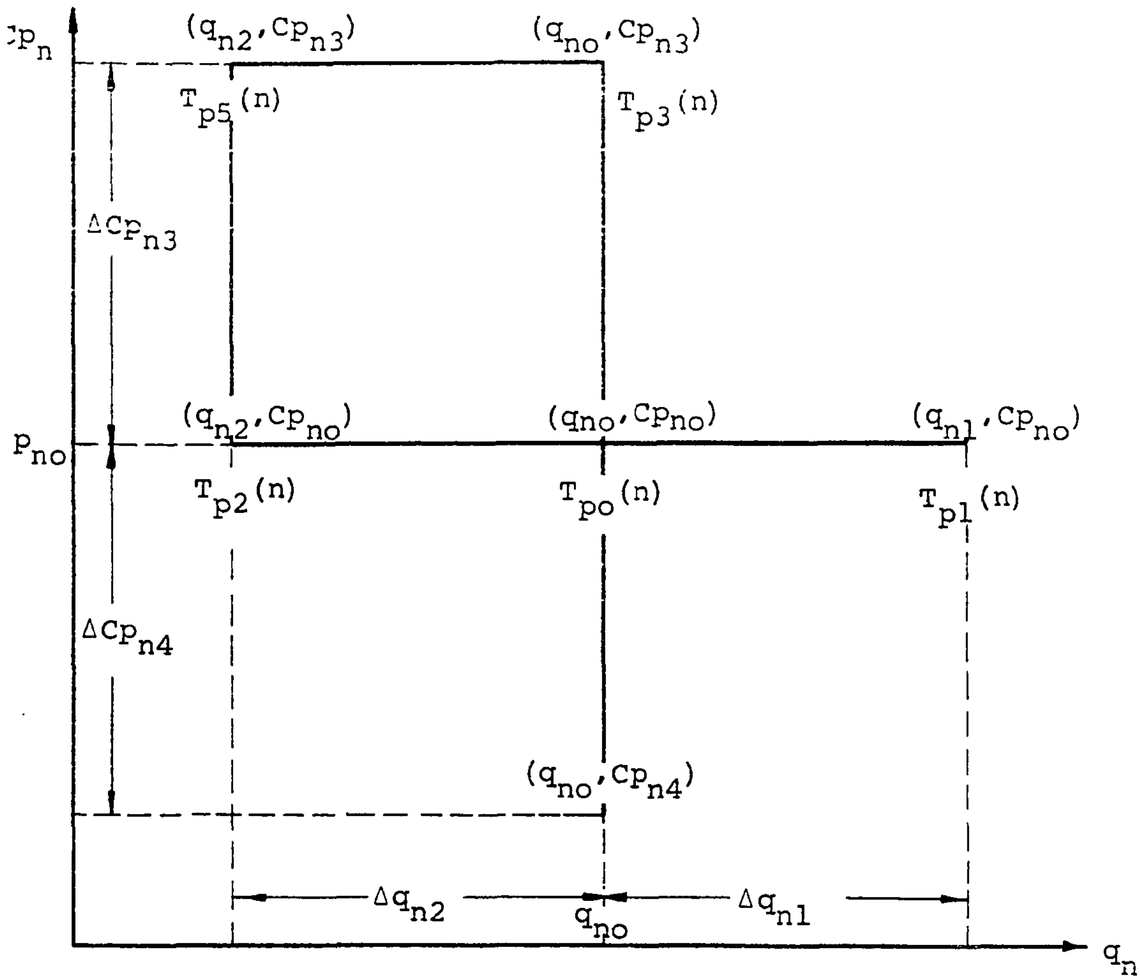

Figure 7. Final Search Experiments with the Interaction Term.

$$(T_{p5}(n) - T_{p5T}(n))/T_{p5}(n).$$

If this ratio is less than some specified value (0.01 was used in this problem), the previously computed maximum point is taken for the true maximum. On the other hand if this ratio is not less than the specified value, it indicates that the variables are interacting so strongly that the simplified expansion cannot be used. The more suitable form of the Taylor series, for approximating purposes, then is

$$\Delta T_p(n) = m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n + \frac{1}{2}\left[m_{q_n^2} \cdot (\Delta q_n)^2 + 2m_{q_n Cp_n} \cdot (\Delta q_n \cdot \Delta Cp_n) + m_{Cp_n^2}(\Delta Cp_n)^2\right].$$

By applying this expansion to the points shown in the previous figure we will have

1. For the point (q_{n1}, Cp_{no}),

$$\Delta T_{p1}(n) = m_{q_n} \cdot \Delta q_{n1} + \frac{1}{2}\left[m_{q_n^2} \cdot (\Delta q_{n1})^2\right].$$

2. For the point (q_{n2}, Cp_{no}),

$$\Delta T_{p2}(n) = m_{q_n} \cdot (-\Delta q_{n1}) + \frac{1}{2}\left[m_{q_n^2} \cdot (\Delta q_{n1})^2\right].$$

3. For the point (q_{no}, Cp_{n3}),

$$\Delta T_{p3}(n) = m_{Cp_n} \cdot (\Delta Cp_{n3}) + \frac{1}{2}\left[m_{Cp_n^2} \cdot (\Delta Cp_{n3})^2 \right].$$

4. For the point (q_{no}, Cp_{n4}),

$$\Delta T_{p4}(n) = m_{Cp_n} \cdot (-\Delta Cp_{n3}) + \frac{1}{2}\left[m_{Cp_n^2} \cdot (-\Delta Cp_{n3})^2 \right].$$

5. For the point (q_{n2}, Cp_{n3}),

$$\Delta T_{p5}(n) = m_{q_n} \cdot (-\Delta q_{n1}) + m_{Cp_n} \cdot (\Delta Cp_{n3}) +$$

$$\frac{1}{2}\left[m_{q_n^2} \cdot (-\Delta q_{n1})^2 + 2m_{q_n Cp_n} \cdot (-\Delta q_{n1} \cdot \Delta Cp_{n3}) + \right.$$

$$\left. m_{Cp_n^2} \cdot (\Delta Cp_{n3})^2 \right].$$

By using the first four expression one can compute the numerical value of m_{q_n}, m_{Cp_n}, $m_{q_n^2}$ and $m_{Cp_n^2}$. Substituting these values in the expression number 5, we will be able to compute the value of $m_{q_n Cp_n}$. Having all of the coefficients of the new Taylor series expansion, a better approximation for the profit function is now available. This new approximate is a fitting of the interacting form of the Taylor series to the criterion function, and it is in the form of

$$\Delta T_p(n) = m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n + \frac{1}{2}\left[m_{q_n^2} \cdot (\Delta q_n)^2 + 2m_{q_n Cp_n} \cdot (\Delta q_n \cdot \Delta Cp_n) + m_{Cp_n^2}(\Delta Cp_n)^2 \right].$$

To find the maximum point of this series, its derivatives with respect to Δq_n and ΔCp_n should be set equal to zero. The results are

$$\frac{\partial \Delta T_p(n)}{\partial \Delta q_n} = m_{q_n} + \frac{1}{2}\left[2\, m_{q_n^2} \cdot \Delta q_n + 2\, m_{q_n Cp_n} \cdot \Delta Cp_n \right] = 0$$

$$\frac{\partial \Delta T_p(n)}{\partial \Delta Cp_n} = m_{Cp_n} + \frac{1}{2}\left[2\, m_{Cp_n^2} \cdot \Delta Cp_n + 2\, m_{q_n Cp_n} \cdot \Delta q_n \right] = 0.$$

or

$$\begin{cases} m_{q_n^2} \cdot \Delta q_n + m_{q_n Cp_n} \cdot \Delta Cp_n = -m_{q_n} \\ m_{q_n Cp_n} \cdot \Delta q_n + m_{Cp_n^2} \cdot \Delta Cp_n = -m_{Cp_n} \end{cases}$$

The answers to this set of simultaneous equations are

$$\Delta q_n = \frac{m_{q_n Cp_n} \cdot m_{Cp_n} - m_{Cp_n^2} \cdot m_{q_n}}{m_{q_n^2} \cdot m_{Cp_n^2} - (m_{q_n Cp_n})^2}$$

and

$$\Delta Cp_n = \frac{m_{q_n Cp_n} \cdot m_{q_n} - m_{q_n^2} \cdot m_{Cp_n}}{m_{q_n^2} \cdot m_{Cp_n^2} - (m_{q_n Cp_n})^2}.$$

The newly computed Δq_n and ΔCp_n are the deviations of the more accurate maximum point from the center point, (q_{no}, Cp_{no}).

The final search operations just described will not be reliable without investigating whether the obtained corrections correspond to the minimum or the maximum point. Theoretically, setting the first derivative equal to zero may identify the coordinates of either of the maximum or the minimum point of a function. The second derivative, however, has a negative value at the maximum point.[51] Therefore it is necessary to compute the numerical value of the second derivative of the profit function at the point in question. Determination of the second derivative can be carried out by following the steps given below.

1. Consider a function F(t) defined by

$$F(t) = T_p(q_n, Cp_n) = T_p(q_{nf}+h \cdot t, Cp_{nf}+k \cdot t),$$

in which h and k are constants and t is a variable. q_{nf} and Cp_{nf} are the coordinates of the point at which the final search began.

2. The first two derivatives of F(t) can be obtained by differentiating with respect to t.

$$F'(t) = \frac{\partial}{\partial q_n}\left[T_p(q_n, Cp_n)\right] \cdot \frac{dq_n}{dt} + \frac{\partial}{\partial Cp_n}\left[T_p(q_n, Cp_n)\right] \frac{dCp_n}{dt}$$

$$F'(t) = \frac{\partial T_p(q_n, Cp_n)}{\partial q_n} \cdot h + \frac{\partial T_p(q_n, Cp_n)}{\partial Cp_n} \cdot k.$$

and

$$F''(t) = \left\{ \frac{\partial}{\partial q_n} \left[\frac{\partial T_p(q_n, Cp_n)}{\partial q_n} \right] \cdot \frac{dq_n}{dt} + \frac{\partial}{\partial Cp_n} \left[\frac{\partial T_p(q_n, Cp_n)}{\partial q_n} \right] \frac{\partial Cp_n}{dt} \right\} \cdot h + \left\{ \frac{\partial}{\partial q_n} \left[\frac{\partial T_p(q_n, Cp_n)}{\partial Cp_n} \right] \frac{dq_n}{dt} + \frac{\partial}{\partial Cp_n} \left[\frac{\partial T_p(q_n, Cp_n)}{\partial Cp_n} \right] \frac{dCp_n}{dt} \right\} \cdot k$$

or

$$F''(t) = \frac{\partial^2 T_p(q_n, Cp_n)}{\partial q_n^2} \cdot h^2 + 2h \cdot k \frac{\partial^2 T_p(q_n, Cp_n)}{\partial q_n \partial Cp_n} + k^2 \cdot \frac{\partial^2 T_p(q_n, Cp_n)}{\partial Cp_n^2} .$$

3. The numerical values of q_n and Cp_n at the final point are

$$q_n = q_{nf} + \Delta q_n$$

$$Cp_n = Cp_{nf} + \Delta Cp_n .$$

Δq_n and ΔCp_n in these relations are the corrections obtained via the final search operations.

4. The value of $F''(t)$ at the final point can be calculated by setting

$$t = 1$$

$$h = \Delta q_n$$

$$k = \Delta Cp_n.$$

In this case F"(t) at the final point is equal to F"(1).

$$F''(1) = \frac{\partial^2 T_p(q_{nf} + \Delta q_n, Cp_{nf} + \Delta Cp_n)}{\partial q_n^{\,2}} \cdot (\Delta q_n)^2 +$$

$$2 \cdot \Delta Cp_n \cdot \Delta q_n \, \frac{\partial^2 T_p(q_{nf} + \Delta q_n, Cp_{nf} + \Delta Cp_n)}{\partial q_n \partial Cp_n} +$$

$$\frac{\partial^2 T_p(q_{nf} + \Delta q_n, Cp_{nf} + \Delta Cp_n)}{\partial Cp_n^{\,2}} \cdot (\Delta Cp_n)^2.$$

Having the value of the second derivative at the final point, one can inspect as to whether it is a positive or a negative value. The corrections obtained by the final search will be taken into consideration only if the second derivative is negative.

CHAPTER VI

NUMERICAL ANALYSIS AND CODING

Application of the theoretical considerations of the previous chapter to the actual development of a petroleum production system is not an easy task without access to digital computers. Performing such a heavy load of mathematical operations either by hand or on desk calculators is not feasible. It is necessary, therfore, to make these calculations on the computer. The Fortran coding and numerical analysis of these operations are presented in this chapter. A parallel correspondence between the order of the suggested computations and the sequence of discussions will be maintained.

The coding comprises a main program and two function subprograms as explained below.[38,42]

The beginning statements of the main program read in the following data:

present rate of production, QI

yearly percentage increase in market demand, D

the time increment used for discretization of the study period, DT

number of years in the study period, YRS water influx constants for various stages, WIC(IWIC)

volume of original oil in place, OO

compressibility factor of oil, COMP

cost of drilling a well, CDW

cost of operating a well, COW

coefficient of the initial cost function of a separation unit, CIS

operating cost of a separation station, OCS

$\Delta_o = P_o - P_w$ = DO

constant of productivity, AT.

In order to attach the same values to these codes for the entire program, COMMON statements are included both in the main and subprograms.[22,41]

To enter the optimization routine at the nth stage of development, it is necessary to feed the optimum decisions of the stage, computed previously, into the program. The results of the computations are the optimal decisions pertaining to the (n + 1)th stage. By repeating the same procedure, as shown in Figure 8 similar optimal values for all the other stages are obtained. In the first stage, however, there are no prior computations and estimated values have to be used to start the sequential calculations. The following statements are included in the program for this purpose.

```
QTS = Q1
PTS = P1
```

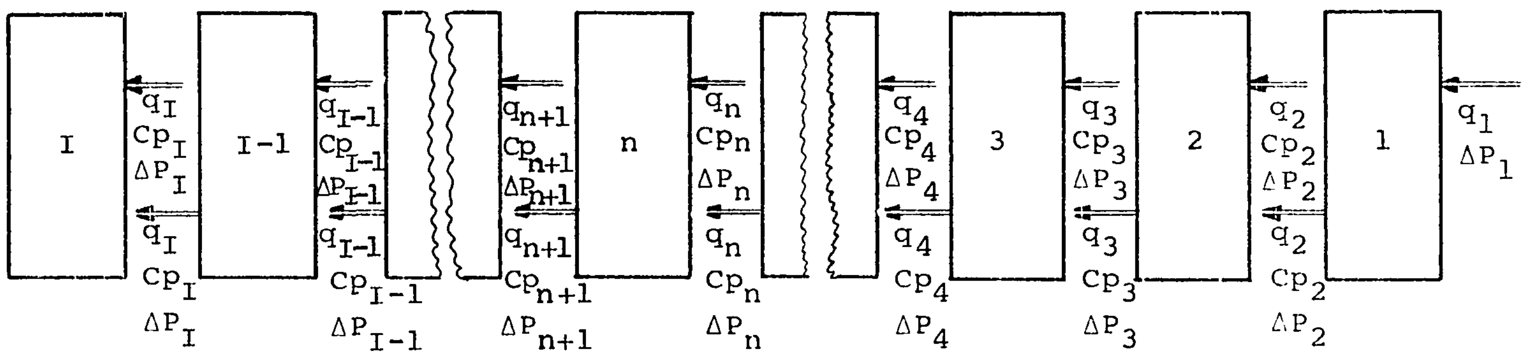

Figure 8. Sequential Decision Making in Various Stages of Development.

Q1 and P1 are the estimated values of the production rate and cumulative pressure drop at the end of the planning horizon.

Preliminary Observations

To begin the optimization at any stage n, the predicted market demand for that period is calculated from

$$MD_n = MD_I \cdot e^{d\cdot(I - n)\cdot\Delta}$$

The upper limit of the production rate of (n + 1)th stage in Fortran language, then becomes

```
Q = QI*EXP(D*DT*(YRS/DT-FN-1.0))
```

Since the capacity of a production unit is always less than the total production rate of the field Q is also considered to be the capacity of the unit in the first trial. The statement

```
C = Q
```

in which C is the capacity of the unit is provided for this purpose. To make an elementary exploration the operating range of the variables is divided into ten equal intervals. The value of the profit function at these points are computed. By comparing the result of these computations the highest profit among the ten values is found. The corresponding point is the suggested location at which to start the beginning experiments. The value of the profit function at any point is obtained by using the subprogram PFT explained in the next section.

Subprogram PFT

The function subprogram PFT supplies the main program with the value of profit made in any stage of development. The necessary parameters which should be transferred to the subprogram are

the stage number, AN

production rate of the nth stage, QK

capacity of the production unit installed at the nth stage, CK

water influx constant of the (n + 1)th stage, SKX

$\Delta P_n = P_o - P_n$ of the nth stage, PK

estimated production rate and the capacity of the (n + 1)th stage, QX and CX

The computed profit pertains to the (n + 1)th stage of development.

A COMMON statement makes the supplied data applicable to both the main program and the subprogram.

The estimated production rate QX has to be within that particular range of values permitted by all of the pertaining restrictions. To keep the value of QX within this range the following provision is made.

Production beyond the limits of the market demand is discouraged by considering a cost which is invoked in the event of excessive production. The cost of overproduction is assumed to be proportional to the square of extra production. When the production is less than or equal to the current estimated demand, it is set equal to zero. The following statements in the PFT function, compute the value of excessive production cost.

```
IF(QX - Q)700,700,701
    700 OVPC = 0.0
    701 OVPC = 365.*DT*OC*((QX-Q)**2.)
```

Other costs include the drilling and operating cost of wells, and the installation and operating costs of separation stations. These terms are computed by the following set of statements

```
     FT = QK/(AT*(DO-PK))
   FUPD = (365.*DT*QX*R-PK)/(365.*DT*FNU-1.0)
     ST = QX/(AT*(DO-FUPD))
     TT = QK/(CK)-QX/CX
    FRT = 182.5*(QK + QX)*DT
    FIT = (CX)**.6
 29 CST = (FT-ST)*CDW+FRT*(COW+OSC)+TT*FIT*CIS+OVPC
```

Statement number 29 calculates the total cost of one stage of development. Defining the value of profit as the difference between the sale of production and the total cost, the profit function becomes

```
PFT = 182.5*DT*PPC*(QX + QK) - CST
```

The expressions for calculation of total cost of operation are slightly different when in the first stage.

In that stage the first five statements are replaced by

```
FT = Q1/(AT*(DO - P1))
FUPD = (365.*DT*QX*R-P1)/(365.*DT*FNU-1.0)
ST = QX/(AT*(DO - FUPD))
TT = (Q1-QX)/CX
FRT = 182.5*(Q1-QX)*DT
```

The sixth statement will remain the same for all stages. A RETURN and an END statement will return the control of computations to the main program. Figure 9 shows a flow chart for the subprogram PFT.

Statement Function DLP

Cumulative pressure drop of the field is shown by ΔP_t and defined by

$$\Delta P_t = P_o - P_t$$

To compute the value of ΔP_t at any stage of development a statement function DLP is provided in the main program.[25] The function is the Fortran equivalent of the numerical solution of the state equation of pressure. This equation, as discussed before, is a linear differential equation which describes the variation of the cumulative pressure drop of the field with time. The state equation of pressure, as derived in Appendices A and B, is in the form of

$$\frac{d\Delta P_t}{dt} + \gamma_t \cdot \Delta P_t = R \cdot q_t .$$

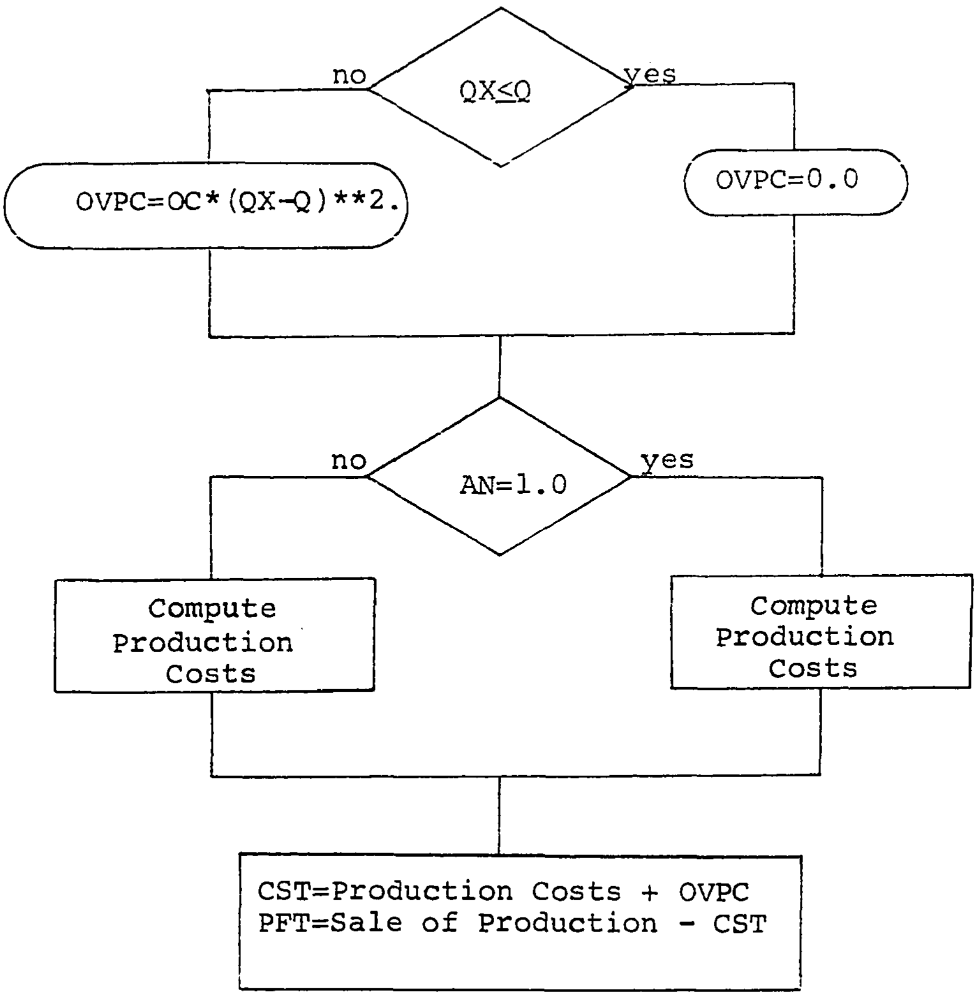

Figure 9. Flow Chart of the PFT Subprogram.

The value of $d\Delta P/dt$ can be approximated by

$$\frac{d\Delta P}{dt} = \frac{\Delta P_n - \Delta P_{n+1}}{\Delta t}$$

Thus the approximate form of the state equation becomes

$$\frac{\Delta P_n - \Delta P_{n+1}}{\Delta t} + \gamma_{n+1} \cdot \Delta P_{n+1} = R \cdot q_{n+1}$$

or

$$\Delta P_n - \Delta P_{n+1} + \gamma_{n+1} \cdot \Delta t \cdot \Delta P_{n+1} = R \cdot \Delta t \cdot q_{n+1} .$$

Solving for ΔP_{n+1} we will have

$$\Delta P_{n+1} = (R \cdot \Delta \cdot q_{n+1} - \Delta P_n)/(\gamma_{n+1} \cdot \Delta T - 1.) .$$

The equivalent statement function is

DLP(G,P,W,) = (365.*DT*G/00/COMP-P)/(365.*DT*W/00/COMP-1.)

To obtain the value of ΔP_t, this function should be supplied with

1. the estimated production rate of the field at (n+1)th stage, G

2. cumulative pressure drop of the field at the nth stage, P

3. water influx constant of the (n+1)th stage, W.

The computed result DLP is the cumulative pressure drop of the field at the (n+1)th stage of development.

The value of ΔP_t, however, has to meet certain requirements as discussed below.

1. To acquire any flow from the reservoir into the wellbore, the reservoir pressure has to be higher than the bottom hole pressure.

$$P_t > P_w$$

or

$$P_t - P_w > 0$$

By manipulating this inequality we obtain

$$P_o - P_w - (P_o - P_t) > 0$$

or

$$\Delta_o - \Delta P_t > 0$$

and finally

$$\Delta P_t < \Delta_o$$

2. Normally there is a decline in reservoir pressure with time and P_o is greater than P_t

$$P_o - P_t > 0$$

$$\Delta P_t > 0$$

If the cumulative pressure drop of the nth stage, fed into the DLP function is unrealistically high it may overpower

365.*DT*G/00/COMP

The result of the computations in this case would be a negative ΔP. To avoid this situation the function subprogram GPTS is developed. This function tests the value of DLP and if it is a negative number a new P will be calculated. GPTS subprogram should be supplied with

1. the estimated rate of production, QIN
2. the pressure drop obtained from the DLP statement function, BP
3. pressure drop of the nth stage, EPTS
4. water influx constant of (n+1)th stage, SK.

The outcome of the subprogram is the modified value of the nth stage pressure drop. Substituting this value in the DLP statement function creates a positive pressure drop.

Beginning Experiments

Beginning experiments are the trials made at the start of a search to decide which zone on the $q_n - C_{p_n}$ plane is favorable for further exploration. Starting at the point which was selected among the ten experiments, one begins to find the slopes of the profit function along both axes. Denoting this starting point by (QOB, COB) the value of its corresponding profit function would be

$$TPOB = PFT(QOB,COB,WIC(ISK),FN,QTS,CTS,PTS)$$

By running two other experiments in the neighborhood of (QOB,COB) and designating the pertaining profit values by TP1B, and TP2B we will have

$$TP1B = PFT(Q1B,COB,WIC(ISK),FN,QTS,CTS,PTS)$$

$$TP2B = PFT(QOB,C2B,WIC(ISK),FN,QTS,CTS,PTS).$$

Then the slopes of the profit surface in this vicinity and along q_n and Cp_n axes are

$$m_{q_n} = SP1QM = (TP1B-TPOB)/(Q1B-QOB)$$

and

$$m_{Cp_n} = SP1CM = (TP2B-TOB)/DCB$$

Having m_{q_n} and m_{Cp_n} the slope of the line,

$$m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n = 0$$

becomes:

$$SPOL = -SP1QM/SP1CM$$

Considering a point on this line with q_n = QOL, its corresponding Cp_n will be

$$COL = COB + SPOL*(QOL-QOB)$$

Raising the value of COL by DCB, while keeping the previous q_n = QOL, a new point (QOL, COL+DCB) will be located. The variation in the value of profit function, DTPB is obtained by the statement

$$DTPB = SP1QM*(QOL-QOB)+SP1CM*(COL+DCB-COB)$$

Based on the sign of SPOL and DTPB one can decide which region of the $q_n - Cp_n$ is favorable for further exploration. Figure 10 is a flow chart of the routine which isolates the favorable segment of the plane. The results are in terms of whether to raise or lower the values of the variables when moving in the direction perpendicular to the line

$$m_{q_n} \cdot \Delta q_n + m_{Cp_n} \cdot \Delta Cp_n = 0 \cdot$$

Middle Strategies

Having obtained the favorable section of the $q_n - Cp_n$ plane for further exploration, one tries to find the direction in which the profit function increases most rapidly. This direction, referred to as the steepest ascent direction, has a slope of

$$TAN = SP1QM/SP1CM$$

The movement along this path should continue as long as any significant improvement in the value of the profit function

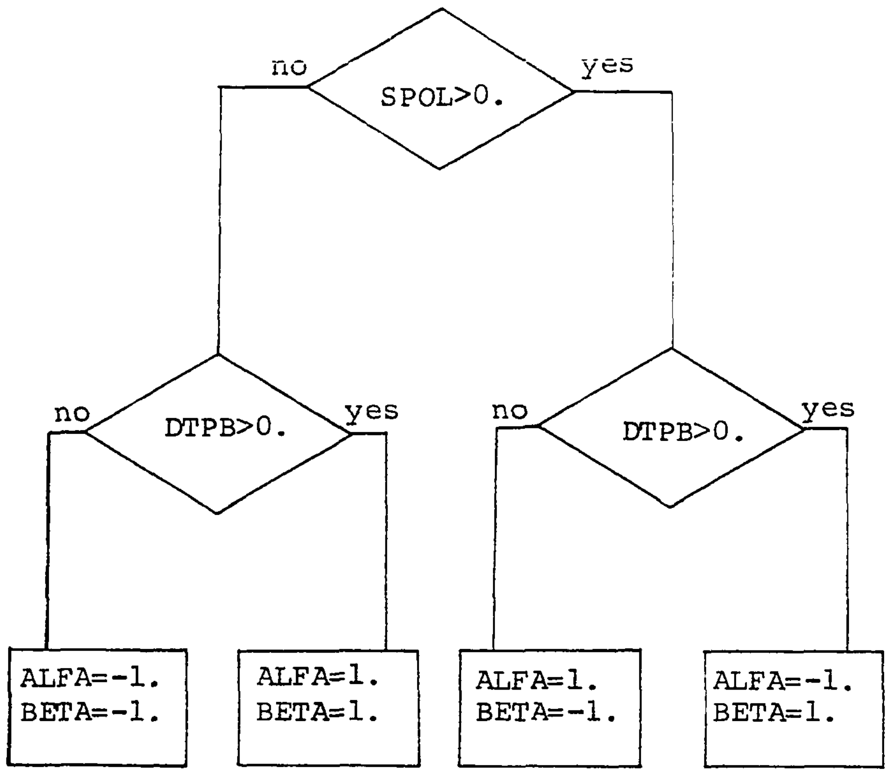

Figure 10. Determination of ALFA and BETA for the Middle Strategies.

Final Search

Final search begins from the point at which the middle strategies were terminated. To create the crosslike arrangement of the final search, an initial point (QF,CF) is located by

$$QF = QM$$

$$CF = CM$$

Two other points, (QF1,CF) and (QF2,CF) having the same horizontal distance from (QF,CF), are also considered. Their coordinates are:

$$QF1 = QF + DQF$$

$$CF1 = CF$$

and

$$QF2 = QF - DQF$$

$$CF2 = CF$$

Adopting new notations, the center of the cross has the coordinates of

$$GRFO = (GRF1 + GRF2)/2.$$

$$CF = CM$$

where

$$GRF1 = QF1$$

$$GRF2 = QF2$$

The value of the profit function at the center point is

$$TPF = PFT(GRFO,CF,WIC(ISK),FN,QTS,CTS,PTS)$$

The coordinates of the end points of the cross will be

(GRF1,CF), (GRF2,CF), (GRFO,CF3), and (GRFO,CF4). CF3 and CF4 in these coordinates are computed from

CF3 = CF + DCF

CF4 = CF - DCF

The value of the profit function at each of the points is obtained from

TPF1 = PFT(GRF1,CF,WIC(ISK),FN,QTS,CTS,PTS)

TPF2 = PFT(GRF2,CF,WIC(ISK,FN,QTS,CTS,PTS)

TPF3 = PFT(GRFO,CF3,WIC(ISK,FN,QTS,CTS,PTS)

TPF4 = PFT(GRFO,CF4,WIC(ISK)FN,QTS,CTS,PTS)

The following statements are used to compute values for ΔT_p at the end points of the cross.

DTPF1 = TPF1 - TPF

DTPF2 = TPF2 - TPF

DTPF3 = TPF3 - TPF

DTPF4 = TPF4 - TPF

Having values for ΔT_p at the end points, various slopes of the profit function can be determined as follows

m_{q_n} = SP1Q = (DTPF1 - DTPF2)/(2.*(GRF1 - GRFO))

m_{Cp_n} = SP1C = (DTPF3 - DTPF4)/(2.*DCF)

$m_{q^2_n}$ = SP2Q = (DTPF1 + DTPF2)/((GRF1-GRFO)*(GRF1-GRFO))

$m_{Cp^2_n}$ = SP2C = (DTPF3 + DTPF4)/(DCF*DCF)

In order to determine which of the noninteracting or the interacting form of the Taylor series is more suitable for approximating the profit function a fifth point (QF5,CF5) is also selected. The coordinates of this point are

QF5 = GRF2

CF5 = CF + DCF

The profit function at this point has a value of

TPF5 = PFT(QF5,CF5,WIC(ISK),FN,QTS,CTS,PTS)

Various terms of the Taylor series expansion of the profit function at this fifth point are

$$m_{q_n} \cdot \Delta q_n = T1 = SP1Q*DQF1$$

$$m_{Cp_n} \cdot \Delta Cp_n = T2 = SP1C*DCF1$$

T3 = .5*SP2Q*DQF1*DQF1

T5 = .5*SP2C*DCF1*DCF1

Values for DQF1 and DCF1 in these statements are obtained from

DQF1 = QF5 - GRFO

DCF1 = CF5 - CF

Approximating the value of the profit function at (QF5,CF5) by the noninteracting form of the Taylor series gives

TPF5T = TPF + T1 + T2 + T3 + T5

When the two values TPF5 and TPF5T are close enough, the noninteracting form of the Taylor series is suitable for purposes of a approximation and the final search corrections are

CDQF1 = - SP1Q/SP2Q

CDCF1 = - SP1C/SP2C

Presence of an intolerable difference, however, is an indication of the necessity of inclusion of the interaction term in the series. The following set of statements compute the final search corrections for this case.

FNUMQ = SPQC*SP1C - SP2C*SP1Q

FNUMC = SPQC*SP1C - SP2Q*SP1C

FDEN = SP2Q*SP2C - SPQC*SPQC

CDQF2 = FNUMQ/FDEN

CDCF2 = FNUMC/FDEN

The variables CDQF2 and CDCF2 are the correction values and SPQC is eqivalent to the slope $m_{q_n Cp_n}$.

The corrections obtained in both of the noninteracting and interacting approximations are valid only if they are confirmed by the maximum point test. In accordance with this test the value of

$$m_{q_n^2} \cdot (\Delta q_{nf})^2 + 2m_{q_n Cp_n} \cdot \Delta q_{nf} \cdot \Delta CP_{nf} + m_{Cp_n^2} \cdot (\Delta Cp_{nf})^2$$

at the maximum point has to be a negative one. Δq_{nf} and

ΔCp_{nf} in this expression are the computed corrections in either of the two cases which are also shown by H and XK. Thus if the value of

$$TST = SP2Q*H*H+2.*SPQC*H*XK+SP2C*XK*XK$$

is negative the more accurate maximum point of the profit function has the coordinates of

$$QF = GRFS(1)+H$$

$$CF = CF+XK$$

The flow chart of Figure 11 pertains to the final search segment of the program.

Sequential Decision Making

All phases of the search procedure described above must be carried out for every stage of the study period. The values QF and CF which are computed at any stage n are the optimal values of the variables pertaining to the (n+1)th time interval of the planning horizon. Feeding these values back into the routine at the (n+1)th stage, one computes optimal values for the (n+2)th stage. By repeating this procedure the optimal values of the variables are obtained for all times. An exception to this general practice is the first stage of the period. Since there are no computations prior to the first stage we have to have estimated values

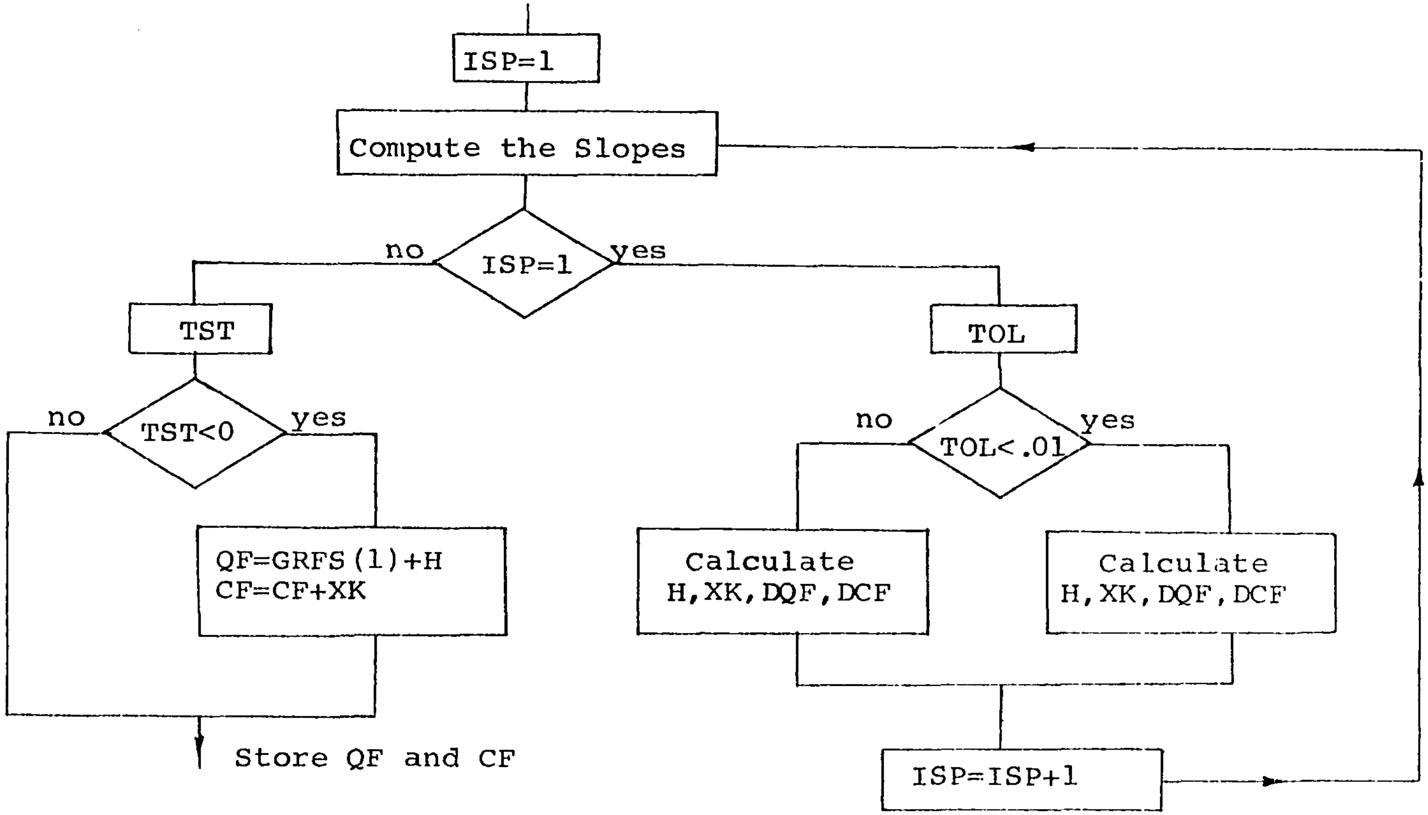

Figure 11. Final Search Flow Chart.

for q_1 and ΔP_1 before the start of optimization. These values must be read into the program in the form of known data, Q1 and P1.

A complete list of the program as executed on IBM 1130 is shown below.

List of the Program

```
// JOB T
// FOR
*ONE WORD INTEGERS
*EXTENDED PRECISION
      FUNCTION GPTS(QIN,BP,EPTS,SK)
      COMMON OO,COMP,DT,PPC,CDW,COW,CIS,OCS,Q1,DO,AT,P1,Q
      GPTS=EPTS
      IF(BP)471,471,472
  471 R=1.0/(OO*COMP)
      FNU=SK/(OO*COMP)
      DESC=365.*DT*FNU-1.0
      IF(DESC)671,671,672
  671 GPTS=1.01*365.*DT*QIN*R
      RETURN
  672 GPTS=0.99*365.*DT*QIN*R
  472 RETURN
      END
// DUP
*STORE      WS  UA   GPTS
// FOR
*ONE WORD INTEGERS
*EXTENDED PRECISION
      FUNCTION PFT(QX,CX,SKX,AN,QK,CK,PK)
      COMMON OO,COMP,DT,PPC,CDW,COW,CIS,OCS,Q1,DO,AT,P1,Q
      R=1.0/(OO*COMP)
      FNU=SKX/(OO*COMP)
      GQPFT=.98*(365.*DT*FNU*DO-DO+PK)/(365.*DT*R)
      IF(QX-GQPFT)475,475,474
  475 XPDC=0.0
      GO TO 481
  474 XPDC=365.*DT*((QX-GQPFT)**2.0)
  481 IF(QX-Q)700,700,701
  700 OVPC=0.0
      GO TO 702
  701 OVPC=365.0*DT*.0022*((QX-Q)**2.0)
  702 IF(AN-1.0)25,25,27
```

```
   25 FT=Q1/(AT*(DO-P1))
      FUPD=(365.*DT*QX*R-P1)/(365.*DT*FNU-1.0)
      ST=QX/(AT*(DO-FUPD))
      TT=(Q1-QX)/CX
      FRT=182.5*(Q1+QX)*DT
      FIT=(CX)**.6
      GO TO 29
   27 FT=QK/(AT*(DO-PK))
      FUPD=(365.*DT*QX*R-PK)/(365.*DT*FNU-1.0)
      ST=QX/(AT*(DO-FUPD))
      TT=QK/CK-QX/CX
      FRT=182.5*(QK+QX)*DT
      FIT=(CX)**.6
   29 CST=(FT-ST)*CDW+FRT*(COW+OCS)+TT*FIT*CIS+OVPC+XPDC
      PFT=182.5*DT*PPC*(QK+QX)-CST
      RETURN
      END
// DUP
*STORE       WS    US    PFT
// FOR
*IOCS(CARD, 1132 PRINTER, DISK)
*ONE WORD INTEGERS
*EXTENDED PRECISION
*NAME OPTMM
**   EBNOLNASSIR  EN100201
      DIMENSION QKPT(40),CKPT(40),PKPT(40),TPBX(10),
     1QBEX(10),CBEX(10),PBEX(10),WIC(20),GRFS(3)
      COMMON OO,COMP,DT,PPC,CDW,COW,CIS,OCS,Q1,DO,AT,P1,Q
      DLP(G,P,W)=(365.*DT*G/OO/COMP-P)/(365.*DT*W/OO/COMP-1.)
C  G=GR,P=PB,ANDW=WIC)
      I=2
      J=3
      READ(I,10)Q1,QI,D,DT,YRS,DPO
   10 FORMAT(2F10.0,F8.3,F10.5,F5.1,F10.2)
      READ(I,30)OO,COMP,P1,PPC
   30 FORMAT(2E10.3,F10.1,F10.3)
      READ(I,40)CDW,COW,CIS,OCS,DO,AT
   40 FORMAT(F10.1,F10.4,F10.1,F10.4,F10.1,F10.2)
      READ(I,50)IC,CMIN
   50 FORMAT(I4,F10.0)
      FWIS=YRS/DT
      IWIS=FWIS
      READ(I,400) (WIC(IWIC),IWIC=1,IWIS)
  400 FORMAT(F10.0)
      CTS=0.0
  200 SUM=0.0
      FN=1.0
      QTS=Q1
      PTS=P1
    1 PWR=D*DT*(YRS/DT-FN-1.0)
```

```
      Q=QI*EXP(PWR)
      C=Q
      QMIN=.2*Q
      WRITE(J,22)Q,C,QMIN,CMIN
   22 FORMAT(10X,15HQ,C,QMIN,CMIN,4E14.6)
      SCHC=FN+1.0
      ISK=SCHC
      SZQ=(Q-QMIN)/10.0
      SZC=(C-CMIN)/10.0
      SSQ=ABS(SZQ)
      SSC=ABS(SZC)
      DO 58 IBX=1,10
      XIBX=IBX
      QBEX(IBX)=Q-XIBX*SSQ
      CBEX(IBX)=C-XIBX*SSC
      BPBEX=DLP(QBEX(IBX),PTS,WIC(ISK))
      GQ=QBEX(IBX)
      PBEX(IBX)=DLP(GQ,PTS,WIC(ISK))
      TPBX(IBX)=PFT(GQ,CBEX(IBX),WIC(ISK),FN,QTS,CTS,PTS)
   58 CONTINUE
      TEMPP=TPBX(1)
      DO 42 JBX=1,10
      IF(TEMPP-TPBX(JBX))203,203,42
  203 IC2=JBX
      TEMPP=TPBX(JBX)
   42 CONTINUE
      WRITE(J,220)QBEX(IC2),CBEX(IC2),PBEX(IC2),TPBX(IC2),IC2
  220 FORMAT(10X,25HQBEX,CBEX,PBEX,TPBX,IC2,4E14.6,I5)
      QGIBX=QBEX(IC2)
      CGIBX=CBEX(IC2)
      DQB=.02*SSQ
      DCB=.02*SSC
      NIPC=0
      IBXC=1
  501 BPOB=DLP(QGIBX,PTS,WIC(ISK))
      QOB=QGIBX
      COB=CGIBX
      PTS=GPTS(QOB,BPOB,PTS,WIC(ISK))
      TPOB=PFT(QOB,COB,WIC(ISK),FN,QTS,CTS,PTS)
      BP1B=DLP(QOB+DQB,PTS,WIC(ISK))
      Q1B=QOB+DQB
      PTS=GPTS(Q1B,BP1B,PTS,WIC(ISK))
      TP1B=PFT(Q1B,COB,WIC(ISK),FN,QTS,CTS,PTS)
      TP2B=PFT(QOB,COB+DCB,WIC(ISK),FN,QTS,CTS,PTS)
      SP1QM=(TP1B-TPOB)/(Q1B-QOB)
      IF(SP1QM)61,54,61
   54 WRITE(J,140)IBXC
  140 FORMAT(10X,28HNO STEEPEST DIRECTION EXISTS,I5)
      QF=QOB
      CF=COB
      GO TO 65
```

```
 61 SP1CM=(TP2B-TPOB)/(DCB)
    SPOL=(-1.0*SP1QM)/SP1CM
    TAN=SP1CM/SP1QM
    WRITE(J,820)TP1B,TP2B,SP1QM,TAN
820 FORMAT(10X,21HTP1B,TP2B,SP1QM,TAN,4E14.6)
    BPOL=DLP(QOB+DQB,PTS,WIC(ISK))
    QOL=QOB+DQB
    PTS=GPTS(QOL,BPOL,PTS,WIC(ISK))
    COL=COB+SPOL*(QOL-QOB)
    DTPB=SP1QM*(QOL-QOB)+SP1CM*(COL+DCB-COB)
    IF(SPOL)151,151,153
151 IF(DTPB)155,155,157
155 ALFA=-1.0
    BETA=-1.0
    GO TO 53
157 ALFA=1.0
    BETA=1.0
    GO TO 53
153 IF(DTPB)159,159,161
159 ALFA=1.0
    BETA=-1.0
    GO TO 53
161 ALFA=-1.0
    BETA=1.0
 53 ISAC=0
    WRITE(J,810)DTPB,PTS,ALFA,BETA
810 FORMAT(10X,20HDTPB,PTS,ALFA,BETA,2E14.6,2F5.2)
 57 TPOM=PFT(QOL,COL,WIC(ISK),FN,QTS,CTS,PTS)
    BQM=QOL+ALFA*DQB
    BPM=DLP(BQM,PTS,WIC(ISK))
    QM=BQM
    PTS=GPTS(QM,BPM,PTS,WIC(ISK))
    ADQ=ABS(QM-QOL)
    POSDC=ABS(ADQ*TAN)
    CM=COL+BETA*POSDC
    CRTR2=PFT(QM,CM,WIC(ISK),FN,QTS,CTS,PTS)
    CRTRN=(CRTR2-TPOM)/TPOM
    IF(CRTRN-.0005)62,55,55
 62 NIPC=NIPC+1
    QBEX(NIPC)=QM
    CBEX(NIPC)=CM
    QM=QBEX(1)
    CM=CBEX(1)
    GO TO 59
 55 ISAC=ISAC+1
    NIPC=0
    QOL=QM
    COL=CM
    IF(ISAC-20)57,57,59
```

```
 59 IBXC=IBXC+1
    IF(IBXC-5)505,505,65
505 QGIBX=QM
    CGIBX=CM
    GO TO 501
 65 DQF=.005*SSQ
    DCF=.005*SSC
    WRITE(J,11)QM,CM,CRTR2,IBXC,ISAC
 11 FORMAT(10X,23HQM,CM,CRTR2,IBXC,ISAC,3E14.6,2I5)
    QF=QM
    CF=CM
    ISP=1
713 QF1=QF+DQF
    CF1=CF
    BPF1=DLP(QF1,PTS,WIC(ISK))
    GRF1=QF1
    PTS=GPTS(GRF1,BPF1,PTS,WIC(ISK))
    TPF1=PFT(GRF1,CF1,WIC(ISK),FN,QTS,CTS,PTS)
    QF2=QF-DQF
    CF2=CF
    BPF2=DLP(QF2,PTS,WIC(ISK))
    GRF2=QF2
    PTS=GPTS(GRF2,BPF2,PTS,WIC(ISK))
    TPF2=PFT(GRF2,CF2,WIC(ISK),FN,QTS,CTS,PTS)
    GRFO=(GRF1+GRF2)/2.0
    GRFS(ISP)=GRFO
    TPF=PFT(GRFO,CF,WIC(ISK),FN,QTS,CTS,PTS)
    DTPF1=TPF1-TPF
    DTPF2=TPF2-TPF
    QF3=QF
    CF3=CF+DCF
    TPF3=PFT(GRFO,CF3,WIC(ISK),FN,QTS,CTS,PTS)
    DTPF3=TPF3-TPF
    QF4=QF
    CF4=CF-DCF
    TPF4=PFT(GRFO,CF4,WIC(ISK),FN,QTS,CTS,PTS)
    DTPF4=TPF4-TPF
    IF(ISP-1)730,731,730
731 WRITE(J,482)GRFO,CF,TPF,BPF1,PTS
482 FORMAT(10X,22HGRFO,CF,TPF,BPF1,PTS,5E14.6)
730 SP2Q=(DTPF1+DTPF2)/((GRF1-GRFO)*(GRF1-GRFO))
    SP1Q=(DTPF1-DTPF2)/(2.0*(GRF1-GRFO))
    SP2C=(DTPF3+DTPF4)/(DCF*DCF)
    SP1C=(DTPF3-DTPF4)/(2.0*DCF)
    QF5=GRF2
    CF5=CF+DCF
    TPF5=PFT(QF5,CF5,WIC(ISK),FN,QTS,CTS,PTS)
    DQF1=QF5-GRFO
    DCF1=CF5-CF
```

```
      T1=SP1Q*DQF1
      T2=SP1C*DCF1
      T3=.5*SP2Q*DQF1*DQF1
      T5=.5*SP2C*DCF1*DCF1
      TPF5T=TPF+T1+T2+T3+T5
      DIFF=TPF5-TPF
      T6=DQF1*DCF1
      SPQC=(DIFF-T1-T2-T3-T5)/T6
      WRITE(J,483)TPF1,TPF2,TPF3,TPF4
  483 FORMAT(10X,21HTPF1,TPF2,TPF3,TPF4,5E14.6)
      WRITE(J,484)SP2Q,SP1Q,SP2C,SP1C
  484 FORMAT(10X,21HSP2Q,SP1Q,SP2C,SP1C ,5E14.6)
      WRITE(J,485)TPF5,TPF5T,DIFF,SPQC
  485 FORMAT(10X,22HTPF5,TPF5T,DIFF,SPQC,5E14.6)
      IF(ISP-1)711,711,712
  711 TOL=(TPF5-TPF5T)/TPF5
      IF(ABS(TOL)-.01)81,81,73
   81 CDQF1=-(SP1Q)/SP2Q
      CDCF1=-(SP1C)/SP2C
      H=CDQF1
      XK=CDCF1
      DQF=H
      DCF=XK
      GO TO 717
   73 FNUMQ=SPQC*SP1C-SP2C*SP1Q
      FNUMC=SPQC*SP1Q-SP2Q*SP1C
      FDEN=SP2Q*SP2C-SPQC*SPQC
      CDQF2=FNUMQ/FDEN
      CDCF2=FNUMC/FDEN
      H=CDQF2
      XK=CDCF2
      DQF=H
      DCF=XK
  717 ISP=ISP+1
      GO TO 713
  712 TST=H*H*SP2Q+2.*H*XK*SPQC+XK*XK*SP2C
      IF(TST)3,5,5
    3 QF=GRFS(1)+H
      CF=CF+XK
      WRITE(J,230)QF,CF,H,XK
  230 FORMAT(10X,12HQF,CF,H,XK,4E14.6)
    5 OPPFT=PFT(  QF,CF,WIC(ISK),FN,QTS,CTS,PTS)
      SUM=SUM+OPPFT
      WRITE(J,20)QF,CF,OPPFT,SUM,FN
   20 FORMAT(10X,20HQF,CF,OPPFT,SUM,FN,4E14.6,F5.1)
      IF(FN-1.)12,12,14
   12 GRFS(3)=PTS
   14 QTS=QF
      CTS=CF
      PTS=DLP(QF,PTS,WIC(ISK))
```

```
      FN=FN+1.0
      SIND=YRS/DT-FN+1.0
      IS=SIND
      QKPT(IS)=QF
      CKPT(IS)=CF
      PKPT(IS)=PTS
      IF(FN-YRS/DT)1,100,100
  100 EIND=YRS/DT
      IE=EIND
      QKPT(IE)=Q1
      CKPT(IE)=CKPT(IE-1)
      PKPT(IE)=GRFS(3)
      DO 291 JWR=1,IE
      IF(CKPT(JWR)-CMIN)815,817,817
  815 CKPT(JWR)=CMIN
  817 WRITE(J,34)QKPT(JWR),CKPT(JWR),PKPT(JWR)
  291 CONTINUE
   34 FORMAT(3E16.6)
      STOP
      END
```

CHAPTER VII

APPLICATION

The theoretical and numerical techniques of the last two chapters will be utilized to study the case described in this Chapter. Most of the numerical values used for this purpose are taken from papers published by Nahai, Ion and Graham on a government owned field in the Middle East.[26,36,45]

The field was discovered with an original reservoir pressure of 3559 psi. The estimated value of the reserves is 8.5 billion barrels of tank oil with an average compressibility factor of $6.8*10^{-6}$ psi^{-1}. Refinery limitations do not allow any water production from the field and artificial lift, and fluid injection are not practiced as as of this date. A pressure drawdown of 50 psi resulted in a production rate of 14,288 BPD from one well. It is estimated that a constant bottom hole pressure of 2200 psi prevails throughout the entire study period. The value of Δ_o is then

$$\Delta_o = P_o - P_w$$

$$\Delta_o = 3559 - 2200 = 1359 \text{ psi}$$

The average value of the productivity constant A_t is estimated to be 10.2 BPD/psi/well. The average reservoir pressure at the beginning of the study period is 3525 psi. Thus the initial pressure drop, $\Delta P_o = P_o - P_i$, becomes

$$\Delta P_o = 3559 - 3525 = 34.0 \text{ psi}$$

The production rate of the field at this time is

$$q_o = 85{,}000 \text{ BPD.}$$

Considering a five year study period and dividing it into ten equal time intervals, we have

$$\Delta t = .5 \text{ years} = 6 \text{ months}$$

In other words there will be ten stages in the dynamic programming optimization in this example. Water influx constants K are estimated for each one of the stages and recorded in Table 3. Cost analysis parameters are

1. cost of drilling a 6000. foot well with a $40.0/ft becomes

$$C_{dw} = \$240{,}000.$$

2. initial cost of a production unit with a capacity of 150,000 BPD is estimated at $500,000. The initial cost of a unit with a capacity of C_{pn} is computed from

$$C_{is} = 500{,}000. * (C_{pn}/150{,}000)^{.6}$$

TABLE 3

ESTIMATED WATER INFLUX

CONSTANTS OF THE VARIOUS STAGES

n	1	2	3	4	5	6	7	8	9	10
K_t	1230	1260	1300	1354	1427	1523	1652	1829	2067	2382

TABLE 4

COMPUTED OPTIMUM VALUES OF THE VARIABLES

Time Period	Production Rate BPD	Optimum Capacity Increase of the Processing Facilities, BPD
1	$.851*10^{5}$	$.873*10^{5}$
2	$.886*10^{5}$	$.968*10^{5}$
3	$.923*10^{5}$	$.982*10^{5}$
4	$.958*10^{5}$	$.101*10^{6}$
5	$.997*10^{5}$	$.105*10^{6}$
6	$.103*10^{6}$	$.109*10^{6}$
7	$.108*10^{6}$	$.119*10^{6}$
8	$.112*10^{6}$	$.279*10^{6}$
9	$.117*10^{6}$	$.500*10^{5}$
10	$.127*10^{6}$	$.500*10^{5}$

or

$$C_{is} = 391 * (C_{pn})^{.6}$$

3. Operating cost of the well is taken to be 80¢ per barrel.

4. Operating cost of the production unit is estimated at 20¢ per barrel.

Market demand is assumed to have a constant annual increase of 8 per cent, and a sale price of two dollars per barrel of crude oil is used in calculations.

To enter the optimization routine, estimated values for the production rate and the cumulative pressure drop of the first stage are required. Following values are used for this purpose.

$$\Delta P_1 = P_o - P_{t1} = 3559\text{-}3449 = 110$$

$$q_1 = q_i * e^{.08*5} = 85{,}000 * e^{.4} = 127{,}000$$

The output is summarized in Table 4. Some of the results computed by the machine were checked by means of hand calculations.

To observe the effect of time increment on the computational results, two separate runs were made, one with a six month and another with a three month time interval. The computed production rates of the two runs were in excellent agreement. However, a mild oscillation in the value of the processing capacity was noticed.

CHAPTER VIII

CONCLUSIONS

1. A mathematical model, based on the system theory notation, for simulation of the development operations of an oil field was devised.

2. Various elements of the development cost of a production system are computed by using a set of derived expressions.

3. To discourage excessive productions beyond the market demand or over the reservoir potential, an expression is introduced to compute the overproduction cost. This cost which acts as a penalty function increases very rapidly with the overproduction and allows only small amounts of excessive production in some of the stages.

4. A function for computation of the total profit of a typical oil field operation is developed.

5. To determine numerical values for the parameters of the profit function suitable expressions, procedures, and sources of information are given.

6. Practical constraints on the variables of the field are pointed out.

7. The recursion equation of dynamic programming for determination of the optimum development plan of the field is obtained.

8. A procedure, based on the steepest ascent concept and the Taylor series expansion of the profit function, is developed to determine the optimum points of the criterion functions.

9. Fortran coding and the numerical analysis pertaining to the entire routine is worked out and presented in the report.

10. A production system of unlimited potential with a constant annual increase in the demand is presented for a numerical example. The procedure proved to be applicable for determination of the optimum values of the variables of the system.

11. Necessity of further research using more critical field variables is hereby acknowledged. The optimum solution to a production system is sensitive to the form of the criterion function and the values of its parameters. The related sensitivity analysis should be made in the future. Maximization of the profit functions containing more than two variables is also recommended as a subject for new investigations.

NOMENCLATURE

A_t = constant in productivity equation

B_g = gas formation volume factor

B_{gi} = initial gas formation volume factor

B_o = oil formation volume factor

B_{oi} = initial oil formation volume factor

B_t = $\frac{A_t}{NC}$

C = the compressibility factor of oil

C_d = cost of drilling

C_{dw} = cost of drilling a well

C_{is} = initial cost of a separation station

C_{ow} = cost of operating a well per unit time

C_p = the capacity of a separation station in stock tank barrel of crude processed per day

C_w = cost of wells

d = constant percentage of yearly increase in the market demand

Δ = time increment between two subsequent stages of the system

Δ_o = original reservoir pressure minus the average bottom hole pressure

ΔP = original reservoir pressure minus the average reservoir pressure at time, t

ΔP_o = value of ΔP at the initial time point, t_o.

$\Delta T_p(n)$ = $T_{p2}(n) - T_{p1}(n)$

$\Delta \bar{V}$ = change in reservoir fluid volume

$$E(P) = \frac{1}{\varphi_W} = B_o - B_{oi} + B_g(R_{si} - R_s) + \frac{mB_{oi}}{B_{gi}}(B_g - B_{gi})$$

$$\eta_t = \lambda_t\left[q_i - q_{wp} + \frac{\gamma\Delta_o}{R}\right]$$

$$E'(P) = \frac{dE(P)}{dP}$$

$$F(P) = \frac{\varphi_Q}{\varphi_W} = B_o - R_s B_g$$

$$F'(P) = \frac{dF(P)}{dP}$$

$$\varphi_G = \frac{B_g}{B_o - B_{oi} + B_g(R_{si} - R_s) + \frac{mB_{oi}}{B_{gi}}(B_g - B_{gi})}$$

$$\varphi_Q = \frac{B_o - R_s \cdot B_g}{B_o - B_{oi} + B_g(R_{si} - R_s) + \frac{mB_{oi}}{B_{gi}}(B_g - B_{gi})}$$

$$\varphi_W = \frac{1}{B_o - B_{oi} + B_g(R_{si} - R_s) + \frac{mB_{oi}}{B_{gi}}(B_g - B_{gi})}$$

G_p = cumulative volume of gas produced, standard conditions

$H(P)$ $= \frac{\varphi_G}{\varphi_W} = B_g$

$H'(P)$ $= \frac{dH(P)}{dP}$

K = water influx constant

λ_t $= n_t\beta_t = n_t \cdot \frac{A_t}{NC}$

m = ratio of initial gas-reservoir volume to initial reservoir oil volume

MD_n = market demand at the nth stage of development

m_{q_n} = the slope of the profit surface in the q_n direction

m_{CP_n} = the slope of the profit surface in the Cp_n direction

N = volume of original oil in place, stock tank conditions

γ $= \frac{K}{NC}$

n_t = number of producing wells at time, t

O_c = proportionality constant of the overproduction function

O_{cs} = operating cost of a separation station per unit production

P_i = reservoir pressure at time, i

P_o = original reservoir pressure

P_{pc} = sale price of one barrel of crude oil

P_t = average reservoir pressure at time, t

Q = cumulative volume of oil produced in terms of stock tank volume

Q_i = cumulative volume of water injected into the reservoir

q_e = dW/dt = the rate of water influx at time, t

q_G = dG_p/dt = the rate of gas production at time, t, standard volume per unit time

q_i = dQ_i/dt = the rate of water injection at time, t

q = dQ/dt = the rate of oil production at time, t

q_{wp} = dW_p/dt = the rate of water production at time, t

R = $\frac{1}{NC}$

R_p = producing gas oil ratio

R_s = solution gas oil ratio

R_{si} = initial solution gas oil ratio

SU_t = number of separation stations in operation at time, t

t = time

T_p = total profit of the entire study period

$T_{p,I}$ = cumulative profit made beginning from the Ith stage to the end of the study period

$T_p(n)$ = profit of the nth stage

max $T_{p,I}$ = maximum value of $T_{p,I}$

W_e = volume of encroached water

W_p = cumulative volume of produced water

ξ_t = $\gamma + \lambda_t - \frac{1}{\lambda_t} \cdot \frac{d\lambda_t}{dt}$

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APPENDIX A

STATE EQUATION OF PRESSURE - ABOVE BUBBLE POINT

Reservoir pore volume, assumed to be a constant, is a container for N stock tank barrels of original oil in place. When the pressure on the oil within this container is reduced the volume of oil changes. Any increase in the volume of oil would be equal to the amount of oil withdrawn (produced) from the container. The definition of compressibility factor of oil at constant temperature is[60]

$$C = \frac{-dV}{VdP} \simeq -\frac{\Delta V}{V \cdot \Delta P}$$

or

$$C = -\frac{V_2 - V_1}{V(P_2 - P_1)} = -\frac{\text{change in the volume of original oil}}{N(P_t - P_o)}$$

Now the volume change of N is really equal to the amount of production minus the volume of the material which has entered the reservoir pore volume. With these details one can conclude that

$$C = -\frac{Q + W_p - Q_i - W_e}{N(P_t - P_o)} = \frac{Q + W_p - Q_i - W_e}{N(P_o - P_t)}$$

from which

$$P_o - P_t = \frac{1}{NC}\,[Q+W_p-Q_i-W_e].$$

Differentiating this last expression with respect to time, t, we will have

$$\frac{d\Delta P}{dt} = \frac{1}{NC}\,[q+q_{wp}-q_i-q_{we}].$$

Water Influx Rate

In may cases a steady state water influx may be assumed.[56] The rate at which water enters a field is directly proportional to the decline in reservoir pressure. Assuming that the pressure in the water bearing strata remains the same, the pressure difference between that and that of the oil and gas reservoir is $P_t - P_o$. P_o is the original reservoir pressure and P_t is any subsequent value. The flow of water to the reservoir, by Darcy's law, is[15,43]

$$q_{we} = \frac{d_{We}}{dt} = K[P_o-P_t] = K\cdot\Delta P$$

where K is the water influx constant.

Using $q_{we}=K\cdot\Delta P$ in the previous state equation of pressure it may be concluded that

$$\frac{d\Delta P}{dt} = \frac{1}{NC}\,[q+q_{wp}-q_i-K\cdot\Delta P]$$

and

$$\frac{d\Delta P}{dt} + \frac{K}{NC} \cdot \Delta P = \frac{q + q_{wp} - q_i}{NC}$$

Let $\frac{K}{NC} = \gamma$ and $\frac{1}{NC} = R$, then

$$\frac{d\Delta P}{dt} + \gamma \cdot \Delta P = R \cdot (q + q_{wp} - q_i)$$

This last expression is the state equation of pressure of an oil field above bubble point.

APPENDIX B

STATE EQUATION OF PRESSURE - BELOW BUBBLE POINT

In order to obtain an expression explaining the variation of pressure with time, which could be applicable to both above and below the bubble point pressure, one may make use of the material balance equation.[44] This equation is a statement of the law of conservation of matter as applied to the gross fluid contents of the oil reservoir.

Upon the start of production from an oil reservoir, there will be changes in volumes of oil, gas and water present in the reservoir. However, the volume of the reservoir is assumed to remain constant. Thus the general material balance equation is a volumetric balance which states that the algebraic sum of the changes in volumes of oil, gas and water must be zero. In the form of an algebraic equation the following relation is valid.

$$\left[NB_{oi} - (N-Q)B_o\right] + \left\{NmB_{oi} - \left[\frac{NmB_{oi}}{B_{gi}} + NR_{si} - QR_p - (N-Q)R_s\right]B_g\right\}$$

$$= \left[W_e + Q_i - Q_p\right] .$$

The symbols used in this balance are defined below.

N = stock tank volume of original oil in place

B_{oi} = initial oil formation volume factor

Q = cumulative stock tank volume of oil produced

B_o = oil formation volume factor[6,57]

m = ratio of reservoir gas-cap volume to reservoir oil volume

B_{gi} = initial gas formation volume factor

R_{si} = initial solution gas-oil ratio

R_p = producing gas-oil ratio

R_s = solution gas-oil ratio

B_g = gas formation volume factor

W_e = cumulative volume of encroached water

Q_i = cumulative volume of injected water

Q_p = cumulative volume of produced water

If this expression is solved for N, the following result will be obtained.

$$N = \frac{Q[B_o - R_s B_g] + G_p B_g - [W_e + Q_i - W_p]}{B_o - B_{oi} + B_g(R_{si} - R_s) + \frac{mB_{oi}}{B_{gi}}(B_g - B_{gi})}$$

Now according to Tracy, we can define φ_Q, φ_G and φ_W as follows[59]

$$\varphi_Q = \frac{B_o - R_s B_g}{B_o - B_{oi} + B_g(R_{si} - R_s) + \frac{mB_{oi}}{B_{gi}}(B_g - B_{gi})}$$

$$\varphi_G = \frac{B_g}{B_o - B_{oi} + B_g(R_{si} - R_s) + \frac{mB_{oi}}{Bg_i}(B_g - B_{gi})}$$

$$\varphi_W = \frac{1}{B_o - B_{oi} + B_g(R_{si} - R_s) + \frac{mB_{oi}}{B_{gi}}(B_g - B_{gi})}$$

Since φ_Q, φ_G and φ_W are functions of reservoir pressure only, the expression for N could be rewritten as given below.

$$N = Q \cdot \varphi_Q(P) + G_p \cdot \varphi_G(P) - (W_e + Q_i - W_p)\varphi_W(P).$$

If we divide both sides of this equation by φ_W and make some rearrangements the following relation will be obtained

$$[W_e + Q_i - W_p] = Q\frac{\varphi_Q}{\varphi_W} + G_p\frac{\varphi_G}{\varphi_W} - \frac{N}{\varphi_W}.$$

Now let

$$E(P) = \frac{1}{\varphi_W} = B_o - B_{oi} + B_g(R_{si} - R_s) + \frac{mB_{oi}}{B_{gi}}(B_g - B_{gi}),$$

$$F(P) = \frac{\varphi_Q}{\varphi_W} = B_o - R_s \cdot B_g$$

and

$$H(P) = \frac{\varphi_G}{\varphi_W} = B_g$$

then

$$[W_e + Q_i - W_p] = Q \cdot F(P) + G_p \cdot H(P) - N \cdot E(P).$$

Differentiating this expression with respect to time will give

$$\frac{d}{dt}[W_e + Q_i - W_p] = \frac{d}{dP}[Q \cdot F(P) + G_p \cdot H(P) - N \cdot E(P)]\frac{dP}{dt}$$

$$\frac{dW_e}{dt} + \frac{dQ_i}{dt} - \frac{dW_p}{dt} = \left[\frac{dQ}{dP} \cdot F(P) + \frac{dG_p}{dP} \cdot H(P) - \frac{dN}{dP} \cdot E(P) + Q\frac{dF(P)}{dP} + G_p\frac{dH(P)}{dP} - N\frac{dE(P)}{dP}\right]\frac{dP}{dt}.$$

or

$$q_e + q_i - q_{wp} = \frac{dQ}{dt} \cdot F(P) + \frac{dG_p}{dt} \cdot H(P) + \left[Q \cdot \frac{dF(P)}{dP} + G_p\frac{dF(P)}{dp} - N \cdot \frac{dE(P)}{dP}\right] \cdot \frac{dP}{dt}$$

denoting

$\frac{dF(P)}{dP}$, $\frac{dH(P)}{dP}$ and $\frac{dE(P)}{dP}$ by $F'(P)$, $H'(P)$ and $E'(P)$

respectively, this equation becomes

$$\left[N \cdot E'(P) - Q \cdot F'(P) - G_p \cdot H'(P)\right] \cdot \frac{dP}{dt} = q \cdot F(P) + q_G \cdot H(P) - (q_e + q_i - q_{wp})$$

Numerical value of $E'(P)$ could be obtained by plotting $E(P)$ against pressure. The slope of these curves at any particular pressure is the value of $E'(P)$ at that pressure. Values for $F'(P)$ and $H'(P)$ can be computed in a similar fashion.

At a given reservoir pressure let us adopt the following notations

$$E'(P) = a_1$$

$$F'(P) = C_1$$

$$H'(P) = C_2$$

and

$$\text{instantaneous producing water-oil-ratio} = \frac{q_{wp}}{q} = a_3$$

$$\text{instantaneous injecting water-oil-ratio} = \frac{q_i}{q} = a_4$$

$$\text{instantaneous producing gas-oil-ratio} = \frac{q_G}{q} = R_p$$

Substituting these values in the last equation we will have

$$\left[N \cdot a_1 - Q \cdot C_1 - Q \cdot R_p \cdot C_2\right] \cdot \frac{dP}{dt} = q \cdot F(P) + q \cdot R_p \cdot H(P) - q_e - q \cdot a_4 + q \cdot a_3 .$$

or

$$\left[N \cdot a_1 - Q \cdot C_1 - Q \cdot R_p \cdot C_2\right] \cdot \frac{dP}{dt} = q\left[F(P) + R_p \cdot H(P) + a_3 - a_4\right] - q_e$$

Adopting a further notation $\Delta P = P_o - P$ and assuming that q_e could be obtained from the steady state equation $q_e = K \cdot \Delta P$ then the previous differential equation becomes

$$\left[N \cdot a_1 - Q(C_1 + R_p \cdot C_2)\right] \cdot \left(\frac{-d\Delta P}{dt}\right) = q\left[F(P) + R_p \cdot H(P) + a_3 - a_4\right] - K \cdot \Delta P$$

from which

$$\frac{d\Delta P}{dt} = \frac{K}{N \cdot a_1 - Q(C_1 + R_p \cdot C_2)} \cdot \Delta P - \frac{q \cdot [F(P) + R_p \cdot H(P) + a_3 - a_4]}{N \cdot a_1 - Q(C_1 + R_p \cdot C_2)}$$

The form of this equation is the same as the one in the previous appendix, i.e.

$$\frac{d\Delta P}{dt} + \gamma \cdot \Delta P = R(P) \cdot f(q, q_{wp}, q_i, q_G)$$

APPENDIX C

STATE EQUATION OF RATE OF PRODUCTION

The state equation of pressure developed in Appendix A is

$$\frac{d\Delta P}{dt} + \gamma\cdot\Delta P = R\cdot(q + q_{wp} - q_i)$$

The rate of production was assumed to be represented by

$$q = n_t\cdot A_t[P_t - P_w] = n_t\cdot A_t\cdot[P_o - P_w - (P_o - P_t)]$$

$$q = n_t\cdot A_t[\Delta_o - \Delta P]$$

$$q = n_t\cdot N\cdot C\cdot\beta_t\cdot[\Delta_o - \Delta P]$$

$$q = \frac{1}{R}\cdot n_t\cdot\beta_t\cdot[\Delta_o - \Delta P]$$

$$q = (\lambda_t/R)\cdot[\Delta_o - \Delta P]$$

from which

$$\Delta P = \Delta_o - (R\cdot q/\lambda_t)$$

Differentiating the expression for q with respect to time t we will have

$$\frac{dq}{dt} = \frac{1}{R}\left[\frac{d\lambda_t}{dt}[\Delta_o - \Delta P] + \left(\frac{d}{dt}[\Delta_o - \Delta P]\right)\cdot\lambda_t\right]$$

$$\frac{dq}{dt} = \frac{1}{R}\left[\frac{d\lambda_t}{dt}[\Delta_o - \Delta P] + \left(\frac{d\Delta_o}{dt} - \frac{d\Delta P}{dt}\right)\lambda_t\right]$$

or

$$\frac{dq}{dt} = \frac{1}{R}\left[\frac{d\lambda_t}{dt}[\Delta_o - \Delta P] + \frac{d\Delta_o}{dt}\lambda_t - \frac{d\Delta P}{dt}\lambda_t\right]$$

Since it was assumed that the flowing bottom hole pressure remains the same throughout the operation then

$$\frac{d\Delta_o}{dt} = \frac{d}{dt}[P_o - P_w] = \frac{dP_o}{dt} - \frac{dP_w}{dt} = 0$$

and

$$\frac{dq}{dt} = \frac{1}{R}\left[\frac{d\lambda_t}{dt}[\Delta_o - \Delta P] - \frac{d\Delta P}{dt}\cdot\lambda_t\right].$$

Considering

$$\Delta_o = \Delta P + \frac{R\cdot q}{\lambda_t}$$

and from the previous appendix

$$\frac{d\Delta P}{dt} = R\cdot(q + q_{wp} - q_i) - \gamma\cdot\Delta P$$

then

$$\frac{dq}{dt} = \frac{1}{R}\left\{\frac{d\lambda_t}{dt} \circ \frac{R\cdot q}{\lambda_t} - \left[R\cdot(q+q_{wp}-q_i) - \gamma\circ\Delta P\right]\cdot\lambda_t\right\}$$

$$\frac{dq}{dt} = \frac{1}{R}\left\{\frac{R\circ q}{\lambda_t} \cdot \frac{d\lambda_t}{dt} - \lambda_t\left[R(q+q_{wp}-q_i) - \gamma(\Delta_o - \frac{R\circ q}{\lambda_t})\right]\right\}$$

$$\frac{dq}{dt} = \left\{\frac{q}{\lambda_t} \cdot \frac{d\lambda_t}{dt} - \lambda_t\left[(q+q_{wp}-q_i) - \frac{\gamma\Delta_o}{R} + \frac{\gamma q}{\lambda_t}\right]\right\}$$

$$\frac{dq}{dt} = \frac{q}{\lambda_t} \circ \frac{d\lambda_t}{dt} - \lambda_t q - \lambda_t q_{wp} + \lambda_t q_i + \frac{\lambda_t\gamma\Delta_o}{R} - \gamma q.$$

After some rearrangement and factorization

$$\frac{dq}{dt} + \left[\gamma + \lambda_t - \frac{1}{\lambda_t} \circ \frac{d\lambda_t}{dt}\right]q = \lambda_t\left[q_i - q_{wp} + \frac{\gamma\Delta_o}{R}\right].$$

Let

$$\gamma + \lambda_t + \frac{1}{\lambda_t} \circ \frac{d\lambda_t}{dt} = \xi_t$$

and

$$\lambda_t\left[q_i - q_{wp} + \frac{\gamma\Delta_o}{R}\right] = \eta_t,$$

the state equation of rate of production would then become

$$\frac{dq}{dt} + \xi_t \cdot q = \eta_t$$

APPENDIX D

SOLUTION OF THE STATE DIFFERENTIAL EQUATIONS

The state equations developed in the previous appendices are of the general form of

$$\frac{dy}{dx} + y \cdot f(x) = g(x) .$$

The solution of such a differential equation can be obtained by making a total derivative out of the left hand side of this differential equation.[9] To accomplish this task one has to multiply both sides of the equation by a function $\varphi(x)$. The result would be

$$\varphi(x) \frac{dy}{dx} + y \cdot \varphi(x) \cdot f(x) = \varphi(x) \cdot g(x) .$$

In order to have a total derivative in the left hand side, $\varphi(x) \cdot f(x)$ must be equal to $d\varphi(x)/dx$. In an equation form we should have

$$\frac{d\varphi(x)}{dx} = \varphi(x) \cdot f(x)$$

From this last equation one may compute the value of unknown function $\varphi(x)$ as carried out below.

$$\frac{d\varphi(x)}{dx} = \varphi(x) \cdot f(x)$$

or

$$\frac{d\varphi(x)}{\varphi(x)} = f(x) \cdot dx$$

One particular solution to this equation is

$$\ln \varphi(x) = \int f(x)\,dx$$

or

$$\varphi(x) = e^{\int f(x) \cdot dx}$$

Having the exact expression for $\varphi(x)$ the original differential equation can be written as

$$e^{\int f(x)\,dx} \cdot \frac{dy}{dx} + y \cdot e^{\int f(x)\,dx} \cdot f(x) = e^{\int f(x)\,dx} \cdot g(x)$$

or

$$\frac{d}{dx}\left[y \cdot e^{\int f(x)\,dx}\right] = e^{\int f(x)\,dx} \cdot g(x)$$

and

$$d\left[y \cdot e^{\int f(x)\,dx}\right] = \left[e^{\int f(x)\,dx} \cdot g(x)\right] dx.$$

By integrating this expression we will get

$$y \cdot e^{\int f(x)\,dx} = \int g(x) \cdot e^{\int f(x)\,dx} \cdot dx + C$$

from which

$$y = e^{-\int f(x)\,dx} \cdot \int g(x)\, e^{\int f(x)\,dx} \cdot dx + C\, e^{-\int f(x)\,dx}$$

or

$$y = e^{-\int f(x)\,dx} \left[\int g(x)\, e^{\int f(x)\,dx} \cdot dx + C \right]$$

This expression holds for all values of x, i.e.

$$y = e^{\left[-\int f(x)\,dx\right]_{x=x}} \left\{ \left[\int g(\xi)\, e^{\left[\int f(x)\,dx\right]_{x=\xi}} d\xi \right]_{\xi=x} + C \right\}$$

To evaluate the constant of integration C one has to substitute the initial condition in this solution.

$$y_o = e^{\left[-\int f(x)\,dx\right]_{x=x_o}} \left\{ \left[\int g(\xi)\, e^{\left[\int f(x)\,dx\right]_{x=\xi}} d\xi \right]_{\xi=x_o} + C \right\}$$

from which

$$C = y_o\, e^{\left[\int f(x)\,dx\right]_{x=x_o}} - \left\{ \left[\int g(\xi)\, e^{\left[\int f(x)\,dx\right]_{x=\xi}} d\xi \right]_{\xi=x_o} \right\}$$

Having obtained the value of C the general solution would be

$$y = e^{\left[-\int f(x)\,dx\right]_{x=x}} \left\{ \left[\int g(\xi)\, e^{\left[\int f(x)\,dx\right]_{x=\xi}} d\xi\right]_{\xi=x} + \right.$$

$$\left. y_o e^{\left[\int f(x)\,dx\right]_{x=x_o}} - \left[\int g(\xi)\, e^{\left[\int f(x)\,dx\right]_{x=\xi}} d\xi\right]_{\xi=x_o} \right\}$$

One can factorize $y_o e^{\left[\int f(x)\,dx\right]_{x=x_o}}$, and then

$$y = e^{\left[-\int f(x)\,dx\right]_{x=x}} \left\{ y_o e^{\left[\int f(x)\,dx\right]_{x=x_o}} \left[1 + \right.\right.$$

$$\left.\left. \frac{\left[\int g(\xi)\, e^{\left[\int f(x)\,dx\right]_{x=\xi}} d\xi\right]_{\xi=x}}{y_o e^{\left[\int f(x)\,dx\right]_{x=x_o}}} - \frac{\left[\int g(\xi)\, e^{\left[\int f(x)\,dx\right]_{x=\xi}} d\xi\right]_{\xi=x_o}}{y_o e^{\left[\int f(x)\,dx\right]_{x=x_o}}} \right] \right\}$$

Now since $e^{\left[\int f(x)\,dx\right]_{x=x_o}}$ is a constant, it can go under the integral sign of the numerators. Therefore

$$\frac{1}{y_o} \cdot \frac{\left[\int g(\xi)\, e^{\left[\int f(x)\,dx\right]_{x=\xi}} d\xi\right]_{\xi=x}}{e^{\left[\int f(x)\,dx\right]_{x=x_o}}} = \frac{1}{y_o}\left[\int g(\xi) \left\{ e^{\left[\int f(x)\,dx\right]_{x=\xi}} \cdot \right.\right.$$

$$\left.\left. e^{-\left[\int f(x)\,dx\right]_{x=x_o}} \right\} d\xi\right]_{\xi=x} = \frac{1}{y_o}\left[\int g(\xi)\, e^{\int_{x_o}^{\xi} f(x)\,dx} d\xi\right]_{\xi=x}$$

By the same token

$$\frac{1}{y_o} \cdot \frac{\left[\int g(\xi)\, e^{\left[\int f(x)\,dx\right]_{x=\xi}} d\xi\right]_{\xi=x_o}}{e^{\left[\int f(x)\,dx\right]_{x=x_o}}} =$$

$$\frac{1}{y_o}\left[\int g(\xi)\left\{e^{\left[\int f(x)\,dx\right]_{x=\xi}} \cdot e^{-\left[\int f(x)\,dx\right]_{x=x_o}}\right\} d\xi\right]_{\xi=x_o}$$

$$= \frac{1}{y_o}\left[\int g(\xi)\, e^{\int_{x_o}^{\xi} f(x)\,dx} d\xi\right]_{\xi=x_o}$$

Substituting these two expressions in the previous general solution for y we will have

$$y = y_o e^{\left[-\int f(x)\,dx\right]_{x=x}} \left\{ e^{\left[\int f(x)\,dx\right]_{x=x_o}} \left[1 + \frac{1}{y_o}\right.\right.$$

$$\left.\left.\left[\int g(\xi)\, e^{\int_{x_o}^{\xi} f(x)\,dx} d\xi\right]_{\xi=x} - \frac{1}{y_o}\left[\int g(\xi)\, e^{\int_{x_o}^{\xi} f(x)\,dx} d\xi\right]\right]_{\xi=x_o}\right\}$$

or

$$y = y_o e^{-\left\{\left[\int f(x)\,dx\right]_{x=x} - \left[\int f(x)\,dx\right]_{x=x_o}\right\}}$$

$$\left\{ 1 + \frac{1}{y_o} \int_{x_o}^{x} g(\xi)\, e^{\int_{x_o}^{\xi} f(x)\,dx}\, d\xi \right\}.$$

and finally

$$\frac{y}{y_o} = e^{-\int_{x_o}^{x} f(x)\,dx} \left\{ 1 + \frac{1}{y_o} \int_{x_o}^{x} g(\xi)\, e^{\int_{x_o}^{\xi} f(x)\,dx}\, d\xi \right\}.$$

APPENDIX E

SOLUTION OF DIFFERENTIAL EQUATIONS OF ΔP AND q

Appendices A, B and C were devoted to the development of state equations for ΔP and q.

The general format for all of these equations were of the form

$$\frac{dy}{dx} = f(x) \cdot y = g(x) .$$

The solution for ΔP and q therefore, is simply the application of Appendix D to these equations.

1. Solution for ΔP.

The state equation is

$$\frac{d\Delta P}{dt} + \gamma \cdot \Delta P = R \cdot (q + q_{wp} - q_i)$$

whose solution according to Appendix D is

$$\frac{\Delta P}{\Delta P_o} = e^{-\int_{t_o}^{t} \gamma \cdot dt} \left\{ 1 + \frac{1}{\Delta P_o} \int_{t_o}^{t} R \cdot (q + q_{wp} - q_i) \cdot e^{\int_{t_o}^{\tau} \gamma \cdot dt} \, d\tau \right\} .$$

2. Solution for the Rate of Production, q.

The state equation of q, as developed in Appendix C is

$$\frac{dq}{dt} + \xi_t \cdot q = \eta_t .$$

The solution to this differential equation is

$$\frac{q}{q_o} = e^{-\int_{t_o}^{t} \xi_t \cdot dt} \left\{ 1 + \frac{1}{q_o} \int_{t_o}^{t} \eta_\tau \cdot e^{\int_{t_o}^{\tau} \xi_t \cdot dt} \, d\tau \right\} .$$

Substituting for the variables ξ_t and η_t one will have

$$\frac{q}{q_o} = \frac{\left\{ 1 + \frac{1}{q_o} \int_{t_o}^{t} \lambda_\tau \left(q_i - q_{wp} + \frac{\gamma \Delta_o}{R} \right) e^{\left[\int_{t_o}^{\tau} (\gamma + \lambda_t) dt - \int_{t_o}^{\tau} \frac{1}{\lambda_t} \cdot \frac{d\lambda_t}{dt} dt \right]} d\tau \right\}}{e^{\left[\int_{t_o}^{t} (\gamma + \lambda_t) dt - \int_{t_o}^{t} \frac{1}{\lambda_t} \cdot \frac{d\lambda_t}{dt} dt \right]}}$$

now since[64]

$$e^{\int_{t_o}^{t} \frac{1}{\lambda} \frac{d\lambda}{dt} dt} = e^{\int_{t_o}^{t} \frac{d\lambda}{\lambda}} = e^{\ln \lambda \Big]_{t_o}^{t}} = e^{\ln \frac{\lambda}{\lambda_o}} = \frac{\lambda}{\lambda_o}$$

and

$$e^{-\int_{t_o}^{t} \frac{1}{\lambda}\frac{d\lambda}{dt}\,dt} = e^{-\int_{t_o}^{t} \frac{d\lambda}{\lambda}} = e^{-\ell n\,\lambda\Big]_{t_o}^{t}} = e^{\ell n \frac{\lambda_o}{\lambda}} = \frac{\lambda_o}{\lambda}$$

then

$$\frac{q}{q_o} = \frac{\lambda_t}{\lambda}\, e^{-\int_{t_o}^{t}(\gamma+\lambda_t)dt} \left\{ 1 + \frac{1}{q_o}\int_{t_o}^{t} \lambda_\tau (q_i - q_{wp} + \frac{\gamma\Delta_o}{R}) \cdot e^{\int_{t_o}^{\tau}(\gamma+\lambda_t)dt}\, d\tau \right\}$$

or

$$\frac{q}{q_o} = \frac{\lambda_t}{\lambda_o}\, e^{-\int_{t_o}^{t}(\gamma+\lambda_t)dt} \left\{ 1 + \frac{\lambda_o}{q_o}\int_{t_o}^{t} (q_i - q_{wp} + \frac{\gamma_\tau\Delta_o}{R})\, e^{\int_{t_o}^{\tau}(\gamma+\lambda_t)dt}\, d\tau \right\}$$

doing some substitutions, the expression for q/q_o finally becomes

$$\frac{q}{q_o} = \frac{n_t\beta_t}{n_o\beta_o}\, e^{-\int_{t_o}^{t}(\gamma+n_t\beta_t)dt} \left\{ 1 + \frac{n_o\beta_o}{q_o}\int_{t_o}^{t} (q_i - q_{wp} + \frac{\gamma_\tau\Delta_o}{R})\, e^{\int_{t_o}^{\tau}(\gamma+n_t\beta_t)\cdot dt}\, d\tau \right\}$$

This is a relation between the rate of production and the number of wells drilled at any time t.