PADÉ APPROXIMANTS

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CHAPTER I

INTRODUCTION

In 1892 a thesis entitled, "Sur La Representation Approchie D'une Fonction Par Des Fractions Rationnelles", Par M. H. Pade', Ancien E'le've De L E'cole Normale Supe'ri eure, Professeur Agrege De L Universite was published in the Annales Scientifiques de S'Ecole Normale Supe'rieure , Serie 3, Volume 9, pages 1-93 in the supplement. This paper was the inception of what is now called "Pade' approximants" which are rational function approximations of functions known by their power series. In his thesis he looks at several examples and proves theorems which have been used as a basis for recent studies that appear in the literature, some of which, are in this paper.

Though this thesis is about Pade' approximants which started with Pade' and his thesis, there was another mathematician, T. J. Stieltjes, who was interested in summing divergent series and influenced greatly that which has been done relative to the subject of Pade' approximants. Hence, some of the basic results printed in this paper, which appeared first in the Annales Faculte' Sciences Toulouse, Volume 9, 1894, are due to T. J. Stieltjes.

Also G. A. Baker, who has been publishing articles on Pade' approximants recently, must be credited with many of the results in this paper.

Pade' approximants are usually studied relative to two objectives.

The physicist or applied mathematician uses Pade′ approximants to provide rational function approximation to mathematical functions that describe certain physical phenomona. For instance, the physicist uses Pade′ approximants in studying perturbation series associated with wave propogation. Also, there is effort toward abstracting quantitative information about functions that are known only from their power series and qualitative behavior. For example, mathematicians have sought for a long time to sum divergent series or to find methods of interpreting divergent series.

In the second chapter an algebraic setting is developed in which the Pade′ approximants are explicitly defined. Then two theorems are presented which show when the Pade′ approximants are invariant relative to linear fractional transformations. In realizing their proofs some lemmas are proved which facilitate understanding the concept of Pade′ approximants. Also in this chapter are theorems on what can be said when the Pade′ approximants are uniformly bounded. In effecting such answers the concepts of polynomial functions and power series functions are important as is the concept of analytic continuation. Hence the analogy between the algebraic setting and the symbology often used in studying these concepts in analysis is made so that these concepts may be treated and utilized in the proofs of certain theorems. Along with these results there is presented a method of calculating the Pade′ approximants if the power series is given and a method of calculating what the coefficients of the power series are if the Pade′ approximants are given.

In the third chapter a very important type of power series is studied. This series is a series of Stieltjes and is important because many power series occur in this form. In fact there are many different power series which describe physical phenomena that may also be reduced to a series of Stieltjes. See (4). One important result presented in this chapter is the solution to the Stieltjes moment problem which must be used in proving some of the theorems pertaining to the series of Stieltjes. An interesting result in this exposition relative to a series of Stieltjes, the proofs of which are detailed, is that the poles of a given sequence of Padé approximants interlace. Furthermore, relative to these Padé approximants, it is shown that the residues are all positive and that the zeros of the numerators interlace those of the denominators.

Relative to the nonnegative real axis the Padé approximants will be seen as a monotonic sequence which converges to an analytic function. This function is above one sequence and below another so that for a series of Stieltjes upper and lower bounds are obtained for the limit function whether the series converges or not. Finally, it is shown that the Padé approximants, under suitable restrictions, converge to an analytic function in the cut plane $(-\infty < z \leq 0)$.

The theorems that have been proved in the dissertation do not in any sense exhaust the questions that one might ask relative to the subject of Padé approximants. In fact they add to the questions that one might ask and point to the infancy of the subject. The theory presented then motivates the last chapter in which some of these questions are discussed.

The initial problems, that were solved in effecting the thesis just mentioned, were those of bringing to some logical order, the recent publications in the area of Padé approximants, the proving of certain assertions, by supplying the details to skeleton type proofs, that had been made in the literature, and the writing of these proofs so that, at the high undergraduate level or beginning graduate level, a seminar can be profitably conducted. As a result of this type of research new facts sometimes unfold as does the discarding of old facts which are found to be in error. In the last chapter the former is exemplified and a counter example to a statement used in proving that certain of the diagonal sequences converge to the same limit function is offered.

The preceeding gives some historical facts about Pade' approximants and gives some information about three men who influenced their development. Some of the problems that have been solved were discussed and a brief introduction to the content of the thesis has been presented. Now a look at some of the general results in the thesis will be taken.

CHAPTER II

PADE′ APPROXIMANTS IN GENERAL

Some Preliminary Remarks

An algebraic setting will now be developed in preparation to defining the Pade′ approximants.

$$L = \{(\cdots,0,a_m,\cdots,a_{-1},a_0,a_1,\cdots) : a_i \in F \text{ a field}\}$$

will be called a field of formal meromorphic series. If

$$f = (\cdots,0,a_m,\cdots) \in L$$

and

$$g = (\cdots,0,b_n,\cdots) \in L$$

then

$$f + g = \{a_k + b_k\}_k$$

and

$$f \cdot g = \sum_{n=-\infty}^{\infty} a_n b_{k-n}\}_k.$$

Notice in the definition of multiplication that each element is a finite sum.

In this field

$$1 = (\cdots,0,a_0,0,\cdots)$$

where $a_0 = 1$ and $0 = (\cdots,0,\cdots)$.

The additive inverses are obvious but since the multiplicative inverses are not quite so obvious a method for calculating them will be presented. Let

$$f = (\cdots,0,a_m,\cdots) \in L$$

where it is being assumed that if $n < m$ then $a_n = 0$ but that $a_m \neq 0$.

Also set $g = (\cdots, 0, b_{-m}, \cdots)$ and make $a_m b_{-m} = 1$. Observe that

$$\sum_{n=-\infty}^{\infty} a_n b_{k-n} = 0 \quad \text{if } k < 0,$$

$$\sum_{n=-\infty}^{\infty} a_n b_{k-n} = 1 \quad \text{if } k = 0,$$

$$\sum_{n=-\infty}^{\infty} a_n b_{k-n} = a_m b_{-m+1} b_{-m} \quad \text{if } k = 1.$$

If $a_m b_{-m+1} + a_{m+1} b_{-m} = 0$ then b_{-m+1} can be uniquely solved for since $a_m \neq 0$. Now for any $k > 1$

$$\sum_{n=-\infty}^{\infty} a_n b_{k-n} = a_m b_{k-m} + a_{m+1} b_{k-m-1} + \cdots + a_{m+k} b_{-m} = 0$$

implies that b_{k-m} can be uniquely solved for provided the b_i, $i < k - m$, have already been determined.

Define P as $\{(\cdots, 0, a_0, a_1, a_2, \cdots, a_n, 0, \cdots)\}$. Then $P \subset L$. If $f \in P$, f is often written as $(a_0, a_1, \cdots, a_n)$.

If $$L_m = \{a : a \in L \text{ and for } n < m \ a_n = 0\}$$

and

$$L^m = \{a : a \in L \text{ and for } n > m \ a_n = 0\}$$

define

$$L_n^m = L^m + L_n$$

and define a function γ^n on L to L such that if

$$f = (\cdots, 0, a_r, \cdots, a_n, a_{n+1}, \cdots) \in L$$

then $$\gamma^n(f) = (\cdots, 0, a_r, \cdots, a_n, 0, \cdots).$$

For nonnegative integers M and N let $A_M = \{\gamma^M(f) : f \in L_0\}$ and

$$B_N = \{\gamma^N(f) : f \in \widetilde{L}_0 = \{f : f \in L_0, a_0 \neq 0\}\}.$$

Denote arbitrary elements of A_M as $P_M = (a_0, a_1, \cdots, a_M)$ and arbitrary elements of B_N as $Q_N = (b_0, b_1, \cdots, b_N)$, where it is realized that many of the a_i and b_i may be zero. In concluding these preliminary remarks define

$$R_N^M = \{P_M/Q_N : P_M \in A_{M_1}, Q_N \in B_N\}.$$

The Pade′ Approximant

The Pade′ approximants are rational function approximations to a formal power series. The algebraic analogues are discussed first. After a brief discussion of the approximants for elements in $\tilde{L}_0$ it will be easy to see how to handle all other members of L.

For every pair of integers N and M define a function π_N^M on L_0 to R_N^M as follows: If $f \in L_0$, then $\pi_N^M(f) = P_M/Q_N$ is called the [M,N] Pade′ approximant of f, where Q_N is some non-zero element of B_N such that $Q_N \cdot f \in L_{N+M+1}^M$ and $P_M = \gamma^M(Q_N \cdot f)$. That at least one non-zero Q_N exists from the fact that N equations in N + 1 unknowns always has a non-trivial solution. See Corollary 5.6.3, page 119 in (16).

Notice that $f \cdot Q_N - P_M = g \in L_{M+N+1}$ or $f-(P_M/Q_N) = h \in L_{M+N+1}$ where if

$$g = (\cdots,0,g_{M+N+1},g_{M+N+2},\cdots)$$

then

$$h = (\cdots,0,h_{M+N+1},h_{M+N+2},\cdots)$$

is obtained recursively by taking $\lambda \in L_{M+N+1}$ and setting $Q_N\lambda = g$ after which λ is calculated and is h.

Now a shifting function and another cutoff function will be defined so that for any $f \in L$ the Pade′ approximants can be calculated.

Let $m > 0$, $f \in L_m$, $a_m \neq 0$ and define $\psi_m(f) = (\cdots,b_0,b_1,\cdots)$ where $b_k = a_{k+m}$ which is the shifting function and is in $\tilde{L}_0$.

If $f = (\cdots,0,a_m,\cdots,a_0,\cdots) \in L$ with $a_m \neq 0$, $a_0 \neq 0$ define $\gamma_0(f) = (a_0,a_1,\cdots)$ which is in $\widetilde{L}_0$.

The Pade′ approximants relative to the former f will be those calculated from $\psi_m(f)$ but in the latter case $(a_m,\cdots,a_{-1}) + \pi_N^M(\gamma_0(f))$ will be called the Pade′ approximants for f. Henceforth in this paper the Pade′ approximants will be calculated relative to $f \in \widetilde{L}_0$ even though, using the preceeding, the theorems would follow for $f \in L$.

It cannot be guaranteed that $\pi_N^M(f)$ the [M,N] Pade′ approximants of f will agree with f for more than the first N + M + 1 places but it does agree with f for at least this many places. If Q_N is any element in A_N such that P_M/Q_N agree with f for the first N + M + 1 places then for any other such

$$Q_N^1, \; P_M^1/Q_N^1 = P_M/Q_N.$$

Another way of saying this is that the Pade′ approximants agree with a given $f \in \widetilde{L}_0$ for more places than any other such element of R_N^M and furthermore $\pi_N^M(f)$ is unique. Whence the following theorem.

<u>Theorem 1</u>. If n and m are nonnegative integers and $f \in \widetilde{L}_0$ there exists one and only one $[n,m] = q \in R_n^m$ such that $\pi_n^m(f) = q$ or equivalently $f - q \in L_{m+n+1}$.

<u>Proof</u>. Assume that $(P_m^1/Q_n^1) \in R_n^m$ and $(P_m^2/Q_n^2) \in R_n^m$. Further assume that

$$f - (P_m^1/Q_n^1) \in L_{m+n+1},$$

and

$$f - (P_m^2/Q_n^2) \in L_{m+n+1}.$$

Then

$$f - (P_m^2/Q_n^2) - (f - (P_m^1/Q_n^1)) = (P_m^1/Q_n^1) - (P_m^2/Q_n^2) \in L_{m+n+1}.$$

Now if $g \in L_{m+n+1}$ then $P_m^1 Q_n^2\, g \in L_{m+n+1}$ and hence $(P_m^1/Q_n^1) - (P_m^2/Q_m^2)$ is an element of L_{m+n+1} if and only if

$$P_m^1 Q_n^2 - Q_m^1 P_m^2 \in L_{m+n+1}.$$

But

$$P_m^1 Q_n^2 - Q_n^1 P_m^2 \in A_{m+n}$$

and

$$(P_m^1/Q_n^1) - (P_m^2/Q_n^2) \text{ is } \{0\}$$

since

$$A_{m+n+1} \cap L_{m+n+1} = \{0\}.$$

Let $f = \{a_n\}_n$ and $g = \{b_k\}_k$ be elements of $\widetilde{L}_0$ and define a mapping b_k^n for each pair of nonnegative integers n and k on $\widetilde{L}_0$ to F as follows:

$$b_k^n(g) = b_0^0 = 1 \quad \text{if } n,k = 0$$

$$b_k^n(g) = b_k^n = \sum_{m_1+\cdots+m_n=k} b_{m_1}\cdots b_{m_n}, \text{ otherwise.}$$

For each $(\cdots,0,f_0,f_1,\cdots) = f \in \widetilde{L}_0$ consider $L_f = \{g = (\cdots,0,g_0,g_1,\cdots) : \sum_{n=0}^{\infty} g_n = z$ is in the circle of convergence of $f\}$. For each nonnegative integer k and $g \in L_f$ define a mapping from L_f to F as follows:

$$f_k g = \sum_{n=0}^{\infty} f_n b_k^n(g).$$

Now for each $(f,g) \in \widetilde{L}_0 \times L_0$ define a mapping, called a composition mapping, $f*g$ on $\widetilde{L}_0 \times L_0$ to L_0 as

$$(f*g)(f,g) = f*g = \{f_k g\}_{k=0}^{\infty} = \{f_i(g)^i\}_{i=0}^{\infty}.$$

In notation $(P_N/Q_M)*g = P_N*g/Q_M*g = \pi_M^N(f)*g$.

Now it is possible to prove some lemmas after which two basic theorems will easily follow.

Lemma 1. If $g \in L_1$, $g = \{g_n\}$, $g_1 \neq 0$ and $h \in L_0$ then there exists one and only one $f \in L_0$ such that $f*g = h$ and if $h_0, h_1, \cdots, h_k$ only are given there exists one and only one $\{f_0, f_1, \cdots, f_k\}$ such that for every $f \in L_0$ of the form $\{f_0, f_1, \cdots, f_k, f_{k+1}, \cdots\}$ $(f*g)_i = h_i$, $i = 0, 1, \cdots, k$.

Proof. Now $$f*g = \{f_k g\}_k = \{\sum_{n=0}^{\infty} f_n b_k^n\}_k$$

$$= \{(f_0 b_0^0 + f_1 b_0^1 + \cdots), (f_0 b_1^0 + f_1 b_1^1 + \cdots), \cdots, (f_0 b_k^0 + f_1 b_k^1 + \cdots), \cdots\}$$

$$= \{f_0, f_1 b_1^1, (f_1 b_2^1 + f_2 b_2^2), \cdots, (f_1 b_k^1 + \cdots + f_k b_k^k), \cdots\}$$

$$= \{f_0, f_1 g_1, f_1 g_2 + f_2 g_1^2, \cdots, f_1 g_k + \cdots + f_k g_1^k, \cdots\}$$

where it is seen that upon equating to h that the f_i are uniquely determined. Also an observation of the above brings to mind that if only the h_i, $0 \leq i \leq k$, are given then first of all $f_0 = h_0$ and, since $g_1 \neq 0$, f_1 can be uniquely solved for. Continuing in this manner f_k can be uniquely solved for since $g_1^k \neq 0$. But this means that $\{f*g\}_i = h_i$ for $i = 0, 1, \cdots, k$ which completes the proof.

Lemma 2. If $g \in L_1$, $g_1 \neq 0$, $f, h \in L_0$ then $(f*g)_i = (h*g)_i$ for $i = 0, 1, \cdots, k$ if and only if $f_i = h_i$ for $i = 0, 1, \cdots, k$.

Proof. As in Lemma 1, write $f*g$ as

$$\{f_0, f_1 g_1, f_1 g_2 + f_2 g_1^2, \cdots, f_1 g_k + \cdots + f_k g_1^k, \cdots\}$$

and $h*g$ as $$\{h_0, h_1 g_1, h_1 g_2 + h_2 g_1^2, \cdots, h_1 g_k + \cdots + h_k g_1^k, \cdots\}.$$

Hence if $(f*g)_0 = (h*g)_0$, then $f_0 = h_0$ and conversely. Since $g_1 \neq 0$ and $h_1 g_1 = f_1 g_1$ if and only if $h_1 = f_1$ the theorem is true for $k = 1$.

Assume the first $k - 1$ f_i and h_i are equal and consider the following equation

$$h_1 g_k + \cdots + h_k g_1^k = f_1 g_k + \cdots + f_k g_1^k$$

which is true if and only if $h_k = f_k$ and concludes the proof.

Lemma 3. If $f,g,h \in L_0$, $g*f$ and $g*h$ are defined and $f_i = h_i$ for $i = 0,1,\ldots,k$ then $(g*f)_i = (g*h)_i$ for $i = 0,\ldots,k$.

Proof. Since
$$g*f = \{ \sum_{n=0}^{\infty} g_n b_0^n, \sum_{n=0}^{\infty} g_n b_1^n, \ldots \}$$

$$= \{ \sum_{n=0}^{\infty} g_n \sum_{m_1 + \cdots + m_n = 0} f_{m_1} \cdots f_{m_n}, \sum_{n=0}^{\infty} g_n \sum_{m_1 + \cdots + m_n = 1} f_{m_1} \cdots f_{m_n}, \ldots \}$$

$$= \{ \sum_{n=0}^{\infty} g_n (f_0)^n, \sum_{n=0}^{\infty} g_n \sum_{m_1 + \cdots + m_n = 1} f_{m_1} \cdots f_{m_n}, \ldots, \sum_{n=0}^{\infty} g_n \sum_{m_1 + \cdots + m_n = k} f_{m_1} \cdots f_{m_n}, \ldots \}$$

$$= \sum_{n=0}^{\infty} g_n (h_0)^n, \sum_{n=0}^{\infty} g_n \sum_{m_1 + \cdots + m_n = 1} h_{m_1} \cdots h_{m_n}, \ldots, \sum_{n=0}^{\infty} g_n \sum_{m_1 + \cdots + m_n = k} h_{m_1} \cdots h_{m_n}, \ldots \}$$

because $h_i = f_i$ if $i \le k$ and in each of the first k elements of $g*f$ the second sum is composed of $h_i = f_i$.

Now these three lemmas will be used in proving two theorems which are basic to the study of Pade' approximants.

Theorem 2. If $f \in L_0$, $g = (0,A)/(1,B) \in L_1$, and $\pi_N^N(f) = P_N/Q_N$ is the [N,N] Pade' approximant of f then $\pi_N^N(f)*g = (P_N*g)/(Q_N*g)$ is the [N,N]

Pade′ approximant of f*g.

Proof. Now $h^N(P_N*g)/(Q_N*g)h^N = (P_N*g)/(Q_N*g)$ where $h = (1,B)$, which is

$$(1,B)^N\{a_i(0,A)^i/(1,B)^i\}_{i=0}^N/(1,B)^N\{b_i(0,A)^i/(1,B)^i\}_{i=0}^N$$

$$= [\{a_i(0,A)(1,B)^{N-i}\}_{i=0}^N/\{b_i(0,A)^i(1,B)^{N-i}\}_{i=0}^N] \in R_N^N,$$

and by Lemma 2, $\pi_N^N(f)*g$ is the [N,N] Pade′ approximant of f*g.

Theorem 3. If $f,g \in L_0$, $g = (A,B)/(C,D)$, $(C,D) \neq 0$ and $\pi_N^N(f)$ is the [N,N] Pade′ approximant of f then $g*\pi_N^N(f)$ is the [N,N] Pade′ approximant of g*f.

Proof. Since $\pi_N^N(f)$ agrees with f for the first 2N + 1 places $g*\pi_N^N(f)$ agrees with g*f for the first 2N + 1 places by Lemma 3. If it can be shown that $g*\pi_N^N(f) \in R_N^N$ then the proof will be complete. Now

$$g*(P_N/Q_N) = (A,B(P_N/Q_N))/(C,D(P_N/Q_N))$$

$$= Q_N(A,B(P_N/Q_N))/Q_N(C,D(P_N/Q_N))$$

$$= (Q_NA,BP_N)/(CQ_N,DP_N) \in R_N^N$$

and the proof is complete.

Now a lemma will be presented after which a generalization of Theorem 2 will follow.

Lemma 1. Let $T = (0,\ldots,h_r,\ldots) \in \tilde{L}_r$ where $r \geq 0$. If $f = (f_0,f,\cdots)$, $g = (g_0,g,\ldots) \in L_0$ and agree for q places then f*T agrees with g*T for (q+1)n-1 places.

Proof. In f*T the (k + 1)th element is

$$c_k = \sum_{n=1}^{\infty} f_n \sum_{m_1+\cdots+m_n=k} h_{m_1}\cdots h_{m_n}$$

if $k > 0$ where $m_1 \geq r$ and $k \geq nr$ or $n \leq k/r$. Hence

$$c_k = \sum_{n=1}^{[k/r]} f_n \sum_{m_1+\cdots+m_n=k} h_{m_1}\cdots h_{m_n} .$$

If $[k/r] \leq q$ then $k \leq rq + t$ where $0 \leq t < r$. Since f_i agrees with g_i, if $i \leq [k/r] \leq q$, then

$$c_k = \sum_{n=1}^{[k/r]} q_i \sum_{m_1+\cdots+m_n=k} h_{m_1}\cdots h_{m_n}$$

and agreement will be realized for all $k \leq rq + t$ or through $k = rq + r - 1 = r(q + 1) - 1$.

Theorem 4. Let

$$T = \frac{A_n w^n}{p_n(W)} = \frac{(0,0,\ldots,A_n)}{(a_0,a_1,\ldots,a_n)} = (0,\ldots,h_r,\ldots) \in \tilde{L}_r$$

and P_N/Q_N be the [N,N] Pade' approximants of $f \in L_0$. Then the $[nN + r, nN + \xi]$ Pade' approximants of $q = f*T$ are P_n*T/Q_n*T.

Proof. Now

$$\left[P_N \left(\frac{A_n w^n}{p_n(w)}\right) / Q_n \left(\frac{A_n w^n}{p_n(w)}\right)\right]\cdot\frac{p_n(w)}{p_n(w)}$$

is a Pade' approximant in R_{nN}^{nN} but by the lemma with $P_N/Q_N = q$ it is seen that P_N*T/Q_N*T agrees with $g = f*T$ for $2nN + r + s$ where $r + s \leq n - 1$ so that its $[nN + r, nN + s]$ Pade' approximants to g are given by P_N*T/Q_N*T.

The preceeding definitions are generalizations to some extent when contrasted with the usual approach of defining the Pade' approximant.

And they offer insight as to what the Pade′ approximant is provided one is just in the process of reading the current literature relative to the subject. In the next two theorems, the latter one of which is a result of several lemmas, the concepts of uniform boundedness and analytic continuation are of much importance. Relative to these concepts the concepts of polynomials and of power series facilitates such a discussion. Hence a mapping will be defined so that power series and polynomials will be synonymous with the algebraic elements already defined.

For every $f = (a_0, a_1, \ldots, a_n, \ldots) \in \tilde{L}_0$ let $\varphi(f) = \sum_{n=0}^{\infty} a_i z^i$. Notice that $\varphi(f)$ makes sense for at least one z, namely $z = 0$. Of course the radius of convergence is not necessarily zero. If $f \in P$ then $\varphi(f)$ is a polynomial. In the future $f(z) = \varphi(f)$ and f will be used interchangably.

Relative to the image under φ, the Pade′ approximants are given by the equations $f(z)\,Q_N(z) - P_M(z) = Az^{N+M+1} + Bz^{N+M+2} + \ldots,$ (1)

and $$Q_N(0) = 1. \quad (2)$$

The following remarks are important. As noted earlier there always exists a nontrivial solution for the coefficients in $Q_N(z)$. By Theorem 1, P_M/Q_N is unique and agrees with $f(z)$ for at least the first $N + M + 1$ terms. So if the nontrivial solution for $Q_n(z)$ has $z = 0$ as a zero of multiplicity M then also $P_M(z)$ must have $z = 0$ as a zero of multiplicity n for otherwise it is impossible for P_M/Q_N to agree with $f(z)$ even for one place since $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $a_0 \neq 0$. That is $Q_N(0)$ can always be made 1.

A series is normal if the

$$D(n,m) = \begin{vmatrix} a_n & \cdots & a_{m+n} \\ \vdots & \ddots & \vdots \\ a_{n+m} & \cdots & a_{n+2m} \end{vmatrix}$$

are not zero and like determinants formed with the coefficients of the reciprical series are different from 0.

If a normal power series $f(z) = \sum_{i=0}^{\infty} a_i z^i$, is given then $a_0 \neq 0$ and the Pade′ approximants are given by

$$\frac{P_M(z)}{Q_N(z)} = \frac{\begin{vmatrix} a_{M-N+1} & a_{M-N+2} & \cdots & a_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_M & a_{M+1} & & a_{M+N} \\ \sum_{N=N}^{M} a_{J-N} z^J & \sum_{J=N-1}^{M} a_{J-N+1} z^J & \cdots & \sum_{J=0}^{M} a_J z^J \end{vmatrix}}{\begin{vmatrix} a_{M-N+1} & a_{M-N+2} & \cdots & a_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_M & a_{M+1} & \cdots & a_{M+N} \\ z^N & z^{N-1} & \cdots & 1 \end{vmatrix}} \quad (3)$$

where $a_J \equiv 0$ if $J < 0$ and where sums with the initial number larger than the terminal number are also zero because equations 1 and 2 are satisfied by the array 3. The proof that 3 satisfies 1 and 2 follows.

If $M + K = N$ and A_i is the array

$$\begin{vmatrix} a_{M-N+1} & a_{M-N+2} & \cdots & a_{M+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_M & a_{M+1} & \cdots & a_{M+N} \end{vmatrix}$$

with the $(N + 1 - i)$th column deleted, then equation 1 becomes

$$a_0A_0+(-a_0A_1+a_1A_0)z+(a_0A_2+(-a_1)A_1+a_2A_0)z^2+\dots+(a_MA_0-a_{M-1}A_1+\dots\pm a_0A_M)z^M$$

$$+\dots+(a_NA_0-a_{N-1}A_1+\dots\pm a_0A_N)z^N+\dots+(a_{M+N}A_0-a_{M+N-1}A_1+\dots\pm a_0A_{M+N})z^{N+M}$$

$$+\dots-\{(a_0+a_1z+a_2z^2+\dots+a_Mz^M)A_0-(a_0z+a_1z^2+\dots+a_{N-1}z^M)A_1+(a_0z^2+\dots+a_{M-2}z^M)A_2$$

$$-\dots\pm(a_0z^{M-1}-a_1z^M)A_{N-k+1}\pm a_0z^M A_{N-k=M}\}=a_0A_0-a_0A_0+(a_1A_0-a_0A_1-a_1A_0+a_0A_1)z$$

$$+\dots+(a_MA_0-a_{M-1}A_1+\dots\pm a_0A_m-a_MA_0+a_{M-1}A_1-\dots\pm a_0A_M)z^M$$

$$+(a_{M+1}A_0-a_MA_1+\dots+a_{M-N+1}A_N-a_{M+1}A_0+\dots\pm a_{M-N+1}A_N)z^{M+1}$$

$$+\dots+(a_{M+N}A_0-a_{M+N-1}A_1+\dots\pm a_MA_N-a_{M+N}A_0+a_{M+N-1}A_1-\dots\pm a_MA_N)z^{M+N}+Az^{M+N+1}$$

$$+Bz^{M+N+2}+\dots=0+Az^{M+N+1}+Bz^{M+N+2}+\dots$$

where it is noticed that the constant term and the coefficients of $z,\dots,z^N$ all add to zero but that the reason for the coefficients of $z^{M+1},\dots,z^{M+N}$ vanishing is because in each case the coefficient is a matrix with 2 identical rows. See Theorem 7.6, page 17 in (16).

If $P_M(z)/Q_N(z)$ is given such that $P_M(0) = a_0$ and $Q_N(0) = 1$ then a power series for which $P_M(z)/Q_N(z)$ is the Pade' approximant can be calculated. If

$$P_M(z)/Q_N(z) = \sum_{n=0}^{\infty} c_nz^n$$

then

$$a_0+a_1z+\dots+a_Mz^M = t_0c_0+(t_1c_0+t_0c_1)z+(\sum_{J+k=2}k_Jc_k)z^2$$

$$+(\sum_{J+k=3}t_Jc_K)z^3+\dots+(\sum_{J+k=M}t_Jc_k)z^M$$

$$+(\sum_{J+k=M+1}t_Jc_k)z^{M+1}+\dots+(\sum_{J+k=N}t_Jc_k)z^N+\dots+(\sum_{J+k=M+N}t_Jc_k)z^{M+N}+\dots.$$

Since $t_0 = 1$, $c_0 = a_0$. Assume that c_i, $i < M$ has been uniquely determined. Then $t_{i+1}c_0 + t_i c_1 + \dots + t_1 c_i + t_0 c_{i+1} = a_{i+1}$ uniquely determines c_{i+1}. Hence the c_i for $i \leq M$ can be uniquely calculated. Let $U_0, U_1, \dots, U_M$ be the result of these calculations. Now $t_0 c_{M+1} + t_1 c_M + \dots + t_M c_1 + t_{M+1} c_0 = 0 \dots t_0 c_{N-1} + t_1 c_{N-2} + \dots + t_{N-1} c_0 = 0$ uniquely determines $U_{M+1}, \dots, U_{N-1}$ and

$$\begin{vmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_{N-1} \\ u_N \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \dots 0 & & 0 \\ 0 & 0 & 1 \dots 0 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \dots 0 & & 1 \\ -t_N & -t_{N-1} & -t_{N-2} & -t_2 & -t_1 \end{vmatrix} \begin{vmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{N-2} \\ u_{N-1} \end{vmatrix} \qquad (4)$$

in a matrix in which U_N is uniquely determined. Now notice that for $t \geq 1$

$$\begin{vmatrix} u_{0+t} \\ u_{1+t} \\ \cdot \\ \cdot \\ \cdot \\ u_{N-2+t} \\ u_{N-1+t} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \dots 0 & & 0 \\ 0 & 0 & 1 \dots 0 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 \dots 0 & & 1 \\ -t_N & -t_{N-1} & -t_{N-2} \dots -t_2 & & -t_1 \end{vmatrix}^t \begin{vmatrix} u_0 \\ u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_{N-2} \\ u_{N-1} \end{vmatrix} \qquad (5)$$

which then determines the u_k for $k \geq N$ and $\sum_{k=0}^{\infty} c_k z^k$ becomes $\sum_{k=0}^{\infty} u_k z^k$ relative to which the given $P_M(z)/Q_N(z)$ is the [M,N] Pade' approximant.

Although certain assumptions about how M and N are related have been assumed in the preceeding proof, a proof of the alternative is similar and for this reason is omitted.

For the moment a look at sequences of the form [N,N+k] with k finite will be taken. The study of such a sequence can be confined to the study of the [N,N] Pade′ approximants by simply looking at

$$g(z) = z^{-k} f(z) \quad \text{if } k \leq 0$$

or

$$g(z) = [f(z) - \sum_{J=0}^{k-1} f_J z^J] z^{-k} \quad \text{if } k > 0,$$

where the f_J are the coefficients in the power series expansion of $f(z)$, because if $k \leq 0$ and $N + k = M$ or $N = M - k$ then $z^{-k}P_{N+k}(z)/Q_N(z)$ is a rational function with numerator and denominator polynomials of degree no more than N and they approximate $z^{-k}f(z)$ in the same sense that $P_{N+k}(z)/Q_N(z)$ approximate $f(z)$. Since

$$f(z)\, Q_N(z) - P_{N+k}(z) = Az^{2N+k+1} + \cdots$$

then

$$z^{-k}f(z)\, Q_N(z) - z^{-k}\, P_{N+k}(z) = Az^{2N+1} + \cdots$$

and

$$z^{-k}P_{N+k}(z)/Q_N(z)$$

is the [N,N] Pade′ approximant for $z^{-k}f(z)$. Notice that $P_{N+k}(z)/Q_N(z)$ converges if and only if $z^{-k}P_{N+k}(z)/Q_N(z)$ does. Furthermore if for a given $f(z)$ the [N,N+k] approximant of $f(z)$, $k > 0$, is gotten by equating like coefficients in $Q_N(z)\, f(z) - P_{N+k}(z) = Az^{2N+k+1} + \cdots$ then

$$Q_N(z)\, f(z) - P_{N+k}(z) - Q_N(z) \sum_{J=0}^{k-1} f_J z^J + Q_N(z) \sum_{J=0}^{k-1} f_J z^J = Az^{2N+k+1} + \cdots$$

or

$$Q_N(z)[f(z) - \sum_{J=0}^{k-1} f_J z^J] + [Q_N(z) \sum_{J=0}^{k-1} f_J z^J - P_{N+k}(z)] = Az^{2N+k+1} + \cdots$$

or

$$Q_N(z)[f(z) - \sum_{J=0}^{k-1} f_J z^J] z^{-k} + [Q_N(z) \sum_{J=0}^{k-1} f_J z^J - P_{N+k}(z)] z^{-k} = Az^{2N+1} + Bz^{2N+2} + \cdots$$

or

$$Q_N(z)[g(z)] + P_N^1(z) = Az^{2N+1} + Bz^{2N+2} + \cdots$$

where

$$g(z) = [f(z) - \sum_{J=0}^{k-1} f_J z^J]z^{-k} = f_k + f_{k+1}\, z + \cdots$$

and $P_N^1(z)$ is a polynomial of degree not more than N because first

$$(Q_N(z) \sum_{J=0}^{k-1} f_J\; z^J - P_{N+k}(z))z^{-k}$$

is at most an expression that has no degree of z greater than N and secondly if $i \leq k - 1$ then the coefficient of z^i in $P_{N+k}(z)$ is exactly the coefficient of z^i in $Q_N(z) \sum_{J=0}^{k-1} f_J z^J$ since this is the way $P_{N+k}(z)$ was determined initially. Finally

$$P_N^1(z)/Q_N^1(z) = \{[Q_N(z) \sum_{J=0}^{k-1} f_J z^J - P_{N+k}(z)]z^{-k}\}/Q_N(z)$$

$$(\sum_{J=0}^{k-1} f_J z^J - P_{N+k}(z)/Q_N(z))z^{-k}$$

which implies that $P_N^1(z)/Q_N^1(z)$ converges if and only if $P_{N+k}(z)/Q_N(z)$ does.

The next theorem is basic to the theory and it will be proved in the classical setting.

Theorem 5. Let $P_k(z)$ be any infinite sequence of [N,M] Pade' approximants to a formal power series where N + M tends to infinity with k. If the absolute value of the P_k is uniformly bounded for $|z| \leq R$, then the P_k converge uniformly for $|z| \leq \tau < R$ to an analytic function f(z) whose power series has a radius of convergence of at least R.

Proof. If any $|P_k(z)|$ is bounded by W for $|z| \leq R$ then since $P_k(z)$ is a rational function of z it is analytic in $|z| \leq R$, and its power series must have a radius of convergence of at least R. Now consider the region $|z| \leq r < R$. The power series coefficients, $a_k(n)$, of z^n must, by Cauchys' inequalities, see Theorem 5.15.1, page 187 in (18), be bounded by W/R^n uniformly in k. Hence

$$|P_k(z) - \sum_{n=0}^{\infty} a_k(n)\ z^n| \leq \sum_{n=J+1}^{\infty} |a_k(n)\ z^n| \leq \sum_{n=J+1}^{\infty} W(r/R)^n = \frac{W(r/R)^{J+1}}{1-r/R}.$$

Now

$$\lim_{J\to\infty}(r/R)^{J+1} = \lim_{J\to\infty} e^{(J+1)\ln(r/R)} = 0$$

since $\ln(r/R)$ is negative. This means that there exists a J such that

$$|(P_k(z) - \sum_{n=0}^{J} a_k(n)z^n| < \epsilon/2$$

uniformly in k. As N + M tends to infinity with k there exists K such that if $k > K$, $N + M \geq J$ and therefore by equation 1 the first J + 1 power series terms are identical for all such k. Hence if J and k are greater than K and $|z| \leq r$,

$$|P_k(z)-P_J(z)| \leq |P_k(z)-\sum_{n=0}^{J} a_k(n)z^n| + |P_J(z)-\sum_{n=0}^{J} a_J(n)z^n| \leq \epsilon/2+\epsilon/2=\epsilon$$

which is the Cauchy criterion for convergence of the sequence $\{P_k\}$. By a theorem on uniform convergence of a sequence of analytic functions, see page 226 in (18), the limiting function f(z) is analytic for all $|z| < R$. By Taylor's Theorem, see page 201 in (18), f(z) can be written as a power series which has a radius of convergence of at least R which completes the proof.

Incidently this is a generalization of a proof for continued fractions due to Van Vleck in 1901. Before going into the next theorem certain definitions and helpful lemmas will be detailed which will be helpful in an analytical continuation process.

Elementary transformations of power series is of fundamental importance in the study of holomorphic functions and analytic continuation in which the binomial theorem

$$z^n = (z - a + a)^n = \sum_{k=0}^{n} \binom{n}{k} (z - a)^k a^{n-k} \tag{6}$$

will be used. Suppose that

$$\sum_{n=0}^{\infty} a_n z^n = f(z) \tag{7}$$

has the radius of convergence $R > 0$ and let $|a| < R$. Then for $|z| < R$

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (z - a)^k a^{n-k}. \tag{8}$$

Let $\Omega_{nk} = a_n \binom{n}{k} (z - a)^k a^{n-k}$ if $k \leq n$ and $\Omega_{nk} = 0$ if $k > n$. Then equation 8 can be regarded as a double series $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \Omega_{nk}$. Three lemmas are needed at this time which will come after some definitions.

Let $\varphi(j,k)$ be a nonnegative integer defined for $j,k = 0,1,\cdots,$ such that:

$$\begin{aligned} &\text{i} \quad \varphi(0,0) = 0; \\ &\text{ii} \quad \varphi(j,k) \leq \varphi(j + 1,k),\ \varphi(j,k) \leq \varphi(j,k + 1); \\ &\text{iii} \quad \varphi(j,k) \to \infty \text{ with } j + k. \end{aligned} \tag{9}$$

Examples of such functions are $\varphi(j,k) = \max(j,k)$ and $\varphi(j,k) = j + k$. Pick any such φ and define $A_n(\varphi) = \sum_{j,k} a_{jk}$, $\varphi(j,k) \leq n$. The sequence $\{A_n(\varphi)\}$ is monotone increasing. Suppose that it is bounded, then it has a limit, see (14), call it A. Let $\psi(j,k)$ be another function of the type of 9. It is desired that $\{A_n(\psi)\} \to A$. Since for all $n, A_n(\varphi) \leq A$ it would be nice if the same is true for $A_n(\psi)$. Let $\max \varphi(j,k) = m(n)$ for $\psi(j,k) \leq n$, then $A_{m(n)}(\varphi)$ is summed over all the a_{jk} occurring in $A_n(\psi)$ and possibly other a_{jk} which means that

$A_n(\psi) \leq A_{m(n)}(\varphi) \leq A$ as desired. Hence it follows that $\lim_{n\to\infty} A_n(\psi) = A^* \leq A$. Now exchange ψ for φ and repeat the argument to find that $A \leq A^*$ so that $A = A^*$.

The double series $\Sigma\,\Sigma\, a_{jk}$, $a_{jk} \geq 0$, is said to be convergent if for a particular choice of an admissable function $\varphi(j,k)$ the corresponding sequence $\{A_n(\varphi)\}$ is bounded. In this case $\lim A_n(\varphi) = A$ is called the sum of the series. The series is said to be divergent if one such sequence is unbounded. The series $\Sigma\,\Sigma\,\Omega_{jk}$ is said to be absolutely convergent if $\Sigma\,\Sigma\, a_{jk}$, $a_{jk} = |\Omega_{jk}|$ is convergent.

The preceeding gives a method of exhaustion which may be applied to double series with complex terms. If $\Sigma\,\Sigma\,\Omega_{jk}$ is the given series and $\varphi(j,k)$ is a function as defined in equation 9 and $W_n(\varphi) = \Sigma_{j,k}\,\Omega_{jk}$, $\varphi(j,k) \leq n$ then it may happen that $W_n(\varphi)$ exists but that $\lim W_n(\varphi)$ does not equal $\lim W_n(\psi)$ where the latter may or may not exist. However under suitable restrictions something can be said.

<u>Lemma 1.</u> If $\Sigma\,\Sigma\,\Omega_{jk}$ is absolutely convergent, then $\lim W_n(\varphi) = W$ exists for every choice of $\varphi(j,k)$, and the limit is independent of φ.

<u>Proof.</u> $|W_{n+k}(\varphi) - W_n(\varphi)| = |\Sigma\,\Omega_{ij}| \leq \Sigma|\Omega_{ij}|$ where $n < \varphi\,(i,j) \leq n + k$ and $\Sigma|\Omega_{ij}| = A_{n+k}(\varphi) - A_n(\varphi)$ which goes to zero by Cauchys' criterion and because it is given that $\Sigma\,\Sigma\,\Omega_{ij}$ is absolutely convergent. Hence $\lim_{n\to\infty} W_n(\varphi) = W$ exists. If $\epsilon > 0$ and another function $\psi(j,k)$ are given, select an n such that $|A - A_n(\psi)| < \epsilon$. If $m(n)$ is given as before then $A_{m(n)}(\varphi) - A_n(\psi) < \epsilon$. Since $|W_{m(n)}(\varphi) - W_n(\psi)| = |\Sigma\,\Omega_{ij}| \leq \Sigma|\Omega_{ij}| = A_{m(n)}(\varphi) - A_n(\psi)$ where $\Sigma\,\Omega_{ij}$ is comprised of those terms that appear whenever $\max \varphi(j,k) = m(n)$ but not those terms that appear whenever $\psi(j,k) \leq n$ it is seen that $\lim W_n(\psi) = W$.

Lemma 2. If the double series is absolutely convergent, then

$$\sum_{j=0}^{\infty}\{\sum_{k=0}^{\infty}\Omega_{jk}\} = \sum_{k=0}^{\infty}\{\sum_{j=0}^{\infty}\Omega_{jk}\} = W.$$

Proof. It must be shown that each of the series $S_j = \sum_{k=0}^{\infty}\Omega_{jk}$, $j = 0,1,$ $\cdots$, is convergent and that $\sum_{j=0}^{\infty} S_j = W$ and that similar results hold for columns. For any n,m

$$\sum_{j=0}^{m}\{\sum_{k=0}^{n} a_{jk}\} < A \tag{10}$$

and upon letting $n \to \infty$ it is noticed that the first m series S_j are absolutely convergent and that

$$\sum_{j=0}^{m} S_j < A \tag{11}$$

for every m which implies that equation 11 is absolutely convergent with sum not greater than A. Let $\varphi(j,k) = \max(j,k)$ and $\epsilon > 0$ and choose n such that

$$|W - W_n(\varphi)| \leq A - A_n(\varphi) < \epsilon \tag{12}$$

where

$$W_n(\varphi) = \sum_{j=0}^{n}\{\sum_{k=0}^{n}\Omega_{jk}\}.$$

It is also known that

$$|\sum_{j=0}^{n} S_j - W_n(\varphi)| < A - A_n(\varphi) < \epsilon. \tag{13}$$

Now adding the appropriate parts of equations 12 and 13 gives

$$|\sum_{j=0}^{n} S_j - W| < 2\,\epsilon. \tag{14}$$

Columns are handled in the same way after which the lemma follows.

Now with respect to the double series in equation 8 and being cognizant of the preceeding lemmas it is noticed that the order of summation can be altered if the series is absolutely convergent.

This will be true if and only if

$$\sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{n} \binom{n}{k} |z - a|^k \, |a|^{n-k} = \sum_{n=0}^{\infty} (|z - a| + |a|)^n \qquad (15)$$

converges, that is, if $|z - a| < R - |a|$. Let this condition be satisfied and sum 8 by columns instead to obtain

$$f(z) = \sum_{k=0}^{\infty} (z - a)^k \, 1/k! \sum_{n=k}^{\infty} n(n - 1)\cdots(n - k + 1)a_n a^{n-k}. \qquad (16)$$

For $|a| < R$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z - a)^k \qquad (17)$$

since

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n - 1)\cdots(n - k + 1)a_n z^{n-k}. \qquad (18)$$

(For the proof of equation 18 see Theorem 5.5.3 in (14)).

Formula 17 represents $f(z)$ in $|z - a| < R - |a|$ and the right side has a radius of convergence R_a. The sum of the series is a holomorphic function $f(z;a)$ in the circle $|z - a| < R_a$. Since $|z - a| < R - |a|$ is in the circle with radius R, $f(z;a) = f(z)$ for $|z - a| < R - |a|$ and a lower and upper bound can be given for R_a.

<u>Lemma 3</u>. $R - |a| \leq R_a \leq R + |a|$.

<u>Proof</u>. Since $f(z;a) = f(z)$ for $|z - a| < R - |a|$ the left inequality holds. For the other inequality suppose that $R_a > |a|$ for if $R_a \leq |a|$ then $R_a \leq R + |a|$ and there is nothing to prove. Hence $z = 0$ is in both regions of convergence where $f(z;a) = f(z)$. That is,

$$f^{(n)}(0) = \sum_{k=n}^{\infty} \frac{f^{(k)}(a)}{(k-n)!} (-a)^{k-n} \qquad (19)$$

for every n. Now the power series $f(z;a)$ was derived from the power series $f(z)$ by setting $z = (z - a) + a$ and using equation 6 to rearrange the resulting double series in powers of $(z - a)$ all of which was done under the assumption that the double series is absolutely convergent for $|z| < R_a - |a|$. Now apply the same technique to

$$f(z;a) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z - a)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} \sum_{n=0}^{k} \binom{k}{n} z^n (-a)^{k-n} \qquad (20)$$

which can be thought of as a double series

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{f^{(k)}(a)}{k!} \frac{k!}{n!(k-n)!} z^n (-a)^{k-n} \qquad (21)$$

where $\binom{k}{n} z^n (-a)^{k-n}$ vanishes for $n > k$. Since it is assumed that the double series is absolutely convergent expression 21 becomes

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=n}^{\infty} \frac{f^{(k)}(a)}{(k-n)!} (-a)^{k-n} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, z^n}{n!} = f(z). \qquad (22)$$

The last equality follows from equation 19. Hence the rearranged series at $z = a$ rearranged again at $z = 0$ gives the original series. Since the radius of convergence of the original series is R, $R_a - |a| \leq R$ which is what was to be proved.

That the two bounds for R_a are the best possible follows from a consideration of

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n. \qquad (23)$$

Here $f(z;a) = \sum_{n=0}^{\infty} \frac{(z - a)^n}{(1 - a)^{n+1}}$, $R_a = |1 - a|$ and $R_a = R - |a|$ if $0 \leq a < 1$ while $R_a = R + |a|$ if $-1 < a < 0$.

Lemma 4. The functions $f(z)$ and $f(z;a)$, defined by the equations 7 and 17 respectively, coincide in their common domain of definition.

Proof. It is obvious that $f(z)$ and $f(z;a)$ coincide in $|z - a| < R - |a|$. It will now be shown that $f(z)$ and $f(z;a)$ coincide in a neighborhood of b where $|b| < R$, $R - |a| \leq |b - a| < R_a$. Let

$$\delta = \min[\min_z(R - |z|), \min_z(R_a - |z - a|)$$

where z ranges over [a,b] and choose points $a = b_0, b_1, \ldots, b_n = b$ on the line segment [a,b] such that $|b_k - b_{k-1}| < \delta$ for $k = 1, \ldots, n$. Since $|b_1 - a| < \delta \leq R - |a|$, $f(z) = f(z;a)$ in some neighborhood of b_1. Now take equations 7 and 17 and rearrange about b_1 obtaining

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(b_1)}{k!} (z - b_1)^k \text{ and } \sum_{k=0}^{\infty} f^{(k)} \frac{(b_1;a)}{k!} (z - b_1)^k \tag{24}$$

respectively. The first series converges and represent f(z) for $z - b_1 < R - |b_1|$ while the second converges and represents f(z;a) for $|z - b_1| < R_a - |b_1 - a|$. Yet the two power series are identical in that they have the same coefficients because f(z) and f(z;a) coincide in some neighborhood of b_1 and have derivatives of all others. That is $f^{(k)}(b_1) = f^{(k)}(b_1;a)$ for all k. This power series represents f(z) in one circular disk and f(z;a) in another concentric disk. The two functions then have to coincide on the smaller of the two disks which will be denoted C_1. Its radius is at least δ and $b_2 \in C_1$. Now rearrange the power series for f(z) and for f(z;a) about b_2. The rearranged series have identical coefficients which means that f(z) and f(z;a) coincide in a disk C_2 about b_2. After this has been done n times f(z) will agree with f(z;a) in a disk C_n about $b = b_n$. Since b was an arbitrary point within each circle of convergence the theorem is proved.

If it happens that $R_a > R - |a|$ and if $C_a : |z - a| < R_a$, $C_0 : |z| < R$ then

$$F(z) = \begin{cases} f(z), & z \in C_0; \\ f(z;a), & z \in C_a \setminus C_0 \cap C_a \end{cases} \tag{25}$$

defines a holomorphic function in $C_0 \cup C_a$. It is said that $f(z;a)$ gives an analytic continuation of $f(z)$ in $C_a \setminus C_0 \cap C_a$.

Theorem 6. Let $P_k(z)$ be any sequence (infinite) of [N,M] Padé approximants to a formal power series where N + M tend to infinity with k. If the $|P_k(z)|$ is uniformly bounded in any closed, simply-connected domain D_1 which contains the origin as an interior point and $|P_k(z)|^{-1}$ is uniformly bounded in any closed, simply-connected domain D_2, which contains the origin as an interior point, then the P_k converge to a meromorphic function $f(z)$ in the interior of $D_1 \cup D_2$.

Proof. D_1 will be considered first. If ${}_0C_R$ denotes an open disk with radius R about the origin then there exists a ${}_0C_{R_1} \subset D_1$. That is if $|z| < R_1$ then $z \in D_1$. The P_k are uniformly bounded rational functions in D_1 and therefore analytic in D_1 and ${}_0C_{R_1}$. By Theorem 5 the $P_k(z)$ converge to an analytic function f_1 in ${}_0C_{R_1}$. Now pick any a such that $|a| < R_1$. Since the $P_k(z)$ are analytic in a neighborhood of a, they can be expanded in a power series about a. See Theorem 6.4.1, page 201 in (18). This power series

$$P_k(z) = \sum_{n=0}^{\infty} \frac{P_k^{(n)}(a)}{n!} (z - a)^n \tag{26}$$

has a radius of convergence R_2. Let ${}_aC_{R_2}$ denote the open disk with center a and radius R_2, then by Lemma 4, the power series representation of $P_k(z)$ about a agrees with P_k on ${}_aC_{R_2} \cap D_1$. As in Theorem 5 let $|z - a| \leq r < R_2$ where it is being assumed that ${}_aC_{R_2} \subset D_1$. Since, as in the proof of Theorem 5, the

$$P_k^{(n)}(a) \leq \frac{Wn!}{R_2^n}$$

and the coefficients

$$a_k(n) = \frac{P_k^{(n)}(a)}{n!}$$

in equation 26 are bounded uniformly in k,

$$|P_k(z) - \sum_{n=0}^{j} a_k(n)(z-a)^n| \leq \sum_{n=j+1}^{\infty} |a_k(n)(z-a)^n| \leq \sum_{n=j+1}^{\infty} W(r/R_2)^n$$

$$= W(r/R_2)^{j+1} \cdot 1/\ (1 - r/R_2) \qquad (27)$$

which is less than $\epsilon/2$ if j is sufficiently large. There exists a K such that $k \geq K$ implies that $N + M \geq j$ and by equation 1 the first $j + 1$ power series terms are identical for all $k \geq K$. Hence if $k \geq K$ and $j \geq K$ and $|z - a| < r$ then

$$|P_k(z) - P_j(z)| \leq |P_k(z) - \sum_{n=0}^{j} a_k(n)(z - a)^n|$$

$$+ |P_j(z) - \sum_{n=0}^{j} a_j(n)(z - a)^n| \leq \epsilon/2 + \epsilon/2 = \epsilon$$

which is again the Cauchy criterion for the convergence of the sequence so that the $P_k(z)$ converge to a $f_2(z)$ analytic in $|z - a| < R_2$. If $R_2 > R_1 - |a|$ then $f_2(z)$ is an extension of $f_1(z)$ in ${}_aC_{R_2} - {}_0C_{R_1}$ because they obviously (see Lemma 4) are equal in ${}_aC_{R_2} \cap {}_0C_{R_1}$ since both are limits of the same sequence P_k in this common domain.

Now suppose that $b \in D_1$ such that $[0,b]$ is in the interior of D_1. For each $a \in [0,b]$ choose an R_a such that ${}_aC_{R_a}$ is contained in the interior of D_1. Think of $[0,b]$ as the line segment with length $|b|$ and cover $[0,b]$ with $\{(a - R_a, a + R_a) : a \in [0,b]\}$. Since $[0,b]$ is compact there exists a finite subcover of $[0,b]$. Denote the subcover

$$\bigcup_{i=1}^{n} {}_{b_i}C_{R_i}.$$

The concept detailed above will be used now. Relative to ${}_{b_1}C_{R_1}$ the P_k are uniformly bounded and admit to a Taylor series expansion about b_1.

As shown above $P_k(z;b_1) = P_k(z)$ in ${}_0C_{R_0} \cap {}_{b_1}C_{R_1}$ and the limit function of $P_k(z_1,b_1)$ and $P_k(z)$ must agree on this common domain. Let

$$\lim_{k\to\infty} P_k(z) = f_1(z) \text{ and } \lim_{k\to\infty} P_k(z_1,b_1) = f_2(z).$$

Then if $z \in {}_0C_{R_0} \cap {}_{b_1}C_{R_1}$ one has $f_1(z) = f_2(z)$ and the function F_1 defined to be f_1 in ${}_0C_{R_0}$ and f_2 in ${}_{b_1}C_{R_1} \setminus ({}_0C_{R_0} \cap {}_{b_1}C_{R_1})$ is an analytic function in ${}_0C_{R_0} \cup {}_{b_1}C_{R_1}$.

Now consider ${}_{b_2}C_{R_2}$ in which the P_k are uniformly bounded. Hence the P_k can be expanded about b_2 using Taylor series and in this representation agreement with respect to $P_k(z_1;b_1)$ is a state of being. Since the $P_k(z_1;b_2)$ converge uniformly in ${}_{b_2}C_{R_2}$ and agree with $P_k(z_1;b_1)$ in ${}_{b_2}C_{R_2} \cap {}_{b_1}C_{R_1}$ the limit function of $P_k(z_1;b_2)$, call it $f_2(z)$, must agree with $f_1(z)$ in ${}_{b_2}C_{R_2} \cap {}_{b_1}C_{R_1}$. Hence the function F_2 defined to be F_1 in ${}_0C_{R_0} \cup {}_{b_1}C_{R_1}$ and f_2 in ${}_{b_2}C_{R_2} \setminus ({}_0C_{R_0} \cup {}_{b_1}C_{R_1})$ is analytic in ${}_0C_{R_0} \cup {}_{b_1}C_{R_1} \cup {}_{b_2}C_{R_2}$. Continuing until ${}_{b_n}C_{R_n}$ is treated analogously one gets that the $P_k(b)$ converges to $F(b)$ where $F(b)$ is an analytic continuation of $f_1(z)$ in ${}_0C_{R_0}$. For any $b \in \text{int}D_1$ there exists a polygonal line in $\text{int}D_1$ joining zero to b so that after a finite number of computations, treatment of the finite number of line segments in the polygonal line, the $P_k(b) \to F(b)$ where F is an analytic continuation of f_1. Since b was arbitrary the assertion follows.

Now consider D_2 and notice that $[P_k(z)]^{-1} = [M,N]$ Pade′ approximant. The $[P_k(z)]^{-1}$ are uniformly bounded in D_2 and hence analytic throughout D_2. By repeating the proof above but this time relative to D_2 and the $[P_k(z)]^{-1}$ one can show that the $[P_k(z)]^{-1}$

converge uniformly in the interior of D_2 to an analytic function $g(z)$. Now the $[P_k(z)]^{-1}$ will converge uniformly to $1/f(z)$ in the interior of $D_1 \cap D_2$ and $1/f(z)$ is analytic herein. The $[P_k(z)]^{-1}$ will converge uniformly in the interior of $D_2 \setminus D_1 \cap D_2$ to a function $g(z)$ given by a power series which is an analytic continuation of $1/f(z)$. Hence the $P_k(z)$ will converge uniformly to $f(z)$ in the interior of D_1 and to $1/g(z)$, which is at worst meromorphic in the interior of $D_2 \setminus D_1 \cap D_2$. See page 233 in (18). This completes the proof.

Theorem 6 gives a rule of procedure in practice. Let a region be given in the complex plane in which it is known (based on physical grounds) that the function is meromorphic. For instance the region might be a neighborhood of a part of the real axis. If an infinite sequence of [N,M] Padé approximants is given select an infinite subsequence which satisfies the conditions of Theorem 6 in the given region. Then they will converge to the meromorphic function by Theorem 6. But in practice, finding this subsequence is a problem whenever M and N tend to infinity.

This theorem is the last one in this chapter on the general theory of the Padé approximant method of approximation. Attention will now be given to a particular power series in conjunction with the Padé method.

CHAPTER III

APPROXIMATION OF THE SERIES OF STIELTJES

Introduction

Physical phenomenon in a mathematics setting can often be reduced to a series of Stieltjes which offers motivation for the study of the series of Stieltjes.

Definition of Series of Stieltjes

A series of Stieltjes is defined as

$$f(z) = \sum_{j=0}^{\infty} f_j(-z)^j \tag{1}$$

if and only if there exists a bounded non-decreasing function $\varphi(u)$ taking on infinitely many values in $[0,\infty)$ such that

$$f_j = \int_0^{\infty} u^j d\varphi(u). \tag{2}$$

The series is not assumed convergent. Define

$$D(m,n) = \det \begin{vmatrix} f_m & f_{m+1} & \cdots & f_{m+n} \\ f_{m+1} & f_{m+2} & \cdots & f_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n} & f_{m+n+1} & \cdots & f_{m+2n} \end{vmatrix}. \tag{3}$$

Now 2 implies that

$$D(0,n) > 0, \quad D(1,n) > 0, \quad n = 0,1,2,\cdots. \tag{4}$$

To see this consider the quadratic form

$$\sum_{p,q=0}^{n} f_{p+q+m} y_p y_q = \int_0^\infty u^m (y_0 + y_1 u + \cdots + y_n u^n)^2 d\varphi(u) > 0 \qquad (5)$$

which can be well considered by definition of the Stieltjes integral.

It is known that if a quadratic form is positive definite then all of the principal minor determinants are greater than zero and hence $D(m,n) > 0$. See Theorem 9.13.2, page 256 in (16).

<u>Theorem 1.</u> Let $f(x) = 1/\psi(x) = f_0 + f_1 x + \cdots + f_m x^n + \cdots$, $\psi(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_m x^m + \cdots$. If $(-1)^n D(1,n) < 0$ and $(-1)^n D(2,n) < 0$ then if $\overline{D}$ are the D's formed by the reciprical series $\psi(x)$ the following relations hold:

$$\begin{aligned} \overline{D}(1,n) &= (-1)^{n+1} \quad D(1,n)/f_0^{2n+2} > 0 \\ \overline{D}(0,n) &= (-1)^n \quad D(2,n-1)/f_0^{2n+1} > 0 \end{aligned} \qquad (6)$$

which means that $\psi(x)$ is a series of Stieltjes. **See** Theorem 5.

Proof. Consider the equations

$$\begin{aligned} &C_{m+n} + A^1 C_{m+n-1} + \cdots + A_m^h C_{m+n-h} + \cdots + A_m^n C_m = 0 \\ &C_{m+n+1} + A_m^1 C_{m+n} + \cdots + A_m^h C_{m+n-h+1} + \cdots + A_m^n C_{m+1} = 0 \\ &\vdots \\ &C_{m+2+2n-1} + A_m^1 C_{m+2n-2} + \cdots + A_m^h C_{m+2p-1-h} + \cdots + A_m^n C_{m+p-1} = 0 \end{aligned} \qquad (7)$$

and set $H = C_{m+2n} + A_m^1 C_{m+2n-1} + \cdots + A_m^n C_{m+n}$. (8)

Now use Cramers rule with equations 7 and 8 in solving for $A_m^1, \cdots, A_m^n$ to get, upon substitution into 8,

$$H = \overline{D}(m,n)/\overline{D}(m,n-1). \qquad (9)$$

Now for a moment think that the f_i in $f(x)$ are given and consider $f(x)\psi(x) = 1$, that is,

$$\begin{aligned} 1 &= f_0 C_0 \\ 0 &= f_0 C_1 + f_1 C_0 \\ &\vdots \\ 0 &= f_0 C_m + f_1 C_{m-1} + \cdots + f_m C_0 . \end{aligned} \tag{10}$$

Again like in the preceeding it is seen that

$$C_m = \frac{\begin{vmatrix} f_0 & 0 & \cdots & 0 & 1 \\ f_1 & f_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1} & f_{m-2} & \cdots & f_0 & 0 \\ f_m & f_{m-1} & \cdots & 1 & 0 \end{vmatrix}}{\begin{vmatrix} f_0 & 0 & \cdots & 0 & 0 \\ f_1 & f_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{m-1} & f_{m-2} & \cdots & f_0 & 0 \\ f_m & f_{m-1} & \cdots & f_1 & f_0 \end{vmatrix}} = \frac{\begin{vmatrix} f_1 & f_0 & \cdots & 0 \\ f_2 & f_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{m-1} & f_{m-2} & \cdots & f_0 \\ f_m & f_{m-1} & \cdots & f_1 \end{vmatrix}}{f_0^{n+1}(-1)^m} . \tag{11}$$

The equation 11 will be used soon. Regress again and suppose that $(1 + A_m^1(x) + \cdots + A_m^n(x))\, \psi(x) = (1 + A_m^1(x) + \cdots + A_m^n(x^n))/f(x)$. Now by equation 8 the coefficient of x^{m+2n} is $H = \overline{D}(m,n)/\overline{D}(m,n-1)$. Consider the product $f(x)\, \Sigma\, a_n^1 x^m = 1 + A_m^1(x) + \cdots + A_m^n(x)$ with the added requirements that $H = a_{m+2n}^1$ and $a_{m+n}^1 = a_{m+n+1}^1 = \cdots = a_{m+2n-1}^1 = 0$. Now the a_i^1 for $1 \leq i \leq n$ are solved for by equating the coefficients of $x, \cdots, x^n$ with $A_m^1, \cdots, A_m^n$ respectively. For the $m + n + 1$ remaining equations, that is, (the system of equations below will be referred to as (12))

$$
\begin{array}{lllllll}
1 = f_0 a'_0 & + 0 & + \cdots + 0 & + 0 & + \cdots + 0 & + 0 \\
0 = f_{n+1} a'_0 & + f_n a'_1 & + \cdots + f_0 a'_{n+1} & + 0 & + \cdots + 0 & + 0 \\
0 = f_{n+2} a'_0 & + f_{n+1} a'_1 & + \cdots + f_1 a'_{n+1} & + f_0 a'_{n+2} & + \cdots + 0 & + 0 \\
\vdots & \vdots & \ddots\ \vdots & \vdots & \ddots\ \vdots & \vdots \\
0 = f_{m+n-1} a'_0 & + f_{m+n-2} a'_1 & + \cdots + f_{m-2} a'_{n+1} & + f_{m-3} a'_{m+2} & + \cdots + f_0 a'_{m+n-1} & + 0 \\
0 = f_{m+n} a'_0 & + f_{m+n-1} a'_1 & + \cdots + f_{m-3} a'_{m+1} & + f_{m-4} a'_{m+2} & + \cdots + f_1 a'_{m+n-1} & + 0 \\
\vdots & \vdots & \ddots\ \vdots & \vdots & \ddots\ \vdots & \vdots \\
0 = f_{m+2n-1} a'_0 & + f_{m+2n-2} a'_1 & + \cdots + f_{m+n-2} a'_{m+1} & + f_{m+n-3} a'_{m+2} & + \cdots + f_n a'_{m+n-1} & + 0 \\
0 = f_{m+2n} a'_0 & + f_{m+2n-1} a'_1 & + \cdots + f_{m+n-3} a'_{m+1} & + f_{m+n-4} a'_{m+2} & + \cdots + f_{n+1} a'_{m+n-1} & + H f_0 .
\end{array}
$$

H is solved for (as before) to get

$$
H = \frac{\begin{vmatrix}
f_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \\
f_{n+1} & f_n & \cdots & f_0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_{m+2n} & f_{m+2n-1} & \cdots & f_{m+n-3} & f_{m+n-4} & \cdots & f_{n+1} & 0
\end{vmatrix}}{\begin{vmatrix}
f_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
f_{n+1} & f_n & \cdots & f_0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_{m+2n} & f_{m+2n-1} & \cdots & f_{m+n-3} & f_{m+n-4} & \cdots & f_{n+1} & f_0
\end{vmatrix}}
$$

$$
= \frac{(-1)^{m+n+2} E(m,n)}{f_0^2 E(m,n-1)} .
$$

But

$$
\overline{D}(m,n)/\overline{D}(m,n-1) = (-1)^{m+n} E(m,n)/f_0^2 E(m,n-1)
$$

or

$$f_0^{2n}\bar{D}(m,n)/(-1)^{mn+\frac{n(n+1)}{2}} E(m,n)$$

$$= f_0^{2(n-1)}\bar{D}(m,n-1)/(-1)^{m(n-1)+\frac{n(n-1)}{2}} E(m,n-1) \qquad (13)$$

and it is noticed that the equality holds when n is increased or decreased by unity. So if $n = 0$, then, by equation 11, the following is true:

$$f_0^0\bar{D}(m,0)/(-1)^0 f_m(-1)^m C_0^{m+1} = C_m/C_m(-1)^m f_0^{m+1} = (-1)^m/f_0^{m+1}. \qquad (14)$$

Hence, from 13

$$\bar{D}(m,n) = (-1)^{mn+\frac{n(n+1)}{2}} E(m,n)(-1)^n/f_0^{2n}f_0^{m+1}$$

$$= (-1)^{mn+\frac{n(n+1)}{2}} E(m,n)/f_0^{m+2n+1}. \qquad (15)$$

Also $$\bar{D}(1,n) = (-1)^{n+1} (-1)^{\frac{n(n+1)}{2}} E(1,n)/f_0^{1+2n+1}. \qquad (16)$$

Now

$$E(1,n) = \begin{vmatrix} f_{n+1} & f_n & \cdots & f_1 \\ f_{n+2} & f_{n+1} & \cdots & f_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ f_{2n+1} & f_{2n} & \cdots & f_{n+1} \end{vmatrix} = (-1)^{[\frac{n+1}{2}]} \begin{vmatrix} f_1 & \cdots & f_{n+1} \\ f_2 & \cdots & f_{n+2} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ f_{n+1} & \cdots & f_{2n+1} \end{vmatrix} \qquad (17)$$

$$= (-1)^{[\frac{n+1}{2}]} D(1,n)$$

where $[(n + 1)/2]$ is the greatest integer less than or equal $(n + 1)/2$.

Hence

$$\bar{D}(1,n) = (-1)^{n+1} (-1)^{\frac{n(n+1)}{2}} (-1)^{[\frac{n+1}{2}]} D(1,n)/f_0^{2n+z}.$$

If $n + 1$ is odd then $[(n + 1)/2] = n/2$ and $(-1)^{(n/2)(n+1)}\cdot(-1)^{[(n+1)/2]}$ is $(-1)^{n(n+2)/2} = ((-1)^{n+2})^{n/2} = 1$. On the other hand if $n + 1$ is even $(-1)^{n(n+1)/2}\cdot(-1)^{[(n+1)/2]} = (-1)^{n((n+1)/2)}\cdot(-1)^{(n+1)/2} = (-1)^{(n+1)(\frac{n}{2} + \frac{1}{2})}$ $= 1$ and $\bar{D}(1,n)$ becomes $\bar{D}(1,n) = (-1)^{n+1}\cdot D(1,n)/f_0^{2n+2} > 0.$ (18)

While if $m = 0$ and $a'_n = a'_{n+1} = \dots = a'_{2n-1} = 0$ and $a'_{2p} = H$, then

$$E(0,n) = \begin{vmatrix} f_{n+1} & \cdots & f_2 \\ f_{n+2} & \cdots & f_3 \\ \vdots & \ddots & \vdots \\ f_{2n} & \cdots & f_{n+1} \end{vmatrix} = (-1)^{[n/2]} \begin{vmatrix} f_2 & \cdots & f_{n+1} \\ f_3 & \cdots & f_{n+2} \\ \vdots & \ddots & \vdots \\ f_{n+1} & \cdots & f_{2n} \end{vmatrix} \qquad (19)$$

$$= (-1)^{[n/2]} D(2,n-1).$$

Hence

$$\overline{D}(0,n) = (-1)^{n(n+1)/2} E(0,n)/f_0^{2n+1}$$

$$= (-1)^{n(n+1)/2} (-1)^{[n/2]} D(2,n-1)/f_0^{2n+1} \qquad (20)$$

where it is seen that if n is even $[n/2] = n/2$ and $(-1)^{n(n+1)/2}\cdot(-1)^{(n/2)}$ $= (-1)^{(n/2)(n+2)} = 1$. But whenever n is odd, $[n/2] = (n-1)/2$ and $(-1)^{(n/2)(n+1)}\cdot(-1)^{n-1/2} = (-1)^{(n^2+2n-1)/2} = -1$. The last equality follows from the fact that if $n = 2k+1$ an odd number then $((n^2+2n-1)/2) =$ $(4k^2+4k+1+4k+2-1)/2$ which is $[2(2k^2+2k+2k+1)/2] =$ $2(k^2+2k)+1$ an odd number. Therefore $(-1)^{(n/2)(n+1)}\cdot(-1)^{n/2} = (-1)^n$ and equation 20 becomes

$$\overline{D}(0,n) = (-1)^n D(2,n-1)/f_0^{2n+1} \qquad (21)$$

which becomes, if $k = n-1$, $\overline{D}(0,k+1) = (-1)^{k+1}\cdot D(2,k)/f_0^{2k+3}$. Since $\overline{D}(0,n+1) = (-1)^{n+1}\cdot D(2,n)/f_0^{2n+3} > 0$, holds by hypothesis, 21 is greater than zero. Therefore under the hypothesis it has been shown that the reciprical series is a series of Stieltjes. Most of the above was done by Hadamard. See pages 101 to 186 in (10).

The importance of the inequalities in this theorem follows from the fact that $D(M-N+1, N-1) > 0$ for $M \geq N-1 \geq 0$ for a series of

Stieltjes. Now it will be shown that these determinantal inequalities determine the location of all poles of any $[N, N+j]$ Padé approximant where $j \geq -1$.

Theorem 2. If $\Sigma f_j(-z)^j$ is a series of Stieltjes then the poles of the $[N, N+j]$, where $j \geq -1$, Padé approximants are on the negative real axis. Also, the poles of successive approximants interlace, and all of the residues are positive. Furthermore the roots of the numerator also interlace with those of the denominator.

Proof. Multiply the second column of the denominator in equation 3, Chapter II by x and then subtract it from the first column. Continue in this manner until this has been done with the next to last and the last column. In this fashion the denominator of equation 3, Chapter II can be reduced to (22)

$$(-1)^{(N(N+j)} \begin{vmatrix} f_{1+j} + xf_{2+j} & f_{2+j} + xf_{3+j} & \cdots & f_{N+j} + xf_{N+j+1} \\ f_{2+j} + xf_{3+j} & f_{3+j} + xf_{4+j} & \cdots & f_{N+j+1} + xf_{N+j+2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{N+j} + xf_{N+1+j} & f_{N+j+1} + xf_{N+j+2} & \cdots & f_{2N+j-1} + xf_{2N+j} \end{vmatrix}.$$

To prove this one might procede as follows. The denominator of equation 3, Chapter II is

$$\begin{vmatrix} \pm f_{1+j} & \pm f_{2+j} & \cdots & \pm f_{N+j+1} \\ \mp f_{2+j} & \pm f_{3+j} & \cdots & \mp f_{N+j+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mp f_{N+j} & \pm f_{N+j+1} & \cdots & + f_{2N+j} \\ x^N & x^{N-1} & \cdots & 1 \end{vmatrix}.$$

After performing the operations mentioned this becomes

$$\begin{vmatrix} \pm(f_{1+j} + xf_{2+j}) & \dots & \mp(f_{N+j} + xf_{N+j+1}) & \pm f_{N+j+1} \\ \vdots & \ddots & \vdots & \vdots \\ \mp(f_{N+j} + xf_{N+j+1}) & \dots & -(f_{2N+j-1} + xf_{2N+j}) & f_{2N+j} \\ 0 & \dots & 0 & 1 \end{vmatrix},$$

or expanding about the last row one gets (23)

$$\begin{vmatrix} \pm(f_{1+j} + xf_{2+j}) & \dots & \mp(f_{N+j} + xf_{N+j+1}) \\ \vdots & \ddots & \vdots \\ \mp(f_{N+j} + xf_{N+j+1}) & \dots & -(f_{2N+j-1} + xf_{2N+j}) \end{vmatrix}.$$

In order to get rid of the $\pm$ in expression 23 which will therefore give the $(-1)^{N(N+j)}$ factor in expression 22 consider first that $j + 1$ is even and that $N + j + 1$ is even. There are N rows in the expression 23 and by assumption N is even and $N + j$ is odd. Hence there will be exactly N/2 different rows that begin with negative signs in expression 23. Likewise there will be the same number of columns with like beginnings and after (-1) is factored out of 23 in the N/2 fore-mentioned rows each column will have a constant sign for its elements. That is, expression 23 becomes (24)

$$(-1)^{N/2} \begin{vmatrix} (f_{1+j} + xf_{2+j}) & -(f_{2+j} + xf_{3+j}) & \dots & -(f_{N+j} + xf_{N+j+1}) \\ (f_{2+j} + xf_{3+j}) & -(f_{3+j} + xf_{4+j}) & \dots & -(f_{N+j+1} + xf_{N+j+2}) \\ \vdots & \vdots & \ddots & \vdots \\ (f_{N+j} + xf_{N+j+1}) & -(f_{N+j+1} + xf_{N+j+2}) & \dots & -(f_{2N+j-1} + xf_{2N+j}) \end{vmatrix}.$$

Now factor (-1) N/2 times which will make the totality of signs in expression 24 positive to get

$$(-1)^{N/2 + N/2} \begin{vmatrix} (f_{1+j} + xf_{2+j}) & \cdots & -(f_{N+j} + xf_{N+j+1}) \\ \vdots & \ddots & \vdots \\ (f_{N+j} + xf_{N+j+1}) & \cdots & -(f_{2N+j-1} + xf_{2N+j}) \end{vmatrix} \tag{25}$$

$$= (-1)^N D = D.$$

But $(-1)^{N(N+j)} = 1$ since N is even and 23 is 22 in this case.

Now suppose in 23 that $j + 1$ is even and $N + j + 1$ is odd. Then $N + j$ is even and N is odd. The number of rows beginning with negative signs is $(N - 1)/2$ and there are after the above operations have been performed $(N - 1)/2$ (-1)'s to be factored from the columns. Hence expression 23 becomes

$$(-1)^{N-1} \cdot D = D = (-1)^{N(N+j)} \cdot D \tag{26}$$

If in expression 23 it is assumed that $j + 1$ is odd and $N + j + 1$ is odd then N is even and as before 23 becomes

$$(-1)^{N/2} \begin{vmatrix} (f_{1+j} + xf_{2+j}) & \cdots & -(f_{N+j} + xf_{N+j+1}) \\ \vdots & \ddots & \vdots \\ (f_{N+j} + xf_{N+j-1}) & \cdots & -(f_{2N+j-1} + xf_{2N+j}) \end{vmatrix}$$

$$= (-1)^{N/2+N/2} D = (-1)^N D = 1 \cdot D = (-1)^{N(N+j)} \cdot D .$$

And in the last case let $j + 1$ be odd and $N + j + 1$ be even. Then N is odd and $N + j$ is odd which means that $N(N + j)$ is too and $(-1)^{N(N+j)} = -1$. Also expression 23 becomes

$$(-1)^{N+1/2} \cdot (-1)^{N-1/2} \cdot D = (-1)^{2N/2} D = (-1) D = (-1)^{N(N+j)} D. \tag{27}$$

Hence 22 is valid.

Denote expression 22 without the sign A_N and the coaxial minor formed by striking off the last r rows and columns by A_{N-r}. Note that

A_{N-r} is the denominator for $[N - r, N + j - r]$. Therefore the sequence $A_N, A_{N-1}, \ldots, A_0 = 1$ is a Strum sequence, that is, if $A_k = 0$ then A_{k-1} and A_{k+1} have opposite signs. In order to see this a lemma will be proved.

Lemma 2.1. The product of a determinant and any one of its minors M is expressible as a sum of products of pairs of minors. The first factors of the products are obtained by taking q rows in which the rows of M are included and forming from them every minor of the qth order which contains M and the second factor of any product is that minor which includes M and the complementary of the first factor. The sign of any product is gotten by transforming the second factor so as to have its principal diagonal coincident with those of the two minors which it was formed to include, and then taking plus or minus according as the sum of the numbers indicating the rows and columns from which the first factor was formed is even or odd.

Proof. Let $|a_{1n}|$ be the given determinant and let the minor M be denoted $|a_{pp}\ a_{p+1,p+1} \cdots a_{qq}|$. Let the q rows be the first q rows that include M.

By the Laplace expansion, see Theorem 2.13.4, page 56 in (16), the product of $|a_{1n}|$ and M can be written as the following determinant, where the determinant on the next page is a continuation of the determinant below,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots a_{1p} & \cdots a_{1q} & \cdots a_{1n} & 0 & \cdots 0 \\ a_{21} & a_{22} & \cdots a_{2p} & \cdots a_{2q} & \cdots a_{2n} & 0 & \cdots 0 \\ \vdots & \vdots & \ddots \vdots & \ddots \vdots & \ddots \vdots & \vdots & \ddots \vdots \\ a_{p-1,1} & a_{p-1,2} & \cdots a_{p-1,p} & \cdots a_{p-1,q} & \cdots a_{p-1,n} & 0 & \cdots 0 \end{vmatrix} \qquad (28)$$

$$\begin{vmatrix} a_{p1} & a_{p2} & \cdots & a_{pp} & \cdots & a_{pq} & \cdots & a_{pn} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & \cdots & a_{qp} & \cdots & a_{qq} & \cdots & a_{qn} & 0 & \cdots & 0 \\ a_{q+1,1} & a_{q+1,2} & \cdots & a_{q+1,p} & \cdots & a_{q+1,q} & \cdots & a_{q+1,n} & a_{q+1,p} & \cdots & a_{q+1,q} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} & \cdots & a_{nq} & \cdots & a_{nn} & a_{np} & \cdots & a_{nq} \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & a_{pp} & \cdots & a_{pq} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & a_{qp} & \cdots & a_{qq} \end{vmatrix}$$

where $|a_{1n}|$ is in the first n rows and columns with nothing but zero elements below it. Expression 28 will also be called A. Notice that M is the compliment of $|a_{1n}|$ in A and that the first q rows of $|a_{1n}|$ are extended with zeros but that from row q + 1 to n the extension is comprised of the elements in columns p to q. In A the minor M occurs twice. Now add each element of the first row where M occurs first to each element of the first row where M occurs second and continue this process until all the rows in which M lies has so been operated on. Now subtract each element in the first column of where the second M begins from the corresponding element in the first column where the first M begins. Continue until all such columns have been taken care of. The result is that A becomes

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,p} & \cdots & a_{1q} & \cdots & a_{1n} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2p} & \cdots & a_{2,q} & \cdots & a_{2n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{p-1,1} & a_{p-1,2} & \cdots & a_{p-1,p} & \cdots & a_{p-1,q} & \cdots & a_{p-1,n} & 0 & \cdots & 0 \\ a_{p,1} & a_{p,2} & \cdots & a_{p,p} & \cdots & a_{p,q} & \cdots & a_{p,n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{q,1} & a_{q,2} & \cdots & a_{q,p} & \cdots & a_{q,q} & \cdots & a_{n,n} & 0 & \cdots & 0 \end{vmatrix} \qquad (29)$$

$$\begin{vmatrix} a_{q+1,1} & a_{q+1,2} & \cdots & 0 & \cdots & 0 & \cdots & a_{q+1,n} & a_{q+1,p} & \cdots & a_{q+1,q} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 & \cdots & 0 & \cdots & a_{n,n} & a_{np} & \cdots & a_{nq} \\ a_{p1} & a_{p2} & \cdots & 0 & \cdots & 0 & \cdots & a_{pn} & a_{pp} & \cdots & a_{pq} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & \cdots & a_{q,p-1} & \cdots & 0 & \cdots & a_{qn} & a_{qp} & \cdots & a_{qq} \end{vmatrix}.$$

Now if relative to the first q rows every minor of the qth order is formed in preparation to finding the expansion of A as a sum of products of complementary minors (Laplaces expansion) it is seen that, although the totality of such minors is exactly the same as for $|a_{1n}|$, the only ones relevant are those that include M because all of the others have complementaries which vanish. Now each of the complementaries of those thus taken includes or contains the complementary of the same minor in $|a_{1n}|$ and the selected minor besides and each complementary itself is a minor of $|a_{1n}|$ being formed from those n - P + 1 rows of $|a_{1n}|$ which are made from the n - q rows not included in the chosen q rows and the q - P + 1 rows in which M is located. But this sum of products is the sum of products specified in the theorem and since it is the equivalent of A and of $|a_{1n}|\cdot M$ the lemma is proved.

Now if A is symmetric then by the lemma

$$A\cdot A_{12,12} = A_{11}\, A_{22} - A_{12}^2 \tag{30}$$

where A is a given determinant and A_{11}, A_{12}, A_{22} are the respective minors of a_{11}, a_{12}, and a_{22}. $A_{12,12}$ is the minor formed by deleting the first and second columns and rows. Now if $A_{11} = 0$ then A and $A_{12,12}$ must have opposite signs.

In order to locate the roots notice the following remark. If $(-1)^{N(N+j)}\cdot Q_N^j(z)$ is the denominator given by equation 3 in Chapter II for the [N,N + j] approximant then $Q_N^j(0) = D(1 + j, N - 1) > 0$ for all N. Now $Q_1^j(z) = f_{1+j} + xf_{2+j}$ and

$$f_k = \int_0^\infty u^k \, d\varphi(k) = D(k,0) > 0$$

which implies that the root of $Q_1^j(z)$ is real and negative. Call it x_1^1. Also

$$Q_2^j(0) = \begin{vmatrix} f_{1+j} + 0f_{2+j} & f_{2+j} + 0f_{3+j} \\ f_{2+j} + 0f_{3+j} & f_{3+j} + 0f_{4+j} \end{vmatrix} = f_{1+j} \cdot f_{3+j} - f_{2+j}^2 > 0$$

and by expression 22 and equation 30 is $-(f_{2+j} + zf_{3+j})^2 < 0$ where z is the root of $Q_1^j(z)$. $Q_2^j(z)$ is a polynomial of degree 2 and must therefore vanish between zero and the root of $Q_1^j(z)$. The coefficient of x^2 is

$$\begin{vmatrix} f_{2+j} & f_{3+j} \\ f_{3+j} & f_{4+j} \end{vmatrix} = D(j + 2,1)$$

and, since N is even, is greater than zero by expression 22. Hence as x goes to $-\infty$ the second degree equation, with leading coefficient positive, must become positive which means there are 2 negative real roots. Let x_2^1 and x_2^2 denote the roots for $Q_2^j(z)$. If

$$Q_3^j(x) = \begin{vmatrix} f_{1+j} + xf_{2+j} & f_{2+j} + xf_{3+j} & f_{3+j} + xf_{4+j} \\ f_{2+j} + xf_{3+j} & f_{3+j} + xf_{4+j} & f_{4+j} + xf_{5+j} \\ f_{3+j} + xf_{4+j} & f_{4+j} + xf_{5+j} & f_{5+j} + xf_{6+j} \end{vmatrix}$$

then $Q_3^j(0) = D(j + 1,2) > 0$ and if $f_{5+j} + xf_{6+j}$ is put in the first column and first row and $f_{1+j} + xf_{2+j}$ is put in the last row and column then $Q_3^j(x)$ remains exactly the same. Denote this ${}_TQ_3^j(x)$. Assume that $x_2^2 < x_1^1 < x_2^1$. Let $x = x_2^1$. Then

$${}_TQ_3^j(x_2^1)(f_{1+j}+x_2^1 f_{2+j}) = {}_TQ_3^j(x_2^1)_{11} \cdot {}_TQ^j(x_2^1)_{22} - ({}_TQ_3^j(x_2^1)_{12})^2 = -({}_TQ^j(x_2^1)_{12})^2,$$

since

$$ {}_TA_{11} = \begin{vmatrix} f_{1+j} + x_2^1 f_{2+j} & f_{2+j} + x_2^1 f_{3+j} \\ f_{2+j} + x_2^1 f_{3+j} & f_{3+j} + x_2^1 f_{4+j} \end{vmatrix} = 0. $$

Now since $f_{1+j} + x_2^1 f_{2+j} > 0$ (this is because $x_2^1 > x_1^1$) ${}_T Q_0^j(x_2^1)$ must be negative. Hence it must have a root x_3^1 such that $x_2^1 < x_3^1 < 0$. Yet when $x = x_2^2$ then $f_{1+j} + xf_{2+j} < 0$ and ${}_T Q_3^j(x) > 0$ and there must be a root for ${}_T Q_3^j(x)$ between x_2^1 and x_2^2. Then the limit as x goes to minus infinity of the third degree polynomial $Q_3^j(x)$ becomes negative and this is the last root (call it x_3^3) of $Q_3^j(x)$. Now if $(-1)^{N(N+j)} > 0$, then the denominator which is a third degree equation must become negative as x goes to $-\infty$. Of course if $(-1)^{N(N+j)} < 0$ then the denominator is negative at zero and positive at x_2^1 and negative at x_2^2 and the trinomial will become positive as x goes to negative infinity so that again the third root will occur some place smaller than x_2^2. Assume for $N = k$ that the roots are as for $N = 2$ or $N = 3$ above and consider $N = k + 1$. Without loss of generality assume that $(-1)^{(N+1)(N+1+j)} > 0$. Put $f_{k+N+1} + xf_{N+2+j}$ in the first column and row so that in ${}_T Q_{N+1}^j(x)$ one has $Q_{N+1}^j(x)_{11} = Q_N^j(x)$ and $Q_{N+1}^j(x)_{12,12} = Q_{N-1}^j(x)$ where it is realized that $f_{1+j} + xf_{2+j}$ is in the last row and column. The determinant stays the same in fulfilling this obligation, in that starting with an interchange of the first and last columns and continuing with the second and next to last column being interchanged, it is noticed that there will be [n/2] interchanges, depending on whether n is odd. Now do the same thing with the rows. Altogether there have been [n/2] + [n/2] interchanges and this means that the sign of the determinant is the same.

Now for $x = x_N^1$, $Q_{N+1}^j(x)_{11} = Q_N^j(x) = 0$ and $Q_{N+1}^j(x)_{12,12} = Q_{N-1}^j(x) > 0$ since the first root of $Q_{N-1}^j(x)$ is to the left of x_N^1. Hence $Q_{N+1}^j(x)$ is negative and has a root between x_N^1 and 0. Now at x_N^2, $Q_N^j = 0$ and $Q_{N-1}^n < 0$ since $x_N^2 < x_{N-1}^1$ which defines a root $x_{N+1}^2 \in (x_N^2, x_N^1)$. Continuing in this manner until the last of the x_N^i have been passed it is noted that the leading coefficient (coefficient of x^{N+1}) is positive and if N + 1 is odd, N is even, and under the assumption that $(-1)^{N(N+j)} > 0$ one has that $P_{N+1} = (-1)^{(N+1)(N+1+j)}Q_{N+1}^j > 0$ and as x goes to $-\infty$, P_{N+1} will become negative. While if N + 1 is even P_{N+1} is negative and still a polynomial of degree N + 1 = 2k which means that, as x tends to $-\infty$, P_{N+1} will become positive which gives the last root of P_{N+1}. This was done under the assumption that $(-1)^{(N+1)(N+1+j)} > 0$ but should it be otherwise an identical arguement will validate the assertion.

In showing the positiveness of the residues it is helpful to show first that if $(-1)^{N(N+j)}P_N^j(z)$ is the numerator given in equation 3 of Chapter II for the [N,N + j] Pade′ approximant then

$$P_{N+1}^j(z)Q_N^j(z) - P_N^j(z)Q_{N+1}^j(z) = (-z)^{2N+1+j}[D(1+j,N)]^2. \tag{31}$$

By equation 1, Chapter II

$$[P_{N+1}(z)/Q_{N+1}^j(z)]Q_N^j(z) - P_N^j(z) = O(z^{2N+1+j}) \tag{32}$$

because the [N,N] Pade′ approximant is considered in the determination of the [N+1,N+1] approximant. Multiply by $Q_{N+1}^j(z)$ to obtain a polynomial of degree 2N + 1 + j and notice, on examination of this result, that only the coefficient of z^{2N+1+j} does not vanish. To establish equation 31 this coefficient must be evaluated. The (2N + 1 + j)th coefficient in [N+1,N+1+j] is $(-1)^{1+j}f_{2N+1+j}$ since this approximant must agree with f(z) through the coefficient of z^{2N+1+j}. In

$$f(z)Q_N^j(z) - P_N^j(z) - P_N^j(z) = A^{2N+j+1} + \ldots, \qquad (33)$$

it is realized that $P_N^j(z)/Q_N^j(z)$ agrees with f(z) for all coefficients through z^{2N+j}. Now in $f(z)Q_N^j(z)$, the coefficient of z^{2N+j+1} is given by

$$(-1)^{j+1}f_{2N+j+1}\ D(1+j,N-1) + (-1)^{1+j}\begin{vmatrix} f_{1+j} & \cdots f_{N+j} & f_{N+j+1} \\ \vdots & \ddots & \vdots \\ f_{N+j} & \cdots f_{2N+j-1} & f_{2N+j} \\ f_{N+j+1} & \cdots f_{2N+j} & 0 \end{vmatrix} = A \qquad (34)$$

where A is in 33 and $(-1)^{1+j}$ exists in the last addend to give $f_{N+j+1},\ldots,f_{2N+j}$ their appropriate sign. Notice that $(-1)^{N(N+j)}$ has been divided out. And also be aware of, in the computation of A, that z in $Q_N^j(z)$ can be considered zero.

Therefore

$$f(z) - \frac{P_N^j(z)}{Q_N^j(z)} = \frac{Az^{2N+j+1} + Bz^{2N+j+2} + \cdots}{Q_N^j(z)}$$

$$= \frac{\frac{A}{D(1+j,N-1)}z^{2N+j+1} + \frac{B}{D(1+j,N-1)}z^{2N+j+2} + \cdots}{(\text{Polynomial of degree N with constant term 1})}$$

$$= \frac{z^{2N+j+1}\left((-1)^{j+1}f_{2N+j+1}D(1+j,N-1)+(-1)^{1+j}\begin{vmatrix} f_{1+j} & \cdots f_{N+j} & f_{N+j+1} \\ \vdots & \ddots & \vdots \\ f_{N+j} & \cdots f_{2N+j+1} & f_{2N+j} \\ f_{N+j+1} & \cdots f_{2N+j} & 0 \end{vmatrix}\right)}{D(1+j,N-1)}$$

(Plus a power series beginning with z^{2N+j+2}).

Now the coefficient of z^{2N+j+1} of the left side must be the same as the right side. Hence the coefficient of z^{2N+j+1} in $P_N^j(z)/Q_N^j(z)$ is

$$(-1)^{1+j} f_{2N+j+1} D(1+j,N-1) - (-1)^{j+1} f_{2N+j+1} D(1+j,N-1) - (-1)^{1+j} T$$

which is

$$-(-1)^{1+j} \begin{vmatrix} f_{1+j} & \cdots & f_{N+j} & f_{N+j+1} \\ \vdots & \ddots & \vdots & \vdots \\ f_{N+j} & \cdots & f_{2N+j-1} & f_{2N+1} \\ f_{N+j+1} & \cdots & f_{2N+j} & 0 \end{vmatrix} /D(1+j,N-1). \qquad (35)$$

This means that the coefficient of z^{2N+j+1} in [N,N-j] must be expression 35. After subtracting [N,N+j] from [N+1,N+j+1] the coefficient of z^{2N+j+1} becomes

$$(-1)^{j+1} f_{2N+j+1} D(1+j,N-1) + (-1)^{1+j} \begin{vmatrix} f_{1+j} & \cdots & f_{N+j+1} \\ \vdots & \ddots & \vdots \\ f_{N+j} & \cdots & f \\ f_{N+j+1} & \cdots & 0 \end{vmatrix} /D(1+j,N-1)$$

$$= (-1)^{j+1} \frac{D(1+j,N)}{D(1+j,N-1)}.$$

So from equation 32

$$\frac{P_{N+1}^j(z)}{Q_{N+1}^j(z)} - \frac{P_N^j(z)}{Q_N^j(z)} = (-1)^{j+1} \frac{D(1+j,N)}{D(1+j,N-1)} z^{2N+j+1} + (z^{2N+2+j})$$

where (z^{2N+2+j}) stands for a power series beginning with a term containing no power of z less than z^{2N+2+j}. Now multiply by $Q_{N+1}^j(z)Q_N^j(z)$ to get

$$P_{N+1}^j(z)Q_N^j(z) - Q_{N+1}^j(z) = (-1)^{j+1} \frac{D(1+j,N)}{D(1+j,N-1)} Q_{N+1}^j(z)Q_N^j(z)z^{2N+j+1} \quad (36)$$

where the rest of the right portion of the equality vanishes because the left side is a polynomial of at degree most z^{2N+1+j}. Now for all

$z \neq 0$ $Q^j_{N+1}(z)$ or $Q^j_N(z)$ must vanish so that $Q^j_{N+1}(z)Q^j_N(z)$ is meaningful only if $z = 0$. But this implies that

$$Q^j_N(z)Q^j_{N+1}(z) = D(1 + j,N)D(1 + j,N-1)$$

and equation 36 becomes

$$P^j_{N+1}(z)Q^j_N(z) - Q^j_{N+1}(z)P^j_N(z) = (-z)^{2N+1+j}[D(1 + j,N)]^2$$

which is equation 31. It is convenient to write 31 as

$$\frac{P^j_{N+1}(z)}{Q^j_{N+1}(z)} - \frac{P^j_N(z)}{Q^j_N(z)} = \frac{(-z)^{2N+1+j}[D(1+j,N)]^2}{Q^j_{N+1}(z)Q^j_N(z)} . \tag{37}$$

The interlacing property implies that the residue at the first (going from right to left starting at the origin) root of Q^j_{N+1} must be positive on the right because $[D(1 +j,N)]^2 > 0$ and Q^j_N is positive to the right of its first root $x^1_N < x^1_{N+1}$. Now $Q^j_{N+1}(x)$ is a polynomial of degree $N + 1$ which has for all its roots negative distinct values. By a theorem, see (5), $Q^j_{N+1}(x) = A(x - x^1_{N+1})\ldots(x - x^{N+1}_{N+1})$. In a sufficiently small neighborhood N of x^1_{N+1} equation 37 is analytic except at x^1_{N+1}. For all $x \in N$, $(x - x^i_{N+1}) > 0$ for $i = 2,\ldots,N + 1$. This means that A is positive, for if $x - x^1_{N+1} > 0$ then $A(x - x^1_{N+1})\ldots(x - x^{N+1}_{N+1}) > 0$ or $A > 0$. Also if $z^1_{N+1} < 0$ then $-z^1_{N+1} > 0$ and by page 239 in (18) the residue

$$\text{Res}\left[\frac{(-z)^{2N+1+j}[D(1 +j,N)]^2}{Q^j_{N+1}(z)Q^j_N(z)} , z^1_{N+1}\right]$$

which is

$$\lim_{z\to z^1_{N+1}} (z - z^1_{N+1})\frac{(-z)^{2N+1+j}[D(1+j,N)]^2}{A(z-z^1_{N+1})\ldots(z-z^{N+1}_{N+1})Q^j_N(z)} \tag{38}$$

$$= \frac{(-z^1_{N+1})^{2N+1+j}[D(1+j,N)]^2}{A(z^1_{N+1}- z^1_{N+1})\ldots(z^1_{N+1}- z^{N+1}_{N+1})Q^j_N(z^1_{N+1})} > 0. \tag{39}$$

Hence the left side of equation 37 has a positive residue at z^{11}_{N+1}. Now go by the first root of $Q^j_N(z)$ and only the first root until the root z^2_{N+1} is reached. In the region from z^1_N to z^2_{N+1} $Q^j_N(z) < 0$ as is $Q^j_{N+1}(z)$. Using the same residue theorem one gets

$$\frac{(-z^2_{N+1})^{2N+j+1}[D(1+j,N]^2}{A(z^2_{N+1} - z^1_{N+1})(z^2_{N+1} - z^3_{N+1})\ldots(z^2_{N+1} - z^{N+1}_{N+1})Q^j_N(z^2_{n+1})} > 0.$$

Continuing in this fashion one realizes that all of the residues are positive. Since

$$\frac{P^j_{k+1}(z)}{Q^j_{k+1}(z)} = \frac{P^j_k(z)}{Q^j_k(z)} + \frac{(-z)^{2N+j+1}[D(1+j,N)]^2}{Q^j_{N+1}(z)Q^j_N(z)}$$

and $\lim_{z\to z^1_{N+1}} \frac{P^j_{k+1}(z)}{Q^j_{k+1}(z)}(z - z^1_{k+1})$ is the residue given in expression 39.

the residues of all Padé approximants are positive. Consider $P^j_N(z)/Q^j_N(z)$. $Q^j_N(0)$ is positive and $\lim_{z\to z^1_N} (z - z^1_N)\frac{P^j_N(z)}{Q^j_N(z)} > 0$

which means that $P^j_N(z^1_N) > 0$ because

$$(z - z^1_N)/Q^j_N(z) = 1/A(z - z^2_N)\ldots(z - z^N_N) > 0.$$

Now at z^2_N, $(z - z^2_N)/Q^j_N(z)$ is negative and the residue is positive which means that $P^j_N(z^2_N) < 0$. Hence $P^j_N(z)$ must have a negative real root between z^1_N and z^2_N. Continue the argument to get that the first $N - 1$ roots of P^j_N interlace those first N roots of Q^j_N.

It is now convenient to prove a theorem which provides upper and lower bounds for the exact sum of a series of Stieltjes whether the series is convergent or not.

Theorem 3. The Pade' approximants for a series of Stieltjes satisfy the following inequalities where f(z) is the sum of the series $\Sigma f_j(-z)^j$, $j \geq -1$, and z is real and nonnegative:

$$(-1)^{1+j}\{[N+1,N+1+j] - [N,N+j]\} \geq 0 \tag{40}$$

$$(-1)^{1+j}\{[N,N+j] - [N-1,N+j+1]\} \geq 0 \tag{41}$$

$$[N,N] \geq f_1(z) \geq [N,N-1] \tag{42}$$

$$[N,N]^1 \geq f_1^1(z) \geq [N,N-1]^1. \tag{43}$$

These inequalities have the consequence that the [N,N] and [N,N-1] sequences form the best upper and lower bounds obtainable from the [N,N + j] approximants with a given number of coefficients and that the use of additional coefficients (higher N) improves the bounds. Expressions 40 and 41 are valid when differentiated if $j \geq 0$ in 40.

Proof. By Theorem 2 and 37 whenever $z \geq 0$ $Q_{N+1}^j(z) > 0$ and $Q_N^j(z) > 0$ so that

$$(-1)^{1+j} \frac{(-z)^{2N+1+j}[D(1+j,N)]^2}{Q_{N+1}^j(z)Q_N^j(z)} \geq 0$$

or

$$(-1)^{1+j}\{[N+1,N+1+j] - [N,N+j]\} \geq 0.$$

In proving 41 first consider the following equation.

$$Q_{N-1}^{j+2}(z) \frac{P_N^j(z)}{Q_N^j(z)} - P_{N-1}^{j+2}(z) = O(z^{2N+j+1})$$

or

$$Q_{N-1}^{j+2}(z)\ P_N^j(z) - Q_N^j(z)\ P_{N-1}^{j+2}(z) = Q_N^j(z)\ \ O(z^{2N+j+1})$$

which is a polynomial of degree 2N + j + 1 where only the coefficient of z^{2N+j+1} does not vanish. Since [N,N+j] and [N-1,N-1+j+2] both agree with f(z) for the first 2N+j places only their coefficients relative to z^{2N+j+1} must be calculated. Now if equation 1 in Chapter II is to be satisfied, that is,

$$Q_N^j(z)f(z) - P_N^j(z) = Az^{2N+j+1} + \dots$$

then

$$f(z) - \frac{P_N^j(z)}{Q_N^j(z)} = \frac{\{(-1)^{j+1} f_{2N+j+1} D(1+j,N-1) + (-1)^{1+j} \begin{vmatrix} f_{1+j} & \cdots & f_{N+j+1} \\ \vdots & \ddots & \vdots \\ f_{N+j+1} & \cdots & 0 \end{vmatrix} \}}{D(1+j,N-1)}$$

times z^{2N+j+1} plus a power series beginning with z^{2N+j+2}.

Hence the coefficient in $P_N^j(z)/Q_N^j(z)$ of z^{2N+j+1} is as before

$$-(-1)^{1+j} \begin{vmatrix} f_{1+j} & \cdots & f_{N+j+1} \\ \vdots & \ddots & \vdots \\ f_{N+j+1} & \cdots & 0 \end{vmatrix} / D(1+j,N-1). \qquad (44)$$

Now

$$f(z) - \frac{P_{N-1}^{j+2}(z)}{Q_{N-1}^{j+2}(z)} = \{(-1)^{j+1} f_{2N+j+1} D(j+3,N-2) + (-1)^{1+j} \begin{vmatrix} f_{3+j} & \cdots & f_{N+j+2} \\ \vdots & \ddots & \vdots \\ f_{N+j+2} & \cdots & 0 \end{vmatrix} \}$$

times $(z^{2N+j+1}/D(j+3, N-2)) + (z^{2N+j+2})$.

Hence the coefficient for z^{2N+j+1} in $P_{N-1}^{j+2}(z)/Q_{N-1}^{j+2}(z)$ is

$$-(-1)^{1+j} \begin{vmatrix} f_{j+3} & \cdots & f_{N+j+1} & f_{N+j+2} \\ \vdots & \ddots & \vdots & \vdots \\ f_{N+j+1} & \cdots & f_{2N+j-1} & f_{2N+j} \\ f_{N+j+2} & \cdots & f_{2N+j} & 0 \end{vmatrix} / D(j+3, N-2). \qquad (45)$$

Therefore

$$\frac{P_N^j(z)}{Q_N^j(z)} - \frac{P_{N-1}^{j+2}(z)}{Q_{N-1}^{j+2}(z)} = [(44) - (45)]\, z^{2N+j+1} + (z^{2N+j+2}) \qquad (46)$$

or

$$P_N^j(z)Q_{N-1}^j(z) - Q_N^j(z)P_{N-1}^{j+2}(z) = Q_N^j(z)Q_{N-1}^{j+2}(z)[(44) - (45)]z^{2N+j+1} \qquad (47)$$

where it is necessary to consider only $Q_N^j(0)Q_{N-1}^j(0)$ since otherwise 47 would not be a polynomial with one term in z^{2N+j+1}. As was done for expression 22 one gets

$$Q_N^j(0) = D(1 + j, N - 1), \quad Q_{N-1}^{j+2}(0) = D(j+3,N-2)$$

and 47 becomes

$$(-1)^{1+j} \frac{\begin{vmatrix} f_{j+3} & \cdots & f_{n+j+1} & f_{N+j+2} \\ \vdots & \ddots & \vdots & \vdots \\ f_{N+j+1} & \cdots & f_{2N+j-1} & f_{2N+j} \\ f_{N+j+2} & \cdots & f_{2N+j} & 0 \end{vmatrix}}{D(j+3,N-2)} - (-1)^{1+j} \frac{\begin{vmatrix} f_{1+j} & \cdots f_{N+j} & f_{N+j+1} \\ \vdots & \ddots \vdots & \vdots \\ f_{N+1} & \cdots f_{2N+j-1} & f_{2N+j} \\ f_{N+j+1} & \cdots f_{2N+j} & 0 \end{vmatrix}}{D(1+j,N-1)}$$

times $$D(1+j,N-1)\, D(j+3,N-2)z^{2N+j+1}$$

$$= [(-1)^{1+j}D(1+j,N-1)\, T - (-1)^{1+j}D(j+3,N-2)\, L]z^{2N+j+1}. \tag{48}$$

Now $$T = D(j+3,N-1)-f_{2N+j+1}D(j+3,N-2)$$

and $$L = D(j+1,N) - f_{2N+j+1}D(1+j,N-1)$$

so that 48 becomes

$$z^{2N+j+1}(-1)^{1+j}[D(1+j,N-1)D(j+3,N-1)-f_{2N+j+1}D(j+3,N-2)D(1+j,N-1)$$
$$- D(j+3,N-2)D(j+1,N) + f_{2N+j+1}D(1+j,N-1)D(j+3,N-3)]$$
$$= (-1)^{1+j}[D(1+j,N-1)D(j+3,N-1) - D(j+3,N-2)D(j+1,N)]z^{2N+j+1}. \tag{49}$$

And from 47 one has

$$\frac{P_N^j(z)}{Q_N^j(z)} - \frac{P_{N-1}^{j+2}(z)}{Q_{N-1}^{j+2}(z)} = (-z)^{2N+j+1}\, \frac{[D(1+j,N-1)D(3+j,N-1)-D(j+3,N-2)D(j+1,N)]}{Q_N^j(z)Q_{N-1}^{j+2}(z)}. \tag{50}$$

If

$$A = D(1+j,N) = \begin{vmatrix} f_{1+j} & f_{2+j} & f_{3+j} & \cdots & f_{N-1+j} & f_{N+j} & f_{N+j+1} \\ f_{2+j} & f_{3+j} & f_{4+j} & \cdots & f_{N+j} & f_{N+j+1} & f_{N+j+2} \\ f_{3+j} & f_{4+j} & f_{5+j} & \cdots & f_{N+j+1} & f_{N+j+2} & f_{N+j+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$\begin{vmatrix} f_{N+j-1} & f_{N+j} & f_{N+j+1} & \cdots & f_{2N+j-4} & f_{2N+j-3} & f_{2N+j-2} \\ f_{N+j} & f_{N+j+1} & f_{N+j+2} & \cdots & f_{2N+j-3} & f_{2N+j-2} & f_{2N+j-1} \\ f_{N+j+1} & f_{N+j+2} & f_{N+j+3} & \cdots & f_{2N+j-2} & f_{2N+j-1} & f_{2N+j} \end{vmatrix} \quad (51)$$

then by changing the second and last column and then the second and last row 51 becomes

$$\begin{vmatrix} f_{1+j} & f_{N+j+1} & f_{3+j} & \cdots & f_{N-1+j} & f_{N+j} & f_{2+j} \\ f_{N+j+1} & f_{2N+j} & f_{N+j+3} & \cdots & f_{2N+j-2} & f_{2N+j-1} & f_{N+j+2} \\ f_{3+j} & f_{N+j+3} & f_{5+j} & \cdots & f_{N+j+1} & f_{N+j+2} & f_{4+j} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ f_{N+j-1} & f_{2N+j-2} & f_{N+j+1} & \cdots & f_{2N+j-4} & f_{2N+j-3} & f_{N+j} \\ f_{N+j} & f_{2N+j-1} & f_{N+j+2} & \cdots & f_{2N+j-3} & f_{2N+j-2} & f_{N+j+1} \\ f_{2+j} & f_{N+j+2} & f_{4+j} & \cdots & f_{N+j} & f_{N+j+1} & f_{3+j} \end{vmatrix} . \quad (52)$$

The value of A has not changed and likewise for A_{11}. Now by exchanging the last and next to last column and so on until the next to the last column is in the last column there will have been $N - 2$ interchanges for both A and A_{11}. Now perform identical transformation on the rows. There will be $N - 2$ of these. And again the A and A_{11} have had $N - 2$ interchanges of rows. Hence A and A_{11} have had $2(N - 2)$ sign changes and remain the same. Notice that in 51, $A_{11} = D(3+j,N-1)$, and after all of these transformations it still is but now $A_{22} = D(3+j,N-2)$ and also after these transformations $A_{12,12}$ is $D(3+j,N-2)$. Using 30, that is, $0 < A_{12}^2 = A_{11} A_{22} - A \cdot A_{12,12}$ the quantity in brackets of 50 is seen to be positive. The Q's are positive for $z \geq 0$ so that $(-1)^{1+j}$ multiplied on the right of 50 is a nonnegative number. This proves 41.

Now repeated application of 41 shows one how to bound [N,N+j] by the first 2N + j power series terms. That is

$$(-1)^{1+j}\{[N,N+j] - [N-1,N+j+1]\} \geq 0$$

$$(-1)^{1+j+2}\{[N-1,N+j+1] - [N-2,N+j+2]\} \geq 0$$

$$(-1)^{1+j+4}\{[N-2,N+j+2] - [N-3,N+j+3]\} \geq 0$$

$$\vdots \qquad\qquad \vdots$$

$$(-1)^{1+j+2(N-1)}\{[1,2N+j-1] - [0,2N+j]\} \geq 0$$

and upon adding, the terms telescope and become

$$(-1)^{1+j}\{[N,N+j] - [0,2N+j]\} \geq 0.$$

If j is even $[N,N+j] - [0,N2+j] \leq 0$ or $[N,N+j] \leq [0,2N+j]$ and it is noticed that the right side of the inequality is the first 2N+j terms of the power series of which the approximants are calculated. Now

$$P_N^0(z)Q_N^{-1}(z) - Q_N^0(z)P_N^{-1}(z) \tag{53}$$

is a polynomial with one term in z^{2N} and $P_N^0(z)/Q_N^0(z)$ has for its coefficient f_{2N}. Now the coefficient of z^{2N} in $P_N^{-1}(z)/Q_N^{-1}(z)$ will be computed as before from 1 in Chapter II. Since

$$Q_N^{-1}(z)f(z) - P_N^{-1}(z) = Az^{2N} + Bz^{2N+1} + \cdots$$

the coefficient A of z^{2N} is

$$f_{2N}D(0,N-1) + \begin{vmatrix} f_0 & \cdots & f_{N-1} & f_N \\ \vdots & \ddots & \vdots & \vdots \\ f_N & \cdots & f_{2N-1} & 0 \end{vmatrix}.$$

Now $(f(z) - P_N^{-1}(z)/Q_N^{-1}(z)) = (Az^{2N} + Bz^{2N+1}+\ldots)/Q_N^{-1}(z)$ so that the coefficient of z^{2N} in $P_N^{-1}(z)/Q_N^{-1}(z)$ is as before

$$- \begin{vmatrix} f_0 & \cdots & f_N \\ \vdots & \ddots & \vdots \\ f_N & \cdots & 0 \end{vmatrix} / D(0,N-1).$$

Since $$(P_N^0(z)/Q_N^0(z))Q_N^{-1}(z) - P_N^{-1}(z) = (z^{2N}),$$

$$\frac{P_N^0(z)}{Q_N^0(z)} - \frac{P_N^{-1}(z)}{Q_N^{-1}(z)} = (f_{2N} + \frac{\begin{vmatrix} f_0 & \cdots & f_N \\ \vdots & \ddots & \vdots \\ f_N & \cdots & 0 \end{vmatrix}}{D(0,N-1)}) z^{2N} + (z^{2N+1})$$

$$= \frac{D(0,N)}{D(0,N-1)} z^{2N} + (z^{2N+1})$$

or

$$P_N^0(z)Q_N^{-1}(z) - Q_N^0(z)P_N^{-1}(z) = (Q_N^0(z)Q_N^{-1}(z)D(0,N)/D(0,N-1))z^{2N}. \quad (54)$$

Now $Q_N^{-1}(z)Q_N^0(z)D(0,N)/D(0,N-1)$ cannot have a term in z. Therefore it is only necessary to evaluate $Q_N^{-1}(z)Q_N^0(z)$ at zero for otherwise there would be z terms which would raise the power of z^{2N} so that a contradiction to 53 arises, that is, the product in the coefficient must vanish whenever z is involved but will not necessarily so vanish when z is not involved and this is when

$$Q_N^0(z)Q_N^{-1}(z) = D(1,N-1)D(0,N-1).$$

Hence 54 is

$$P_N^0(z)Q_N^{-1}(z) - Q_N^0(z)P_N^{-1}(z) = D(1,N-1)D(0,N)z^{2N}$$

or (55)

$$(P_N^0(z)/Q_N^0(z))-(P_N^{-1}(z)/Q_N^{-1}(z))=((D(0,N)D(1,N-1)z^{2N})/Q_N^0(z)Q_N^{-1}(z)) \geq 0$$

or $[N,N] \geq [N,N-1]$. Now from 40, when $j = -1$, $[N+1,N] \geq [N,N-1]$ so that $\{[N,N-1]\}$ is monotonic increasing. Hence $[2,1] \geq [1,0]$ or

$[N,N-1] \geq 0$ for every N which makes $[N,N] \geq [1,0]$ for every N. For every $a > 0$, and since by Theorem 2, Chapter II the $P_N^0(z-a)/Q_N^0(z-a)$ is the $[N,N]$ approximant to $f(z-a)$ and since the $[N,N]$ are monotone decreasing by 40 with $j = 0$ and uniformly bounded in $|z-a| \leq a$ since $[N,N] \leq [1,1]$ for all z the Padé approximants converge to an analytic function $g(z)$ by Theorem 5 in Chapter II. Likewise $[N,N-1]$ evaluated at $(z-a)$ where $|z-a| \leq a$ is uniformly bounded by $[1,1]$ and must also converge to an analytic function $h(z)$. By 41 $[N,N] \leq [0,2N]$ and $[N,N-1] \geq [0,2N-1]$. If $[0,2N] \to f_1(z)$ and $[0,2N-1] \to f_1(z)$ then since $[0,2N-1] \leq [N,N-1] \leq [N,N] \leq [0,2N]$ one gets $h(z) = f_1(z) = g(z)$. Hence $[N,N-1] \leq f_1(z) \leq [N,N]$.

Now differentiate 55 and obtain

$$\frac{Q_N^0(z)Q_N^{-1}(z) \quad Nz^{2N-1}D(0,N)D(1,N-1)-z^{2N}D(0,N)D(1,N-1)[Q_N^0(z)(Q_N^{-1}(z))']}{[Q_N^0(z)Q_N^{-1}(z)]^2}$$

$$+\frac{Q_N^0(z)Q_N^{-1}(z) \quad Nz^{2N-1}D(0,N)D(1,N-1)-z^{2N}D(0,N)D(1,N-1)[Q_N^{-1}(z)(Q_N^0(z))']}{[Q_N^0(z)Q_N^{-1}(z)]^2}$$

the numerator of which is

$$= D(0,N)D(1,N-1)[2Nz^{2N-1}Q_N^0Q_N^{-1}-z^{2N}Q_N^0(z)(Q_N^{-1}(z))'+Q_N^{-1}(z)(Q_N^0(z))'z^{2N}]$$

$$=D(0,N)D(1,N-1)z^{2N-1}[2NQ_N^0Q_N^{-1}-Q_N^0(z)(Q_N^{-1}(z)-D(0,N))-Q_N^{-1}(z)(Q_N^0(z)-D(1,N+1))]$$

$$=D(0,N)D(1,N-1)z^{2N-1}[Q_N^0Q_N^{-1}(N-1)+Q_N^0D(0,N)+Q_N^0Q_N^{-1}(N-1)+Q_N^{-1}D(1,N+1)]$$

which is greater than or equal to zero so that in using 55 one has

$$\frac{P_N^0(z)}{Q_N^0(z)} - \frac{P_N^{-1}(z)}{Q_N^{-1}(z)} \geq 0.$$

Now differentiate $[N,N+j] - [N-1,N+j+1]$ to obtain

$$(-1)^{1+j}\ \frac{k[Q_N^j(z)Q_{N-1}^{j+2}(z)2Nz^{2N-1}-z^{2N}(Q_N^j(z)(Q_{N-1}^{j+2}(z))' + Q_{N-1}^{j+2}(z)(Q_N^j(z))']}{[Q_N^j(z)Q_{N-1}^{j+2}(z)]^2}$$

$$=(-1)^{1+j}k^{11}z^{2N-1}[Q_N^j(z)NQ_{N-1}^{j+2}(z)-Q_N^j(z)z(Q_{N-1}^{j+2}(z))'+Q_{N-1}^{j+2}(z)NQ_N^j(z) -Q_{N-1}^{j+2}(z)z(Q_N^j(z))'] \quad (56)$$

$$=(-1)^{1+j}k^{11}z^{2N-1}[Q_N^j(z)(NQ_{N-1}^{j+2}(z)-z(Q_{N-1}^{j+2}(z))')+Q_{N-1}^{j+2}(z)(NQ_N^j(z)-z(Q_N^j(z))')].$$

By Theorem 7 on page 17 in (8) the coefficients of both $Q_{N-1}^{j+2}(z)$ and $Q_N^j(z)$ are all positive so that $NQ_{N-1}^{j+2}(z) - z(Q_{N-1}^{j+2}(z))^1 \geq 0$ where it is realized that the coefficients of $Q_{N-1}^{j+2}(z)$ are multiplied by N and $z(Q_{N-1}^{j+2}(z))'$ is $Q_{N-1}^{j+2}(z)$ minus its constant term with the exception that each coefficient in $z(Q_{N-1}^{j+2}(z))'$ is i/N of the coefficient in $Q_{N-1}^{j+2}(z)$ minus its constant where it is assumed that $i \leq N$. Hence $(-1)^{1+j}$ times 56 is positive and

$$(-1)^{1+j}\{[N,N+j]' - [N-1,N+j+1]'\} \geq 0. \quad (57)$$

Now differentiate $[N+1,N+1+j]-[N,N+j]$ to obtain

$$\frac{(-1)^{1+j}k[Q_{N+1}^j(z)Q_N^j(z)2Nz^{2N-1}-z^{2N}(Q_{N+1}^j(z)(Q_N^j(z))'+Q_N^j(z)(Q_{N+1}^j(z))')]}{[Q_N^j(z)Q_{N+1}^j(z)]^2}$$

$$=(-1)^{1+j}k^1z^{2N-1}[Q_{N+1}^j(z)(NQ_N^j(z)-z(Q_N^j(z))')+Q_N^j(z)(NQ_N^j(z)-z(Q_{N+1}^j(z))')]$$

which by an identical arguement as above implies that

$$(-1)^{1+j}\{[N+1,N+j+1]' - [N,N+j]'\} \geq 0 \text{ if } j \geq 0. \quad (58)$$

If $j = -1$ and $N = 0$ then upon differentiating as above one gets $[k/Q_0^{-1}(z)Q_{N+1}^{-1}(z)][0 - \text{positive number}]$ and 58 would not be valid.

As argued for the approximants the $\{[N,N]'\}$ is monotonically decreasing as long as $N \geq 1$ and since they converge to $g'(z)$ by Theorem 6.8.2, page 226 in (18) $[N,N]' \geq f_1'(z)$. Also as argued for $\{[N,N-1]\}$, $\{[N,N-1]'\}$ is asserted to be monotonic increasing and they too must converge to $h'(z)$. Hence

$$[N,N]' \geq f_1'(z) \geq [N,N-1]' \; f_1'(z)=h'(z) \text{ or } f_1'(z)=g'(z)$$

This completes the proof of Theorem 3.

The following theorem is the last theorem directly related to Pade′ approximants in this chapter and it is the climax of what has already actually been proved relative to the Pade′ method. Although it must quickly be added that there are some conjectures with partial proofs, which is the substance of the last chapter, which will follow after the rest of the big moment problem is solved which is Theorem 5.

Theorem 4. Any sequence of $[N,N+j]$ Pade′ approximants for series of Stieltjes converges to an analytic function in the cut complex plane $(-\infty \leq z \leq 0)$. If the f_p are a convergent series with a radius of convergence R, then any $[N,N+j]$ sequence converges in the cut plane $(-\infty \leq z \leq -R)$ to the analytic function defined by the power series.

Proof. The first thing to notice is that the sequence $[N,N+j]$ is strictly monotonic decreasing for j even and for real positive z by 40.

If j is even then from 40 and 41 one has

$$[N+1,N+1+j] \leq [N,N+j] \leq [N-1,N+j+1] = [N-1,N-1+j+2]. \tag{59}$$

So for a given $[M,M+k]$ with k even one can well consider $N - 1 = M$ and $j + 2 = k$. In this case

$$[M,M+k] = [N-1,N-1+j+2] = [N-1,N+j+1] \geq [N+1,N+1+j]$$

since j is even and 59 is valid. Now $[N+1,N+1+j] = [N+1,N+3+j-2]$ and j - 2 is even. Therefore, again using 59,

$$[N+3,N+3+j-2] \leq [N+1,N+1+j] \leq [N-1,N-1+j+2] = [M,M+k].$$

Similarly one gets

$$[N+5,N+5+j-4] \leq [N+3,N+3+j-2] \leq \ldots \leq [M,M+k].$$

Since j is even, after a finite number of steps one will have $[N+N_1,N+N_1 + j-j] = [N+N_1,N+N_1] \leq [M,M+k]$. By 51

$$[N+N_1,N+N_1] = [N,N] \geq [N,N-1] \geq [1,0]$$

where $N = N + N_1$ and the monotonicity of $[N,N-1]$ is used. Hence the $[N,N+j]$ are decreasing and bounded below by $[1,0]$.

On the other hand if j is odd, j + 1 is even and

$$[N+1,N+1+j] \geq [N-1,N+j+1] = [N-1,N-1+j+2]. \tag{60}$$

Let $[M,M+k]$ be given and set $M = N - 1$ and $j + 2 = k$ where k is odd and $j + 2$ is odd. Then from 58 $[M,M+k] = [N-1,N-1+j+2] \leq [N+1,N+1+j]$ and $[N+1,N+1+j] \leq [N+3,N+3+j-2] \leq \ldots \leq [N+N_1,N+N_1-1]$ the $N+N_1-1$ coming about since in each case an even number 2 is being subtracted from an odd number j. By 42 $[N,N-1] \leq [N,N]$ and $[N,N]$ is monotonic decreasing which means $[N,N] \leq [1,1]$ or that $[M,M+k] \leq [1,1]$. Hence if j is odd $[M,M+k]$ is monotonic increasing and bounded above by $[1,1]$. By the Bolzano-Weierstrass Theorem for every real point these sequences converge. The assertion

$$\left|[N,N+j]\right| < \left|\sum_{p=0}^{j} f_p(-z)^p\right| + |z|^{1+j} f_{1+j} \text{ for } R(z) \geq 0 \tag{61}$$

will be shown to follow from Theorem 2. Write

$$[N,N+j] = \sum_{p=0}^{j} f_p(-z)^p + (-z)^{1+j} \sum_{p=1}^{N} \beta_p/(1+\gamma_p z),\ \beta_p > 0,\ \gamma_p > 0, \tag{62}$$

which can be done by partial fractions and Theorem 2, which states that the residues are positive and the poles are negative. Notice that

$[N, N+j]$ agrees with f for $2N+j+1$ places so that $P_N^j(z)/Q_N^j(z)$ can be written as $\sum_{p=0}^{j} f_p(-z)^p+(-z)^{1+j}[a_0+a_1z+ \dots +a_{N-1}z^{N-1}]/Q_N^j(z)$. The second factor of the last term is the ratio of a polynomial of degree less than the degree of the donominator to the denominator which by the theory of rational fractions, as can be found in undergraduate calculus books, can be written as $\sum_{p=1}^{N} \beta_p/1+\gamma_p z$ where the $\beta_p>0$ by Theorem 2.

That is,

$$\lim_{z\to -1/\gamma_1} (z+1/\gamma_1)\sum_{p=0}^{j} f_p(-z)^p+(-z)^{1+j}(z+1/\gamma_1)\beta_1/\gamma_1(z+1/\gamma_1)+(-z)^{1+j}$$

times $(z+1/\gamma_1)\sum_{p=2}^{N}\beta_p/(1+\gamma_p z) = |1/\gamma_1|^{1+j}\beta_1/\gamma_1 > 0$ and $\gamma_p > 0$ by Theorem 2 too. Hence,

$$|[N,N+j]| \leq |\sum_{p=0}^{j} f_p(-z)^p|+|z|^{1+j}\sum_{p=1}^{N}\beta_p/|1+\gamma_p z|$$

and since $R(z) \geq 0$ this means that

$$|[N,N+j]| \leq |\sum_{p=0}^{j} f_p(-z)^p+|z|^{1+j}\sum_{p=1}\beta_p. \qquad (63)$$

Now $$([N,N+j] - \sum_{p=0}^{j} f_p^p(-z))/(-z)^{j+1} = \sum_{p=1}^{N}\beta_p/(1+\gamma_p z)$$

and $$\sum_{p=j+1}^{2N+1} f_p(-z)^p+(z^{2N+j+1}))/(-z)^{j+1} = \sum_{p=1}^{N}\beta_p/(1+\gamma_p z). \qquad (64)$$

Evaluate 64 at zero to obtain $f_{j+1}=\sum_{p=1}^{N}\beta_p$ so that 63 becomes

$$|\sum_{p=0}^{j} f_p(-z)^p| + |z|^{1+j}f_{j+1}. \qquad (65)$$

If $R(z)<0$ the absolute value of the denominators in the preceeding are larger than $|\gamma_p \text{Im}(z)|$ since if $z=a+bi$ then

$$|1+\gamma_p z| = |1+\gamma_p a+\gamma_p bi| = ((1+\gamma_p a)^2+(\gamma_p b)^2)^{1/2} \geq ((\gamma_p b)^2)^{1/2}$$

which is $|\gamma_p \text{Im}(z)|$. Hence $[N,N+j] \leq |\sum_{p=0}^{j} f_p(-z)^p|+|z|^{1+j}/|\text{Im}(z)|$ plus $\sum_{p=1}^{N}\beta_p/\gamma_p$. Yet for the coefficient of the largest power of z, that is, the coefficient of z^{N+j}, from expression 3 chapter 2 is

$(D(j,N)/D(1+j,N-1)) > 0$ since all $D(N,M) > 0$. Now

$$[N,N+j] = \sum_{p=0}^{j} f_p(-z)^p + (-z)^j \sum_{p=1}^{N} \beta_p(-z)/(1+\gamma_p z)$$

$$= \sum_{p=0}^{j} f_p(-z)^p + (-z)^j \sum_{p=1}^{N} (-\beta_p/\gamma_p + (\beta_p/\gamma_p)/(1+\gamma_p z))$$

$$= \sum_{p=0}^{j-1} f_p(-z)^p + (-z)^j (f_j - \sum_{p=1}^{N} \beta_p/\gamma_p) + \sum_{p=1}^{N} (-z)\ (\beta_p/\gamma_p)/(1+\gamma_p z)$$

so that the coefficient of $(-z)^j$ which is the highest power of z in $[N,N+j]$ is $f_j - \sum_{p=1}^{N} \beta_p/\gamma_p = D(j,N)/D(1+j,N-1) > 0$. Therefore if $R(z) < 0$

$$|[N,N+j]| \leq \sum_{p=0}^{j} f_p(-z)^p + f_j|z|^{1+j}/|\text{Im}(z)|. \tag{66}$$

If $j = -1, [N,N-1] = \sum_{p=1}^{N} \beta_p/(1+\gamma_p z) < f_0/\text{Im}(z)$. And if $\text{Im}(z) \geq \delta$, when $R(z) < 0$, then $|[N,N+j]| \leq |\sum_{p=0}^{j} f_p(-z)^p| + f_j|z|^{1+j}/\delta$ and if $R(z) \geq 0$ then $|[N,N+j]| \leq |\sum_{p=0}^{j} f_p(-z)^p| + |z|^{1+j} f_{1+j}$ so that in $A - \{z: \text{Im}(z) \leq \delta$ and $R(z) < 0\}$ where A is compact the $\{[N,N+j]\}$ is uniformly bounded independent of N. Consider $0 < x_0 < \infty$ so that in $|z-x_0| \leq x_0$ every $[N,N+j]$ approximant is uniformly bounded by say W. As was shown in Theorem 5 chapter 2 the $[N,N+j]$, since they are uniformly bounded rational functions which must thus be analytic and admit to a Taylor series expansion in $|z-x_0| \leq x$, can be made to satisfy the following inequality uniformly in N: $|[N,N+j] - \sum_{n=0}^{k} a_n(z)^n| < \frac{\varepsilon}{6} > 0$. In this inequality $\sum_{n=0}^{k} a_n(z)^n$ is the first k terms of the power series expansion of $[N,N+j]$. In using Taylors theorem with remainder one has $g(x) = [N,N+j](x) = [N,N+j](x_0) + \sum_{k=1}^{n} [N,N+j]^{(k)}(x_0)(x-x_0)^k/k!$ plus $[N,N+j]^{(n+1)}(\xi)(x-x_0)^{n+1}/(n+1)!$ where ξ is between x and x_0.

Hence $[N,N+j](x_0) = g(x_0)$. Now if t is allowed to vary, then

$$g(x_0+t) = g(x_0)+tg^{(1)}(x_0)+g^{(2)}(x_0+\theta t)t^2/2 \text{ where}$$

$0 < \theta < 1$. Take the first difference of both sides, see pages 2 and 58 in (9) and Theorem 4 page 8 in (12), to obtain

$$g(x_0+h+t)-g(x_0+t) = g^{(1)}(x_0)+[g^{(2)}(x_0+\theta(t+h)(t+h)^2-g^{(2)}$$

of $\quad (x_0+\theta t)t^2]/2. \qquad (67)$

Notice that $\nabla t g^{(1)}(x_0) = g^{(1)}(x_0)\nabla(t) = g^{(1)}(x_0)$ and that the brackets follow from differencing $g^2(x_0+\theta t)t^2/2$. Now let $t = 0$ and 67 becomes

$$g(x_0+h)-g(x_0) = g^{(1)}(x_0)+g^{(2)}(x_0+\theta h)h/2 = g^{(1)}(x_0)+g^{(2)}(x_0+\theta h)h/2.$$

In general $\qquad [N,N+j](x_0+t) = g(x_0+t)$

$$= g(x_0)+tg^{(1)}(x_0)+(t^2/2)g^{(2)}(x_0)+ \dots+(t^n/n!)g^{(n)}(x_0)+(t^{n+1}/(n+1)!) \text{ times } g^{n+1}(x_0+\theta t) \qquad (68)$$

where $0 < \theta < 1$. Now take the nth difference of both sides where the difference is h and t is thought of as a variable to get:

$$\nabla^n g(x_0+t) = \nabla^n(t^n/n!)g^{(n)}(x_0)+\nabla^n(t^{n+1})/(n+1)!)g^{(n+1)}(x_0+\theta t) \quad (69)$$

since $\nabla^n cx^k = c\nabla^n x^k = c.0 = 0$ if $k < n$. Also $\nabla^n t^n = n!$ so that 69 becomes

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}g(x_0+t+(n-r)h)$$

$$= g^{(n)}(x_0)+\sum_{r=0}^{n}(-1)^r\binom{n}{r}g^{n+1)}(x_0+\theta(t+(n-r)h)(n-r)/(n+1)!)h)^{n+1}.$$

Let $t = 0$ to obtain

$$g^n(x_0) = \sum_{r=0}^{n}(-1)^r\binom{n}{r}g(x_0+(n-r)h)-(h^{n+1}/(n+1)!)\sum_{r=0}^{n}(-1)^r\binom{n}{r} \text{ times } g^{(n+1)}(x_0+\theta(n-r)h)(n-r)^{n+1}$$

or $\quad |g^n(x_0) - \sum_{r=0}^{n} (-1)^r \binom{n}{r} g(x\ t(n-r)h)| \leq (h^{n+1}/(n+1)!) \sum_{r=0}^{n} \binom{n}{r}$

$$\text{times } |g^{(n+1)}(x_0+\theta(n-r)h)|\,|n-r|^{n+1}. \qquad (70)$$

By Cauchy's inequality $g^{(n+1)}(x_0+\theta(n-r)h) \leq W(n+1)!/x_0^{\,n+1}$ the proof of which can be found on page 187 Theorem 5,15.1 in (18). Write 70 in the following way

$$g^n(x_0) - \sum_{r=0}^{n} (-1)^r \binom{n}{r} g(x_0+(n-r)h) \leq (W(n+1)!h^{n+1}/(n+1)!x_0^{\,n+1}) \text{ times } \sum_{r=0}^{n} \binom{n}{r} |n-r|^{n+1}$$

$$\leq (Wh^{n+1}/x^{\ n+1}) n^{n+1} \sum_{r=0}^{n} \binom{n}{r}$$

$$\leq (Wh^{n+1}/x_0^{\,n+1}) n^{n+1} 2^n. \qquad (71)$$

Hence $\quad (|(z-x_0)|^n/n!) | \sum_{n=0}^{m(\epsilon)} g^{(n)}(x_0) - \sum_{n=0}^{m(\epsilon)} \sum_{r=0}^{m} (-1)^r \binom{n}{r} g(x_0+(n-r)h)|$

$$\leq W/x_0 \sum_{n=0}^{m(\epsilon)} h^{n+1} 2^n n^{n+1} |z-x_0|^n / x_0^n (1/n!)$$

$$|\sum_{n=0}^{M(\epsilon)} g^{(n)}(x_0) |(z-x_0)^n/n!) - \sum_{n=0}^{M(\epsilon)} \sum_{r=0}^{n} (-1)^r \binom{n}{r} g(x_0+(n-r)h \text{ times } (z-x_0)^n/n!|$$

$$\leq W2^{m(\epsilon)} m(\epsilon)^{m(\epsilon)+1} h \sum_{r=0}^{n} |z-x_0|^n / x_0^{\,n}$$

$$\leq W2^{m(\epsilon)} m(\epsilon)^{m(\epsilon)+1} h/(x_0 - |z-x_0|) \qquad (72)$$

where it has been assumed that $h < 1$ so that $h^{m(\epsilon)+1} < h$ and $|z-x_0| < x_0$ which gives $\sum_{r=0}^{n} |z-x_0|^n/x_0 < \sum_{r=0}^{\infty} |z-x_0|^n/x_0^{\,n} = 1/(1-|z-x_0|/x_0)$ equals $x_0/(x_0-|z-x_0|)$. Now as was seen in the proof of Theorem 5 chapter 2 there exists an $m(\epsilon)$ such that $|[N,N+j](z) - \sum_{n=0}^{m(\epsilon)} g^{(n)}(x_0)(z-x_0)^n/n!|$ is less than $\epsilon/6$ uniformly in N. And

$$|[N,N+j](z) - \sum_{n=0}^{m(\epsilon)} g^{(n)}(x_0)(z-x_0)^n/n! + \sum_{n=0}^{m(\epsilon)} g^{(n)}(x_0)(z-x_0)^n/n! -$$

$$\sum_{n=0}^{m(\epsilon)} \sum_{r=0}^{m} (-1)^r \binom{n}{r} g(x_0+(n-r)h(z-x_0)^n/n!|$$

$$\leq \epsilon/6+\epsilon/6 = \epsilon/3$$

where the second $\epsilon/6$ is gotten by, after having first found $m(\epsilon)$, solving for h in 81 which is $W2^{m(\epsilon)}m(\epsilon)^{m(\epsilon)}h/(x_0-|z-x_0|) < \epsilon/6$.

The $[N,N+j](z) = g(z)$ converge uniformly by Theorem 5 chapter 2 to an analytic function. If $N(\epsilon)$ is picked sufficiently large so that for $t = 0, \ldots, m(\epsilon)$ $|[N_1,N_1+j](x_0+th)-[N_2,N_2+j](x_0+th)|$

$$\leq (\epsilon/3m(\epsilon)2^{m(\epsilon)})2|r|-1/(2|r|^{n+1}/(n+1)-|r|^{n+1}/(n+1))$$

then $$2\epsilon/3 > [N_1,N_1+j](z)-[N_2,N_2+j](z) -$$

$$\sum_{n=0}^{m(\epsilon)} \sum_{n=0}^{m} (g_1^{(n)}(x_0+(n-r)h))-(g_2^{(n)}(x_0+(n-r)h)/(n!)(z-x_0)^n(-1)^r\binom{n}{r}))|$$

$$\geq |[N_1,N_1+j](z)-[N_2,N_2+j](z)| -$$

$$|\sum_{n=0}^{m(\epsilon)} \sum_{r=0}^{m} (-1)^r\binom{n}{r}(z-x_0)^n/n!)(\epsilon/2^{m(\epsilon)}3m(\epsilon))(2|r|/(n+1)-1/(n+1)/$$

$$(2|r|^{n+1}/(n+1)!)-|r|^{n+1}/(n+1)!)|$$

$$= |[N_1,N_1+j](z)-[N_2,N_2+j](z)|-(\epsilon/2^{m(\epsilon)}3m(\epsilon)) \text{ times}$$

$$|\sum_{n=0}^{m(\epsilon)} \sum_{r=0}^{n} (-1)^r\binom{n}{r}(r)^n/n!(2|r|/(n+1)-r/(n+1))/(2|r|^{n+1}-|r|^{n+1})/(n+1)!)|$$

$$|[N_1,N_1+j](z)-[N_2,N_2+j](z)|-(\epsilon/3m(\epsilon)|\sum_{n=0}^{m(\epsilon)} \sum_{r=0}^{n} (-1)^r\binom{n}{r}| \leq (\epsilon/3m(\epsilon)2^{m(\epsilon)}) \text{ times } \sum_{n=0}^{m(\epsilon)} 2^n)$$

$$|[N_1,N_1+j](z)-[N_2,N_2+j](z)| \leq \epsilon \text{ for } N_1,N_2 > N(\epsilon), |z-x_0| \leq r < x_0. \tag{73}$$

Inequality 73 is the Cauchy condition for convergence. Hence the sequence of $[N,N+j]$ Padé approximants converge in $|z-x_0| < x_0$ to a regular function of z. Now proceed as in Theorem 6 chapter 2 in extending the convergence to any point in the cut plane which proves the first statement of Theorem 4.

If $2N+j \geq p$ then $f_p = |[N,N+j]^p/p!|$. Consider $p > 2N+j$ and expand the right side of 37. One can write the right side as

$$k(-z)^{2N+j+1}\prod_{k=1}^{2N+1}(1+\alpha_k z)^{-1} = k\sum_{\gamma=2N+1+j}^{\infty}\left[\sum_{\gamma_1+\ldots+\gamma_{2N+j}=\gamma-2N-j-1}\prod_{k=1}^{2N+1}\alpha_k^{\gamma_k}(-z)^k\right] \quad (74)$$

where k and α_k are positive. From 37 one also has

$$f_p-(-1)^p[N,N+j]^{(p)}/p! = (-1)^p\{[N+1,N+1+j]^{(p)}/p!-[N,N+j]^{(p)}/p!\}+\ldots+ (-1)^p\{(-1)^p f_p-[N_1,N_1+j]^{(p)}/p!\} > 0 \quad (75)$$

where N_1 is sufficiently large to make $(-1)^p f_p = [N_1,N_1+j]^{(p)}/p!$.

Thus 75 is greater than zero because as is seen in 74 the coefficient of z^p has sign $(-1)^p$ for any $(P^j_{N+1}(z)/Q^j_{N+1}(z))-(P^j_N(z)/Q^j_N(z))$ (76)

so that for 75, since the successive coefficients in 75 are nothing more than the coefficients of a particular 76 and are multiplied by $(-1)^p$, are each $(-1)\{([N+i,N+i+j]^{(p)}/p!)-([N+i-1,N+i-1+j]^{(p)})/p!\} \geq 0$. In 62 first write it as, $\sum_{p=0}^{j} f_p(-z)^{(p)}+$ polynomial of degree p+N times polynomial of less degree than p+N times a constant remainder, where the division algorithm was used. See page 105 in (5). Differentiate p times, divide by p! and multiply by $(-1)^p$ which verifies that $(-1)^p[N,N+j]^{(p)}/p! > 0$. Now using 74, 75 and the last statement one gets $f_p \geq |[N,N+j]^{(p)}/p!|$.

Therefore the power series for $[N,N+j]$ is dominated by the one for $f(z)$ and the former must converge whenever the latter one does. Thus the v_p of 62 are such that $0 < \gamma_p < R^{-1}$. From 63 and the immediately following it was seen that

$$|[N,N+j]| \leq |\sum_{p=0}^{j} f_p(-z)^p| + |z|^{1+j} f_{1+j}, \; R(z) \geq 0,$$

$$|[N,N+j]| \leq |\sum_{p=0}^{j} f_p(-z)^p| + f_j|(z)|^{1+j}/|Im(z)|, \; R(z) \leq 0.$$

For all z with $R(x) \geq 0$ and for all z with $R(x) < 0$ such that $Im(y) \geq \delta$,

$$|[N,N+j](z)| \leq |\sum_{p=0}^{j} f_p(-z)^p| + R^{1+j}(f_{1+j} + f_j/\delta).$$

Hence $\{[N,N+j]\}$ is uniformly bounded in the closed region $C = \{z\colon Im(z) < \delta \text{ whenever } R(z) < 0\} \cup C_R$ where C_R is a closed disk with radius R. Now if $|z| \leq r < R$ and $r+d = R$ then, if $z = x+iy$,

$$-r < x < r$$

$$-\gamma_p r < \gamma_p x < \gamma_p r$$

and since

$$\gamma_p < 1/R$$

$$r\gamma_p < r/R$$

$$-r\gamma_p > -r/R$$

$$0 < 1-r/R < 1-r\gamma_p < 1+\gamma_p x$$

or

$$|1-r/R|^2 = (1-r/R)^2 < (1+\gamma_p x)^2 + \gamma_p{}^2 y^2 = |1+\gamma_p z|^2$$

and

$$1/|1+\gamma_p z| < 1/|1-r/R|$$

Hence for all $|z| \leq r$

$$|[N,N+j](z)| \leq |\sum_{p=0}^{j} f_p(-z)^p| + |z|^{1+j} \sum_{p=1}^{N} (\beta_p/|1+\gamma_p z|)$$

$$\leq \sum_{p=0}^{j} f_p(-z)^p + (z)^{1+j} \sum_{p=1}^{N} (\beta_p/|1-r/R|)$$

$$= \sum_{p=0}^{j} f_p(-z)^p + |z|^{1+j}(f_{1+j}/|1-r/R|).$$

Hence by Theorem 6 chapter 3 the last statement of this theorem follows. Now a look at the converse of statement 4 will be taken.

Theorem 5. Conversely, if $D(0,n)$, $D(1,n)$ are both greater than zero then there exists a bounded nondecreasing function ϖ taking on infinitely many values in $[0, \infty]$ such that $f_n = \int_0^\infty t^n \, d\varpi(n)$, $n = 0, 1, 2, \ldots$.

Proof. The proof will follow after six lemmas and five definitions. A good reference is (22). The moment $M[P(t)]$ of a polynomial

$$P(t) = \sum_{k=0}^{n} a_k t^k$$

with respect to a sequence $\{f_n\}$ is

$$M[P(t)] = \sum_{k=0}^{n} a_k f_k.$$

The sequence is said to be positive if the moment of every nonnegative polynomial is nonnegative.

The sequence $\{f_k\}$ is positive definite if the moment of every nonnegative polyndmial which is not identically zero is greater than zero.

Lemma 1. Every real nonnegative polynomial is the sum of the squares of two real polynomials.

Proof. Since the polynomial is real its complex roots appear in conjugate pairs. And since the polynomial is positive the real roots are of multiplicity 2 which is seen by taking the first and second derivatives of a factor of the second degree. Hence its factored form is

$$P(x) = \prod_{i=j}^{n} [(x-a_i) + b_i^2] \prod_{i=0}^{m} (x-c_i)^2 \tag{77}$$

where a_i, b_i, c_i are real numbers. Since

$$|x_1 + iy_1|^2 \, |x_2 + iy_2|^2 = |(x_1 + iy_1)(x_2 + iy_2)|^2$$

or $(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1 x_2 - y_1 y_2)^2 + (y_2 x_1 + y_1 x_2)^2$

one has $[(x - a_i)^2 + b_i^2][(x - a_j)^2 + b_j^2] = [(x - a_i)(x - a_j) - b_i b_j]^2$

$$+ [b_j(x - a_i) + b_i(x - a_j)]^2$$

Let $\{\xi_m\}_1^\infty$ be the set of all rational numbers arranged in some order. Note that the set is dense on the interval $(-\infty, \infty)$. Let

$$h_m(t) = 1 \text{ if } t \leq \zeta_m$$

and

$$= 0 \text{ if } t > \zeta_m.$$

The set of functions E is the set of all linear combinations over the real field of a finite number of the functions

$$\ldots, t^2, t, 1, h_1(t), h_2(t), \ldots.$$

or that the product or the sum of two squares by the sum of two squares is the sum of two squares. Repeated application of this result used in 77 validates the lemma.

Lemma 2. A necessary and sufficient condition that the sequence $\{f_p\}$ should be positive definite in that the quadratic forms

$$\sum_{i=0}^{n} \sum_{j=0}^{n} f_{i+j} \zeta_i \zeta_j \qquad (n = 0, 1, 2, \ldots) \tag{78}$$

should be positive definite.

Proof. Suppose that the quadratic forms in 78 are all positive definite. That is, no form vanishes unless all the variables vanish. Let $P(t)$ be an arbitrary nonnegative polynomial. By the preceeding lemma $P(t)$ is a sum of the squares of two polynomials $P_1(t)$ and $P_2(t)$:

$$P_1(t) = \sum_{i=0}^{n} a_i t^i, \; P_2(t) = \sum_{i=0}^{m} b_i t^i$$

in which the a_i and b_i are not all zero. Hence

$$M[P(t)] = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j f_{i+j} + \sum_{i=0}^{m} \sum_{j=0}^{m} b_i b_j f_{i+j}$$

which is greater than zero by hypothesis.

Now suppose that $\{f_n\}$ is a positive definite sequence. Let n be an arbitrary positive integer and $\zeta_0, \zeta_1, \ldots, \zeta_n$ be arbitrary constants not all zero. It needs to be shown that $\sum_{i=0}^{n} \sum_{j=0}^{n} f_{i+j} \zeta_i \zeta_j > 0$. This follows from the definition of a positive definite sequence since the polynomial $[\sum_{i=0}^{n} \zeta_i t^i]^2$ is nonnegative and not identically zero and

$$0 < M[\sum_{i=0}^{n} \zeta_i t^i]^2 = \sum_{i=0}^{n} \sum_{j=0}^{n} \zeta_i \zeta_j f_{i+j}.$$

Lemma 3. If $\{u_n\}_0^\infty$ is a positive sequence, there exists an operator M which is applicable to the class of functions E, is positive and distributive, and reduces to the moment of a polynomial when applied to a polynomial.

Proof. Define the set of functions E_1 as the set of all linear combinations over the real field of a finite number of the functions

$$h_1(t), 1, t, t^2, \ldots.$$

Define

$$\underline{h}_1 = \sup_{p(t)<h_1(t)} M[p(t)]$$

and

$$\overline{h}_1 = \inf_{h_1(t)<P(t)} M[P(t)].$$

Now

$$\underline{h}_1 < \overline{h}_1$$

because if

$$p(t) < h_1(t) < P(t)$$

then

$$M[p(t)] \leq M[P(t)]$$

since M is a positive distributive operator when applied to polynomials. Polynomials $p(t)$ and $P(t)$ exist less than and greater than $h_1(t)$ respectively which implies that $\underline{h}_1$ and $\overline{h}_1$ are finite numbers and that $\underline{h}_1 \leq \overline{h}_1$. Let

$$M[h_1(t)] = h_1 = (\underline{h}_1 + \overline{h}_1)/2$$

and

$$M[P(t) + c_1\ h_1(t)] = M[P(t)] + c_1\ h_1$$

for any polynomial $P(t)$ and any real constant c. Hence M remains a distributive operator when applied to E_1. To see that it also remains positive, suppose that $p(t)$ is a polynomial and c_1 a constant such that $p(t) + c_1\ h(t) \geq 0$. If $c_1 > 0$, then $-p(t)/c_1 \leq h_1(t)$. By definition of $\underline{h}_1$ and h_1

$$-M[p(t)]/c_1 = M[(-p(t))/c_1] \leq \underline{h}_1 \leq h_1$$

or

$$M[p(t) + c_1\ h_1(t)] \geq 0.$$

A similar proof holds using the definition of $\overline{h}_1$ when $c_1 < 0$.

Now define E_2 as the set of all functions which are linear combinations of a finite number of functions in $E_1 \cup h_2(t)$.

Define $$\underline{h}_2 = \sup_{f_1(t) < h_2(t)} \{M[f_1(t)]\}$$

and

$$\overline{h}_2 = \inf_{h_2(t) < F_1(t)} \{M[F_1(t)]\}$$

where $f_1(t)$ and $F_1(t)$ are functions of E_1 satisfying the indicated inequalities.

Also let

$$M[h_2(t)] = h_2 = (\underline{h}_2 + \overline{h}_2)/2$$

and

$$M[F_1(t) + c_2\ h_2(t)] = M[F_1(t)] + c_2\ h_2$$

for any function $F_1(t)$ of E_1 and any real constant c_2. As before it can be shown that M remains positive and distributive. Continuing validates the theorem.

Lemma 4. A necessary and sufficient condition that there should exist one nondecreasing function $\alpha(t)$ such that

$$u_n = \int_{\infty}^{\infty} t^n \, d\alpha(t) \qquad (n = 0, 1, 2, \ldots) \tag{79}$$

All of the integrals converging, is that the sequence $\{u_n\}$ should be positive.

Proof. First suppose that $\alpha(t)$ is a nondecreasing solution of equations 79 and that P(t) is an arbitrary nonnegative polynomial,

$$P(t) = \sum_{k=0}^{n} a_k \, k^k.$$

Then

$$M[P(t) = \sum_{k=0}^{n} a_k \, u_k = \int_{-\infty}^{\infty} P(t) \, d\,\alpha(t) \geq 0$$

so that the sequence $\{u_n\}$ is positive.

For the sufficiency suppose that the sequence $\{u_n\}$ is positive. A nondecreasing solution of $\alpha(t)$ in 79 will be exhibited. It is defined at the regional points by the equations $\alpha(\zeta_m) = M[h_m(t)]$, $m = 0, 1, 2, \cdots$. If $\zeta_i < \zeta_j$ then from the positive character of M, $h_i(t) \leq h_j(t)$ and $\alpha(\zeta_i) \leq \alpha(\zeta_j)$. Hence $\alpha(t)$ is nondecreasing in as far as it has been defined. To complete the definition define for any irrational number ν.

$$\overline{\varphi} = \inf_{\zeta_m > \nu} \alpha(\zeta_m),$$

$$\underline{\varphi} = \sup_{\zeta_m < \nu} \alpha(\zeta_m),$$

$$\alpha(\nu) = (\overline{m} + \underline{m})/2.$$

Since $\alpha(t)$ is nondecreasing on the rational points it is true that $\underline{m} \leq \overline{m}$ and that $\alpha(t)$ is now completely defined and is nondecreasing.

Now let n be any positive integer. It must be shown that

$$u_n = \int_{\infty}^{\infty} t^n \, d\alpha(t).$$

Since, $\alpha(t)$ is nondecreasing it will be sufficient to show that for a given $\epsilon > 0$ there exists a $T_0 > 0$ such that for every pair of rational numbers T_1 and T_2 greater than T_0

$$\left|\int_{-T_1}^{T_2} \tau^n d\alpha(\tau) - u_n\right| < \epsilon.$$

Let m be an integer such that $2m > n$. Now $t^{2m} > 0$ implies that $u_{2m} > 0$. Assume that $u_0 > 0$ for if $u_0 = 0$ allof the u_n are zero and $\alpha(t) = 0$ is a solution to the problem. To see that $u_1 = 0$ if $u_0 = 0$ consider $(x + c)^2 \geq 0$, which is true for all c. Now

$$M(x^2 + 2xc + c^2) = u_2 + 2\, u_1\, c \geq 0$$

and is true only if $u_1 = 0$. Again and since $(x^2 + c)^2 \geq 0$ one gets $2\, c\, u_2 + u_4 \geq 0$ and $u_2 = 0$. If one considers successively the polynomials $(x^2 + c^x)^2$, $(x^3 + c^x)$, $(x^3 + cx^2)^2$, $(x^4 + cx^2)^2, \cdots$ then one notices that all the $u_n = 0$. Let

$$\epsilon' = \epsilon/2(u_0 + u_{2m}),$$

which can be well done by the comments above, and determine T_0 such that

$$|t^n| < \epsilon' \, t^{2m}, \quad |t| > T_0. \tag{80}$$

Let T_1 and T_2 be any two rational numbers greater than T_0. Divide the interval $(-T_1, T_2)$ into p subintervals by choosing rational points $t_i = \zeta_{k_i}$, that is, $t_0 = -T_1 < t_1 < t_2 < \cdots < t_p = T_2$ where p is large enough and the subintervals sufficiently small so that for t^n

$$\sup \{x^n - y^n : x, y \in I, \text{ an interval}\} < \epsilon'$$

and
$$\left| \sum_{i=0}^{p-1} t_{i+1}^n [\alpha(t_{i+1}) - \alpha(t_i)] - \int_{-T_1}^{T_2} t^n \, d\alpha(t) \right| < \epsilon/2$$

which is possible by the uniform continuity of t^n and by the definition of the Stieltjes integral.

Next let

$$V(t) = 0 \text{ if } t \leq -T_1,\ t > T_2$$

and

$$V(t) = t_{i+1}^n \text{ if } t_i < t < t_{i+1},\ i = 0, 1, \cdots, p - 1.$$

Then

$$V(t) = \sum_{i=0}^{p-1} t_{i+1}^n [h_{k_{i+1}}(t) - h_{k_i}(t)]$$

and

$$|V(t) - t^n| \leq \epsilon' \ t^{2n} \text{ by } 4.$$

Also

$$|V(t) - t^n| \leq \sum_{i=0}^{p-1} t_{i+1}^n [h_{k_{i+1}}(t) - h_{k_i}(t)] - t_n$$

$$= t_{i+1}^n \ h_{k+1}(t) - t_n$$

$$< \epsilon'$$

if $-T_1 < t \leq T_2$ so that $|V(t) - t^n| \leq \epsilon' + c' \ t^{2m}$ on $-\infty < t < \infty$.

$V(t)$ belongs to some E_j since sooner or later one gets to $h_k(t_i)$ for every t_i in the partition and a linear combination of these will vanish and be otherwise in the appropriate intervals. Hence the operator M is positive and distributive when applied to $V(t)$.

Or
$$|M[V(t)] - u_n| \leq \epsilon'(u_0 + u_{2m}) = \epsilon/2.$$

But
$$M[V(t)] = \sum_{i=0}^{p-1} t_{i+1}^n [\alpha(t_{i+1}) - \alpha(t_i)]$$

and from the above

$$\left|\int_{-T_1}^{T_2} d\alpha(t) - u_n\right| = \left|\int_{-T_1}^{T_2} t^n \, d\alpha(t) - u_n + M[V(t)] - M[V(t)]\right|$$

$$\leq \left| M[V(t)] - \int_{-T_1}^{T_2} t^n \, d\alpha(t)\right| + \left| M[V(t)] - u_n\right.$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon$$

Now all but the boundedness of the proof of the theorem will follow: The equivalence of the two forms

$$\{f_n\}_0^\infty, \ \{f_n\}_1^\infty \text{ and } \sum_{i=0}^{n}\sum_{j=0}^{n} f_{i+j} \ \zeta_i \ \zeta_j, \qquad \sum_{x=0}^{n}\sum_{y=0}^{n} f_{i+j+1} \ \zeta_i \ \zeta_j$$

is given by Lemma 2. In proving the theorem the quadratic forms will be used. Consider the quadratic forms positive and the new sequence $\{V\}_0^\infty$ where $V_{2n} = f_n$, $n = 0, 1, \cdots$, and $V_{2n+1} = 0$, $n = 0, 1, \cdots$.

If n is odd

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \gamma_{i+j} \ \zeta_i \ \zeta_j = \sum_{i=0}^{n-1/2}\sum_{j=0}^{n-1/2} f_{i+j} \ \zeta_{2i} \ \zeta_{2j}$$

$$+ \sum_{i=0}^{n-1/2}\sum_{j=0}^{n-1/2} f_{i+j+1} \ \zeta_{2i+1} \ \zeta_{2j+1}$$

and if n is even

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \gamma_{i+j} \ \zeta_i \ \zeta_j = \sum_{i=0}^{n/2}\sum_{j=0}^{n/2} f_{i+j} \ \zeta_{2i} \ \zeta_{2j} + \sum_{i=0}^{n/2-1}\sum_{j=0}^{n/2-1}$$

$$\text{times } f_{i+j+1} \ \zeta_{2i+1} \ \zeta_{2j+1}.$$

Hence $\{\gamma\}_0^\infty$ is positive and by Lemma 4 there exists a nondecreasing function $\beta(t)$ such that

$$\gamma_n = \int_{\infty}^{\infty} t^n \, d\beta(t) \qquad n = 0, 1, 2, \cdots,$$

$$u_n = \int_{\infty}^{\infty} t^{2n} \, d\beta(t) \qquad n = 0, 1, 2, \cdots, \tag{81}$$

and

$$0 = \int_{\infty}^{\infty} t^{2n+1} \, d\beta \ (\tau) \quad n = 0, 1, 2, \cdots.$$

Set

$$\gamma(t) = (\beta(t) - \beta(-t))/2, \ -\infty < t < \infty.$$

This function is odd and satisfies the equations in 81. Also $\gamma(t)$ is nondecreasing. That $\gamma(t)$ is odd and nondecreasing follows by looking at it in conjunction with what is known about $\beta(t)$. To see that it satisfies equations 81 proceed as

$$\int_0^{T_1} t^{2n}\, d\gamma(t) \doteq \sum_{i=0}^{k} \zeta_i^{2n}(((\beta(t_{i+1}) - \beta(t_{i+1}))/2 - (\beta(t_i) - \beta(-t_i))/2)$$

$$\int_{-T_1}^{0} t^{2n}\, d\gamma(t) \doteq \sum_{i=0}^{k} \zeta_i^{2n}(((\beta(\tau_i) - \beta(t_i))/2 - (\beta(-\tau_{i+1}) - \beta(\tau_{i+1}))/2)$$

or

$$\int_{-T_1}^{T_1} t^{2n}\, d\gamma(t) = \int_0^{T_1} t^{2n}\, d\gamma(t) + \int_{-T_1}^{0} t^{2n}\, d\gamma(t)$$

$$= \sum_{i=0}^{k} \zeta_i^{2n}\, \beta(t_{i+1}) - \beta(t_i) + \sum_{i=0}^{k} \zeta_i^{2n}[\beta(-t_i) - \beta(-t_{i+1})]$$

$$\doteq \int_0^{T_1} t^{2n}\, d\beta(t) + \int_{-T_1}^{0} t^{2n}\, d\beta(t)$$

$$= \int_{-T_1}^{T_1} t^{2n}\, d\beta(t).$$

Let $\alpha(\tau) = 2\ \gamma(t^{\frac{1}{2}})$ for $t > 0$. Then

$$u_n = \int_{-\infty}^{\infty} t \quad d\gamma(\tau) = \int_0^{\infty} t^{2n}\, d\gamma(t) + \int_{-\infty}^{\infty} t^{2n}\, d\gamma(t) \tag{82}$$

or upon letting $u = -t$ 82 becomes

$$\int_0^{\infty} t^{2n}\, d\gamma(t) + \int_{\infty}^{\infty} u^{2n}\, d\gamma(-u) = \int_0^{\infty} t^{2n}\, d\gamma(t) - \int_0^{\infty} t^{2n}\, d\gamma(-t). \tag{83}$$

But since γ is odd 83 becomes

$$\int_0^{\infty} t^{2n}\, d\gamma(\tau) + \int_0^{\infty} t^{2n}\, d\gamma(t) = 2\int_0^{\infty} t^{2n}\, d\gamma(t) = \int_0^{\infty} t^n d[2\gamma(t^{\frac{1}{2}})]$$

$$= \int_0^{\infty} t^n\, d\alpha(t).$$

Since $\alpha(t)$ is nondecreasing in $[0, \infty)$ everything except the boundedness of $\alpha(t)$ has been proved. That is, to now it has been shown that if $D(0,M)$, $D(1,M) > 0$ then there is a nondecreasing function ϖ taking on infinitely many values in $[0,\infty]$ such that $f_n = \int_0^\infty t^n \, d\varpi(n)$, $n = 0, 1, \cdots$.

It remains to be shown that ϖ can be made a bounded function.

Lemma 5. A necessary and sufficient condition that a non-singular algebraic quadratic form (a_{ij}), $i, j = 0, 1, \cdots, n-1$ be positive definite is that

$$\delta_1 = a_{00},\ \delta_2 = \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix},\ \cdots,\ \delta_n = |a_{ij}|$$

for $i,j = 0, \cdots, n-1$ be all positive.

Proof. This is given in (6). First consider the form

$$y = \sum_0^2 a_{ij}x_ix_j = a_{00}x_0^2+2(a_{01}x_1+a_{02}x_2)x_0+a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2 \tag{84}$$

which is the quadratic form in x_0. A necessary and sufficient condition that 84 be positive definite is for all real $(x_1,x_2) \neq (0,0)$

$$a_{00}(a_{11}x_1^2 + 2\, a_{12}x_1x_2 + a_{22}\, x_2^2) - (a_{01}x_1 + a_{02}x_2)^2 > 0 \tag{85}$$

and

$$a_{00} > 0. \tag{86}$$

Write 85 as a quadratic in x_1 and get

$$D_{11}\, x_1^2 + 2D_{12}x_1x_2 + D_{22}x_2^2 > 0 \tag{87}$$

where

$$D_{ij} = a_{00}\, a_{ij} - a_{i0}\, a_{0j} \qquad (i,j = 1,2).$$

A necessary and sufficient condition that 87 be satisfied is

$$D_{11} > 0$$

and

$$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} > 0.$$

But

$$D_{11} = \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix} = \delta_2.$$

and $$\begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = \begin{vmatrix} a_{00}\ a_{11} - a_{10}\ a_{01} & a_{00}\ a_{12} - a_{10}\ a_{02} \\ a_{00}\ a_{21} - a_{20}\ a_{01} & a_{00}\ a_{22} - a_{20}\ a_{02} \end{vmatrix}$$

$$= a_{00} \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{vmatrix} .$$

Thus conditions 86 and 85 become

$$a_{00} = \delta_1 > 0, \; D_{11} = \delta_2 > 0, \quad \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} > 0$$

so that $\delta_1 > 0$, $\delta_2 > 0$ and $\delta_3 > 0$.

Now proceed to the general case under the assumption

$$a_{00} = \delta_1 > 0, \; \delta_2 > 0, \; \cdots, \; \delta_n > 0$$

are necessary and sufficient for positive definiteness of a nonsingular quadratic form in n variables. Consider the (n + 1) ary form

$$y = \sum_0^n a_{ij}\, x_i x_j .$$

In order that

$$a_{00}x_0^2 + 2(a_{01}x_1 + \cdots + a_{0n}x_n)x_0 + a_{11}x_1^2 + 2\, a_{12}x_1x_2 + \cdots + a_{nn}x_n^2 > 0$$

it is necessary and sufficient that for all real $(x_1, \cdots, x_n) \neq (0, 0, \cdots 0)$

$$a_{00}(a_{11}x_1^2 + \cdots + a_{nn}x_n^2) - (a_{01}x_1 + \cdots + a_{0n}x_n)^2 > 0, \; a_{00} > 0. \qquad (88)$$

Write 88 as

$$D_{11}x_1^2 + 2\, D_{12}x_1x_2 + \cdots + D_{nn}\, x_n^2 \qquad (89)$$

with D_{ij} $(i, j = 1, \cdots, n)$ defined as before. Now 89 is a form of n variables $(x_1, \cdots, x_n)$. By hypothesis the condition that it is positive is $k_1 > 0, \cdots, k_n > 0$ where the k_i are principal minors of its determinant. But $k_1 = D_{11} > 0$, $k_2 = a_{00}\,\delta_3 > 0, \cdots,$ $k_n = a_{00}^{n-1}\,\delta_{n+1} > 0$. Whence conditions 88 and 89 may be written

$$\delta_1 = a_{00} > 0,\ \delta_2 > 0,\ \delta_3 > 0,\ \cdots,\ \delta_{n+1} > 0.$$

Lemma 6. The equations $u_n = \int_0^\infty t^n\,d\alpha(t)$ $n = 0, 1, \cdots$ always have a solution $\alpha(t)$ of bounded variation for which

$$\int_0^\infty |\,d\alpha(t)| < \infty.$$

Proof. Set up 2 sequences $\{\psi_n\}_0^\infty$ and $\{\gamma_n\}_0^\infty$ such that

$$u_n = \psi_n - \gamma_n, \tag{90}$$

$$\psi_n = \int_0^\infty t^n\,d\beta(t), \tag{91}$$

and

$$\gamma_n = \int_0^\infty t^n\,d\gamma(t) \tag{92}$$

where $\beta(t)$ and $\gamma(t)$ are bounded nondecreasing functions. First $\psi_0, \psi_1, \gamma_0, \gamma_1$ should be chosen so that they satisfy 90. Proceed by induction. Suppose ψ_k and γ_k have already been determined for $k = 0, 1, \cdots, 2n - 1$ in which 90 holds and the following determinants are positive:

$$[\psi_0, \psi_1, \cdots, \psi_{2k}] = \begin{vmatrix} \psi_0 & \psi_1 & \cdots & \psi_k \\ \vdots & & \ddots & \vdots \\ \psi_k & \psi_{k+1} & \cdots & \psi_{2k} \end{vmatrix} \tag{93}$$

$$[\psi_1, \psi_2, \cdots, \psi_{2k+1}] = \begin{vmatrix} \psi_1 & \cdots & \psi_{k+1} \\ \vdots & \ddots & \vdots \\ \psi_{k+1} & \cdots & \psi_{2k+1} \end{vmatrix},$$

and $[\gamma_0, \gamma_1, \cdots, \gamma_{2k}]$, $(\gamma_1, \gamma_2, \cdots, \gamma_{2k+1}]$.

Now $\psi_{2n}, \psi_{2n}, \psi_{2n+1}, \psi_{2n+1}$ will be defined. With undetermined

x_{2n} one has

$$[\psi_0, \cdots, \psi_{2n}] = \psi_{2n}[\psi_0, \cdots, \psi_{2n-2}] + P \tag{94}$$

where P is a polynomial in $\psi_0, \psi_1, \cdots, \psi_{2n-1}$ and similarly for $[\gamma_0, \gamma_1, \cdots, \gamma_{2n}]$. Since $[\psi_0, \psi_1, \cdots, \psi_{2n-2}]$ and $[\gamma_0, \gamma_1, \cdots, \gamma_{2n-2}]$ are both greater than zero one can choose ψ_{2n} and ψ_{2n} positive and large enough so that $\psi_{2n} - \psi_{2n} = u_{2n}$ and $[\psi_0, \cdots, \psi_{2n}] > 0$, $[\gamma_0, \cdots, \gamma_{2n}] > 0$.

It is observed that 94 holds when the subscripts are increased by unity, P then becoming a polynomial in $\psi_1, \psi_2, \cdots, \psi_{2n}$ with a similar equation holding for the γ_k. Now that ψ_{2n} and γ_{2n} have been determined proceed as before ψ_{2n+1} and γ_{2n+1} which completes the induction.

By Lemma 5, if the determinants in 93 are positive for k = 0,1, $\cdots$, equations 91 and 92 have bounded nondecreasing solutions $\beta(t)$ and $\gamma(t)$ respectively, so if $\alpha(t) = \beta(t) - \gamma(t)$ the proof is complete as this is a characterization of a function of bounded variation. See (1).

Hence there always exists a nondecreasing function $\alpha(t)$ as given in the theorem. That is, in light of the last 2 lemmas, $\alpha(t)$ can be made bounded.

Not only in this chapter has it been shown what has been proved relative to series of Stieltjes but a solution to the Stieltjes moment problem has been given.

CHAPTER IV

SUMMARY

In studying the theorems presented one might ask if there is a relation between the given divergent series of Stieltjes, call it $f(z)$, and the $f_1(z)$ given in Theorem 3 of Chapter III. For instance the series $1 - 1 ! z + 2 ! z^2 - 3 ! z^3 + \cdots$ is a series of Stieltjes that diverges. In this series $\varphi(t) = \int_0^t e^{-t} dt$ and $f_n = \int_0^\infty t^n c^{-t} dt = n !$ It can be shown that

$$\lim_{z \to 0} \frac{\left| \int_0^\infty \frac{e^{-t} dt}{1+zt} - \sum_{k=0}^{n} (-1)^k k ! z^k \right|}{|z|^n} = 0$$

see (12) and (19) for a treatment of this, which is by definition of what is meant when it is said that

$$f(z) = \int_0^\infty \frac{e^{-t} dt}{1+zt} \quad \text{is asymptotic to} \sum_{k=0}^{\infty} (-1)^k k ! z^k.$$

Formally, it is said that $f(z)$ is asymptotic to $S_n(z)$ if for every n

$$\lim_{z \to 0} \frac{|f(z) - S(z)|}{|z|^n} = 0.$$

Thus one might ask the question, is $f_1(z)$ of Theorem 3 Chapter 3 equal to $\int_0^\infty \frac{e^{-t} dt}{1+zt}$ where the power series is given by $f(z) = \sum_{k=0}^{\infty} (-1)^k k ! z^k$. At $z = 0$, $f_1(z)$ thus defined, is 1. The [1,1] and [1,0] approximants are 1 when evaluated at 0 since $[1,0](0) = \frac{1}{1+1x}$ and

$$[1,1](0) = \frac{f_0 f_1 + f_1^{2}(x) - f_2 f_0 x}{f_1 - f_2\, x} = 1,$$

it being noticed that $f_0 = 1$, $f_1 = -1$, and $f_2 = 2!$. Hence by Theorem 3 in Chapter 3, since 1 is real, the [N, N+j] Padé approximants evaluated at z = 1 must all be 1. Baker in (3) shows that $\lim_{z\to\infty} [1,1](z) = \frac{1}{2}$ and $\lim_{z\to\infty} [5,5](z) = \frac{1}{6}$ and that the $\lim_{\substack{N\to\infty \\ z\to\infty}} [N,N](z) = 0$

where the Padé approximants are calculated with respect to $\sum_{k=0}^{\infty} (-1)^k k! z^k$. This can be compared to $\lim_{z\to\infty} f(z) = \int_0^\infty \frac{e^{-t}dt}{1+zt}$. He also notes in passing that the exact value of $\int_0^\infty \frac{e^{-t}}{1+t}\, dt$ is .5963 and that Lacroix's calculated, using three Euler transformations and 13 terms of the divergent series, this function evaluated at z = 1 to be 0.5992. The [6,6] Padé approximant, which incidentally is obtained by using the first 13 terms of the divergent series, when evaluated at 1 is 0.5968. These examples then might promote effort toward answering what $f_1(z)$ really is.

In Theorem 4 of Chapter III it was proved, for each fixed j, that the [N,N+j] Padé approximants relative to a series of Stieltjes, converge to an analytic function in the cut plane $(-\infty \leq z \leq 0)$. It would be nice to have a condition that is sufficient for all such [N,N+j] sequences to have a common limit. If it can be shown for a series of Stieltjes that

$$f_{p+1} D(1,p)/D(D,p+1) \geq 1 \tag{1}$$

then the divergence of $\sum_{p=1}^{\infty} (f_p)^{-1/2p+1)}$ is sufficient for all of the sequences of [N,N+j] Padé approximants, j fixed, to have the same limit function. Most of this is proved in (2). A counter example, due to Professor R. B. Deal at Oklahoma State University, will be given for 1. Now this does not disprove Baker's assertion, because, if for all but a finite number of the p in $\sum_{p=1}^{\infty} f_p(-z)^p$ 1 holds then the assertion can be verified.

Let

$$\varphi(x) = \begin{cases} 0 & \text{if } x < D \\ x^{\alpha} & \text{if } 0 \leq x \leq 1,\ \alpha > 0, \\ 1 & \text{if } 1 < x . \end{cases}$$

Then

$$f_n = \int_0^{\infty} t^n \, d\varphi(t)$$
$$= \int_0^1 t^n \, \alpha \, t^{\alpha-1} \, dt$$
$$= \frac{\alpha}{n+\alpha} .$$

Now

$$D(0,2) = \begin{vmatrix} 1 & \frac{\alpha}{(1+\alpha)} & \frac{\alpha}{(2+\alpha)} \\ \frac{\alpha}{(1+\alpha)} & \frac{\alpha}{(2+\alpha)} & \frac{\alpha}{(3+\alpha)} \\ \frac{\alpha}{(2+\alpha)} & \frac{\alpha}{(3+\alpha)} & \frac{\alpha}{(4+\alpha)} \end{vmatrix}$$

$$= \alpha^2 \begin{vmatrix} 1 & \frac{1}{(1+\alpha)} & \frac{1}{(2+\alpha)} \\ \frac{\alpha}{(1+\alpha)} & \frac{1}{(2+\alpha)} & \frac{1}{(3+\alpha)} \\ \frac{\alpha}{(2+\alpha)} & \frac{1}{(3+\alpha)} & \frac{1}{(4+\alpha)} \end{vmatrix}$$

$$= \alpha^2 \psi(\alpha)$$

and

$$f_2 \, D(1,1) = \frac{\alpha^3}{2+\alpha} \begin{vmatrix} \frac{1}{(1+\alpha)} & \frac{1}{(2+\alpha)} \\ \frac{1}{(2+\alpha)} & \frac{1}{(3+\alpha)} \end{vmatrix} .$$

Hence
$$f_2 \frac{D(1,1)}{D(0,2)} = \frac{\frac{\alpha^3}{(2+\alpha)} \begin{vmatrix} \frac{1}{(1+\alpha)} & \frac{1}{(2+\alpha)} \\ \frac{1}{(2+\alpha)} & \frac{1}{(3+\alpha)} \end{vmatrix}}{\alpha^2 \begin{vmatrix} 1 & \frac{1}{(1+\alpha)} & \frac{1}{(2+\alpha)} \\ \frac{\alpha}{(1+\alpha)} & \frac{1}{(2+\alpha)} & \frac{1}{(3+\alpha)} \\ \frac{\alpha}{(2+\alpha)} & \frac{1}{(3+\alpha)} & \frac{1}{(4+\alpha)} \end{vmatrix}} = \frac{\alpha}{(2+\alpha)} \frac{\chi(\alpha)}{\psi(\alpha)} .$$

Now $\lim_{\alpha\to 0} \psi(\alpha) = \frac{1}{72}$ and $\lim_{\alpha\to 0} \chi(\alpha) = \frac{1}{24}$ or $\lim_{\alpha\to 0} \frac{\chi(\alpha)}{\psi(\alpha)} = 3$ so that $\frac{\chi\alpha}{2+\alpha}$ times $\frac{\chi(\alpha)}{\psi(\alpha)} \to 0$. In fact if $\alpha = \frac{1}{10}$, $\frac{\alpha}{1+\alpha} \frac{\chi(\alpha)}{\psi(\alpha)} < 0.4$ which contradicts $f_2\, D(1,1)/D(0,2) \geq 1$.

Yet another question or Padé conjecture that is yet to be proved or disproved probably evolved from a theorem proved by de Montessus de Balloire which will now be stated.

Let $P(z)$ be a power series representing a function which is regular for $|z| \leq R$ except for m poles within this circle. Then the $(m + 1)$ st horizontal file of the Padé table

$$\begin{array}{llll} [0,0], & [0,1], & [0,2], & \cdots \\ [1,0], & [1,1], & [1,2], & \cdots \\ [2,0], & [2,1], & [2,2], & \cdots \\ \vdots & \vdots & \vdots & \\ [n,0], & [n,1], & [n,2], & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

for $P(z)$ converges to $P(z)$ uniformly in the domain obtained from

$|z| \leq R$ by removing the interiors of small circles with centers at these poles. See page 112 in Wall.

Now notice the likeness of the theorem due to de Montessus de Balloire and the Padé conjecture that was mentioned.

<u>Padé Conjecture</u>. If $P(z)$ is a power series representing a function which is regular for $|z| \leq 1$, except for m poles within this circle and except for $z = 1$ at which point the function is assumed continuous when only points $|z| \leq 1$ are considered then at least a subsequence of the [N,N] Padé approximants converge uniformly to the function (as N tends to infinity) in the domain formed by removing the interiors of small circles with centers at these poles.

In order to prove some theorems that are results of this conjecture, define a function $f(z)$ as one of type I if a straight line can be drawn in the z plane, such that $f(z)$ is regular every place, except for a finite number of poles, in the open half-plane containing the origin and is continuous in this half-plane at infinity. Let $R_2(f)$ be a half plane, except for the interiors of certain small circles containing the finite number of holes, contained in this half plane such that its edge is parallel to the edge of $R_2(f)$ and between it and the origin.

<u>Quasitheorem A</u>. If $f(z)$ is of type I, then the [N,N] Padé approximants converge to it uniformly in $R_2(f)$.

Proof. By Theorem 2 Chapter II $g(w) = f(A^{-1}w(1-w))$ may equivalently be studied. If the line that bounds $R_2(f)$ passes through the point $-1/2A$, then the transformation $w = Az/(1+Az)$ maps $R_2(f)$ onto the unit circle. The point at infinity is mapped onto $w = 1$ and the origin goes to the origin under this mapping. Thus $g(w)$ is regular in the unit circle except for the finite number of poles in $|w| \leq 1$ and continuous at $w = 1$ from theinterior of $|w| \leq 1$. Hence the conjecture implies that the [N,N] Padé approximants for $g(w)$ converge uniformly to $g(w)$ in $|w| \leq 1$ except for the interior of small circles centered at the given poles and the theorem follows.

Quisicorollary A. The [N,N] Padé approximants converge uniformly to any function $f(z)$ in any circle containing the origin as an interior point minus the interiors of a finite number of small circles with centers which are the only poles of $f(z)$ in the circle and which are the only places where $f(z)$ is not regular except at one boundary point where $f(z)$ is continuous.

Proof. By Theorem 2 in Chapter 2 and the Quasitheorem A the proof will follow if it can be shown that any such circle can be mapped onto the unit circle under a transformation as in Theorem 2 in Chapter II. Now for a circle with center at A and radius R,

$$w = \frac{e^{i\theta} z}{(R(1 - |A/R|^2) + (\bar{A}z)R)}$$

will accomplish this transformation where it is realized that θ is such that the nonregular boundary point goes to $w = 1$. The corollary thus follows as was shown in the theorem.

Quasicorollary B. The [N,N] Padé approximants converge uniformly to $f(z)$ in the union $R_3(f)$ of any finite number of regions $R_2(f)$ and circles as given in Corollary A.

Proof. By Quasitheorem A and Quasicorollary A, [N,N] converges at every point in this union. Since the union of a finite number of closed sets is closed the [N,N] converge uniformly in $R_3(f)$.

The conjecture and the Quasistatements that followed are desirable results relative to the Padé approximants and it is interesting that the Quasistatements can be gotten more cheaply. That is, the Padé conjecture can be weakened as Baker proved in (2). This weakened conjecture will be presented which concludes the dissertation.

Conjecture. If $P(z)$ is a power series representing a function which is regular for $|z| \leq 1$, except for m poles within this circle and except for $z = 1$, at which point the function is assumed continuous when only points $|z| \leq 1$ are considered, then at least a subsequence of the [N,N] Padé approximants are uniformly bounded in N, in the domain formed by removing the interiors of small circles with centers at these poles and uniformly continuous at $z = 1$ for $|z| \leq 1$.

To see that this conjecture implies the Padé conjecture will be shown. Now $f(z)$ is regular at every boundary point of the convergence domain, except at $z = 1$, in the Padé conjecture. If $\delta > 0$ and L is the set of boundary points such that $\delta \leq \arg z < 2\pi - \delta$ then the closed set can be finitely covered by the interiors of circles in which $f(z)$ is regular. Hence there exists a circle containing L in the interior in which f is regular where it is realized that there is a small piece of the circle at one excluded. Yet for any point not 1 there exists such a circle with piece missing in which the point is interior. Its tacitly assumed that points on the interiors of the small circles are excluded too. This circle can be mapped onto the unit circle by a linear fractional transformation under which by Theorem 2 in Chapter II the [N,N] Padé approximants are invariant. Since as already stated any point in the Padé conjecture is interior to some such circle by the conjecture and Theorem 5 in Chapter II there exists an infinite subsequence of Padé approximants which converge to a limit function that is regular everywhere in the domain of the Padé conjecture except at $z = 1$. But by the uniform continuity of the subsequence the convergence is realized at $z = 1$.

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