# ELASTO-STATIC PROBLEM OF A RECTANGULAR PLATE WITH A CIRCULAR HOLE

By

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## NOMENCLATURE

| | |
|---|---|
| a | Half length of the plate |
| $a_n$, $b_n$, $c_n$, $d_n$ | General coefficients in Airy stress function |
| b | Half width of the plate |
| c | Radius of hole |
| $F_n(r)$ | Functions of r in the Airy stress function |
| f(y) | Arbitrary normal stress on the boundary |
| K | Stress concentration factor |
| $r, \theta$ | Polar coordinates |
| x, y | Cartesian coordinates |
| $\alpha$ | Angle measured from the x-axis to the corner of the plate |
| $\beta$ | Ratio of plate length to plate width |
| $\gamma$ | Ratio of hole diameter to plate width |
| $\nabla^2$ | Laplacian operator |
| $\sigma_{rr}$, $\sigma_{\theta\theta}$, $\tau_{r\theta}$ | Stresses in polar coordinates |
| $\sigma_{xx}$, $\sigma_{yy}$, $\tau_{xy}$ | Stresses in rectangular coordinates |
| $\overline{\sigma}$ | Uniform end traction on plate |
| $\varphi$ | Airy stress function |

# CHAPTER I

# INTRODUCTION

## 1-1 General

For many years, it has been known that the introduction of a geometrical discontinuity into a stressed elastic solid may greatly alter the stress distribution. This is true for a plate under uniaxial tension also, because due to the introduction of a very small hole in the center of the plate, a high stress concentration will occur at the edge of the hole. Since the nominal area of the cross-section has not been reduced appreciably, this indicates that some relation other than the applied force divided by the net area of the cross-section must be used in order to accurately determine the stress distribution in the vicinity of the hole. Thus, in this investigation the equations of plane elasticity are solved by employing an Airy stress function.

Most of the work done in calculating stress concentration factors has been done using an infinite strip while the related problem of a finite plate with a large circular hole has received much less attention.

## 1-2 Historical Notes

G. Kirsch (1) first presented the solution for an infinite plate with a small circular hole under the action of all round tension. Muskhelishvili (2), by using complex variables, presents the solution for a perforated infinite plate under various loading conditions. His results included problems involving displacement boundary conditions as well as those involving stress boundary conditions.

The solution for an infinite strip under uniaxial tension was investigated by Howland and Stevenson (3) but their results were confined to hole diameters less than half the plate width. Koiter (4) used elementary beam theory to solve the case of a hole diameter almost equal to the width of the plate. Combining Howland's and Koiter's results yields the curve labeled $\beta=\infty$ in Fig. 2. Jeffery (5) introduced bipolar coordinates in 1921 leading to solutions of plates with more than one hole or eccentrically located holes in strips.

Of the various photoelastic tests conducted in this area (6-9), Wahl (10) conducted very precise tests in which the stress concentration factor is determined for a complete range of hole sizes. For hole sizes less than half the plate width, his results agree very well with the mathematical results of Howland. Wahl's results indicate also, that for large holes having a diameter nearly equal to the width of the bar the stress concentration factor is not far from 2 in cases where the lateral displacements of the minimum section of the bar are small compared to the thickness of the section. This agrees with the results of Koiter.

The stress distribution around the edge of a circular hole in a square plate was solved by an approximate method by Hengst (11) ~~but~~ the accuracy of his solution decreases with increased hole size, due to numerical difficulties and Hengst restricts his anaylsis to hole diameters of less than one-half the plate dimension. Little and Schlack (12) obtained the solution for a square plate under uniaxial tension for a complete range of hole sizes. Their results are plotted in Fig. 2 as $\beta$=1.0. Thus, from Fig. 2, it is observed that for a hole diameter greater than four-tenths the plate width the stress concentration factor for a square plate varies considerably from that for an infinite plate. The purpose of this investigation is to fill in the gap between the two curves shown in Fig. 2 by using an approach very similar to that taken by Little and Schlack. Thus, the problem in this investigation is that of obtaining a solution for a rectangular perforated plate under uniaxial tension. Numerical results will be obtained for hole diameters of four-tenths the plate width and larger.

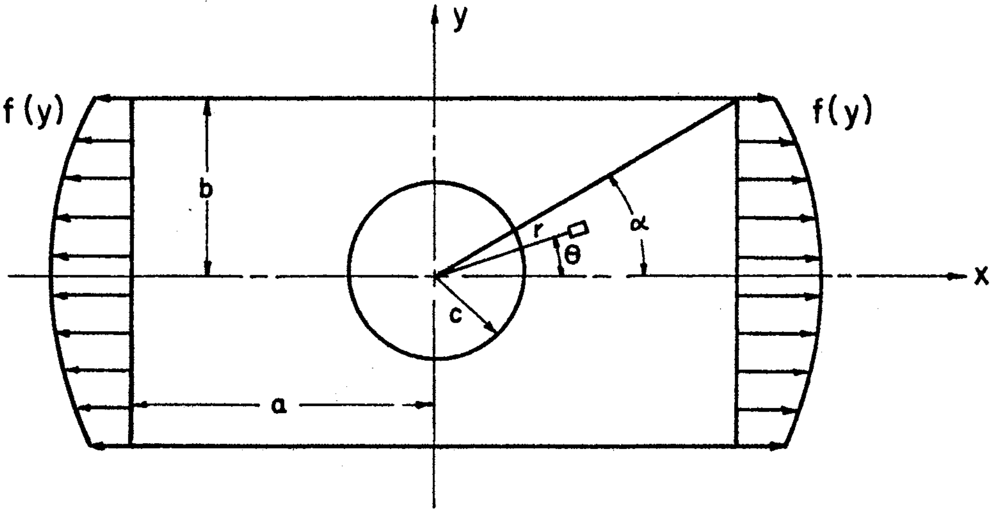


FIG. 1 - RECTANGULAR PLATE WITH END TRACTIONS

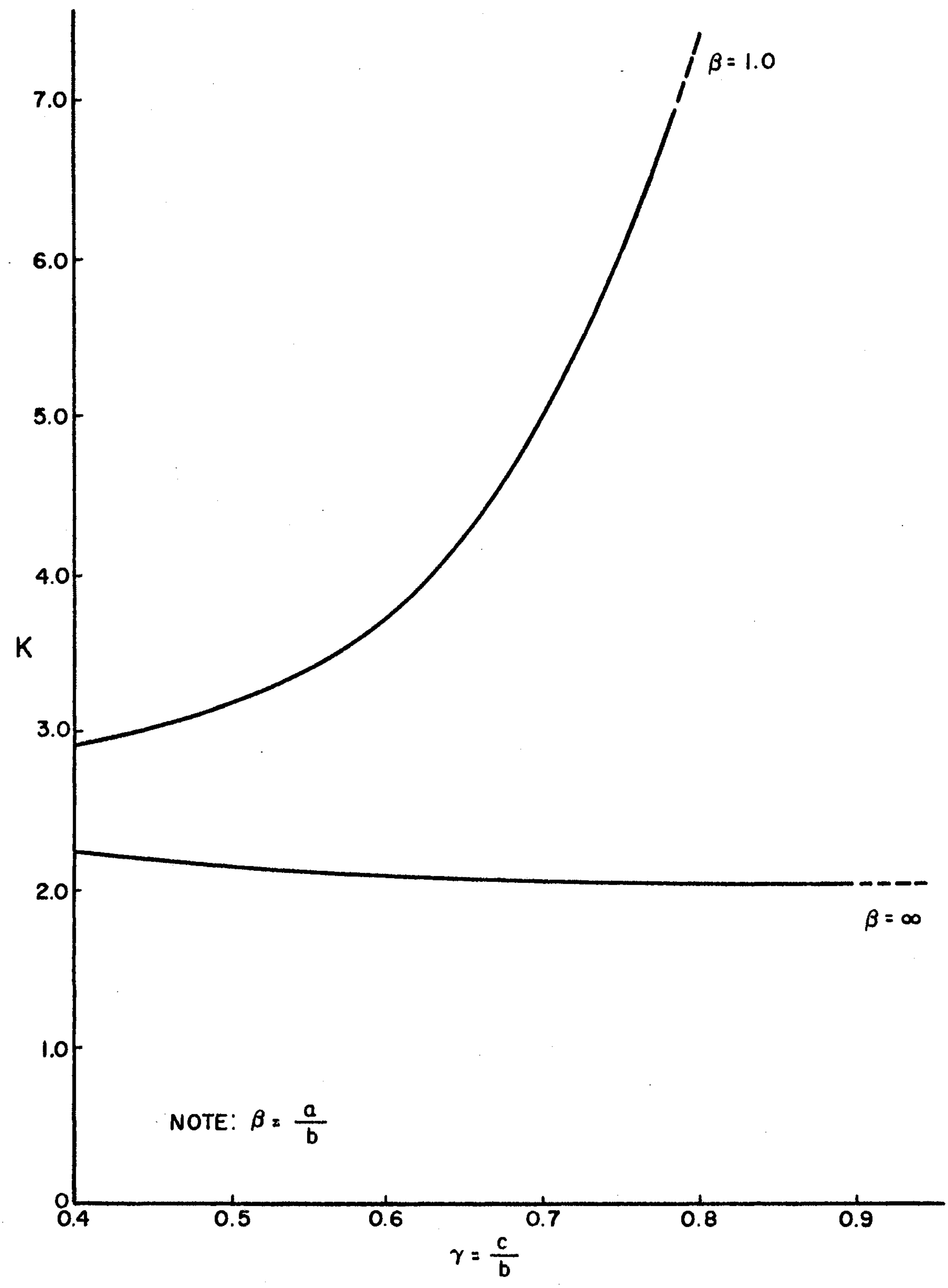


FIG. 2 – STRESS CONCENTRATION FACTOR FOR AN INFINITE STRIP AND FOR A SQUARE PLATE.

# CHAPTER II

## FORMULATION OF PROBLEM

The stresses in polar coordinates are related to the Airy stress function, $\varphi$, by the equations:

$$\sigma_{rr} = \frac{1}{r}\varphi_{,r} + \frac{1}{r^2}\varphi_{,\theta\theta} \tag{2-1}$$

$$\sigma_{\theta\theta} = \varphi_{,rr} \tag{2-2}$$

$$\tau_{r\theta} = \frac{1}{r^2}\varphi_{,\theta} - \frac{1}{r}\varphi_{,r\theta} \tag{2-3}$$

where $\varphi_{,r}$ denotes the partial derivative of $\varphi$ with respect to r. Assuming plane stress conditions, the equilibrium equations of elasticity are satisfied, and the defining equation for $\varphi$ is:

$$\nabla^2\nabla^2\varphi = 0 \tag{2-4}$$

The boundary conditions at the edge of the hole for the rectangular plate shown in Fig. 1 are:

$$\sigma_{rr}(c, \theta) = 0 \tag{2-5}$$

$$\tau_{r\theta}(c, \theta) = 0 \tag{2-6}$$

On the outer boundary, the conditions become:

a) for $\theta < \alpha$:

$$\sigma_{xx}(a, y) = f(y) = 1.0 \tag{2-7}$$

$$\tau_{xy}(a, y) = 0 \tag{2-8}$$

b) for $\theta > \alpha$:

$$\sigma_{yy}(x, b) = 0 \tag{2-9}$$

$$\tau_{xy}(x, b) = 0 \tag{2-10}$$

Considering the state of stress at a point, the stress components in rectangular coordinates are related to those in polar coordinates at any position $(r, \theta)$ by the equations:

$$\sigma_{xx}(r, \theta) = \cos^2 \theta \, \sigma_{rr}(r, \theta) + \sin^2 \theta \, \sigma_{\theta\theta}(r, \theta) - 2 \sin \theta \cos \theta \, \tau_{r\theta}(r, \theta) \tag{2-11}$$

$$\sigma_{yy}(r, \theta) = \sin^2 \theta \, \sigma_{rr}(r, \theta) + \cos^2 \theta \, \sigma_{\theta\theta}(r, \theta) + 2 \sin \theta \cos \theta \, \tau_{r\theta}(r, \theta) \tag{2-12}$$

$$\tau_{xy}(r, \theta) = \sin \theta \cos \theta \left[\sigma_{rr}(r, \theta) - \sigma_{\theta\theta}(r, \theta)\right] + (\cos^2 \theta - \sin^2 \theta) \, \tau_{r\theta}(r, \theta) \tag{2-13}$$

Restricting the problem to one in which $f(y)$ in Fig. 1 is an even function in y, the general solution for $\varphi$, in Eq. (2-4), may be taken as:

$$\varphi = \sum_{n}^{\infty} F_n(r) \cos n\theta \tag{2-14}$$

where $n = 0, 2, 4 \ldots$ because of symmetry of $\sigma_{rr}$ at $\theta$ and $(180^\circ + \theta)$, and due to symmetry at $\theta$ and $(180^\circ - \theta)$. Then with the above restriction upon n, it is necessary to work only with the portion of the plate that lies in the first quadrant. Substituting the above expression for $\varphi$ into Eq. (2-4) yields the following differential equation that must be satisfied:

$$r^4 \frac{d^4 F_n}{dr^4} + 2r^3 \frac{d^3 F_n}{dr^3} - (1 + 2n^2) r^2 \frac{d^2 F_n}{dr^2}$$

$$+ (1 + 2n^2) r \frac{dF_n}{dr} + (n^4 - 4n^2) F_n = 0 \qquad (2\text{-}15)$$

Assuming a solution of the form:

$$F_n = C_n r^\lambda$$

leads to the solution of Eq. (2-15) as:

$$F_0 = a_0' + b_0' r^2 + c_0' \ln r + d_0' r^2 \ln r \qquad (2\text{-}16)$$

$$F_1 = a_1' r + b_1'/r + c_1' r \ln r + d_1' r^3 \qquad (2\text{-}17)$$

and, for $n \geq 2$:

$$F_n = a_n' r^n + b_n' r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2} \qquad (2\text{-}18)$$

However, for single-valued displacements $d_0'$ must be set equal to zero. Nondimensionalizing the variable yields for:

$$\varphi = \sum_n^\infty F_n\left(\frac{r}{b}\right) \cos n\theta , \qquad (2\text{-}19)$$

the relations,

$$F_0 = a_0 + b_0\left(\frac{r}{b}\right)^2 + c_0 \ln\left(\frac{r}{b}\right) \qquad (2\text{-}20)$$

$$F_1 = a_1\left(\frac{r}{b}\right) + b_1/\left(\frac{r}{b}\right) + c_1\left(\frac{r}{b}\right)\ln\left(\frac{r}{b}\right) + d_1\left(\frac{r}{b}\right)^3 \qquad (2\text{-}21)$$

$$(n \geq 2) \qquad F_n = a_n\left(\frac{r}{b}\right)^n + b_n\left(\frac{r}{b}\right)^{n+2} + c_n\left(\frac{r}{b}\right)^{-n} + d_n\left(\frac{r}{b}\right)^{-n+2} \qquad (2\text{-}22)$$

Noting the restrictions on n, $F_1$ may be dropped. Thus, the Airy stress function, $\varphi$, becomes:

$$\varphi = a_0 + b_0\left(\frac{r}{b}\right)^2 + c_0 \ln\left(\frac{r}{b}\right) + \sum_{n=2,4,6\ldots}^{\infty}\left[a_n\left(\frac{r}{b}\right)^n + b_n\left(\frac{r}{b}\right)^{n+2} + c_n\left(\frac{r}{b}\right)^{-n} + d_n\left(\frac{r}{b}\right)^{-n+2}\right]\cos n\theta \qquad (2\text{-}23)$$

Differentiating with respect to r yields:

$$\varphi_{,r} = 2b_0\left(\frac{r}{b}\right)\left(\frac{1}{b}\right) + c_0\frac{1}{\left(\frac{r}{b}\right)}\frac{1}{b} + \sum_{n=2,4,6\ldots}^{\infty}\left[na_n\left(\frac{r}{b}\right)^{n-1}\left(\frac{1}{b}\right) + (n+2)b_n\left(\frac{r}{b}\right)^{n+1}\left(\frac{1}{b}\right) - nc_n\left(\frac{r}{b}\right)^{-n-1}\left(\frac{1}{b}\right) + (-n+2)d_n\left(\frac{r}{b}\right)^{-n+1}\left(\frac{1}{b}\right)\right]\cos n\theta \qquad (2\text{-}24)$$

From Eq's. (2-1), (2-3), (2-5), and (2-6), the boundary conditions at the edge of the hole are:

$$r\varphi_{,r} + \varphi_{,\theta\theta} = 0 \qquad (2\text{-}25)$$

$$\varphi_{,\theta} - r\varphi_{,r\theta} = 0 \qquad (2\text{-}26)$$

for all values of $\theta$.

Letting $\varphi_{,r}(c, \theta) = 0$, the first term of Eq. (2-25) and the second term of Eq. (2-26) drop out of the equations. Letting $\varphi(c, \theta) = 0$ then sets the second term of Eq. (2-25) and the first term of Eq. (2-26) to zero. Thus, the boundary stresses at the edge of the hole may be written as:

$$\varphi(c, \theta) = 0 \qquad (2\text{-}27)$$

$$\varphi_{,r}(c, \theta) = 0 \qquad (2\text{-}28)$$

Imposing these conditions upon Eq's. (2-23) and (2-24) yields, where $\gamma = \frac{c}{b}$:

$$\left[a_0 + b_0\gamma^2 + c_0 \ln \gamma\right] + \sum_{n=2,4,6\ldots}^{\infty} \left[a_n\gamma^n + b_n\gamma^{n+2} + c_n\gamma^{-n} + d_n\gamma^{-n+2}\right] \cos n\theta = 0 \qquad (2\text{-}29)$$

$$\left[2b_0\gamma^2 + c_0\right] + \sum_{n=2,4,6\ldots}^{\infty} \left[n a_n\gamma^n + (n+2) b_n\gamma^{n+2} - n c_n\gamma^{-n} + (2-n) d_n\gamma^{-n+2}\right] \cos n\theta = 0 \qquad (2\text{-}30)$$

For the above equations to be satisfied for all values of $\theta$, the terms inside each of the four brackets must be zero; therefore the following relations must hold:

$$a_0 + b_0\gamma^2 + c_0 \ln \gamma = 0 \tag{2-31}$$

$$2b_0\gamma^2 + c_0 = 0 \tag{2-32}$$

$$n a_n\gamma^n + (n+2) b_n\gamma^{n+2} - n c_n\gamma^{-n} + (-n+2) d_n\gamma^{-n+2} = 0 \tag{2-33}$$

$$a_n\gamma^n + b_n\gamma^{n+2} + c_n\gamma^{-n} + d_n\gamma^{-n+2} = 0 \tag{2-34}$$

Thus,

$$a_0 = c_0\left[\frac{1}{2} - \ln \gamma\right] \tag{2-35}$$

$$b_0 = -c_0\left[\frac{1}{2\gamma^2}\right] \tag{2-36}$$

and, for $n \geq 2$:

$$a_n = -c_n(n+1)\gamma^{-2n} - d_n n\gamma^{-2(n-1)} \tag{2-37}$$

$$b_n = c_n n\gamma^{-2(n+1)} + d_n(n-1)\gamma^{-2n} \tag{2-38}$$

By using Eq's. (2-35) through (2-38), the stress components that satisfy the equations of elasticity and the boundary conditions at the edge of the hole may be written:

In polar coordinates,

$$\sigma_{rr}(r,\theta) = c_0\left[\frac{1}{\left(\frac{r}{b}\right)^2} - \frac{1}{\gamma^2}\right] + \sum_{n=2,4\ldots}^{\infty} \left\{c_n\left[n(n-1)(n+1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n-2}\right.\right.$$

$$\left. - n(n+1)(n-2)(\gamma)^{-2(n+1)}\left(\frac{r}{b}\right)^n - n(n+1)\left(\frac{r}{b}\right)^{-(n+2)}\right] \cos n\theta$$

$$+ d_n\left[n^2(n-1)(\gamma)^{-2(n-1)}\left(\frac{r}{b}\right)^{n-2} - (n+1)(n-1)(n-2)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n}\right.$$

$$\left. - (n+2)(n-1)\left(\frac{r}{b}\right)^{-n}\right]\cos n\theta\Big\} \qquad (2\text{-}39)$$

$$\sigma_{\theta\theta}(r,\theta) = -c_0\left[\frac{1}{\left(\frac{r}{b}\right)^2} + \frac{1}{\gamma^2}\right] + \sum_{n=2,4,6\ldots}^{\infty}\Big\{c_n\left[n(n+1)(n+2)(\gamma)^{-2(n+1)}\left(\frac{r}{b}\right)^{n}\right.$$

$$\left. + n(n+1)\left(\frac{r}{b}\right)^{-(n+2)} - n(n-1)(n+1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n-2}\right]\cos n\theta$$

$$+ d_n\left[(n+1)(n+2)(n-1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n} + (n-1)(n-2)\left(\frac{r}{b}\right)^{-n}\right.$$

$$\left. - n^2(n-1)(\gamma)^{-2(n-1)}\left(\frac{r}{b}\right)^{n-2}\right]\cos n\theta\Big\} \qquad (2\text{-}40)$$

$$\tau_{r\theta}(r,\theta) = \sum_{n=2,4,6\ldots}^{\infty}\Big\{c_n\left[-n(n-1)(n+1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n-2} + n^2(n+1)(\gamma)^{-2(n+1)}\left(\frac{r}{b}\right)^{n}\right.$$

$$\left. - n(n+1)\left(\frac{r}{b}\right)^{-(n+2)}\right]\sin n\theta + d_n\left[-n^2(n-1)(\gamma)^{-2(n-1)}\left(\frac{r}{b}\right)^{n-2}\right.$$

$$\left. + n(n+1)(n-1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n} - n(n-1)\left(\frac{r}{b}\right)^{-n}\right]\sin n\theta\Big\} \qquad (2\text{-}41)$$

In rectangular coordinates,

$$\sigma_{xx}(r,\theta) = c_0\left[\frac{\cos 2\theta}{\left(\frac{r}{b}\right)^2} - \frac{1}{\gamma^2}\right] + \sum_{n=2,4,6\ldots}^{\infty} c_n\Big\{\Big[n(n-1)(n+1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n-2}$$

$$- n^2(n+1)(\gamma)^{-2(n+1)}\left(\frac{r}{b}\right)^{n}\Big]\cos(n-2)\theta - n(n+1)\left(\frac{r}{b}\right)^{-(n+2)}\cos(n+2)\theta$$

$$+ 2n(n+1)(\gamma)^{-2(n+1)}\left(\frac{r}{b}\right)^{n}\cos n\theta\Big\} + d_n\Big\{\Big[n^2(n-1)(\gamma)^{-2(n-1)}\left(\frac{r}{b}\right)^{n-2}$$

$$- n(n+1)(n-1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n}\Big]\cos(n-2)\theta - n(n-1)\left(\frac{r}{b}\right)^{-n}\cos(n+2)\theta$$

$$+ \Big[2(n+1)(n-1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n} - 2(n-1)\left(\frac{r}{b}\right)^{-n}\Big]\cos n\theta\Big\} \qquad (2\text{-}42)$$

$$\sigma_{yy}(r,\theta) = -c_0\left[\frac{\cos 2\theta}{\left(\frac{r}{b}\right)^2} + \frac{1}{\gamma^2}\right] + \sum_{n=2,4,6\ldots}^{\infty} c_n\Big\{\Big[n^2(n+1)(\gamma)^{-2(n+1)}\left(\frac{r}{b}\right)^{n}$$

$$- n(n-1)(n+1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n-2}\Big]\cos(n-2)\theta + n(n+1)\left(\frac{r}{b}\right)^{-(n+2)}\cos(n+2)\theta$$

$$+ 2n(n+1)(\gamma)^{-2(n+1)}\left(\frac{r}{b}\right)^{n}\cos n\theta\Big\} + d_n\Big\{\Big[n(n+1)(n-1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n}$$

$$- n^2(n-1)(\gamma)^{-2(n-1)}\left(\frac{r}{b}\right)^{n-2}\Big]\cos(n-2)\theta + n(n-1)\left(\frac{r}{b}\right)^{-n}\cos(n+2)\theta$$

$$+ \Big[2(n+1)(n-1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n} - 2(n-1)\left(\frac{r}{b}\right)^{-n}\Big]\cos n\theta\Big\} \qquad (2\text{-}43)$$

$$\tau_{xy}(r,\theta) = c_0\frac{\sin 2\theta}{\left(\frac{r}{b}\right)^2} + \sum_{n=2,4\ldots}^{\infty} c_n\Big\{\Big[n^2(n+1)(\gamma)^{-2(n+1)}\left(\frac{r}{b}\right)^{n}$$

$$- n(n-1)(n+1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n-2}\Big]\sin(n-2)\theta$$

$$- n(n+1)\left(\frac{r}{b}\right)^{-(n+2)}\sin(n+2)\theta\Big\} + d_n\Big\{\Big[n(n+1)(n-1)(\gamma)^{-2n}\left(\frac{r}{b}\right)^{n}$$

$$- n^2(n-1)(\gamma)^{-2(n-1)}\left(\frac{r}{b}\right)^{n-2}\Big] \sin(n-2)\theta - n(n-1)\left(\frac{r}{b}\right)^{-n} \sin(n+2)\theta\Big\} \qquad (2\text{-}44)$$

Along the outer boundary, the relations between (x, r, $\theta$) and (y, r, $\theta$) from Fig. 1 are:

For $0 \le \theta \le \alpha$,

$$r = a/\cos\theta \qquad (2\text{-}45)$$

For $\alpha \le \theta \le 90^\circ$,

$$r = b/\sin\theta \qquad (2\text{-}46)$$

Substituting these expressions into Eq's. (2-42) through (2-44), the expressions for the stresses along the boundary become:

$$\sigma_{xx}\Big]_{x=a} = c_0\left[\frac{\cos 2\theta\cos^2\theta}{\beta^2} - \frac{1}{\gamma^2}\right] + \sum_{n=2,4,\ldots}^{\infty} c_n\Big\{n(n-1)(n+1)(\gamma)^{-2n}\left(\frac{\beta}{\cos\theta}\right)^{n-2}$$

$$- n^2(n+1)(\gamma)^{-2(n+1)}\left(\frac{\beta}{\cos\theta}\right)^{n}\Big] \cos(n-2)\theta - n(n+1)\left(\frac{\beta}{\cos\theta}\right)^{-(n+2)} \cos(n+2)\theta$$

$$+ 2n(n+1)(\gamma)^{-2(n+1)}\left(\frac{\beta}{\cos\theta}\right)^{n}\cos n\theta\Big\} + d_n\Big\{\Big[n^2(n-1)(\gamma)^{-2(n-1)}\left(\frac{\beta}{\cos\theta}\right)^{n-2}$$

$$- n(n+1)(n-1)(\gamma)^{-2n}\left(\frac{\beta}{\cos\theta}\right)^{n}\Big]\cos(n-2)\theta - n(n-1)\left(\frac{\beta}{\cos\theta}\right)^{-n}\cos(n+2)\theta$$

$$+ \Big[2(n+1)(n-1)(\gamma)^{-2n}\left(\frac{\beta}{\cos\theta}\right)^{n} - 2(n-1)\left(\frac{\beta}{\cos\theta}\right)^{-n}\Big]\cos n\theta\Big\} \qquad (2\text{-}47)$$

$$\tau_{xy}\Big]_{x=a} = c_0 \frac{\sin 2\theta \cos^2\theta}{\beta^2} + \sum_{n=2,4,\ldots}^{\infty} c_n\Big\{\Big[n^2(n+1)(\gamma)^{-2(n+1)}\Big(\frac{\beta}{\cos\theta}\Big)^n$$

$$- n(n-1)(n+1)(\gamma)^{-2n}\Big(\frac{\beta}{\cos\theta}\Big)^{n-2}\Big]\sin(n-2)\theta$$

$$- n(n+1)\Big(\frac{\beta}{\cos\theta}\Big)^{-(n+2)}\sin(n+2)\theta\Big\} + d_n\Big\{\Big[n(n+1)(n-1)(\gamma)^{-2n}\Big(\frac{\beta}{\cos\theta}\Big)^n$$

$$- n^2(n-1)(\gamma)^{-2(n-1)}\Big(\frac{\beta}{\cos\theta}\Big)^{n-2}\Big]\sin(n-2)\theta$$

$$- n(n-1)\Big(\frac{\beta}{\cos\theta}\Big)^{-n}\sin(n+2)\theta\Big\} \qquad (2\text{-}48)$$

$$\sigma_{yy}\Big]_{y=b} = -c_0\Big[\cos 2\theta \sin^2\theta + \frac{1}{\gamma^2}\Big] + \sum_{n=2,4,\ldots}^{\infty} c_n\Big\{\Big[n^2(n+1)(\gamma)^{-2(n+1)}\Big(\frac{1}{\sin\theta}\Big)^n$$

$$- n(n-1)(n+1)(\gamma)^{-2n}\Big(\frac{1}{\sin\theta}\Big)^{n-2}\Big]\cos(n-2)\theta + n(n+1)\Big(\frac{1}{\sin\theta}\Big)^{-n-2}\cos(n+2)\theta$$

$$+ 2n(n+1)(\gamma)^{-2(n+1)}\Big(\frac{1}{\sin\theta}\Big)^n\cos n\theta\Big\} + d_n\Big\{\Big[n(n+1)(n-1)(\gamma)^{-2n}\Big(\frac{1}{\sin\theta}\Big)^n$$

$$- n^2(n-1)(\gamma)^{-2(n-1)}\Big(\frac{1}{\sin\theta}\Big)^{n-2}\Big]\cos(n-2)\theta + n(n-1)\Big(\frac{1}{\sin\theta}\Big)^{-n}\cos(n+2)\theta$$

$$+ \Big[2(n+1)(n-1)(\gamma)^{-2n}\Big(\frac{1}{\sin\theta}\Big)^n - 2(n-1)\Big(\frac{1}{\sin\theta}\Big)^{-n}\Big]\cos n\theta\Big\} \qquad (2\text{-}49)$$

$$\tau_{yx}\Big]_{y=b} = c_0 \sin 2\theta \sin^2\theta + \sum_{n=2,4,\ldots}^{\infty} c_n\Big\{\Big[n^2(n+1)(\gamma)^{-2(n+1)}\Big(\frac{1}{\sin\theta}\Big)^n$$

$$- n(n-1)(n+1)(\gamma)^{-2n}\Big(\frac{1}{\sin\theta}\Big)^{n-2}\Big]\sin(n-2)\theta$$

$$- n(n+1)\left(\frac{1}{\sin\theta}\right)^{-(n+2)} \sin(n+2)\theta\Big\} + d_n\Big\{\Big[n(n+1)(n-1)(\gamma)^{-2n}\left(\frac{1}{\sin\theta}\right)^{n}$$

$$- n^2(n-1)(\gamma)^{-2(n-1)}\left(\frac{1}{\sin\theta}\right)^{n-2}\Big] \sin(n-2)\theta$$

$$- n(n-1)\left(\frac{1}{\sin\theta}\right)^{-n} \sin(n+2)\theta\Big\} \qquad (2\text{-}50)$$

The boundary conditions on the outer boundary will be satisfied by the method of least squares by minimizing the integral:

$$\int_0^{\alpha}\left\{\left[1.0 - \sigma_{xx}\left(\frac{a}{\cos\theta}, \theta\right)\right]^2 + \left[\tau_{xy}\left(\frac{a}{\cos\theta}, \theta\right)\right]^2\right\} d\theta$$

$$+ \int_{\alpha}^{\pi/2} \left\{\left[\sigma_{yy}\left(\frac{b}{\sin\theta}, \theta\right)\right]^2 + \left[\tau_{yx}\left(\frac{b}{\sin\theta}, \theta\right)\right]^2\right\} d\theta = \text{minimum} \qquad (2\text{-}51)$$

The procedure used for minimizing Eq. (2-51) is discussed in detail in Hildebrand (13). This consists of specifying the stresses at discrete points along the outer boundary. The number and the spacing of the discrete points along the boundary required for the numerical integration of Eq. (2-51), and the number of constants $c_n$ and $d_n$ were selected according to the hole size and rate of convergence. In all cases, an equal number of c's and d's under the summation was chosen so the governing differential equation is satisfied. The numerical results were all obtained from an IBM 7090 digital computer.

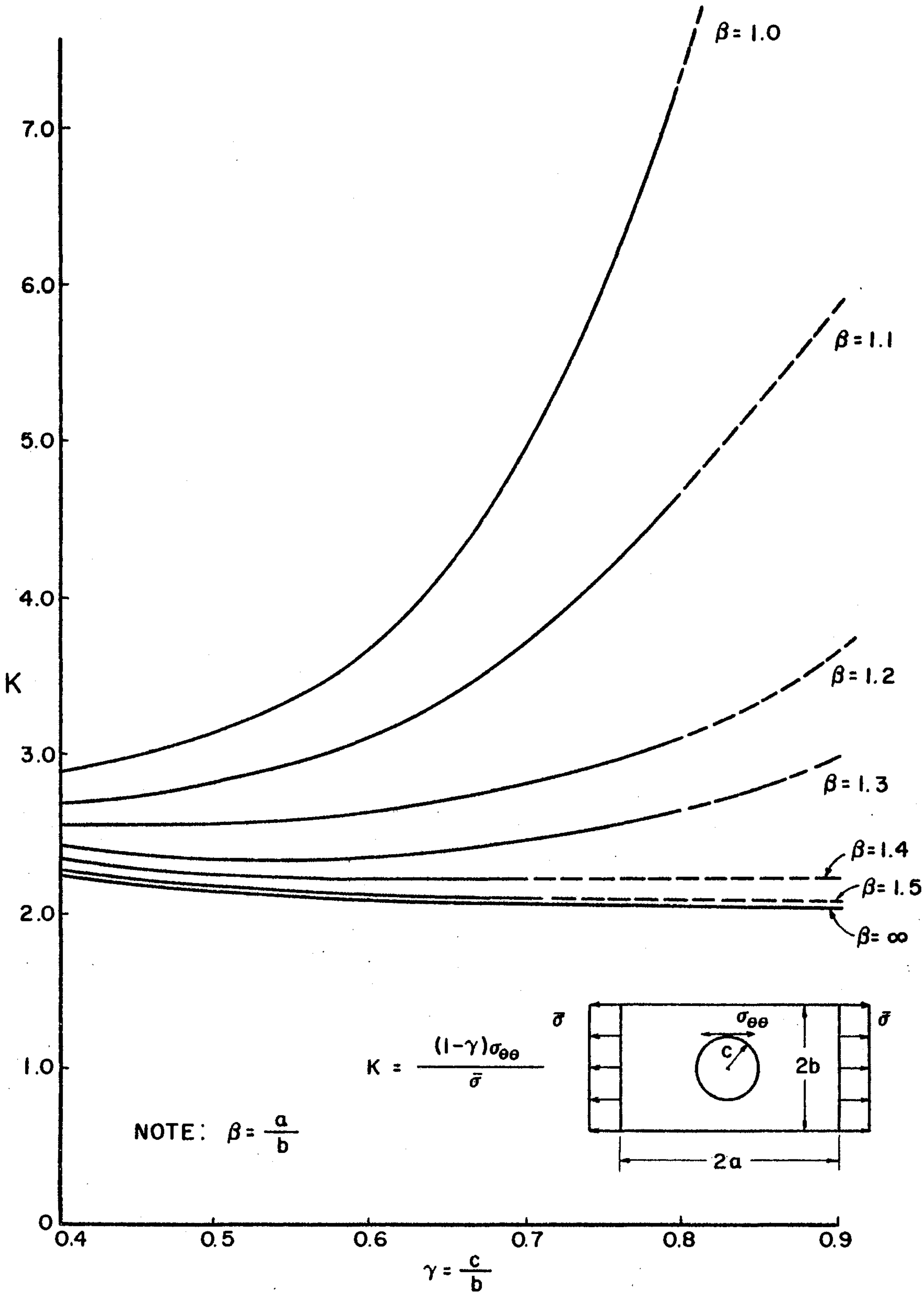


FIG. 3 - STRESS CONCENTRATION FACTOR AT EDGE OF HOLE FOR $\Theta = 90°$

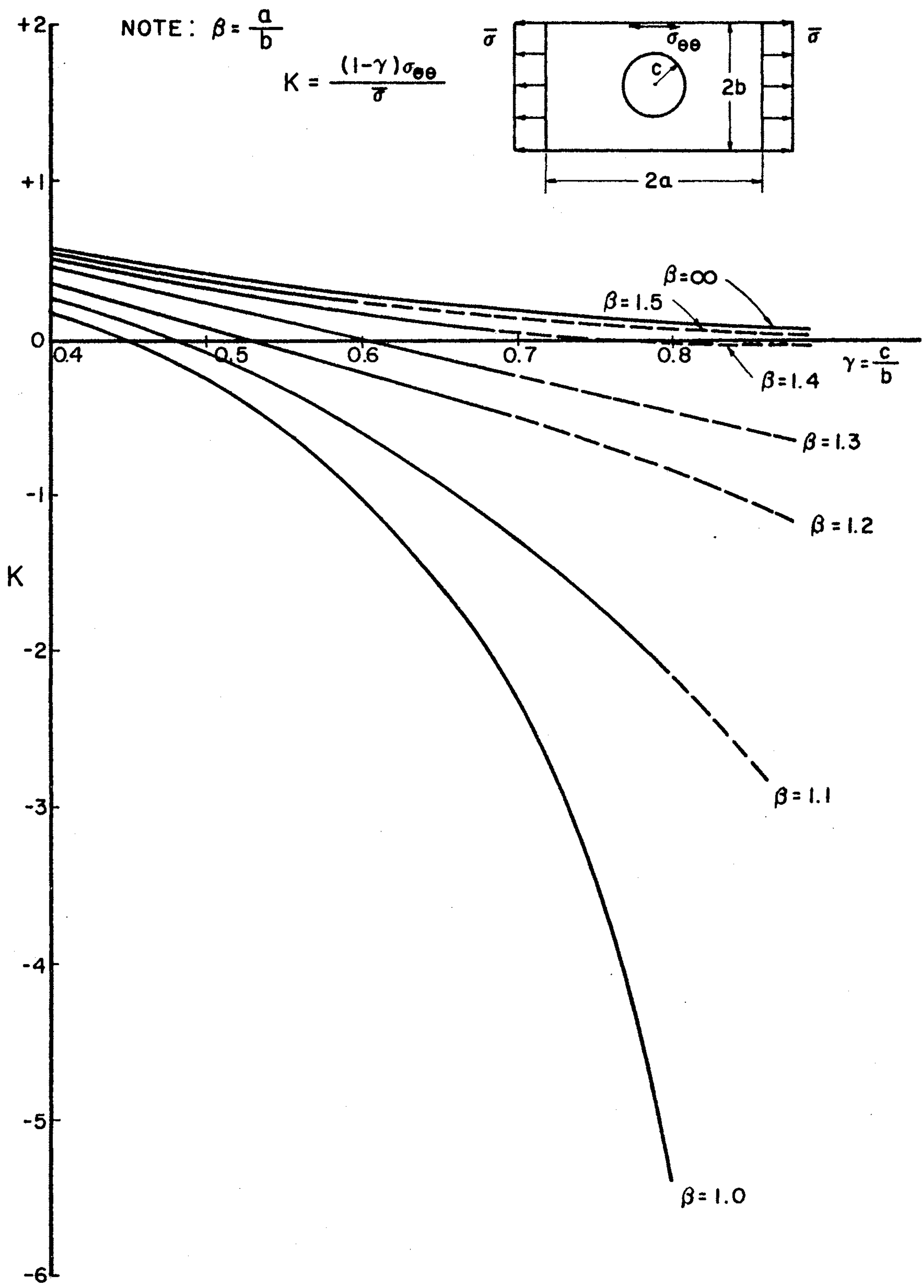


FIG. 4 - STRESS CONCENTRATION FACTOR AT OUTER BOUNDARY FOR $\theta = 90°$.

## CHAPTER III

## RESULTS AND CONCLUSIONS

For all numerical calculations, the number of terms and the points specified along the outer boundary were picked in an attempt to optimize satisfaction of the outer boundary stresses. The number of terms required varied from nineteen to twenty-nine and the number of points specified varied from thirty-two to forty. In most cases, the boundary stresses converged to within 3% of the desired stress, but for large hole sizes with $\beta$ greater than 1.3, the best results obtained were within approximately 10%. Thus, the results for this case are not exactly correct but do give a good approximation. Approximated results are shown in the figures as dotted lines.

The stress concentration factors at the edge of the hole are plotted in Fig. 3. The curve labeled $\beta=1.0$ represents the results as obtained by Little and Schlack; the curve labeled $\beta=\infty$ represents the combination of Howland's and Koiter's results. From the trend of the intermediate curves it is observed that a plate under uniaxial tension with length to width ratio of approximately 1.5 can be treated essentially as an infinite plate. It is believed that this is due to the increased stiffness against bending as $\beta$ increases.

In the investigation by Little and Schlack for a square plate, it is shown that a thick ring analogy is appropriate when considering large holes, because the maximum stresses occur at the minimum sections and are primarily caused by flexure as can be seen from the distribution of stress across the minimum section at $\theta=90^\circ$. For hole sizes of $\gamma=0.4$ and 0.5, the stress concentration factor due to the presence of the hole is more pronounced, but for larger hole sizes the axial and bending effects are more prominent, making the ring solution more applicable. However, an increase in length will increase the in plane bending stiffness due to the additional amount of material at $\theta=0^\circ$. Thus, the bending stress would be decreased, and it is this bending effect that tends to increase the stress concentration factor for large holes in square plates. Thus, an increase in plate length should obviously decrease the stress concentration factor for large hole sizes. This is the trend shown in Fig. 3 and it is shown that this decrease is quite rapid.

The stress concentration factors at the outer boundary, r=b, are plotted in Fig. 4. From these curves it is noted that the stress may be as much as five times the nominal stress and of opposite sign for a square plate with a large hole. Thus, it may be necessary when designing to check the stress concentration at the outer boundary as well as at the edge of the hole. From these curves, the transition from a square plate to an infinitely long plate is again shown. For $\beta=1.5$ and $\gamma=0.8$, the stress concentration factor approaches zero. Koiter predicts a zero stress at the outer boundary for an infinitely long plate with a very large hole, and it is again verified that when the length to width ratio of the plate approaches approximately 1.5, the plate may be treated as one infinite in length.

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