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CHAPTER I

INTRODUCTION

The analysis of variance is a versatile statistical tool which enjoys wide usage in many fields of study. One type of analysis of variance design which is especially appropriate for many experimental situations in the medical field is the repeated measures design [Winer, 1960]. Although there are many different repeated measures designs, they all have in common the fact that each experimental unit is used under all levels of at least one factor. The suitability of this design is due to the fact that the model accounts for correlation between repeated observations on the same experimental unit. This situation may arise, for example, in psychological testing or in studies of hearing defects, in which each person is subjected to a battery of tests.

Unfortunately the standard univariate analysis of variance test for equality of means among correlated treatment groups requires that the within-treatment group variances be equal, and that all the pairwise correlations between treatment groups be equal [Box, 1954b; Danford and Hughes, 1957]. That is, the covariance matrix must have the following form:

$$\begin{bmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \rho\sigma^2 & \cdots & \cdots & \sigma^2 \end{bmatrix} .$$

A covariance matrix of this form is called a uniform covariance matrix. Undoubtedly the sample covariance matrix obtained in many experimental situations does not meet this requirement. As is the case with other violated model assumptions for this and other design models, the statistician must then consider alternative and perhaps more appropriate methods of analysis. A decision as to whether to use a method other than the usual analysis of variance would be based on such considerations as the consequence of using the usual test when an underlying assumption is violated, relative power of the available tests, presentability of the results, and complexity of alternative techniques.

Several authors have proposed a variety of procedures for analyzing experiments with repeated measures, taking into consideration the fact that the covariance matrices may not be uniform.

In an experiment on growth and wear, Box [1950] differenced the data to put it in a form which would give equal variances and covariances. In that procedure, as the amount of growth or wear increased with time, the observations were taken to be the amount of change within successive time periods rather than the accumulated growth or wear. However, Box pointed out that if interaction were present between time and other factors in the experiment, the method would not be completely successful in yielding a uniform covariance matrix. The author further stated that the validity of the usual univariate analysis which he used must be checked by testing the covariance matrix for uniformity. He gave a test statistic which is approximately distributed as χ^2 that could be used. This statistic is discussed in Chapter III.

In another paper, Box [1954b] derived the distribution of entries in the univariate analysis of variance table for an n-by-t data set in which the rows and columns represented fixed effects and the columns were correlated. This work was based on general theorems given in an earlier paper [Box, 1954a] which dealt with exact and approximate distributions of quadratic forms and their ratios. In the second 1954 paper, Box showed that the null distribution of the ratio of the column mean square to the error mean square is approximately that of $F[(t-1)\theta, (t-1)(n-1)\theta]$, where

$$\theta = t^2(\bar{v}_{ii}-\bar{v}_{..})^2/(t-1)\left[\sum_{i=1}^{t}\sum_{j=1}^{t} v_{ij}^2 - 2t\sum_{i=1}^{t}\bar{v}_{i.}^2 + t^2\bar{v}_{..}^2\right],$$

$\bar{v}_{ii}$ is the mean of the diagonal elements of the covariance matrix, $\bar{v}_{i.}$ is the mean of the elements in the i^{th} row, and $\bar{v}_{..}$ is the mean of all the elements in the matrix.

Box then used the approximate distribution to study the effect of special cases of serial correlation on the Type I error rate for the usual F-test. A positive bias was shown, but he concluded that moderate correlation has little effect on probability levels. However, it should be noted that the cases Box selected for study yield very low values for the χ^2 test statistic, given in his 1950 paper as a measure of nonuniformity, so his conclusions here are not unexpected.

Geisser and Greenhouse [1958] extended Box's approximate test for among-column effects to the case where there was within-cell replication. These authors also gave the lower bound of θ, θ_L, as $\frac{1}{t-1}$ where t is the number of levels of the correlated factor. They then gave a conservative test of among-column effects which is the same as Box's approximate test, but which uses θ_L instead of θ. The authors recommended the approximate or the conservative test for situations in which the covariance matrix is unknown and when a more appropriate multivariate analysis is unavailable (e.g., when the experiment does not provide enough degrees of freedom for the error term of a multivariate test). However, the authors pointed out that the conservative test may be too conservative.

In a 1959 paper on the analysis of profile data, Greenhouse and Geisser illustrated several methods of analysis for data of the type described above. Hotelling's generalized T^2 was used to illustrate a multivariate analysis of the data for two groups. A generalization to g groups using Roy's largest root criterion was also given. If univariate analysis is to be used, the authors pointed out that the effect produced on the approximate F distributions of Box's test by estimating θ from the sample data is unknown. They therefore recommended the conservative test, which is independent of the covariance matrix. Scheffé [1956] also criticized use of the approximate test for this reason, and recommended Hotelling's T^2.

Elsewhere in the 1959 paper, Greenhouse and Geisser recommended the following procedure if it is decided that a univariate analysis of variance should be used:

First, the usual analysis of variance is performed. If the computed F-ratio is nonsignificant, all other univariate tests will yield nonsignificant results, and testing stops. If the usual test is significant, the computed F-ratio is then compared to the tabulated F value for the conservative test. If the result is still significant, an exact test would give significance also, so testing stops. If the result of the conservative test is not significant, then the approximate test should be carried out.

Cole and Grizzle [1966] developed another multivariate procedure which was based on the largest root criterion. The authors pointed out that the method has several disadvantages. These are that a computer would be needed for the calculations, the tests of hypotheses are not independent, and the method is not as powerful as the univariate tests.

It is seen that both univariate and multivariate tests may be used in the analysis of data from repeated measures experiments, but both have disadvantages. While multivariate procedures almost always are appropriate, they have a number of disadvantages. Not the least among these is the relative difficulty of computation, since these procedures require inversion of the sample covariance matrix. Also important is the fact that the multivariate tests are less powerful than the univariate tests, when the assumptions for both designs are met [Danford and Hughes, 1957; Cole and Grizzle, 1966]. Also, multivariate procedures cannot be used unless the sample size exceeds the number of treatment groups. In addition, for some repeated measures designs [Danford, Hughes, and McNee, 1960] the usual F-test will be appropriate for a portion of the analysis.

When the univariate analysis is being considered for data which have an unknown covariance matrix, two general approaches are seen to be available. One is always to use either the approximate or the conservative test. The possible disadvantages of this procedure were pointed out earlier. The second approach is to use Box's test for the uniformity of the covariance matrix and then to select among the usual, approximate, and conservative tests, depending on the outcome of Box's test.

However, the practice of preliminary testing of assumptions is not without pitfalls. Box [1953] and Box and Andersen [1955] discussed the case of preliminary tests for equality of variances when the main test is to be a test for equality of means. They pointed out that while tests for equality of means are quite robust for heteroscedasticity and nonnormality, tests for equality of variances are very sensitive to nonnormality. Thus a preliminary test of variances might be highly significant when in fact the main test would have been disturbed very little. It seems reasonable that the same would be true for preliminary testing of covariance matrices. Another problem with preliminary testing is selection of the size of the preliminary test while preserving the size of the main analysis. In a paper by Bancroft [1964] the recommended size was as high as 0.8 for some types of preliminary analyses.

It is evident then that selection of a procedure for the analysis of repeated measures experiments can be difficult, with no clear-cut criteria for selection. In addition, it is doubtful whether any of these procedures offer a distinct advantage over the usual F-test.

The purpose of the present study is to examine the robustness of the usual F-test for a variety of violations of the assumption of a uniform covariance matrix. Computer simulation methods were used to examine balanced, one-way designs with three and four correlated treatment groups. In order that both the Type I error rates and the power of the tests could be studied, a variety of mean vectors and sample sizes were used. For some cases, the results of the conservative F-test were compared to the results of the usual F-test.

CHAPTER II

METHODOLOGY

The basic underlying model for the one-way repeated measures design is

$$y_{ij} = \mu + \tau_i + \pi_j + e_{ij}, \quad \begin{array}{l} i=1,2,\ldots,t \\ j=1,2,\ldots,n. \end{array}$$

The τ_i are fixed treatment effects, the π_j are random person effects, and the e_{ij} are random errors. Assumptions for the model are that $\sum_{i=1}^{t} \tau_i = 0$, the π_j are normally and independently distributed with mean 0 and variance σ_π^2, the e_{ij} are normally and independently distributed with mean 0 and variance σ_e^2, and the π_j and e_{ij} are independent.

In this paper, y_{ij} denotes an observation on the j^{th} person under the i^{th} treatment, $y_{i.}$ is the mean of the observations on n persons under the i^{th} treatment, $y_{.j}$ is the mean of the observations under t treatments on the j^{th} person, and $y_{..}$ is the grand mean. For this model, it can be seen from the following equations that the covariance matrix which is appropriate for a vector of observations on a person is uniform.

Now,

$$
\begin{aligned}
E(y_{ij}) &= \mu + \tau_i \\
Var(y_{ij}) &= var\ (\pi_j + e_{ij}) \\
&= var\ (\pi_j) + var\ (e_{ij}) \\
&= \sigma_\pi^2 + \sigma_e^2 \\
&= \sigma^2,
\end{aligned}
$$

$$
\begin{aligned}
Cov(y_{ij}, y_{kj}) &= E\ (\pi_j + e_{ij})\ (\pi_j + e_{kj}) \\
&= \sigma_\pi^2 \\
&= \rho\sigma^2,\ i \neq k,
\end{aligned}
$$

and

$$Cov(y_{ij}, y_{i\ell}) = 0,\ j \neq \ell.$$

Therefore,

$$
\begin{bmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{tj} \end{bmatrix} = Y_j \sim N \left(\begin{bmatrix} \mu + \tau_1 \\ \mu + \tau_2 \\ \vdots \\ \mu + \tau_t \end{bmatrix} , \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \rho\sigma^2 & \cdots & \cdots & \rho\sigma^2 \end{bmatrix} \right)
$$

For the repeated measures design analysis of variance appropriate for the above model, the treatment mean square, TMS, is

$$\frac{1}{t-1} \sum_{i=1}^{t} \sum_{j=1}^{n} (y_{i.} - y_{..})^2$$

and

$$E(TMS) = \frac{1}{t-1} \sum_{i=1}^{t} \sum_{j=1}^{n} (\tau_i - \bar{\tau}_.)^2 + \sigma_e^2.$$

The error mean square, EMS, is

$$\frac{1}{(n-1)(t-1)} = \sum_{i=1}^{t} \sum_{j=1}^{n} (y_{ij} - y_{i.} - y_{.j} + y_{..})^2$$

and

$$E(EMS) = \sigma_e^2 .$$

If TSS and ESS are respectively the treatment and error sums of squares, it can be shown that

$$TSS/\sigma_e^2 = \sum_{i=1}^{t} \sum_{j=1}^{n} (y_{i.} - y_{..})^2/\sigma_e^2$$

is distributed as

$$\chi'^2(t-1,\lambda),\ \lambda = \frac{n}{2\sigma_e^2} \sum_{i=1}^{t} (\tau_i - \tau_.)^2$$

and that

$$ESS/\sigma_e^2 = \sum_{i=1}^{t} \sum_{j=1}^{n} (y_{ij} - y_{i.} - y_{.j} + y_{..})^2/\sigma_e^2$$

is distributed as

$$\chi^2(n-1)(t-1).$$

Therefore the usual Central F-ratio of TMS/EMS may be used to test the hypothesis that $\tau_1 = \tau_2 = \ldots = \tau_t$. The above distribution for this ratio is derived in Appendix A.

There are other models which give the same distribution for the Y_j's. One alternative model is

$$y_{ij} = \mu + \tau_i + \beta_j + e_{ij}.$$

The τ_i and β_j represent fixed treatment effects and the e_{ij}

are random errors. The assumptions for this model are that $\sum_{i=1}^{t} \tau_i = 0$, $\sum_{j=1}^{n} \beta_j = 0$, and the e_{ij} are distributed normally with mean 0 and variance σ^2. Also,

$$\text{cov } (e_{ij}, e_{kj}) = \rho\sigma^2, \ i \neq k$$

and

$$\text{cov } (e_{ij}, e_{i\ell}) = 0, \ j \neq \ell.$$

This model was discussed by Box [1954b].

Both of the above models may be used to describe a randomized complete-block experiment also. In the first model, blocks would be represented by the π_j. In the second model, blocks would be represented by the β_j, and the errors may be correlated or uncorrelated [Steel and Torrie, 1960].

The first model described does not allow negative covariances for the Y_j. Accordingly, the second model, which does allow negative covariances, was included to extend the generality of the results of this study. In any case, a more general covariance matrix is appropriate for a study of violation of covariance assumptions.

When the covariance matrix of Y_j is not like the one shown above, but takes the form

$$V = \begin{bmatrix} \sigma^2 & \sigma_{12} & \cdots & \sigma_{1t} \\ \sigma_{12} & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{1t} & & \cdots & \sigma_t^2 \end{bmatrix},$$

then the covariance matrix is not uniform, and the ratio

TMS/EMS

is not distributed as F [Box, 1954a,b].

To study empirically the robustness of the usual test for equality of treatment means when the covariance matrix is not uniform, the following general procedure was used. From a population of size 16,000 with a specified covariance matrix and mean vector, 1000 data sets of size nt were drawn. Each data set was subjected to the usual F-test and the result was compared to the appropriate criterion value in the central F-table. The percent of the computed F-ratios which exceeded the tabulated value was taken as an estimate of the true Type I error rate or power of the test under the specified conditions. These results were then examined. The details of the procedure follow.

First a pool of 16,000 random normal deviates was stored on an IBM 1810 random access disk after construction in the following manner. 16,000 pseudo-random (hereafter called random) numbers distributed uniformly on the interval (0,1) were generated using an IBM 1800 random number generator and the IBM Subroutine RANDU [Scientific Subroutine Package, 1968]. This generator uses the multiplicative congruential method of generating random numbers, according to the formula

$$W_{n+1} = 899W_n \pmod{2^{15}}.$$

The period for this generator is 8192 and two different sequences of this length may be obtained through proper selection of the numbers initially supplied to the generator [Jansson, 1966]. For convenience, the first 8000 numbers of each sequence generated were used in this study.

As each pair of numbers was generated, it was used to construct a pair of normal random deviates according to the formulas

$$x_1 = (-2 \log_e u_1)^{\frac{1}{2}} \cos 2\pi u_2$$
$$x_2 = (-2 \log_e u_1)^{\frac{1}{2}} \sin 2\pi u_2$$

where u_1 and u_2 are the uniformly distributed random variables and x_1 and x_2 are independent normally distributed random variables with mean 0 and variance 1 [Box and Muller, 1958; Muller, 1959].

For a single analysis of variance the main computer program, SMAOV (Appendix B), first constructed the x_{ij}, in part according to an algorithm given by Scheuer and Stoller [1962]. The method uses the following theorem:

Let X be distributed $N(0,I_t)$ and let $Z = CX$. Then Z is distributed $N(0,CC')$. In this case, $CC' = V$. Now,

$$Z + \begin{bmatrix} \mu + \tau_1 \\ \cdot \\ \cdot \\ \cdot \\ \mu + \tau_t \end{bmatrix} = Y_j = \begin{bmatrix} y_{1j} \\ \cdot \\ \cdot \\ \cdot \\ y_{tj} \end{bmatrix} .$$

The same process was carried out for each of the Y_j, $j = 1,2,\ldots,n$, in the experiment. The usual repeated measures analysis of variance was then performed.

The covariance matrices studied were of order 3 or 4. Only populations with equal within-treatment variances were considered. With no loss of generality, the covariance matrix was taken to be equal to the population correlation matrix

$$\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1t} \\ \rho_{12} & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ \rho_{1t} & \cdots & \cdots & 1 \end{bmatrix} .$$

The statement that the covariances are (a, b, c) is taken to mean that $\rho_{12} = a$, $\rho_{13} = b$, and $\rho_{23} = c$. Similarly, the statement that the covariances are (a, b, c, d, e, f) means that $\rho_{12} = a$, $\rho_{13} = b$, $\rho_{14} = c$, $\rho_{23} = d$, $\rho_{24} = e$, and $\rho_{34} = f$. All matrices used were positive definite.

For each covariance matrix studied, two types of mean vectors were used. First the null mean vector was used to estimate α, the probability of a Type I error. The second type was generally of the form (0, 0, m) or (0, 0, 0, m). The values of m used were 0.5, 1.0, and 2.0. In some cases only 1 or 2 of these mean vectors were used. The second type of mean vector was used to study the power of the F-test. In this study power is defined as the probability of rejecting the null hypothesis (equality of treatment means) when in fact the null hypothesis is not true.

In all cases analyses were performed with 3, 6, 10, and 15 observations in each treatment group. The same data sets were used to estimate the percent significant at α-levels of both .01 and .05 for all cases.

It should be noted that, in the case of a uniform covariance matrix, for a given mean vector, the power of the test changes as ρ changes. In order to make statements as to whether the power of the test is altered by nonuniformity of the covariance matrix when compared to the uniform case, empirical determination of power was made for a variety of uniform covariance matrices. The null mean vector together with the uniform covariance matrices were used to judge the precision and accuracy of the method in estimating α-levels.

CHAPTER III

RESULTS AND ANALYSIS

Examination of the tables and graphs in this chapter is more meaningful after consideration of the sampling variation. In the offset graphs in Figure 1, for the indicated uniform matrices and the null mean vector, percent significant is shown at $\alpha = .01$ and $\alpha = .05$ for each value of n. Since the off-diagonal elements are all equal in a uniform matrix, a single covariance value is given to indicate a matrix on the ordinate. The broken lines represent the expected values, 1.0% or 5.0% for each case. Variances computed according to the formula

$$\hat{\sigma}^2 = (100)^2 \sum_{i=1}^{k} (\hat{\alpha}_i - \bar{\hat{\alpha}})^2/k - 1$$

are given in Table 1 for each value of α and n.

TABLE 1

VARIANCES FOR 10 UNIFORM CASES

	n = 3	n = 6	n = 10	n = 15
$\alpha = .01$	.0951	.0766	.0582	.0622
$\alpha = .05$	.4423	.4227	.3490	.2929

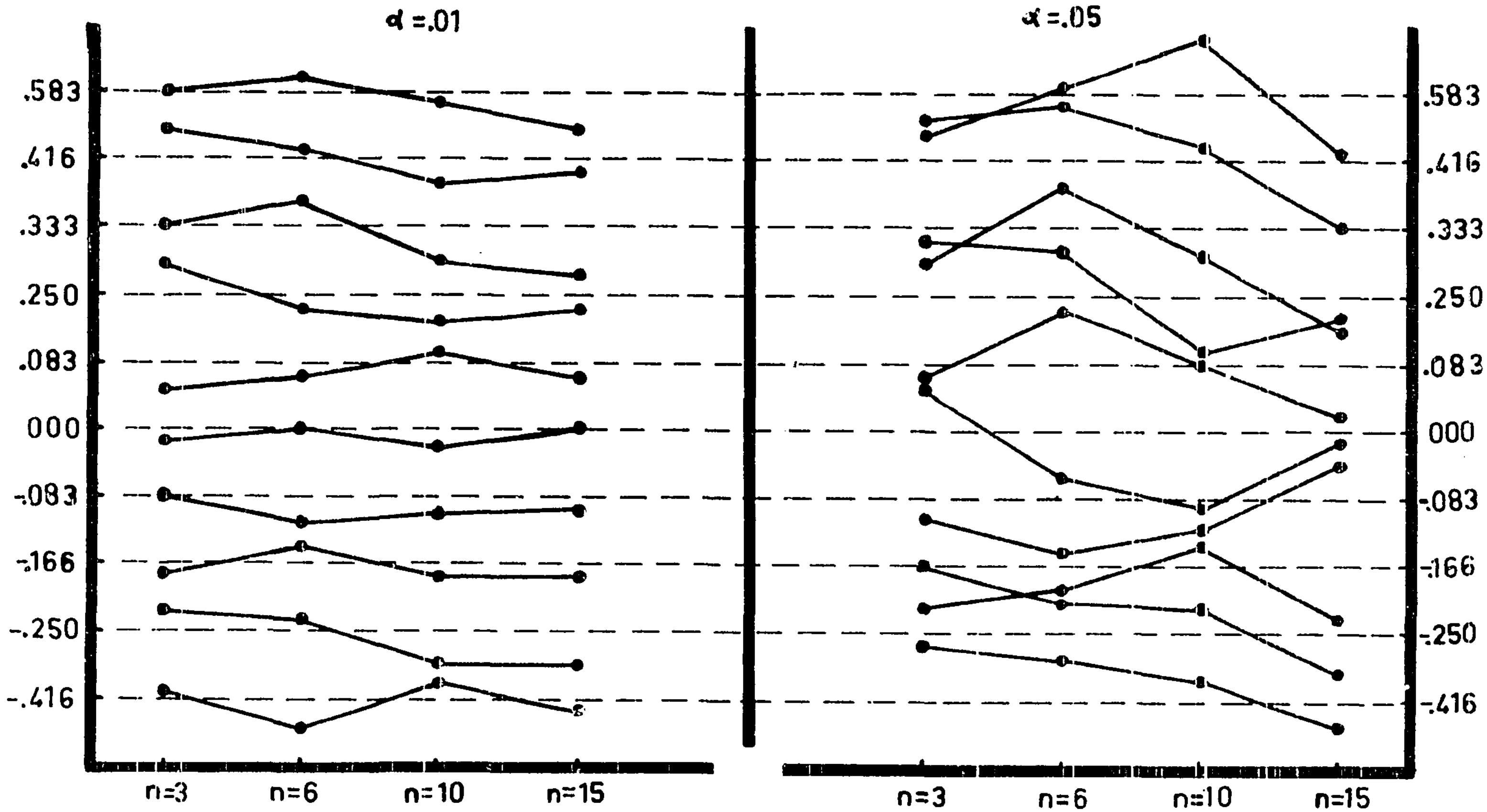


FIGURE 1. VARIATION IN PERCENT SIGNIFICANT FOR UNIFORM MATRICES OF ORDER 3 .

These compare well with expected variances based on the formula

$$\sigma^2 = (100)^2\ pq/1000$$

which gives $\sigma^2 = .099$ when $\alpha = .01$ and $\sigma^2 = .475$ when $\alpha = .05$.

In the power studies, 19 analyses yielding power in the range of .100 to .900 had been duplicated. The differences between duplicates were pooled over mean vectors, α-levels, and sample sizes to give a variance of .00016 for the differences.

To grade the nonuniform matrices as to relative nonuniformity, the statistic given by Box [1950] and mentioned in CHAPTER I was used. The statistic is

$$T = (1-C)M$$

where

$$C = t(t+1)^2\ (2t-3)/6(n-1)(t-1)(t^2+t-4)$$

and

$$M = -(n-1)\ \ell og_e\ (|V_o|/|V|).$$

V_o is a matrix in which all the diagonal elements are equal to the average of the diagonal elements of V and the off-diagonal elements are equal to the average of the off-diagonal elements of V. The degrees of freedom for the statistic are given by $(t^2+t-4)/2$. The values of the statistic for the matrices used in this study are given in Appendix C. Note that increased differences among the covariances do not account entirely for the increased nonuniformity.

In the power studies, a covariance matrix where

$\rho = (a+b+c)/3$

was chosen as the uniform case against which to compare a nonuniform case with covariances (a, b, c). The uniform matrices when t = 4 were determined analogously. The uniform matrix for each nonuniform matrix used in the study is given in Appendix D.

Results are presented for covariance matrices of order 3 first. For matrices of orders 3 or 4, power studies are presented before robustness for the α-levels is shown.

Some results of comparing power for nonuniform cases to power for the appropriate uniform cases are seen in Tables 2 through 9. In each of the tables a mean vector is specified and the empirically determined power is given for a group of nonuniform covariances for each sample size and value of α. The matrices within each table are tabulated in order of increasing nonuniformity.

In Tables 2, 3, and 4, the covariances are all of the form (a, a, a $\pm$ .25) but with various values of a. In Tables 5, 6, and 7, the covariances take the form (a, a, a $\pm$ .5) and the range in location is larger. In Tables 8 and 9, the discrepancies among the covariances are even greater. While differences in power are seen in all the tables, a noticeable and fairly consistent increase appears in Tables 8 and 9.

For the mean vectors shown, power is in general greater for the nonuniform cases than for the uniform cases. However, note that in each case the higher mean value belongs

TABLE 2

POWER FOR COVARIANCES (a, a, a ± .25); μ = (0, 0, .5)

	NONUNIFORM		UNIFORM	
		n = 3		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.25)	.016	.065	.017	.088
(.25,.25,.5)	.022	.077	.017	.073
(.5,.5,.75)	.025	.107	.028	.101
		n = 6		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.25)	.028	.133	.032	.109
(.25,.25,.5)	.040	.137	.038	.144
(.5,.5,.75)	.079	.199	.064	.215
		n = 10		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.25)	.048	.180	.044	.182
(.25,.25,.5)	.081	.250	.068	.235
(.5,.5,.75)	.140	.359	.141	.370
		n = 15		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.25)	.081	.260	.103	.256
(.25,.25,.5)	.153	.368	.163	.367
(.5,.5,.75)	.280	.560	.319	.577

TABLE 3

POWER FOR COVARIANCES (a, a, a ± .25); μ = (0, 0, 1)

	NONUNIFORM		UNIFORM	
		n = 3		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,-.25)	.027	.126	.027	.111
(0.0,0.0,.25)	.029	.132	.033	.140
(.25,.25,.5)	.040	.164	.047	.175
(.5,.5,.75)	.080	.265	.081	.258
		n = 6		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,-.25)	.110	.307	.093	.281
(0.0,0.0,.25)	.120	.330	.137	.325
(.25,.25,.5)	.182	.458	.177	.450
(.5,.5,.75)	.361	.703	.348	.668
		n = 10		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,-.25)	.260	.490	.224	.500
(0.0,0.0,.25)	.330	.601	.305	.591
(.25,.25,.5)	.489	.766	.469	.747
(.5,.5,.75)	.759	.933	.735	.931
		n = 15		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,-.25)	.489	.731	.461	.713
(0.0,0.0,.25)	.550	.788	.599	.834
(.25,.25,.5)	.773	.934	.767	.923
(.5,.5,.75)	.969	.996	.952	.993

TABLE 4

POWER FOR COVARIANCES (a, a, a ± .25); μ = (0, 0, 2)

	NONUNIFORM		UNIFORM	
		n = 3		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.25)	.132	.416	.120	.421
(.25,.25,.5)	.183	.542	.208	.537
(.5,.5,.75)	.360	.769	.324	.745
		n = 6		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.25)	.652	.899	.663	.905
(.25,.25,.5)	.844	.988	.843	.968
(.5,.5,.75)	.981	.999	.962	.999
		n = 10		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.25)	.968	.997	.966	.997
(.25,.25,.5)	.998	.999	.996	.999
(.5,.5,.75)	1.000	1.000	1.000	1.000
		n = 15		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.25)	1.000	1.000	1.000	1.000
(.25,.25,.5)	1.000	1.000	1.000	1.000
(.5,.5,.75)	1.000	1.000	1.000	1.000

TABLE 5

POWER FOR COVARIANCES (a, a, a ± .5); μ = (0, 0, .5)

	NONUNIFORM		UNIFORM	
	$n = 3$			
	$\alpha = .01$	$\alpha = .05$	$\alpha = .01$	$\alpha = .05$
(-.25,-.25,.25)	.007	.066	.014	.071
(.25,.25,.75)	.030	.099	.023	.098
	$n = 6$			
	$\alpha = .01$	$\alpha = .05$	$\alpha = .01$	$\alpha = .05$
(-.25,-.25,.25)	.020	.112	.023	.088
(.25,.25,.75)	.052	.153	.040	.155
	$n = 10$			
	$\alpha = .01$	$\alpha = .05$	$\alpha = .01$	$\alpha = .05$
(-.25,-.25,.25)	.053	.180	.053	.187
(.25,.25,.75)	.107	.258	.090	.243
	$n = 15$			
	$\alpha = .01$	$\alpha = .05$	$\alpha = .01$	$\alpha = .05$
(-.25,-.25,.25)	.070	.231	.093	.267
(.25,.25,.75)	.162	.385	.194	.416

TABLE 6

POWER FOR COVARIANCES (a, a, a ± .5); μ = (0, 0, 1)

	NONUNIFORM		UNIFORM	
		n = 3		
	α = .01	α = .05	α = .01	α = .05
(-.25,-.25,.25)	.033	.127	.027	.111
(-.25,-.25,-.75)	.029	.111	.034	.123
(.25,.25,.75)	.065	.232	.040	.181
		n = 6		
	α = .01	α = .05	α = .01	α = .05
(-.25,-.25,.25)	.100	.292	.093	.281
(-.25,-.25,-.75)	.072	.220	.068	.218
(.25,.25,.75)	.223	.529	.217	.537
		n = 10		
	α = .01	α = .05	α = .01	α = .05
(-.25,-.25,.25)	.271	.545	.224	.500
(-.25,-.25,-.75)	.173	.412	.186	.429
(.25,.25,.75)	.561	.864	.537	.800
		n = 15		
	α = .01	α = .05	α = .01	α = .05
(-.25,-.25,.25)	.460	.745	.461	.713
(-.25,-.25,-.75)	.346	.584	.355	.638
(.25,.25,.75)	.854	.983	.838	.963

TABLE 7

POWER FOR COVARIANCES (a, a, a ± .5); μ = (0, 0, 2)

	NONUNIFORM		UNIFORM	
		n = 3		
	α = .01	α = .05	α = .01	α = .05
(-.25,-.25,.25)	.096	.385	.107	.377
(.25,.25,.75)	.263	.621	.240	.603
		n = 6		
	α = .01	α = .05	α = .01	α = .05
(-.25,-.25,.25)	.585	.867	.548	.831
(.25,.25,.75)	.900	.995	.411	.908
		n = 10		
	α = .01	α = .05	α = .01	α = .05
(-.25,-.25,.25)	.938	.994	.924	.991
(.25,.25,.75)	1.000	1.000	.998	.999
		n = 15		
	α = .01	α = .05	α = .01	α = .05
(-.25,-.25,.25)	1.000	1.000	1.000	1.000
(.25,.25,.75)	1.000	1.000	1.000	1.000

TABLE 8

POWER FOR COVARIANCES (a, a, b); μ = (0, 0, .5)

	NONUNIFORM		UNIFORM	
		n = 3		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.75)	.019	.095	.019	.087
(.05,.05,.9)	.041	.116	.017	.073
		n = 6		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.75)	.053	.155	.032	.130
(.05,.05,.9)	.080	.176	.038	.144
		n = 10		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.75)	.069	.203	.066	.227
(.05,.05,.9)	.093	.213	.068	.235
		n = 15		
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,.75)	.125	.328	.135	.321
(.05,.05,.9)	.153	.369	.163	.367

TABLE 9

POWER FOR COVARIANCES (a, a, b); μ = (0, 0, 1)

	NONUNIFORM		UNIFORM	
	n = 3			
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,-.75)	.038	.139	.032	.142
(-.5,-.5,.5)	.042	.134	.031	.129
(0.0,0.0,.75)	.060	.183	.033	.162
(.05,.05,.9)	.086	.264	.047	.175
	n = 6			
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,-.75)	.092	.271	.075	.245
(-.5,-.5,.5)	.092	.255	.079	.249
(0.0,0.0,.75)	.170	.445	.163	.431
(.05,.05,.9)	.226	.517	.177	.450
	n = 10			
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,-.75)	.198	.444	.210	.444
(-.5,-.5,.5)	.225	.486	.235	.469
(0.0,0.0,.75)	.421	.744	.422	.699
(.05,.05,.9)	.495	.826	.469	.747
	n = 15			
	α = .01	α = .05	α = .01	α = .05
(0.0,0.0,-.75)	.395	.620	.393	.635
(-.5,-.5,.5)	.475	.752	.461	.707
(0.0,0.0,.75)	.713	.923	.673	.882
(.05,.05,.9)	.807	.977	.767	.923

to a treatment group which has the more extreme covariance with another treatment group. To study the effect of permuting the means, two covariance matrices were selected. One matrix, with covariances (.5, .5, .75), is moderately nonuniform and the other matrix, with covariances (0.0, 0.0, .75), is highly nonuniform. The new mean vectors are (1,0,0) and (2,0,0). In Figures 2 and 3, the result of comparing the uniform case with both permutations of the two mean vectors is shown for α = .01 and .05. For the moderately nonuniform case in Figure 2, it appears that placing the higher mean with a treatment group that has the lower covariance with the other groups results in a decrease in power. For the highly nonuniform case in Figure 3, the same is true for $n = 10$ and $n = 15$, but power continues to be greater when $n = 3$. For $n = 6$, the results are mixed.

Heretofore, only matrices with covariances of the form (a, a, b) have been shown. The behavior of power for matrices with covariances of the form (a, b, c) is shown in Figure 4 for α = .01 and in Figure 5 for α = .05. In both figures the mean vector is (0,0,1). Here a and c are constant, while b varies. These figures show that in general the nonuniform case has greater power and that the discrepancy increases as the nonuniformity of the covariance matrix increases. For comparison, the case with zero covariances is also included.

Attention is given now to the robustness of α-levels for covariance matrices of order 3. In Figures 6, 7, 8, and

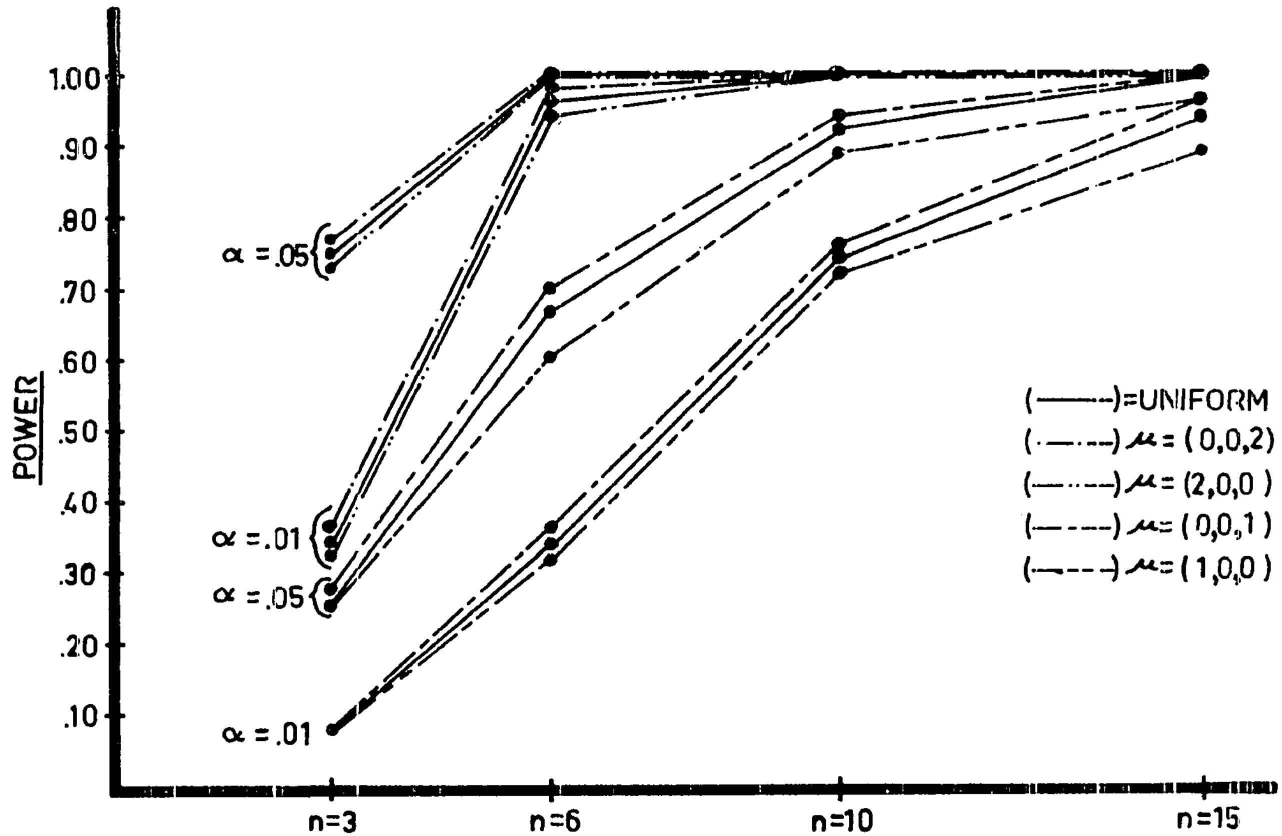


FIGURE 2. EFFECT OF PERMUTING THE MEANS; COVARIANCES= (.5, .5, .75).

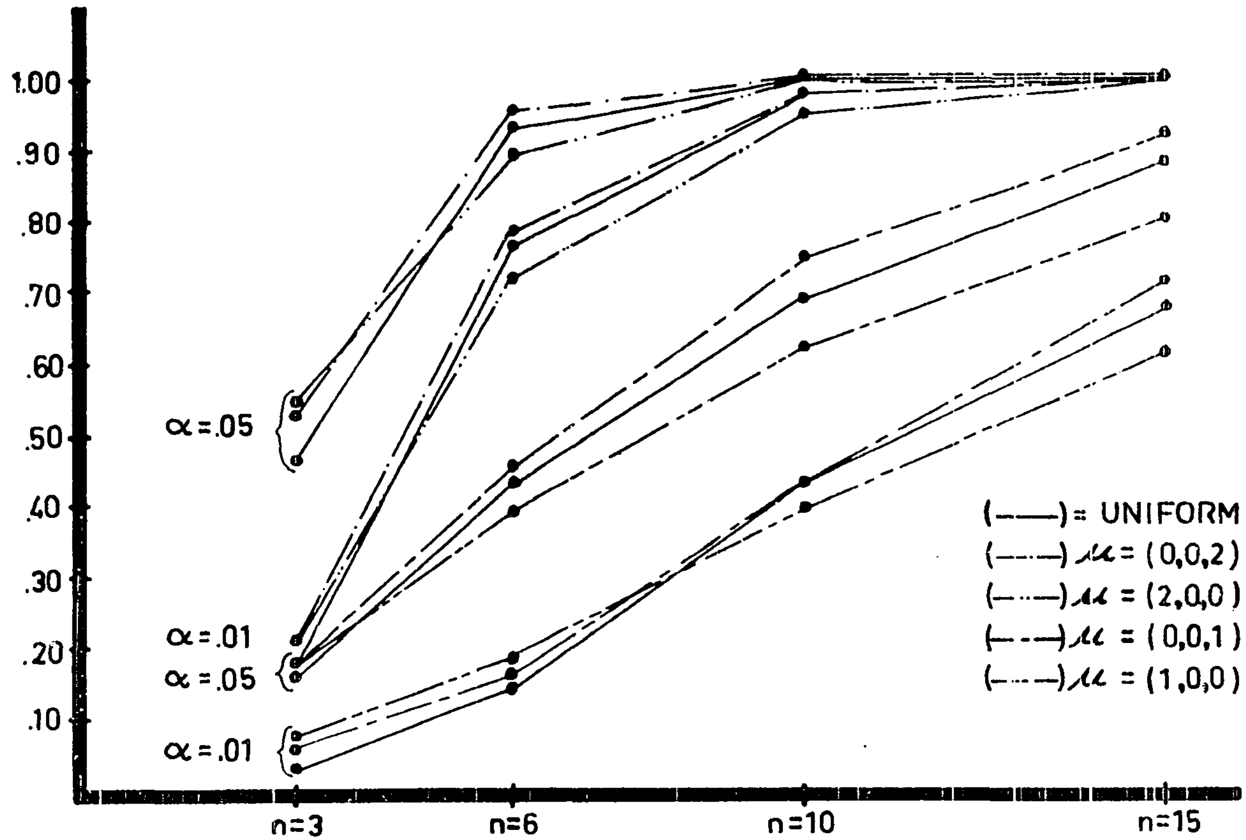
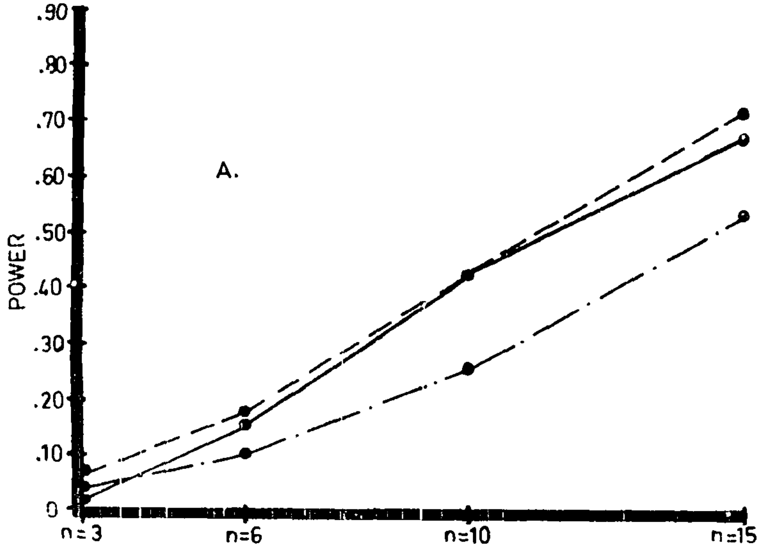


FIGURE 3. EFFECT OF PERMUTING THE MEANS ; COVARIANCES = (0.0, 0.0, .75).

FIGURE 4. POWER CURVES FOR MATRICES OF THE FORM (a,b,c) ; μ = (0,0,1) ; α = .01 .



— — — NON-UNIFORM
——— UNIFORM
— · — NO COVARIANCE
A. = (0.0, 0.0, .75)
B. = (0.0, .25, .75)
C. = (0.0, .5, .75)

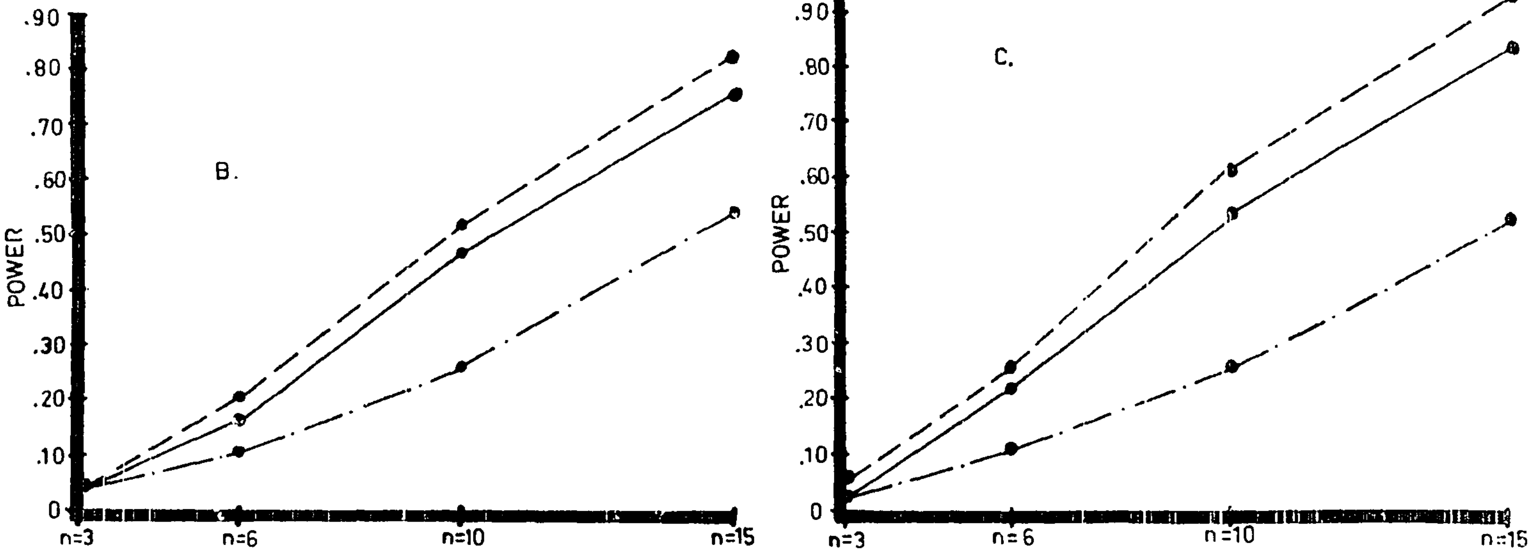
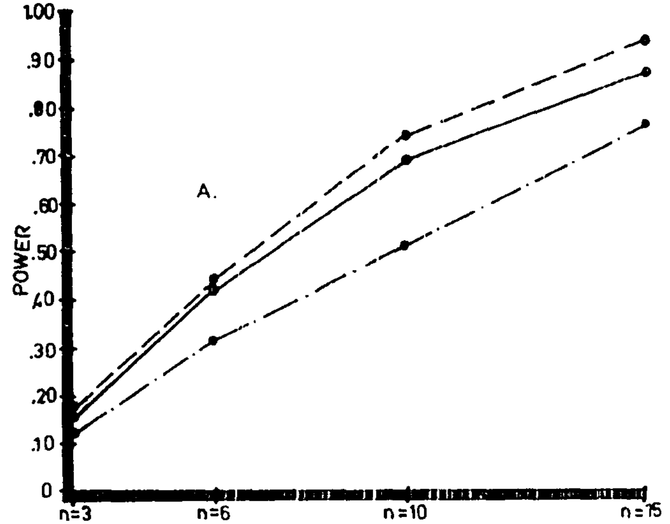
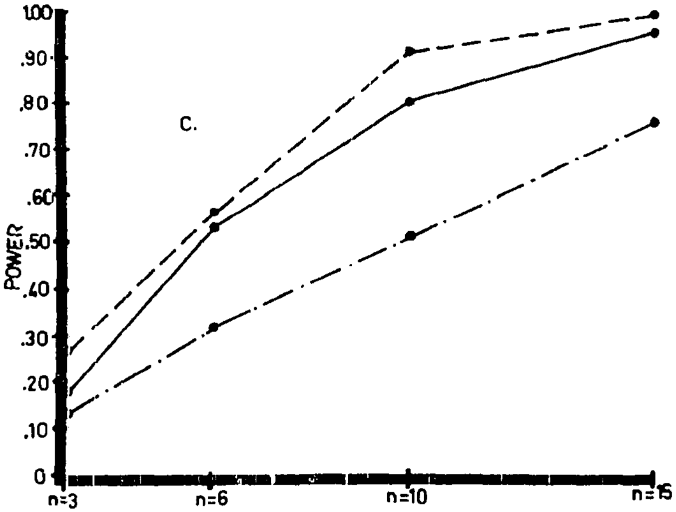
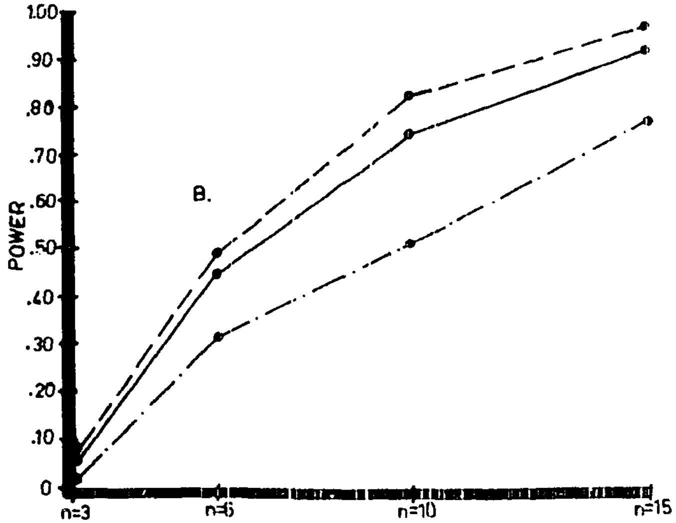


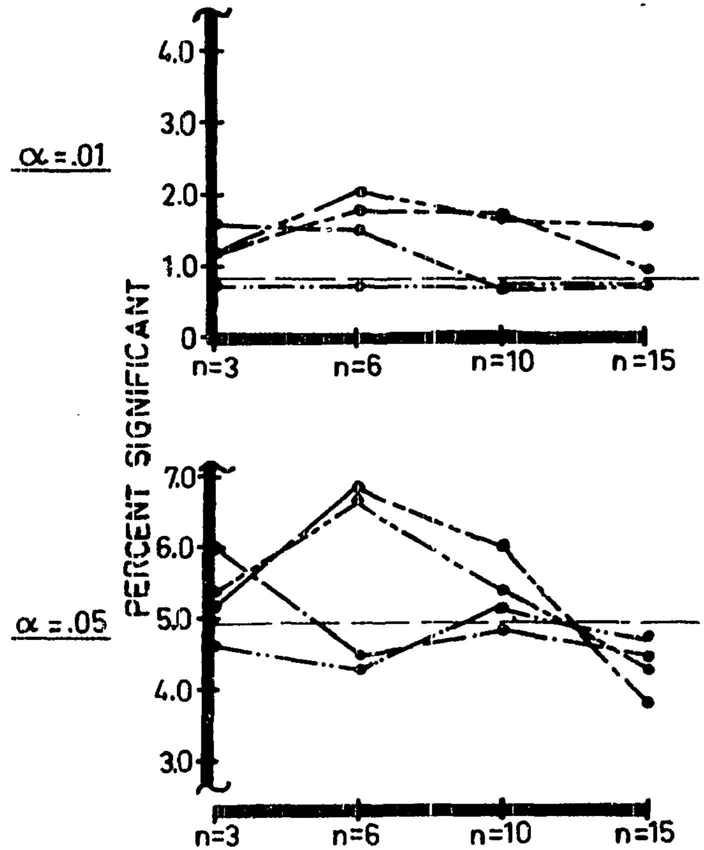
FIGURE 5. POWER CURVES FOR MATRICES OF THE FORM (a,b,c); $\mu=(0,0,1)$; $\alpha=.05$.

— — — NON-UNIFORM
———— UNIFORM
—·— NO COVARIANCE
A. = (0.0, 0.0, .75)
B. = (0.0, .25, .75)
C. = (0.0, .5, .75)



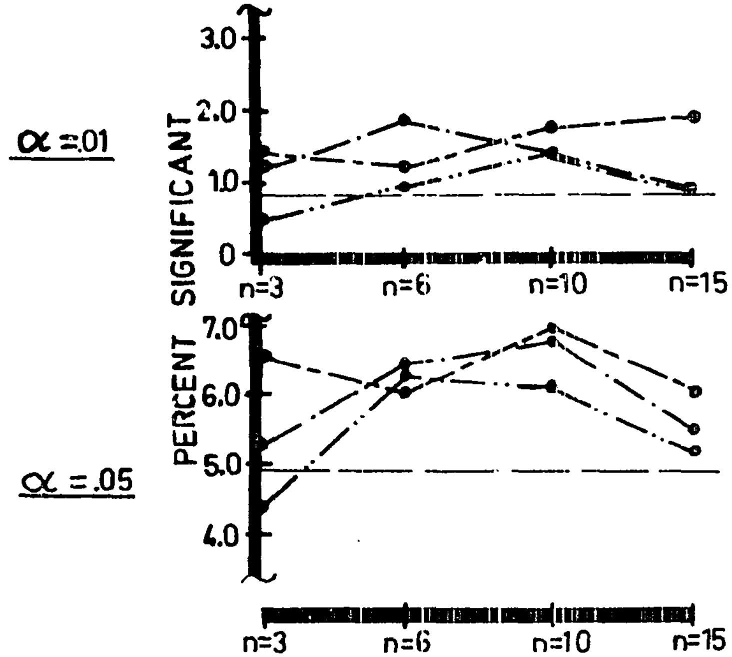






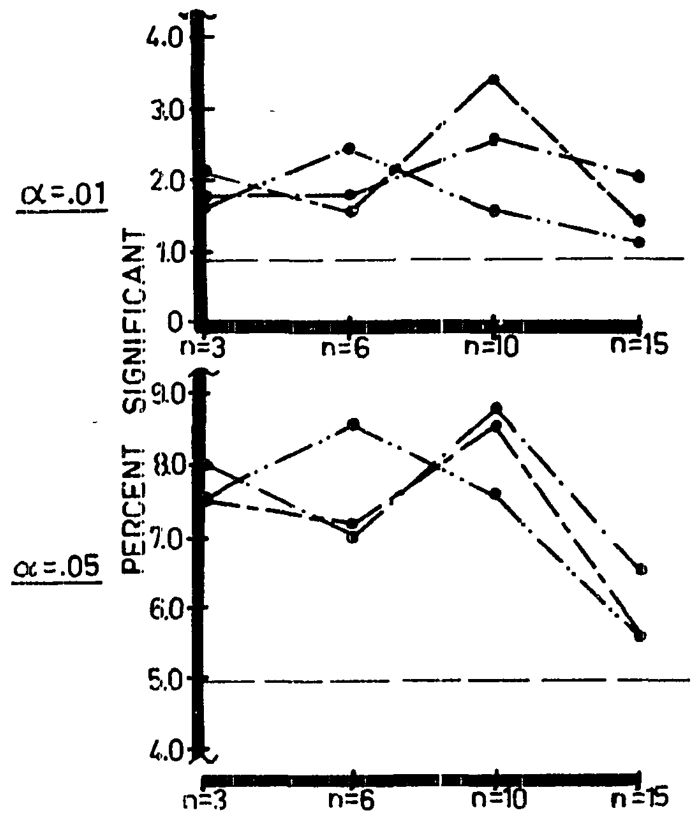
(—·—)=(0.0,0.0,-.25) (—··—)=(0.0, 0.0, .25)
(—–—)=(.25,.25, .5) (—---)=(.5, .5 , .75)
—— —— EXPECTED

FIGURE 6. PERCENT SIGNIFICANT FOR COVARIANCES= (a,a,a ± .25); μ =(0,0,0).



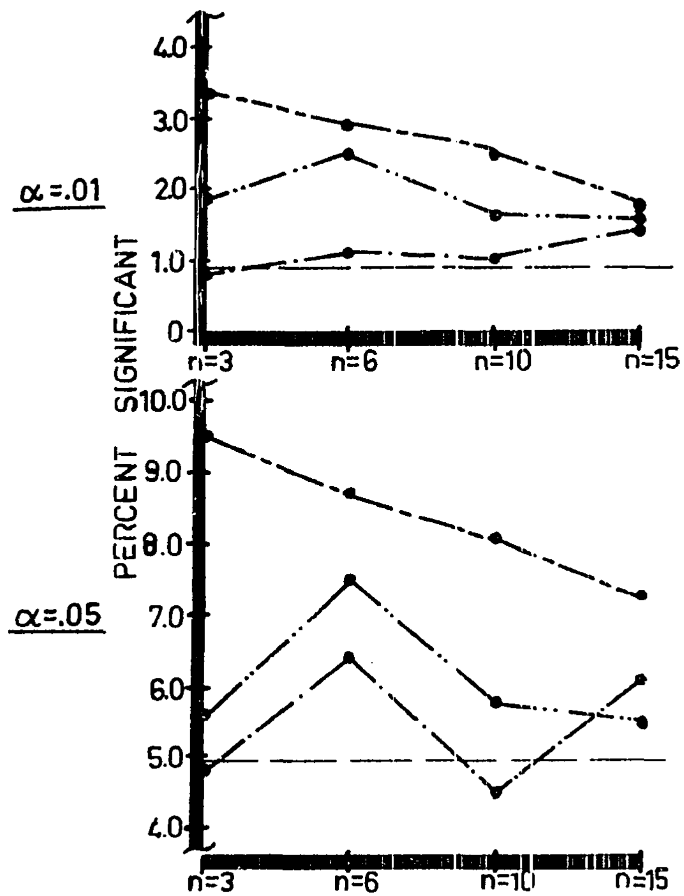
(—·—)=(-.25,-.25, .25)
(—··—)=(-.25,-.25,-.75)
(—–—)=(.25, .25, .75)
—— —— EXPECTED

FIGURE 7. PERCENT SIGNIFICANT FOR COVARIANCES= (a,a,a ± .5); μ = (0,0,0).



(---·—)=(0.0 , 0.0 , .75)
(—··—)=(0.0, .25 , .75)
(— — —)=(0.0, .5 , .75)
—— —— EXPECTED

FIGURE 8. PERCENT SIGNIFICANT FOR COVARIANCES = (a,b,c); μ = (0,0,0).



(—·—)=(0.0, 0.0, -.75) (—··—)=(.05, .05, .9)
(—··—)=(-.5, -.5, .5) —— —— EXPECTED

FIGURE 9. PERCENT SIGNIFICANT FOR COVARIANCES = (a,a,b); μ = (0,0,0).

9, for the indicated nonuniform matrices and the null mean vector, percent significant is shown at α = .01 and α = .05 for each value of n. The expected values shown in each figure were obtained by averaging the means for the 10 uniform cases in Figure 1 over sample size. For several matrices in each figure, the estimated α-levels are considerably higher than the levels obtained for the uniform matrices.

Since the estimated α-levels should not be dependent on sample size, the values were averaged over sample size for each nonuniform matrix. These averages are shown in Figure 10, in order of increasing nonuniformity of the covariance matrices. The average values over all sample sizes for all the uniform matrices are indicated by broken lines. It is clear that percent significant tends to increase as nonuniformity increases.

Power for cases with nonuniform matrices of order 4 was first examined by varying the range of the covariances while the average covariance remained constant. The results are shown in Figure 11. Little difference in power is seen, except for some increase for the case with the larger range of covariances, at n = 10 and n = 15. For comparison, power for the case with zero covariances is also shown in Figure 11.

In Figures 12 and 13, the matrices all have the same uniform covariance matrix and range, but the extent of nonuniformity is varied. In general these nonuniform cases show an increase in power over the uniform. However, the amount of

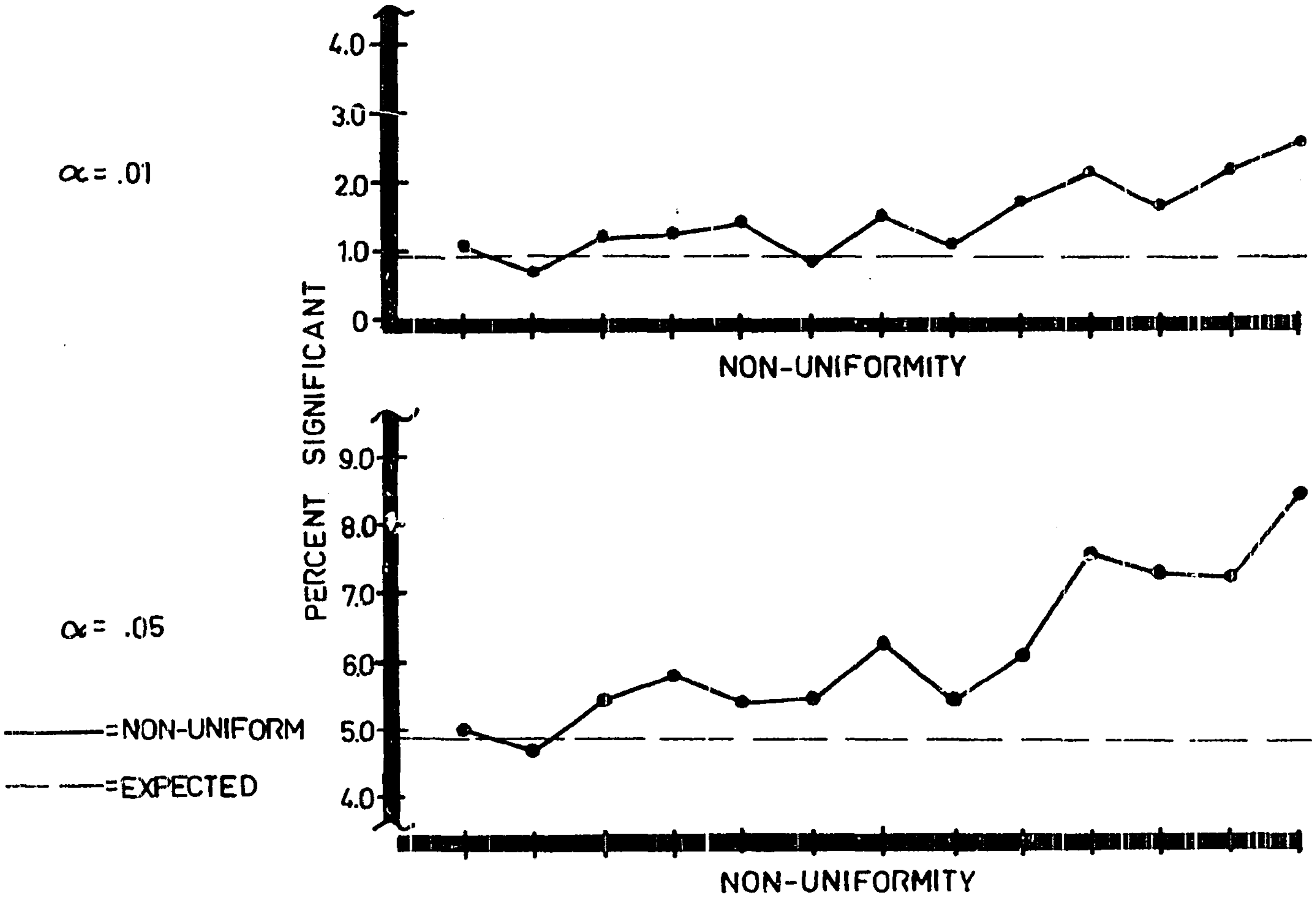


FIGURE 10. PERCENT SIGNIFICANT, AVERAGED OVER SAMPLE SIZE; $\mu = (0,0,0)$.

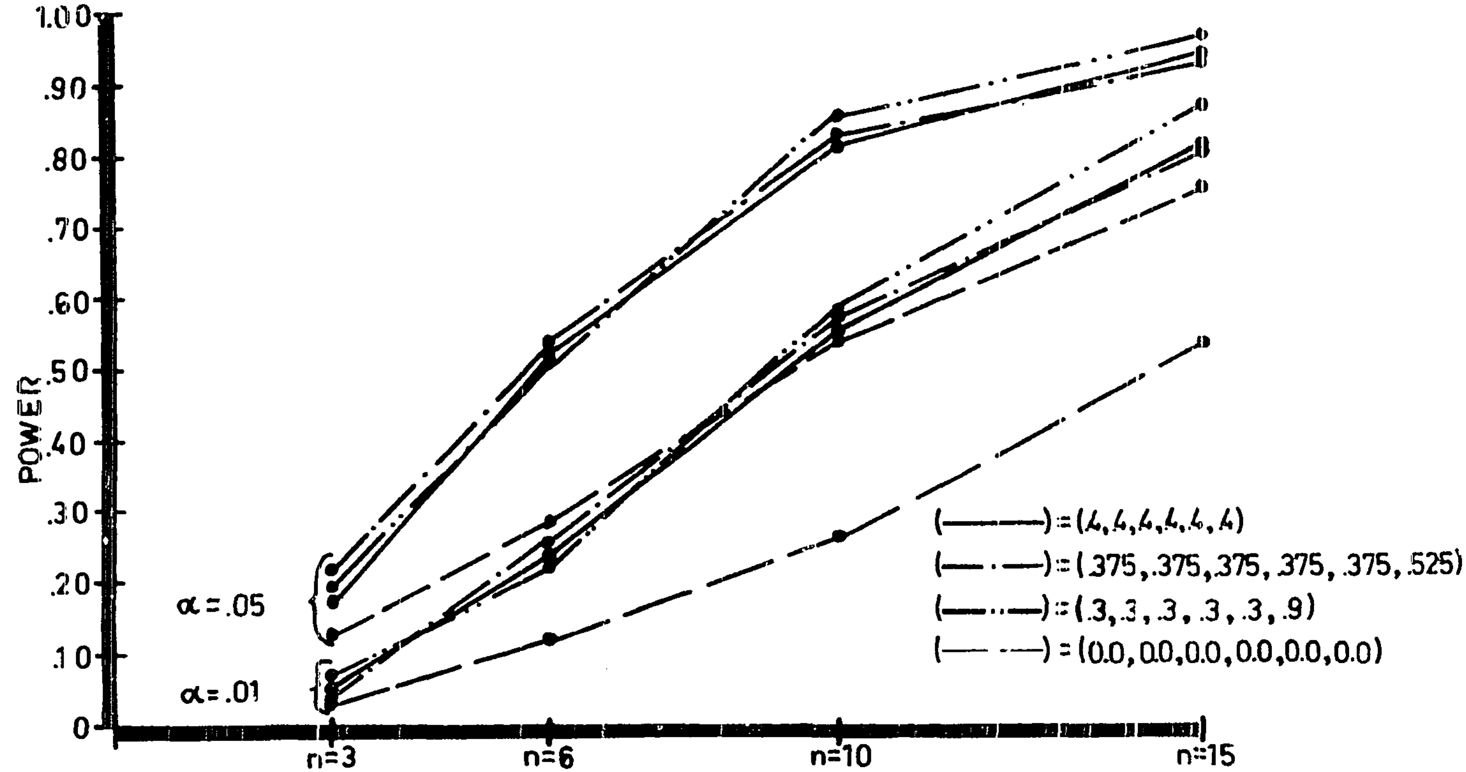


FIGURE 11. EFFECT OF INCREASING RANGE; $\mu=(0,0,0,1)$.

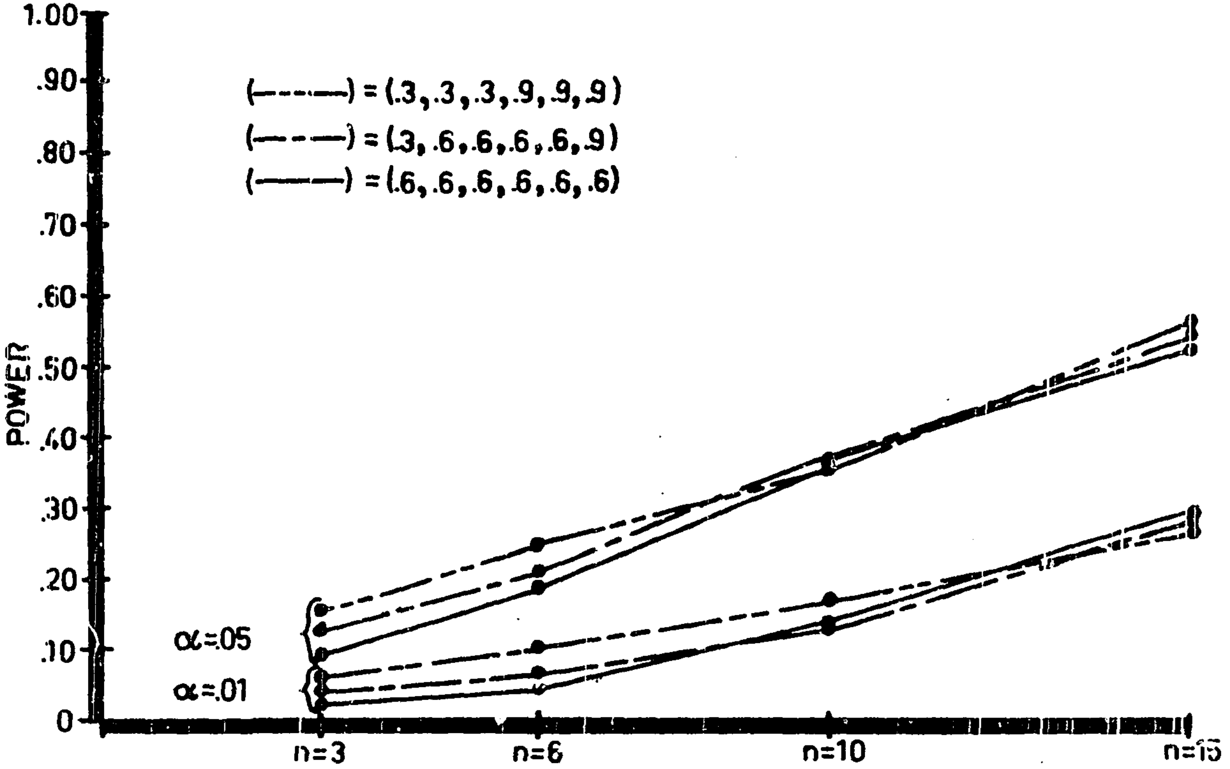


FIGURE 12. POWER FOR A CONSTANT RANGE OF COVARIANCES; μ=(0,0,0,.5).

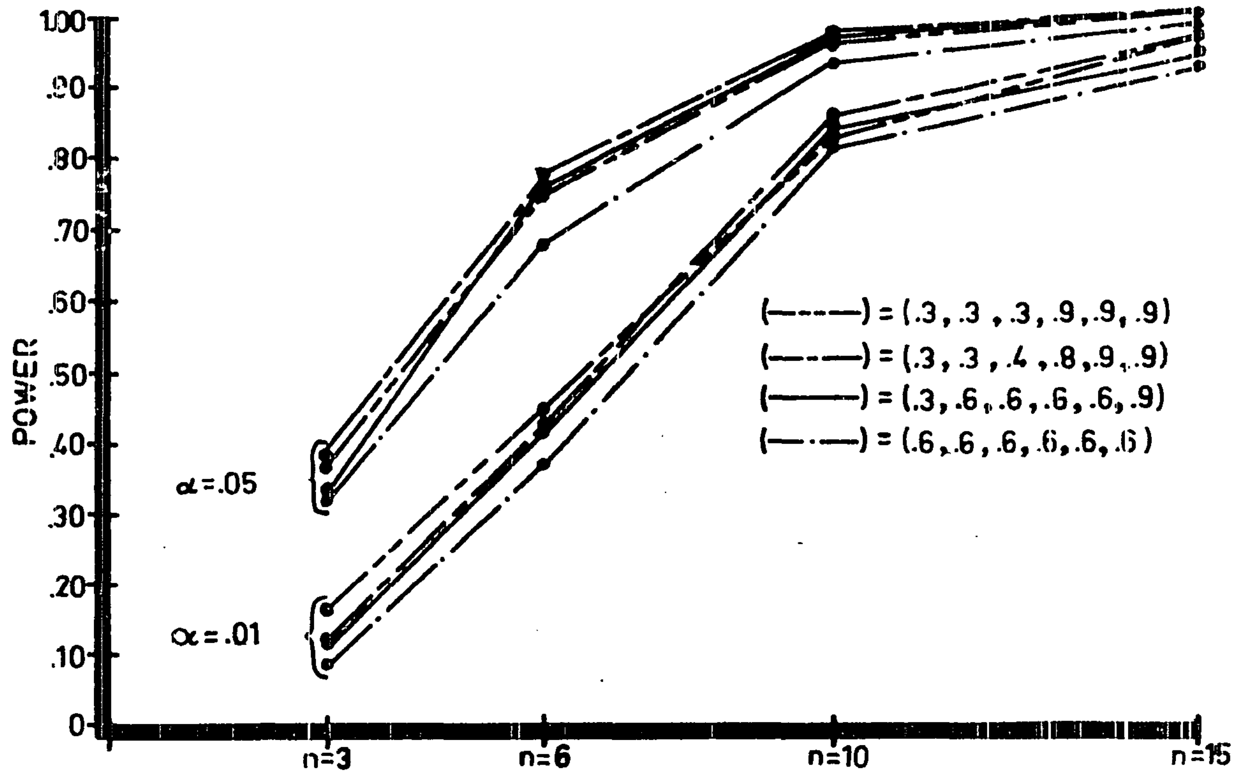


FIGURE 13. POWER FOR A CONSTANT RANGE OF COVARIANCES; $\mu = (0,0,0,1)$.

change does not consistently increase when nonuniformity increases.

The effect of permuting the means is shown in Figure 14 for the moderately nonuniform matrix (.3, .3, .3, .3, .3, .9) and in Figure 15 for the highly nonuniform matrix (.3, .3, .3, .9, .9, .9). For the larger values of n, the power is reduced by the permutation. For the smaller sample sizes, power is still increased when compared to the appropriate uniform case if $\alpha = .01$. When $\alpha = .05$, power continues to be increased only when $n = 3$.

To examine robustness of the α-levels, percent significant is shown in Figure 16 for several cases in order of increasing nonuniformity. Each point represents the average over sample size, for $\alpha = .01$ or $.05$. The broken lines represent average estimates of the α-levels obtained. It is apparent that use of the criterion for the usual F-test results in underestimation of the correct α-level, even for only moderately nonuniform cases.

The results of using the conservative test given by Greenhouse and Geisser [1958] are shown in Tables 10 and 11. Table 10 shows the reduction in power from the usual test which is obtained. Table 11 shows the reduction in average (over sample size) percent significant obtained for several cases when the mean is the null vector.

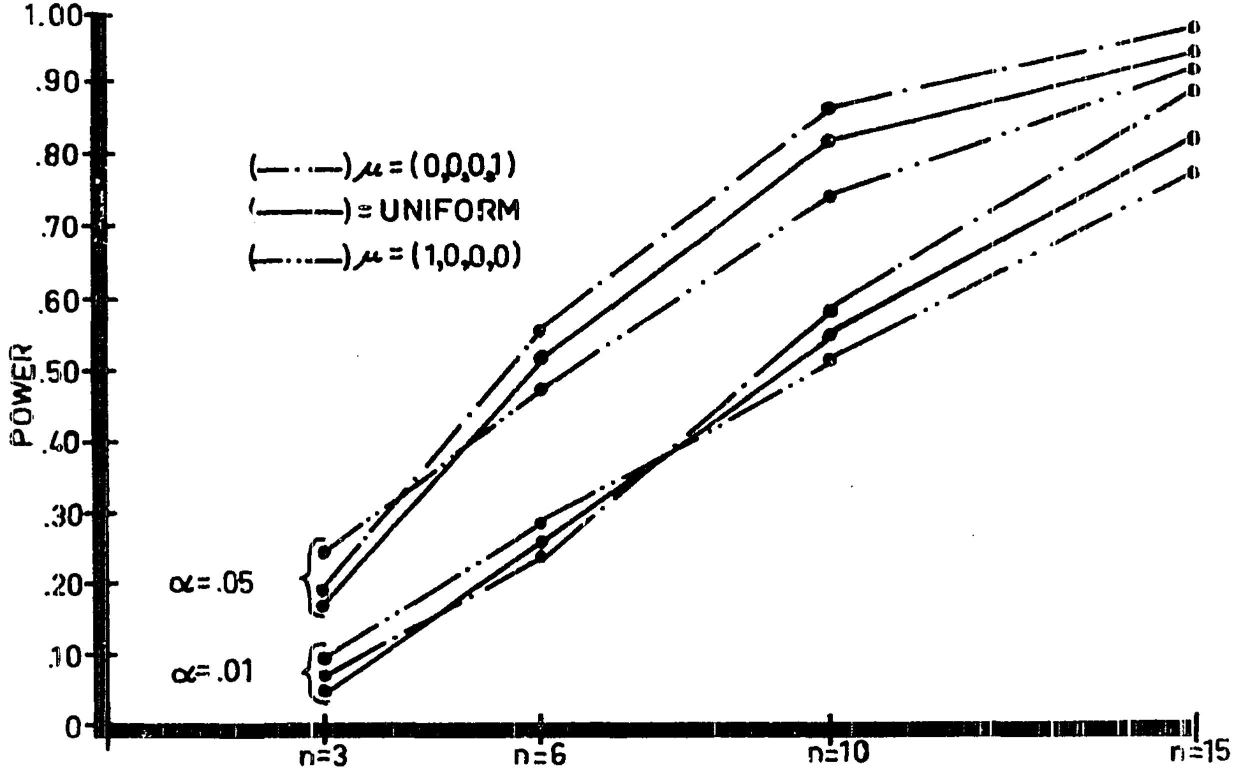


FIGURE 14. EFFECT OF PERMUTING THE MEANS; COVARIANCES = (.3,.3,.3,.3,.3,.9).

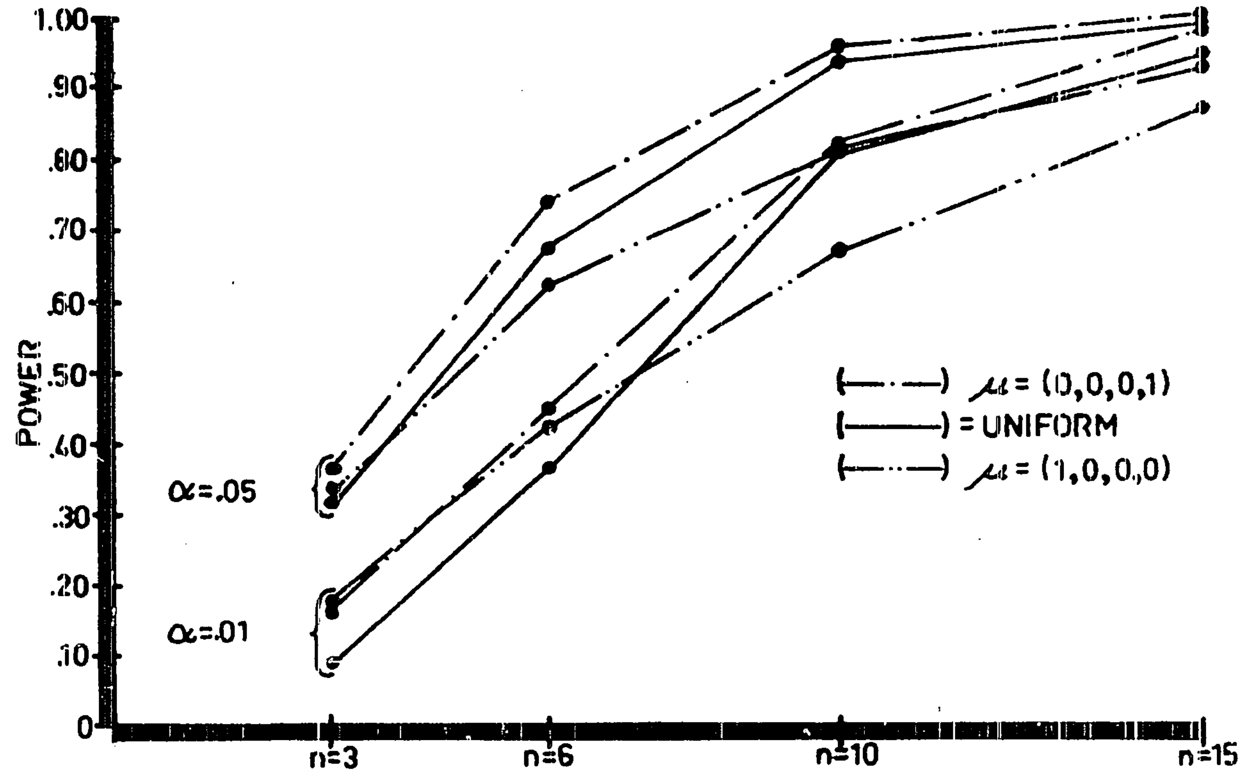
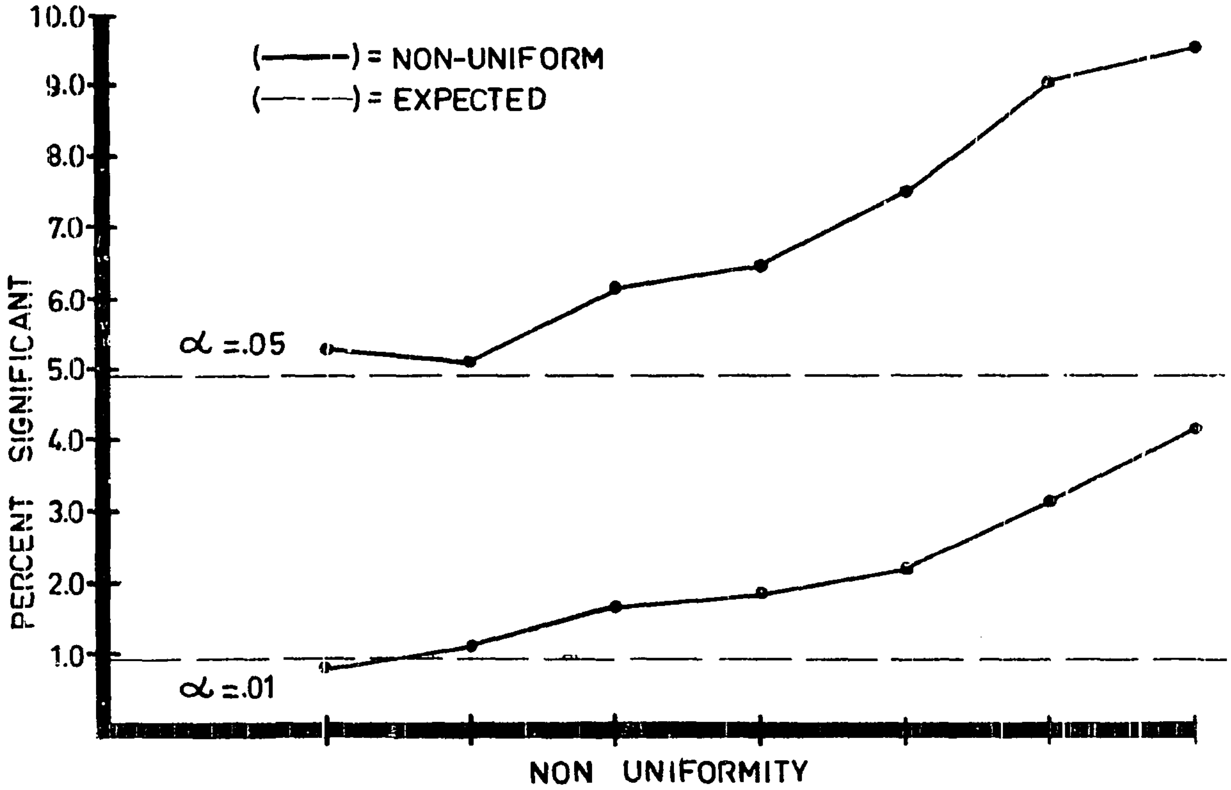


FIGURE 15. EFFECT OF PERMUTING THE MEANS; COVARIANCES = (.3,.3,.3,.9,.9,.9).



<u>FIGURE 16.</u> PERCENT SIGNIFICANT, AVERAGED OVER SAMPLE SIZE; μ = (0,0,0,0).

TABLE 10

COMPARISON OF POWER; COVARIANCES = (.25, .25, .75); μ = (0, 0, 1)

	n = 3		n = 6		n = 10		n = 15	
	α=.01	α=.05	α=.01	α=.05	α=.01	α=.05	α=.01	α=.05
Usual Test	.065	.232	.223	.529	.561	.864	.854	.983
Conservative Test	.032	.064	.049	.288	.216	.673	.535	.924

TABLE 11

COMPARISON OF PERCENT SIGNIFICANT: μ = (0, 0, 0) OR (0, 0, 0, 0)

	(.25,.25,.75)		(.05,.05,.9)		(.3,.6,.6, .6,.6,.9)		(.3,.3,.3, .9,.9,.9)	
	α= 1%	α= 5%	α= 1%	α= 5%	α= 1%	α= 5%	α= 1%	α= 5%
Usual Test	1.550	6.325	2.625	8.400	1.800	6.450	4.100	9.500
Conservative Test	.250	2.025	.300	3.400	.000	1.000	.400	2.775

CHAPTER IV

DISCUSSION AND CONCLUSIONS

The results of the power studies presented in Chapter III show that nonuniformity of the covariance matrices may increase or decrease the power of the usual F-test. The direction and amount of change seem to depend on the degree of nonuniformity, the permutation of the means relative to the covariances, the significance level, and the magnitude of t and n. In general, the change in power is not large, and if the usual F-test is used for data in which the assumption of uniformity is untenable, tables for the Non-central Beta distribution [Graybill, 1961] could be used in conjunction with the appropriate uniform case to give a rough approximation of the power of the test. It should be noted that, except for small negative correlations, the F-test is more powerful for correlated data than for uncorrelated data from this design.

The case where $n = 3$ is of particular interest. Here, there is no alternative multivariate procedure, since $n \leq t$ for $t = 3$ or 4. The only exceptionable results obtained for $n = 3$ occurred when the means were permuted (Figures 2, 3, 14, and 15). For the highly nonuniform case when $t = 3$ and for both nonuniform cases when $t = 4$, power continued to be increased when the means were permuted. This also occurred for n = 6 at the lower levels of power.

Before consideration of the effect of nonuniformity on α-levels, the accuracy of the sampling procedure in estimating these probability levels is demonstrated in Table 12. Here, the average percent significant obtained with the ten uniform matrices of order 3 and with the three uniform matrices of order 4 is shown. It is seen that these values tend to decrease as n increases, and the over-all effect is underestimation of α.

TABLE 12

AVERAGE PERCENT SIGNIFICANT FOR UNIFORM MATRICES; $\mu = (0,0,0)$ OR $(0,0,0,0)$

		n = 3	n = 6	n = 10	n = 15	$\bar{\bar{X}}$
t=3	α = 1%	1.080	.990	.760	.700	.883
	α = 5%	5.170	5.240	4.930	4.380	4.930
t=4	α = 1%	.933	1.200	.700	.933	.942
	α = 5%	5.200	5.433	4.533	4.300	4.867

Figures 10 and 16 show the increase in percent significant as nonuniformity increases. A clear-cut increase is seen even for relatively nonuniform cases. The results for n = 3 were essentially the same as results for higher values of n. When t = 3, the most nonuniform case, with covariances (.05, .05, .9), showed 2.625 percent and 8.400 percent significant. When $t = 4$, the most nonuniform case, with covariances (.3, .3, .3, .9, .9, .9), showed 4.100 percent and 9.500 percent significant. These values may be acceptable in many experimental situations.

The results of using the conservative test are shown in Tables 10 and 11. It is evident that use of this test can seriously reduce power and α-levels, even after consideration of the tendency of the sampling method to underestimate the α-levels.

It should be noted that the results given in this paper are applicable only to tests for equality of means when the observations are correlated among treatments. If observations are correlated within treatments, Box [1954b] has shown that severe disturbance in the α-levels may occur, when the underlying model is like the second model given in Chapter III.

In conclusion, it is found that the power of the usual F-test is not significantly affected by nonuniformity of the covariance matrices. Marked changes do occur in the α-levels, but the differences are such that use of the usual F-test may still be acceptable in many instances. If this test is used, the α-level may be estimated by one of the cases presented in this paper. When the usual F-test is used for other cases, it should be noted that the tabulated α-level is too low. In any case, the usual F-test appears to be a desirable alternative to the conservative test.

CHAPTER V

SUMMARY

The repeated measures design analysis of variance is a statistical technique which has wide applicability in medical research. The experimental design models for univariate repeated measures analyses allow for correlation among observations on the same experimental unit; however, one assumption for most of these models is that all the pairwise correlations must be equal. This assumption is not met in many experimental situations, so the standard univariate analyses of variance may not be appropriate. There are no decisive criteria for selection of an alternative analysis, and the standard analyses of variance may be insensitive to violation of this assumption.

The present study is an investigation of the robustness of the standard F-test for equality of treatment means when the observations are correlated among treatments. Computer simulation techniques were used to investigate balanced one-way designs with correlated observations for 3 and 4 treatment groups. The number of observations per treatment group were 3, 6, 10, and 15. A variety of treatment mean vectors were used, and both the power of the test and the stability of the α-levels were investigated.

The results showed that the power of the F-test is altered very little by inequality of the covariances. The α-levels increase considerably as inequality of the covariances increases. The highest α-levels found were 0.084 and 0.041, when 0.05 and 0.01 respectively were expected.

It was concluded that, for a test of equality of treatment means when the observations among treatments are correlated, the standard analysis of variance may be used if it is noted that the tabulated p-value is too low.

BIBLIOGRAPHY

Bancroft, T. A. 1964 Analysis and Inference for Incompletely Specified Models Involving the Use of Preliminary Test(s) of Significance. Biometrics, 20:3, 427-442.

Box, G. E. P. 1950 Problems in the Analysis of Growth and Wear Curves. Biometrics, 6, 362-389.

Box, G. E. P. 1953 Non-normality and Tests on Variance. Biometrika, 40, 318-335.

Box, G. E. P. 1954a Some Theorems on Quadratic Forms Applied in the Study of Analysis of Variance Problems, I. Effect of Inequality of Variance in the One-way Classification. Annals of Mathematical Statistics, 25, 290-302.

Box, G. E. P. 1954b Some Theorems on Quadratic Forms Applied in the Study of Analysis of Variance Problems, II. Effects of Inequality of Variance and of Correlation Between Errors in the Two-way Classification. Annals of Mathematical Statistics, 25, 484-498.

Box, G. E. P., and Andersen, S. L. 1955 Permutation Theory in the Derivation of Robustness Criteria, and Studies of Departures from Assumption. Journal of the Royal Statistical Society, Series B, 17:1, 1-34.

Box, G. E. P., and Muller, M. E. 1958 A Note on the Generation of Random Normal Deviates. Annals of Mathematical Statistics, 29, 610-611.

Cole, J. W. L., and Grizzle, J. E. 1966 Applications of Multivariate Analysis of Variance to Repeated Measurements Experiments. Biometrics, 22, 810-828.

Danford, M. B. and Hughes, H. M. 1957 Mixed Model Analysis of Variance, Assuming Equal Variances and Covariances. Report 57-144, Randolph Air Force Base.

Danford, M. B., Hughes, H. M., and McNee, R. C. 1960 On the Analysis of Repeated-Measurements Experiments. Biometrics, 16, 547-565.

Geisser, S., and Greenhouse, S. W. 1958 An Extension of Box's Results on the Use of the F Distribution in Multivariate Analysis. Annals of Mathematical Statistics, 29, 885-891.

Graybill, F. A. 1961 An Introduction to Linear Statistical Models, Volume I. McGraw-Hill Book Company, New York.

Greenhouse, S. W. and Geisser, S. 1959 On Methods in the Analysis of Profile Data. Psychometrika, 24:2, 95-112.

International Business Machines Corporation 1966 IBM Application Program, 1130 Scientific Subroutine Package (1130-CM-02X) Programmer's Manual, 60.

Jansson, B. 1966 Random Number Generators. Victor Pettersons Bokindustri Aktiebolag, Stockholm.

Marcus, M. 1960 Basic Theorems in Matrix Theory. National Bureau of Standards Applied Mathematics Series, 57, 4-5.

Muller, M. E. 1959 A Comparison of Methods for Generating Normal Deviates on Digital Computers. Journal of the Association for Computing Machinery, 6:3, 376-383.

Scheffé, H. 1956 A Mixed Model for the Analysis of Variance. Annals of Mathematical Statistics, 27, 23-36.

Scheuer, E. M., and Stoller, D. S. 1962 On the Generation of Normal Random Vectors. Technometrics, 4:2, 278-281.

Steel, R. G. D., and Torrie, J. H. 1960 Principles and Procedures of Statistics. McGraw-Hill Book Company, New York.

Winer, B. J. 1962 Statistical Principles in Experimental Design. McGraw-Hill Book Company, New York.

APPENDIX A

Distribution of the Test Statistic for the Uniform Case

Distribution of the Test Statistic for the Uniform Case

The following method of proof was suggested by Dr. R. B. Deal, Jr.

Consider any model for which the vectors $Y_1, \ldots, Y_n$ are independent and have the same t-dimensional multivariate distribution N(M,V). If

$$Y_j = \begin{bmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{tj} \end{bmatrix}$$

then the distribution for the statistic

$$T = \frac{\sum_{i=1}^{t} \sum_{j=1}^{n} (y_{i.} - y_{..})^2/(t-1)}{\sum_{i=1}^{t} \sum_{j=1}^{n} (y_{ij} - y_{i.} - y_{.j} + y_{..})^2/(n-1)(t-1)}$$

is found by looking at the numerator and denominator separately.

It is convenient to use the Kronecker product $B \otimes C$ of square matrices, $B = (b_{ij})$ for $i,j = 1,\ldots,m$ and $C = (c_{ij})$ for $i,j = 1,\ldots,n$ defined by the mn matrix given in the following n x n blocks:

$$B \otimes C = \begin{bmatrix} b_{11}C & b_{12}C & \cdots & b_{1m}C \\ \vdots & & & \vdots \\ b_{m1}C & \cdots & \cdots & b_{mm}C \end{bmatrix}$$

Elementary properties are listed in Marcus [1960].

Two additional facts needed are that the left distributive law holds and, for the case in which the elements have no divisors of zero, as pertains here, if $B \otimes C = 0_{mn}$, then $B = 0_m$ or $C = 0_n$. This type of product is easily generalized to matrices not necessarily square, and if J_k is the k-dimensional column vector of all 1's, then

$$Y = \begin{bmatrix} Y_i \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ Y_n \end{bmatrix}$$

has the distribution $N(J_n \otimes M, I_n \otimes V)$.

If

$$Y\cdot = \frac{1}{n} \sum_{j=1}^{n} Y_j ,$$

then the quantity

$$\sum_{i=1}^{t} \sum_{j=1}^{n} (y_{i.}-y_{..})^2$$

$$= n(Y\cdot - \frac{1}{t} J_t J_t' Y\cdot)' (Y\cdot - \frac{1}{t} J_t J_t' Y\cdot)$$

$$= nY\cdot' (I_t - \frac{1}{t} J_t J_t') Y\cdot$$

$$= \frac{1}{n} \sum_{r=1}^{n} \sum_{s=1}^{n} Y_r (I_t - \frac{1}{t} J_t J_t') Y_s$$

can be written as

$$Y'E_n \otimes A_t Y$$

where E_n is the idempotent matrix $\frac{1}{n} J_n J_n'$ and A_t is the idempotent matrix (I_t-E_t).

Theorem 4.9 in Graybill [1961] says that if X is distributed N(M,V) then X'BX is distributed as a non-central chi-square $\chi'^2(k,\lambda)$ where k is the rank of B and $\lambda = \frac{1}{2} M'BM$ if and only if BV is idempotent. Now

$$(C \otimes D)\ (U \otimes V) = (CU) \otimes (DV),$$

and it is easy to see that if C is idempotent then $C \otimes D$ is idempotent if and only if D is. Thus

$$(E_n \otimes A_t)\ (I_n \otimes V) = E_n \otimes (A_t V)$$

is idempotent if and only if $A_t V$ is. For the repeated measures model, V is uniform and can be written as

$$\sigma^2(1-\rho)I_t + \sigma^2 \rho t E_t$$

and

$$\begin{aligned} A_t V &= \sigma^2[I-E_t]\ [(1-\rho)\ I_t + \rho t E_t] \\ &= \sigma^2(1-\rho)\ (I_t - E_t) \\ &= \sigma^2(1-\rho)\ A_t. \end{aligned}$$

Except for the constant $\sigma^2(1-\rho)$, this matrix is idempotent.

In the denominator,

$$\begin{aligned} &\sum_{i=1}^{t} \sum_{j=1}^{n} (y_{ij} - y_{i.} - y_{.j} + y_{..})^2 \\ &= \sum_{j=1}^{n} [Y_j - Y.- \tfrac{1}{t} J_t J_t' (Y_j - Y.)]'\ [Y_j - Y.- \tfrac{1}{t} J_t J_t' (Y_j - Y.)] \\ &= \sum_{j=1}^{n} (Y_j - Y.)'\ A_t (Y_j - Y.) \\ &= \sum_{q=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} (\delta_{qs} - \tfrac{1}{n})\ Y_s'\ A_t\ (\delta_{qr} - \tfrac{1}{n})\ Y_r \\ &= Y' A_n \otimes A_t\ Y. \end{aligned}$$

Now,

$$(A_n \otimes A_t)\ (I_n \otimes V) = A_n \otimes (A_t V)$$

which except for the constant $\sigma^2(1-\rho)$ is idempotent.

The rank $\rho(B \otimes C) = \rho(B)\rho(C)$ so the rank of the numerator is $\rho(E_n)\rho(A_t) = (1)\ (t-1) = t-1$ and the rank of the denominator is $\rho(A_n)\rho(A_t) = (n-1)(t-1)$.

Theorem 4.22 in Graybill says that if X is distributed N(M,V) then X'AX and X'BX are independent if and only if AVB = 0.

Here,

$$(E_n \otimes A_t)\ (I_n \otimes V)\ (A_n \otimes A_t)$$

$$= (E_n A_n) \otimes (A_t V A_t) = 0\ .$$

Thus, T is distributed

$$F'[(t-1),(n-1)(t-1);\ \lambda = \frac{1}{2} M'VM].$$

APPENDIX B

Computer Program

APPENDIX B

COMPUTER PROGRAM

THIS PROGRAM IS WRITTEN IN BASIC FORTRAN IV
FOR AN IBM 1800 COMPUTER

```
      REAL MEANS(10)
      DIMENSION A(10,10),B(10,10),C(10,10),YDATA(15,10)
      DIMENSION F(1000),TOBAD(150),PEOPL(15),TRTMT(10)
      DIMENSION TABF(21),ITAB(21),XDATA(15,10)
      DEFINE FILE 1(16000,3,U,IEND)
C
C MEANS IS THE ASSIGNED MEAN VECTOR, READ FROM CARDS
C A IS THE POPULATION VARIANCE-COVARIANCE MATRIX.  IT IS
C   READ FROM CARDS
C B CONTAINS CC-TRANSPOSE
C C IS A DIAGONAL MATRIX SUCH THAT CC-TRANSPOSE EQUALS A
C   AND SUCH THAT XDATA-TRANSPOSE EQUALS C*YDATA-TRANSPOSE
C YDATA CONTAINS RANDOM NUMBERS DISTRIBUTED N(0,I),
C   READ FROM DISK
C XDATA CONTAINS RANDOM NUMBERS DISTRIBUTED N(M,A)
C XDATA CONSTITUTES ONE SET OF SIMULATED EXPERIMENTAL DATA
C F CONTAINS THE 1000 COMPUTED F VALUES
C TOBAD IS A WORK ARRAY FOR PASSING NUMBERS FROM DISK
C   TO YDATA
C PEOPL IS A WORK ARRAY USED IN THE ANALYSIS OF VARIANCE
C TRTMT IS A WORK ARRAY USED IN THE ANALYSIS OF VARIANCE
C TABF CONTAINS VALUES FROM AN F TABLE, READ FROM CARDS
C ITAB CONTAINS THE FREQUENCY DISTRIBUTION OF THE 1000
C   COMPUTED F VALUES
C
C
C INITIAL INPUT/OUTPUT
C
      READ(2,401)
      READ(2,402)KSIZE,NCELL,IRAND
      READ(2,403)((A(I,J),J=1,KSIZE),I=1,KSIZE)
      READ(2,404)(MEANS(J),J=1,KSIZE)
      READ(2,405)(TABF(I),I=1,21)
  401 FORMAT('  THIS IS THE USER LABEL.   CAN BE  UP TO',
     1' 80 CARD COLUMNS                                    ')
  402 FORMAT(2I2,I5)
  403 FORMAT(7F11.8)
  404 FORMAT(10F8.3)
  405 FORMAT(8F10.5)
      WRITE(3,401)
      WRITE(3,4)
    4 FORMAT(////30H THIS IS THE COVARIANCE MATRIX)
      WRITE(3,5)((A(I,J),J=1,10),I=1,10)
      WRITE(3,9)(MEANS(I),I=1,KSIZE)
    9 FORMAT(///' THE VECTOR OF MEANS IS ',/('0',10F10.6))
      WRITE(3,10)KSIZE
   10 FORMAT(///' THE NUMBER OF TREATMENT GROUPS IS',I3)
      WRITE(3,21)NCELL,IRAND
   21 FORMAT(' THE SAMPLE SIZE IS',I3,
     1/' THE STARTING RANDOM NUMBER IS',I6)
C
C FIND C SUCH THAT CC-TRANSPOSE EQUALS A
C
      DO 11 IR=1,KSIZE
   11 C(IR,1)=A(IR,1)/SQRT(A(1,1))
      DO 16 IR=2,KSIZE
      DO 16 IC=2,IR
      IF(IC-IR)14,12,999
   12 SUM=0.0
```

```
      L=IR-1
      DO 13 M=1,L
   13 SUM=SUM+C(IR,M)**2
      C(IR,IR)=SQRT(A(IR,IR)-SUM)
      GO TO 16
   14 SUM=0.0
      L=IC-1
      DO 15 M=1,L
   15 SUM=SUM+C(IR,M)*C(IC,M)
      C(IR,IC)=(A(IR,IC)-SUM)/C(IC,IC)
   16 CONTINUE
      DO 17 I=1,10
      DO 17 J=1,10
   17 A(I,J)=0.0
C
C GMTRA FINDS THE TRANSPOSE  OF C
C GMPRD FINDS CC-TRANSPOSE
C
      CALL GMTRA(C,B,10,10)
      CALL GMPRD(C,B,A,10,10,10)
      WRITE(3,6)
    5 FORMAT(/,10(/' ',10F11.6)////)
    6 FORMAT(///17H THIS MATRIX IS C)
    7 FORMAT(' THIS MATRIX IS CC-TRANSPOSE')
      WRITE(3,5)((C(I,J),J=1,10),I=1,10)
      WRITE(3,7)
C
C ORIGINAL A-MATRIX WAS DESTROYED BY GMPRD
C A NOW CONTAINS CC-TRANSPOSE
C
      WRITE(3,5)((A(I,J),J=1,10),I=1,10)
      GO TO 18
  999 WRITE(3,8)
   18 CONTINUE
    8 FORMAT(18H0PROGRAMMING ERROR)
C
C SAMPLE RANDOM NUMBERS ON DISK
C
      IDMEN=KSIZE*NCELL
      CALL RANDU(IRAND,IX,XNOT)
      DO 150 IOVER=1,1000
      DO 119 I=1,NCELL
      DO 119 J=1,KSIZE
  119 XDATA(I,J)=0.0
  101 CALL RANDU(IX,IY,XNOT)
      IX=IY
C
C IS IY SUITABLE FOR INDEXING DISK FILE OF
C   N(0,I) RANDOM NUMBERS
C
      XIY=IY
      FINDX=(XIY+2.)/2.
      INDX=FINDX
      IF(INDX-16000)104,104,102
  102 WRITE(3,103)IY
  103 FORMAT(20H RANDOM INDEX NUMBER,I6,12H IS REJECTED)
      GO TO 101
C
C INSURE LOOP FROM FILE NUMBER 16000 TO FILE NUMBER 00001
C
  104 L=0
      ICAN=0
      ITBAD=0
      DO 120 I=1,IDMEN
  120 TOBAD(I)=0.0
      ICAN=16001-INDX-IDMEN
      IF(ICAN)105,111,111
  105 ITBAD=16001-INDX
      READ(1'INDX)(TOBAD(I),I=1,ITBAD)
```

```
      ITBAD=ITBAD+1
      READ(1'1)(TOBAD(I),I=ITBAD,IDMEN)
      DO 106 I=1,NCELL
      DO 106 J=1,KSIZE
      L=(J-1)*NCELL+I
  106 YDATA(I,J)=TOBAD(L)
      GO TO 112
  111 READ(1'INDX)((YDATA(I,J),I=1,NCELL),J=1,KSIZE)
C
C TRANSFORM RANDOM NUMBERS FROM N(0,I) TO N(0,A)
C THE FORMULA FOR THE TRANSFORMATION IS X(I,J)=SUM ON K
C   FROM 1 TO J OF C(J,K)*Y(I,K)
C
  112 DO 113 I=1,NCELL
      DO 113 J=1,KSIZE
      DO 113 K=1,J
      XDATA(I,J)=XDATA(I,J)+C(J,K)*YDATA(I,K)
  113 CONTINUE
      DO 114 I=1,NCELL
      DO 114 J=1,KSIZE
      XDATA(I,J)=XDATA(I,J)+MEANS(J)
  114 CONTINUE
C
C THE ANALYSIS OF VARIANCE
C
      DO 145 I=1,NCELL
  145 PEOPL(I)=0.0
      DO 146 J=1,KSIZE
  146 TRTMT(J)=0.0
      TOTAL=0.0
      SSPEO=0.0
      SSTRT=0.0
      SSTOT=0.0
C
C FIND AND SUBTRACT GRAND MEAN FROM ALL OBSERVATIONS
C
      DO 151 I=1,NCELL
      DO 151 J=1,KSIZE
  151 TOTAL=TOTAL+XDATA(I,J)
      GRAND=TOTAL/IDMEN
      DO 152 I=1,NCELL
      DO 152 J=1,KSIZE
      XDATA(I,J)=XDATA(I,J)-GRAND
C
C FIND ALL SUMS OF SQUARES
C
  152 SSTOT=SSTOT+XDATA(I,J)**2
      DO 153 I=1,NCELL
      DO 153 J=1,KSIZE
      PEOPL(I)=PEOPL(I)+XDATA(I,J)
  153 TRTMT(J)=TRTMT(J)+XDATA(I,J)
      DO 154 I=1,NCELL
      PEOPL(I)=PEOPL(I)/KSIZE
  154 SSPEO=SSPEO+PEOPL(I)**2
      DO 155 J=1,KSIZE
      TRTMT(J)=TRTMT(J)/NCELL
  155 SSTRT=SSTRT+TRTMT(J)**2
      SSPEO=KSIZE*SSPEO
      SSTRT=NCELL*SSTRT
      SSRES=SSTOT-SSPEO-SSTRT
C
C FIND MEAN SQUARES
C
      TSTRT=SSTRT/(KSIZE-1.)
      TSRES=SSRES/((NCELL-1.)*(KSIZE-1.))
      F(IOVER)=TSTRT/TSRES
      WRITE(3,156)IOVER,IY,GRAND,TSRES,F(IOVER)
  156 FORMAT(I5,' RIV = ',I5,' XBAR = ',F9.5,' SSQUARE = ',
     1F13.8,10X,'F = ',F14.8)
```

```
  150 CONTINUE
      DO 203 I=1,21
  203 ITAB(I)=0
C
C BEGINNING OF LOOP TO ORDER AND TABULATE THE 5*200
C   COMPUTED F VALUES
C
      DO 250 MALL=1,5
      K=200*(MALL-1)+1
      KK=200*MALL
C
C ORDER AND WRITE 200 F VALUES
C ORDERING METHOD --- TEST F(I) AGAINST ALL LOWER VALUES.
C   WHEN PROPER POSITION FOR F(I) IN ARRAY IS FOUND,MOVE
C   ALL HIGHER VALUES UP ONE STEP IN THE ARRAY AND PUT
C   FORMER F(I) IN THE REMAINING POSITION
C
      DO 210 I=K,KK
      ILIM=I-1
      HOLD=F(I)
      DO 209 J=K,ILIM
      IF(HOLD-F(J))201,209,209
  201 MOVE=I+1
      MEND=I-J
      DO 202 M=1,MEND
      MOVE=MOVE-1
  202 F(MOVE)=F(MOVE-1)
      F(J)=HOLD
      GO TO 210
  209 CONTINUE
  210 CONTINUE
      WRITE(3,211)K,KK
      WRITE(3,212)(F(I),I=K,KK)
  211 FORMAT(///'     ORDERED F VALUES, F(',I3,') TO F(',
     1I4,')')
  212 FORMAT(/(1H ,10F11.6))
C
C COMPARE THE ABOVE 200 ORDERED F VALUES WITH THE TABULATED
C   F VALUES AND COUNT THE NUMBER IN EACH INTERVAL
C
C TABF VALUES ARE READ IN INVERSE ORDER---
C   HIGHEST TO LOWEST
C TABF(1) IS THE TABULATED F VALUE FOR WHICH THE PROBABILITY
C   OF A SMALLER VALUE IS .9999
C TABF(2) IS THE SAME FOR .9995
C TABF(3) IS THE SAME FOR .999
C TABF(4) IS THE SAME FOR .99
C TABF(5) IS THE SAME FOR .975
C TABF(6) IS THE SAME FOR .95
C     ETC.
C
      IT=21
      IF(F(K)-TABF(21))301,301,260
  260 DO 265 JJ=1,21
      IT=IT-1
      IF(F(K)-TABF(IT))301,301,265
  265 CONTINUE
  301 DO 309 I=K,KK
  310 IF(F(I)-TABF(IT))309,309,302
  302 ITAB(IT)=ITAB(IT)+I-200*(MALL-1)-1
      IT=IT-1
      IF(IT)250,250,310
  309 CONTINUE
      IF(IT)250,250,320
  320 DO 321 I=1,IT
  321 ITAB(I)=ITAB(I)+200
  250 CONTINUE
C
      WRITE(3,311)
```

```
      WRITE(3,312)
      WRITE(3,313)(ITAB(I),TABF(I),I=1,21)
      WRITE(3,314)
  311 FORMAT(////' NUMBER OF VALUES LESS THAN OR',12X,
     1'TABULATED VALUE')
  312 FORMAT(' EQUAL TO THE TABULATED VALUE')
  313 FORMAT(/(13X,I4,22X,F10.5))
  314 FORMAT(///' END OF ANALYSIS'////////////////)
      CALL EXIT
      END
```

APPENDIX C

Box's Statistic for Nonuniformity

Box's Statistic for Nonuniformity

The computed value of the statistic is tabulated for each covariance matrix and value of n.

t=3 (4 degrees of freedom)

covariances	n=3	n=6	n=10	n=15
(0.0,0.0,-.25)	.021	.148	.317	.529
(0.0,0.0,.25)	.022	.156	.335	.558
(.25,.25,.5)	.037	.261	.559	.932
(-.25,-.25,.25)	.074	.517	1.108	1.846
(.5,.5,.75)	.093	.649	1.391	2.318
(-.25,-.25,-.75)	.212	1.486	3.185	5.309
(.25,.25,.75)	.215	1.501	3.217	5.362
(0.0,0.0,-.75)	.290	2.029	4.349	7.248
(-.5,-.5,.5)	.298	2.086	4.470	7.450
(0.0,0.0,.75)	.328	2.299	4.926	8.210
(0.0,.25,.75)	.340	2.383	5.105	8.509
(0.0,.5,.75)	.601	4.207	9.016	15.027
(.05,.05,.9)	.682	4.771	10.225	17.041

t=4 (8 degrees of freedom)

covariances	n=3	n=6	n=10	n=15
(.375,.375,.375,.375,.375, .525)	.011	.138	.307	.519
(.35,.35,.35,.35,.35,.65)	.049	.601	1.338	2.258
(.325,.325,.325,.325,.325, .775)	.128	1.578	3.512	5.929
(.3,.6,.6,.6,.6,.9)	.240	2.972	6.613	11.166
(.3,.3,.3,.3,.3,.9)	.308	3.814	8.488	14.331
(.3,.3,.4,.8,.9,.9)	.482	5.967	13.279	22.420
(.3,.3,.3,.9,.9,.9)	.517	6.390	14.221	24.009

APPENDIX D

Uniform Matrix for Each Nonuniform Matrix

Uniform Matrix for Each Nonuniform Matrix

Nonuniform	Uniform
t=3	
(0.0,0.0,-.25)	-.08333
(0.0,0.0,.25)	.08333
(.25,.25,.5)	.33333
(-.25,-.25,.25)	-.08333
(.5,.5,.75)	.58333
(-.25,-.25,-.75)	-.41666
(.25,.25,.75)	.41666
(0.0,0.0,-.75)	-.25000
(-.5,-.5,.5)	-.16666
(0.0,0.0,.75)	.25000
(0.0,.25,.75)	.33333
(0.0,.5,.75)	.41666
(.05,.05,.9)	-.16666
t=4	
(.375,.375,.375,.375,.375,.525)	.4
(.35,.35,.35,.35,.35,.65)	.4
(.325,.325,.325,.325,.325,.775)	.4
(.3,.6,.6,.6,.6,.9)	.6
(.3,.3,.3,.3,.3,.9)	.4
(.3,.3,.4,.8,.9,.9)	.6
(.3,.3,.3,.9,.9,.9)	.6