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CHAPTER I

INTRODUCTION

A. The Problem

The fundamental axiom in classical electrodynamics is that moving charged particles affect each other's motions by means of so-called electric and magnetic fields. There are, therefore, three basic problems in classical electrodynamics.

(i) The determination of these fields at every point in space and time for a given system of interacting charges.

(ii) The determination of the motions of the charges which occur in response to the fields present.

(iii) The determination in so far as is possible of the structure of the fundamental charged particles.

Problem (i) has been solved since the days of Maxwell, who first formulated the field equations which bear his name. Problem (ii), the determining of the equations of motion for charges was first treated by Lorentz[1] and later by Dirac.[2] It is the problem with which we are primarily concerned in this work. Problem (iii) is generally regarded today as being beyond the scope of classical electrodynamics. The current attitude is summarized by Rohrlich[3]. He says, "The problem is to find a formulation of classical charged particle theory which does not require any reference to or assumptions about the particle structure, its charge distribution and its size."(3)

We will present some preliminary evidence that when problem (ii) is treated in a way which is both physically meaningful and mathematically sound, the resulting formulation has statements about the structure of the classical charge already built into it and thus classical electrodynamics may indeed be capable of making meaningful statements about particle structure. In the next sections, we review the important elements of the current theory. The notation and development closely follow Rohrlich.

B. The Maxwell-Lorentz Equations

The electric field $\vec{E}$, the magnetic induction $\vec{B}$, the charge density ρ, and the current density $\vec{j}$, are related by

$$\begin{aligned} \nabla \times \vec{B} - 1/c \frac{\partial \vec{E}}{\partial t} &= \frac{4\pi}{c} \vec{j} \\ \nabla \cdot \vec{E} &= 4\pi\rho \\ \nabla \times \vec{E} + 1/c \frac{\partial \vec{B}}{\partial t} &= 0 \\ \nabla \cdot \vec{B} &= 0, \text{ where} \end{aligned} \tag{1.1}$$

c is the speed of light.

These equations may also be written in terms of a vector potential $\vec{A}$ and a scalar potential ϕ. These are introduced through the equations

$$\vec{B} = \nabla \times \vec{A}$$

$$- \nabla\phi = \vec{E} + 1/c \frac{\partial \vec{A}}{\partial t} \quad . \tag{1.2}$$

A and ϕ are not unique but are defined only to within a gauge transformation

$$\vec{A} \rightarrow \vec{A}' = A + \nabla\Lambda$$

$$\phi \rightarrow \phi' = \phi - 1/c \frac{\partial \Lambda}{\partial t} \quad , \tag{1.3}$$

with Λ an arbitrary function.

If Λ is required to satisfy the wave equation

$$\nabla^2\Lambda = 1/c^2 \frac{\partial^2\Lambda}{\partial t^2} \quad , \tag{1.4}$$

the expression

$$I \equiv \nabla\cdot\vec{A} + 1/c \frac{\partial\phi}{\partial t} \quad , \tag{1.5}$$

is invariant under all gauge transformations. If we choose I = 0, the Maxwell-Lorentz equations take the form

$$(\nabla^2 - 1/c^2 \frac{\partial^2}{\partial t^2})\vec{A} = - \frac{4\pi}{c} \vec{j} \ ,$$

$$(\nabla^2 - 1/c^2 \frac{\partial^2}{\partial t^2})\phi = - 4\pi\rho \quad . \tag{1.6}$$

We now introduce four-vectors in a Minkowski space. Our convention is

$$b^\mu = (b^0, b^1, b^2, b^3) \ . \tag{1.7}$$

b^0 is the component of b along the $x^0(= ct)$ axis and b^1, b^2, b^3 are the space components of b. When we write a quantity as a four vector, we are implying that the quantity transforms as

$$\bar{b}^{\mu} = b^{\alpha} \frac{\partial \bar{x}^{\mu}}{\partial \bar{x}^{\alpha}} , \tag{1.8}$$

under Lorentz transformations:

$$\begin{aligned} \bar{x}^{i} &= \gamma(x^{i} - \frac{v^{i}}{c} x^{0}) - (\gamma - 1)(x^{i} - \frac{v^{i} x^{j} v_{j}}{v^{2}}) \\ \bar{x}^{0} &= \gamma(x^{0} - \frac{x^{j} v_{j}}{c^{2}}) \qquad \gamma = (1 - v^{2}/c^{2})^{-\frac{1}{2}} \\ i,j &= 1,2,3 \qquad v^{j} x_{j} \equiv \sum_{j=1}^{3} v^{j} x_{j} \end{aligned} \tag{1.9}$$

$\vec{v}$ is the velocity of one reference frame relative to the other.

$$j^{\mu} \equiv (c\rho, \vec{j}) \quad , \tag{1.10}$$

and four-vector potential

$$A^{\mu} = (\phi, \vec{A}) \quad . \tag{1.11}$$

Our notation is simplified by defining

$$\partial_{\mu} \equiv (1/c \frac{\partial}{\partial t} , \nabla) \tag{1.12}$$

and using the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.13}$$

and its contravariant associate to raise or lower indices.

The Maxwell-Lorentz equations then become

$$\partial_\alpha \partial^\alpha A^\mu = -\frac{4\pi}{c} j\mu \quad . \tag{1.14}$$

The quantities $F^{\mu\nu}$ defined by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ , \tag{1.15}$$

from an antisymmetric tensor of the second rank under Lorentz transformations. They are given explicitly in terms of the fields by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \quad . \tag{1.16}$$

The field equations become

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j\nu$$

$$\partial\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial\nu F\lambda\mu = 0. \tag{1.17}$$

In order to consider the solution to these equations, we imagine a charged particle moving along its trajectory or world line. We define an increment of proper time by the equation

$$- c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \ . \tag{1.18}$$

τ plays the role of arc length along the world line and is a monotone increasing quantity. Let $z^\mu = z^\mu(\tau)$ be the position of the charge. Its four-velocity vector is given by

$$v^\mu(\tau) = \frac{dz^\mu(\tau)}{d\tau} , \qquad v^\mu v_\mu = - c^2 . \tag{1.19}$$

Let x^μ be an arbitrary space-time point. The vector $R^\mu(\tau)$ is defined as

$$R^\mu = x^\mu - z^\mu(\tau) . \tag{1.20}$$

For a given point x^μ there are two values of, τ, $\tilde{\tau}$ and $\bar{\tau}$, $\tilde{\tau} < \bar{\tau}$ such that

$$R^\mu(\tau)R_\mu(\tau)\Big|_{\tau=\tilde{\tau},\bar{\tau}} = 0 \tag{1.21}$$

Now define a unit vector u_μ orthogonal to v^μ

$$u_\mu u^\mu = 1 , \qquad u^\mu v_\mu = 0 . \tag{1.22}$$

If a tilde over each quantity refers to the quantity evaluated at $\tau = \tilde{\tau}$, we have

$$\tilde{R}^\mu = \tilde{\rho}(\tilde{u}^\mu + \frac{\tilde{v}^\mu}{c}) \tag{1.23}$$

$$\tilde{\rho} = \tilde{u}_\mu R^\mu = -\frac{\tilde{v}^\mu}{c} \tilde{R}_\mu \tag{1.24}$$

The solutions to Eqs. (1.17) for the field due to a particle with charge e can be written

$$F_{ret}(x)^{\mu\nu} = \frac{e}{\tilde{\rho}^2 c}(\tilde{v}^\mu \tilde{u}^\nu - \tilde{v}^\nu \tilde{u}^\mu)$$

$$+ \frac{e}{\tilde{\rho} c^2}[(\tilde{a}^\mu \tilde{v}^\nu - \tilde{a}^\nu \tilde{v}^\mu)/c - \tilde{u}^\mu(\frac{\tilde{v}^\nu}{c}\tilde{a}_u + \tilde{a}^\nu) \tag{1.25}$$

$$+ \tilde{u}^\nu(\frac{\tilde{v}^\mu}{c}\tilde{a}_u + \tilde{a}^\mu)]$$

$$\tilde{a}_u \equiv \tilde{a}_\lambda \tilde{u}^\lambda .$$

This is the so-called retarded field. There is also another solution, the advanced field given by

$$F_{adv}(x)^{\mu\nu} = \frac{e}{\bar{\rho}^2 c}(\bar{v}^\mu \bar{u}^\nu - \bar{v}^\nu \bar{u}^\mu) \tag{1.26}$$

$$+ \frac{e}{\bar{\rho} c^2}[(\bar{a}^\mu \bar{v}^\nu - \bar{a}^\nu \bar{v}^\mu)/c - \bar{u}^\mu(\frac{\bar{v}^\nu}{c}\bar{a}_u - \bar{a}^\nu) + \bar{u}^\nu(\frac{v^\mu}{c}\bar{a}_u - \bar{a}^\mu)]$$

and

$$\bar{v}^\mu = v^\mu(\tau = \bar{\tau}).$$

The expressions for $F^{\mu\nu}_{ret}$ and $F^{\mu\nu}_{adv}$ can be put in equivalent forms which are more convenient for our purposes.

Take Eq. (1.20) as the definition of R^μ and define

$$\rho\ (\tau) = -\ v_\mu(\tau)R^\mu(\tau)/c \tag{1.27}$$

and .

$$\rho_*(\tau) = +\ v_\mu(\tau)R^\mu(\tau)/c \tag{1.28}$$

(Note:

$$\begin{aligned}\tilde{\rho} &= \rho_*(\tau)\Big|_{\tau=\tilde{\tau}} \\ \bar{\rho} &= \rho_+(\tau)\Big|_{\tau=\bar{\tau}}\ )\end{aligned} \tag{1.29}$$

Then it can be shown that

$$F^{\mu\nu}_{ret} = \left[\frac{e}{\rho\ c^2}\frac{d}{d\tau}\left(\frac{v^\mu(\tau)R^\nu(\tau)-v^\nu(\tau)R^\mu(\tau)}{\rho}\right)\right]\Big|_{\tau=\tilde{\tau}} \tag{1.30}$$

$$F^{\mu\nu}_{adv} = \left[\frac{e}{\rho_* c^2}\frac{d}{d\tau}\left(\frac{v^\mu(\tau)R^\nu(\tau)-v^\nu(\tau)R^\mu(\tau)}{\rho_*}\right)\right]\Big|_{\tau=\bar{\tau}} \tag{1.31}$$

C. Momentum of the Field

We next introduce the energy-momentum tensor $\Theta^{\mu\nu}$ defined by

$$\Theta^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}_{\alpha} + \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{1.32}$$

$\Theta^{\mu\nu}$ is divergenceless:

$$\partial_\mu\Theta^{\mu\nu} = 0\ . \tag{1.33}$$

Using Eq. (1.30), $\Theta^{\mu\nu}$ can be expressed in terms of the kinematics of the charge

$$\Theta^{\mu\nu}_{ret} = \frac{e^2}{4\pi\tilde{\rho}^4}\left(\tilde{u}^\mu\tilde{u}^\nu - \frac{\tilde{v}^\mu\tilde{v}^\nu}{c^2} - \tfrac{1}{2}g^{\mu\nu}\right)$$

$$+ \frac{e^2}{2\pi\tilde{\rho}^3 c^2}\left[\tilde{a}_u \frac{\tilde{R}^\mu\tilde{R}^\nu}{\tilde{\rho}^2} - \left(\frac{\tilde{v}^{(\mu}}{c}\tilde{a}_u + \tilde{a}^{(\mu}\right)\frac{\tilde{R}^{\nu)}}{\tilde{\rho}}\right] \tag{1.34}$$

$$+ \frac{e^2}{4\pi\tilde{\rho}^2 c^4}\left(\tilde{a}_u^2 - \tilde{a}_\lambda\tilde{a}^\lambda\right)\frac{\tilde{R}^\mu\tilde{R}^\nu}{\tilde{\rho}^2}$$

where

$$a^{(\mu}b^{\nu)} \equiv \tfrac{1}{2}(a^\mu b^\nu + a^\nu b^\mu). \tag{1.35}$$

A space-like plane in Minkowski space is defined as a three-dimensional surface such that the distance between any two points x and y satisfies

$$(x_\mu - y_\mu)(x^\mu - y^\mu) > 0 \ . \tag{1.36}$$

It is specified geometrically by a unit normal vector n such that

$$n_\mu n^\mu = -1 \ . \tag{1.37}$$

We can define an element of area for such a plane

$$d\sigma^\nu = n^\nu d\sigma \tag{1.38}$$

with

$$d\sigma = dxdydz. \tag{1.39}$$

Where x, y, z is the coordinate system in which

$$n^{\mu} = (1,0,0,0). \tag{1.40}$$

Consider the expression

$$P^{\mu} = \frac{1}{c}\int \Theta^{\mu\nu} d\sigma_{\nu} \tag{1.41}$$

where the integration is carried out over the plane which contains the charge, excepting the portion of the plane which the charge itself occupies.

The surface element is given by

$$d\sigma^{\mu} = \frac{v^{\mu}}{c}\, d\sigma\ . \tag{1.42}$$

Thus the plane is always normal to the world line of the particle. It is obvious from its definition that P^{μ} is a four vector. We shall take Eq. (1.41) as the definition for the momentum vector of the electromagnetic field.

D. The Dirac Equation

We are now in a position to ask the question which was first asked by Dirac. What is the change in the momentum of the field as the particle moves along its trajectory? To answer this we surround the worldline with a mathematical surface, the world tube, of invariantly defined radius $\hbar$. If 1 and 2 are points on the world line separated by an increment of proper time $d\tau$, then

$$P_2^\mu - P_1^\mu = \frac{1}{c}\int_{\sigma_2} \Theta^{\mu\nu} d\sigma_\nu - \frac{1}{c}\int_{\sigma_1} \Theta^{\mu\nu} d\sigma_\nu \tag{1.43}$$

is the change in momentum, where σ_2, σ_1 are space-like planes simultaneous with the charge at points 2 and 1, respectively, and the integration is carried out outside the charge. By Gauss's theorem, however, this difference in integrals over the space-like planes is equivalent to an integration over the surface of the world tube. The element of surface area for this tube is known to be

$$d\sigma^\mu = u^\mu(1 + \hbar a_u) d\tau \hbar^2 d\Omega \ . \tag{1.44}$$

Where $d\Omega$ is an element of solid angle. Therefore

$$\frac{dP^\mu}{d\tau} = \int \Theta^{\mu\nu} u_\nu \hbar^2 (1+\hbar a_u) d\Omega \ . \tag{1.45}$$

This is the rate of change of electromagnetic momentum.

In order to calculate this explicitly, $\Theta^{\mu\nu}$ must be expressed not in terms of retarded quantities but rather in terms of instantaneous ones. Let $\tau = 0$ be the point of interest on the world line and denote $\tilde{\tau}$ by $-\tau$. The technique used by Dirac and others was to make expansions of the form

$$\tilde{v}^{\mu} = v^{\mu} - \tau a^{\mu} + \frac{\tau^2}{2}\dot{a}^{\mu} + \cdots$$

$$\tilde{R}^{\mu} = \hbar u^{\mu} + \tau v^{\mu} - \frac{\tau^2}{2} a^{\mu} + \cdots \tag{1.46}$$

$$\tilde{\rho} = \tau(1+\hbar a_u) - \frac{\hbar\tau^2}{2}\dot{a}_u + \cdots$$

where the quantities without the tidle refer to instantaneous values. The second series contains both τ and $\hbar$. In order to eliminate τ, one makes use of

$$\tilde{R}^{\mu}\tilde{R}_{\mu} = 0 \tag{1.47}$$

to express τ as a function of $\hbar$. Since one only needs a few terms in each series, these techniques are not too unwieldy. The result of carrying out the expansion and integration is

$$\frac{dp^{\mu}}{d\tau} = -\frac{2}{3}e^2(\dot{a}^{\mu}-a^2v^{\mu}) + \frac{e^2}{2\hbar}a^{\mu} - F^{\mu\nu}_{ext}v_{\nu} + A_1\hbar + A_2\hbar^2 + \cdots \tag{1.48}$$

The A's represent the coefficients of an infinite series in the radius of the world tube which depend on the kinematics of the charge, and F_{ext} represents fields which may be present due to sources other than the charge of interest.

The mechanical momentum of the charged particle is

$$\frac{dp^\mu}{d\tau} = ma^\mu. \tag{1.49}$$

The total conservation of momentum requires

$$\frac{dP^\mu}{d\tau} + \frac{dp^\mu}{d\tau} = 0 \tag{1.50}$$

$$(m + \frac{e^2}{2\hbar})a^\mu = -\frac{2}{3}e^2(\dot{a}^\mu - a^2v^\mu) - F^\mu_{ext} + A_1\hbar + A_2\hbar^2 + \cdots \tag{1.51}$$

$$F^\mu_{ext} \equiv F^{\mu\nu}_{ext}v_\nu. \tag{1.52}$$

At this point, Dirac let

$$m_{obs} = m + \frac{e^2}{2\hbar} \tag{1.53}$$

be the experimentally observed mass. Then writing

$$m_{obs}a^\mu = -\frac{2}{3}e^2(\dot{a}^\mu - a^2v^\mu) - F^\mu_{ext} + A_1\hbar + \cdots \tag{1.54}$$

he took the limit as $\hbar \to 0$, to obtain

$$m_{obs}a^\mu = -\frac{2}{3}(\dot{a}^\mu - a^2v^\mu) - F^\mu_{ext} . \tag{1.55}$$

This is known as the Lorentz-Dirac equation.

It is clear that while it is convenient to renormalize the mass so as to be rid of the infinite series, the renormalization itself is totally unjustified, mathematically. The question therefore arises as to the true significance of the infinite series; are there values of $\hbar$ other than zero such that the series vanishes? The major objective of the present work is to investigate the infinite series which relate the retarded and advanced quantities in electrodynamics to the corresponding instantaneous ones. This formalism will allow the investigation of questions such as the one raised above.

CHAPTER II

THE CALCULATIONS IN OUTLINE

A. The Objectives

This treatise is concerned with developing all the mathematical machinery necessary to derive the Lorentz-Dirac equation of motion in a manner which is physically cogent and mathematically sound.

By physically cogent, we mean that all the assumptions used are either well-accepted in the current understanding of the problem or else are natural and appear practically self-evident. By mathematically sound, we mean that no ill-defined mathematical techniques such as renormalization need be utilized.

The work naturally divides itself into three major endeavors. First, the conversion of the usual retarded and advanced formalism into an "instantaneous" or "co-present" formalism. Second, the use of this formalism to obtain general formulas for the quantities appearing in the expression for the electromagnetic momentum. Finally the drawing of conclusions about the equation of motion.

B. Heuristic Considerations

We shall take Rohrlich's interpretation of Dirac's derivation seriously. That is, by assuming conservation of the total momentum associated with the charge (mechanical plus electromagnetic), we shall lay the groundwork for being able to deduce the equation of motion without having to employ renormalization.

To begin however, we must have a justifiable definition of electromagnetic momentum. This definition is not as natural to come by as one might suppose. Since integrating a second rank tensor over the world tube of the particle is equivalent (if the field vanishes sufficiently rapidly in space-like directions) to forming the difference of the integrals of the tensor over two adjacent space-like planes which are perpendicular to the world line (the integration excludes the volume which the particle itself occupies), it seems natural to look for a tensor $\Theta^{\mu\nu}$ whose integral over such a space-like plane can be considered "the" four-momentum. As we mentioned in Chapter I, Dirac chose $\Theta^{\mu\nu}$ to be $\Theta^{\mu\nu}_{ret}$ where

$$\Theta^{\mu\nu}_{ret} = \frac{1}{4\pi} (F^{\mu\alpha}_{ret} F^{\nu}_{ret\alpha} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta}_{ret} F_{ret\alpha\beta}) \ .$$

The general definition of electromagnetic four-momentum is then (1.41).

This choice has some support: let a charge initially be at $\tau = 0$ and consider the electromagnetic field produced in the interval from $\tau = 0$ to $\tau = d\tau$ by the arbitrary motion of the charge. If an observer moves along between the light cones which have their apexes at $\tau = 0$, and $\tau = d\tau$ he finds that the fields produced in the interval $d\tau$ are more and more confined to the region between the cones until, in the region of the field vary for how the changes location during $d\tau$, there is no field outside this region due to the motion considered. Let $d\sigma_\infty^\nu$ be an element of a timelike surface σ_∞ which the fields intersect at $\tau = \infty$. Then

$$dP^\mu_{rad} \equiv -\frac{1}{c}\int_{\sigma\infty} \Theta^{\mu\nu}_{ret} \; d\sigma_{\infty\nu}$$

can be shown to be the momentum associated with the radiation emitted in the interval $d\tau$. This definition depends on the "asymtotic" field, where there are no charge singularities. *It is not obvious however, that when the integration (1.41) includes that portion of the field <u>near</u> the charge (singularity) the physical meaning of the integral remains unaltered. We do not choose to discuss here other choices and their merits but instead will adopt (1.41) for the simple reason that we are specifically interested in extending Dirac's notions to their proper limit.

The second important point and the most fascinating is, if one believes the Dirac equation but does not believe in renormalization,

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*See Rohrlich for a discussion of this problem.

what logical positions are left to him? One is that the equation is right but the derivation is wrong. This is not an untenable position but it is not very interesting either. The alternative seems to be that the rest of the series, which Dirac threw away, must be important and in fact its behavior must really be the reason why Dirac's technique was successful. In particular, one could suppose that there are values of the world tube radius r for which the series in the right hand member of (1.57) vanishes, thus giving the Dirac equation rigorously. The value(s) of r would be associated with the dimension of the charge. Thus in this theory, <u>classical electrodynamics would imply something about the structure of the (classical) electron</u>. It is this possibility which we want to explore and in the present work, we lay the foundation for a study of it.

We make three assumptions:

(i) The electromagnetic field of the particle is described by the Lienard-Wiechert potentials [Eqs. (1.25) and (1.26)].

(ii) The motion of the charge is such that all power series expansions of field quantities have a radius of convergence which is at least as large as the radius of the electron.

(iii) The electromagnetic four-momentum of the electron is given by Eq. (1.41).

Assumption (i) actually defines our model of the electron and so is an assumption only in the weakest sense of the word.

Assumption (ii) is indispensible. Without it, the entire Dirac formalism collapses. Note however, that we are assuming that the power series are valid only in a very small region of space time and are not assuming that the various quantities are analytic everywhere.

Assumption (iii), we have already discussed.

The remainder of this chapter is devoted to an outline for and a commentary on, the detailed calculations which are present in Chapter III.

C. The Geometry

Consider the situation in which a field point P, with coordinates x^μ is simultaneous with a point Q [coordinates z^μ $(\tau=0)$] on the world line of a charged particle (Fig. 1). The retarded field at x^μ is generated by the particle when it is at the point $\tilde{Q}$ [coordinates z^μ $(\tilde{\tau})$], $\tilde{\tau}<0$. The advanced field at x^μ is generated when the particle is at the point $\bar{Q}$ [coordinates z^μ $(\bar{\tau})$, $\bar{\tau}>0$]. What we want is an expression for the field quantities, e.g. $F^{\mu\nu}_{ret}$ (x), in terms of the "co-present" quantities

$$v^\mu(\tau=0) \equiv v^\mu \qquad a^\mu(\tau=0) \equiv a^\mu \qquad \text{etc.},$$

and the distance r, where

$$r^2 = (x^\mu - z^\mu)(x_\mu - z_\mu) \tag{2.1}$$

(we are following the convention that quantities not evaluated at $\tau=0$, are accompanied by explicit reference to τ). The old technique was to expand the relevant quantities

$$R^{\mu}(\tilde{\tau}) \equiv x^{\mu} - z^{\mu}(\tilde{\tau}), \qquad v^{\mu}(\tilde{\tau}), \quad \text{etc.}$$

in a power series in $\tilde{\tau}$. Then, making use of the fact that

$$R^{\mu}(\tilde{\tau})R_{\mu}(\tilde{\tau}) = 0 \tag{2.2}$$

one obtained a relation between $\hbar$ and $\tilde{\tau}$ to only a few orders. Using this relation one then aliminated $\tilde{\tau}$ from the expansions and obtained power series in terms of $\hbar$ and the kinematical quantities evaluated at $\tau = 0$.

In general we shall also follow this approach but we shall make explicit use of the fact that we are expanding all quantities of interest in terms of $\hbar$. This will simplify the procedure enormously. We proceed to find a technique which will allow us to differentiate directly with respect to $\hbar$. This is an approach which, to the author's knowledge has not appeared in the literature, and in fact, provides the sought after relation between $\tilde{\tau}$ and $\hbar$, to all orders.

D. The Differentiation Process

The differentiation that we are interested in, is to be found by considering the change introduced in the function of interest when the distance $\hbar$ is infinitesimally varied. This variation must be carried out in such a way that

(i) the point z^{μ} is unvaried

(ii) the point P' (coordinates x'^{μ}) which P is mapped into by the variation of λ is still simultaneous with z^{μ}.

(iii) account is taken of the fact that a variation $\delta\lambda$ will map $\tilde{\tau}\to\tilde{\tau}'$ in such a way, that $R'^{\mu}(\tilde{\tau}')\equiv x'^{\mu}-z'^{\mu}(\tilde{\tau}')$ obeys the relation

$$R'^{\mu}R'_{\mu} = 0 \ .$$

Requirement (i) is necessary since z^{μ} is the given point of interest. Requirement (ii) is necessary because all field points must be simultaneous with z^{μ}. Requirement (iii) guarantees that the field at x'^{μ} is really the retarded field of interest.

These conditions are sufficient to specify uniquely the differentiation process. Since

$$x^{\mu} - z^{\mu} = \lambda u^{\mu} \tag{2.3}$$

we have

$$\delta x^{\mu} = \delta\lambda u^{\mu} \tag{2.4}$$

or

$$\frac{dx^{\mu}}{d\lambda} = u^{\mu} \ . \tag{2.5}$$

Now we want to consider the derivative of a function of $\tilde{\tau}$, say $f(\tilde{\tau})$. Then

$$\frac{df}{d\lambda} = \frac{df}{d\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial x^{\mu}}\frac{dx^{\mu}}{d\lambda} \tag{2.6}$$

and since, setting c = 1, (much later we shall allow c to enter the formulas explicitly)

$$\frac{\partial \tilde{\tau}}{\partial x^{\mu}} = -\,(\tilde{u}_{\mu} + \tilde{v}_{\mu}) \tag{2.7}$$

we have

$$\frac{d\mathfrak{f}(\tilde{\tau})}{d\hbar} = -\,\dot{\mathfrak{f}}(\tilde{\tau})\,(\tilde{u}_{\mu} + \tilde{v}_{\mu})u^{\mu}; \qquad \dot{\mathfrak{f}} \equiv \frac{d\mathfrak{f}}{d\tau} \quad . \tag{2.8}$$

From this formula, we see that in order to calculate the nth derivative of $\mathfrak{f}(\tilde{\tau})$, we need the (n-1) derivative of $\tilde{u}^{\mu} + \tilde{v}^{\mu}$.

We assume that $\tilde{u}^{\mu} + \tilde{v}^{\mu}$ can be expanded in a Taylor series and take its form to be

$$\tilde{u}^{\mu} + \tilde{v}^{\mu} = \sum_{n=0}^{\infty} \frac{\hbar^{n}}{n!} A_{n}{}^{\mu}. \tag{2.9}$$

Now, since

$$\frac{d\tilde{\tau}}{d\hbar} = \frac{\partial \tilde{\tau}}{\partial x^{\mu}} u^{\mu} = -\,(\tilde{u}^{\sigma} + \tilde{v}^{\sigma})u_{\sigma} \tag{2.10}$$

$$= \sum_{n=0}^{\infty} \frac{A_{n}{}^{\sigma}u_{\sigma}}{n!}\,\hbar^{n} \tag{2.11}$$

we have the set of equations

$$\left.\frac{d\tilde{\tau}}{d\hbar}\right|_{\hbar=0} = -\,A_{o}^{\sigma}u_{\sigma}$$

$$\left.\frac{d^{2}\tilde{\tau}}{d\hbar^{2}}\right|_{\hbar=0} = -\,A_{1}^{\sigma}u_{\sigma}$$

$$\vdots \qquad\qquad \vdots \tag{2.12}$$

$$\frac{d^n\tilde{\tau}}{d\kappa^n} = -A_{n-1}{}^{\sigma}u_{\sigma}$$

$$\therefore \tilde{\tau}(\kappa) = -\sum_{n=1}^{\infty} \frac{A_{n-1}{}^{\sigma}u_{\sigma}}{n!}\kappa^n . \tag{2.13}$$

The quantities $A_n{}^{\sigma}u_{\sigma}$ turn out to be of fundamental significance and so we introduce some convenient notation. Define

$$\begin{aligned} a_n(1) &\equiv A_{n-1}{}^{\sigma}u_{\sigma} \\ a_o(1) &\equiv 0 \end{aligned} \tag{2.14}$$

then

$$\tilde{\tau}(\kappa) = -\sum_{n=0}^{\infty} \frac{\kappa^n}{n!} a_n(1) . \tag{2.15}$$

If we define

$$a_n(p) \equiv \prod_{r=1}^{p} \sum_{k_r=0}^{n-\Sigma^{r-1} k_i} \binom{n - \sum_{1=0}^{r-1} k_i}{k_r} a_{k_r} \delta(n-\sum^{p-1} k_i, k_p) \tag{2.16}$$

[so that $a_n(1)$ agrees with (2.14)]

$$\begin{aligned} a_k &\equiv a_k(1) \qquad\qquad k_o \equiv 0 \\ a_{\ell}(0) &= a_o(\ell) = \delta(0,\ell) \end{aligned} \tag{2.17}$$

where $\delta(0,\ell)$ is the Kronecker delta, then we show in section A of Chapter III, that

$$\tilde{\tau}p = (-1)^p \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a_n(p) \ . \tag{2.18}$$

In a similar way, starting from the equation

$$\frac{\partial\bar{\tau}}{\partial x^\mu} = \bar{u}_\mu - \bar{v}_\mu \tag{2.19}$$

we have the following relations

$$\begin{aligned}
\bar{u}^\mu - \bar{v}^\mu &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} A_n^{*\mu} \\
A^{*}_{n-1}{}^{\sigma} u_\sigma &\equiv a_n^*(1) \\
\bar{\tau} &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a_n^*(1) \\
\bar{\tau}p &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a_n^*(p)
\end{aligned} \tag{2.20}$$

we can then deduce the important relations

$$\begin{aligned}
\left.\frac{d^n f(\tilde{\tau})}{d\hbar^n}\right|_{\hbar=0} &= \sum_{p=0}^{n} \frac{(-1)^p}{p!} a_n(p) \left[\frac{d^p}{d\tilde{\tau}^p} f(\tilde{\tau})\right]_{\tilde{\tau}=0} \\
\left.\frac{d^n f(\bar{\tau})}{d\hbar^n}\right|_{\hbar=0} &= \sum_{p=0}^{n} \frac{1}{p!} a_n^*(p) \left[\frac{d^p}{d\bar{\tau}} f(\bar{\tau})\right]_{\bar{\tau}=0} \ .
\end{aligned} \tag{2.21}$$

E. The Recursion Relations

As mentioned above, the $a_n(p)$ are the fundamental quantities in this retarded field theory. Once they are known, all other retarded quantities may be expressed in terms of them. The $a_n(p)$ do not seem to have been investigated in the literature and their determination forms one endeavor of this work. This determination is detailed in section B of Chapter III.

There are several approaches which one could use to find the a's. One method consists of deriving a differential equation which the quantity $\tilde{u} + \tilde{v}^{\mu}$ must satisfy and then using the method of Frobenius to find the solution in the form of a power series in κ. The coefficients of this series will be a_n. The differential equation can be shown to be

$$\frac{d}{d\kappa}(\tilde{u}^{\mu} + \tilde{v}^{\mu}) = -\tilde{a}_u(\tilde{u}^{\mu} + \tilde{v}^{\mu})(\tilde{u}_{\nu} + \tilde{v}_{\nu})u^{\nu} - \frac{1}{\tilde{\rho}}(\tilde{u}^{\mu} + \tilde{v}^{\mu})\tilde{u}_{\nu}u^{\nu} + \frac{1}{\tilde{\rho}}\{u^{\mu} + \tilde{v}^{\mu}(\tilde{u}_{\nu} + \tilde{v}_{\nu})u^{\nu}\} \tag{2.22}$$

where, as in Chapter I, quantities with a tilde over them are to be evaluated at $\tau = \tilde{\tau}$. Because of the non-linearity of the equation ($\tilde{\rho}$ must be expressed in terms of the assumed solution), the solution is quite difficult.

A second approach is to write the vector $\tilde{R}^{\mu}$ as

$$\tilde{R}^{\mu} = \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} Q_n^{\ \mu} \tag{2.23}$$

with

$$Q_n{}^\mu = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} a_n(k+1)[\delta(1,n)u^\mu + v^\mu] \qquad (2.24)$$

(The details important in arriving at Eq. (2.24) are discussed in Chapter III). It can also be shown that the expansion for $\tilde{\rho}^{-1}$ takes the form

$$\frac{1}{\tilde{\rho}} = \frac{1}{\hbar} \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} g_n$$

where the g_n are functions of $a_n(1)$, $a_n(2)$, ... Therefore, because

$$\tilde{u}^\mu + \tilde{v}^\mu = \frac{\tilde{R}^\mu}{\tilde{\rho}}$$

we obtain

$$\tilde{u}^\mu + \tilde{v}^\mu = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} g_k Q_{n+1-k}{}^\mu$$

which results in

$$A_n{}^\mu = \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} g_k Q_{n+1-k}{}^\mu . \qquad (2.25)$$

Because the g_n depend on the a_n, the resulting equation is all but intractible.

The approach which seems most straightforward is to begin with Eq. (2.2). Equations (2.23), (2,24) give

$$\sum_{k=0}^{n} \binom{n}{k} Q_k{}^\mu Q_{n-k\mu} = 0 \qquad Q_o{}^\mu \equiv 0$$

as the necessary condition which the a's have to satisfy. This equation, after a certain amount of disagreeable algebra results in the recursion relation

$$\sum_{p=0}^{n-4} \left[\sum_{k=0}^{p} (-1)^p \left[\frac{1}{(k+2)!(p+2-k)!} - \frac{2(p+1)!}{(k+1)!(p-k)!(p+3)!} \right] a_\mu^{(k)} a^{(p-k)\mu} \right] \cdot$$

$$a_n(p+4) + 2n \sum_{p=0}^{n-2} \frac{(-1)^p}{(p+1)!} \cdot \tag{2.26}$$

$$a_{n-1}(p+1) v^{(p)\mu} u_\mu = a_n(2) \qquad n > 2 .$$

Coupling this equation with

$$a_{n-1} = \frac{1}{2n} [a_n(2) - \sum_{k=2}^{n-2} \binom{n}{k} a_k a_{n-k}] \tag{2.27}$$

gives an explicit expression of a_{n-1} in terms of a_{n-2}, a_{n-3}, ..., a_1, where $a_1 = 1$.

The recursion relation [Eqs. (2.26) and (2.27)] does not give the a's in a form which is convenient for integration over a spacetime surface [such an integration must be performed in order e.g. to evaluate P^μ in Eq. (1.41)]. Therefore, we want to find a way of representing a's which will allow us to do such integrations. This representation is based on what we choose to call form coefficients jC^n_ℓ. If one calculates a typical a_n say a_3:

$$a_3 = \dot{a}^\lambda u_\lambda - \frac{1}{4} g_{\sigma\lambda} a^\sigma a^\lambda + \frac{9}{4} a^\sigma a^\lambda u_\sigma u_\lambda$$

he notices that it is composed of a sum of terms, each of which is a product of kinematic terms $\dot{a}^{\mu}$, a^{μ}, etc, contracted on products of powers of the metric tensor $g_{\mu\nu}$ and powers of the vector u_{μ}. What we will show is that there is a general form for the a_n in which the jC^n_ℓ determine exactly what combinations of powers appear; or to be more precise, the jC^n_ℓ are the coefficients of a linear combination of invariants formed on the tensor quantities $a^{(p)\mu}$, u^{μ}, $g^{\mu\nu}$ which determine the a_n. The jC^n_ℓ determine the invariant a_n just as the metric tensor $g_{\mu\nu}$ completely determines the invariant $d\tau^2$. In section D of Chapter III, we obtain the recursion relation for the jC^n_ℓ. Since their form is rather complicated, we postpone displaying it until Chapter III.

F. Further Considerations

In section E, we solve the following problem: suppose one has a power series for the arbitrary function $f(\lambda)$ i.e. suppose

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} f_n \qquad f_0 \neq 0 \tag{2.28}$$

What is the series expansion for the inverse series $1/f(\lambda)$? The beauty of the final expression for the inversion formula, which we find, is that its form does not depend in an essential way on the value of n, the order of the term. In section F, we obtain the bivariate series for $\tau/\rho(\tau)$, where τ is not necessarially equal to $\bar{\tau}$ but remains arbitrary. This series is important because it allows us to find the expression for $u^{\mu}(\tau) \div v^{\mu}(\tau) = \frac{R^{\mu}(\tau)}{\rho(\tau)}$ which is used in

Eq. (1.30). Referring to Eq. (1.30), it will be seen that the value of τ is set equal to $\bar{\tau}$ only after differentiation and for this reason $\frac{\tau}{\rho(\tau)}$ is expressed in terms of arbitrary τ.

After the differentiation in Eq. (1.30) is performed, certain inverse powers of $\bar{\tau}$ appear. To deal with these terms, we chose to find the series expansions for the quantities $(\frac{\hbar}{\tau})^{\ell}$ where ℓ is an arbitrary positive integer. These series are found in section G. The resulting formula is elegantly simple and stands out among similar quantities whose series expansions are far more complex. Using these formulae and Eq. (2.18) $\bar{\tau}$ can now be eliminated from the series.

Having gone to some length to develop the machinery necessary to express all the field quantities directly in terms of series depending on $\hbar$, the most direct approach might seem the simple combination of all these series to calculate the required results. It turns out, however, to be more elegant to work in terms of double powers series in the variables $\bar{\tau}$ and $\hbar$, and only express $\bar{\tau}$ in terms of $\hbar$ at the end of the calculations.

Now that all the machinery has been developed, the series expansion for $\Theta^{\mu\nu}_{\text{ret}}u_{\nu}$, (which is the essential portion of the differential flux of momentum through the world tube) can be found. The rather intricate calculations which lead to this series are carried out in section H. Finally, in section I, all that remains is the calculation of $\frac{1}{4\pi}\int\Theta^{\mu\nu}_{\text{ret}}u_{\nu}\hbar^{2}\cdot(1+\hbar a_{u})d\Omega$. In order to accomplish this, a general formula is derived for the tensor $\int u^{\mu}\ldots u^{\nu}d\Omega$, where the dots indicate an arbitrary even number of vectors u^μ (the tensor is zero for an odd number of vectors). When this formula is used in conjunction with the results of H, the general series expansion for the momentum flux is obtained.

CHAPTER III

THE CALCULATIONS IN DETAIL

A. The Quantities $a_n(p)$

In this chapter, we derive in detail the results discussed in Chapter II. First, consider the product of two power series $B = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} B_n$, and $C = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} C_n$. Their direct product is

$$BC = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\hbar^{m+n}}{m!n!} B_m C_n \quad . \tag{3.1}$$

It is clear that formally at least, we can group the terms together which have $\hbar$ to the same power. Thus we have

$$BC = \sum_{n=0}^{\infty} \hbar^n \sum_{k=0}^{n} \frac{B_k}{k!} \frac{C_{n-k}}{(n-k)!} = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B_k C_{n-k} \quad . \tag{3.2}$$

Refining the binomial coefficient as

$$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!} \quad . \tag{3.3}$$

We obtain

$$BC = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{k=0}^{n} \binom{n}{k} B_k C_{n-k} \quad . \tag{3.4}$$

This is the usual form for the Cauchy product of two power series. If, in particular, the two series are identical, we obtain the square of a series in the form

$$B^2 = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k} . \tag{3.5}$$

Equation (3.5) is the special case p = 2 of

$$B^p = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \prod_{r=1}^{p} \sum_{k_r=0}^{n-\sum_{i=0}^{r-1} k_i} \binom{n-\sum_{i=0}^{r-1} k_i}{k_r} B_{k_r} \delta(n-\sum_{i=0}^{p-1} k_i, k_p) \tag{3.6}$$

$$k_o \equiv 0 .$$

This general expression for the pth power of a power series has not, to the author's knowledge appeared in the literature.

If we define

$$a_n(p) \equiv \prod_{r=1}^{p} \sum_{k_r=0}^{n-\Sigma^{r-1} k_i} \binom{n-\sum^{r-1} k_i}{k_r} a_{k_r} \delta(n-\sum^{p-1} k_i, k_p)$$

$$a_k \equiv a_k(1) \qquad a_\ell(0) = a_o(\ell) = \delta(0,\ell) \tag{3.7}$$

It will follow from Eq. (2.) and Eq. (3.5) that

$$\tau^p = (-1)^p \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a_n(p) . \tag{3.8}$$

Before proceeding, let us prove two properties of the a(p)'s:

$$\text{(i)} \qquad a_n(p+s) = \sum_{k=0}^{n} \binom{n}{k} a_k(p)\, a_{n-k}(s)$$

$$\text{(ii)} \qquad a_n(p) = 0 \qquad\qquad p > n$$

The following simple proofs should suffice: To begin, suppressing the minus sign, we have

$$\tilde{\tau}^p = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a_n(p)$$

$$\tilde{\tau}^s = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a_n(s)$$

$$\tilde{\tau}^p \cdot \tilde{\tau}^s = \tilde{\tau}^{(p+s)} = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a_n(p+s) \quad .$$

But considering the Cauchy product of

$$\sum \frac{\hbar^n}{n!} a_n(p) \quad \text{and} \quad \sum \frac{\hbar^n}{n!} a_n(s)$$

we have the result

$$\sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{k=0}^{n} \binom{n}{k} a_k(p)\, a_{n-k}(s) \ . \tag{3.9}$$

Therefore, equating the two expressions, we have

$$a_n(p+s) = \sum_{k=0}^{n} \binom{n}{k} a_k(p)\, a_{n-k}(s) \tag{3.10}$$

As for (ii), notice that since the first non-zero term in the expansion of $\tilde{\tau}$ is proportional to $\hbar$, $\tilde{\tau}^p$ has as its first non-zero term something that goes as $\hbar^p$. Thus

$$a_n(p) = 0 \qquad p > n \,. \tag{3.11}$$

Let us now obtain the expression for the nth derivative of $f(\tilde{\tau})$. Expand $f(\tilde{\tau})$ as a power series in $\tilde{\tau}$,

$$f(\tilde{\tau}) = \sum_{p=0}^{\infty} \frac{\tilde{\tau}^p}{p!} \left[\frac{d^p}{d\tilde{\tau}^p} f(\tilde{\tau}) \right]_{\tilde{\tau}=0} \,. \tag{3.12}$$

Use Eq. (3.8) for $\tilde{\tau}^p$; then

$$f(\tilde{\tau}) = \sum_{p=0}^{\infty} \frac{1}{p!} (-1)^p \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a_n(p) \left[\frac{d^p}{d\tilde{\tau}^p} f \right]_{\tilde{\tau}=0} \tag{3.13}$$

which, in view of our assumptions about power series convergence along with property (ii) of the a's gives, after interchange of the order of summation

$$f(\tilde{\tau}) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{p=0}^{n} \frac{(-1)^p}{p!} a_n(p) \left[\frac{d^p}{d\tilde{\tau}^p} f(\tilde{\tau}) \right]_{\tilde{\tau}=0} \,. \tag{3.14}$$

Therefore, the nth derivative of a function of $\tilde{\tau}$ is given by

$$\left[\frac{d^n}{d\hbar^n} f(\tilde{\tau}) \right] \Big|_{\hbar=0} = \sum_{p=0}^{n} \frac{(-1)^p}{p!} a_n(p) \left[\frac{d^p}{d\tilde{\tau}^p} f(\tilde{\tau}) \right]_{\tilde{\tau}=0} \,. \tag{3.15}$$

It may be reassuring to know that Eq. (3.15) can be derived in a manner which does not depend on the validity of interchanging summations. In fact it was originally discovered by the author using a formula due to Schwatt[4], for the nth implicit derivative. Indeed, while the method of interchanging summations will be employed freely throughout these derivations, the results do not depend on the interchange as one who wants to do sufficient algebra can show.

B. The Recursion Relation

As discussed in Chapter II, the a's play a central role in the retarded field theory. We want therefore to give a systematic method for finding them. Let us apply the results of the previous section by expanding $z^\mu(\tilde{\tau})$ in powers of $\hbar$.

$$\tilde{z}^\mu = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{p=0}^{n} \frac{(-1)^p}{p!} a_n(p) \left[\frac{d^p}{d\tau^p} z^\mu(\tau) \right]_0 \tag{3.16}$$

or

$$\tilde{z}^\mu = z^\mu + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \sum_{p=0}^{n} \frac{(-1)^p}{p!} a_n(p) \left[\frac{d^p}{d\tau^p} z^\mu(\ ) \right]_0 \tag{3.17}$$

Not let

$$p = p' + 1 .$$

Then we have, suppressing the prime,

$$\tilde{z}^\mu = z^\mu + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \sum_{p=0}^{n-1} \frac{(-1)^{p+1}}{(p+1)!} a_n(p+1) \left[\frac{d^{p+1}}{d\tau^{p+1}} z^\mu(\tau) \right]_0$$

or

$$\tilde{z}^{\mu} = \check{z}^{\mu} + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \sum_{p=0}^{n-1} \frac{(-1)^{p+1}}{(p+1)!} a_n(p+1) v^{(p)\mu} . \qquad (3.18)$$

Since $a_n(0) = 0$ when $n \neq 0$. Now from Eq. (1.20),

$$\tilde{R}^{\mu} = x^{\mu} - \tilde{z}^{\mu} = x^{\mu} - z^{\mu} - \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \sum_{p=0}^{n-1} \frac{(-1)^{p+1}}{(p+1)!} a_n(p+1) v^{(p)\mu}. \qquad (3.19)$$

Then using

$$x^{\mu} - z^{\mu} = \hbar u^{\mu}. \qquad (3.20)$$

We have

$$\tilde{R}^{\mu} = \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} \sum_{p=0}^{n-1} \frac{(-1)^{p}}{(p+1)!} a_n(p+1) [\delta(1,n)u^{\mu} + v^{(p)\mu}] . \qquad (3.21)$$

For convenience in manipulation we define

$$Q_n^{\ \mu} \equiv \sum_{k=0}^{n-1} \frac{(-1)^{k}}{(k+1)!} a_n(k+1) [\delta(1,n)u^{\mu} + v^{(k)\mu}]$$

$$Q_o^{\ \mu} \equiv 0 \quad . \qquad (3.22)$$

Then we may write

$$\tilde{R}^{\mu} = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} Q_n^{\ \mu}. \qquad (3.23)$$

It can be shown that $\tilde{\rho}^{-1}$ can be expanded in the form

$$\frac{1}{\tilde{\rho}} = \frac{1}{\hbar} \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} g_n \qquad (3.24)$$

Therefore because

$$(\tilde{u}^{\mu} + \tilde{v}^{\mu}) = \frac{\tilde{R}^{\mu}}{\tilde{\rho}} \quad . \tag{3.25}$$

We have,

$$\tilde{u}^{\mu} + \tilde{v}^{\mu} = \frac{1}{\hbar} \sum_{n=0}^{\infty} \frac{\hbar^{n}}{n!} \sum_{k=0}^{n} \binom{n}{k} g_k \, Q_{n-k}{}^{\mu} \quad . \tag{3.26}$$

Let

$$n = n' + 1 \; .$$

Then suppressing the prime and making use of the fact that

$$Q_o{}^{\mu} = 0$$

we have

$$\begin{aligned} \tilde{u}^{\mu} + \tilde{v}^{\mu} &= \frac{1}{\hbar} \sum_{n=0}^{\infty} \frac{\hbar^{n+1}}{(n+1)!} \sum_{k=0}^{n+1} \binom{n+1}{k} g_k \, Q_{n+1-k}{}^{\mu} \\ &= \sum_{n=0}^{\infty} \frac{\hbar^{n}}{n!} \left(\frac{1}{n+1}\right) \sum_{k=0}^{n+1} \binom{n+1}{k} g_k \, Q_{n+1-k}{}^{\mu} \end{aligned} \tag{3.27}$$

Thus we obtain the result

$$A_n{}^{\mu} = \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} g_k \, Q_{n+1-k}{}^{\mu} \quad . \tag{3.28}$$

As discussed in Chapter II, we could use this as an equation for finding the a's. It turns out to be simpler to follow an approach in which we begin with instead

$$\tilde{R}^{\mu}\tilde{R}_{\mu} = 0 \; . \tag{3.29}$$

In terms of Eq. (3.23) this condition is

$$\sum_{k=0}^{n} \binom{n}{k} Q_k{}^{\mu} Q_{n-k_{\mu}} = 0$$
$$Q_o{}^{\mu} = 0 \tag{3.30}$$

Writing this out, and using Eq. (3.22), we have

$$\sum_{k=1}^{n-1} \sum_{p=0}^{k-1} \sum_{q=0}^{n-k-1} \binom{n}{k} \frac{(-1)^{p+q}}{(p+1)!(q+1)!} a_k(p+1) a_{n-k}(q+1) \cdot$$
$$[\delta(1,k)u^{\mu} + v^{(p)\mu}][\delta(1,n-k)u_{\mu} + v^{(q)}{}_{\mu}] = 0 \; . \tag{3.31}$$

When the kronecker deltas have done their work, the result is

$$\sum_{p=0}^{n-1} \sum_{q=0}^{n-1} \frac{(-1)^{p+q}}{(p+1)!(q+1)!} v^{(p)\mu} v^{(q)}{}_{\mu} \sum_{k=0}^{n} \binom{n}{k} a_k(p+1) a_{n-k}(q+1)$$
$$+ 2n \sum_{p=0}^{n-2} \frac{(-1)^p}{(p+1)!} a_{n-1}(p+1) a_1(1) \; v^{(p)\mu} u_{\mu} = 0 \qquad (n>2). \tag{3.32}$$

But

$$\sum_{k=0}^{n} \binom{n}{k} a_k(p+1) a_{n-k}(q+1) = a_n(p+q+2) \; . \tag{3.33}$$

So

$$\sum_{p=0}^{n-1} \sum_{q=0}^{n-1} \frac{(-1)^{p+q}}{(p+1)!(q+1)!} v^{(p)\mu} v^{(q)}{}_{\mu} \; a_n(p+q+2)$$
$$+ 2n \sum_{p=0}^{n-2} \frac{(-1)^p}{(p+1)!} a_{n-1}(p+1) v^{(p)\mu} u_{\mu} = 0 \; . \tag{3.34}$$

Now transform the first term,

$$Y = \sum_{p'=0}^{n-1} \sum_{q'=0}^{n-1} \frac{(-1)^{p'+q'}}{(p'+1)!(q'+1)!} v^{(p')\mu}{}_{v}{}^{(q')}{}_{\mu} a_n(p'+q'+a) \tag{3.35}$$

by letting

$$p = p' + q' \qquad q' = p-p'$$

then

$$Y = \sum_{p=0}^{n-2} \sum_{p'=0}^{p} \frac{(-1)^{p}}{(p'+1)!(p+1-p')!} v^{(p')\mu}{}_{v}{}^{(p-p')}{}_{\mu} a_n(p+2) \tag{3.36}$$

since $a_n(p) = 0$ for $p > n$. Thus the equation determining the a's is

$$\sum_{p=0}^{n-2} \sum_{q=0}^{p} \frac{(-1)^{p}}{(q+1)!(p+1-q)!} v^{(q)\mu}{}_{v}{}^{(p-q)}{}_{\nu} a_n(p+2)$$
$$+ 2n \sum_{p=0}^{n-2} \frac{(-1)^{p}}{(p+1)!} a_{n-1}(p+1) v^{(p)\mu} u_{\mu} = 0 \qquad \text{for } n>2. \tag{3.37}$$

It is more convenient for later work to transform the first term in the equation. First consider

$$v^{\mu}(\tau) = \sum_{s=0}^{\infty} \frac{v^{(s)\mu}}{s!} \tau^{s}$$
$$a^{\mu}(\tau) = \sum_{t=0}^{\infty} \frac{a^{(t)\mu}}{t!} \tau^{t} . \tag{3.38}$$

Since v^μ is orthagonal to a^μ,

$$0 = v^\mu(\tau)a_\mu(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{k=0}^{n} \binom{n}{k} v^{(k)}{}_\mu a^{(n-k)\mu} \tag{3.39}$$

or

$$\sum_{k=0}^{n} \binom{n}{k} v^{(k)\mu} a_\mu{}^{(n-k)} = 0 \tag{3.40}$$

$$\therefore\ v^\mu a_\mu{}^{(n)} + \sum_{k=1}^{n} \binom{n}{k} v^{(k)\mu} a_\mu{}^{(n-k)} = 0$$

$$\text{for } n \geq 1, \tag{3.41}$$

let

$$k = k' + 1$$

giving

$$v^\mu a_\mu{}^{(n)} = -\sum_{k=0}^{n-1} \binom{n}{k+1} a^{(k)\mu} a_\mu{}^{(n-k-1)} \quad . \tag{3.42}$$

Changing indices again gives

$$v^\mu v_\mu{}^{(n)} = -\sum_{k=0}^{n-2} \binom{n-1}{k+1} a^{(k)\mu} a_\mu{}^{(n-s-k)} - \delta(o,n) \tag{3.43}$$

Without further qualification, this equation is valid only for $n \geq 2$. At this point we introduce the following important conventions:

(i) Whenever the upper summation limit is less than the lower summation limit, the sum is identically zero.

(ii) The integer arguments of all non-zero quantities are non-negative, i.e. negative integer arguments imply the quantity is zero.

Convention (i) makes Eq. (3.43) valid for all n. Convention (ii) will be valuable throughout this chapter.

Now write Y as

$$Y = \sum_{p=0}^{1} \sum_{q=0}^{p} \frac{(-1)^p}{(q+1)!(p+1-q)!} v^{(q)\mu} v^{(p-q)}{}_{\mu} a_n(p+2)$$
$$+ \sum_{p=2}^{n-2} \sum_{q=0}^{p} \frac{(-1)^p}{(q+1)!(p+1-q)!} v^{(q)\mu} v^{(p-q)}{}_{\mu} a_n(p+2) \; . \tag{3.44}$$

The first sum is just

$$- a_n(2), \text{ since } v^{\mu} a_{\mu} = 0 \; .$$

When we separate the q = 0 and q = p terms from the rest of the summation, the second term in Eq. (3.44) becomes

$$\sum_{p=2}^{n-2} \Big[\sum_{q=1}^{p-1} \Big(\frac{(-1)^p}{(q+1)!(p+1-q)!} v^{(q)\mu} v^{(p-q)}{}_{\mu} \Big)$$
$$+ \frac{2(-1)^p}{(p+1)!} v^{\mu} v^{(p)}{}_{\mu} \Big] a_n(p+2) \tag{3.45}$$

Carrying out another transformation similar to those used already, and use of the formula for $v^{\mu} v^{(n)}{}_{\mu}$ yields for the above equation

$$\sum_{p=2}^{n-2}\Big[\sum_{k=0}^{p-2}\Big\{\frac{(-1)^p}{(k+2)!(p-k)!}-\frac{2(-1)^p(p-1)!}{(k+1)!(p-2-k)!(p+1)!}\Big\}\cdot$$

$$a^{(k)\mu}\,a^{(p-2-k)}_{\mu}\Big]a_n(p+2)\ . \tag{3.46}$$

One final transformation gives then for Eq. (3.37) the form

$$\sum_{p=0}^{n-4}\Big[\sum_{k=0}^{p}(-1)^p\Big\{\frac{1}{(k+2)!(p+2-k)!}-\frac{2(p+1)!}{(k+1)!(p-k)!(p+3)!}\Big\}\cdot a^{(k)}_{\mu}\,a^{(p-k)\mu}\Big]\cdot$$

$$a_n(p+4)\ +\ 2n\sum_{p=0}^{n-2}\frac{(-1)^p}{(p+1)!}\,a_{n-1}(p+1) \tag{3.47}$$

$$v^{(p)\mu}u_{\mu}\ =\ a_n(2)\qquad\qquad n>2$$

Using the definition of $a_n(2)$ we have

$$a_n(2)\ =\ 2n\ a_{n-1}\ +\ \sum_{k=2}^{n-2}\binom{n}{k}a_k a_{n-k}\ . \tag{3.48}$$

Combining Eq. (3.47 and Eq. (3.48) gives at last, the basic recursion relation as

$$a_{n-1}\ =\ \frac{1}{2n}\sum_{p=0}^{n-4}\Big[\sum_{k=0}^{p}(-1)^p\Big\{\frac{1}{(k+2)!(p+2-k)!}-\frac{2(p+1)!}{(k+1)!(p-k)!(p+3)!}\Big\}$$

$$a^{(k)}_{\mu}\,a^{(p-k)\mu}\Big]a_n(p+4)\ +\ \sum_{p=0}^{n-2}\frac{(-1)^p}{(p+1)!}\,a_{n-1}(p+1)\cdot \tag{3.49}$$

$$v^{(p)\mu}u_{\mu}\ -\ \frac{1}{2n}\sum_{k=2}^{n-2}\binom{n}{k}a_k a_{n-k}\ .$$

Similar considerations, which we shall omit, give the recursion relations for the advanced quantities in the form

$$a^*_{n-1} = \frac{1}{2n} \sum_{p=0}^{n-4} \left[\sum_{k=0}^{p} \left\{ \frac{1}{(k+2)!(p+2-k)!} - \frac{2(p+1)!}{(k+1)!(p-k)!(p+3)!} \right\} \cdot \right.$$

$$\left. \alpha^{(k)}_{\mu} \alpha^{(p-k)\mu} \right] a^*_n (p+4) + \tag{3.50}$$

$$\sum_{p=0}^{n-2} \frac{1}{(p+1)!} a^*_{n-1} (p+1) v^{(p)\mu} u_\mu - \frac{1}{2n} \sum_{k=2}^{n-2} \binom{n}{k} a^*_k a^*_{n-k}$$

C. <u>Algebraic Description of Field Quantities</u>

It will become evident in the following development that all the quantities which appear in the current classical charged particle theory can be cast into a form which is in one-one correspondence with a form which depends only on algebraic quantities for its definition. This result will allow us to derive explicit expressions for the propertime rate of change of the four-momentum associated with a charge, which would otherwise possibly be unobtainable.

To demonstrate this, we begin by writing down the most general form which the a's could have and still depend on the kinematic terms:

$$a_{n+1}(1) \equiv A_n{}^{\mu} u_\mu =$$

$$\sum_{j=0}^{n} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\textstyle\sum \alpha^{(g_r)}\} \prod_{r=1}^{j-\ell} \{\textstyle\sum \alpha^{(p_r)}\}_j C^n_\ell \cdot \tag{3.51}$$

$$G_\ell U_{j-\ell}\, \delta(n-j-\sum^{\ell-1} q_i - \sum^{j-\ell} p_i, q_\ell)$$

with the following definitions

$$
\begin{aligned}
U_\ell &\equiv u_{\mu_1} u_{\mu_2} \ldots u_{\mu_\ell} \\
G_{2\ell} &\equiv g_{\mu_1\nu_1} g_{\mu_2\nu_2} \ldots g_{\mu_\ell\nu_\ell} \\
a^{(p_r)} &\equiv \left[\frac{d^{p_r}}{d\tau^{p_r}}\, a \right]_{\tau=0}
\end{aligned}
\tag{3.52}
$$

and where the form coefficients have the following properties

$$
\begin{aligned}
{}_oC^n_p &\equiv \delta^n_o\, \delta_{o,p} \\
{}_jC^n_p &= 0 \text{ whenever } \begin{array}{l} p>j \\ j>n \\ p \text{ odd integer.} \end{array}
\end{aligned}
\tag{3.53}
$$

We also define $\sum^{j-1}(q+p) \equiv \sum_{l=0}^{\ell-1} q_i - \sum_{l=0}^{j-\ell} p_i$, where $q_o = p_o = 0$ and

$$\prod_{r=1}^{j=0} \sum a^{(p_r)} \delta[n-j- \sum^{j-1} (q+p)\ q\ell] = 1\ .$$

While we have not explicitly indicated it, ${}_jC^n_\ell$ is a function of q_1, q_2, ..., q_ℓ p_1,p_2, ... $p_{j-\ell}$. Finally the a's which have the indices q_i are contracted on the G's and those which have the indices p_i are contracted on the U's. The range of summation on the p_i and q_i is n-j. Two important points require discussion. The condition ${}_jC^n_\ell = 0$ j>n, and the limitations imposed by the kronecker delta in Eq. (3.51) are necessary. To see this, consider the following dimensional argument.

$\tilde{\tau}$ has the dimension of time. The dimension of λ is length. Thus, keeping Eq. (2.15) in mind, we see that only those combinations of kinematic terms can appear in the $a_n(1)$ which, when divided by the appropriate power of the speed of light, convert (length) into (time). Let [T] denote dimensions of time, [L] dimensions of length. The dimension of the nth term of the series for $\tilde{\tau}$ is thus

$$\underbrace{\frac{[L]^\sigma}{[T]^q}}_{a_n} \underbrace{[L]^n}_{\lambda^n} \frac{[T]^{n+p}}{[L]^{n+p}}$$

where the last factor is the appropriate power of c. The necessary condition is

$$n + p - q = 1 \quad \text{and} \quad \sigma = p \quad . \tag{3.54}$$

Now it must be that $q \geq 2p$, when $n > 1$; for if $q < 2p$, then there must be at least one factor which is v^μ (if all of the factors were a^μ or higher derivatives then $q \geq 2p$). When v^μ is contracted on u_μ it gives no contribution. When it is contracted on $g_{\mu\nu}$ with some other kinematic term then either the resulting term is expressible as a term in which $g \geq 2p$ since for this C^2 are produced [see Eq. (3.43)] or else one has either the product

$$v_\mu a^\mu \ldots \qquad \text{or} \qquad v_\mu v^\mu \ldots$$

In the first case the result is zero; in the second, the factor $v_\mu v_\mu = 1$ no longer depends on kinematic terms. Thus in any case

$$q \geq 2p \quad \text{when } n > 1 \, . \tag{3.55}$$

Equations (3.54) and (3.55) together imply that the maximum that p can be is n - 1. Thus the restriction ${}_jC^n_\ell = 0$, $j>n$. A simple extension of these arguments results in the further limitations imposed by the kronecker delta.

We now want to show that $a_{n+1}(2)$, $a_{n+1}(3)$, etc. have the same form as $a_{n+1}(1)$. We will show that $a_{n+1}(2)$ can be written as

$$a_{n+1}(2) = \sum_{j=0}^{n-1} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\} \cdot$$
$$ {}_jC^{n-1}_\ell(2) \; G_\ell U_{j-\ell} \; \delta(n-1-j-\sum^{j-1}(q+p), q_\ell) \tag{3.56}$$

and will determine the expression for the constants ${}_jC^{n-1}_\ell(2)$.

To show this, we begin with

$$a_{n+1}(2) = \sum_{k=0}^{n-1} \binom{n+1}{k+1} a_{k+1} \, a_{n-k}$$

which is just Eq. (3.7) with a minor transformation of indices. In accordance with Eq. (3.51), we have

$$a_{k+1} = \sum_{j_1=0}^{k} \sum_{\ell_1=0}^{j_1} \prod_{r=1}^{\ell_1} \{\sum a^{(q_r)}\} \prod_{r=1}^{j_1-\ell_1} \{\sum a^{(p_r)}\}_{j_1} C^k_{\ell_1} G_{\ell_1} U_{j_1-\ell_1} \delta[k-j_1-\sum^{j_1-\ell_1} (q+p), q_{\ell_1}]$$

and

$$a_{n-k} = \sum_{j_2=0}^{n-1-k} \sum_{\ell_2=0}^{j_2} \prod_{r=1}^{\ell_2} \{\sum a^{(q'_r)}\} \prod_{r=1}^{j_2-\ell_2} \{\sum a^{(p'_r)}\}_{j_2} C^{n-1-k}_{\ell_2} G_{\ell_2} U_{j_2-\ell_2} \cdot$$

$$\delta[n-1-k-j_2-\sum^{j_2-1} (q'+p'), q'_{\ell_2}]$$

and

$$a_{n+1}(2) = \sum_{k=0}^{n-1} \binom{n+1}{k+1} \sum_{j_1=0}^{k} \sum_{\ell_1=0}^{j_1} \prod_{r=1}^{\ell_1} \{\sum a^{(q_r)}\} \prod_{r=1}^{j_1-\ell_1} \{\sum a^{(p_r)}\}_{j_1} C^{k_1}_{\ell_1} G_{\ell_1}$$

$$U_{j_1-\ell_1} \sum_{j_2=0}^{n-1-k} \sum_{\ell_2=0}^{j_2} \prod_{r=1}^{\ell_2} \{\sum a^{(q'_r)}\} \prod_{r=1}^{j_2-\ell_2} \{\sum a^{(p'_r)}\}_{j_2} C^{n-1-k}_{\ell_2} \cdot$$

$$G_{\ell_2} U_{j_2-\ell_2} \delta[k-j_1-\sum^{j_1-1} (q+p), q_{\ell_1}]\ \delta[n-1-k-j_2-\sum^{j_2-1} (q'+p'), q'_{\ell_2}]\ .$$

We now do the following three things:

(i) since ${}_{j_1}C^k_{\ell_1} = 0$ when $j_1 > k$ we may raise the upper limits of j_1 and j_2 to n. Then we may interchange the summations over j_1, j_2 with that over k.

(ii) note that $G_{\ell_1} \cdot G_{\ell_2} = G_{\ell_1+\ell_2}$, $U_{\ell_1} \cdot U_{\ell_2} = U_{\ell_1+\ell_2}$

(iii) relable the q_r' and p_r' according as

$$q_r' \to q_{r+\ell_1}$$

$$p_r' \to p_{r+j_1-\ell_1} \quad .$$

These manipulations result in

$$a_{n+1}(2) = \sum_{j_1=0}^{n} \sum_{j_2=0}^{n} \sum_{\ell_1=0}^{j_1} \sum_{\ell_2=0}^{j_2} \prod_{r=1}^{\ell_1+\ell_2} \{\sum a^{(q_r)}\} \cdot$$

$$\prod_{r=1}^{j_1+j_2-\ell_1-\ell_2} \{\sum a^{(p_r)}\} \sum_{k=0}^{n-1} \binom{n+1}{k+1} {}_{j_1}C^{k}_{\ell_1}\, {}_{j_2}C^{n-1-k}_{\ell_2} \cdot \tag{3.57}$$

$$G_{\ell_1+\ell_2} U_{j_1+j_2-\ell_1-\ell_2}\ \delta[k-j_1-\sum^{j_1-1} (q+p), q_{\ell_1}] \cdot$$

$$\delta(n-1-k-j_2-\sum_{\ell_1\ell_1}^{\ell_2} q_i - \sum_{j_1-\ell_1}^{j_1+j_2-\ell_1-\ell_2} p_i, q_{\ell_1+\ell_2}) \ .$$

It will be seen that the summation over k gives a contribution only when

$$k = \sum^{\ell_1} q_i + \sum^{j_1-\ell_1} p_i + j_1 \ . \tag{3.58}$$

Thus the sum over k and the product of the two kronecker deltas may be replaced by

$$\delta(n-1-j_1-j_2-\sum^{\ell_1} q_i - \sum^{j_1-\ell_1} p_i - \sum_{\ell_1}^{\ell_2-1} q_i - \sum_{j_1-\ell_1}^{j+j_2-\ell_1-\ell_2} p_i, q_{\ell_1+\ell_2}) \ . \tag{3.59}$$

Finally, carry out the sum over ℓ_1, ℓ_2 in such a way that the terms are grouped according to the value of $\ell_1+\ell_2$, in the same way that we grouped the terms in the double series (3.1) according to the powers of $\hbar$. This suggests that we introduce

$$\ell = \ell_1+\ell_2 \quad .$$

In the same way, we carry out the sum over j_1, j_2 and define

$$j = j_1+j_2 \quad .$$

Then we obtain

$$\sum_{j=0}^{n-1} \sum_{\ell=0}^{j} \sum_{j_1=0}^{j} \sum_{\ell_1=0}^{\ell} \prod_{r=1}^{\ell} \{a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{a^{(p_r)}\} \cdot$$

$$\begin{pmatrix} n+1 \\ \\ j_1+1+\sum^{\ell_1} q_i + \sum^{j_1-\ell_1} p_i \end{pmatrix} {}_{j_1}C_{\ell_1}^{j_1+\sum q+\sum p} \; {}_{j-j_1}C_{\ell-\ell_1}^{n-1-j_1-\sum q-\sum p} \tag{3.60}$$

$$G_\ell U_{j-\ell}\,\delta(n-1-j-\sum^{\ell-1} q_i-\sum^{j-\ell} p_i, q_\ell)$$

with

$${}_jC_\ell^{n-1}(2) = \sum_{j_1=0}^{j} \sum_{\ell_1=0}^{\ell} \begin{pmatrix} n+1 \\ j_1+1+\sum^{\ell_1} q_i + \sum^{j_1-\ell_1} p_i \end{pmatrix}$$

$$\tag{3.61}$$

$${}_{j_1}C_{\ell_1}^{j_1+1+\sum q+\sum p} \; {}_{j-j_1}C_{\ell-\ell_1}^{n-1-j_1-\sum q-\sum p_1}$$

Then

$$a_{n+1}(2) = \sum_{j=0}^{n-1} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\textstyle\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\textstyle\sum a^{(p_r)}\}$$

$$ {}_jC_\ell^{n-1}(2)\; G_\ell U_{j-\ell}\; \delta(n-1-j-\sum^{\ell-1} q_i - \sum^{j-\ell} p_i, q_\ell) \quad . \tag{3.62}$$

For purposes of generalization, we make the definitions

$$\lambda(r) \equiv \sum_{i=0}^{r} \ell_i [1-\delta(0,i)]$$

$$K(r) \equiv \sum_{i=0}^{r} (j_i-\ell_i)[1-\delta(0,i)] \tag{3.63}$$

$$\textstyle\sum(r)q \equiv \displaystyle\sum_{i=\lambda(r-1)}^{\lambda(r)} q_i \{1 - \delta[\lambda(r-1),\lambda(r)]\}$$

$$\textstyle\sum(r)p \equiv \displaystyle\sum_{i=K(r-1)}^{K(r)} p_i \{1 - \delta[K(r-1),K(r)]\} \quad . \tag{3.64}$$

Now our previous result can be written

$${}_jC_\ell^{n-1}(2) = \sum_{j_1=0}^{j} \sum_{\ell_1=0}^{\ell} \binom{n+1}{j_1+1+\sum(1)q+\sum(1)p}$$

$${}_{j_1}C_{\ell_1}^{j_1+\sum(1)q+\sum(1)p} \quad {}_{j-j_1}C_{\ell-\ell_1}^{n-1-j_1-\sum(1)q-\sum(1)p} \quad . \tag{3.65}$$

Next we consider $a_{n+1}(3)$.

$$a_{n+1}(3) = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1-k_1} \binom{n+1}{k+1} \binom{n-k_1}{k_2+1} a_{k_1+1}\, a_{k_2+1} \cdot a_{n-1-k_1-k_2} \quad (3.66)$$

We just write the results since there is essentially nothing new in the calculation

$$a_{n+1}(3) = \sum_{j=0}^{n-2} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\textstyle\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\textstyle\sum a^{(p_r)}\} \cdot {}_jC_\ell^{n-2}(3)\; G_\ell U_{j-\ell}\; \delta(n-2-j-\sum^{\ell-1} q_i - \sum^{j-\ell} p_i, q_\ell)$$

$${}_jC_\ell^{n-2}(3) = \sum_{j_1=0}^{j} \sum_{j_2=0}^{j-j_1} \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\ell-\ell_1} \binom{n+1}{j_1+1+\sum(1)q+\sum(1)p} \binom{n-j_1-\sum(1)q-\sum(1)p}{j_2+1+\sum(2)q+\sum(2)p} {}_{j_1}C_{\ell_1}^{j_1+\sum(1)q+\sum(1)p}\; {}_{j_2}C_{\ell_2}^{j_2+\sum(2)q+\sum(2)p}\; {}_{j-j_1-j_2}C_{\ell-\ell_1-\ell_2}^{n-2-j_1-j_2-\sum(1)q-\sum(1)p-\sum(2)q-\sum(2)p} \,. \quad (3.67)$$

We can easily generalize if we define

$$\sum(r)(q+p) \equiv \sum(r)q + \sum(r)p$$

$$\sigma(r) \equiv j_r + 1 + \sum(r)(q+p) \qquad r>0 \tag{3.68}$$

$$\sigma(r=0) \equiv 0 .$$

Now we may write

$$a_{n+1}(s) = \sum_{j=0}^{n+1-s} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\} \cdot$$

$$ {}_jC_\ell^{n+1-s}(s) \; G_\ell U_{j-\ell} \; \delta[n+1-s-j- \sum^{j-1} (q+p), q_\ell] \tag{3.69}$$

with

$$ {}_jC_\ell^{n+1-s}(s) = \prod_{r=1}^{s} \sum_{j_r=0}^{j-\sum^{r-1} j_i} \sum_{\ell_r=0}^{\ell-\sum^{r-1}\ell_i} \begin{pmatrix} n+1-\sum^{r-1}\sigma(i) \\ \sigma(r) \end{pmatrix} \cdot$$

$$ {}_{j_r}C_{\ell_r}^{\sigma(r)-1} \; \delta(j-\sum^{s-1} j_i, j_s) \; \delta(\ell-\sum^{s-1} \ell_i, \ell_s) \; \delta[n+1-\sum^{s-1}\sigma(i), \sigma(s)] \tag{3.70}$$

D. Form Coefficient Recursion Relation

Since we are interested in developing the algebraic description of field quantities, we want to obtain a recursion relation for the form coefficients. To do this, consider the fundamental recursion relation on the a's, which we shall transform piece by piece. Starting with the last term we have

$$\sum_{p=0}^{n-2} \frac{(-1)^p}{(p+1)!} a_{n-1}(p+1) v^{(p)\mu} u_\mu = \sum_{p=1}^{n-2} \frac{(-1)^p}{(p+1)!} a_{n-1}(p+1) a^{(p-1)\mu} u_\mu$$

This term becomes, upon transformation

$$\sum_{p=0}^{n-3} \frac{(-1)^{p+1}}{(p+2)!} a_{n-1}(p+2)\ a^{(p)\mu} u_\mu \quad .$$

Therefore

$$\sum_{p=0}^{n-3} \frac{(-1)^{p+1}}{(p+2)!} a_{n-1}(p+2) a^{(p)\mu} u_\mu =$$

$$\sum_{p=0}^{n-3} \frac{(-1)^{p+1}}{(p+2)!} a^{(p)} U_1 \sum_{j=0}^{n-3-p} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \cdot \tag{3.71}$$

$$\prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\}\ {}_jC_\ell^{n-3-p}(p+2)\ G_\ell U_{j-\ell}\ \delta[n-3-j-\sum^{j-1}(q+p), q_\ell] \ ,$$

where we have used Eq. (3.69). If we relable $p = p_{j+1-\ell}$, then the definition

$${}_{j+1}D_\ell^{n-3} \equiv \frac{(-1)^{p_{j+1-\ell}+1}}{(p_{j+1-\ell} + 2)!}\ {}_jC_\ell^{\,n-3-p_{j+1-\ell}}(p_{j+1-\ell}+2) \tag{3.72}$$

allows us to write the right hand side of Eq. (3.71) as

$$\sum_{j=0}^{n-3} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j+1-\ell} \{\sum a^{(p_r)}\} \cdot \tag{3.73}$$

$${}_{j+1}D_{\ell}^{n-3} \; G_{\ell} U_{j+1-\ell} \; \delta[n-3-j-\sum^{j}(q+p), q_{\ell}] \quad .$$

Now let

$$\tilde{\Lambda}_k^p \equiv (-1)^p \left\{\frac{1}{(k+2)!(p+2-k)!} - \frac{2(p+1)!}{(k+1)!(p-k)!(p+3)}\right\} \quad . \tag{3.74}$$

This gives the first term in Eq. (3.49) the simpler form

$$\sum_{p=0}^{n-4} [\sum_{k=0}^{p} \tilde{\Lambda}_k^p a^{(k)\mu} a_{\mu}^{(p-k)} \; a_n(p+4)] \quad . \tag{3.75}$$

Now use Eq. (3.69) to write $a_n(s+4)$ as

$$\sum_{j=0}^{n-4-s} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\} \cdot \tag{3.76}$$

$${}_{j}C_{\ell}^{n-4-s}(s+4) \; G_{\ell} U_{j-\ell} \; \delta(n-4-j-s-\sum^{j-1}(q+p), q_{\ell}) \quad .$$

So we have for the first term in Eq. (3.49)

$$\sum_{s=0}^{n-4} \sum_{k=0}^{s} \tilde{\Lambda}_k^s \, a^{(k)\mu} \, a_\mu^{(s-k)} \, a_n(s+4) =$$

$$\sum_{s=0}^{n-4} \sum_{k_1=0}^{s} \sum_{k_2=0}^{s-k_1} \tilde{\Lambda}_{k_1}^s \, a^{(k_1)} \, a^{(k_2)} \, G_2 \, \delta(s-k_1,k_2) \cdot \tag{3.77}$$

$$\sum_{j=0}^{n-4-s} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\}_j C_\ell^{n-4-s}(s+4) \cdot$$

$$G_\ell U_{j-\ell} \, \delta[n-4-j-s-\sum^{j-1}(q+p), q_\ell] \ .$$

We introduce the eta function with the definition

$$\eta(k) = \begin{matrix} 1 \\ 0 \end{matrix}\left\{ \begin{matrix} k \geq 0 \\ k < 0 \end{matrix} \right. \tag{3.78}$$

$$\Lambda_k^p \equiv \tilde{\Lambda}_k^p \eta(p-k) \tag{3.79}$$

With this definition, we may interchange the order of summation on the k's and s. Relabling the kinematic term, letting the kronecker delta do its work and making the transformation $\ell+2 = \ell'$ and, suppressing the prime, we have for the right hand side of Eq. (3.77).

$$\sum_{j=0}^{n-1} \sum_{\ell=0}^{j+2} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j+2-\ell} \{\sum a^{(p_r)}\}_{j+2} F_\ell^{n-4}$$

$$G_\ell U_{j+2-\ell} \delta[n-4-j-\sum^{j+1}(q+p), q_\ell] \tag{3.80}$$

where

$$ {}_{j+2}F_{\ell}^{n-4} = \Lambda_{q_{\ell-1}}^{q_{\ell-1}+q_{\ell}}\ {}_{j}C_{\ell-2(q_{\ell-1}+q_{\ell}+4)}^{n-4-q_{\ell-1}-q_{\ell}} . \tag{3.81} $$

So the fundamental equation becomes

$$ \begin{aligned} a_n(2) = & \sum_{j=0}^{n-4} \sum_{\ell=0}^{j+2} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j+2-\ell} \{\sum a^{(p_r)}\} \cdot \\ & {}_{j+2}F_{\ell}^{n-4}\ G_{\ell} U_{j+2-\ell}\ \delta[n-4-j-\sum^{j+1}(q+p), q_{\ell}] \\ & + 2n \sum_{j=0}^{n-3} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j+1-\ell} \{\sum {}^{(p_r)}\} \\ & {}_{j+1}D_{\ell}^{n-3}\ G_{\ell} U_{j+1-\ell}\ \delta[n-3-j-\sum^{j}(q+p), q_{\ell}] . \end{aligned} \tag{3.82} $$

Now make the substitutions $j' = j+2$, $j' = j+1$ in the first and second terms respectively. Then we have, suppressing the primes

$$ \begin{aligned} a_n(2) = & \sum_{j=0}^{n-2} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\}\ {}_{j}F_{\ell}^{n-4} \\ & G_{\ell} U_{j-\ell}\ \delta[n-2-j-\sum^{j-1}(q+p), q_{\ell}] + 2n \sum_{j=0}^{n-2} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \\ & \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\}\ {}_{j}D_{\ell}^{n-3}\ G_{\ell} U_{j-\ell}\ \delta[n-2-j-\sum^{j-1}(q+p), q_{\ell}] \end{aligned} \tag{3.83} $$

with

$$
{}_jF_\ell^{n-4} = \Lambda_{q_{\ell-1}}^{q_{\ell-1}+q_\ell} \; {}_{j-2}C_{\ell-2(q_{\ell-1}+q_\ell+4)}^{n-4-q_{\ell-1}-q_\ell}
$$

$$
{}_jD_\ell^{n-3} = \frac{(-1)^{p_{j-\ell}}}{(p_{j-\ell}+2)!} \; {}_{j-1}C_{\ell(p_{j-\ell}+2)}^{n-3-p_{j-\ell}} \tag{3.84}
$$

comparing this with the general expression for $a_n(2)$ from Eq. (3.69), we have

$$
{}_jC_\ell^{n-2}(2) = \{\Lambda_{q_{\ell-1}}^{q_{\ell-1}+q_\ell} \; {}_{j-2}C_{\ell-2(q_{\ell-1}+q_\ell+4)}^{n-4-q_{\ell-1}-q_\ell} + 2n \frac{(-1)^{p_{j-\ell}+1}}{(p_{j-\ell}+2)!} \cdot
$$

$$
{}_{j-1}C_{\ell(p_{j-\ell}+2)}^{n-3-p_{j-\ell}} \} \, \delta[n-2-j-\sum^{j-1} (q+p), q_\ell] \; . \tag{3.85}
$$

Since we are not explicitly interested in the ${}_jC_p^m(2)$, but rather the ${}_jC_p^m(1)$, we proceed to solve for them.

In general, from Eq. (3.70)

$$
{}_jC_\ell^{n-2}(2) = \sum_{j_1=0}^{j} \sum_{\ell_1=0}^{\ell} \binom{n}{j_1+1+\sum(1)(q+p)} \; {}_{j_1}C_{\ell_1}^{j_1+\sum(1)(q+p)}
$$

$$
{}_{j-j_1}C_{\ell-\ell_1}^{n-2-j_1-\sum(1)(q+p)} \quad \delta[n-2-j-\sum^{j-1} (q+p), q_\ell] \; . \tag{3.86}
$$

Solving this for ${}_jC_p^{n-2}$, with the by now familiar techniques, we obtain from Eq. (3.85) and (3.86)

$$
{}_jC_\ell^{n-2} = \Big\{\frac{1}{2n}\Lambda_{q_{\ell-1}}^{q_{\ell-1}+q_\ell}\ {}_{j-2}C^{n-4-q_{\ell-1}-q_\ell}_{-2(q_{\ell-1}+q_\ell+4)}
+ \frac{(-1)^{p_{j-\ell}+1}}{(p_{j-\ell}+2)!}\ {}_{j-1}C^{n-3-p_{j-\ell}}_{\ell(p_{j-\ell}+2)} - \frac{1}{2n}\sum_{j_1=1}^{j-1}\sum_{\ell_1=0}^{\ell}
\tag{3.87}
$$

$$
\binom{n}{j_1+1+\sum(1)(q+p)}\ {}_{j_1}C_{\ell_1}^{j_1+\sum(1)(q+p)}\ {}_{j-j_1}C_{\ell-\ell_1}^{n-2-j_1-\sum(1)(q+p)}\Big\}\cdot
$$

$$
\delta[n-2-j-\sum^{j-1}(q+p),q_\ell) \quad .
$$

We can put this equation in a more accessible form with the following definitions

$$
\begin{aligned}
\Delta(j;s) &\equiv \delta(j-\sum^{s-1} j_i,j_s)\\
\Delta'(\ell;s) &\equiv \delta(\ell-\sum^{s-1} \ell_i,\ell_s)\\
\Delta^*(m;s) &\equiv \delta[m-\sum^{s-1}\sigma(i),\sigma(s)] \quad .
\end{aligned}
\tag{3.88}
$$

Now the general form for the C's becomes, from Eq. (3.70),

$$
{}_jC_\ell^{n+1-s}(s) = \prod_{r=1}^{s}\sum_{j_r=0}^{j-\sum^{r-1}j_i}\sum_{\ell_r=0}^{\ell-\sum^{r-1}\ell_i}\binom{n+1-\sum^{r-1}\sigma(i)}{\sigma(r)}
\tag{3.89}
$$

$$
{}_{j_r}C_{\ell_r}^{\sigma(r)-1}\ \Delta(j;s)\ \Delta'(\ell;s)\ \Delta^*(n+1;s)\ \delta[n+1-s-j-\sum^{j-1}(q+p),q_\ell] \ .
$$

Using this equation on the separate terms of Eq. (3.89) gives after the transformation $n = n'+2$,

$$
\begin{aligned}
{}_jC^n_\ell = \Big\{ & \frac{1}{2(n+2)} \Lambda^{q_{\ell-1}+q_\ell}_{q_{\ell-1}} \prod_{r=1}^{q_\ell+q_{\ell-1}+4} \sum_{j_r=0}^{j-2-\sum^{r-1} j_1} \\
& \sum_{\ell_r=0}^{\ell-2-\sum^{r-1}\ell_i} \binom{n+2-\sum^{r-1}\sigma(i)}{\sigma(r)} {}_{j_r}C^{\sigma(r)-1}_{\ell r} \, \Delta(j-2;q_\ell+q_{\ell-1}+4)\cdot \\
& \Delta(\ell-2;q_\ell+q_{\ell-1}+4)\ \Delta^*(n+2;q_\ell+q_{\ell-1}+4) \\
& + \frac{(-1)^{p_{j-\ell}+1}}{(p_{j-\ell}+2)!} \prod_{r=1}^{p_{j-\ell}+2} \sum_{j_r=0}^{j-1-\sum^{r-1} j_i} \sum_{\ell_r=0}^{\ell-\sum^{r-1}\ell_i} \binom{n+1-\sum^{r-1}\sigma(i)}{\sigma(r)} \\
& {}_{j_r}C^{\sigma(r)-1}_{\ell_r} \, \Delta(j-1;p_{j-\ell}+2)\ \Delta'(\ell;p_{j-\ell}+2)\ \Delta^*(n+1;p_{j-\ell}+2)\ - \\
& \sum_{j_1=1}^{j-1} \sum_{\ell_1=0}^{\ell} \frac{1}{2(n+2)} \binom{n+2}{j_1+1+\sum(1)(q+p)} {}_{j_1}C^{j_1+1+\sum(1)(q+p)}_{\ell_1} \\
& {}_{j-j_1}C^{n-j_1-\sum(1)(q+p)}_{\ell-\ell_1} + \delta^n_o\ \delta_{o,j}\ \delta_{o,\ell}\Big\}\delta\big[n-j-\sum^{j-1}(Q+p),q_\ell\big) \ ,
\end{aligned}
\tag{3.90}
$$

which is the recursion relation for the form coefficients, i.e. the relation existing between various ${}_jC^r_m \div {}_jC^r_m(1)$.

Similarly the recursion relation for the advanced quantities is

$$
\begin{aligned}
{}_jC_\ell^{n*} = \Bigg\{ & \frac{1}{2(n+2)} \Lambda^{*\,q_{\ell-1}+q_\ell}_{q_{\ell-1}} \prod_{r=1}^{q_\ell+q_{\ell-1}+4} \sum_{j_r=0}^{j-2-\sum^{r-1} j_i} \sum_{\ell_r=0}^{\ell-2-\sum^{r-1} \ell_i} \cdot \\
& \binom{n+2-\sum^{r-1}\sigma(i)}{\sigma(r)} {}_{j_r}C_{\ell_r}^{*\sigma(r)-1} \Delta(j-2;q_\ell+q_{\ell-1}+4)\Delta(\ell-2;q_\ell+q_{\ell-1}+4) \cdot \\
& \Delta^*(n+2;q_\ell+q_{\ell-1}+4) + \frac{1}{(p_{j-\ell}+2)!} \prod_{r=1}^{p_{j-\ell}+2} \sum_{j_r=0}^{j-1-\sum^{r-1} j_i} \sum_{\ell_r=0}^{\ell-\sum^{r-1} \ell_i} \cdot \\
& \binom{n+1-\sum^{r-1}\sigma(i)}{\sigma(r)} {}_{j_r}C_{\ell_r}^{*\sigma(r)-1} \Delta(j-1;p_{j-\ell}+2)\Delta'(\ell;p_{j-\ell}+2) \cdot \\
& \Delta^*(n+1;p_{j-\ell}+2) - \sum_{j_1=1}^{j-1} \sum_{\ell_1=0}^{\ell} \frac{1}{2(n+2)} \binom{n+2}{j_1+1+\Sigma(1)(q+p)} \cdot \\
& {}_{j_1}C_{\ell_1}^{*j_1+1+\Sigma(1)(q+p)} \cdot {}_{j-j_1}C_{\ell-\ell_1}^{*n-j_1-\Sigma(1)(q+p)} + \delta_o^n \delta_{o,j} \delta_{o,\ell} \Bigg\} \cdot \\
& \delta[n-j-\textstyle\sum^{j-1}(q+p), q_\ell]
\end{aligned}
\tag{3.91}
$$

where

$$
\Lambda^{*p}_k \equiv (-1)^p \Lambda^p_k \ .
$$

E. The Inversion Formula

It will be necessary in what follows to find the power series expansion coefficients for the reciprocal of a function when the coefficients are known for the function itself. Therefore, suppose

$$f(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} b_n \tag{3.92}$$

$$b_o \neq 0$$

and let

$$1/f(\tau) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} W_n \quad . \tag{3.93}$$

The necessary condition follows from Eq. (3.4)

$$\sum_{k=0}^{n} \binom{n}{k} b_k W_{n-k} = \delta_{o,n} \tag{3.94}$$

or

$$b_o W_n + \sum_{k=1}^{n} \binom{n}{k} b_k W_{n-k} = \delta_{o,n} \quad . \tag{3.95}$$

Thus

$$W_n = b_o^{-1} \delta_{o,n} - b_o^{-1} \sum_{k_1=1}^{n} \binom{n}{k_1} b_{k_1} W_{n-k_1} \quad . \tag{3.96}$$

Use this recursively

$$W_{n-b_1} = b_o^{-1} \delta_{o,n-k_1} - b_o^{-1} \sum_{k_2=1}^{n-k_1} \binom{n-k_1}{k_2} b_{k_2} W_{n-k_1-k_2}$$

Then

$$W_n = b_o^{-1}\delta_{o,n} - b_o^{-1}\sum_{k_1=1}^{n}\binom{n}{k_1}b_{k_1}\Big\{b_o^{-1}\delta_{o,n-k_1}$$

$$- b_o^{-1}\sum_{k_2=1}^{n-k_1}\binom{n-k_1}{k_2}b_{k_2}W_{n-k_1-k_2}\Big\} \tag{3.97}$$

$$W_n = b_o^{-1}\delta_{o,n} - b_o^{-2}\sum_{k_1=1}^{n}\binom{n}{k_1}b_{k_1}\delta_{o,n-k_1}$$

$$- (-1)^2 b_o^{-2}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n-k_1}\binom{n}{k_1}\binom{n-k_1}{k_2}b_{k_1}b_{k_2}W_{n-k_1-k_2} .$$

This gives

$$W_n = b_o^{-1}\delta_{o,n} + b_o^{-1}\sum_{j=1}^{n}(-1)^j\, b_o^{-j}\prod_{r=1}^{j}$$

$$\sum_{k_r=1}^{n-\sum^{r-1} k_i}\binom{n-\sum^{r-1} k_i}{k_r}b_{k_r}\,\delta(n-\sum^{r-1} k_i, k_j) \quad . \tag{3.98}$$

It is more convenient to alter the form a little by defining

$$C_k \equiv -\frac{b_k}{b_o} \qquad k > 0 \tag{3.99}$$

$$C_o \equiv 0 \ .$$

Then we have

$$W_n = b_o^{-1}\delta_{o,n} + b_o^{-1}\sum_{j=1}^{n}\prod_{r=1}^{j}\sum_{k_r=0}^{n-\sum^{r-1} k_i}\binom{n-\sum^{r-1} k_i}{k_r}C_{k_r}\,\delta(n-\sum^{j-1} k_i, k_j). \tag{3.100}$$

F. The Series for $\tau/\rho(\tau)$

In this section and the next, we develop the remaining formalism necessary to obtain the general expression for the electromagnetic four-momentum. Let us begin with Eq. (1.20)

$$
\begin{aligned}
R^{\mu}(\tau) &= x^{\mu} - z^{\mu}(\tau) \\
&= x^{\mu} - \sum_{k=0}^{\infty} \frac{\tau^{k}}{k!} \left[\frac{d^{k}}{d^{\,k}} z^{\mu}(\tau) \right]_{o} \\
&= x^{\mu} - z^{\mu} - \sum_{k=1}^{\infty} \frac{\tau^{k}}{k!} v^{(k-1)\mu} .
\end{aligned}
$$

The τ under concern here is <u>any</u> $\tau < 0$. Using Eq. (2.3) this is

$$
R^{\mu}(\tau) = \hbar u^{\mu} - \sum_{k=1}^{\infty} \frac{\tau^{k}}{k!} v^{(k-1)\mu} .
$$

Since

$$
\rho(\tau) = - v_{\mu}(\tau) R^{\mu}(\tau)
$$

multiplying the two series together gives

$$
\begin{aligned}
\rho(\tau) &= -\hbar \sum_{k=1}^{\infty} \frac{\tau^{k}}{k!} v^{(k)\mu} u_{\mu} \\
&\quad + \sum_{n=1}^{\infty} \frac{\tau^{n}}{n!} \sum_{k=0}^{n} \binom{n}{k} v^{(k)\mu}\, v_{\mu}^{(n-1-k)} .
\end{aligned}
$$

Or

$$
\rho(\tau) = \sum_{n=1}^{\infty} \frac{\tau^{n}}{n!} \sum_{k=0}^{n} \binom{n}{k} v^{(k)\mu} \left\{ v_{\mu}^{(n-1-k)} - \hbar\delta(k,n) u_{\mu} \right\} . \qquad (3.101)
$$

Note that the expression $v_\mu^{(n-1-k)}$ in Eq. (3.101) would be undefined for n,k = 1 were it not for convention (ii) in Section B of this chapter. I.e., this convention implies that $v_\mu^{(-1)} = 0$. To proceed, from Eq. (3.101) we have

$$\frac{\rho(\tau)}{\tau} = \sum_{n=1}^{\infty} \frac{\tau^{n-1}}{(n-1)!} \sum_{k=0}^{n} \frac{1}{n} \binom{n}{k} v^{(k)\mu} \{ v_\mu^{(n-1-k)} - \hbar\delta(k,n) u_\mu \}.$$

Make the transformation n = n'+1, then suppressing the prime

$$\frac{\rho(\tau)}{\tau} = \sum_{n=0}^{\infty} \frac{\tau^{n}}{n!} \sum_{k=0}^{n+1} \frac{1}{n+1} \binom{n+1}{k} v^{(k)\mu} \{ v_\mu^{(n-k)} - \hbar\delta(k,n+1) u_\mu \}. \tag{3.102}$$

We define

$$c''_n \equiv \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} v^{(k)\mu} \{ v_\mu^{(n-k)} - \hbar\delta(k,n+1) u_\mu \} \tag{3.103}$$

so that

$$\frac{\rho(\tau)}{\tau} = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} c''_n \quad . \tag{3.104}$$

We see that

$$c''_o = -(1 + \hbar a_u); \quad \text{where } a_u = a_\sigma u^\sigma \; . \tag{3.105}$$

With Eq. (3.99) in mind, we consider

$$c'_n \equiv - \frac{c''_n}{c''_n} = \frac{1}{1+a_u\hbar} \sum_{k=0}^{n+1} \frac{1}{n+1} \binom{n+1}{k} v^{(k)\mu} \cdot \{ v_\mu^{(n-k)} - \hbar\delta(k,n+1) u_\mu \} \tag{3.106}$$

Using Eq. (3.43), it is not difficult to show that,

$$\sum_{k=0}^{n+1} \binom{n+1}{k} v^{(k)\mu} v_{\mu}^{(n-k)} = -\,\delta(o,n)$$

$$+ \sum_{k=0}^{n-2} \left\{ \binom{n+1}{k+1} - (n+2)\binom{n-1}{k+1} \right\} a^{(k)\mu} a_{\mu}^{(n-2-k)} \tag{3.107}$$

Similarly

$$\sum_{k=0}^{n+1} \binom{n+1}{k} v^{(k)\mu} u_{\mu} = \sum_{k=0}^{n} \binom{n+1}{k+1} a^{(k)\mu} u_{\mu}, \text{ since } v^{(k)\mu} = a^{(k-1)\mu}. \tag{3.108}$$

If we define

$$T_k^{n-2} \equiv \frac{1}{n+1} \left\{ \binom{n+1}{k+1} - (n-2)\binom{n-1}{k+1} \right\} \quad n \geq 2$$

$$T_k^{n-2} \equiv 0 \quad n<2 \tag{3.109}$$

Eq. (3.106) then takes the form

$$c'_n = \frac{1}{1+a_u \hbar} \left\{ \sum_{k=0}^{n-2} T_k^{n-2} a^{(k)\mu} a_{\mu}^{(n-2-k)} \right.$$

$$\left. - \hbar \sum_{k=0}^{n} \frac{1}{n+1} \binom{n+1}{k+1} a^{(k)\mu} u_{\mu}\, \delta(k,n) - \delta(o,n) \right\} . \tag{3.110}$$

With our experience in dealing with the a's, we seek to write the c'_n in the form

$$c'_p = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} e'_n(p) . \tag{3.111}$$

Expanding $(1 + \hbar a_u)^{-1}$, we find

$$(1+\hbar a_u)^{-1} = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} Bn$$

$$Bn = (-1)^n n' (a_u)^n \tag{3.112}$$

Developing this series in c'_p gives, after a little algebra

$$e'_n(p) = (-1)^{n+1}\{n;(a_u)^n \delta(o,p) - (n-1)!\cdot$$

$$(a_u)^{n-1} \sum_{k=0}^{p} \frac{1}{p+1} \binom{p+1}{k+1} a^{(k)\mu} u_\mu \ \delta(k,p)\} \tag{3.113}$$

$$+ (-1)^n n!(a_u)^n \sum_{k=0}^{p-2} T_k^{p-2} \ a_\mu^{(k)} a^{(p-2-k)\mu} \quad .$$

The first term we can write as

$$\prod_{r=1}^{n} \sum_{p_r} a^{(p_r)} \{(-1)^{n+1} n! \ \delta(o,\sum^n p_i) \ \delta(o,p) U_n$$

$$+ (-1)^n (n-1)! \ \frac{1}{p+1} \binom{p+1}{p_n+1} \delta(o, \sum^{n-1} p_i) \delta(p_n,p) \ U_n\} \ . \tag{3.114}$$

The second term is

$$\sum_{q_1} \sum_{q_2} a^{(q_1)} a^{(q_2)} \prod_{r=1}^{n} \sum a^{(p_r)} \{(-1)^n n! T_{q_1}^{p-2} \delta(o,\sum^n p_i) \delta(p-2-q_1,q_2)\} G_2 U_n . \tag{3.114'}$$

Now we can write

$$e_n'(p) = \sum_{j=0}^{n+2} \sum_{=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\}\ {}_jD_\ell'(n,p) \cdot G_\ell U_{j-\ell} \ . \tag{3.115}$$

Where, in the above, extra indices were introduced and then removed with δ-factors in order to achieve the above familiar form.

$$\begin{aligned} {}_jD_\ell'(n,p) \equiv \ & \delta(j,n)\delta(o,\ell)(-1)^{n+1}\{n! \ \delta(o,p)\delta(o,\sum^{n} p_i) \\ & - (n-1)! \ \frac{1}{p+1} \binom{p+1}{p_n+1} \delta(o, \sum^{n-1} p_i)\delta(p_n,p)\} \\ & + \delta(j,n+2)\delta(2,\ell)(-1)^n n! \ {}^{p-2}_{q_1}\delta(o,\sum^{n} p_i) \cdot \delta(p-2-q_1,q_2) \ . \end{aligned} \tag{3.116}$$

In order to meet the second requirement in Eq. (3.99), which is necessary in order for the inversion formula to be valid we now define

$$\begin{aligned} e_n(p) &= e_n'(p) \qquad p \neq 0 \\ e_n(p=0) &= 0 \ . \end{aligned} \tag{3.117}$$

This calls for the definition

$$\begin{aligned} {}_jD_\ell(n,p) \equiv \ & \delta(j,n)\delta(o,\ell)(-1)^{n+1}(n-1)!\{\delta(o,p)\cdot\delta(o,\sum^{n} p_i) \\ & = \frac{1}{p+1} \binom{p+1}{p_n+1} \delta(o, \sum^{n-1} p_i)\delta(p_n,p)\} + \delta(j,n+2)\delta(\ell,2)(-1)^n \\ & n! \ T^{p-2}_{q_1}\delta(o,\sum^{n} p_i)\cdot\delta(p-2-q_1,q_2) . \end{aligned} \tag{3.118}$$

Note that the definition Eq. (3.118) implies

$$
\begin{aligned}
{}_jD_\ell(n,p) &= {}_jD'_\ell(n,p) \qquad p>0 \\
{}_jD_\ell(n,p=0) &= 0 \quad .
\end{aligned}
\tag{3.119}
$$

Thus, the coefficients which are to be used in the inversion formula Eq. (3.100) are

$$c_p = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} e_n(p) \tag{3.120}$$

$$
\begin{aligned}
e_n(p) = \sum_{j=0}^{n+2} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\textstyle\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\textstyle\sum a^{(p_r)}\} \cdot \\
{}_jD_\ell(n,p) G_\ell U_{j-\ell}
\end{aligned}
\tag{3.121}
$$

and the form of Eq. (3.115) is preserved.

Now consider the essential part of the inversion formula

$$\prod_{r=1}^{j} \sum_{k_r=0}^{n-\sum^{r-1} k_1} \binom{n-\sum^{r-1} k_i}{k_r} c_{k_r} \delta(n-\sum^{j-1} k_i, k_j) \quad .$$

Look at e.g. the case for $j = 2$,

$$\sum_{j=0}^{n} \binom{n}{p} c_p c_{n-p} = \sum_{p=0}^{n} \binom{n}{p} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} e_m(p) \sum_{\ell=0}^{\infty} \frac{\hbar^\ell}{\ell!} e_\ell(n-p) \quad . \tag{3.122}$$

Taking the Cauchy product gives

$$\sum_{p=0}^{n} \binom{n}{p} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \left\{ \sum_{\ell=0}^{m} \binom{m}{\ell} e_\ell(p) e_{m-\ell}(n-p) \right\} \quad . \tag{3.123}$$

Let us fix our attention just on

$$\sum_{\ell=0}^{m} \binom{m}{\ell} e_\ell(p) e_{m-\ell}(n-p) =$$

$$\sum_{\ell=0}^{m} \binom{m}{\ell} \sum_{j_1=0}^{\ell+2} \sum_{s_1=0}^{j_1} \prod_{r=1}^{s_1} \{\sum a^{(q_r)}\} \prod_{r=1}^{j_1-s_1} \{\sum a^{(p_r)}\}_{j_1}D_{s_1}(\ell,p) \tag{3.124}$$

$$\sum_{j_2=0}^{m+2-\ell} \sum_{s_2=0}^{j_2} \prod_{r=1}^{s_2} \{\sum a^{(q_r')}\} \prod_{r=1}^{j_2-s_2} \{\sum a^{(p_r)}\}_{j_2}D_{s_2}(m-\ell,n-p) \cdot G_{s_1+s_2} U_{j_1+j_2-s_1-s_2}$$

Since

$$_jD_s(\ell,p) = 0 \qquad j > \ell + 2 \tag{3.125}$$

the $_jD_s$ behave formally just like the $_jC_s^n$. The only difference in behavior is exhibited in Eq. (3.125). Thus we may carry out the same manipulations with the $_jD_s$ that we did with the $_jC_s$. For example, if we write

$$\sum_{m=0}^{\infty} \frac{\kappa^m}{m!} e_m^{(2)}(n) \equiv \sum_{p=0}^{n} \binom{n}{p} C_p C_{n-p} \tag{3.126}$$

then in the same way that we arrived at the corresponding result for $_jC_p^n(2)$, we can show

$$e_m^{(2)}(n) = \sum_{j=0}^{m+2\cdot 2} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{a^{(p_r)}\} \cdot$$

$$_jD_\ell^{(2)}(m,n) G_\ell U_{j-\ell} \tag{3.127}$$

where

$$
{}_{j}D_{q}^{(2)}(m,n) = \sum_{j'=0}^{j} \sum_{s=0}^{q} \sum_{\ell=0}^{m} \sum_{p=0}^{n} \binom{m}{\ell}\binom{n}{p} {}_{j'}D_{s}(\ell,p) \cdot
$$

$$
{}_{j-j'}D_{q-q'}(m-\ell,n-p) \quad . \tag{3.128}
$$

Generalizing, we have

$$
\sum_{m=0}^{\infty} \frac{\lambda^m}{m!} e_m^{(v)}(n) = \prod_{r=1}^{v} \sum_{k_r=0}^{n-\sum^{r-1} k_i} \binom{n-\sum^{r-1} k_i}{k_r} C_{k_r} \delta(n-\sum^{v-1} k_i, k_v) \tag{3.129}
$$

with

$$
e_m^{(v)}(n) = \sum_{j=0}^{m+2v} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\} \cdot
$$

$$
{}_{j}D_{\ell}^{(v)}(m,n) G_{\ell} U_{j-\ell} \tag{3.130}
$$

and

$$
{}_{j}D_{q}^{(v)}(m,n) = \prod_{r=1}^{v} \sum_{j_r=0}^{j-\sum^{r-1} j_i} \sum_{q_r=0}^{q-\sum^{r-1} q_i} \sum_{\ell_r=0}^{m-\sum^{r-1} \ell_i} \sum_{p_r=0}^{n-\sum^{r-1} p_i} .
$$

$$
\binom{m-\sum^{r-1}\ell_i}{\ell_r}\binom{n-\sum^{r-1}p_i}{p_r} {}_{j_r}D_{q_r}(\ell_r,p_r) \delta(j-\sum^{v-1} j_i, j_v) \delta(q-\sum^{v-1} q_i, q_v) \cdot \tag{3.131}
$$

$$
\delta(m-\sum^{v-1}\ell_i, \ell_v) \delta(n-\sum^{v-1} p_i, p_v) \quad .
$$

Therefore from Eqs. (3.100), (3.129), (3.130), we have

$$W_n = \frac{-1}{1+a_u h}\{\delta_{o,n} + \sum_{m=0}^{\infty}\frac{h^m}{m!}\sum_{v=1}^{n} e_m^{(v)}(n)\} \tag{3.132}$$

where

$$\frac{\tau}{\rho(\tau)} = \sum_{n=0}^{\infty}\frac{\tau^n}{n!}W_n \quad . \tag{3.133}$$

Since, as follows from Eqs. (3.125) and (3.131),

$${}_jD_\ell^{(v)}(m,n) = 0 \qquad j>\ell + 2v \tag{3.134}$$

we may raise the upper limit on the j-summation to m+2n and then interchange summations. This gives

$$W_n = \frac{1}{1+a_u h}\sum_{m=0}^{\infty}\frac{h^m}{m!}G(m,n) \tag{3.135}$$

where

$$G(m,n) = \sum_{j=0}^{m+2n}[\sum_{\ell=0}^{j}\prod_{r=1}^{\ell}\{\sum a^{(q_r)}\}\prod_{r=1}^{j-\ell}\{\sum a^{(p_r)}\}\cdot$$

$$\sum_{v=1}^{n} {}_jD_q^{(v)}(m,n)G_\ell U_{j-\ell}] + \delta_{o,m}\delta_{o,n} \quad . \tag{3.136}$$

Now define

$$\frac{-1}{1+\hbar a_u} = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \; T_n$$

$$_n = (-1)^{n+1} n! (a_u)^n \quad . \tag{3.137}$$

If we let

$$_jT^n_\ell \equiv (-1)^{n+1} n! \delta(o,\ell)\delta(j,n)\delta(o,\Sigma^n p_i) \tag{3.138}$$

then

$$T_n = \sum_{j=0}^{n} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\} \cdot$$

$$_jT^n_\ell \; G_\ell U_{j-\ell} \quad . \tag{3.139}$$

$$W_n = \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \quad {}_kG(m-k,n) \quad . \tag{3.140}$$

Carrying out the manipulations we finally have

$$\frac{\tau}{\rho(\tau)} = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} s(m,n) \tag{3.141}$$

with

$$S(m,n) = \sum_{j=0}^{m+2n} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\} \cdot$$

$$_jS_\ell(m,n) G_\ell U_{j-\ell} \tag{3.142}$$

and

$$_jS_\ell(m,n) = \sum_{j'=0}^{j} \sum_{\ell'=0}^{\ell} \sum_{k=0}^{m} {}_{j'}T^k_{q'} \{ \sum_{v=1}^{n} \prod_{r=1}^{v} \cdot$$

$$\sum_{j_r=0}^{j-j'-\sum^{r-1} j_i} \sum_{\ell_r=0}^{\ell-\ell'-\sum^{r-1} \ell_i} \sum_{\delta_r=0}^{m-k-\sum^{r-1} \delta_i} \sum_{p_r=0}^{n-\sum^{r-1} p_i} \binom{m}{k} \binom{m-k-\sum^{r-1}\delta_i}{\delta_r} \cdot \tag{3.143}$$

$$\binom{n-\sum^{r-1} p_i}{p_r} {}_{j_r}D_{\ell_r}(\delta_r,p_r)\,\delta(j-j'-\sum^{v-1} j_i, j_v)\,\delta(\ell-\ell'-\sum^{v-1}\ell_i,\ell_v) \cdot$$

$$\delta(m-k-\sum^{v-1}\delta_i,\delta_v)\,\delta(n-\sum^{v-1}p_i,p_v) + \delta_{o,n}\delta_{j,j'}\delta_{\ell,\ell}\,\delta_{m,k}\}$$

G. The Series For $(\frac{\hbar}{\tilde{\tau}})^\ell$

Let us begin with Eq. (2.15)

$$\tilde{\tau} = - \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} a_n(1) \tag{3.144}$$

$$= -\hbar \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \frac{a_{n+1}(1)}{n+1}$$

$$\frac{\tilde{\tau}}{\hbar} = - \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \frac{a_{n+1}(1)}{n+1} \quad . \tag{3.145}$$

Set

$$B_n \equiv \frac{a_{n+1}(1)}{n+1} \qquad n > 0$$
$$B_o \equiv 0 \tag{3.146}$$

which we can write explicitly as

$$B_n = \frac{a_{n+1}(1)}{n+1} [1-\delta(o,n)] \quad . \tag{3.147}$$

Let us define

$$b_n(p) \equiv (-1)^p \prod_{r=1}^{p} \sum_{k_r=0}^{n-\overset{r-1}{\Sigma} k_i} \binom{n-\overset{r-1}{\Sigma} k_i}{k_r} B_{k_r} \delta(n-\overset{p-1}{\Sigma} k_i, k_p) \tag{3.148}$$

Note that $b_n(p) = 0$ for $p > n$. Consider e.g. the case for $p = 3$, then this expression becomes

$$b_n(3) = (-1)^3 \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \binom{n}{k_1}\binom{n-k_1}{k_2} B_{k_1} B_{k_2} B_{n-k_1-k_2} \quad . \tag{3.149}$$

Look first at this portion of the expression:

$$(-1)^3 \sum_{k_1=0}^{n} \sum_{k_2=0}^{n-k_1} \frac{n!}{k_1!(n-k_1)!} \frac{(n-k_1)!}{k_2!(n-k_1-k_2)!} \frac{a_{k_1+1}}{k_1+1} \frac{a_{k_2+1}}{k_2+1} \cdot$$
$$\frac{a_{n+1-k_1-k_2}}{n+1-k_1-k_2} \quad . \tag{3.150}$$

A little algebra and the transformation

$$k+1 = k'$$

yields

$$(-1)^3 \frac{n!}{(n+3)!} \sum_{k_1=1}^{n+3} \sum_{k_2=1}^{n+3-k_1} \binom{n+3}{k_1} \binom{n+3-k_1}{k_2} a_{k_1} a_{k_2} a_{n+3-k_1-k_2} \tag{3.151}$$

using Eq. (3.7) this becomes

$$(-1)^3 \frac{n!}{(n+3)!} a_{n+3}(3) \quad .$$

Now look at the terms which result from considering the product,

$$[1-\delta(1,k_1)][1-\delta(1,k_2)][1-\delta(1,n+3-k_1-k_2)] \quad , \tag{3.152}$$

which appears as a consequence of Eq. (3.147). Proceeding as before, we find that the three terms linear in the Kronecker deltas give

$$(-1)^2 \frac{3\ n!}{(n+2)!} a_{n+2}(2)$$

while the quadraic terms result in

$$(-1) \frac{3\ n!}{(n+1)!} a_{n+1} \quad .$$

Therefore,

$$b_n(3) = \sum_{j=0}^{3} \binom{3}{j} \frac{n!}{(n+j)!} (-1)^j a_{n+j}(j) \quad .$$

Generalizing, gives

$$b_n(p) = \sum_{j=0}^{p} \binom{p}{j} \frac{n!}{(n+j)!} (-1)^j a_{n+j}(j)\eta(p-j) \qquad (3.153)$$

Where we have appended the η-function because it will allow us to raise the upper summation limit on j without changing the value of $b_n(p)$.

We can now find the coefficients h_n in the series

$$\frac{h}{\tilde{\tau}} = \sum_{n=0}^{\infty} \frac{h^n}{n!} h_n \ . \qquad (3.154)$$

Observing that Eq. (3.145) implies

$$h_o = -1 \qquad (3.155)$$

we apply Eq. (3.100) and (3.148). Thus

$$h_n = \sum_{p=0}^{n} - b_n(p) \ . \qquad (3.156)$$

Define

$$S_j^n \equiv \frac{n!}{(n+j)!} (-1)^j a_{n+j}(j)\eta(p-j) \ . \qquad (3.157)$$

Then

$$h_n = - \sum_{p=0}^{n} \sum_{j=0}^{p} \binom{p}{j} S_j^n \ . \qquad (3.158)$$

Raise the upper summation limit on j to n. Then we can interchange the order of summation but due to the η function, there will be no contribution to the sum over p until $p \geq j$ i.e.

$$h_n = - \sum_{j=0}^{n} \sum_{p=j}^{n} \binom{p}{j} S_j^n \tag{3.159}$$

the transformation

$$p = p'+j$$

results in

$$h_n = - \sum_{j=0}^{n} \sum_{p=0}^{n-j} \binom{p+j}{j} S_j^n \quad . \tag{3.160}$$

It is known that[5]

$$\sum_{p=0}^{n} \binom{p+j}{j} = \binom{n+j+1}{j+1} \quad . \tag{3.161}$$

Thus we obtain

$$h_n = \sum_{j=0}^{n} \binom{n+1}{j+1} \frac{n!}{(n+j)!} (-1)^{j+1} a_{n+j}(j)\eta(n-j) \quad . \tag{3.162}$$

Now turning our attention to $(\frac{h}{\tau})^{\ell}$, $\ell > 1$, we have, using Eq. (3.6)

$$(\tfrac{h}{\tau})^{\ell} = \sum_{n=0}^{\infty} \frac{h^n}{n!} \; {}^{(\ell)}_{n} ,$$

$${}^{(\ell)}_{n} = \prod_{r=1}^{\ell} \sum_{k_r=0}^{n-\sum^{r-1} k_i} \binom{n-\sum^{r-1} k_i}{k_r} \sum_{p_r=0}^{k_r} (-1) b_{k_r}(p_r) \delta(n-\sum^{\ell-1} k_i , k_\ell) \tag{3.163}$$

As a consequence of the η-function in Eq. (3.153),

$$b_k(p) = 0 \qquad p > k \quad \text{(See Eq. 3.148)} \tag{3.164}$$

we may raise the upper limits on the p_r summations and thus interchange the summations

$$h_n^{(\ell)} = (-1)^{\ell} \prod_{r=1}^{\ell} \sum_{p_r=0}^{n} \sum_{k_r=0}^{n-\sum^{r-1} k_i} \binom{n-\sum^{r-1} k_i}{k_r} b_{k_r}(p_r) \cdot \delta(n-\sum^{\ell-1} k_i, k_\ell) \tag{3.165}$$

$$= (-1)^{\ell} \sum_{p_1=0}^{n} \cdots \sum_{p_\ell=0}^{n} b_n\left(\sum_{i=1}^{\ell} p_i\right)^* \tag{3.166}$$

Let us consider the sum

$$\sum_{i=1}^{\ell} p_i$$

there is more than one set of numbers $p_1, p_2, \ldots, p_\ell$, such that $\sum p_i$ equals a given number, say p. Let $N(\ell,p)$ be the number of distinct sets of ℓ numbers whose sum is p. Then Eq. (3.166) can be rewritten

*This results follows from the definition of $b_n(p)$ which includes the second of Eq. (3.146) as well as Eq. (3.148). Compare Eqs. (3.7) and (3.10) with the above.

as

$$h_n^{(\ell)} = (-1)^\ell \sum_{p=0}^{n} N(\ell,p) b_n(p) \tag{3.167}$$

To find $N(\ell,p)$, we recognize that $N(\ell,p)$ is the number of ways that p balls can be placed in ℓ boxes and thus

$$N(\ell,p) = \frac{(p+\ell-1)!}{p!(\ell-1)!} = \frac{(p+1)(p+2)\cdots(p+\ell-1)}{(\ell-1)!} . \tag{3.168}$$

So Eq. (3.167) becomes

$$h_n^{(\ell)} = (-1)^\ell \sum_{p=0}^{n} \sum_{j=0}^{p} \frac{(p+1)(p+2)\cdots(p+\ell-1)}{(\ell-1)!} \binom{p}{j} S_j^n . \tag{3.169}$$

Interchanging summations as we did above we have

$$(-1)^\ell \sum_{j=0}^{n} \sum_{p=j}^{n} \frac{(p+1)(p+2)\cdots(p+\ell-1)}{(\ell-1)!} \binom{p}{j} S_j^n , \tag{3.170}$$

which is

$$(-1)^\ell \sum_{j=0}^{n} \sum_{p=0}^{n-j} \frac{(p+j+1)(p+j+2)\cdots(p+j+\ell-1)}{(\ell-1)!} \binom{p+j}{j} S_j^n . \tag{3.171}$$

Some algebra puts this into the form

$$h_n^{(\ell)} = (-1)^\ell \sum_{j=0}^{n} \binom{j+\ell-1}{j} \left[\sum_{p=0}^{n-j} \binom{p+j+\ell+1}{j+\ +1} \right] S_j^n , \tag{3.172}$$

and using Eq. (3.161) one obtains

$$h_n^{(\ell)} = (-1)^\ell \sum_{j=0}^{n} \binom{j+\ell-1}{j} \binom{n+\ell}{j+\ell} \frac{n!}{(n+j)!} (-1)^j a_{n+j}(j) \cdot \tag{3.173}$$

$$\eta(n-j)$$

The corresponding relations for the advanced quantities are

$$\left(\frac{r}{\bar{\tau}}\right)^\ell = \sum_{n=0}^{\infty} \frac{r^n}{n!} h^{*(\ell)} \tag{3.174}$$

$$h^{*(\ell)} = (-1)^\ell \sum_{j=0}^{n} \binom{j+\ell-1}{j} \binom{n+\ell}{j+\ell} a^*_{n+j}(j) \tag{3.174'}$$

with $a^*_n(j)$ given by Eq. (3.50).

H. The Series for $\Theta^{\mu\nu}(\tau)u_\nu$

In the first paragraph of Section F of this chapter, we obtained the equation

$$R^\mu(\tau) = \hbar u^\mu - \sum_{k=1}^{\infty} \frac{\tau^k}{k!} v^{(k-1)\mu} \quad . \tag{3.175}$$

Another result which we have previously derived is

$$\frac{1}{\rho(\tau)} = \frac{1}{\tau} \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} S(m,n) \quad . \tag{3.176}$$

This is essentially just Eq. (3.141). To shorten our notation, let us define

$$r^\mu(\tau) \equiv u^\mu(\tau) + v^\mu(\tau) = \frac{R^\mu(\tau)}{\rho(\tau)} \; . \tag{3.177}$$

Using Eq. (3.175) and Eq. (3.176) we can obtain the series for the electro-magnetic field tensor $F^{\mu\nu}$. To begin with, we have

$$\begin{aligned} r^\mu(\tau) &= \frac{\hbar}{\tau} \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} S(m,n) u^\mu \\ &\quad - \frac{1}{\tau} \sum_{k=1}^{\infty} \frac{\tau^k}{k!} v^{(k-1)\mu} \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} S(m,n) \; . \end{aligned} \tag{3.178}$$

Now, by Eq. (1.30) we have that

$$F^{\mu\nu}_{ret} = \left\{ \frac{e}{\rho} \frac{d}{d\tau} \left(\frac{v^\mu(\tau) R^\nu(\tau) - v^\nu(\tau) R^\mu(\tau)}{\rho} \right) \right\} \Big|_{\tau=\tilde{\tau}} \; ,$$

which we can condense through the definition

$$v^{[\mu}r^{\nu]} \equiv \frac{1}{2}(v^\mu r^\nu - v^\nu r^\mu)$$

to the compact expression

$$F^{\mu\nu}_{ret} = \left\{\frac{2e}{\rho}\frac{d}{d\tau}\left(v^{[\mu}(\tau)r^{\nu]}(\tau)\right)\right\}\Big|_{\tau=\tilde{\tau}} \ . \tag{3.179}$$

Let us momentarily work on the second term of Eq. (3.178). Make the transformation

$$k' = k-1 \ .$$

Then multiplying the two series in τ together, the second term of Eq. (3.178) becomes

$$-\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\bar{\lambda}^m}{m!}\sum_{k=0}^{n}\binom{n}{k}\frac{v^{(k)\mu}}{k+1}S(m,n-k) \ . \tag{3.180}$$

Using

$$v^\mu(\tau) = \sum_{n=0}^{\infty}\frac{\tau^n}{n!}v^{(n)\mu}$$

we have, after some rewriting

$$\begin{aligned} v^\mu(\tau)r^\nu(\tau) = {} & \frac{\bar{\lambda}}{\tau}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\bar{\lambda}^m}{m!}\sum_{k=0}^{n}\binom{n}{k}S(m,n-k)\cdot \\ & v^{(k)\mu}u^\nu - \sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\bar{\lambda}^m}{m!}\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\binom{n}{k}\binom{n-k}{\ell}S(m,n-k-\ell)\frac{v^{(k)\mu}v^{(\ell)\nu}}{\ell+1} \ . \end{aligned} \tag{3.181}$$

Carrying out the multiplication the other way gives

$$r^{\mu}(\tau)v^{\nu}(\tau) = \frac{\hbar}{\tau} \sum_{n=0}^{\infty} \frac{\tau^{n}}{n!} \sum_{m=0}^{\infty} \frac{\hbar^{m}}{m!} \sum_{k=0}^{n} \binom{n}{k} S(m,k) \cdot$$

$$u^{\mu}v^{(n-k)\nu} - \sum_{n=0}^{\infty} \frac{\tau^{n}}{n!} \sum_{m=0}^{\infty} \frac{\hbar^{m}}{m!} \sum_{k=0}^{n} \sum_{\ell=0}^{k} \binom{n}{k}\binom{k}{\ell} \cdot \qquad (3.182)$$

$$S(m,k-\ell)\, \frac{v^{(\ell)\mu} v^{(n-k)\nu}}{\ell+1} .$$

Now make the transformations

$$j' = n-k, \qquad k' = n-j .$$

The result is

$$r^{\mu}(\tau)v^{\nu}(\tau) = \frac{\hbar}{\tau} \sum_{n=0}^{\infty} \frac{\tau^{n}}{n!} \sum_{m=0}^{\infty} \frac{\hbar^{m}}{m!} \sum_{j=0}^{n} \binom{n}{n-j} S(m,n-j) \cdot$$

$$u^{\mu}v^{(j)\nu} - \sum_{n=0}^{\infty} \frac{\tau^{n}}{n!} \sum_{m=0}^{\infty} \frac{\hbar^{m}}{m!} \sum_{j=0}^{n} \sum_{\ell=0}^{n-j} \binom{n}{n-j}\binom{n-j}{\ell} \cdot$$

$$S(m,n-j-\ell)\, \frac{v^{(\ell)\mu}}{\ell+1}\, v^{(j)\nu} .$$

Since

$$\binom{n}{n-j} = \binom{n}{j}$$

we have therefore,

$$r^{\mu}(\tau)v^{\nu}(\tau) = \frac{\hbar}{\tau}\sum_{n=0}^{\infty}\frac{\tau^{n}}{n!}\sum_{m=0}^{\infty}\frac{\hbar^{m}}{m!}\sum_{j=0}^{n}\binom{n}{j}S(m,n-j)\cdot \tag{3.183}$$

$$u^{\mu}{}_{v}{}^{(j)\nu} - \sum_{n=0}^{\infty}\frac{\tau^{n}}{n!}\sum_{m=0}^{\infty}\frac{\hbar^{m}}{m!}\sum_{j=0}^{n}\sum_{\ell=0}^{n-j}\binom{n}{j}\binom{n-j}{\ell}\cdot S(m,n-j-\ell)\frac{v^{(\ell)\mu}}{\ell+1}\,v^{(j)\nu}.$$

Then

$$v^{[\mu}(\tau)r^{\nu]}(\tau) = \frac{\hbar}{\tau}\sum_{n=0}^{\infty}\frac{\tau^{n}}{n!}\sum_{m=0}^{\infty}\frac{\hbar^{m}}{m!}\sum_{j=0}^{n}\binom{n}{j}\cdot S(m,n-j)v^{(j)[\mu}{}_{u}{}^{\nu]}$$

$$- \sum_{n=0}^{\infty}\frac{\tau^{n}}{n!}\sum_{m=0}^{\infty}\frac{\hbar^{m}}{m!}\sum_{j=0}^{n}\sum_{\ell=0}^{n-j}\binom{n}{j}\binom{n-j}{\ell}S(m,n-j-\ell)\cdot\frac{v^{(j)[\mu}{}_{v}{}^{(\ell)\nu]}}{\ell+1}\,. \tag{3.184}$$

With Eq. (3.179) in mind, we now differentiate Eq. (3.184) giving

$$\frac{d}{d\tau}\left\{v^{[\mu}(\tau)r^{\nu]}(\tau)\right\} = -\frac{\hbar}{\tau^{2}}\sum_{n=0}^{\infty}\frac{\tau^{n}}{n!}\sum_{m=0}^{\infty}\frac{\hbar^{m}}{m!}\sum_{j=0}^{n}\binom{n}{j}\cdot$$

$$S(m,n-j)v^{(j)[\mu}{}_{u}{}^{\nu]}$$

$$+\frac{\hbar}{\tau}\sum_{n=1}^{\infty}\frac{\tau^{n-1}}{(n-1)!}\sum_{m=0}^{\infty}\frac{\hbar^{m}}{m!}\sum_{j=0}^{n}\binom{n}{j}S(m,n-j)v^{(j)[\mu}{}_{u}{}^{\nu]} \tag{3.185}$$

$$-\sum_{n=1}^{\infty}\frac{\tau^{n-1}}{(n-1)!}\sum_{m=0}^{\infty}\frac{\hbar^{m}}{m!}\sum_{j=0}^{n}\sum_{\ell=0}^{n-j}\binom{n}{j}\binom{n-j}{\ell}S(m,n-j-\ell)\cdot\frac{v^{(j)[\mu}{}_{v}{}^{(\ell)\nu]}}{\ell+1}\,.$$

We can work on the second term above, writing it as

$$\frac{\hbar}{\tau^2}\sum_{n=1}^{\infty}\frac{\tau^n}{(n-1)!}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{n}\binom{n}{j}S(m,n-j)v^{(j)[\mu_u\nu]} ,$$

which transforms into

$$\frac{\hbar}{\tau^2}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\,n\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{n}\binom{n}{j}S(m,n-j)v^{(j)[\mu_u\nu]} .$$

This gives

$$\frac{d}{d\tau}\{v^{[\mu}(\tau)r^{\nu]}(\tau)\} = \frac{\hbar}{\tau^2}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{n}\binom{n}{j}[n-1]\cdot S(m,n-j)v^{(j)[\mu_u\nu]} \tag{3.186}$$

$$-\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{n+1}\sum_{\ell=0}^{n+1-j}\binom{n+1}{j}\binom{n+1-j}{\ell}S(m,n+1-j-\ell)\cdot\frac{v^{(j)[\mu_v(\ell)\nu]}}{\ell+1} .$$

Using Eq. (3.176), we have

$$\frac{1}{\rho(\tau)}\frac{d}{d\tau}\{v^{[\mu}(\tau)r^{\nu]}(\tau)\} = \frac{\hbar}{\tau^3}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{k=0}^{n}\sum_{\ell=0}^{m}\sum_{j=0}^{k}\binom{n}{k}\binom{m}{\ell}\binom{k}{j}\cdot$$

$$(k-1)S(\ell,k-j)S(m-\ell,n-k)v^{(j)[\mu_u\nu]} - \frac{1}{\tau}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{k=0}^{n}\sum_{\ell=0}^{m}\cdot \tag{3.187}$$

$$\sum_{j=0}^{k+1}\sum_{s=0}^{k+1-j}\binom{n}{k}\binom{m}{\ell}\binom{k+1}{j}\cdot\binom{k+1-j}{s}S(\ell,k+1-j-s)S(m-\ell,n-k)\frac{v^{(j)[\mu_v(s)\nu]}}{s+1} .$$

It is convenient for the manipulations below to define

$$F(m,n;j) \equiv \sum_{k=0}^{n} \sum_{\ell=0}^{m} \binom{n}{k}\binom{m}{\ell}\binom{k}{j}(k-1)S(\ell,k-j)\cdot S(m-\ell,n-k)\eta(k-j) \tag{3.188}$$

$$\begin{aligned}\bar{F}(m,n;js) \equiv \sum_{\ell=0}^{m} \sum_{k=0}^{n} \binom{n}{k}\binom{m}{\ell}\binom{k+1}{j}\binom{k+1-j}{s}S(\ell,k+1-j-s)\cdot \\ S(m-\ell,n-k)\left(\frac{1}{s+1}\right)\eta(k+1-j)\eta(k+1-j-s).\end{aligned} \tag{3.189}$$

Now we can write Eq. (3.179) using Eq. (Eq. 3.187) and these definitions as

$$\begin{aligned}F^{\mu\nu}(\tau) = \frac{2\hbar}{\tau^3} \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \sum_{j=0}^{n} F(m,n;j)v^{(j)[\mu}u^{\nu]} \\ - \frac{2}{\tau} \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \sum_{j=0}^{n+1} \sum_{s=0}^{n+1-j} \bar{F}(m,n;j,s)\cdot v^{(j)[\mu}v^{(s)\nu]*}.\end{aligned} \tag{3.190}$$

Since our aim in this section is to calculate $\Theta^{\mu\nu}u_\nu$, we will not express $F^{\mu\nu}$ completely in terms of $\hbar$ here but proceed directly toward the expression for $\Theta^{\mu\nu}u_\nu$. Since $\Theta^{\mu\nu}$ is given by Eq. (1.32) we first compute

$$F^{\mu\nu}(\tau)\ F_{\nu\sigma}(\tau) = \frac{4\hbar^2}{\tau^6} \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \sum_{k=0}^{m} \sum_{\ell=0}^{n} \cdot$$

$$\sum_{j=0}^{\ell} \sum_{s=0}^{n-\ell} F(k,\ell;j)F(m-k,n-\ell,s)v^{(j)[\mu}u^{\nu]}v^{(s)}_{[\nu}u_{\sigma]}$$

---

*Note that this is not $F^{\mu\nu}_{ret}$ because as yet we have not set $\tau=\tilde{\tau}$.

$$- \frac{8\hbar}{\tau^4} \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \sum_{k=0}^{m} \sum_{\ell=0}^{n} \sum_{j=0}^{\ell} \sum_{s=0}^{n+1-\ell} \sum_{p=0}^{n+1-s-\ell} \binom{m}{k}\binom{n}{\ell} \cdot$$

$$F(k,\ell;j)F(m-k,n-\ell;s,p)v^{(j)[\mu}{}_{u}{}^{\nu]}v^{(s)}{}_{[\nu}v^{(p)}{}_{\sigma]} \tag{3.191}$$

$$+ \frac{4}{\tau^2} \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \sum_{k=0}^{m} \sum_{\ell=0}^{n} \sum_{j=0}^{\ell+1} \sum_{s=0}^{\ell+1-j} \sum_{q=0}^{n+1-\ell} \sum_{p=0}^{n+1-\ell-q} \cdot$$

$$\binom{m}{k}\binom{n}{\ell}F(k,\ell;j,s)F(m-k,n-\ell;q,p)v^{(j)[\mu}{}_{v}{}^{(s)\nu]} \cdot v^{(q)}{}_{[\nu}v^{(p)}{}_{\sigma]}$$

with

$$v^{(s)}{}_{[\mu}v^{(p)}{}_{\sigma]} \equiv g_{\mu\lambda}g_{\nu\sigma}v^{(s)[\lambda}v^{(p)\nu]} ,$$

which follows straightforwardily from Eq. (3.190). Now we need the following identities, which one verifies by direct computation

$$v^{(j)[\mu}{}_{u}{}^{\nu]}v^{(s)}{}_{[\nu}u_{\sigma]}u^{\sigma} = \tfrac{1}{4}u^{\mu}[v^{(j)\nu}u_{\nu}v^{(s)\sigma}u_{\sigma}-v^{(j)\nu}v^{(s)}{}_{\nu}] \tag{3.192}$$

$$v^{(j)[\mu}{}_{u}{}^{\nu]}v^{(s)}{}_{[\mu}u_{\nu]} = \tfrac{1}{2}[v^{(j)\sigma}v^{(s)}{}_{\sigma}-v^{(j)\sigma}u_{\sigma}\cdot v^{(s)\nu}u_{\nu}] \tag{3.193}$$

$$v^{(j)[\mu}{}_{u}{}^{\nu]}v^{(s)}{}_{[\nu}v^{(p)}{}_{\sigma]}u^{\sigma} = \tfrac{1}{4}u^{\mu}v^{(j)\nu}[v^{(s)\sigma}u_{\sigma}v^{(p)}{}_{\nu}-v^{(p)\sigma}u_{\sigma}v^{(s)}{}_{\nu}] \tag{3.194}$$

$$v^{(j)[\mu}{}_{u}{}^{\nu]}v^{(s)}{}_{[\mu}v^{(p)}{}_{\nu]} = \tfrac{1}{2}[v^{(j)\sigma}v^{(s)}{}_{\sigma}v^{(p)\nu}u_{\nu}-v^{(j)\sigma}v^{(p)}{}_{\sigma}v^{(s)\nu}u_{\nu}] \tag{3.195}$$

$$v^{(j)[\mu}v^{(s)\nu]}v^{(q)}{}_{[\nu}v^{(p)}{}_{\sigma]}u^{\sigma} = \tfrac{1}{4}[\{v^{(j)\mu}v^{(s)\nu}\cdot v^{(q)}{}_{\nu}-v^{(s)\mu}v^{(j)\nu}\cdot$$
$$v^{(q)}{}_{\nu}\}v^{(p)\sigma}u_{\sigma} + \{v^{(s)\mu}v^{(j)\nu}v^{(p)}{}_{\nu} - v^{(j)\mu}v^{(s)\nu}v^{(p)}{}_{\nu}\}v^{(q)\sigma}u_{\sigma}] \tag{3.196}$$

$$v^{(j)[\mu}v^{(s)\nu]}v^{(q)}{}_{[\mu}v^{(p)}{}_{\nu]} = \tfrac{1}{2}[v^{(j)\sigma}v^{(q)}{}_{\sigma}v^{(s)\nu}v^{(p)}{}_{\nu} - v^{(j)\sigma}v^{(p)}{}_{\sigma}\cdot$$
$$v^{(s)\nu}v^{(q)}{}_{\nu}] \ . \tag{3.197}$$

Using these identities we find that

$$F^{\mu\nu}(\tau)F_{\nu\sigma}(\tau)u^{\sigma} + \tfrac{1}{4}u^{\mu}F^{\alpha\beta}F_{\alpha\beta} = \frac{4\hbar^2}{\tau^6}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\sum_{j=0}^{\ell}\sum_{s=0}^{n-\ell}\cdot$$
$$F(k,\ell;j)F(m-k,n-\ell;s)[\tfrac{1}{4}u^{\mu}\{v^{(j)\nu}u_{\nu}v^{(s)\sigma}u_{\sigma}-v^{(j)\nu}v^{(s)}_{\nu}\} + \frac{1}{8}u^{\mu}\{v^{(j)\nu}\cdot$$
$$v^{(s)}_{\nu}-v^{(j)\sigma}u_{\sigma}v^{(s)\nu}u_{\nu}\}] - \frac{8\hbar}{\tau^4}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{k=0}^{m}\sum_{\ell=0}^{n}\sum_{j=0}^{\ell}\sum_{s=0}^{n+1-\ell}\sum_{p=0}^{n+1-s-\ell}\cdot$$
$$\binom{m}{k}\binom{n}{\ell}\cdot F(k,\ell;j)F(m-k,n-\ell;s,p)[\tfrac{1}{4}u^{\mu}v^{(j)\nu}\{v^{(s)\sigma}u_{\sigma}v^{(p)}_{\nu}-v^{(p)\sigma}u_{\sigma}v^{(s)}_{\nu}\}$$
$$+ \frac{1}{8}u^{\mu}\{v^{(j)\sigma}v^{(s)}_{\sigma}v^{(p)\nu}u_{\nu}-v^{(j)\sigma}v^{(p)}_{\sigma}v^{(s)\nu}u_{\nu}\}] + \frac{4}{\tau^2}\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{k=0}^{m}\cdot \tag{3.198}$$
$$\sum_{\ell=0}^{n}\sum_{j=0}^{\ell+1}\sum_{s=0}^{\ell+1-j}\sum_{q=0}^{n+1-\ell}\sum_{p=0}^{n+1-\ell-q}\binom{m}{k}\binom{n}{\ell}F(k,\ell;j,s)F(m-k,n-\ell;q,p)[\tfrac{1}{4}\{v^{(j)\mu}\cdot$$
$$v^{(s)\nu}v^{(q)}_{\nu}-v^{(s)\mu}v^{(j)\mu}v^{(q)}_{\nu}\}v^{(p)\sigma}u_{\sigma} + \tfrac{1}{4}\{v^{(s)\mu}v^{(j)\nu}v^{(p)}_{\nu}-v^{(j)\mu}v^{(s)\nu}\cdot$$
$$v^{(p)}_{\nu}\}v^{(q)\sigma}u_{\sigma} + \frac{1}{8}u^{\mu}\{v^{(j)\sigma}v^{(q)}_{\sigma}v^{(s)\nu}v^{(p)}_{\nu}-v^{(j)\sigma}v^{(p)}_{\nu}v^{(s)\nu}v^{(q)}_{\nu}\}] \ .$$

We now want to eliminate τ from Eq. (3.198). To do this, we use Eq. (3.8) and the following relations which follow from Eqs. (3.163) and (3.173)

$$\frac{\hbar^2}{\tau^6} = \frac{1}{\hbar^4} \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} h_n^{(6)} \tag{3.199}$$

with

$$h_n^{(6)} = \sum_{j=0}^{n} \binom{j+5}{j} \binom{n+6}{j+6} \frac{n!}{(n+j)!} (-1)^j a_{n+j}(j)\eta(n-j)$$

$$\frac{\hbar}{\tau^4} = \frac{1}{\hbar^3} \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} h_n^{(4)} \tag{3.200}$$

with

$$h_n^{(4)} = \sum_{j=0}^{n} \binom{j+3}{j} \binom{n+4}{j+4} \frac{n!}{(n+j)!} (-1)^j a_{n+j}(j)\eta(n-j)$$

and

$$\frac{1}{\tau^2} = \frac{1}{\hbar^2} \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} h_n^{(2)} \tag{3.201}$$

with

$$h_n^{(2)} = \sum_{j=0}^{n} \binom{j+1}{j} \binom{n+2}{j+2} \frac{n!}{(n+j)!} (-1)^j a_{n+j}(j)\eta(n-j) \ .$$

When the lengthy but standard algebra is done, $F_{ret}^{\mu\nu} F_{ret\mu\sigma} u^{\sigma} + \frac{1}{4} u^{\mu} F_{ret}^{\alpha\beta} F_{ret\alpha\beta}$ can be expressed as the sum of three terms, the first of which is

$$\frac{1}{\hbar^4}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{k_1=0}^{m}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_1-k_2}\sum_{k_4=0}^{k_2}\sum_{k_5=0}^{k_3}\sum_{s_1=0}^{k_5}\sum_{s_2=0}^{k_3-k_5}\sum_{k_6=0}^{m-k_1}\binom{m}{k_1}\binom{k_1}{k_2}\cdot$$

$$\frac{(-1)^{k_3+k_6}}{k_3!}\binom{k_6+5}{k_6}\binom{m-k_1+6}{k_6+6}\frac{(m-k_1)!}{(m-k_1+k_6)!}F(k_4,k_5;s_1)F(k_2-k_4,k_3-k_5;s_2)\cdot$$

$$a_{m-k_1+k_6}(k_6)\,n(m-k_1-k_6)\,[\tfrac{1}{2}u^{\mu}\{v^{(s_1)\nu}u_{\nu}v^{(s_2)\sigma}u_{\sigma}-v^{(s_1)\nu}v_{\nu}^{(s_2)}\}]\ , \tag{3.202}$$

the second term is

$$\frac{1}{\hbar^3}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{k_1=0}^{m}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_1-k_2}\sum_{k_4=0}^{k_2}\sum_{k_5=0}^{k_3}\sum_{s_1=0}^{k_5}\sum_{s_2=0}^{k_3+1-k_5}\sum_{s_3=0}^{k_3+1-s_2-k_5}\sum_{k_6=0}^{m-k_1}\cdot$$

$$\frac{(-1)^{k_3+k_6}}{k3!}\binom{m}{k_1}\binom{k_1}{k_2}\binom{k_2}{k_4}\binom{k_3}{k_5}\binom{k_6+3}{k_6}\binom{m-k_1+4}{k_6+4}\frac{(m-k_1)!}{(m-k_1+k_6)!}\ a_{k_1-k_2}(k_3)\cdot$$

$$a_{m-k_1+k_6}(k_6)F(k_4,k_5;s_1)F(k_2-k_4,k_3-k_5;s_2,s_3)\,n(m-k_1-k_6)\,[u^{\mu}\{v^{(s_1)\nu}\cdot$$

$$v_{\nu}^{(s_2)}v^{(s_3)\sigma}u_{\sigma}-v^{(s_1)\nu}v_{\nu}^{(s_3)\sigma}u_{\sigma}]\ . \tag{3.203}$$

The third term is

$$\frac{1}{\hbar^2}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{k_1=0}^{m}\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_1-k_2}\sum_{k_4=0}^{k_2}\sum_{k_5=0}^{k_3}\sum_{k_6=0}^{m-k_1}\sum_{s_1=0}^{k_5+1}\sum_{s_2=0}^{k_5+1-s_1}\sum_{s_3=0}^{k_3+1-k_5}\cdot$$

$$\sum_{s_4=0}^{k_3+1-k_5-s_3}\frac{(-1)^{k_3+k_6}}{k_3!}\binom{m}{k_1}\binom{k_1}{k_2}\binom{k_2}{k_4}\binom{k_3}{k_5}\binom{k_6+1}{k_6}\binom{m-k_1+2}{k_6+2}\frac{(m-k_1)!}{(m-k_1+k_6)!}\cdot$$

$$a_{m-k_1}(k_6)a_{k_1-k_2}(k_3)F(k_4,k_5,s_1,s_2)F(k_2-k_4,k_3-k_5,s_3,s_4)\eta(m-k_1-k_6)\cdot$$

$$[\tfrac{1}{2}u^{\mu}\{v^{(s1)\sigma}v_{\sigma}^{(s3)}v^{(s2)\nu}v_{\nu}^{(s4)}-v^{(s1)\sigma}v_{\sigma}^{(s4)}v^{(s2)\nu}v_{\nu}^{(s3)}\}+v^{(s4)\sigma}u_{\sigma}\cdot$$

$$\{v^{(s1)\mu}v^{(s2)\nu}v_{\nu}^{(s3)}-v^{(s2)\mu}v^{(s1)\nu}v_{\nu}^{(s3)}\}+v^{(s3)\sigma}u_{\sigma}\{v^{(s2)\mu}v^{(s1)\nu}v_{\nu}^{(s4)}$$

$$-v^{(s1)\mu}v^{(s2)\nu}v_{\nu}^{(s4)}\}]. \tag{3.204}$$

Since we want to integrate

$$F^{\mu\nu}_{ret}F^{\sigma}_{ret\nu}+\tfrac{1}{4}g^{\mu\sigma}F^{\alpha\beta}_{ret}F_{ret\alpha\beta}$$

over the surface of the world-tube, we want to express it entirely in terms of products of u_{μ} and $g_{\mu\nu}$. To do this, we use the definitions (3.188) and (3.189) to express $F(m,n;p)$ and $F(m,n;p,q)$ in the form

$$F(m,n;p) = \sum_{j=0}^{m+2n}\sum_{\ell=0}^{j}\prod_{r=1}^{\ell}\{\sum a^{(q_r)}\}\prod_{r=1}^{j-\ell}\{\sum a^{(p_r)}\}\cdot {}_jF_{\ell}(m,n;p)G_{\ell}U_{j-\ell} \tag{3.205}$$

with

$${}_jF_{\ell}(m,n;p) = \sum_{j_1=0}^{j}\sum_{\ell_1=0}^{\ell}\sum_{k=0}^{n}\sum_{q=0}^{m}\binom{n}{k}\binom{m}{q}\binom{k}{q}(k-1)\,{}_{j_1}S_{\ell_1}(q,k-p)\cdot {}_{j-j_1}S_{\ell-\ell_1}(m-q,n-k)\eta(k-p) \tag{3.206}$$

and

$$F(m,n;p,q) = \sum_{j=0}^{m+2n} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\} {}_jF_\ell(m,n;p,q) \cdot \tag{3.207}$$

$$G_\ell U_{j-\ell}$$

(again preserving the now familiar form)

$${}_jF_\ell(m,n;p,q) = \sum_{j_1=0}^{j} \sum_{\ell_1=0}^{\ell} \sum_{k=0}^{n} \sum_{s=0}^{m} \binom{n}{k}\binom{m}{s}\binom{k+1}{p}\binom{k+1-p}{q} {}_{j_1}S_{\ell 1}(s,k+1-p-q) \cdot \tag{3.208}$$

$${}_{j-j_1}S_{\ell-\ell_1}(m-s,n-k)\ \frac{1}{q+1}\ \eta(k+1-p)\eta(k+1-p-q),$$

with ${}_jS_\ell$ defined in Eq. (3.143). Now using Eq. (3.69), Eq. (3.207), and Eq. (3.208) we find after considerable calculation that Eq. (3.202) can be written

$$\frac{1}{\hbar^4} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \sum_{s_1=0}^{m} \sum_{s_2=0}^{m} \sum_{j=0}^{3m} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\sum a^{(q_r)}\} \prod_{r=1}^{j-\ell} \{\sum a^{(p_r)}\} \cdot \tag{3.209}$$

$${}_j\Xi_\ell^1(m,s_1,s_2) G_\ell U_{j-\ell} [\tfrac{1}{2} u^\mu \{v^{(s1)\nu} u_\nu v^{(s2)\sigma} u_\sigma - v^{(s1)\nu} v_\nu^{(s2)}\}] ,$$

with

$${}_j\Xi_\ell^1(m,s_1,s_2) \equiv \sum_{j_1=0}^{j} \sum_{j_2=0}^{j-j_1} \sum_{\ell_1=0}^{\ell} \sum_{\ell_2=0}^{\ell-\ell_1} \sum_{k_1=0}^{m} \sum_{k_2=0}^{k_1} \sum_{k_3=0}^{k_1-k_2} \sum_{k_4=0}^{k_2} \sum_{k_5=0}^{k_3} \cdot$$

$$\sum_{k_6=0}^{m-k_1} \binom{m}{k_1}\binom{k_1}{k_2} \frac{(-1)^{k_3+k_6}}{k_3!} \binom{k_6+5}{k_6}\binom{m-k_1+6}{k_6+6} \frac{(j-k_1)!}{(m-k_1+k_6)!}\ {}_{j_1}C_{\ell_1}^{m-k_1}(k_6) \cdot$$

$${}_{j_2}F_{\ell_2}(k_4,k_5;s_1)\,{}_{j-j_1-j_2}F_{\ell-\ell_1-\ell_2}(k_2-k_4,k_3-k_5;s_2)\delta(m-k_1-j_1-\sum^{j_1-1}(q+p),\cdot$$

$$q_{\ell_1})\eta(k_4+2k_5-j_2)\eta(m-k_1-k_6)\eta(k_2-k_4+2k_3-2k_5+j_1+j_2-j)\eta(k_5-s_1)\eta(k_3-k_5-s_2)$$

(3.210)

Similarly, Eq. (3.203) becomes

$$\frac{1}{\hbar^3}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{3m}\sum_{\ell=0}^{j}\sum_{s_1=0}^{m}\sum_{s_2=0}^{m}\sum_{s_3=0}^{m}\prod_{r=1}^{\ell}\{\sum a^{(q_r)}\}\prod_{r=1}^{j-\ell}\{\sum a^{(p_r)}\}\cdot \tag{3.211}$$

$${}_j\Xi_\ell^2(m,s_1,s_2,s_3)[u^\mu\{v^{(s_1)\nu}v_\nu^{(s_2)}v^{(s_3)\sigma}u_\sigma-v^{(s_1)\nu}v_\nu^{(s_3)}v^{(s_2)\sigma}u_\sigma\}]$$

with

$${}_j\Xi_\ell^2(m,s_1,s_2,s_3) = \sum_{j_1=0}^{j}\sum_{j_2=0}^{j-j_1}\sum_{j_3=0}^{j-j_1-j_2}\sum_{\ell_1=0}^{\ell}\sum_{\ell_2=0}^{\ell-\ell_1}\sum_{\ell_3=0}^{\ell-\ell_1-\ell_2}\sum_{k_1=0}^{m}\cdot$$

$$\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_1-k_2}\sum_{k_4=0}^{k_2}\sum_{k_5=0}^{k_3}\sum_{k_6=0}^{m-k_1}\frac{(-1)^{k_3+k_6}}{k_3!}\binom{m}{k_1}\binom{k_1}{k_2}\binom{k_2}{k_4}\binom{k_3}{k_5}\binom{k_6+3}{k_6}\cdot$$

$$\binom{m-k_1+4}{k_6+4}\frac{(m-k_1)!}{(m-k_1+k_6)!}\,{}_{j_1}C_{\ell_1}^{k_1-k_2-k_3}(k_3)\,{}_{j_2}C_{\ell_2}^{m-k_1}(k_6)\,{}_{j_3}F_{\ell_3}(k_4,k_5,s_1)\cdot$$

$${}_{j-j_1-j_2-j_3}F_{\ell-\ell_1-\ell_2-\ell_3}(k_2-k_4,k_3-k_5;s_2,s_3)\delta(k_1-k_2-k_3-j_1-\sum^{j_1-1}(q+p),q_{\ell_1})\cdot$$

$$\delta(m-k_1-j_2-\sum^{j_2-1}(q+p),q_{\ell_2})\eta(k_4+2k_5-j_3)\eta[k_2-k_4+2(k_3-k_5+1)+j_1+j_2+j_3-j)\cdot$$

$$\eta(k_5-s_1)\eta(k_3+1-k_5-s_2)\eta(k_3+1-k_5-s_2-s_3) \tag{3.212}$$

and Eq. (3.204) becomes

$$\frac{1}{\hbar^2}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{3m}\sum_{\ell=0}^{j}\sum_{s_1=0}^{m}\cdots\sum_{s_4=0}^{m}\prod_{r=1}^{\ell}\{\sum a^{(q_r)}\}\prod_{r=1}^{j-\ell}\{\sum a^{(p_r)}\}\,{}_j\Xi_\ell^3\cdot$$

$$(m,s_1,s_2,s_3,s_4)G_\ell U_{j-\ell}[\tfrac{1}{2}u^\mu\{v^{(s1)\sigma}v_\sigma^{(s3)}v^{(s2)\nu}v_\nu^{(s4)}-v^{(s1)\sigma}\cdot$$

$$v_\sigma^{(s4)}v^{(s2)\nu}v_\nu^{(s3)}\}+v^{(s4)\sigma}u_\sigma\{v^{(s1)\mu}v^{(s2)\nu}v_\nu^{(s3)}-v^{(s2)\mu}\cdot \tag{3.213}$$

$$v^{(s1)\nu}v_\nu^{(s3)}\}+v^{(s3)\sigma}u_\sigma\{v^{(s2)\mu}v^{(s1)\nu}v_\nu^{(s4)}-v^{(s1)\mu}v^{(s2)\nu}v_\nu^{(s4)}\}]$$

where

$${}_j\Xi_\ell^3(m,s_1,s_2,s_3,s_4)\equiv\sum_{j_1=0}^{j}\sum_{j_2=0}^{j-j_1}\sum_{j_3=0}^{j-j_1-j_2}\sum_{\ell_1=0}^{\ell}\sum_{\ell_2=0}^{\ell-\ell_1}\sum_{\ell_3=0}^{\ell-\ell_1-\ell_2}\sum_{k_1=0}^{m}\cdot$$

$$\sum_{k_2=0}^{k_1}\sum_{k_3=0}^{k_1-k_2}\sum_{k_4=0}^{k_2}\sum_{k_5=0}^{k_3}\sum_{k_6=0}^{m-k_1}\frac{(-1)^{k_3+k_6}}{k_3!}\binom{m}{k_1}\binom{k_1}{k_2}\binom{k_2}{k_4}\binom{k_3}{k_5}\cdot$$

$$\binom{k_6+1}{k_6}\binom{m-k_1+2}{k_6+2}\frac{(m-k_1)!}{(m-k_1+k_6)}\,{}_{j_1}C_{\ell_1}^{m-k_1-k_6}(k_6)\,{}_{j_2}C_{\ell_2}^{k_1-k_2-k_3}(k_3)\cdot$$

$${}_{j_3}F_{\ell_3}(k_4,k_5;s_1,s_2)\,{}_{j-j_1-j_2-j_3}F_{\ell-\ell_1-\ell_2-\ell_3}(k_2-k_4,k_3-k_5,s_3,s_4)\cdot$$

$$\eta(m-k_1-k_6)\delta[m-k_1-k_6-j_1-\sum^{j_1-1}(q+p),q_{\ell_1}]\delta[k_1-k_2-k_3-j_2-\sum^{j_2-1}(q+p)\cdot$$

$$q_{\ell_2}]\eta[k_4+2(k_5+10-j_3]\eta[k_2-k_4+2(k_3-k_5+1)+j_1+j_2+j_3-j]\eta(k_5+1-s_1)\cdot$$

$$\eta(k_5+1-s_1-s_2)\eta(k_3+1-k_5-s_3)\eta(k_3+1-k_5-s_3-s_4) \tag{3.214}$$

As they stand, Eqs. (3.209), (3.211) and (3.213) are not in the standard form which we have worked with throughout this development in that they depend on the a^{μ} and v^{μ}. Consider e.g. the essential part of Eq. (3.209).

$$\sum_{k_1=0}^{m} \sum_{k_2=0}^{m} {}_j\Xi_\ell^1(m,k_1,k_2)[v^{(k_1)\nu} u_\nu v^{(k_2)\sigma} u_\sigma - v^{(k_1)\nu} v_\nu^{(k_2)}] .$$

The first term can be written

$$\sum_{p_{j-1}=0}^{m-1} \sum_{p_j=0}^{m-1} {}_j\Xi_\ell^1(m,p_{j+1-\ell}+1,p_{j+2-\ell}+1) a^{(p_{j+1-\ell})} a^{(p_{j+2-\ell})} U_2 .$$

The second we first write as

$$\sum_{k_1=0}^{m-1} \sum_{k_2=0}^{m-1} {}_j\Xi_\ell^1(m,k_1+1,k_2+1) a^{(k_1)} a^{(k_2)} G_2 + \sum_{k_1=0}^{m-1} {}_j\Xi_\ell^1(m,k_1+1,0)\cdot \tag{3.215}$$

$$a^{(k_1)\nu} v_\nu + \sum_{k_2=0}^{m-1} {}_j\Xi_\ell^1(m,0,k_2+1) a^{(k_2)\nu} v_\nu .$$

Then using

$$a^{(k_1)\nu} v_\nu = - \sum_{k_2=0}^{k_1-1} \binom{k_1-1}{k_2+1} a^{(k_2)\mu} a_\mu^{(k_1-k_2-2)}$$

we can write Eq. (3.215) as

$$\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-1} {}_j\Xi_\ell^1(m,k_1+1,k_2+1)a^{(k_1)}a^{(k_2)}G_2 - \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{k_1-1}\binom{k_1-1}{k_2+1}a^{(k_2)\mu}\cdot$$

$$a^{(k_1-k_2-2)}\,{}_j\Xi_\ell^1(m,k_1+1,0) - \sum_{k_2=0}^{m-1}\sum_{k_1=0}^{k_2-1}\binom{k_2-1}{k_1+1}a^{(k_1)}a^{(k_2-k_1-2)}\,{}_j\Xi_\ell^1\cdot \tag{3.216}$$

$$(m,0,k_2+1) \ .$$

Next, we want to transform $a^{(k_1-k_2-2)}$ to $a^{(k_1)}$, so let

$$k_1' = k_1-k_2-2$$

or

$$k_1 = k_1'+k_2+2 \quad .$$

This gives the term

$$a^{(k_2)\mu}a^{(k_1')} \quad .$$

Notice that the sum of k_1' and k_2 is k_1-2 which ranges from zero to m-3 so we can write the second term in Eq. (3.216) as

$$\sum_{k_1'=0}^{m-1}\sum_{k_2=0}^{m-1}\sum_{q=0}^{m-3}\binom{k_1+k_2+1}{k_2+1}a^{(k_1)}a^{(k_2)}G_2\cdot$$

$${}_j\Xi_\ell^1(m,k_1'+k_2+3,0)\,\delta(k_1+k_2,q)$$

$$\sum_{k_1=0}^{m}\sum_{k_2=0}^{m} {}_j\Xi_\ell^1(m,k_1,k_2)v^{(k_1)\nu}v_\nu^{(k_2)} = \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-1} a^{(k_1)}a^{(k_2)}G_2\cdot$$

$$\Big\{{}_j\Xi^1_\ell(m,k_1+1,k_2+1) - \sum_{q=0}^{m-3}\binom{k_1+k_2+1}{k_2+1}{}_j\Xi^1_\ell(m,k_1+k_2+3,0)\,\delta(k_1+k_2,q)$$

$$- \sum_{q=0}^{m-3}\binom{k_1+k_2+1}{k_1+1}{}_j\Xi^1_\ell(m,0,k_1+k_2+3)\,\delta(k_1+k_2,q)\Big\} \ . \tag{3.217}$$

To keep the formulas from becoming too unwieldy, it is convenient to introduce the substitution operator

$$\Gamma(p_1,p_2;k_1,k_2) = \delta(p_1,k_1+1)\delta(p_2,k_2+1) - \sum_{q=0}^{m-3}\binom{k_1+k_2+1}{k_1+1}\cdot$$

$$\delta(p_1,0)\delta(p_2,k_1+k_2+3)\delta(k_1+k_2,q) - \sum_{q=0}^{m-3}\binom{k_1+k_2+1}{k_2+1}\delta(p_1,k_1+k_2+3)$$

$$\delta(p_2,0)\delta(k_1+k_2,q) \ , \tag{3.218}$$

this gives

$$\sum_{k_1=0}^{m}\sum_{k_2=0}^{m}{}_j\Xi^1_\ell(m,k_1,k_2)v^{(k_1)}{}_\nu v^{(k_2)}_\nu = \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-1}{}_j\Xi^1_\ell(m,k_1,k_2)\cdot$$

$$\sum_{p_1=0}^{m-1}\sum_{p_2=0}^{m-1}\Gamma(k_1,k_2;p_1,p_2)a^{(p_1)}a^{(p_2)}G_2 \ . \tag{3.219}$$

Now the Eq. (3.209) takes the form

$$\frac{1}{\hbar^4}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{3m}\sum_{\ell=0}^{j}\prod_{r=1}^{\ell}\{\sum a^{(q_r)}\}\prod_{r=1}^{j-\ell}\{\sum a^{(p_r)}\}{}_j\Theta^1_\ell(m)G_\ell U_{j-\ell}u^\mu \tag{3.220}$$

with

$$ {}_j\Theta^1_\ell(m) \equiv \tfrac{1}{2}\{{}_{j-2}\Xi^1_\ell(m,p_{j-1-\ell}+1,p_{j-\ell}+1) - \sum_{s_1=0}^{m}\sum_{s_2=0}^{m} \cdot $$

$$ {}_{j-2}\Xi^1_\ell(m,s_1,s_2)\Gamma(m,s_1,s_2,q_{\ell-1}q_\ell)\} \; . \tag{3.221} $$

Similarly, Eq. (3.211) becomes

$$ \frac{1}{\hbar^3}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{3m}\sum_{\ell=0}^{j}\prod_{r=1}^{\ell}\{\textstyle\sum a^{(q_r)}\}\prod_{r=1}^{j-\ell}\{\textstyle\sum a^{(p_r)}\}{}_j\Theta^2_\ell(m)G_\ell U_{j-\ell}u^\mu \tag{3.222} $$

with

$$ {}_j\Theta^2_\ell(m) \equiv \sum_{s_1=0}^{m}\sum_{s_2=0}^{m} {}_{j-3}\Xi^2_\ell(m,s_1,s_2,p_j+1)\Gamma(s_1,s_2;q_{\ell-1},q_\ell) $$

$$ - \sum_{s_1=0}^{m}\sum_{s_3=0}^{m} {}_{j-3}\Xi^2_\ell(m,s_2,p_{j-\ell}+1,s_3)\Gamma(s_1,s_3;q_{\ell-1}q_\ell) \; . \tag{3.223} $$

Lastly, Eq. (3.213) is

$$ \frac{1}{\hbar^2}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{3m}\sum_{\ell=0}^{j}\prod_{r=1}^{\ell}\{\textstyle\sum a^{(q_r)}\}\prod_{r=1}^{j-\ell}\{\textstyle\sum a^{(p_r)}\}{}_j\Theta^3_\ell(m)\cdot $$

$$ G_\ell U_{j-\ell}u^\mu + \frac{1}{\hbar^2}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{s=0}^{m} v^{(s)\mu}\sum_{j=0}^{3m}\sum_{\ell=0}^{j}\prod_{r=1}^{\ell}\{\textstyle\sum a^{(q_r)}\}\prod_{r=1}^{j-\ell}\cdot $$

$$ \{\textstyle\sum a^{(p_r)}\}{}_j\Theta^4_\ell(m,s)G_\ell U_{j-\ell} \; , \tag{3.224} $$

with

$$
{}_j\Theta_\ell^3(m) \equiv \tfrac{1}{2} \sum_{s_1=0}^{m} \sum_{s_2=0}^{m} \sum_{s_3=0}^{m} \sum_{s_4=0}^{m} {}_{j-4}\Xi_\ell^3(m,s_1,s_2,s_3,s_4) \cdot
$$

$$
[\Gamma(s_1,s_3;q_{\ell-3},q_{\ell-2})\Gamma(s_2,s_4;q_{\ell-1},q_\ell) - \Gamma(s_1,s_4;q_{\ell-3},q_{\ell-2}) \cdot
$$

$$
\Gamma(s_2,s_3;q_{\ell-1},q_\ell)] \ , \tag{3.225}
$$

and

$$
{}_j\Theta_\ell^4(m,s) \equiv \sum_{s_2=0}^{m} \sum_{s_3=0}^{m} {}_{j-3}\Xi_\ell^3(m,s,s_2,s_3,p_{j-\ell}+1)\Gamma(s_2,s_3;q_{j-1},q_j)
$$

$$
- \sum_{s_1=0}^{m} \sum_{s_3=0}^{m} {}_{j-3}\Xi_\ell^3(m,s_1,s,s_3,p_{j-\ell}+1)\Gamma(s_1,s_3,q_{\ell-1},q_\ell) + \sum_{s_1=0}^{m} \cdot
$$

$$
\sum_{s_4=0}^{m} {}_{j-3}\Xi_\ell^3(m,s_1,s,p_{j-\ell}+1,s_4)\Gamma(s_1,s_4,q_{\ell-1},q_\ell) - \sum_{s_2=0}^{m} \sum_{s_4=0}^{m} \cdot
$$

$$
{}_{j-3}\Xi_\ell^3(m,s,s_2,p_{j-\ell}+1,s_4)\Gamma(s_2,s_4;q_{\ell-1},q_\ell) \ . \tag{3.226}
$$

Making use of the standard manipulations, we finally obtain from Eqs. (3.220), (3.222), and (3.224)

$$
\Theta_{\text{ret}}^{\mu\nu} u_\nu = F_{\text{ret}}^{\mu\nu} F_{\text{ret}\nu}{}^{\sigma} u_\sigma + \tfrac{1}{4} u^\mu F_{\text{ret}}^{\alpha\beta} F_{\text{ret}\alpha\beta}
$$

$$
= \frac{1}{\hbar^4} \sum_{m=0}^{\infty} \frac{\hbar^m}{m!} \sum_{j=0}^{3m} \sum_{\ell=0}^{j} \prod_{r=1}^{\ell} \{\textstyle\sum a^{(q_r)}\} \cdot
$$

$$\prod_{r=1}^{j-\ell}\left\{\sum a^{(p_r)}\right\}\left\{u^\mu\left[{}_j\Theta^1_\ell(m) + m\,{}_j\Theta^2_\ell(m-1) + m(m-1)\,{}_j\Theta^3_\ell(m-2)\right]\right.$$

$$\left. + \sum_{s=0}^{m} v^{(s)\mu}\left[m(m-1)\,{}_j\Theta^4_\ell(m-2,s)\right]\right\}G_\ell U_{j-\ell} \quad (3.227)$$

From Eq. (3.227), we can obtain the expression for the flux of electromagnetic momentum leaving the world tube. The complete expression for the world-tube surface element is given by Eq. (1.44). A simple calculation gives

$$\Theta^{\mu\nu}_{\mathrm{ret}}u_\nu r^2(1+ra_u) = \frac{1}{r^2}\sum_{m=0}^{\infty}\frac{r^m}{m!}\sum_{j=0}^{3m}\sum_{\ell=0}^{j}\prod_{r=1}^{\ell}\left\{\sum a^{(q_r)}\right\}\prod_{r=1}^{j-\ell}\left\{\sum a^{(p_r)}\right\}\cdot$$

$$\left[{}_j\Phi^1_\ell(m)u^\mu + \sum_{s=0}^{m} v^{(s)\mu}\,{}_j\Phi^2_\ell(m,s)\right]G_\ell U_{j-\ell}\ , \quad (3.228)$$

where

$${}_j\Phi^1_\ell(m) \equiv {}_j\Theta^1_\ell(m) + m\,{}_j\Theta^2_\ell(m-1) + m(m+1)\,{}_j\Theta^3_\ell(m-2) + \left\{{}_{j-1}\Theta^1_\ell(m-1)\right.$$

$$\left. + (m-1)\,{}_{j-1}\Theta^2_\ell(m-2) + m(m-1)\,{}_{j-1}\Theta^3_\ell(m-3)\right\}\delta(p_j,0)\ . \quad (3.229)$$

$${}_j\Phi^2_\ell(m,s) = m(m-1)\,{}_j\Theta^4_\ell(m-2,s) + (m-1)(m-2)\ {}_{j-1}\Theta^4_\ell(m-3,s)\cdot$$

$$\delta(p_{j-\ell},0) \quad (3.230)$$

I. The Integration of U_n

In this section, we will find general expressions for integrals of the form

$$\int Q^{\mu\nu\ldots\lambda} u_\mu u_\nu \ldots u_\lambda d\Omega$$

where Q is a constant vector with respect to the integration. Using this general expression, we will at last be able to obtain the expression for the electromagnetic momentum.

In the rest frame of the charge

$$u^\mu = (0;\xi^i)$$

Where the semicolon separates the time component from the space components and ξ^i are the components of the unit position vector which points from the charge to the field point of interest. To begin our investigation, let us consider the integral

$$\int \xi_i \xi_j d\Omega \ .$$

The integral is a symmetric tensor and thus is proportional to the Kronecker delta

$$\int \xi_i \xi_j d\Omega = b_2\ \delta_{ij} \qquad b_2 \text{ a constant.} \tag{3.231}$$

Since

$$\xi^i \xi_i = 1 \qquad \text{(i summed)}$$

when we contract on i and j, we find

$$b_2 = \frac{4\pi}{3} \ .$$

The integral of an odd number of products of ξ^i is zero. Consider next

$$\int \xi_i \xi_j \xi_k \xi_\ell d\Omega \ .$$

This integral is completely symmetric in the indices i, j, k, ℓ. Moreover, the integral can be nonzero only if the indices are "paired". Thus

$$\int \xi_i \xi_j \xi_k \xi_\ell d\Omega = b_4 \{ \delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} \} \ . \qquad (3.232)$$

Now contract j on i and k on ℓ.

$$4\pi = b_4 \{ (3)(3)+3+3 \}$$

$$b_4 = \frac{4\pi}{15} \ .$$

Now let us consider the general case. If we have an integral over $d\Omega$ of 2m position vectors, then there will be $\frac{(2m)!}{2^m m!}$ ways of pairing the indices as above and so there will be this many terms in the three-tensor expression for the integral. Symbolically

$$\int \xi_i \cdots \xi_k \ d\Omega = b_{2m} \{ \delta_{ij} \cdots \delta_{\ell k} + \text{other arrangements} \} \ . \qquad (3.233)$$

To find the proportionality factor b_{2m}, contract on the indices in an arbitrary order. The left hand side is

$$\int d\Omega = 4\pi \ .$$

In the right hand member there is one arrangement in which every Kronecker delta is contracted on itself. This arrangement we call completely faithful. In all other arrangements, there are deltas in which the partners of the pair of indices belonging to the delta are not contracted on each other but are contracted on the indices of other pairs belonging to other deltas. The contribution which such an "unfaithful configuration" makes to the right hand member of Eq. (3.233) is $3^{m+1-p} \equiv (3^{m-p})\cdot(3)$ where p is the number of "unfaithful pairs" in the arrangement.

For m pairs let us find the number of arrangements in which there are p unfaithful pairs. The smallest number of unfaithful pairs is two. These pairs can be rearranged in

$$\frac{(2.2)!}{2^2 2!}$$

ways but one of these ways is the faithful pairing. Thus the number of unfaithful arrangements of two pairs are

$$\frac{(2.2)!}{2^2 2!} - 1 \ .$$

There are $\binom{m}{2}$ ways of choosing two pairs from m pairs. Hence, among m pairs, there are

$$N_2(m) \equiv \binom{m}{2}\left[\frac{(2.2)!}{2^2 2!} - 1\right] \tag{3.234}$$

arrangements in which there are precisely two unfaithful pairs. For three unfaithful pairs there are

$$\frac{(2.3)!}{2^3 3!}$$

arrangements of three pairs but along these are the arrangements containing no unfaithful pairs and two unfaithful pairs. Hence there are

$$N_3(m) = \binom{m}{3}\left[\frac{(2.3)!}{2^3 3!} - N_2(3) - 1\right] \tag{3.235}$$

arrangements with precisely three unfaithful pairs. The number of arrangements of m pairs which contain j unfaithful pairs are

$$N_j(m) = \binom{m}{j}\left[\frac{(2j)!}{2^j j!} - 1\right] - \binom{m}{j}\sum_{k=0}^{j-1} N_k(j)$$

$$N_o(m) \equiv 1 \tag{3.236}$$

Using this recursively gives

$$\begin{aligned} N_j(m) = {} & \binom{m}{j}\left[\frac{(2j)!}{2^j j!} - 1\right] \\ & - \binom{m}{j}\sum_{k=0}^{j-1}\binom{j}{k}\left[\frac{(2k)!}{2^k k!} - 1\right] \\ & + \binom{m}{j}\sum_{k=0}^{j-1}\sum_{\ell=0}^{k-1}\binom{j}{k}N_\ell(k) \ . \end{aligned}$$

Rewrite it as

$$N_{k_o}(m) = \binom{m}{k_o}\left[\frac{(2k_o)!}{2^{k_o}k_o!} - 1\right] - \binom{m}{k_o}\sum_{k_1=0}^{k_o-1}\binom{k_o}{k_1}\cdot$$

$$\left[\frac{(2k_1)!}{2^{k_1}k_1!} - 1\right] + \binom{m}{k_o}\sum_{k_1=0}^{k_o-1}\sum_{k_2=0}^{k_1-1}\binom{k_o}{k_1}N_{k_2}(k_1) .$$

Generalizing we have

$$N_{k_o}(m) = \sum_{p=1}^{k_o-1}\binom{m}{k_o}\prod_{r=1}^{p}\left\{(-1)^{p+1}\sum_{k_r=0}^{k_{r-1}}\binom{k_{r-1}}{k_r}\right\}\prod_{r=1}^{p-1}\cdot$$

$$\left\{\eta(k_{r-1}-1-k_r)\right\}\left[\frac{(2^{k_p})!}{2^{k_p}k_p!} - 1\right]\delta(k_{p-1},k_p) \tag{3.237}$$

where η is defined in Eq. (3.78). The expression for b_{2m} is

$$b_{2m} = \frac{4\pi}{3^m + \sum_{p=1}^{m} 3^{m+1-p}N_p(m)} \tag{3.238}$$

with $N_p(m)$ given by Eq. (3.237).

If we define

$$s^{\mu\nu} \equiv \int u^\mu u^\nu d\Omega$$

we know from Eq. (3.231) that

$$s^{ij} = b_2\delta^{ij}$$
$$s^{oj} = s^{jo} = 0 \tag{3.239}$$

in the rest frame of the charge. Also since, in the rest frame

$$\int u^{\mu}u^{\nu}d\Omega = \int \xi^{i}\xi^{j}d\Omega$$

$$s^{oo} = 0$$

in the rest frame. Let us now prove that

$$s^{\mu\nu} = s^{(1)\mu\nu} \equiv b_2(g^{\mu\nu}+v^{\mu}v^{\nu}) \ . \tag{3.240}$$

From Eq. (1.13), we see that $s^{(1)\mu\nu}$ satisfies Eq. (3.239). Also, since

$$v^{\mu} = (1;0,0,0)$$

in the rest frame, we see that

$$T^{\mu\nu} \equiv s^{\mu\nu} - s^{(1)\mu\nu} = 0$$

in the rest frame. But since $s^{\mu\nu}$ and $s^{(1)\mu\nu}$ are both tensors by their definitions, $T^{\mu\nu}$ is a tensor which is zero in one reference frame and hence zero in all frames, which proves Eq. (3.240). For

$$s^{\mu\nu\sigma\lambda} \equiv \int u^{\mu}u^{\nu}u^{\sigma}u^{\lambda}d\Omega$$

Eq. (3.232) suggests

$$s^{\mu\nu\sigma\lambda} = b_4\{(g^{\mu\nu}+v^{\mu}v^{\nu})(g^{\sigma\lambda}+v^{\sigma}v^{\lambda}) + (g^{\mu\sigma}+v^{\mu}v^{\sigma})(g^{\nu\lambda}+v^{\nu}v^{\lambda})$$

$$+ (g^{\mu\lambda}+v^{\mu}v^{\lambda})(g^{\sigma\nu}+v^{\sigma}v^{\nu})\} \ ,$$

which can be proven in precisely the same manner.

We can write this result in a short hand way as

$$\int u^{\mu}u^{\nu}u^{\sigma}u^{\lambda}d\Omega = b_4 \underset{\sim}{P}\{[G_2+v_{(2)}]^2\}^{\mu\nu\sigma\lambda}$$

where the notation $\underset{\sim}{P}\{\ldots\}$ means from the sum of all the possible distinct terms resulting from different arrangements of two pairs of indices. In general

$$\int U_n d\Omega = b_n \underset{\sim}{P}\{[G_2+v_{(2)}]^{n/2}\} \tag{3.241}$$

where the tensor indices have been suppressed, and where it is understood that the right hand member is zero if $\frac{n}{2}$ is not an integer. Using Eq. (3.241) to integrate Eq. (3.229), we find at last

$$-\frac{dP^{\mu}}{d\tau} = \int \Theta^{\mu\nu}_{ret}u_{\nu}\hbar^2(1+\hbar a_u) = \frac{1}{\hbar^2}\sum_{m=0}^{\infty}\frac{\hbar^m}{m!}\sum_{j=0}^{3m}\sum_{\ell=0}^{j}\prod_{r=1}^{\ell}\cdot$$

$$\{\sum a^{(q_r)}\}\prod_{r=1}^{j-\ell}\{\sum a^{(p_r)}\}[{}_j\Phi^1_\ell(m)\ \frac{1}{4\pi}b_{j+1-\ell}\underset{\sim}{P}\{[G_2+v_{(2)}]\cdot$$

$$\frac{j+1-\ell}{2}\}^{\mu} + \sum_{s=0}^{m}v^{(s)\mu}{}_j\Phi^2_\ell(m,s)\ \frac{b_{j-\ell}}{4\pi}\ \underset{\sim}{P}\{[G_2+v_{(2)}]\cdot$$

$$\frac{j-\ell}{2}\}]G_\ell\ .$$

CHAPTER IV

A. The Dirac Technique

The Dirac Technique is far from simply being a gimmick which works for electric charges. Bhahba[6] has used it to derive the equations of motion which a "classical meson" obeys, while Bhahba[7], Mathisson[8], and others have also applied it to the study of spinning charges. Never-the-less, until the present work, the theory was open to serious objections due to renormalization. The general formulation which has been presented here disposes of all reasonable questions concerning rigor. Moreover, our model of the electron is among the most general imaginable, with no restrictive assumptions about structure. In short, the author's formalism carries Dirac's technique to its rigorous limit.

B. Applications of the Current Investigation

Nearly all of the results derived in the last few years for classical charges apply only in the limit $\hbar \to 0$. One of the most interesting examples is the problem of the Coulomb momentum. The Coulomb momentum, as defined by Rohrlich is

$$P^{\mu}_{coul} = \int \tilde{\Theta}^{\mu\nu}_{coul}\, d\sigma_{\nu} \tag{4.1}$$

where

$$\tilde{\Theta}^{\mu\nu}_{coul} \equiv \frac{e^2}{4\pi\tilde{\rho}4} (\tilde{u}^\mu\tilde{u}^\nu - \tilde{v}^\mu\tilde{v}^\nu - \tfrac{1}{2}g^{\mu\nu}) \ . \qquad (4.2)$$

If the charge has always been in uniform motion, then it can be shown that in the limit $\hbar\to 0$,

$$\frac{dP^\mu_{coul}}{d\tau} = m_{elm} a^\mu \qquad (4.3)$$

with

$$m_{elm} = \frac{e^2}{2\hbar} \ .$$

However, Professor Cohn[9] has shown that for arbitrary motion, the Coulomb momentum based on Eq. (4.2) is not given by Eq. (4.3) Cohn has been able (again in the limit $\hbar\to 0$), to define a Coulomb momentum which is given by Eq. (4.3) for arbitrary motions. The question naturally arises as to what happens when one is not restricted to the limit $\hbar\to 0$. With the author's work, it is now possible to investigate all questions of this nature.

The most exciting problem of all is what significance the infinite series Eq. (3.242) holds for the equations of motion. That the series <u>is</u> significant is clear from many points of view. The most direct way of regarding it, is to imagine comparing the solution of the Dirac equation to the solutions obtained for equations of motion when more and more of the higher order terms are considered. To illustrate this, the author has calculated the equation of motion resulting from keeping terms in Eq. (3.242) which are linear in $\hbar$. This equation is

$$(m_{bare} + \frac{e^2}{2\hbar c^2})a^\mu = \frac{2}{3}\frac{e^2}{c^3}(\mathring{a}^\mu - a^2 v^\mu)$$

$$+ \frac{e^2\hbar}{c^4}(-\frac{1}{3}\ddot{a}^\mu + \frac{7}{12}a^2 a^\mu + \frac{7}{6}\mathring{a}\cdot a a^\mu)$$

$$+ F^{\mu\nu}_{ext} v_\nu . \tag{4.4}$$

To compare a solution of this equation with one of the Dirac equation, assume that

$$F^\mu_{ext} \equiv F^{\mu\nu}_{ext} v_\nu$$

is not a function of τ and that the particle moves so that

$$v^\mu = \alpha^\mu e^{\lambda\tau} + \beta^\mu e^{-\lambda\tau} . \tag{4.5}$$

Since

$$v^\mu v_\mu = - c^2$$

we have that

$$\alpha^\mu \alpha_\mu = 0 = \beta^\mu \beta_\mu ; \qquad 2\alpha^\mu \beta_\mu = - c^2 .$$

Otherwise α, and β, are arbitrary four-vectors. Because the rate at which the charge radiates energy in the form of radiation is

$$R = \frac{2}{3}\frac{e^2}{c^3}a^\mu a_\mu$$

we find, by differentiating Eq. (4.5) with respect to τ, that

$$\lambda^2 = \frac{3}{2}\frac{cR}{e^2} .$$

For the motion Eq. (4.5) the Dirac equation forces that conclusion that a^μ is τ-independent and that

$$a^\mu = \frac{1}{m_{bare}} F^\mu_{ext} . \tag{4.6}$$

This is the so-called constant intrinsic acceleration case. When the same motion is substituted in Eq. (4.4) the result is

$$a^\mu = \frac{F^\mu_{ext}}{m_{bare}\left(1 - \frac{3}{8}\frac{\hbar R}{mc^3}\right)}$$

i.e. the intrinsic acceleration is constant for the same physical situation that produced constant acceleration for the Dirac particle. The ratio of magnitudes for the two different accelerations in Eqs. (4.6) and (4.7) respectively is

$$1 - \frac{3}{8}\hbar\frac{R}{mc^3} ,$$

Thus, in principle, an experiment could distinguish between the Dirac equation and Eq. (4.4).

The general treatment of the series Eq. (3.242) is beyond the scope of this investigation. The question which is begging to be answered is whether there exist values of $\hbar>0$ such that one recovers the Dirac equation exactly from the series Eq. (3.242). If at least one such value of $\hbar$ exists then the Dirac technique is completely justified, and we obtain the remarkable result that the classical theory is capable of predicting the size of the (classical) electron. If no such value of $\hbar$ exists, then the Dirac tehhnique is utterly without rigorous foundation. In either case, the present work has made the investigation of this question possible.

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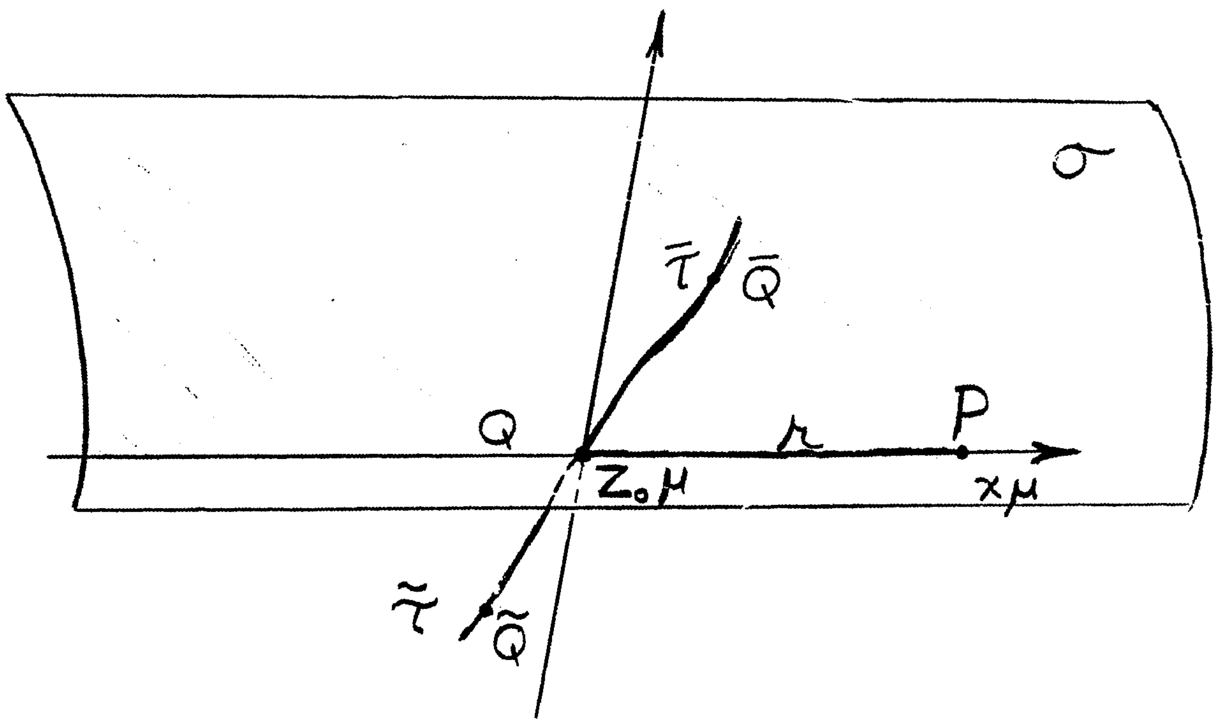

Figure 1. Particle World-line and Associated Surfaces.