FREQUENCY ANALYSIS OF PLANAR FRAMES WITH DISTRIBUTED MASS — RECTANGULAR FRAMES BY METHOD OF STIFFNESS COEFFICIENTS

By

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PREFACE

The method presented in this thesis has been shown by Laursen, Shübinski and Clough. (14) Further generalization and modification of the method was given in lectures by Dr. J. W. Gillespie at Oklahoma State University.

Sincere appreciation is expressed to Dr. James W. Gillespie for his guidance and assistance throughout the preparation of this thesis.

Special gratitude is expressed to the following:

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D. J. C.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

General

The purpose of this thesis is to present a method for finding the natural frequencies of a one dimensional distributed mass system. A general stiffness matrix is presented for a typical member, taking into account the distribution of mass. By establishing equilibrium of forces at nodes and compatibility with the boundary conditions, the defining dynamic equations for the frame are obtained.

It is assumed that each element of the structure is of constant cross-section and has uniform distribution of mass per unit length. All end forces (including moments) and end displacements (including rotations) are denoted by vectors, and are designated positive in accordance with the selected reference axes.

The general theory is applied to typical rectangular frames with the bases either fixed or pinned. A typical frame is analyzed with both sets of boundary conditions, and the first five frequencies for each case are tabulated for comparison. All symbols are defined where they first appear.

A lumped mass approximation to the distributed mass system for the dynamic analysis of structures yields reasonably good results for the lower modes of vibration(1); however, in order to find theoretically exact solutions the distributed mass properties of actual structures need to be considered. Darney (2) found the natural frequencies of continuous beams using determinants. In 1933, Hohenemser and Prager(3) developed a method for dynamic analysis closely allied to the slope-deflection method for static analysis. Holzer(4) presented a method for finding torsional frequencies, which has been generalized and extended to flexural problems by Myklestad (5) and Thomson(6).

The application of transfer matrices to vibration analysis has been presented by Marguerre(7) and Pestel(8). A relaxation-type solution has been shown by Gaskell(9), which utilizes some principles of moment distribution as set forth by Cross(10). Looney(11) and Veletsos and Newmark(12, 13) made further contributions in this area. An approach, similar to the presentation in this thesis, has been presented by Laursen, Shubinski, and Clough(14).

CHAPTER II

BASIC THEORY

Longitudinal Vibrations

A general axial stiffness equation is assumed, relating the end forces and end displacements for a typical member m (Fig. 2.1)

$$\begin{bmatrix} {}_mP^m_{ix} \\ \hline {}_mP^m_{jx} \end{bmatrix} = \left[\begin{array}{c|c} {}^m_mk^m_{iixx} & {}^m_mk^m_{ijxx} \\ \hline {}^m_mk^m_{jixx} & {}^m_mk^m_{jjxx} \end{array}\right] \begin{bmatrix} {}_m\delta^m_{ix} \\ \hline {}_m\delta^m_{jx} \end{bmatrix}$$

or

$$\left[{}_mP^m_x \right] = \left[{}^m_mK^m_{xx} \right] \left[{}_m\delta^m_x \right] \tag{2.1}$$

where

${}_mP^m_{ix}$ = x-component of force at end i of member m in the m-reference system.

${}_m\delta^m_{ix}$ = x-component of displacement at end i of member m in the m-reference system.

${}^m_mk^m_{ijxx}$ = element of stiffness matrix; x-component of force at end i of member m in the m-reference system due to a unit displacement in the x-direction at end j of member m in the m-reference system.

The remaining elements are similarly defined.

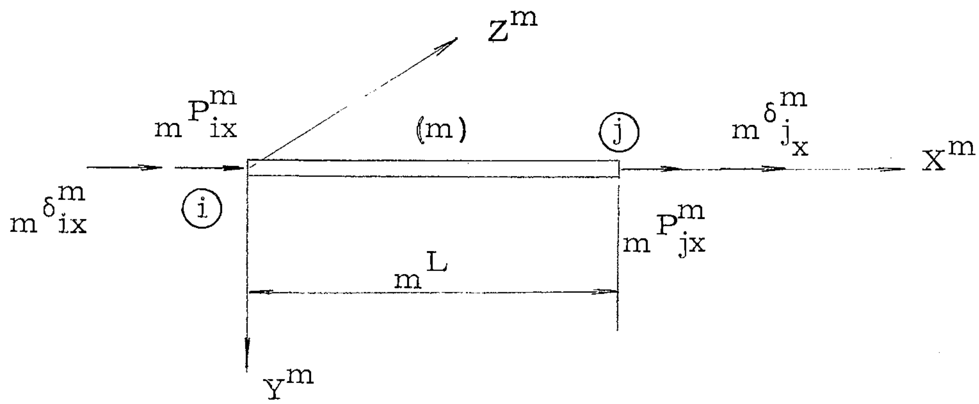

FIG. 2.1 - TYPICAL ELEMENT SHOWING CONVENTION FOR AXIAL FORCES AND DISPLACEMENTS

Considering longitudinal forces and displacements of a typical member, the wave equation for longitudinal vibrations is (1)

$$EA \frac{\partial^2 u}{\partial x^2} = m \frac{\partial^2 u}{\partial t^2} \tag{2.2}$$

where

A = cross-sectional area

E = modulus of elasticity

m = mass per unit length

u = longitudinal displacement.

Separation of variables yields two uncoupled equations. Utilizing duplicate sets of boundary conditions, the stiffness matrix is evaluated.

$$\left[{}_mK^m_{xx} \right] = \frac{EA}{L} \left[\begin{array}{c|c} \gamma L \cot \gamma L & -\gamma L \csc \gamma L \\ \hline -\gamma L \csc \gamma L & \gamma L \cot \gamma L \end{array} \right] \tag{2.3}$$

where

$$\gamma = \left(\frac{mp^2}{EA}\right)^{\frac{1}{2}}$$

and each parameter is understood to be for member m.

Transverse Vibrations

Similarly to the consideration for longitudinal vibrations, a flexural stiffness equation is assumed for member m (Fig. 2.2).

$$\begin{bmatrix} {}_mP^m_{iy} \\ {}_mP^m_{iz} \\ \hline {}_mP^m_{jy} \\ {}_mP^m_{jz} \end{bmatrix} = \begin{bmatrix} {}^m_mk^m_{iiyy} & {}^m_mk^m_{iiyz} & {}^m_mk^m_{ijyy} & {}^m_mk^m_{ijyz} \\ {}^m_mk^m_{iizy} & {}^m_mk^m_{iizz} & {}^m_mk^m_{ijzy} & {}^m_mk^m_{ijzz} \\ \hline {}^m_mk^m_{jiyy} & {}^m_mk^m_{jiyz} & {}^m_mk^m_{jjyy} & {}^m_mk^m_{jjyz} \\ {}^m_mk^m_{jizy} & {}^m_mk^m_{jizz} & {}^m_mk^m_{jjzy} & {}^m_mk^m_{jjzz} \end{bmatrix} \begin{bmatrix} {}_m\delta^m_{iy} \\ {}_m\delta^m_{iz} \\ \hline {}_m\delta^m_{jy} \\ {}_m\delta^m_{jz} \end{bmatrix}$$

or

$$\begin{bmatrix} {}_mP^m_i \\ {}_mP^m_j \end{bmatrix} = \begin{bmatrix} {}^m_mK^m_{ii} & {}^m_mK^m_{ij} \\ {}^m_mK^m_{ji} & {}^m_mK^m_{jj} \end{bmatrix} \begin{bmatrix} {}_m\delta^m_i \\ {}_m\delta^m_j \end{bmatrix}$$

In compact form

$$\left[{}_mP^m\right] = \left[{}^m_mK^m\right]\left[{}_m\delta^m\right] \qquad (2.4)$$

where

${}_m k^m_{ijzy}$ = element of stiffness matrix; z-component of force at end i of member m in the m-reference system due to a unit y-component of displacement at end j of member m in the m-reference system.

The remaining elements are defined as for Eq. (2.1).

For the case of pure bending, a fourth-order partial differential equation is obtained.

$$\frac{\partial^4 y}{\partial x^4} + \frac{m}{EI}\frac{\partial^2 y}{\partial t^2} = 0 \tag{2.5}$$

Separating variables and utilizing the duplicate boundary conditions, the force-displacement relations are established.

$$\begin{bmatrix} {}_m P^m_{iy} \\ {}_m P^m_{iz} \\ {}_m P^m_{jy} \\ {}_m P^m_{jz} \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12\,\beta_1 & 6L\,\beta_2 & -12\,\beta_3 & 6L\,\beta_4 \\ 6L\,\beta_2 & 4L^2\,\beta_5 & -6L\,\beta_4 & 2L^2\,\beta_6 \\ -12\,\beta_3 & -6L\,\beta_4 & 12\,\beta_1 & -6L\,\beta_2 \\ 6L\,\beta_4 & 2L^2\,\beta_6 & -6L\,\beta_2 & 4L^2\,\beta_5 \end{bmatrix} \begin{bmatrix} {}_m\delta^m_{iy} \\ {}_m\delta^m_{iz} \\ {}_m\delta^m_{jy} \\ {}_m\delta^m_{jz} \end{bmatrix} \tag{2.6}$$

where

$$\beta_1 = \frac{(\lambda L)^3}{12}\;\frac{\sin\lambda L\;\cosh\lambda L + \cos\lambda L\;\sinh\lambda L}{1 - \cos\lambda L\;\cosh\lambda L}$$

$$\beta_2 = \frac{(\lambda L)^2}{6}\;\frac{\sin\lambda L\;\sinh\lambda L}{1 - \cos\lambda L\;\cosh\lambda L}$$

$$\beta_3 = \frac{(\lambda L)^3}{12}\;\frac{\sin\lambda L + \sinh\lambda L}{1 - \cos\lambda L\;\cos\lambda L}$$

$$\beta_4 = \frac{(\lambda L)^2}{6} \frac{\cos \lambda L - \cos \lambda L}{1 - \cos \lambda L \cosh \lambda L}$$

$$\beta_5 = \frac{3\theta}{4\theta^2 - \psi^2} = \frac{\lambda L}{4} \frac{\cosh \lambda L \sin \lambda L - \sinh \lambda L \cos \lambda L}{1 - \cos \lambda L \cosh \lambda L}$$

$$\beta_6 = \frac{3\psi}{4\theta^2 - \psi^2} = \frac{\lambda L}{2} \frac{\sin \lambda L - \sin \lambda L}{1 - \cos \lambda L \cosh \lambda L}$$

$$\theta = \frac{3/2}{\lambda L} (\coth \lambda L - \cot \lambda L)$$

$$\psi = \frac{3}{\lambda L} (\csc \lambda L - \operatorname{csch} \lambda L)$$

and

$$\lambda = \left(\frac{mp^2}{EI}\right)^{\frac{1}{4}}$$

Also

E = modulus of elasticity

I = moment of inertia of the cross-section with respect to the axes of bending

m = mass per unit length

L = length of member

p = circular frequency

λ = shape parameter.

All parameters are understood to be for member m.

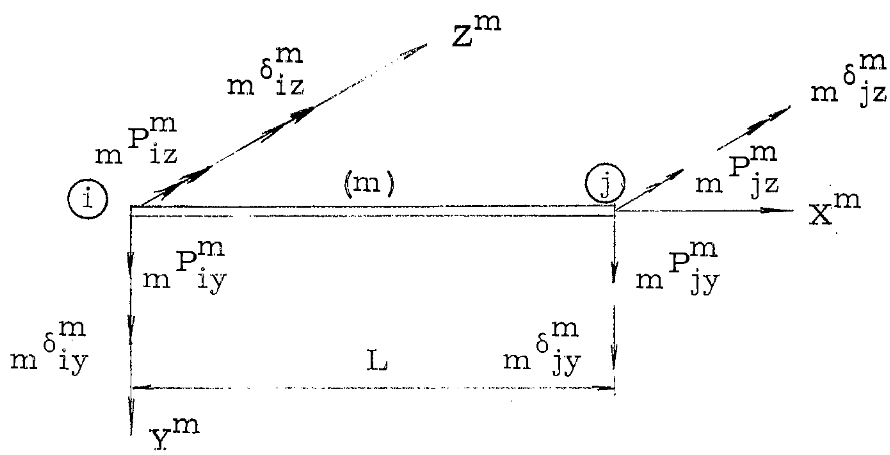

FIG. 2.2 - TYPICAL ELEMENT SHOWING CONVENTION FOR FLEXURAL FORCES AND DISPLACEMENTS

Combining Equations (2.3) and (2.6), a final dynamic stiffness matrix for member m is obtained.

$$\left[{}_m^m K^m\right] =$$

$$\frac{EI}{L^3}\begin{bmatrix} (\frac{L}{r})^2\alpha_1 & 0 & 0 & -(\frac{L}{r})^2\alpha_2 & 0 & 0 \\ 0 & 12\beta_1 & 6L\beta_2 & 0 & -12\beta_3 & 6L\beta_4 \\ 0 & 6L\beta_2 & 4L^2\beta_5 & 0 & -6L\beta_4 & 2L^2\beta_6 \\ -(\frac{L}{r})^2\alpha_2 & 0 & 0 & (\frac{L}{r})^2\alpha_1 & 0 & 0 \\ 0 & -12\beta_3 & -6L\beta_4 & 0 & 12\beta_1 & -6L\beta_2 \\ 0 & 6L\beta_4 & 2L^2\beta_6 & 0 & -6L\beta_2 & 4L^2\beta_5 \end{bmatrix} \qquad (2.7)$$

where

$$\alpha_1 = \gamma L \cot \gamma L$$

$$\alpha_2 = \gamma L \csc \gamma L$$

$$\gamma = \left(\frac{mp^2}{EA}\right)^{1/2} = \lambda^2 r$$

$$r = \left(\frac{I}{A}\right)^{\frac{1}{2}}$$

Transformations

For planar structures, the necessary transformations are confined to angular transformations only. This type of transformation is common (15); thus, its derivation is not shown here.

Consider the final matrix equation of a typical member m in the m-reference system.

$$\begin{bmatrix} {}_mP_i^m \\ \\ {}_mP_j^m \end{bmatrix} = \begin{bmatrix} {}^m_mK_{ii}^m & {}^m_mK_{ij}^m \\ \\ {}^m_mK_{ji}^m & {}^m_mK_{jj}^m \end{bmatrix} \begin{bmatrix} {}_m\delta_i^m \\ \\ {}_m\delta_j^m \end{bmatrix}$$

or

$$\left[{}_mP^m\right] = \left[{}^m_mK^m\right] \left[{}^m_m\delta^m\right] \tag{2.8}$$

If the displacements are expressed in a different reference system "0", Eq. (2.8) becomes

$$\begin{bmatrix} {}_mP_i^m \\ \\ {}_mP^m \end{bmatrix} = \begin{bmatrix} {}^m_mK_{ii}^o & {}^m_mK_{ij}^o \\ \\ {}^m_mK_{ji}^o & {}^m_mK_{jj}^o \end{bmatrix} \begin{bmatrix} {}_m\delta_i^o \\ \\ {}_m\delta_j^o \end{bmatrix}$$

or

$$\left[{}_{m}P^{m}\right] = \left[{}_{m}^{m}K^{o}\right] \left[{}_{m}\delta^{o}\right] \tag{2.9}$$

where

$\left[{}_{m}^{m}K_{ij}^{o}\right]$ = element of stiffness matrix; effect at end i of member m in the m-reference system due to a unit cause at end j of member m in the o-reference system.

$\left[{}_{m}P_{i}^{m}\right]$ = end force vector at end i of member m in the m-reference system

$\left[{}_{m}P_{j}^{m}\right]$ = end force vector at end j of member m in the m-reference system

$\left[{}_{m}\delta_{i}^{o}\right]$ = deformation vector at end i of member m in the o-reference system

$\left[{}_{m}\delta_{j}^{o}\right]$ = deformation vector at end j of member m in the o-reference system

$\left[{}_{m}P^{m}\right]$ = composite force matrix of member m in the m-reference system

$\left[{}_{m}^{m}K^{o}\right]$ = composite stiffness matrix of member m in the mixed m o-reference system

$\left[{}_{m}\delta^{o}\right]$ = composite deformation matrix of member m in the o-reference system.

If the force vectors are also transformed to the o-reference system, Eq. (2.9) takes the form

$$\begin{bmatrix} {}_{m}P_{i}^{o} \\ {}_{m}P_{i}^{o} \end{bmatrix} = \begin{bmatrix} {}_{m}^{o}K_{ii}^{o} & {}_{m}^{o}K_{ij}^{o} \\ {}_{m}^{o}K_{ji}^{o} & {}_{m}^{o}K_{jj}^{o} \end{bmatrix} \begin{bmatrix} {}_{m}\delta_{i}^{o} \\ {}_{m}\delta_{j}^{o} \end{bmatrix}$$

or

$$\left[{}_{m}P^{o}\right] = \left[{}_{m}^{o}K^{o}\right] \left[{}_{m}\delta^{o}\right] \tag{2.10}$$

where

$\left[{}_{m}^{o}K_{ij}^{o}\right]$ = element of stiffness matrix; effect at end i of member m in the o-reference system due to a unit cause at the end j of member m in the o-reference system.

$\left[{}_{m}P_{i}^{o}\right]$ = end force vector at end i of member m in the o-reference system.

$\left[{}_{m}P^{o}\right]$ = composite force matrix of member m in the o-reference system.

$\left[{}_{m}^{o}K^{o}\right]$ = composite stiffness matrix of member m in the o-reference system.

For member m, the angular transformation matrix from the o- to the m-reference system is

$$\left[{}^{m}\Omega^{o}\right] = \begin{bmatrix} {}^{m}\pi^{o} & 0 \\ 0 & {}^{m}\pi^{o} \end{bmatrix} = \left[{}^{o}\Omega^{m}\right]^{T} = \left[{}^{o}\Omega^{m}\right]^{-1} \tag{2.11}$$

where in this case

$$\left[{}^{o}\pi^{m}\right] = \begin{bmatrix} \cos {}^{o}\varphi^{m} & \sin {}^{o}\varphi^{m} & 0 \\ -\sin {}^{o}\varphi^{m} & \cos {}^{o}\varphi^{m} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$\left[{}_{m}P^{o}\right] = \left[{}^{o}\Omega^{m}\right] \left[{}_{m}P^{m}\right]$$

$$\left[{}_{m}\delta^{o}\right] = \left[{}^{o}\Omega^{m}\right]\left[{}_{m}\delta^{m}\right]$$

Substituting

$$\left[{}_{m}P^{o}\right] = \left[{}^{o}\Omega^{m}\right]\left[{}^{m}_{m}K^{m}\right]\left[{}^{m}\Omega^{o}\right]\left[{}_{m}\delta^{o}\right] \qquad (2.12)$$

which implies

$$\left[{}^{o}_{m}K^{o}\right] = \left[{}^{m}\Omega^{o}\right]^{T}\left[{}^{m}_{m}K^{m}\right]\left[{}^{m}\Omega^{o}\right] \qquad (2.13)$$

For a structure composed of several members, a matrix equation is written in the primary (member) reference systems, which includes the force-displacement relations for each member.

$$\begin{bmatrix} \vdots \\ \left[{}_{i}P^{i}\right] \\ \left[{}_{j}P^{j}\right] \\ \left[{}_{k}P^{k}\right] \\ \left[{}_{l}P^{l}\right] \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & & \\ & \left[{}^{i}_{i}K^{i}\right] & & & & \\ & & \left[{}^{j}_{j}K^{j}\right] & & & \\ & & & \left[{}^{k}_{k}K^{k}\right] & & \\ & & & & \left[{}^{l}_{l}K^{l}\right] & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \left[{}_{i}\delta^{i}\right] \\ \left[{}_{j}\delta^{j}\right] \\ \left[{}_{k}\delta^{k}\right] \\ \left[{}_{l}\delta^{l}\right] \\ \vdots \end{bmatrix}$$

or

$$\left[P^{m}\right] = \left[{}^{m}K^{m}\right]\left[\delta^{m}\right] \qquad (2.14)$$

After transformation to an absolute reference system (all elements referred to the same system)

$$\begin{bmatrix} \vdots \\ \left[{}_{i}P^{o}\right] \\ \left[{}_{j}P^{o}\right] \\ \left[{}_{k}P^{o}\right] \\ \left[{}_{l}P^{o}\right] \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & & \\ & \left[{}^{o}_{i}K^{o}\right] & & & & \\ & & \left[{}^{o}_{j}K^{o}\right] & & & \\ & & & \left[{}^{o}_{k}K^{o}\right] & & \\ & & & & \left[{}^{o}_{l}K^{o}\right] & \\ & & & & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \left[{}_{i}\delta^{o}\right] \\ \left[{}_{j}\delta^{o}\right] \\ \left[{}_{k}\delta^{o}\right] \\ \left[{}_{l}\delta^{o}\right] \\ \vdots \end{bmatrix}$$

or

$$\left[P^{o}\right] = \left[{}^{o}K^{o}\right]\left[\delta^{o}\right] \tag{2.15}$$

where each element of Eq. (2.15) is transformed in accordance with Eq. (2.12).

In order to establish equilibrium at the nodes (joints), the corresponding force components in Eq. (2.15) are summed by adding appropriate rows. For compatibility of displacements, certain displacement components in Eq. (2.15) are the same; thus, the appropriate columns are superimposed. The resulting matrix equation represents equilibrium and compatibility of the structure. Thus

$$\left[\overset{*}{P}{}^{o} \right] = \left[{}^{o}\overset{*}{K}{}^{o} \right] \left[\overset{*}{\delta}{}^{o} \right] = \left[0 \right] \qquad (2.16)$$

where no external forces are applied. Then

$$\left[{}^{o}\overset{*}{K}{}^{o} \right] \left[\overset{*}{\delta}{}^{o} \right] = \left[0 \right]$$

for non-trivial solutions

$$\left[\overset{*}{\delta}{}^{o} \right] \neq \left[0 \right]$$

thus, the determinant of the coefficient matrix must vanish

$$\left| {}^{o}\overset{*}{K}{}^{o} \right| = 0 \qquad (2.17)$$

CHAPTER III

APPLICATION

General

The application of the theory developed in this thesis is shown for a rectangular planar frame with distributed mass (Fig. 3.1). Two types of boundary conditions are considered: both ends fixed, and both ends pinned.

In general, a transformation of joint displacements for the frame can be written

$$\begin{bmatrix} [{}_1\delta_0^1] \\ [{}_1\delta_1^1] \\ [{}_2\delta_1^2] \\ [{}_2\delta_2^2] \\ [{}_3\delta_2^3] \\ [{}_3\delta_3^3] \end{bmatrix} = \begin{bmatrix} [{}^1\pi^0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [{}^1\pi^0] & [0] & [0] & [0] & [0] \\ [0] & [0] & [{}^2\pi^0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [{}^2\pi^0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [{}^3\pi^0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [{}^3\pi^0] \end{bmatrix} \begin{bmatrix} [{}_1\delta_0^0] \\ [{}_1\delta_1^0] \\ [{}_2\delta_1^0] \\ [{}_2\delta_2^0] \\ [{}_3\delta_2^0] \\ [{}_3\delta_3^0] \end{bmatrix} \qquad (3.1)$$

A modified transformation matrix can be obtained by introducing displacement compatibility at all joints, including compatibility with support conditions. Thus,

$$\left[\delta^{m}\right] = \left[m\overset{*}{\Omega}o\right]\left[\overset{*}{\delta}o\right] \tag{3.2}$$

where

$\left[\delta^{m}\right]$ = joint displacement matrix in the member reference system.

$\left[m\overset{*}{\Omega}o\right]$ = modified angular transformation matrix.

$\left[\overset{*}{\delta}o\right]$ = modified displacement matrix in the o-reference system.

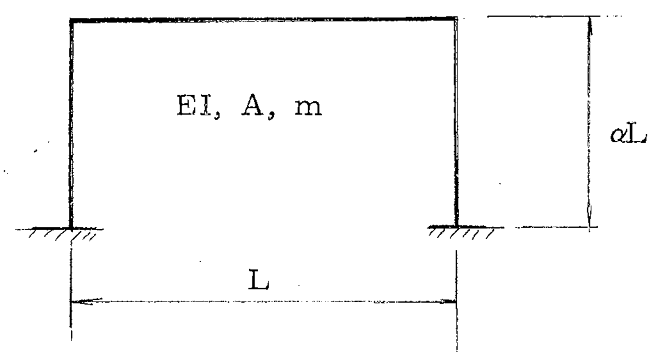

FIG. 3.1 - RECTANGULAR FRAME WITH DISTRIBUTED MASS

For the frame with bases fixed, support conditions require that

$$\left[{}_{1}\delta_{0}^{1}\right] = \left[{}_{3}\delta_{3}^{3}\right] = \left[\delta_{0}^{0}\right] = \left[\delta_{3}^{0}\right] = \left[0\right] \tag{3.3}$$

Modifying the transformation in accordance with these boundary conditions, the following relation is obtained.

$$\begin{bmatrix} \left[{}_1\delta^1_1\right] \\ \left[{}_2\delta^2_1\right] \\ \left[{}_2\delta^2_2\right] \\ \left[{}_3\delta^3_2\right] \end{bmatrix} = \begin{bmatrix} \left[{}^1\pi^0\right] & \left[\ 0\ \right] \\ \left[{}^2\pi^0\right] & \left[\ 0\ \right] \\ \left[\ 0\ \right] & \left[{}^2\pi^0\right] \\ \left[\ 0\ \right] & \left[{}^3\pi^0\right] \end{bmatrix} \begin{bmatrix} \left[\delta^0_1\right] \\ \left[\delta^0_2\right] \end{bmatrix} \qquad (3.4)$$

Since no external forces exist, joint equilibrium is expressed by

$$\left[\overset{*}{P}{}^o\right] = \left[{}^m\overset{*}{\Omega}{}^o\right]^T \left[{}_mP^m\right] = \left[\ 0\ \right] \qquad (3.5)$$

thus,

$$\left[{}^o\overset{*}{K}{}^o\right]\left[\delta^o\right] = \left[{}^m\overset{*}{\Omega}{}^o\right]^T \left[{}^m\overset{*}{K}{}^m\right]\left[{}^m\overset{*}{\Omega}{}^o\right]\left[\overset{*}{\delta}{}^o\right] = \left[\ 0\ \right] \qquad (3.6)$$

The natural frequencies are determined by evaluating the shape parameter when the determinant of the coefficient matrix vanishes.

For the frame with bases pinned, the stiffness matrix in the m-reference system for the bar with one end pinned can be modified. A bar m with end i pinned is shown (Fig. 3.2). The force-displacement relation at end j can be written

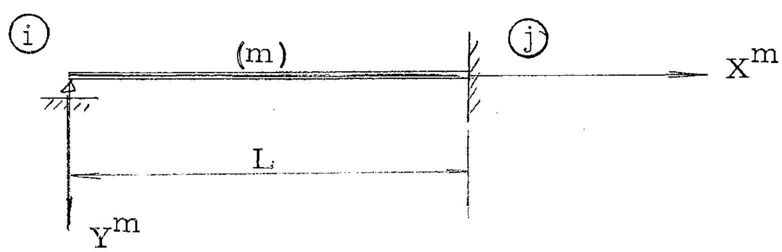

FIG. 3.2 - BAR m WITH END i PINNED

$$\begin{bmatrix} {}_m P_{jx}^m \\ {}_m P_{jy}^m \\ {}_m P_{jz}^m \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} (\frac{L}{r})^2 \alpha_1 & 0 & 0 \\ 0 & 3\beta_7 & -3L\beta_8 \\ 0 & -3\beta_8 & 3L^2\beta_9 \end{bmatrix} \begin{bmatrix} {}_m \delta_{jx}^m \\ {}_m \delta_{jy}^m \\ {}_m \delta_{jz}^m \end{bmatrix} \quad (3.7)$$

where

$$\beta_7 = \frac{2(\lambda L)^3}{3} \frac{\cos \lambda L \cosh \lambda L}{\sin \lambda L \cosh \lambda L - \cos \lambda L \sinh \lambda L}$$

$$\beta_8 = \frac{(\lambda L)^2}{3} \frac{\sin \lambda L \cosh \lambda L + \cos \lambda L \sinh \lambda L}{\sin \lambda L \cosh \lambda L - \cos \lambda L \sinh \lambda L}$$

$$\beta_9 = \frac{2(\lambda L)}{3} \frac{\sin \lambda L \sinh \lambda L}{\sin \lambda L \cosh \lambda L - \cos \lambda L \sinh \lambda L}$$

This modification permits the end displacement vector for the pinned ends to be eliminated from the formulation of joint equilibrium equations. With these modifications and the same approach as previously discussed, Eq. (3.6) can be obtained for the frame with pinned ends.

Numerical Examples

A rectangular frame with fixed bases is considered. Each member of the frame is assumed to be a uniform bar with constant circular cross-section, mass per unit length and moment of inertia. These properties are as follow:

$E = 30.6 \times 10^6$ lb/in.^{2}

$I = 34.22822 \times 10^{-6}$ in.^{4}

$m = 15.2174 \times 10^{-6}$ lb-sec^2/in.^{2}

$A = 0.02074$ in.^{2}

$L = 8$ in.

${}_1L = \alpha L = 4$ in., $\alpha = 0.5$

$r = 0.0406$ in.

Diameter = 0.1625 in.

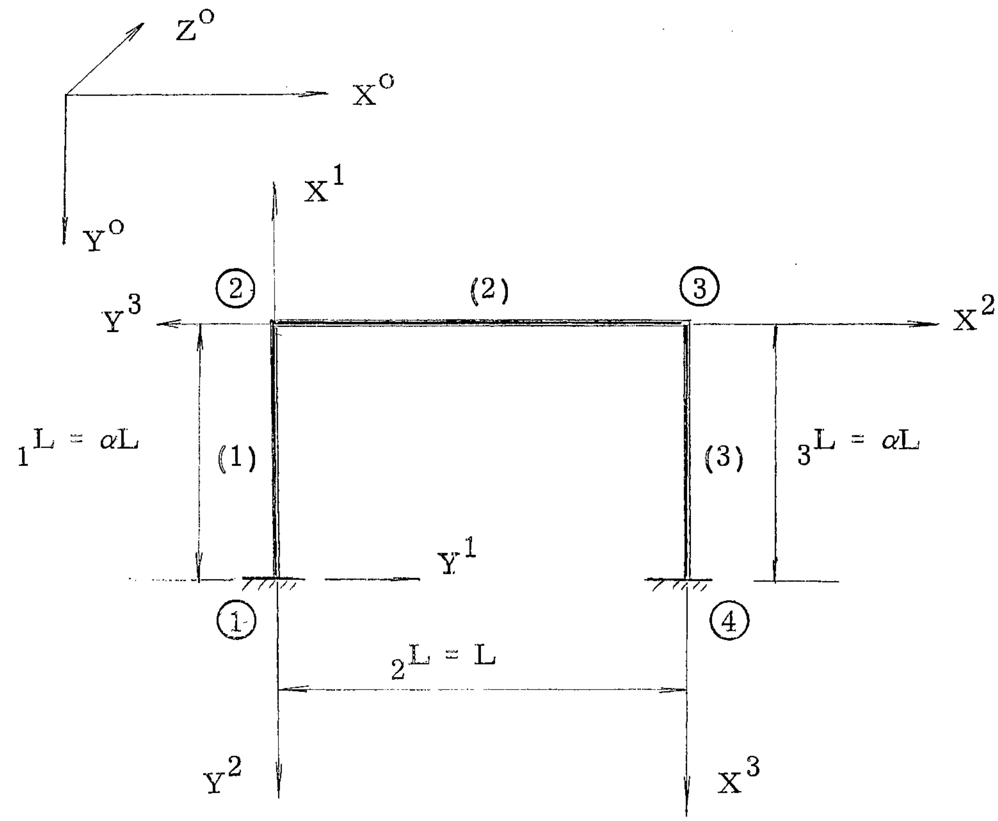

FIG. 3.3 - RECTANGULAR FRAME WITH FIXED BASES

The modified angular transformation matrix (Eq. (3.)) is shown in numerical form by Eq. (3.8).

$$\left[m_{\Omega}^{*} o \right] = \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]_{(12)(6)} \qquad (3.8)$$

The stiffness matrix of the frame in the member reference systems is shown by Eq. (3.9). After transformation, the modified stiffness matrix for the frame in the o-reference system is given by Eq. (3.10), which is the matrix utilized to determine the natural frequencies.

The same problem as previously formulated, except that the end conditions are changed to pinned bases, is presented. Utilizing the modifications of the dynamic stiffness factors for the column members, the stiffness matrix for the frame in the m-reference systems, is written (Eq. (3.11)) according to Eq. (2.14). This matrix is transformed using the same angular transformation matrix Eq. (3.8). The final stiffness matrix in the o-reference system is shown by Eq. (3.12).

$$\left[{}^{m}K^{m} \right] = \frac{EI}{L^3} \begin{bmatrix}
\frac{1}{\alpha^3}\left(\frac{\alpha L}{r}\right)^2 ({}_{1}\alpha_{1}) & & & & & & & & & & & \\
& \frac{12}{\alpha^3}({}_{1}\beta_{1}) & -\frac{6\alpha L}{\alpha^3}({}_{1}\beta_{2}) & & & & & & & & & \\
& \frac{6\alpha L}{\alpha^3}({}_{1}\beta_{2}) & \frac{4(\alpha L)^2}{\alpha^3}({}_{1}\beta_{5}) & & & & & & & & & \\
& & & \left(\frac{L}{r}\right)^2 ({}_{2}\alpha_{1}) & & & -\left(\frac{L}{r}\right)^2 ({}_{2}\alpha_{2}) & & & & & \\
& & & & 12({}_{2}\beta_{1}) & 6L({}_{2}\beta_{2}) & & -12({}_{2}\beta_{3}) & 6L({}_{2}\beta_{4}) & & & \\
& & & & 6L({}_{2}\beta_{2}) & 4L^2({}_{2}\beta_{5}) & & -6L({}_{2}\beta_{4}) & 2L^2({}_{2}\beta_{6}) & & & \\
& & & -\left(\frac{L}{r}\right)^2 \alpha_2 & & & \left(\frac{L}{r}\right)^2 ({}_{2}\alpha_{1}) & & & & & \\
& & & & -12({}_{2}\beta_{3}) & -6L({}_{2}\beta_{4}) & & 12({}_{2}\beta_{1}) & -6L\beta_2 & & & \\
& & & & 6L({}_{2}\beta_{4}) & 2L^2({}_{2}\beta_{6}) & & -6L({}_{2}\beta_{2}) & 4L^2({}_{2}\beta_{5}) & & & \\
& & & & & & & & & \frac{1}{\alpha}\left(\frac{\alpha L}{r}\right)^2 ({}_{3}\alpha_{1}) & & \\
& & & & & & & & & & \frac{12}{\alpha^3}({}_{3}\beta_{1}) & \frac{6\alpha L}{\alpha^3}({}_{3}\beta_{2}) \\
& & & & & & & & & & \frac{6\alpha L}{\alpha^3}({}_{3}\beta_{2}) & \frac{4(\alpha L)^2}{\alpha^3}({}_{3}\beta_{5})
\end{bmatrix}_{12 \times 12} \tag{3.9}$$

$$\left[{}^{o}K^{o}\right] = \frac{EI}{L^3} \begin{bmatrix} \frac{12}{\alpha^3}({}_1\beta_1) + (\frac{L}{r})^2 ({}_2\alpha_1) & 0 & -\frac{6\alpha L}{\alpha^3}({}_1\beta_2) & -(\frac{L}{r})^2 ({}_2\alpha_2) & 0 & 0 \\ 0 & \frac{1}{\alpha^3}(\frac{\alpha L}{r})^2 ({}_1\alpha_1) + 12({}_2\beta_1) & 6L({}_2\beta_2) & 0 & -12({}_2\beta_3) & 6L({}_2\beta_4) \\ -\frac{6\alpha L}{\alpha^3}({}_1\beta_2) & 6L({}_2\beta_2) & \frac{4(\alpha L)^2}{\alpha^3}({}_1\beta_5) + 4L^2({}_2\beta_5) & 0 & -6L({}_2\beta_4) & 2L^2({}_2\beta_6) \\ -(\frac{L}{r})^2 ({}_2\alpha_2) & 0 & 0 & \frac{12}{\alpha^3}({}_3\beta_1) + (\frac{L}{r})^2 ({}_2\alpha_1) & 0 & -\frac{6\alpha L}{\alpha^3}({}_3\beta_2) \\ 0 & -12({}_2\beta_3) & -6L({}_2\beta_4) & 0 & \frac{1}{\alpha^3}(\frac{L}{r})^2 ({}_3\alpha_1) + 12({}_2\beta_1) & 6L({}_2\beta_2) \\ 0 & 6L({}_2\beta_4) & 2L^2({}_2\beta_6) & -\frac{6\alpha L}{\alpha^3}({}_3\beta_2) & -6L({}_2\beta_2) & 4L^2({}_2\beta_5) + \frac{4(\alpha L)^2}{\alpha^3}({}_3\beta_5) \end{bmatrix}_{6 \times 6} \tag{3.10}$$

$$
\left[{}_{m}K_{m}\right] = \frac{EI}{L^3}
\begin{bmatrix}
\frac{1}{\alpha^3}\left(\frac{L}{r}\right)^2({}_1\alpha_1) & & & & & & & & & & & \\
 & 3({}_1\beta_7) & -3L({}_1\beta_8) & & & & & & & & & \\
 & -3L({}_1\beta_8) & 3L^2({}_1\beta_9) & & & & & & & & & \\
 & & & \left(\frac{L}{r}\right)^2({}_2\alpha_1) & & & -\left(\frac{L}{r}\right)^2({}_2\alpha_2) & & & & & \\
 & & & & 12({}_2\beta_1) & 6L({}_2\beta_2) & & -12({}_2\beta_3) & 6L({}_2\beta_4) & & & \\
 & & & & 6L({}_2\beta_2) & 4L^2({}_2\beta_5) & & -6L({}_2\beta_4) & 2L^2({}_2\beta_6) & & & \\
 & & & -\left(\frac{L}{r}\right)^2({}_2\alpha_2) & & & \left(\frac{L}{r}\right)^2\alpha_1 & & & & & \\
 & & & & -12({}_2\beta_3) & -6L({}_2\beta_4) & & 12({}_2\beta_1) & -6L({}_2\beta_2) & & & \\
 & & & & 6L({}_2\beta_4) & 2L^2({}_2\beta_6) & & -6L({}_2\beta_2) & 4L^2({}_2\beta_5) & & & \\
 & & & & & & & & & \frac{1}{\alpha^3}\left(\frac{L}{r}\right)^2({}_3\alpha_1) & & \\
 & & & & & & & & & & 3({}_3\beta_7) & 3L({}_3\beta_8) \\
 & & & & & & & & & & 3L({}_3\beta_8) & 3L^2({}_3\beta_9)
\end{bmatrix}_{12\times 12}
\qquad (3.11)
$$

$$
\left[{}^{o}K^{o} \right] = \frac{EI}{L^3}
\begin{bmatrix}
\frac{3}{\alpha^3}({}_1\beta_7) + (\frac{L}{r})^2 \alpha_1 & 0 & -3L({}_1\beta_8) & -(\frac{L}{r})^2 ({}_2\alpha_2) & 0 & 0 \\
0 & 12({}_2\beta_1) + \frac{1}{\alpha^3}(\frac{L}{r})^2 ({}_1\alpha_1) & 6L({}_2\beta_2) & 0 & -12({}_2\beta_3) & 6L({}_2\beta_4) \\
-3L({}_1\beta_8) & 6L({}_2\beta_2) & 3L^2({}_1\beta_9) + 4L^2({}_2\beta_5) & 0 & -6L({}_2\beta_4) & 2L^2({}_2\beta_6) \\
-(\frac{L}{r})^2 ({}_2\alpha_2) & 0 & 0 & \frac{3}{\alpha^3}({}_3\beta_7) + (\frac{L}{r})^3 ({}_2\alpha_2) & 0 & -3L({}_3\beta_8) \\
0 & -12({}_2\beta_3) & -6L({}_2\beta_4) & 0 & 12({}_2\beta_1) + \frac{3}{\alpha^3}\beta_9 & -6L({}_2\beta_2) \\
0 & 6L({}_2\beta_4) & 2L^2({}_2\beta_6) & -3L({}_3\beta_8) & -6L({}_2\beta_2) & 4L^2({}_2\beta_5) + 3L^2({}_3\beta_9)
\end{bmatrix}_{6 \times 6}
\qquad (3.12)
$$

Eqs. (3.10) and (3.12) are the coefficient matrices utilized to find the natural frequencies, in accordance with Eq. (2.17). The numerical calculations are performed by the IBM 1410 Electronic Computer. Crout's method is used to expand the determinants of the characteristic matrices. The lowest five natural frequencies are obtained for each case, from the values of the shape parameter λL which makes the determinant in Eq. (2.17) vanish(16). The final values of the shape parameters λL and natural frequencies for each case are tabulated for comparison (Table 1).

TABLE 1. NATURAL FREQUENCIES-RECTANGULAR FRAME

Mode	Fixed Base		Pinned Base	
	λL	f (cps)	λL	f (cps)
1	3.0280	180	1.4233	41
2	4.0580	339	4.0430	337
3	4.6864	452	4.6842	451
4	6.7901	951	7.8351	1266
5	7.7114	1227	7.8406	1268

CHAPTER IV

SUMMARY AND CONCLUSIONS

Summary

The dynamic analysis of a rectangular planar frame, considering uniform distribution of mass and constant cross-section, is presented. A stiffness formulation is utilized to obtain the final equilibrium equations.

A general stiffness matrix for a single member is derived, including the effects of axial and flexural deformations. An expression is obtained relating all end forces and displacements for the frame by combining these independent equations into a composite matrix equation. From the compatibility conditions of joints and supports, a transformation matrix is obtained. Using this transformation matrix, joint equilibrium is established.

The frequencies are obtained by evaluating the shape parameters λL which cause the determinant of the characteristic matrix to vanish. Numerical examples are worked out for symmetrical rectangular frames with fixed bases and pinned bases. The first five frequencies for these cases are tabulated for comparison.

Conclusions

The stiffness method can be used directly for the vibration analysis of planar frames. Displacement compatibility and force equilibrium can be established using basic transformations. The necessary angular transformation matrices are the same as for static analysis.

The accuracy of this method depends on the increment of the shape parameter in the incremental process of evaluating the determinant, and the computing facilities available. The stiffness matrix elements are quite complex; however, they can be evaluated with the aid of an electronic computer.

The theory presented in this thesis is directly applicable to the frequency analysis of any planar structure. Extension to include the analysis of three dimensional structures can be attained with minor additional complexities.

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