FREQUENCY ANALYSIS OF PLANAR FRAMES WITH DISTRIBUTED MASS - GABLE FRAMES BY METHOD OF STIFFNESS COEFFICIENTS

By

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1962

Submitted to the Faculty of the Graduate School
of the Oklahoma State University
in partial fulfillment of
the requirements for
the degree of
MASTER OF SCIENCE
May, 1965

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ACKNOWLEDGMENT

The writer wishes to express his sincere appreciation to some of the people who have made his graduate education possible.

To Dr. James W. Gillespie, for his assistance and guidance in the preparation of this thesis, and also for his instruction during graduate work;

To Professor Jan J. Tuma and the Faculty of the School of Civil Engineering for the awarding of a graduate assistantship and for the instruction received;

To his wife, Kay, for her encouragement and understanding, and for her typing of this thesis.

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CHAPTER I

INTRODUCTION

General

Interest in the dynamic analysis of structural frames has been increasing over the past several years. In many cases, the standard method of static analysis of a structure undergoing dynamic loading is no longer considered adequate. For these cases new methods of analysis have been derived, or the methods of static analysis have been adapted for use in dynamic analysis. Some structural loadings for which dynamic analysis is being used are the vibrational loads produced by mechanical equipment, blast forces, and forces due to earthquakes.

In general, the two approaches to the dynamic analysis of frames are the lumping of the mass of the frame at discrete points and the distribution of mass along the frame members.

The purpose of this thesis will be to discuss the application of the method of stiffness coefficients for finding the natural vibrational frequencies of planar frames with distributed mass.

The method of stiffness coefficients, in theory, is a practical and simple approach for finding natural frequencies of gable frames. The only difficulty in applying the theory to practical examples is the evaluation of complex mathematical expressions. This difficulty can be partly overcome by making use of electronic computers in evaluating the mathematical expressions.

Scope of Discussion

The basic theory, presented in the second chapter, is based on the following conditions and assumptions:

(1) Frame members have their mass distributed along the length of the members.

(2) Properties of the members; cross-sectional area, mass per unit length, moment of inertia, and modulus of elasticity are constant between joints of the frame.

(3) Joints of the frame are taken at the base points and around the perimeter of the frame where there are changes in slope.

(4) Axial and bending deformations are considered in the derivation of the basic expressions.

(5) Shearing deformations and rotatory inertia effects are considered negligible.

(6) No damping forces are considered.

In the third chapter the application of the basic theory is given for two gable frames. Both are symmetrical, but one has fixed bases and the other has pinned bases. Four numerical examples are given for each frame. The mass per unit length, cross-sectional area, moment of inertia, and modulus of elasticity are constant values for all the frame members and for all the example problems. In general, the lower natural frequencies are the most important; so, for each example, only the first four frequencies are found. A table is given with the frequencies for the example problems so that a comparison of the natural frequencies for the different base conditions and different ratios of column height and gable height to span length can be made.

The last chapter gives a brief summary with some conclusions and limitations of the method. Also included is a brief discussion of possible extensions of the method.

Historical

According to historical information presented by Laursen, Shubinski, and Clough (1), the basic differential equations governing the natural frequencies of framed structures have been known for many years.(2) However, it is difficult to obtain solutions for the differential equations except for the simplest of structures. A widely used approach which gives approximate solutions is lumping the mass of a frame at discrete points. This reduces the partial differential equations to ordinary differential equations, which can be solved much more readily.

Among the earliest investigators to use the distributed mass properties of a frame were Hohenemser and Prager (3). Their approach is closely allied to slope-deflection analysis of static frames. Since this approach gives large systems of simultaneous equations to solve, its usefulness was limited before the era of electronic computers. The next step in the dynamic analysis of frames was to adapt iterative, relaxation type procedures, corresponding to moment distribution in static analysis. This approach was investigated first by Gaskell (4) and then developed further by Veletsos and Newmark (5).

In addition to this historical material, Laursen, Shubinski, and Clough (1), present a method of dynamic matrix analysis for the solution of natural frequencies of frames which takes into account the distribution of mass.

CHAPTER II

BASIC THEORY

General

The basic theory and general expressions required for evaluating natural frequencies of planar, single bay gable frames are set forth in several steps. First, the stiffness matrix for axial deformations and the stiffness matrix for flexural deformations of a typical frame member are given. The two matrices are then combined to form a general stiffness matrix for any member of the frame undergoing axial and flexural deformations. Also given in each of the above steps is a matrix force equation involving end forces, deformations, and the stiffness matrix for the corresponding element. Matrix force equations are written for each member of the frame, and these equations are combined into a single matrix force equation for the frame.

In the last step of the basic theory development, a transformation matrix and procedure is given which will transform end forces, deformations, and the stiffness matrix of a member in the member-oriented reference system to some base reference system. The frame matrix force equation is then rewritten so all terms in the equation are with respect to the same coordinate system.

Stiffness Matrix - Axial Deformations

The stiffness matrix for dynamic axial deformations will be given for a typical member n, shown in Fig. 1. As previously stated, the cross-sectional area, mass per unit length, modulus of elasticity, and moment of inertia are constant over the length of the member.

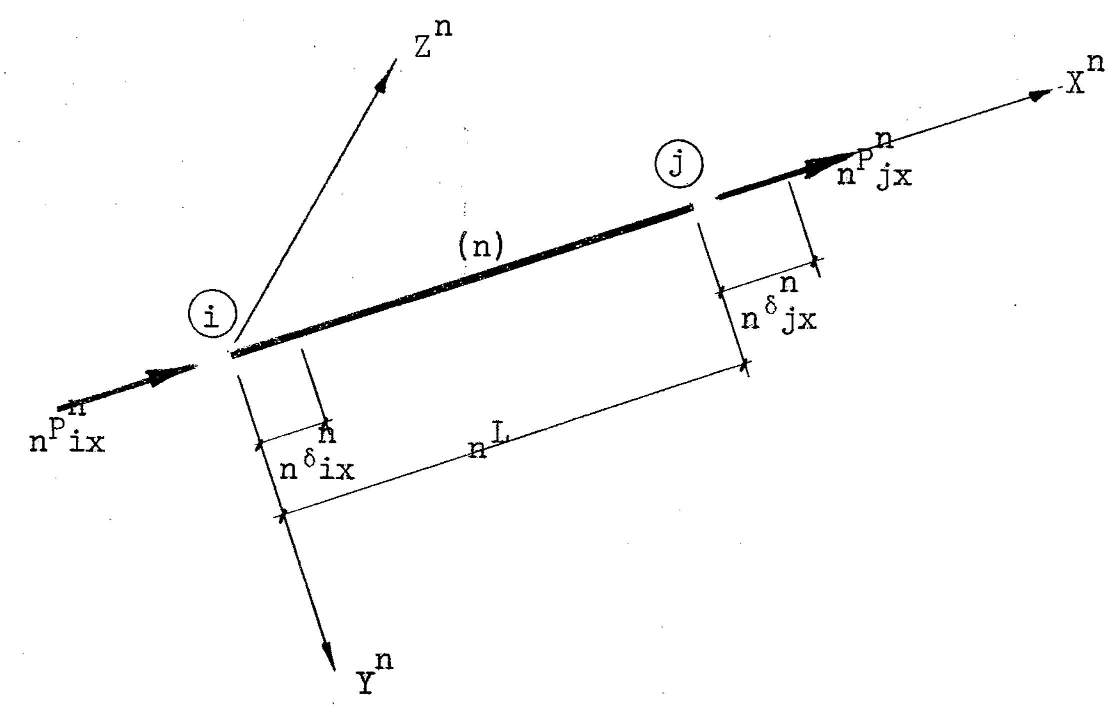

Fig. 1

Axial Forces and Deformations of a Typical Member

The resulting matrix equation for axial forces in terms of stiffness coefficients and axial deformations is as follows:

$$\begin{bmatrix} {}_n P^n_{ix} \\ {}_n P^n_{jx} \end{bmatrix} = \begin{bmatrix} {}^n_n K_{iixx} & {}^n_n K_{ijxx} \\ {}^n_n K_{jixx} & {}^n_n K_{jjxx} \end{bmatrix} \begin{bmatrix} {}_n \delta^n_{ix} \\ {}_n \delta^n_{jx} \end{bmatrix} \tag{2.1}$$

where the stiffness coefficients for axial deformations are defined as

${}^n_n K_{iixx}$ = force developed at i in the x direction due to a unit deformation at i in the x direction.

${}^n_n K_{ijxx}$ = force developed at i in the x direction due to a unit deformation at j in the x direction.

${}^n_n K_{jixx}$ = force developed at j in the x direction due to a unit deformation at i in the x direction.

${}^n_n K_{jjxx}$ = force developed at j in the x direction due to a unit deformation at j in the x direction.

Equation 2.1 can be rewritten using shorter notation. This gives Eq. 2.2,

$$\begin{bmatrix} {}_n P^n_{x} \end{bmatrix} = \begin{bmatrix} {}^n_n K_{xx} \end{bmatrix} \begin{bmatrix} {}_n \delta^n_{x} \end{bmatrix} \tag{2.2}$$

where

$$\left[{}_{n}^{n}K_{xx}\right] = \text{axial stiffness matrix for member n.}$$

To develop expressions for the stiffness coefficients, a differential segment is taken from anywhere along member n. This segment, with the axial forces and deformations existing on the segment, is shown in Fig. 2.

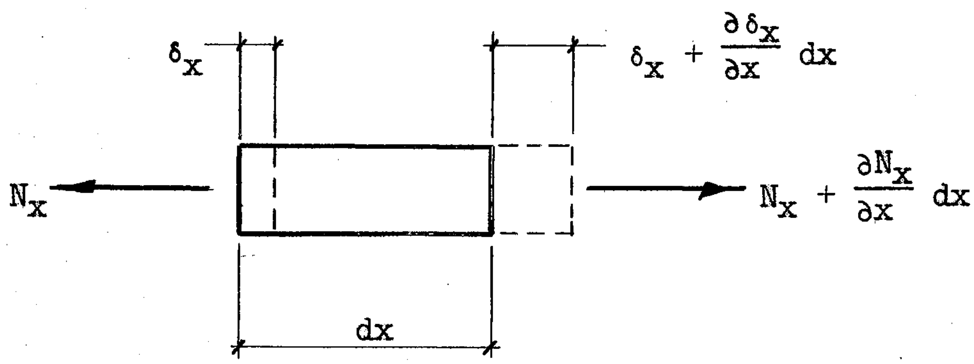

Fig. 2

Axial Forces and Deformations Existing on a Differential Segment of a Typical Member

If the segment dx^n is from a member in a frame vibrating at some natural frequency, the only applied force on the segment will be the inertia force due to acceleration of the segment. Thus, force equilibrium can be expressed as

$$\frac{\partial N_x}{\partial x} dx = m_x dx \frac{\partial^2 \delta_x}{\partial t^2} \tag{2.3}$$

Using Eq. 2.3 and the equation of deformation for the segment dx^n the following differential equation can be derived (2).

$$EA \frac{\partial^2 \delta x}{\partial x^2} = m \frac{\partial^2 \delta x}{\partial t^2} \tag{2.4}$$

Solving the differential equation and applying the available force and displacement boundary conditions, the stiffness matrix for axial deformations is found to be;

$$\left[{}^{n}_{n}K^{n}_{xx}\right] = \frac{EA}{L} \begin{bmatrix} \gamma L\cot\gamma L & -\gamma L\csc\gamma L \\ -\gamma L\csc\gamma L & \gamma L\cot\gamma L \end{bmatrix} \tag{2.5}$$

Equation 2.5 can also be written as

$$\left[{}^{n}_{n}K^{n}_{xx}\right] = \frac{EI}{L^3} \begin{bmatrix} \frac{L^2}{a^2}(\alpha_1) & \frac{L^2}{a^2}(-\alpha_2) \\ \frac{L^2}{a^2}(-\alpha_2) & \frac{L^2}{a^2}(\alpha_1) \end{bmatrix} \tag{2.6}$$

where

$$\gamma = \left(\frac{mp^2}{EA}\right)^{\frac{1}{2}}$$

$$\alpha_1 = \gamma L\cot\gamma L$$

$$\alpha_2 = \gamma L\csc\gamma L$$

$$a^2 = \frac{I}{A}$$

and all parameters are understood to be for member n.

Stiffness Matrix - Flexural Deformations

The stiffness matrix for dynamic flexural deformations will be given for a typical member n shown in Fig. 3. The section properties for the member are the same as for the member used for axial deformations.

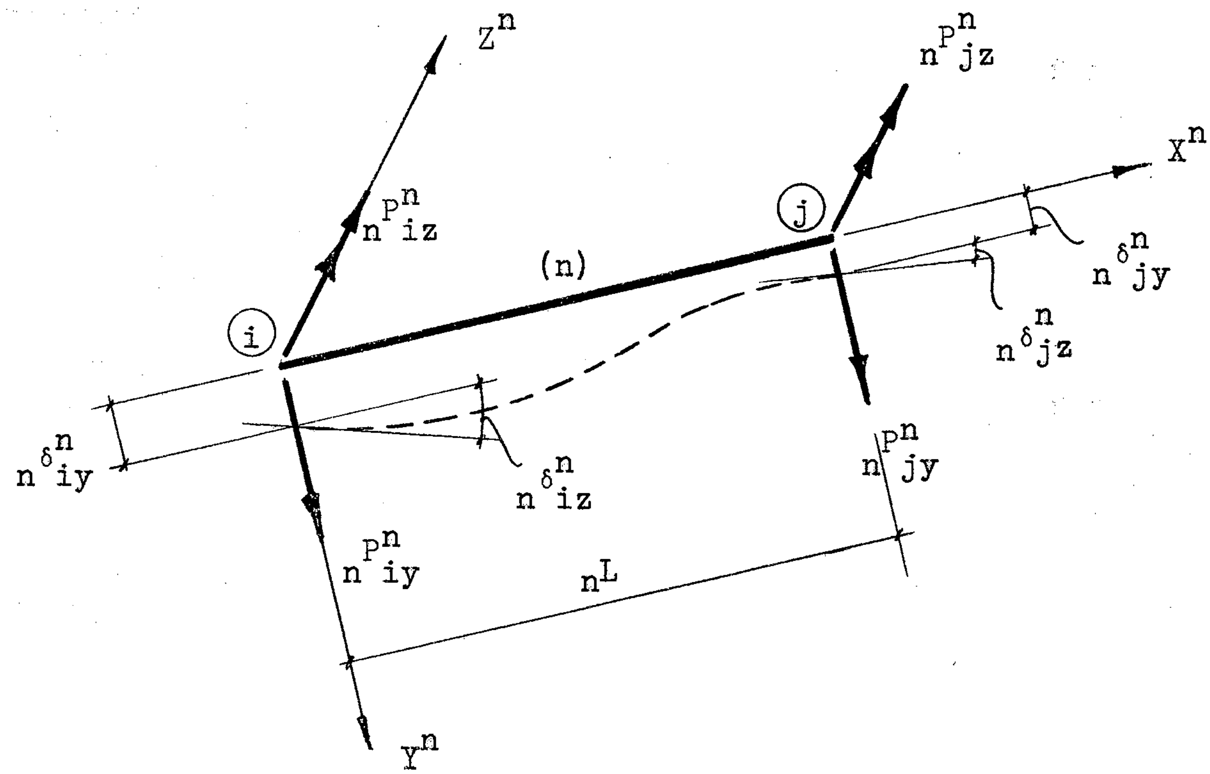

Fig. 3

Flexural Forces and Deformations of a Typical Member

The resulting equation for flexural forces in terms of a stiffness matrix and flexural deformations can be written as

$$\left[{}_nP^n\right]^* = \left[{}_n^nK^n\right]^* \cdot \left[{}_n\delta^n\right]^* \qquad (2.7)$$

where the asterisk superscript denotes flexural quantities only. Eq. 2.7 can be expanded to the following:

$$\begin{bmatrix} {}_{n}^{n}P_{i} \\ \hline {}_{n}^{n}P_{j} \end{bmatrix}^{*} = \left[\begin{array}{c|c} {}_{n}^{n}K_{ii} & {}_{n}^{n}K_{ij} \\ \hline {}_{n}^{n}K_{ji} & {}_{n}^{n}K_{jj} \end{array}\right]^{*} \begin{bmatrix} {}_{n}^{n}\delta_{i} \\ \hline {}_{n}^{n}\delta_{j} \end{bmatrix}^{*} \tag{2.8}$$

where the submatrices of the stiffness matrix are defined as

${}_{n}^{n}K_{ii}$ = flexural forces at i due to unit deformations at i

${}_{n}^{n}K_{ij}$ = flexural forces at i due to unit deformations at j

${}_{n}^{n}K_{ji}$ = flexural forces at j due to unit deformations at i

${}_{n}^{n}K_{jj}$ = flexural forces at j due to unit deformations at j.

Eq. 2.8 can be further expanded to give

$$\begin{bmatrix} {}_{n}^{n}P_{iy} \\ {}_{n}^{n}P_{iz} \\ \hline {}_{n}^{n}P_{jy} \\ {}_{n}^{n}P_{jz} \end{bmatrix} = \left[\begin{array}{cc|cc} {}_{n}^{n}K_{iiyy} & {}_{n}^{n}K_{iiyz} & {}_{n}^{n}K_{ijyy} & {}_{n}^{n}K_{ijyz} \\ {}_{n}^{n}K_{iizy} & {}_{n}^{n}K_{iizz} & {}_{n}^{n}K_{ijzy} & {}_{n}^{n}K_{ijzz} \\ \hline {}_{n}^{n}K_{jiyy} & {}_{n}^{n}K_{jiyz} & {}_{n}^{n}K_{jjyy} & {}_{n}^{n}K_{jjyz} \\ {}_{n}^{n}K_{jizy} & {}_{n}^{n}K_{jizz} & {}_{n}^{n}K_{jjzy} & {}_{n}^{n}K_{jjzz} \end{array}\right] \begin{bmatrix} {}_{n}^{n}\delta_{iy} \\ {}_{n}^{n}\delta_{iz} \\ \hline {}_{n}^{n}\delta_{jy} \\ {}_{n}^{n}\delta_{jz} \end{bmatrix} \tag{2.9}$$

Defining only random terms from the stiffness matrix,

${}_{n}^{n}K_{iiyz}$ = force at i in the y direction due to unit deformation at i in the z direction.

${}_n^nK_{ijzy}^n$ = force at i in the z direction due to unit deformation at j in the y direction.

${}_n^nK_{jiyz}^n$ = force at j in the y direction due to unit deformation at i in the z direction.

${}_n^nK_{jjzz}^n$ = force at j in the z direction due to unit deformation at j in the z direction.

To develop expressions for the stiffness coefficients, once again a differential segment in length is taken from anywhere along the length of member n. This segment, with the forces and moments which exist on the segment, is shown in Fig. 4.

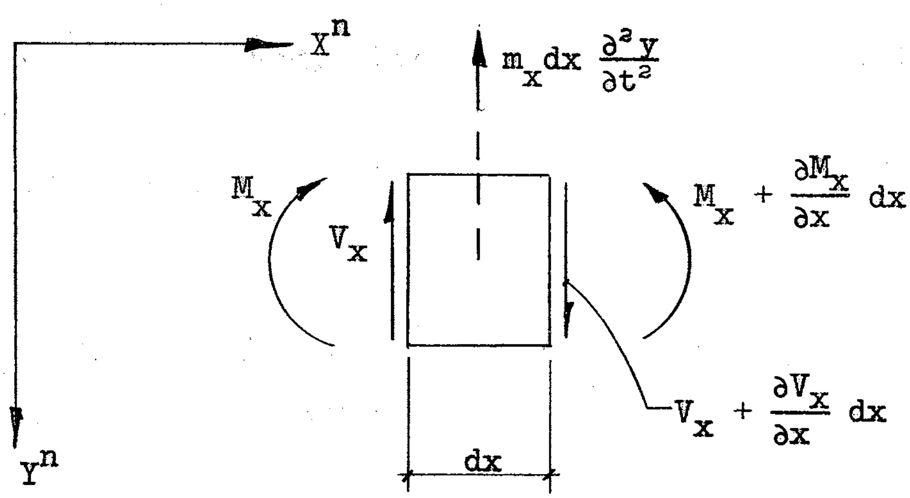

Fig. 4

Forces and Moments Existing on a Differential Segment of a Typical Member

As in the case for axial deformation, if the segment of length dx^n is from a member in a frame which is vibrating at a natural frequency, the only forces acting on the segment will be those induced by the vibration. If forces in the y direction are summed, moments about the z axis summed, and the condition of deformation given by Eq. 2.10 used, the differential equation for transverse vibrations of member n (Eq. 2.11) can be determined. This differential equation is developed by Timoshenko (2) and also by Rogers (6). Rogers also gives the expression for the stiffness coefficients of Eqs. (2.12) and (2.13).

$$\frac{\partial^2 y}{\partial x^2} = -\frac{M_x}{EI} \tag{2.10}$$

$$EI\,\frac{\partial^4 y}{\partial x^4} + m\,\frac{\partial^2 y}{\partial t^2} = 0 \tag{2.11}$$

Solving the differential equation and using the available force and displacement boundary conditions to eliminate the arbitrary constants, the stiffness matrix for flexural deformations becomes:

$$\left[{}_{n}^{n}K\right]^{*} = \frac{EI}{L^3}\begin{bmatrix} 12\beta_1 & 6L\beta_2 & -12\beta_3 & 6L\beta_4 \\ 6L\beta_2 & 4L^2\beta_5 & -6L\beta_4 & 2L^2\beta_6 \\ -12\beta_3 & -6L\beta_4 & 12\beta_1 & -6L\beta_2 \\ 6L\beta_4 & 2L^2\beta_6 & -6L\beta_2 & 4L^2\beta_5 \end{bmatrix} \tag{2.12}$$

where

$$\beta_1 = \frac{(\lambda L)^3}{12} \frac{(\sin\lambda L\cosh\lambda L + \sinh\lambda L\cos\lambda L)}{(1 - \cos\lambda L\cosh\lambda L)}$$

$$\beta_2 = \frac{(\lambda L)^2}{6} \frac{(\sin\lambda L\sinh\lambda L)}{(1 - \cos\lambda L\cosh\lambda L)}$$

$$\beta_3 = \frac{(\lambda L)^3}{12} \frac{(\sin\lambda L + \sinh\lambda L)}{(1 - \cos\lambda L\cosh\lambda L)}$$

$$\beta_4 = \frac{(\lambda L)^2}{6} \frac{(\cos\lambda L - \cosh\lambda L)}{(1 - \cos\lambda L\cosh\lambda L)}$$

$$\beta_5 = \frac{3\theta}{4\theta^2 - \Psi^2}$$

$$\beta_6 = \frac{3\Psi}{4\theta^2 - \Psi^2}$$

$$\theta = \frac{3}{2\lambda L} (\coth\lambda L - \cot\lambda L)$$

$$\Psi = \frac{3}{\lambda L} (\csc\lambda L - \operatorname{csch}\lambda L)$$

and

$$\lambda = \left(\frac{mp^2}{EI}\right)^{\frac{1}{4}}$$

A modification of the stiffness matrix for flexural deformations can be made for members which have one end simply supported. The modified stiffness coefficients can be found from the solution of the differential equation for transverse vibrations (Eq. 2.11) when the appropriate force and displacement boundary conditions are used.

For Fig. 2.3 with the moment ${}_{n}P^{n}_{jz}$ equal to zero the modified stiffness matrix for flexural deformations is

$$\left[{}_{n}K^{n'}\right]^{*} = \frac{3EI}{L}\begin{bmatrix} \beta_1' & L\beta_2' & -\beta_3' \\ L\beta_2' & L^2\beta_5' & -L\beta_4' \\ -\beta_3' & -L\beta_4' & \beta_1' \end{bmatrix} \tag{2.13}$$

where

$$\beta_1' = \frac{2(\lambda L)^3}{3} \frac{(\cos\lambda L\cosh\lambda L)}{(\sin\lambda L\cosh\lambda L - \cos\lambda L\sinh\lambda L)}$$

$$\beta_2' = \frac{(\lambda L)^2}{3} \frac{(\sin\lambda L\cosh\lambda L + \cos\lambda L\sinh\lambda L)}{(\sin\lambda L\cosh\lambda L - \cos\lambda L\sinh\lambda L)}$$

$$\beta_3' = \frac{(\lambda L)^3}{3} \frac{(\cos\lambda L + \cosh\lambda L)}{(\sin\lambda L\cosh\lambda L - \cos\lambda L\sinh\lambda L)}$$

$$\beta_4' = \frac{(\lambda L)^2}{3} \frac{(\sin\lambda L + \sinh\lambda L)}{(\sin\lambda L\cosh\lambda L - \cos\lambda L\sinh\lambda L)}$$

$$\beta_5' = \frac{1}{\theta}$$

$$\theta = \frac{3}{2\lambda L}(\coth\lambda L - \cot\lambda L)$$

Member Stiffness Matrix

The stiffness matrix for a typical member n is found by combining the stiffness matrices for axial and flexural deformations. The equation for end forces in terms of the stiffness matrix and end deformations is similar to the two equations written for axial forces and flexural forces in the previous topics and is given by

$$\left[{}_{n}P^{n}\right] = \left[{}_{n}K^{n}\right]\left[{}_{n}\delta^{n}\right] \tag{2.14}$$

The subscript and superscript notation that is used will be defined as follows. The left hand subscript on all terms denotes the member to which the term applies. The right hand superscript on force and deformation terms denotes the coordinate system to which the term is referenced. For stiffness coefficient terms the left hand and right hand superscripts denote the coordinate system to which the terms are referenced with respect to forces and deformations, respectively.

Expanding Eq. 2.14, which will hereafter be called the member matrix force equation, into submatrices gives,

$$\begin{bmatrix} {}_nP_i^n \\ \hline {}_nP_j^n \end{bmatrix} = \left[\begin{array}{c|c} {}_n^nK_{ii}^n & {}_n^nK_{ij}^n \\ \hline {}_n^nK_{ji}^n & {}_n^nK_{jj}^n \end{array}\right] \begin{bmatrix} {}_n\delta_i^n \\ \hline {}_n\delta_j^n \end{bmatrix} \tag{2.15}$$

The definitions of the submatrices in the stiffness matrix are the same as the definitions given for the submatrices of the stiffness matrix in Eq. 2.8 except both axial and flexural stiffness coefficients are included in the stiffness matrix of Eq. 2.15.

Expanding the stiffness matrix of Eq. 2.14 into individual terms gives

$$\left[{}_{n}K^{n}\right] = \frac{EI}{L^3}\left[\begin{array}{ccc|ccc} \frac{L^2}{a^2}(\alpha_1) & 0 & 0 & \frac{-L^2}{a^2}(\alpha_2) & 0 & 0 \\ 0 & 12\beta_1 & 6L\beta_2 & 0 & -12\beta_3 & 6L\beta_4 \\ 0 & 6L\beta_2 & 4L^2\beta_5 & 0 & -6L\beta_4 & 2L^2\beta_6 \\ \hline -\frac{L^2}{a^2}(\alpha_2) & 0 & 0 & \frac{L^2}{a^2}(\alpha_1) & 0 & 0 \\ 0 & -12\beta_3 & -6L\beta_4 & 0 & 12\beta_1 & -6L\beta_2 \\ 0 & 6L\beta_4 & 2L^2\beta_6 & 0 & -6L\beta_2 & 4L^2\beta_5 \end{array}\right] \tag{2.16}$$

where all terms in the matrix have previously been defined.

Frame Stiffness Matrix

A typical, planar gable frame with a base reference coordinate system and a reference coordinate system for each member is shown in Fig. 5.

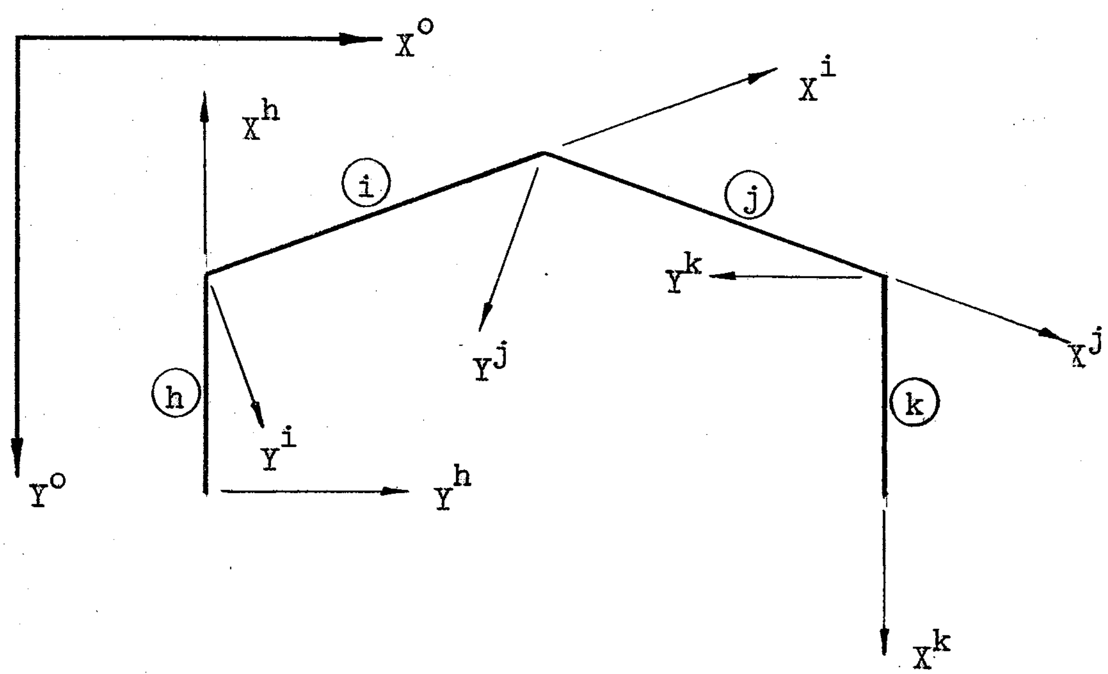

Fig. 5

Base and Member Coordinate Systems of a Typical Gable Frame

The frame stiffness matrix is found by combining the stiffness matrices of the individual members. This can best be shown by first writing the member matrix force equations for each member. These equations are

Member h - Coordinate system h

$$\left[{}_hP^h\right] = \left[{}_h^hK^h\right]\left[{}_h\delta^h\right] \tag{2.17a}$$

Member i - Coordinate system i

$$\left[{}_iP^i\right] = \left[{}_i^iK^i\right]\left[{}_i\delta^i\right] \tag{2.17b}$$

Member j - Coordinate system j

$$\left[{}_jP^j\right] = \left[{}_j^jK^j\right]\left[{}_j\delta^j\right] \tag{2.17c}$$

Member k - Coordinate system k

$$\left[{}_kP^k\right] = \left[{}_k^kK^k\right]\left[{}_k\delta^k\right] \tag{2.17d}$$

Combining the four equations above gives the frame matrix force equation (Eq. 2.18) from which the frame stiffness matrix can readily be identified.

$$\begin{bmatrix} \left[{}_hP^h\right] \\ \left[{}_iP^i\right] \\ \left[{}_jP^j\right] \\ \left[{}_kP^k\right] \end{bmatrix} = \begin{bmatrix} \left[{}_h^hK^h\right] & [0] & [0] & [0] \\ [0] & \left[{}_i^iK^i\right] & [0] & [0] \\ [0] & [0] & \left[{}_j^jK^j\right] & [0] \\ [0] & [0] & [0] & \left[{}_k^kK^k\right] \end{bmatrix} \begin{bmatrix} \left[{}_h\delta^h\right] \\ \left[{}_i\delta^i\right] \\ \left[{}_j\delta^j\right] \\ \left[{}_k\delta^k\right] \end{bmatrix} \tag{2.18}$$

Transformation Matrices

In the previous sections the matrix force equation for individual frame members and for a complete frame have been set up using member oriented coordinate systems, although a base reference coordinate system for all members of any given frame has been mentioned. Generally, for the solution of problems of the nature discussed here, it is convenient to have all forces and deformations referenced to the same coordinate system. A transformation matrix will be given which will be used to transform forces and deformations of any typical member from the member system to the base reference system or what is also called the "O" system.

Halfman (7) gives a development of the transformation matrix for the rotation of axes in a plane which is the case for planar frames. Hall and Woodhead (8) gives a brief development of a general transformation matrix for axis rotations in three dimensions, and also discusses the application of axis transformations to the analysis of static frames by stiffness methods.

The transformation for the x, y, and z components of a vector in the XY plane from a general coordinate system n to a base coordinate system o can be written in the following matrix equation form.

$$\left[{}_{n}D^{o}\right] = \left[{}^{o}\omega^{n}\right]\left[{}_{n}D^{n}\right] \qquad (2.19)$$

where

$\left[D^{o} \right] =$ x,y,and z components of a vector in the o system

$\left[{}^{o}\omega^{n} \right] =$ transformation from system n to system o

$\left[D^{n} \right] =$ x,y, and z components of a vector in the n system

Expanding Eq. 2.19 gives:

$$\begin{bmatrix} D_x^o \\ D_y^o \\ D_z^o \end{bmatrix} = \begin{bmatrix} \cos\phi_n & -\sin\phi_n & 0 \\ \sin\phi_n & \cos\phi_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_x^n \\ D_y^n \\ D_z^n \end{bmatrix} \qquad (2.20)$$

where

ϕ_n = the angle between the base coordinate system and the member coordinate system.

For each member of a frame there are two points at which the forces and deformations must be transformed from the member coordinate system to the base coordinate system; these two points are at the member ends. Since the matrix $\left[{}^{o}\omega^{n} \right]$ will only transform forces or deformations at a single point along the member, a new matrix designated $\left[{}^{o}\pi^{n} \right]$ will be defined as follows:

$$\left[{}^{o}\pi^{n} \right] = \begin{bmatrix} \left[{}^{o}\omega^{n} \right] & [0] \\ [0] & \left[{}^{o}\omega^{n} \right] \end{bmatrix} \qquad (2.21)$$

Also to be defined is the matrix $\left[{}^{n}\pi^{o}\right]$ which is the transformation matrix for forces or deformations at the ends of a member in the base coordinate system to forces or deformations in the member coordinate system. It can easily be shown that the matrix transpose of $\left[{}^{o}\omega^{n}\right]$ is equal to the matrix $\left[{}^{n}\omega^{o}\right]$. Using this relationship it follows that

$$\left[{}^{o}\pi^{n}\right]^{T} = \left[{}^{n}\pi^{o}\right] \tag{2.22}$$

Using Eqs. 2.21 and 2.22 a procedure can be developed to transform the stiffness matrix of a member n in the n coordinate system to the base coordinate system.

First, consider the following transformation expressions:

$$\left[{}_{n}P^{o}\right] = \left[{}^{o}\pi^{n}\right]\left[{}_{n}P^{n}\right] \tag{2.23a}$$

$$\left[{}_{n}\delta^{n}\right] = \left[{}^{n}\pi^{o}\right]\left[{}_{n}\delta^{o}\right] \tag{2.23b}$$

Next, it is recalled that the member matrix force equation is given by

$$\left[{}_{n}P^{n}\right] = \left[{}^{n}_{n}K^{n}\right] \left[{}_{n}\delta^{n}\right] \tag{2.14}$$

Substituting the transformation given by Eq. 2.23b into the member matrix force equation gives:

$$\left[{}_{n}P^{n}\right] = \left[{}^{n}_{n}K^{n}\right]\left[{}^{n}\pi^{o}\right]\left[{}_{n}\delta^{o}\right]$$

Premultiplying both sides of the above equation by $\left[{}^{o}\pi^{n}\right]$

$$\left[{}^{o}\pi^{n}\right]\left[{}_{n}P^{n}\right] = \left[{}^{o}\pi^{n}\right]\left[{}^{n}_{n}K^{n}\right]\left[{}^{n}\pi^{o}\right]\left[{}_{n}\delta^{o}\right]$$

Substituting the expressions given by Eqs. 2.22 and 2.23a yields:

$$\left[{}_{n}P^{o}\right] = \left[{}^{o}\pi^{n}\right]\left[{}^{n}_{n}K^{n}\right]\left[{}^{o}\pi^{n}\right]^{T}\left[{}_{n}\delta^{o}\right] \tag{2.24}$$

From Eq. 2.24 it is seen that the transformation of the stiffness matrix of a member from the member coordinate system to the base coordinate system is given by:

$$\left[{}^{o}_{n}K^{o}\right] = \left[{}^{o}\pi^{n}\right]\left[{}^{n}_{n}K^{n}\right]\left[{}^{o}\pi^{n}\right]^{T} \tag{2.25}$$

Table I gives the transformed stiffness matrix $\left[{}^{o}_{n}K^{o}\right]$ in general terms which have been previously defined.

Substituting transformed end forces, end deformations, and stiffness coefficient matrices into the frame matrix force equation (Eq. 2.18) gives the following frame matrix force equation on which the application of the method given in the next chapter is based.

$$\begin{bmatrix} [{}_h P^o] \\ [{}_i P^o] \\ [{}_j P^o] \\ [{}_k P^o] \end{bmatrix} = \begin{bmatrix} [{}^o_h K^o] & [0] & [0] & [0] \\ [0] & [{}^o_i K^o] & [0] & [0] \\ [0] & [0] & [{}^o_j K^o] & [0] \\ [0] & [0] & [0] & [{}^o_k K^o] \end{bmatrix} \begin{bmatrix} [{}_h \delta^o] \\ [{}_i \delta^o] \\ [{}_j \delta^o] \\ [{}_k \delta^o] \end{bmatrix} \qquad (2.26)$$

TABLE I

STIFFNESS MATRIX TRANSFORMATION FROM MEMBER COORDINATE SYSTEM TO BASE COORDINATE SYSTEM FOR A GENERAL MEMBER *

$\frac{EI}{L^3}$	$\frac{L^2}{a^2}\alpha_1\cos^2\phi$ $+12\beta_1\sin^2\phi$	$\frac{L^2}{a^2}\alpha_1\cos\phi\sin\phi$ $-12\beta_1\cos\phi\sin\phi$	$-6L\beta_2\sin\phi$	$-\frac{L^2}{a^2}\alpha_2\cos^2\phi$ $-12\beta_3\sin^2\phi$	$-\frac{L^2}{a^2}\alpha_2\cos\phi\sin\phi$ $+12\beta_3\cos\phi\sin\phi$	$-6L\beta_4\sin\phi$
	$\frac{L^2}{a^2}\alpha_1\cos\phi\sin\phi$ $-12\beta_1\cos\phi\sin\phi$	$\frac{L^2}{a^2}\alpha_1\sin^2\phi$ $+12\beta_1\cos^2\phi$	$6L\beta_2\cos\phi$	$-\frac{L^2}{a^2}\alpha_2\cos\phi\sin\phi$ $+12\beta_3\cos\phi\sin\phi$	$-\frac{L^2}{a^2}\alpha_2\sin^2\phi$ $-12\beta_3\cos^2\phi$	$6L\beta_4\cos\phi$
	$-6L\beta_2\sin\phi$	$6L\beta_2\cos\phi$	$4L^2\beta_5$	$6L\beta_4\sin\phi$	$-6L\beta_4\cos\phi$	$2L^2\beta_6$
	$-\frac{L^2}{a^2}\alpha_2\cos^2\phi$ $-12\beta_3\sin^2\phi$	$-\frac{L^2}{a^2}\alpha_2\cos\phi\sin\phi$ $+12\beta_3\cos\phi\sin\phi$	$6L\beta_4\sin\phi$	$\frac{L^2}{a^2}\alpha_1\cos^2\phi$ $+12\beta_1\sin^2\phi$	$\frac{L^2}{a^2}\alpha_1\cos\phi\sin\phi$ $-12\beta_1\cos\phi\sin\phi$	$6L\beta_2\sin\phi$
	$-\frac{L^2}{a^2}\alpha_2\cos\phi\sin\phi$ $+12\beta_3\cos\phi\sin\phi$	$-\frac{L^2}{a^2}\alpha_2\sin^2\phi$ $-12\beta_3\cos^2\phi$	$-6L\beta_4\cos\phi$	$\frac{L^2}{a^2}\alpha_1\cos\phi\sin\phi$ $-12\beta_1\cos\phi\sin\phi$	$\frac{L^2}{a^2}\alpha_1\sin^2\phi$ $+12\beta_1\cos^2\phi$	$-6L\beta_2\cos\phi$
	$-6L\beta_4\sin\phi$	$6L\beta_4\cos\phi$	$2L^2\beta_6$	$6L\beta_2\sin\phi$	$-6L\beta_2\cos\phi$	$4L^2\beta_5$

* All parameters appearing in this table are for a typical member n.

CHAPTER III

APPLICATION

The application of the matrix force equation (Eq. 2.26) to any particular frame involves forming the necessary equilibrium equations for the solution. The matrix force equation in general form has three force equations for each end of each member of a frame. This gives six force equations for any interior joint, where an interior joint can be any joint of the frame excluding the base joints.

Figure 6 shows the ends of two members comprising a typical joint. The joint is shown before and after end displacements have occured. End rotations are not shown since they are equal regardless of the rotation of the member coordinate axes in the XY plane. From the figure it is obvious that the end displacements along the X and Y axes of the members in their respective coordinate systems are not equal. However, the members do not separate; so the end displacements of the members at the joint in the direction of the X and Y axes of the base coordinate system have to be equal.

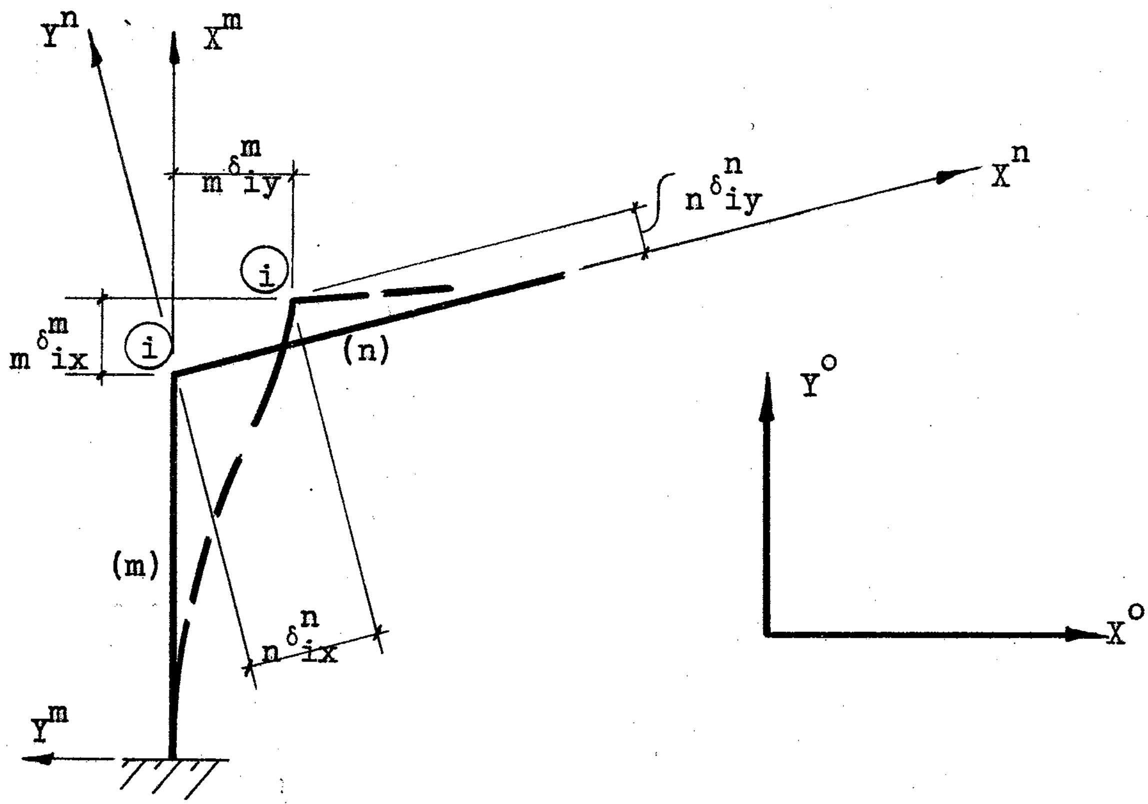

Fig. 6

Displacements of a Typical Joint

Writing joint compatibility equations for the deformations of a typical joint i, in the base coordinate system, connecting members m and n,

$$
\begin{aligned}
{}_m\delta^o_{ix} &= {}_n\delta^o_{ix} \\
{}_m\delta^o_{iy} &= {}_n\delta^o_{iy} \\
{}_m\delta^o_{iz} &= {}_n\delta^o_{iz}
\end{aligned}
\tag{3.1}
$$

Joint equilibrium equations can also be written for the forces at the ends of members m and n at joint i. Remembering that for the stiffness coefficient expressions developed in the basic theory there can be no externally applied forces on a member, the equilibrium equations for end forces at joint i are:

$$
\begin{aligned}
{}_mP^o_{ix} + {}_nP^o_{ix} &= 0 \qquad (3.2)\\
{}_mP^o_{iy} + {}_nP^o_{iy} &= 0\\
{}_mP^o_{iz} + {}_nP^o_{iz} &= 0
\end{aligned}
$$

Equations similar to Eqs. 3.1 and 3.2 can be written for all the interior joints of a frame. These equations can then be applied to the member matrix force equations of the frame matrix force equation to obtain Eq. 3.3. Eq. 3.3, which is for the frame shown in Fig. 7, will hereafter be called the matrix equilibrium equation. It is noted that the member matrix force equations have been expanded into the form of the submatrices of Eq. 2.15. It is also seen that the size of the stiffness coefficient matrix of the matrix equilibrium equation is reduced by order 3s from the size of the stiffness coefficient matrix of the frame matrix equation, where s is the number of interior joints.

$$\begin{bmatrix} [{}_kP_i^o] \\ [0] \\ [0] \\ [0] \\ [{}_nP_j^o] \end{bmatrix} = \begin{bmatrix} [{}_kK_{ii}^o] & [{}_kK_{ij}^o] & [0] & [0] & [0] \\ [{}_kK_{ji}^o] & \left[[{}_kK_{jj}^o]+[{}_lK_{ii}^o]\right] & [{}_lK_{ij}^o] & [0] & [0] \\ [0] & [{}_lK_{ji}^o] & \left[[{}_lK_{jj}^o]+[{}_mK_{ii}^o]\right] & [{}_mK_{ij}^o] & [0] \\ [0] & [0] & [{}_mK_{ji}^o] & \left[[{}_mK_{jj}^o]+[{}_nK_{ii}^o]\right] & [{}_nK_{ij}^o] \\ [0] & [0] & [0] & [{}_nK_{ji}^o] & [{}_nK_{jj}^o] \end{bmatrix} \begin{bmatrix} [{}_k\delta_i^o] \\ [\delta_2^o] \\ [\delta_3^o] \\ [\delta_4^o] \\ [{}_n\delta_j^o] \end{bmatrix} \qquad (3.3)$$

Where δ_2, δ_3, and δ_4 are joint deformations.

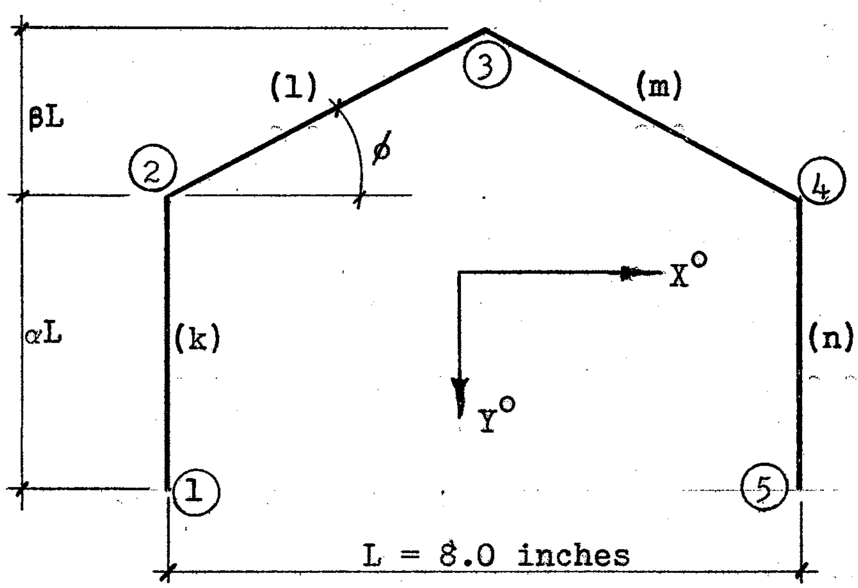

Fig. 7

General Frame Dimensions for Theory Application

Next, consider frames with fixed connections at the bases. The joint equations for deformations of member n at the fixed base 5 are given by:

$$ n^{\delta}\overset{o}{5}x = 0 \qquad (3.4)$$

$$ n^{\delta}\overset{o}{5}y = 0 $$

$$ n^{\delta}\overset{o}{5}z = 0 $$

For frames with hinged connections at the bases, the joint equations for deformations of member n at the pinned base 5 are given by:

$$ {}_n\delta^{o}_{5x} = 0 $$

$$ {}_n\delta^{o}_{5y} = 0 $$

$$ {}_n\delta^{o}_{5z} \neq 0 $$

Using these expressions the stiffness coefficient matrix of the matrix equilibrium equation can be further reduced by eliminating the rows and columns of the stiffness matrix corresponding to the zero displacement terms. This reduces the stiffness matrix of a frame with fixed bases by order six, and of a frame with pinned bases by order four.

When joint equilibrium equations are used and the terms corresponding to zero deformations eliminated, the column matrix of forces on the left hand side of the matrix equilibrium equation is zero. To satisfy the equation the other side of the matrix equilibrium equation, with the stiffness coefficient matrix postmultiplied by the displacement matrix, must also be zero. The column matrix of displacement terms cannot be zero when a frame is vibrating at any natural frequency except for the trival case of the natural frequency equal to zero. Therefore, for the matrix equilibrium equation to be satisfied, the determinant of the stiffness coefficient matrix must be zero. The only frequencies which will give zero values for the determinant of the stiffness coefficient matrix are the natural frequencies of the frame in question.

The numerical examples which follow were evaluated by an IBM 1620 electronic computer. The Appendix gives the Fortran programs used for the numerical examples.

The frame dimensions, angle of rotation of gable members, member section properties, an initial value of frequency, and an incremental value of frequency were the input data to the computer. With this data the coefficients of the stiffness matrix and the determinant of coefficients were evaluated. The first evaluation was for the initial value of frequency. The computer then added the incremental value of frequency to the previous value used and reevaluated the coefficients and the determinant of the coefficients. This cycle was repeated as many times as was needed for the solution of a problem. For each cycle the output data from the computer was the value of frequency used in that cycle and the corresponding value of the determinant of stiffness coefficients. When the value for the determinant changed signs from one frequency value to the next, this was an indication that the determinant was zero between the two frequencies. For the range between these two frequencies the incremental value of frequency was made smaller, and the natural frequency was found to the nearest cycle per second.

The frame shown in Fig. 7 is typical for all the numerical problems worked. The properties of the frame members are the same for all the problems; the values of these properties are as follows:

Cross sectional area (A) = 20740.0×10^{-6} in.^{2}

Moment of inertia (I) = 34.2282×10^{-6} in.^{4}

Modulus of elasticity (E) = 30.6×10^{6} lb./in.^{2}

Mass per unit length (m) = $15.2174 \times 10^{-6} \dfrac{\text{lb.-sec.}^2}{\text{in.}^2}$

The parameters of the frame dimensions, given by α and β in Fig. 7, vary for each problem. These parameters and the first four natural frequencies for each problem are given in Table II.

TABLE II

NATURAL FREQUENCIES FOR EXAMPLE PROBLEMS (CYCLES PER SECOND)

Base Conditions		Fixed				Pinned			
Example No.		1	2	3	4	1	2	3	4
Dimension Parameters	α	0.2	0.2	0.4	0.4	0.2	0.2	0.4	0.4
	β	0.1	0.2	0.1	0.2	0.1	0.2	0.1	0.2
1st Frequency		321	394	307	300	302	278	145	89
2nd Frequency		697	700	312	303	358	376	311	294
3rd Frequency		1146	1031	975	884	1046	940	923	848
4th Frequency		1730	1547	1667	1515	1723	1542	1392	1325

CHAPTER IV

SUMMARY AND CONCLUSIONS

Summary

The method of stiffness coefficients for finding natural frequencies of single bay gable frames is presented in this thesis. The theory given is based on the mass of the frame members being distributed along the length of the members. The mass per unit length and moment of inertia are constant values over the length of each member. Bending and axial deformations are considered while shear deformations and rotatory inertia effects are neglected. No damping forces are considered in the theory development.

The stiffness matrix for a general member is established. The stiffness matrices for all members of a frame are then combined into a single matrix. A transformation matrix and procedure is given to transform the stiffness matrices of members in their own reference systems to a base reference system.

Joint equilibrium equations are formed and the theory applied to the cases of symmetrical gable frames with fixed bases and hinged bases. Four numerical examples are worked for each case and the results tabulated.

Conclusions

The method of stiffness coefficients provides a relatively simple approach to finding natural frequencies of gable frames. If the basic assumptions that shear deformation and rotatory inertia effects are negligible and that no damping forces exist are valid, then in theory, the method gives exact solutions for the natural frequencies of frames. The exactness of the natural frequencies found for a numerical problem depends on how accurately the frame dimensions and properties are known, and on what increment of natural frequency is used in evaluating the determinant of the stiffness coefficients. Since the natural frequencies of a numerical problem are found by evaluating the determinant of stiffness coefficients for various values of frequency, the increment or difference between the values of frequency used in evaluating the determinant should be as small as practical. Due to the nature of the stiffness coefficient expressions, if large increments of frequency are used some natural frequencies which exist for a particular frame might not be discovered.

Two limitations of the method of stiffness coefficients, which are obvious, include the following. First, for frames in general, the size or number of joints which a frame can have and still be analyzed is limited to the available computer facilities. The second limitation is that a frame with members of varying cross section can be analyzed only if the members are assumed to have a constant section over the member lengths, or if the members are broken into segments which have constant cross sections. Breaking the members into segments is not very practical, however, since at every change of section there must be an associated joint. This would require even the smallest of frames with members of varying cross sections to have very large stiffness coefficient matrices.

Extension

The method presented in this thesis is directly applicable to the frequency analysis of many types of structures, several of which are continuous beams, rectangular frames, continuous frames, and complex frames. The method could also be extended to the frequency analysis of space frames.

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APPENDIX

The Fortran statements for the computer program used in finding the natural frequencies of the fixed end frame problems are listed below.

```
C     FREQUENCY ANALYSIS OF A SYMMETRICAL GABLE FRAME
C     WITH FIXED BASES   CHARLES W BRINK   SUMMER 1964
C     A=AREA  E=MODULUS OF ELASTICITY  Z=MOMENT OF INERTIA
C     W=MASS PER UNIT LENGTH  CL=COLUMN HEIGHT
C     BL=BEAM SLANT LENGTH  Y=ANGLE IN RADIANS
      DIMENSION X(9,9)
    5 READ 500,A,E,Z,W
      READ 500,CL,BL,Y
C     COLUMN DYNAMIC FACTOR COEFFICIENTS
      C1=(A*E)/CL
      C2=(E*Z)/(CL**3)
      C3=C2*CL
      C4=4.0*C3*CL
C     BEAM DYNAMIC FACTOR COEFFICIENTS
      CO=COS(Y)
      SI=SIN(Y)
      B0=(A*E)/BL
      B1=B0*CO*CO
      B2=B0*SI*SI
      B3=B0*CO*SI
      B4=(2.0*E*Z)/BL
      B5=2.0*B4
      B0=B5/(4.0*BL)
      B6=B0*CO
      B7=B0*SI
      B0=B0/BL
      B8=B0*CO*CO
      B9=B0*SI*SI
      B10=B0*CO*SI
C     ACCEPT P=FREQUENCY(CPS), DP=INCREMENTAL FREQUENCY
   10 ACCEPT 500,P,DP
   20 CP=6.2831853*P
C     COLUMN DYNAMIC FACTORS
      FC=((W*CP*CP)/(E*Z))**0.25
      F=FC*CL
```

```
      KEY=1
   30 D=SQR(Z/A)*FC*F
      SI1=SIN(F)
      SI2=SIN(D)
      CO1=COS(F)
      CO2=COS(D)
      CO3=EXP(F)
      SH=(CO3-1.0/CO3)*0.5
      CH=(CO3+1.0/CO3)*0.5
      DV=1.0-(CO1*CH)
      TH=(3.0/(2.0*F))*(CH/SH-CO1/SI1)
      PS=(3.0/F)*(1.0/SI1-1.0/SH)
      GO TO(40,50),KEY
   40 CF1=D*(CO2/SI2)
      CF2=F**3*((SI1*CH+CO1*SH)/DV)
      CF3=F*F*((SI1*SH)/DV)
      CF4=(3.0*TH)/(4.0*TH*TH-PS*PS)
C     BEAM DYNAMIC FACTORS
      C=FC*BL
      F=C
      KEY=2
      GO TO 30
   50 B=D
      BF1=B*(CO2/SI2)
      BF2=B*(1.0/SI2)
      BF3=C**3*((SI1*CH+CO1*SH)/DV)
      BF4=C*C*((SI1*SH)/DV)
      BF5=C**3*((SI1+SH)/DV)
      BF6=C*C*((CO1-CH)/DV)
      BF7=(3.0*TH)/(4.0*TH*TH-PS*PS)
      BF8=(3.0*PS)/(4.0*TH*TH-PS*PS)
C     STIFFNESS MATRIX
      DO 60 I=1,9
      DO 60 J=1,9
   60 X(I,J)=0.0
      X(1,1)=C2*CF2+B1*BF1+B9*BF3
      X(1,2)=B3*BF1-B10*BF3
      X(1,3)=C3*CF3-B7*BF4
      X(1,4)=-B1*BF2-B9*BF5
      X(1,5)=-B3*BF2+B10*BF5
      X(1,6)=-B7*BF6
      X(2,2)=C1*CF1+B2*BF1+B8*BF3
      X(2,3)=B6*BF4
      X(2,4)=X(1,5)
      X(2,5)=-B2*BF2-B8*BF5
      X(2,6)=B6*BF6
      X(3,3)=C4*CF4+B5*BF7
      X(3,4)=-X(1,6)
      X(3,5)=-X(2,6)
      X(3,6)=B4*BF8
      X(4,4)=2.0*(B1*BF1+B9*BF3)
      X(4,6)=2.0*(B7*BF4)
      X(4,7)=X(1,4)
      X(4,8)=-X(1,5)
```

```
      X(4,9)=X(3,4)
      X(5,5)=2.0*(B2*BF1+B8*BF3)
      X(5,7)=-X(2,4)
      X(5,8)=X(2,5)
      X(5,9)=X(2,6)
      X(6,6)=2.0*B5*BF7
      X(6,7)=X(1,6)
      X(6,8)=X(3,5)
      X(6,9)=X(3,6)
      X(7,7)=X(1,1)
      X(7,8)=-X(1,2)
      X(7,9)=X(1,3)
      X(8,8)=X(2,2)
      X(8,9)=-X(2,3)
      X(9,9)=X(3,3)
      DO 70 I=1,9
      DO 70 J=1,9
   70 X(J,I)=X(I,J)
C     EVALUATION OF DETERMINATE
      DET=1.0
      DO 80 K=1,8
      DET=DET*X(K,K)
      DO 80 I=K,8
      DO 80 J=K,8
   80 X(I+1,J+1)=X(I+1,J+1)-X(I+1,K)*X(K,J+1)/X(K,K)
      DET=DET*X(9,9)
      PRINT 501,P,DET
      IF(SENSE SWITCH 1)10,90
   90 IF(SENSE SWITCH 2)5,100
  100 P=P+DP
      GO TO 20
  500 FORMAT (E16.8,E16.8,E16.8,E16.8)
  501 FORMAT (8X F10.4, 12X E14.8)
      END
```

The computer program for finding the natural frequencies of the pinned end frame problems is the same as the program for fixed end except the statements defining the column dynamic factors CF2, CF3, and CF4 are deleted and the following statements inserted.

```
      DIV=SI1*CH-CO1*SH
      CF2=2.0*F**3*(CO1*CH)/DIV
      CF3=F*F*(SI1*CH+CO1*SH)/DIV
      CF4=0.75*(1.0/TH)
```

VITA

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Candidate for the Degree of

Master of Science

Thesis: FREQUENCY ANALYSIS OF PLANAR FRAMES WITH DISTRIBUTED MASS - GABLE FRAMES BY METHOD OF STIFFNESS COEFFICIENTS

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