STABILITY OF MULTILAYER

SANDWICH PLATES

By

JULIUS PAN WONG

Diploma in Civil Engineering
Hong Kong Baptist College
Hong Kong
1960

Master of Science
Louisiana Polytechnic Institute
Ruston, Louisiana
1962

Submitted to the Faculty of the Graduate School
of the Oklahoma State University
in partial fulfillment of the requirements
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Thesis Approved:

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Thesis Adviser

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Dean of the Graduate School

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Julius Pan Wong

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NOMENCLATURE

The following symbols have been adopted for use in the thesis:

A, B, C . . . . .   components of first variation of auxiliary functional;

D . . . .   flexural rigidity;

E_i . . . .   modulus of elasticity of ith facing membrane layer;

F_D, F_z, F_S . . .   property coefficients of plate;

G_{ixz}, G_{iyz} . . .   moduli of rigidity of ith core layer;

G, G_x, G_y . . . .   property coefficients of plate;

K_x, K_y . . . . .   property coefficients of plate;

M_x, M_y, M_{xy} . .   moment stress resultants;

N_x, N_y, N_{xy} . .   face-parallel stress resultants;

P . . . .   load function normal to the plate;

Q_x, Q_y . . . . .   transverse shear stress resultants;

R . . . .   total transverse resultant;

S . . . .   property coefficient of plate;

U . . . .   strain energy;

W^* . . . .   work done on the part of boundary where displacements are prescribed;

$\overline{X}, \overline{Y}, \overline{Q}, \overline{M}_x, \overline{M}_y$ total resultants in a specific direction along the boundary;

a, b . . . .   plate dimensions;

c . . . .   boundary;

c_d . . . .   the part of boundary where displacements are specified;;

h_j . . . .   thickness of jth core layer;

i . . . . index, designates ith facing membrane layer;

j . . . . index, designates jth core layer;

n . . . . total number of layers;

s . . . . length;

t_i . . . . thickness of ith facing membrane layer;

u, v, w . . . . . displacements;

x, y, z . . . . . coordinates;

z_i . . . . distance measured from xy-plane to middle plane of ith membrane layer;

$\bar{z}$ . . . . property coefficient, defining position of weighted neutral surface;

π^* . . . . complementary energy;

π^{**} . . . . auxiliary functional;

α, β . . . . generalized displacements;

δ . . . . first variation;

$\lambda_1, \lambda_2, \ldots \lambda_{13}$. Lagrangian multipliers;

ν_i . . . . Poisson's ratio for ith facing membrane layer;

ν_D, φ, ν_S . . . . property coefficients of plate;

$\sigma_{ix}, \sigma_{iy}, \sigma_{ixy}$ . . face-parallel stresses at ith facing membrane layer;

τ_{ixz}, τ_{iyz} . . . transverse shearing stresses at ith core layer;

∇^2 . . . . Laplacian operator;

∇^4 . . . . $\nabla^2 \nabla^2$ and

∇^6 . . . . $\nabla^2 \nabla^2 \nabla^2$.

Additional symbols used in the example problems are defined when they appear and are not listed.

CHAPTER I

INTRODUCTION

1.1 General

Sandwich constructions are characterized by different material properties between facing layers and core layers. The facing materials have relatively high moduli of elasticity and low moduli of rigidity as compared with the core materials. In this investigation the facing materials are considered to be homogeneous and isotropic, and the core materials homogeneous and orthotropic. All displacements are defined on a weighted neutral surface whose location will be given in the next chapter.

In addition to the above general description, the following assumptions are essential for this analysis:

(1) The material of each layer is elastic and follows Hooke's law.

(2) The transverse rigidities of the core materials are relatively high compared to the facing materials, i.e., transverse shear forces are completely taken by the core layers.

(3) The core stiffnesses associated with face-parallel stresses are neglected.

(4) The facing layers are thin compared with the core layers, i.e., the facing layers act as membranes.

(5) The total thickness of the multilayer plate is small compared

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with the other dimensions.

(6) Prior to buckling the deformations are small.

(7) At any section of the plate the transverse deflection of each layer is the same as that of the weighted neutral surface.

(8) No bond failure may occur prior to buckling of the plate.

(9) Local buckling is not considered.

(10) Temperature effect is neglected.

The governing differential equations for the stability of multilayer sandwich plates are derived by means of a variational method and the minimizing principle. The problem is formulated in a complete Lagrange form and the minimizing of the functional maps the general state of stresses to the one which satisfies the compatibility conditions. Once the set of Euler equations is obtained, the stress resultant - generalized displacement relations may be found and henceforth lead to the governing differential equations. A discussion of the effect of unequal Poisson's ratios in the facing layers and the orthotropic moduli of rigidity in the core layers is also included.

The letter symbols adopted for use in this thesis are defined where they first appear and are listed in the Nomenclature.

1.2 Historical Notes

The analytical study of structural members composed of sandwich construction becomes increasingly important with the developments of modern technology and the introduction of new materials. The majority of the past efforts connected with this work have been confined to single core construction with two facing layers.

The general solution for bending of sandwich plates was presented by E. Reissner[5, 6]. He considered a plate consisting of a core with two facing membranes identical both in thickness and elastic properties, and assumed that the face-parallel stresses in the core layer and the variation of the stresses over the thickness of the facing layers were negligible. This assumes that the sandwich is a thin plate composed of a core layer of high transverse rigidity and low stiffnesses associated with the face-parallel stresses and that the facing layers act as membranes. Since Reissner's work many extensions of the theory have been presented[1, 2]. One of the more significant extensions to the theory was the one presented by S. Cheng[3] in which he treated the bending of sandwich plates with orthotropic core. The problems were formulated either by minimizing the complementary energy with the stress resultants taken as variables or by minimizing the potential energy with the displacements taken as variables.

The problem of multilayer sandwich plates was less explored. Recently a theory of bending was presented by B. D. Liaw[4]. Based on Reissner-Cheng assumptions, he formulated the problem in terms of the complementary energy with stresses taken as independent variables and stress resultants as dependent variables, and developed a general Lagrange-Navier type equation for the multilayer plates with equal Poisson's ratios for all facing membrane layers and orthotropic core layers.

CHAPTER II

GENERAL ANALYSIS

2.1 Statement of the Problem

The mathematical model considered is a rectangular plate consisting of two core layers of thickness h_1 and h_2 and three facing membrane layers of thickness t_1, t_2 and t_3. Each facing membrane layer is assumed to be isotropic and homogeneous and possesses different elastic properties, while each core layer is orthotropic and also possesses different elastic properties. Let the xy-plane coincide with the weighted neutral surface at the undeformed position, with z-axis normal to this plane. Also let z_1, z_2 and z_3 be the distances measured from the xy-plane to the middle plane of each facing membrane layer respectively (Fig. 1). The weighted neutral surface of
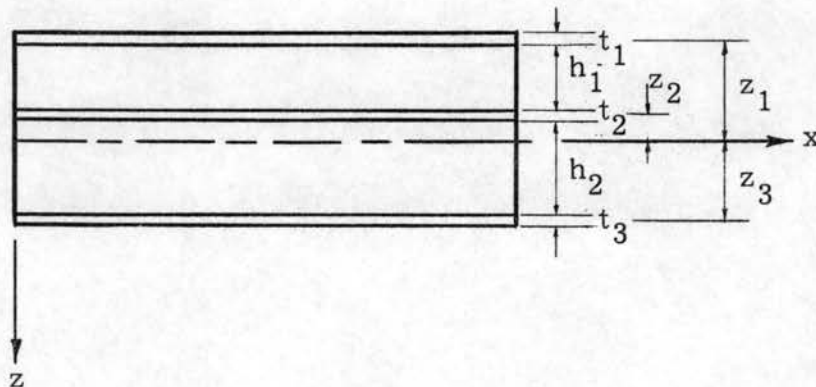
Figure 1 - A Cross Section of Plate

the multilayer sandwich plate is at the position such that

$$\sum_{i=1}^{3} \frac{E_i t_i z_i}{1 - v_i^2} = 0 \tag{1}$$

where E_i and v_i are the modulus of elasticity and Poisson's ratio of the ith facing membrane layer respectively.

The problem is then to develop the governing differential equations for the stability of multilayer sandwich plates. Though the model adopted is a five-layer plate, the final result is valid for any finite number of layers.

2.2 Stress Resultants and Equilibrium Equations

In accordance with the assumptions made previously that face-parallel stresses of core layers and the variations of stresses over the facing membrane layers are negligible, the stress resultants may be defined as follows:

$$M_x = \sum_{i=1}^{3} \sigma_{ix} t_i z_i \tag{2}$$

$$M_y = \sum_{i=1}^{3} \sigma_{iy} t_i z_i \tag{3}$$

$$M_{xy} = \sum_{i=1}^{3} \sigma_{ixy} t_i z_i \tag{4}$$

$$N_x = \sum_{i=1}^{3} \sigma_{ix} t_i \tag{5}$$

$$N_y = \sum_{i=1}^{3} \sigma_{iy} t_i \tag{6}$$

$$N_{xy} = \sum_{i=1}^{3} \sigma_{ixy} t_i \tag{7}$$

$$Q_x = \sum_{j=1}^{2} \tau_{jxz} h_j \tag{8}$$

$$Q_y = \sum_{j=1}^{2} \tau_{jyz} h_j \tag{9}$$

where

$M_x(M_y)$ designates the bending moment about y(x) axis,

$M_{xy}(M_{yx})$ the twisting moment about x(y) axis,

$N_x(N_y)$ the normal force on the x(y) face,

$N_{xy}(N_{yx})$ the shear force parallel to the plane of the plate on the x(y) face,

$Q_x(Q_y)$ the transverse shear force on the x(y) face,

$\sigma_{ix}(\sigma_{iy})$ the normal stress on the ith membrane in x(y) direction,

$\sigma_{ixy}(\sigma_{iyx})$ the shearing stress parallel to the plane of the plate on the ith membrane in y(x) direction,

$\tau_{jxz}(\tau_{jyz})$ the transverse shearing stress on the x(y) face of the jth core (Fig. 2).
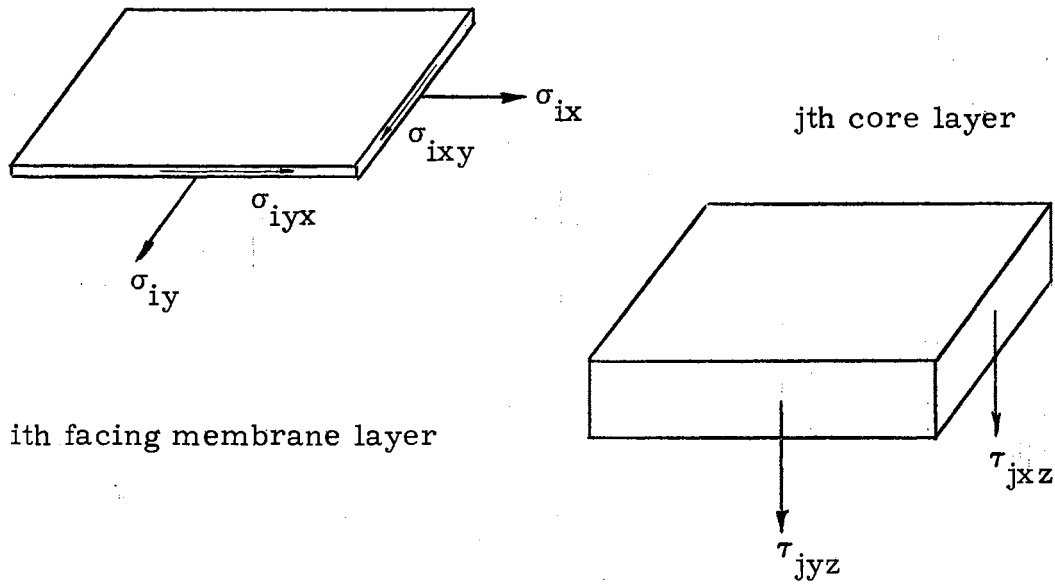
Figure 2 - Stresses on Typical Layers

Also the relations $M_{xy} = M_{yx}$, $N_{xy} = N_{yx}$ and $\sigma_{ixy} = \sigma_{iyx}$ are retained.

Equilibrium of forces and moments acting on a differential plate element (Fig. 3) yield the following equations:

$$N_{x,x} + N_{yx,y} = 0 \tag{10}$$

$$N_{y,y} + N_{xy,x} = 0 \tag{11}$$

$$Q_{x,x} + Q_{y,y} + P + (N_x w_{,x} + N_{yx} w_{,y})_{,x} + (N_y w_{,y} + N_{xy} w_{,x})_{,y} = 0 \tag{12}$$

$$M_{x,x} + M_{yx,y} - Q_x = 0 \tag{13}$$

$$M_{y,y} + M_{xy,x} - Q_y = 0 \tag{14}$$

where comma means to take partial derivative of the quantity in front of it with respect to the following subscripts, and P is the intensity
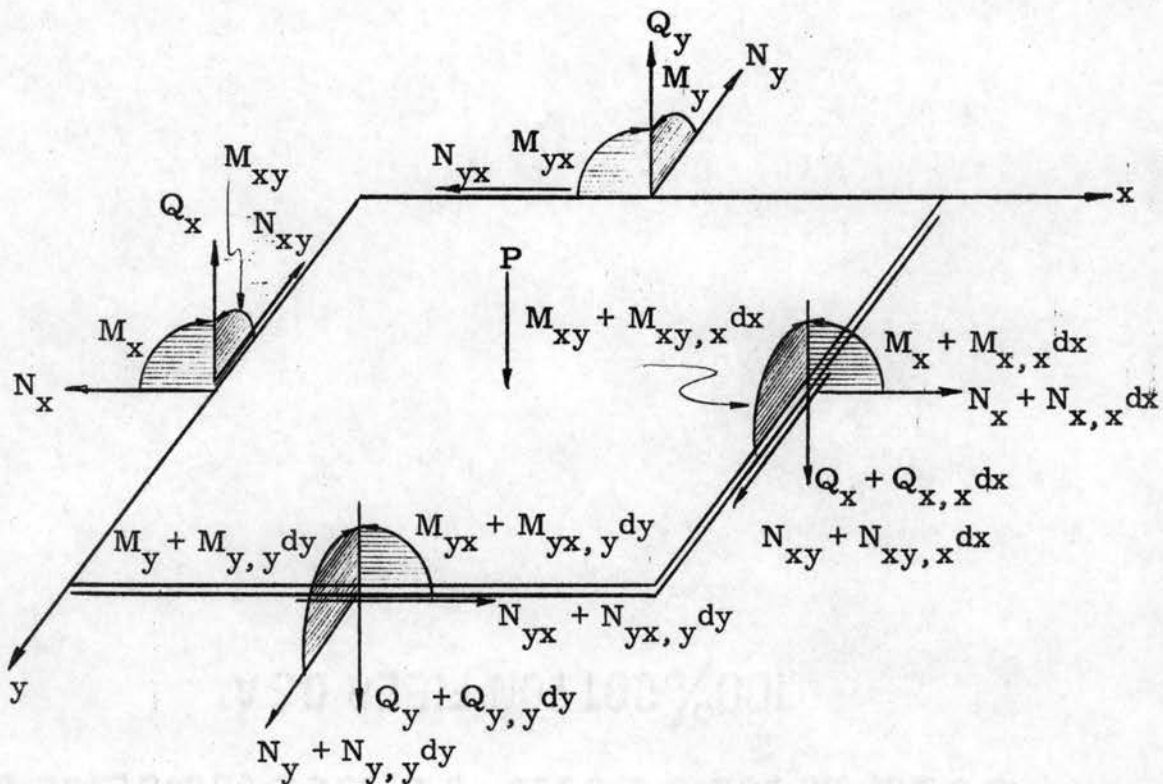
Figure 3 - A Differential Plate Element

of transverse load applied on the differential plate element. Equations
(10), (11) and (12) may be combined into one equation:

$$Q_{x,x} + Q_{y,y} + R = 0 \tag{15}$$

where

$$R = P + N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} \tag{16}$$

2.3 Complementary Energy

Considering a rectangular plate of the dimensions a by b,
the strain energy stored in the system may be expressed in terms of

stresses as:

$$U = \frac{1}{2} \int_0^b \int_0^a \left\{ \sum_{i=1}^{3} \left[\frac{t_i}{E_i} (\sigma_{ix}^2 + \sigma_{iy}^2 - 2\nu_i \sigma_{ix}\sigma_{iy}) + \frac{2t_i(1+\nu_i)}{E_i} \sigma_{ixy}^2 \right] \right.$$

$$\left. + \sum_{j=1}^{2} \left[\frac{h_j}{G_{jxz}} \tau_{jxz}^2 + \frac{h_j}{G_{jyz}} \tau_{jyz}^2 \right] \right\} dxdy \qquad (17)$$

where G_{jxz} (G_{jyz}) is the shear modulus of rigidity in xz(yz) plane of the jth core layer.

Let u, v, w, α and β be the generalized displacements at the boundary; then the work which the surface stresses do over that portion of the surface where the displacements are prescribed is

$$W^* = \int_{c_d} \left[\overline{X}u + \overline{Y}v + (\overline{X}w_{,x} + \overline{Y}w_{,y} + \overline{Q})w \right.$$

$$\left. + \overline{M}_x \alpha + \overline{M}_y \beta \right] ds \qquad (18)$$

where the capital bar letters indicate the total resultants in a specific direction along the boundary, for example, $\overline{X}$ represents N_x on x = 0, x = a and N_{yx} on y = 0, y = b and c_d designates the boundary where displacements are specified.

Then the complementary energy of the plate is

$$\pi^* = U - W^*. \qquad (19)$$

Thus the problem becomes one of seeking the conditions for the extremum of the functional π^* subjected to the constraint conditions of equations (2) to (14).

2.4 Auxiliary Functional

Introducing thirteen Lagrangian multipliers λ_1, λ_2, ..., λ_{13}, the auxiliary functional, whose integrand consisting of the terms of complementary energy and the constraint equations multiplied by the Lagrangian multipliers, can be formulated as follows:

$$\pi^{**} = \int_0^b \int_0^a \left\{ \sum_{i=1}^3 \left[\frac{t_i}{2E_i} (\sigma_{ix}^2 + \sigma_{iy}^2 - 2\nu_i \sigma_{ix} \sigma_{iy}) + \frac{t_i(1 + \nu_i)}{E_i} \sigma_{ixy}^2 \right] \right.$$

$$+ \frac{1}{2} \sum_{j=1}^2 \left[\frac{h_j}{G_{jxz}} \tau_{jxz}^2 + \frac{h_j}{G_{jyz}} \tau_{jyz}^2 \right]$$

$$+ \lambda_1 \left(M_x - \sum_{i=1}^3 \sigma_{ix} t_i z_i \right)$$

$$+ \lambda_2 \left(M_y - \sum_{i=1}^3 \sigma_{iy} t_i z_i \right)$$

$$+ \lambda_3 \left(M_{xy} - \sum_{i=1}^3 \sigma_{ixy} t_i z_i \right)$$

$$+ \lambda_4 \left(N_x - \sum_{i=1}^3 \sigma_{ix} t_i \right)$$

$$+ \lambda_5 \left(N_y - \sum_{i=1}^3 \sigma_{iy} t_i \right)$$

$$+ \lambda_6 (N_{xy} - \sum_{i=1}^{3} \sigma_{ixy} t_i)$$

$$+ \lambda_7 (Q_x - \sum_{j=1}^{2} \tau_{jxz} h_j)$$

$$+ \lambda_8 (Q_y - \sum_{j=1}^{2} \tau_{jyz} h_j)$$

$$+ \lambda_9 (N_{x,x} + N_{yx,y})$$

$$+ \lambda_{10} (N_{y,y} + N_{xy,x})$$

$$+ \lambda_{11} \Big[Q_{x,x} + Q_{y,y} + P + (N_x w_{,x} + N_{yx} w_{,y})_{,x}$$

$$+ (N_y w_{,y} + N_{xy} w_{,x})_{,y} \Big]$$

$$+ \lambda_{12} (M_{x,x} + M_{yx,y} - Q_x)$$

$$+ \lambda_{13} (M_{y,y} + M_{xy,x} - Q_y) \Big\} \, dx dy$$

$$- \int_{C_d} \Big[\overline{X} u + \overline{Y} v + (\overline{X} w_{,x} + \overline{Y} w_{,y} + \overline{Q}) w + \overline{M}_x \alpha + \overline{M}_y \beta \Big] ds \qquad (20)$$

It is obvious that the Lagrangian multipliers have physical meanings.
By the law of dimensional homogenity, the Lagrangian multipliers are
some sort of displacements.

2.5   Compatibility Equations

The well known principle of complementary energy states that for all stresses satisfying the equilibrium conditions, the actual state of stress, i.e., the stresses which satisfy the compatibility equations, is such that the complementary energy π^* assumes a stationary value. For problems with small strains and displacements, it can be shown[20] that π^* is a minimum. In order to have extrema for the functional π^* subjected to the constraint conditions, the necessary and sufficient condition is that the first variation of the auxiliary functional must vanish, i.e., $\delta\pi^{**} = 0$. Thus the condition $\delta\pi^{**} = 0$ furnishes the set of compatibility equations.

Taking the first variation and collecting terms, the following expression is obtained:

$$\delta\pi^{**} = A + B + C = 0 \tag{21}$$

where

$$A = \int_0^b \int_0^a \sum_{i=1}^3 \left\{ \delta\sigma_{ix}\left[\frac{t_i}{E_i}(\sigma_{ix} - \nu_i\sigma_{iy}) - \lambda_1 t_i z_i - \lambda_4 t_i\right]\right.$$

$$+ \delta\sigma_{iy}\left[\frac{t_i}{E_i}(\sigma_{iy} - \nu_i\sigma_{ix}) - \lambda_2 t_i z_i - \lambda_5 t_i\right]$$

$$+ \delta\sigma_{ixy}\left[\frac{2t_i(1+\nu_i)}{E_i}\sigma_{ixy} - \lambda_3 t_i z_i - \lambda_6 t_i\right]\right\}$$

$$+ \sum_{j=1}^2 \left\{ \delta\tau_{jxz}(\frac{h_j}{G_{jxz}}\tau_{jxz} - \lambda_7 h_j)\right.$$

$$+ \delta \tau_{jyz} \left(\frac{h_j}{G_{jyz}} \tau_{jyz} - \lambda_8 h_j \right) \Bigg\} \; dxdy \; , \tag{21a}$$

$$B = \int_0^b \int_0^a \Bigg\{ \lambda_1 \delta M_x + \lambda_{12} \delta M_{x,x}$$

$$+ \lambda_2 \delta M_y + \lambda_{13} \delta M_{y,y}$$

$$+ \lambda_3 \delta M_{xy} + \lambda_{12} \delta M_{xy,y} + \lambda_{13} \delta M_{xy,x}$$

$$+ \lambda_4 \delta N_x + \lambda_9 \delta N_{x,x} + \lambda_{11} (w_{,x} \delta N_x)_{,x}$$

$$+ \lambda_5 \delta N_y + \lambda_{10} \delta N_{y,y} + \lambda_{11} (w_{,y} \delta N_y)_{,y}$$

$$+ \lambda_6 \delta N_{xy} + \lambda_9 \delta N_{xy,y} + \lambda_{10} \delta N_{xy,x}$$

$$+ \lambda_{11} (w_{,y} \delta N_{xy})_{,x} + \lambda_{11} (w_{,x} \delta N_{xy})_{,y}$$

$$+ (\lambda_7 - \lambda_{12}) \delta Q_x + \lambda_{11} \delta Q_{x,x}$$

$$+ (\lambda_8 - \lambda_{13}) \delta Q_y + \lambda_{11} \delta Q_{y,y} \Bigg\} dxdy \; , \tag{21b}$$

$$C = - \int_{C_d} \Bigg[u \delta \overline{X} + v \delta \overline{Y} + w(w_{,x} \delta \overline{X} + w_{,y} \delta \overline{Y} + \delta \overline{Q})$$

$$+ \alpha \delta \overline{M}_x + \beta \delta \overline{M}_y \Bigg] ds \tag{21c}$$

The vanishing of $\delta \pi^{**}$ requires the individual vanishing of A, B, and C. By the fundamental lemma of calculus of variations, the vanishing of A furnishes thirteen Euler equations which are the compatibility equations. They are:

$$\frac{t_i}{E_i} (\sigma_{ix} - \nu_i \sigma_{iy}) - \lambda_1 t_i z_i - \lambda_4 t_i = 0 \qquad i = 1, 2, 3 \qquad (22)$$

$$\frac{t_i}{E_i} (\sigma_{iy} - \nu_i \sigma_{ix}) - \lambda_2 t_i z_i - \lambda_5 t_i = 0 \qquad i = 1, 2, 3 \qquad (23)$$

$$\frac{2t_i(1 + \nu_i)}{E_i} \sigma_{ixy} - \lambda_3 t_i z_i - \lambda_6 t_i = 0 \qquad i = 1, 2, 3 \qquad (24)$$

$$\frac{1}{G_{jxz}} \tau_{jxz} - \lambda_7 = 0 \qquad j = 1, 2 \qquad (25)$$

$$\frac{1}{G_{jyz}} \tau_{jyz} - \lambda_8 = 0 \qquad j = 1, 2 \qquad (26)$$

Thus the stresses may be written in terms of the Lagrangian multipliers:

$$\sigma_{ix} = \frac{E_i}{1 - \nu_i^2} \left[z_i(\lambda_1 + \nu_i \lambda_2) + (\lambda_4 + \nu_i \lambda_5) \right] \quad i = 1, 2, 3 \qquad (27)$$

$$\sigma_{iy} = \frac{E_i}{1 - \nu_i^2} \left[z_i(\lambda_2 + \nu_i \lambda_1) + (\lambda_5 + \nu_i \lambda_4) \right] \quad i = 1, 2, 3 \qquad (28)$$

$$\sigma_{ixy} = \frac{E_i}{2(1 + \nu_i)} (z_i \lambda_3 + \lambda_6) \qquad i = 1, 2, 3 \qquad (29)$$

$$\tau_{jxz} = G_{jxz} \lambda_7 \qquad j = 1, 2 \qquad (30)$$

$$\tau_{jyz} = G_{jyz} \lambda_8 \qquad j = 1, 2 \qquad (31)$$

2.6 Lagrangian Multipliers

Integrating by parts equation (21b), then adding the results to equation (21c), and recalling from equation (21) that the sum should vanish:

$$
\begin{aligned}
B + C = \int_o^b \int_o^a \Big\{ & \delta M_x (\lambda_1 - \lambda_{12,x}) \\
& + \delta M_y (\lambda_2 - \lambda_{13,y}) \\
& + \delta M_{xy} (\lambda_3 - \lambda_{12,y} - \lambda_{13,x}) \\
& + \delta N_x (\lambda_4 - \lambda_{9,x} - \lambda_{11,x} w_{,x}) \\
& + \delta N_y (\lambda_5 - \lambda_{10,y} - \lambda_{11,y} w_{,y}) \\
& + \delta N_{xy} (\lambda_6 - \lambda_{9,y} - \lambda_{10,x} - \lambda_{11,x} w_{,y} - \lambda_{11,y} w_{,x}) \\
& + \delta Q_x (\lambda_7 - \lambda_{11,x} - \lambda_{12}) \\
& + \delta Q_y (\lambda_8 - \lambda_{11,y} - \lambda_{13}) \Big\} dx dy \\
& + \int_c \Big[\lambda_9 \delta \overline{X} + \lambda_{10} \delta \overline{Y} + \lambda_{11} (w_{,x} \delta \overline{X} + w_{,y} \delta \overline{Y} + \delta \overline{Q}) \\
& + \lambda_{12} \delta \overline{M}_x + \lambda_{13} \delta \overline{M}_y \Big] ds \\
& - \int_{c_d} \Big[u \delta \overline{X} + v \delta \overline{Y} + w(w_{,x} \delta \overline{X} + w_{,y} \delta \overline{Y} + \delta \overline{Q}) \\
& + \alpha \delta \overline{M}_x + \beta \delta \overline{M}_y \Big] ds = 0
\end{aligned}
\tag{32}
$$

Therefore, the area integral and the line integrals must vanish individually.

It may be observed that the first line integral of equation (32) vanishes on the part of the boundary where surface stresses are specified, hence it has a non-zero value only at that portion of the boundary where the displacements are prescribed, i.e., at c_d. Then on the boundary:

$$\lambda_9 = u \tag{33}$$

$$\lambda_{10} = v \tag{34}$$

$$\lambda_{11} = w \tag{35}$$

$$\lambda_{12} = \alpha \tag{36}$$

$$\lambda_{13} = \beta \tag{37}$$

Since equation (21) also holds for any part of the plate, it follows that the Lagrangian multipliers throughout the plate are related to the generalized displacements by equations (33) to (37).

Introducing equations (33) to (37) in the area integral of equation (32), the other eight Lagrangian multipliers may be expressed in terms of the generalized displacements:

$$\lambda_1 = \alpha_{,x} \tag{38}$$

$$\lambda_2 = \beta_{,y} \tag{39}$$

$$\lambda_3 = \alpha_{,y} + \beta_{,x} \tag{40}$$

$$\lambda_4 = u_{,x} + w_{,x}w_{,x} \tag{41}$$

$$\lambda_5 = v_{,y} + w_{,y}w_{,y} \tag{42}$$

$$\lambda_6 = v_{,x} + u_{,y} + 2w_{,x}w_{,y} \tag{43}$$

$$\lambda_7 = \alpha + w_{,x} \tag{44}$$

$$\lambda_8 = \beta + w_{,y} \tag{45}$$

2.7   Boundary Conditions

From the line integrals of equation (32), the following quantities should take on the corresponding values at the boundaries $x = 0$ and $x = a$:

$$
\begin{aligned}
&u \ \text{ or } \ N_x \\
&v \ \text{ or } \ N_{xy} \\
&w \ \text{ or } \ N_x w_{,x} + N_{xy}w_{,y} + Q_x \\
&\alpha \ \text{ or } \ M_x \\
&\beta \ \text{ or } \ M_{xy}
\end{aligned}
\tag{46}
$$

and at the boundaries $y = 0$ and $y = b$:

$$
\begin{aligned}
&u \ \text{ or } \ N_{yx} \\
&v \ \text{ or } \ N_y
\end{aligned}
$$

$$w \quad \text{or} \quad N_y w_{,y} + N_{yx} w_{,x} + Q_y \qquad (47)$$

$$\alpha \quad \text{or} \quad M_{yx}$$

$$\beta \quad \text{or} \quad M_y$$

2.8 General Analysis

The purpose of this investigation is to find a general solution for the stability of multilayer sandwich plates. At the beginning, thirteen conditions are known: There are eight equations defining stress resultants in terms of stresses and five equilibrium equations in terms of stress resultants. Thus the problem is essentially to find the compatibility conditions and then the governing equations which satisfy both the thirteen known conditions and the compatibility conditions.

Two approaches appear to be adequate for the task. The first approach calls for the formulation of the complementary energy from which the Euler equations are obtained. Then these compatibility conditions can be solved simultaneously with the thirteen known conditions. The second approach introduces an auxiliary functional containing the complementary energy and the thirteen known conditions each multiplied by a corresponding Lagrangian multiplier. Since the known conditions are treated as constraint conditions, the set of Euler equations provided by the auxiliary functional represents the compatibility conditions subjected to the constraint of the thirteen known conditions; in other words, both the equilibrium equations and the stress resultant-stress relations are incorporated in the Euler equations. The second approach takes full advantage of the elegant Lagrange formulation,

reflects a deeper philosophy of structural mechanics, and is thus adopted.

For an n-layer sandwich plate, n = 3, 5, 7 ..., there are $\frac{1}{2}(5n + 1)$ stresses. Except for the fundamental case, where n = 3, the number of stresses is always more than the number of equations given by the stress resultant-stress relations. A direct inversion of the stress resultant-stress relations for the stress-stress resultant relations is not feasible. The technique of taking stress resultants as independent variables in the formulation of an auxiliary functional, which has been employed by many investigators in the study of three-layer sandwich plates, is not applicable to sandwich plates composed of more than three layers. In this investigation, the stresses are taken as independent variables and the stress resultants as dependent variables.

The auxiliary functional approach is both general and unique. For an n-layer sandwich plate there are $21 + \frac{1}{2}(5n + 1)$ unknowns: $\frac{1}{2}(5n + 1)$ stresses, eight stress resultants and thirteen Lagrangian multipliers. Also there are $21 + \frac{1}{2}(5n + 1)$ conditions: $\frac{1}{2}(5n + 1)$ Euler equations, eight relations between Lagrangian multipliers, eight stress resultant-stress relations and five equilibrium equations.

Introducing the thirteen Lagrangian multipliers has important physical significance. Their presence enables the known conditions to be treated as constraint conditions and also provides information which is necessary for the stress-generalized displacement relations.

The successful definition of a proper reference surface, the weighted neutral surface, is essential to the problem. The location of the weighted neutral surface in a multilayer sandwich plate may not be recognized without elaborate exploration of the problem. Due to the proper choice of a reference surface, the elimination processes that lead to the governing equations are reduced substantially.

CHAPTER III

GOVERNING DIFFERENTIAL EQUATIONS

3.1 Property Coefficients

For convenience, the following constants are defined:

$$D = \sum_{i=1}^{3} \frac{E_i t_i z_i^2}{1 - \nu_i^2} \tag{48a}$$

$$\bar{z} = \sum_{i=1}^{3} \frac{E_i t_i z_i}{1 - \nu_i^2} \tag{48b}$$

$$S = \sum_{i=1}^{3} \frac{E_i t_i}{1 - \nu_i^2} \tag{48c}$$

$$\nu_D = \frac{1}{D} \sum_{i=1}^{3} \frac{E_i t_i z_i^2 \nu_i}{1 - \nu_i^2} \tag{48d}$$

$$\varphi = \sum_{i=1}^{3} \frac{E_i t_i z_i \nu_i}{1 - \nu_i^2} \tag{48e}$$

$$\nu_S = \frac{1}{S} \sum_{i=1}^{3} \frac{E_i t_i \nu_i}{1 - \nu_i^2} \tag{48f}$$

$$F_D = \sum_{i=1}^{3} \frac{E_i t_i z_i^2}{2(1+\nu_i)} \qquad (48g)$$

$$F_z = \sum_{i=1}^{3} \frac{E_i t_i z_i}{2(1+\nu_i)} \qquad (48h)$$

$$F_S = \sum_{i=1}^{3} \frac{E_i t_i}{2(1+\nu_i)} \qquad (48i)$$

$$G_x = \sum_{j=1}^{2} G_{jxz} h_j \qquad (48j)$$

$$G_y = \sum_{j=1}^{2} G_{jyz} h_j \qquad (48k)$$

Recalling the definition of weighted neutral surface, $\bar{z} = 0$.

3.2   Stress Resultants

By substituting equations (27) to (31) in equations (2) to (9), stress resultants may be written in terms of Lagrangian multipliers:

$$M_x = D(\lambda_1 + \nu_D \lambda_2) + \varphi \lambda_5 \qquad (49)$$

$$M_y = D(\lambda_2 + \nu_D \lambda_1) + \varphi \lambda_4 \qquad (50)$$

$$M_{xy} = F_D \lambda_3 + F_z \lambda_6 \qquad (51)$$

$$N_x = \varphi \lambda_2 + S(\lambda_4 + \nu_S \lambda_5) \qquad (52)$$

$$N_y = \varphi\lambda_1 + S(\lambda_5 + \nu_S\lambda_4) \tag{53}$$

$$N_{xy} = F_z\lambda_3 + F_S\lambda_6 \tag{54}$$

$$Q_x = G_x\lambda_7 \tag{55}$$

$$Q_y = G_y\lambda_8 \tag{56}$$

From equations (44), (45), (55), and (56),

$$\alpha = \frac{Q_x}{G_x} - w_{,x} = -(w_{,x} - \frac{Q_x}{G_x}) \tag{57}$$

$$\beta = \frac{Q_y}{G_y} - w_{,y} = -(w_{,y} - \frac{Q_y}{G_y}) \tag{58}$$

Hence the generalized displacements α and β are the negative slopes (excluding shear effect) in x and y directions respectively.

Introducing equations (38) to (45) in equations (49) to (56), the stress resultants may be written in terms of generalized displacements:

$$M_x = D(\alpha_{,x} + \nu_D\beta_{,y}) + \varphi(v_{,y} + w_{,y}w_{,y}) \tag{59}$$

$$M_y = D(\beta_{,y} + \nu_D\alpha_{,x}) + \varphi(u_{,x} + w_{,x}w_{,x}) \tag{60}$$

$$M_{xy} = F_D(\alpha_{,y} + \beta_{,x}) + F_z(v_{,x} + u_{,y} + 2w_{,x}w_{,y}) \tag{61}$$

$$N_x = \varphi\beta_{,y} + S\left[u_{,x} + w_{,x}w_{,x} + \nu_S(v_{,y} + w_{,y}w_{,y})\right] \tag{62}$$

$$N_y = \varphi\alpha_{,x} + S\left[v_{,y} + w_{,y}w_{,y} + \nu_S(u_{,x} + w_{,x}w_{,x})\right] \tag{63}$$

$$N_{xy} = F_z(\alpha_{,y} + \beta_{,x}) + F_S(v_{,x} + u_{,y} + 2w_{,x}w_{,y}) \qquad (64)$$

$$Q_x = G_x(\alpha + w_{,x}) \qquad (65)$$

$$Q_y = G_y(\beta + w_{,y}) \qquad (66)$$

3.3 Governing Differential Equations

The governing differential equations for stability of multilayer sandwich plates may be obtained by substituting equations (59) through (66) into equations (10) to (14):

$$D\alpha_{,xx} + F_D\alpha_{,yy} + (D\nu_D + F_D)\beta_{,xy}$$

$$+ F_z u_{,yy} + (\varphi + F_z)v_{,xy}$$

$$+ 2F_z w_{,x}w_{,yy} + 2(\varphi + F_z)w_{,y}w_{,xy} - G_x(\alpha + w_{,x}) = 0 \qquad (67)$$

$$F_D\beta_{,xx} + D\beta_{,yy} + (D\nu_D + F_D)\alpha_{,xy}$$

$$+ F_z v_{,xx} + (\varphi + F_z)u_{,xy}$$

$$+ 2F_z w_{,y}w_{,xx} + 2(\varphi + F_z)w_{,x}w_{,xy} - G_y(\beta + w_{,y}) = 0 \qquad (68)$$

$$D(\alpha_{,xxx} + \beta_{,yyy}) + (D\nu_D + 2F_D)(\alpha_{,xyy} + \beta_{,xxy})$$

$$+ (\varphi + 2F_z)(u_{,xyy} + v_{,xxy})$$

$$+ 2F_z(w_{,x}w_{,xyy} + w_{,y}w_{,xxy} + 2w_{,xx}w_{,yy})$$

$$+ 2(\varphi + F_z)(w_{,x}w_{,xyy} + w_{,y}w_{,xxy} + 2w_{,xy}w_{,xy}) + R = 0 \qquad (69)$$

$$F_z\alpha_{,yy} + (\varphi + F_z)\beta_{,xy}$$

$$+ Su_{,xx} + F_S u_{,yy} + (S\nu_S + F_S)v_{,xy}$$

$$+ 2w_{,x}(Sw_{,xx} + F_S w_{,yy}) + 2w_{,y}w_{,xy}(S\nu_S + F_S) = 0 \quad (70)$$

$$F_z\beta_{,xx} + (\varphi + F_z)\alpha_{,xy}$$

$$+ F_S v_{,xx} + Sv_{,yy} + (S\nu_S + F_S)u_{,xy}$$

$$+ 2w_{,y}(F_S w_{,xx} + Sw_{,yy}) + 2w_{,x}w_{,xy}(S\nu_S + F_S) = 0 \quad (71)$$

3.4   Approximate Solution

It is possible to solve the set of nonlinear differential equations (67) to (71) by the method of successive approximations. First an approximate solution is obtained by some convenient method, then this solution is introduced into the original equations, and the error may be distributed and minimized. This process may be repeated several times until the error is within a preassigned tolerable range.

An approximation solution is suggested by neglecting certain terms in the governing differential equations. By comparing the order of magnitude, it is obvious that F_z and φ are small for practical materials whose Poisson's ratios fall between 0.30 to 0.34. If F_z and φ are neglected, the approximate equations take the form:

$$D\alpha_{,xx} + F_D\alpha_{,yy} + (D\nu_D + F_D)\beta_{,xy} - G_x(\alpha + w_{,x}) = 0 \quad (67a)$$

$$F_D\beta_{,xx} + D\beta_{,yy} + (D\nu_D + F_D)\alpha_{,xy} - G_y(\beta + w_{,y}) = 0 \quad (68a)$$

$$D(\alpha_{,xxx} + \beta_{,yyy}) + (D\nu_D + 2F_D)(\alpha_{,xyy} + \beta_{,xxy}) + R = 0 \quad (69a)$$

By neglecting F_z and φ, the equations are simplified considerably. This simplification not only uncouples equations (67a), (68a), and (69a) from u and v, but also linearizes the equations. It is felt that equations (67a), (68a), and (69a) would give fairly good approximate results. If the solution furnished by these equations is not satisfactory, then a successive approximation technique may be applied.

CHAPTER IV

SPECIALIZATION FOR EQUAL POISSON'S RATIOS

4.1 Poisson's Ratio

The theory presented so far is for a general case in which each facing membrane layer may have a different Poisson's ratio. Due to the presence of a different Poisson's ratio for each facing membrane layer, the governing differential equations may not be simplified to a favorable form. It is observed that the values of Poisson's ratio may fall in a narrow region for materials with appreciably different moduli of elasticity. This is particularly true for materials that are generally used as facing membrane layers, such as steel ($E = 29 \times 10^6$ psi, $\nu = 0.30$) and aluminum ($E = 10 \times 10^6$ psi, $\nu = 0.33$).

In the following, an analysis is presented for the case of equal Poisson's ratios for all facing membrane layers. It is believed that this specialization represents fairly good approximation for the general case.

4.2 Stress Resultants

For the case of all facing membrane layers having an equal Poisson's ratio,

$$\nu_1 = \nu_2 = \nu_3 = \nu \tag{72}$$

the property coefficients defined by equation (55) become:

$$D = \frac{1}{1 - \nu^2} \sum_{i=1}^{3} E_i t_i z_i^2 \qquad (73a)$$

$$\bar{z} = 0 \qquad (73b)$$

$$S = \frac{1}{1 - \nu^2} \sum_{i=1}^{3} E_i t_i \qquad (73c)$$

$$\nu_D = \nu \qquad (73d)$$

$$\varphi = 0 \qquad (73e)$$

$$\nu_S = \nu \qquad (73f)$$

$$F_D = \frac{1 - \nu}{2} D \qquad (73g)$$

$$F_z = 0 \qquad (73h)$$

$$F_S = \frac{1 - \nu}{2} S \qquad (73i)$$

$$G_x = \sum_{j=1}^{2} G_{jxz} h_j \qquad (73j)$$

$$G_y = \sum_{j=1}^{2} G_{jyz} h_j \qquad (73k)$$

And the stress resultants are:

$$M_x = D(\alpha_{,x} + \nu \beta_{,y}) \qquad (74)$$

$$M_y = D(\beta_{,y} + \nu \alpha_{,x}) \tag{75}$$

$$M_{xy} = \frac{1-\nu}{2} D(\alpha_{,y} + \beta_{,x}) \tag{76}$$

$$N_x = S\left[u_{,x} + w_{,x}w_{,x} + \nu(v_{,y} + w_{,y}w_{,y})\right] \tag{77}$$

$$N_y = S\left[v_{,y} + w_{,y}w_{,y} + \nu(u_{,x} + w_{,x}w_{,x})\right] \tag{78}$$

$$N_{xy} = \frac{1-\nu}{2} S(v_{,x} + u_{,y} + 2w_{,x}w_{,y}) \tag{79}$$

$$Q_x = G_x(\alpha + w_{,x}) \tag{80}$$

$$Q_y = G_y(\beta + w_{,y}) \tag{81}$$

It is readily recognized that the property coefficient D is the equivalent flexural rigidity of the multilayer sandwich plate.

Making use of equations (57) and (58), equations (74), (75), and (76) may be expressed as

$$M_x = D\left[\frac{Q_{x,x}}{G_x} + \nu\frac{Q_{y,y}}{G_y} - (w_{,xx} + \nu w_{,yy})\right] \tag{74a}$$

$$M_y = D\left[\frac{Q_{y,y}}{G_y} + \nu\frac{Q_{x,x}}{G_x} - (w_{,yy} + \nu w_{,xx})\right] \tag{75a}$$

$$M_{xy} = \frac{1-\nu}{2} D\left[\frac{Q_{x,y}}{G_x} + \frac{Q_{y,x}}{G_y} - 2w_{,xy}\right] \tag{76a}$$

4.3 Governing Differential Equations

Differentiate equation (74a) with respect to x, equation (76a) with respect to y and substitute into equation (13),

$$Q_x = D\left[\frac{Q_{x,xx}}{G_x} + \frac{1-\nu}{2}\frac{Q_{x,yy}}{G_x} + \frac{1+\nu}{2}\frac{Q_{y,xy}}{G_y} - \nabla^2 w_{,x}\right] \quad (82)$$

where ∇^2 is the Laplacian operator. Similarly from equations (73a), (74a), and (14),

$$Q_y = D\left[\frac{Q_{y,yy}}{G_y} + \frac{1-\nu}{2}\frac{Q_{y,xx}}{G_y} + \frac{1+\nu}{2}\frac{Q_{x,xy}}{G_x} - \nabla^2 w_{,y}\right] \quad (83)$$

Differentiate equation (82) with respect to x, equation (83) with respect to y and substitute into equation (15),

$$\nabla^4 w = \frac{R}{D} + \nabla^2\left(\frac{Q_{x,x}}{G_x} + \frac{Q_{y,y}}{G_y}\right) \quad (84)$$

Differentiate equation (82) with respect to y and equation (83) with respect to x:

$$Q_{x,y} = D\left[\frac{Q_{x,xxy}}{G_x} + \frac{1-\nu}{2}\frac{Q_{x,yyy}}{G_x} + \frac{1+\nu}{2}\frac{Q_{y,xyy}}{G_y} - \nabla^2 w_{,xy}\right] \quad (85)$$

$$Q_{y,x} = D\left[\frac{Q_{y,xyy}}{G_y} + \frac{1-\nu}{2}\frac{Q_{y,xxx}}{G_y} + \frac{1+\nu}{2}\frac{Q_{x,xxy}}{G_x} - \nabla^2 w_{,xy}\right] \quad (86)$$

Subtract equation (86) from equation (85),

$$Q_{x,y} - Q_{y,x} = \frac{1-\nu}{2} D\nabla^2 (\frac{Q_{x,y}}{G_x} - \frac{Q_{y,x}}{G_y}) \tag{87}$$

Differentiate equation (87) with respect to y, equation (15) with respect to x,

$$Q_{x,yy} - Q_{y,xy} = \frac{1-\nu}{2} D\nabla^2 (\frac{Q_{x,yy}}{G_x} - \frac{Q_{y,xy}}{G_y}) \tag{88}$$

$$-Q_{y,xy} = Q_{x,xx} + R_{,x} \tag{89}$$

Substitute equation (89) into equation (88),

$$\frac{1-\nu}{2G_x} D\nabla^2 Q_{x,yy} + \frac{1-\nu}{2G_y} D\nabla^2 Q_{x,xx} - \nabla^2 Q_x = R_{,x} - \frac{1-\nu}{2G_y} D\nabla^2 R_{,x} \tag{90}$$

Let

$$K_x = \frac{(1-\nu)D}{2G_x} \tag{91a}$$

$$K_y = \frac{(1-\nu)D}{2G_y} \tag{91b}$$

After rearranging, equation (90) may be written as:

$$K_x \nabla^4 Q_x + (K_y - K_x)\nabla^2 Q_{x,xx} - \nabla^2 Q_x = R_{,x} - K_y \nabla^2 R_{,x} \tag{92}$$

and similarly,

$$K_y \nabla^4 Q_y + (K_x - K_y)\nabla^2 Q_{y,yy} - \nabla^2 Q_y = R_{,y} - K_x \nabla^2 R_{,y} \tag{93}$$

Equations (84), (92), and (93) are the governing differential equations.

4.4 Deflection Surface

It is possible to develop a governing differential equation in terms of the deflection surface. Such an equation is of the same form as the Lagrange-Navier equation of classical homogeneous thin plates.

Differentiate equation (82) with respect to x, equation (83) with respect to y and add together,

$$Q_{x,x} + Q_{y,y} = D\left[\frac{1}{G_x} \nabla^2 Q_{x,x} + \frac{1}{G_y} \nabla^2 Q_{y,y} - \nabla^4 w\right]$$

Substituting above expression into equation (15), then multiply by $\frac{1}{2}(1 - \nu)$ and rearrange,

$$(K_x - K_y)\nabla^2 Q_{x,x} = \frac{(1 - \nu)D}{2}(\nabla^4 w - \frac{R}{D}) + K_y \nabla^2 R \tag{94}$$

That is,

$$\nabla^2 Q_{x,x} = \frac{(1 - \nu)D}{2(K_x - K_y)}(\nabla^4 w - \frac{R}{D}) + \frac{K_y}{K_x - K_y}\nabla^2 R \tag{95}$$

Differentiate equation (92) with respect to x, making use of equation (95) and rearranging

$$K_x \nabla^6 w + (K_y - K_x)\nabla^4 w_{,xx} - \nabla^4 w$$

$$= \frac{1}{D}\left[-\frac{2K_x K_y}{1 - \nu}\nabla^4 R + (K_x + \frac{2K_y}{1 - \nu})\nabla^2 R + \frac{1 + \nu}{1 - \nu}(K_x - K_y)R_{,xx} - R\right] \tag{96}$$

Equation (96) is the governing differential equation in terms of the deflection surface. Another form of equation (96) in which x and y are balanced is as follows:

$$K_y \nabla^4 w_{,xx} + K_x \nabla^4 w_{,yy} - \nabla^4 w$$

$$= \frac{1}{D}\left[-\frac{2K_x K_y}{1-\nu}\nabla^4 R + (\frac{2K_x}{1-\nu} + K_y)R_{,xx} + (K_x + \frac{2K_y}{1-\nu})R_{,yy} - R\right] \quad (97)$$

The governing differential equation may be written in a form similar to the Lagrange-Navier equation of classical homogenous thin plates:

$$(1 - K_y \frac{\partial^2}{\partial x^2} - K_x \frac{\partial^2}{\partial y^2})\nabla^4 w$$

$$= \left[1 - (\frac{2K_x}{1-\nu} + K_y)\frac{\partial^2}{\partial x^2} - (K_x + \frac{2K_y}{1-\nu})\frac{\partial^2}{\partial y^2} + \frac{2K_x K_y}{1-\nu}\nabla^4\right]\frac{R}{D} \quad (98)$$

4.5 Isotropic Core Layers

For multilayer sandwich plates with isotropic core layers, the elastic constants become:

$$G_x = G_y = G \quad (99)$$

$$K_x = K_y = \frac{(1-\nu)D}{2G} = K \quad (100)$$

Then from equation (94)

$$\frac{(1-\nu)D}{2}(\nabla^4 w - \frac{R}{D}) + K\nabla^2 R = 0 \quad (101)$$

Thus the governing differential equation is

$$\nabla^4 w = \frac{R}{D} - \frac{1}{G} \nabla^2 R \tag{102}$$

CHAPTER V

A SIMPLY SUPPORTED RECTANGULAR PLATE

Two examples of a simply supported rectangular plate are pre-
sented in this chapter. The first example treats a multilayer sand-
wich plate with unequal Poisson's ratios for each facing membrane
layer by the linearized approximate equations. The second example
illustrates a multilayer sandwich plate with equal Poisson's ratios
for all facing membrane layers by the exact equation.

5.1 Example 1

A rectangular multilayer sandwich plate simply supported
along all edges is subjected to a uniform distributed compressive load
of intensity P_x (Fig. 4). Determine the critical load by the linear-
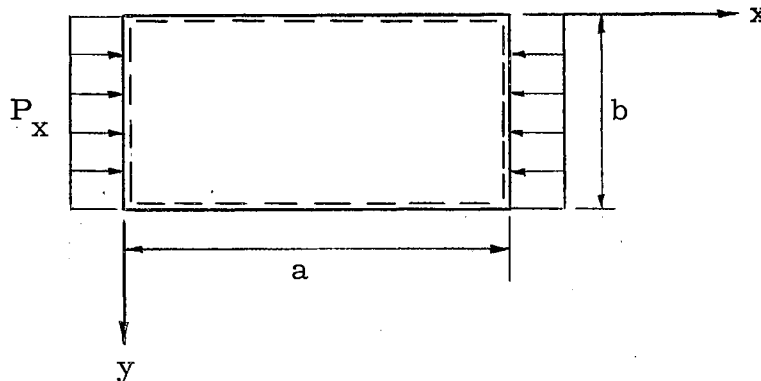ized approximate equations.

Figure 4 - A Simply Supported Rectangular Plate

For this case

$$P = N_y = N_{xy} = 0 \qquad\qquad N_x = - P_x$$

$$R = - P_x w_{,xx}$$

And G_x, G_y, D, ν_D, F_D are constants as defined by equation (48).

The governing equations are

$$D\alpha_{,xx} + F_D\alpha_{,yy} + (D\nu_D + F_D)\beta_{,xy} - G_x(\alpha + w_{,x}) = 0 \qquad (103)$$

$$F_D\beta_{,xx} + D\beta_{,yy} + (D\nu_D + F_D)\alpha_{,xy} - G_y(\beta + w_{,y}) = 0 \qquad (104)$$

$$D(\alpha_{,xxx} + \beta_{,yyy}) + (D\nu_D + 2F_D)(\alpha_{,xyy} + \beta_{,xxy}) + R = 0 \qquad (105)$$

Assume solution of the series form:

$$w = \sum_m \sum_n W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (106)$$

$$\alpha = \sum_m \sum_n A_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad (107)$$

$$\beta = \sum_m \sum_n B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \qquad (108)$$

each term of which satisfies the boundary conditions:

at $\quad x = 0, \quad x = a$

$$w = 0$$

$$M_x = D(\alpha_{,x} + \nu_D\beta_{,y}) = 0 \qquad\qquad (109a)$$

$$\beta = 0$$

at $\quad y = 0, \quad y = b$

$w = 0$

$M_y = D(\beta_{,y} + \nu_D \alpha_{,x}) = 0$ (109b)

$\alpha = 0$

Substitute equations (106), (107), and (108) into equations (103), (104), and (105):

$$\left[D\left(\frac{m\pi}{a}\right)^2 + F_D\left(\frac{n\pi}{b}\right)^2 + G_x\right]A_{mn} + \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)(D\nu_D + F_D)B_{mn}$$

$$+ \frac{m\pi}{a} G_x W_{mn} = 0 \tag{110}$$

$$\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)(D\nu_D + F_D)A_{mn} + \left[D\left(\frac{n\pi}{b}\right)^2 + F_D\left(\frac{m\pi}{a}\right)^2 + G_y\right]B_{mn}$$

$$+ \frac{n\pi}{b} G_y W_{mn} = 0 \tag{111}$$

$$\left[D\left(\frac{m\pi}{a}\right)^3 + (D\nu_D + 2F_D)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2\right]A_{mn}$$

$$+ \left[D\left(\frac{n\pi}{b}\right)^3 + (D\nu_D + 2F_D)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)\right]B_{mn}$$

$$+ \left(\frac{m\pi}{a}\right)^2 P_x W_{mn} = 0 \tag{112}$$

For critical load, the following determinant must vanish:

$$\begin{vmatrix} D\left(\frac{m\pi}{a}\right)^2 + F_D\left(\frac{n\pi}{b}\right)^2 + G_x & \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)(D\nu_D + F_D) & \frac{m\pi}{a} G_x \\ \left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)(D\nu_D + F_D) & D\left(\frac{n\pi}{b}\right)^2 + F_D\left(\frac{m\pi}{a}\right)^2 + G_y & \frac{n\pi}{b} G_y \\ D\left(\frac{m\pi}{a}\right)^3 + (D\nu_D + 2F_D)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2 & D\left(\frac{n\pi}{b}\right)^3 + (D\nu_D + 2F_D)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right) & \left(\frac{m\pi}{a}\right)^2 P_x \end{vmatrix} = 0$$

(113)

For fundamental mode, set $m = n = 1$ in equation (113) and solve for P_x.

5.2 Example 2

A rectangular multilayer sandwich plate simply supported along all edges is subjected to a uniform distributed compressive load of intensity P_x. Assume all facing membrane layers have the same Poisson's ratio. Determine the critical load.

$$P = N_y = N_{xy} = 0 \qquad\qquad N_x = -P_x$$

$$R = -P_x w_{,xx}$$

And D, ν, K_x, K_y are constants which may be calculated from the respective defining equations.

The governing equation is

$$\left[1 - K_y \frac{\partial^2}{\partial x^2} - K_x \frac{\partial^2}{\partial y^2} \right] \nabla^4 w$$

$$+ \left[1 - (\frac{2K_x}{1-\nu} + K_y) \frac{\partial^2}{\partial x^2} - (K_x + \frac{2K_y}{1-\nu}) \frac{\partial^2}{\partial y^2} \right.$$

$$\left. + \frac{2K_x K_y}{1-\nu} \nabla^4 \right] \frac{N_x w_{,xx}}{D} = 0 \tag{114}$$

Assume solution of the series form:

$$w = \sum_m \sum_n W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{b} \tag{115}$$

$$\alpha = \sum_m \sum_n A_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi x}{b} \qquad (116)$$

$$\beta = \sum_m \sum_n B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi x}{b} \qquad (117)$$

each term of which satisfies the boundary conditions of equation (107).

Substitute equations (115), (116), and (117) into equation (114), then

$$\left[1 + K_x (\frac{n\pi}{b})^2 + K_y (\frac{m\pi}{a})^2 \right] \left[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \right]^2 W_{mn}$$

$$= \left\{ \left[1 + K_x (\frac{n\pi}{b})^2 + K_y (\frac{m\pi}{a})^2 \right] \right.$$

$$+ \frac{2}{1-\nu} \left[K_x K_y (\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2})^2 + K_x (\frac{m\pi}{a})^2 \right.$$

$$\left. \left. + K_y (\frac{n\pi}{b})^2 \right] \right\} \frac{P_x}{D} (\frac{m\pi}{a})^2 W_{mn}$$

Hence the critical load is

$$P_x = (\frac{a}{m\pi})^2 \cdot \frac{PA}{PB + PC} \qquad (118)$$

where

$$PA = D \left[1 + K_x (\frac{n\pi}{b})^2 + K_y (\frac{m\pi}{a})^2 \right] \left[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \right]^2$$

$$PB = 1 + K_x \left(\frac{n\pi}{b}\right)^2 + K_y \left(\frac{m\pi}{a}\right)^2$$

$$PC = \frac{2}{1-\nu} \left\{ K_x K_y \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]^2 + K_x \left(\frac{m\pi}{a}\right)^2 + K_y \left(\frac{n\pi}{b}\right)^2 \right\}$$

For fundamental mode, set $m = n = 1$ in equation (118) and calculate P_x.

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Summary and Conclusions

A theory defining the stability of multilayer sandwich plates has been developed in this thesis. The development of the theory consists mainly of the formulation of an auxiliary functional, application of the minimizing principle and the elimination process. The problem is formulated in a complete Lagrange form with stresses taken as independent variables and stress resultants as dependent variables. The elegance of the Lagrange formulation is realized and illustrated in this study.

The successful selection of a reference surface, the weighted neutral surface, leads to the governing equations for the stability of multilayer sandwich plates subjected to assumptions less restricted than most previous works. This weighted neutral surface has not appeared previously in any available literature.

A set of five nonlinear differential equations governing the stability of multilayer sandwich plates is obtained. When specialization for equal Poisson's ratios is made, the governing equation is a sixth order partial differential equation. For isotropic core layers, the governing differential equation becomes a fourth order equation.

Though the mathematical model used is a five-layer sandwich plate, the approach is perfectly general. For sandwich plates composed

of more than five layers, the governing equations are also valid, provided that the indices of property coefficients are adjusted accordingly.

6.2   Comparison of Results

Equation (96) is the governing differential equation for the stability of multilayer sandwich plates with equal Poisson's ratios for all facing membrane layers and orthotropic core layers. For pure bending, replace R by P, then equation (98) reduces to

$$\left[1 - K_y \frac{\partial^2}{\partial x^2} - K_x \frac{\partial^2}{\partial y^2}\right] \nabla^4 w$$

$$= \left[1 - (\frac{2K_x}{1-\nu} + K_y)\frac{\partial^2}{\partial x^2} - (K_x + \frac{2K_y}{1-\nu})\frac{\partial^2}{\partial y^2} + \frac{2K_x K_y}{1-\nu} \nabla^4\right]\frac{P}{D} \tag{119}$$

which is the same as given by B. D. Liaw[4].

For the bending of a three-layer sandwich plate with identical facing layers and an orthotropic core, the equation given by S. Cheng[3] is a special case of equation (119).

For the buckling of three-layer sandwich plates with isotropic core, the equation developed by E. Reissner[6] is a special case of equation (102).

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VITA

Julius Pan Wong

Candidate for the Degree of

Doctor of Philosophy

Thesis: STABILITY OF MULTILAYER SANDWICH PLATES

Major Field: Engineering

Biographical:

  Personal Data: Born May 8, 1937, in Shanghai, China, the son
      of K. S. and P. L. Wong.

  Education: Graduated from Bethel High School, Hong Kong in
      June, 1956. Received the degree of Diploma in Civil
      Engineering from Hong Kong Baptist College, Hong Kong,
      June, 1960. Received the Degree of Master of Science
      in Civil Engineering from Louisiana Polytechnic Institute,
      Ruston, Louisiana, May, 1962. Completed the require-
      ments for the degree of Doctor of Philosophy in May, 1966.

  Professional Experience: Surveying work for T. L. James Co.,
      Ruston, Louisiana, summer, 1961. Graduate Assistant
      at Oklahoma State University, 1962-1966. Associate
      member of ASCE and member of ACI.