A PROBABILISTIC SEQUENCING-DELIVERY MODEL

By

GLADSTONE T. STEVENS, JR.

Bachelor of Science
University of Oklahoma
Norman, Oklahoma
1956

Master of Science
Case Institute of Technology
Cleveland, Ohio
1962

Submitted to the Faculty of the Graduate College
of Oklahoma State University
in partial fulfillment of
the requirements for
the Degree of
DOCTOR OF PHILOSOPHY
May, 1966

A PROBABILISTIC SEQUENCING-DELIVERY
MODEL

Thesis Approved:

James E. Shamblin
Thesis Adviser

Dean of the Graduate College

PREFACE

This dissertation is concerned with obtaining decision rules for a probabilistic sequencing-delivery model. The specific model considered consists of a shipping point with "n" orders waiting for shipment to "m" destinations. Each order has a penalty for lateness and a number of days before it becomes late. Also, each order has a space requirement which may limit the number of orders that can be loaded in a particular vehicle and the number of available vehicles is limited. In addition, the time required to travel between any two points in the system is assumed to be distributed normally.

An optimum solution is obtained for the preceding model. Also, in the case where computation requirements become excessive, an approximate solution to the preceding model is given. These solutions were accomplished by comparing the expected cost of shipping an order at a particular time and the expected cost that would result from delaying shipment for a period of time. In addition, the simplex method of solving linear programming problems was employed.

I would like to take this opportunity to express my appreciation for the assistance and guidance given me by the following members of my committee: Professor Wilson J. Bentley, who was always available for counsel and encouragement; Dr. James E. Shamblin, who gave so generously of his time and whose suggestions and directions were of great value; Dr. Paul E. Torgersen, for his personal interest and encouragement; Dean Rollo E. Venn and Dr. William W. Thompson, for their interest and assistance.

In addition, I would like to thank Velda Davis for her typing excellence and advice.

Finally, I would like to express appreciation to my wife, Jane, and son, Bob, whose understanding, encouragement, and sacrifice were instrumental in the preparation of this dissertation.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

English Symbols

C_m = Cost per mile

D = Number of days late

e = Actual error

e_m = Maximum error

EC_i = Expected cost of order number "i" due to penalty for lateness

EC'_i = Expected cost of order number "i" due to penalty for lateness if shipment is delayed

i = Order number

K = Normal deviate

k = Combination of orders shipped in a particular load

L = Receiving point for a particular route

M = Number of miles

m = Number of destinations

n = Number of orders waiting for shipment

P_i = Penalty of order "i" for being late

R = Receiving point

r_i = Volume ratio of order "i"

S = Shipping point

t = Number of days between available vehicles

T_i = Number of days to due date of order "i"

U = Slack variable

w_i = Weight ratio of order "i"

x_i = Proportion of order "i" shipped

$\bar{x}$ = Mean time to travel between points in the system

z = Objective function

z_i = Route profit

Greek Symbols

α = Area under the normal curve from $-\infty$ to K

α_D = Exact probability of being D days late

ΔEC_i = $EC'_i - EC_i$

σ = Standard deviation of time to travel between points in the system

σ^2 = Variance of time to travel between points in the system

CHAPTER I

INTRODUCTION

Sequencing, as defined by Churchman, Arnoff, and Ackoff (1), refers to the order in which units requiring service are processed, and is a problem that occurs in many operational situations. For example, consider the situation of a number of facilities (machines) and a number of commodities (jobs) which must be processed through some or all of these facilities. The problem in this situation, assuming the processing time for a particular commodity through each specific facility is known, is the determination of the order or sequence by which the commodities at each facility should be processed such that it optimizes the use of the facilities.

According to Starr (2), sequencing models have not been generalized to the extent of other types of scheduling models. Sisson (3) points out that the most common and frequently referred to example of a sequencing problem is the job shop. In a job shop, there is a requirement for processing "n" jobs on "m" facilities. Some of the earlier work on this problem was done by Johnson (4) and Akers and Friedman (5). Johnson developed a procedure for minimizing the total elapsed time for "n" jobs being processed by two machines. Akers and Friedman developed a procedure for minimizing total elapsed time for the case of two jobs and "m" facilities. Neither of these two models permits alternate job routing. Other authors have considered versions of these problems. For example, Mitten (6) (7) determined an analytical solution that minimizes total time to process "n" jobs through two machines with arbitrary start and stop lags, and a common sequence. Sisson (8) provides an excellent review, through 1958, of the work done on the sequencing problem but presents no new concepts.

In the period of 1960-61, the problem of sequencing "n" jobs on "m" facilities received attention from the following writers. Giffler and Thompson (9) developed algorithms for minimizing the length of production schedules by generating and evaluating all possible schedules. This is not generally practical for commercial applications. Thompson (10) considered some of the computational feasibilities of the general problem of "n" jobs and "m" machines but did not present an optimum solution. Heller (11) presented the results from some numerical experiments for a "n" by "m" flow shop. From these experiments, Heller concluded that schedule times are approximately normally distributed for large numbers of jobs. Heller and Logeman (12) developed an algorithm for the generation of feasible schedules and the determination of completion times of the job operations. Rowe (13) developed selective priority rules for processing jobs and a formula for determining the start date of each job for the case of "n" jobs and "m" machines.

More recently, Giffler (14) developed procedures, for an "n" by "m" system, which determines facilities on which tasks should be performed and the time when each task should start. His analysis also considers the delaying of tasks as a result of congestion and the idleness of facilities due to a shortage of tasks. Dudek and Tueton (15) reported the development of an algorithm that yields an optimum sequence of "n" jobs requiring processing through "m" machines when passing is not allowed. However, these authors point out that their algorithm requires additional verification.

From the preceding references, it can be seen that sequencing models have been, primarily, defined in terms of the "n" by "m" job shop and are often treated in combination with related scheduling problems. Optimum analytical solutions have not been reported for the more complex problems in which alternate routes are permissible and especially where machine times and/or costs are of a probabilistic nature, or where machines are subject to breakdowns.

Despite the large amount of reported research on the job-shop sequencing problem, there are other areas where sequencing models are of equal importance but have very little pertinent, reported work. It is the intention of this dissertation to consider one of these undeveloped areas and to formulate appropriate sequencing models to obtain optimal solutions, or, in the case where computation requirements become excessive, a good solution.

Consider the following situation. At some shipping point "s", there are "n" orders waiting for shipment to "R_m" ($m = 1, 2, \ldots$) destinations (receiving points). Associated with each order, there is a penalty, "P_i" ($i = 1, 2, \ldots, n$), for being late which may be independent of the penalties imposed on other orders. These penalties are assessed on a cost per unit time period late basis. Each order has a space and/or weight requirement, "r_i" ($i = 1, 2, \ldots, n$). Unless otherwise stated, this requirement will be considered to be only a space limitation for discussion purposes. This requirement is expressed as a ratio of the total volume available in a vehicle. Each order has some known number of days before it becomes late "T_i" ($i = 1, 2, \ldots, n$). It is assumed, where partial shipment of orders is allowed, that the penalty is some function of the proportion of the order that is not shipped. In this dissertation, a linear relationship is used. Further, it is assumed that the volume ratio varies directly with the proportion of the order shipped. There is some known time period "t" between available vehicles; thus, the decision to delay shipment of an order will result in a delay of at least "t" days duration. In addition, the time required to travel between any two points in the system is distributed according to some known probability distribution. In this dissertation, these values are assumed to be normally distributed with a known mean and standard deviation. It is assumed that as the mileage between points increases the mean and deviation increase. The problem is to determine a procedure which provides a sequencing decision that selects the orders which should be loaded in a particular vehicle and, in some cases, what route should be taken in delivering these orders. An example of the required information to solve a typical problem is shown in Table I and Figure 1.

It will be noted that the described model is, in addition to being a sequencing model, a delivery problem. In general, the solution of one problem is dependent upon the solution of the other problem. Consequently, the most generalized version of the proposed model is actually a sequencing-delivery model.

The delivery type problem has received attention from the following writers. Ferguson and Dantzig (16) considered the problem of allocating several types of aircraft over a number of routes having deterministic flight times. In their problem, they minimize the cost of performing the transportation and the loss of revenue due to the inability to supply the entire demand. Dantzig and Ramser (17)

TABLE I

DATA FOR THE GENERAL MODEL

1 i Order Number	2 Destination	3 P_i Penalty $/Day	4 r_i Volume Ratio	5 T_i Available Days to Due Date
1	A	300	0.3	5
2	B	200	0.1	5
3	A	200	0.3	6
4	E	400	0.1	9
5	A	600	0.1	3
6	D	400	0.1	1
7	B	400	0.2	2
8	E	700	0.1	9
9	D	900	0.8	4
10	D	300	0.3	5
11	E	200	0.5	8
12	A	500	0.5	1
13	E	100	0.6	1
14	C	300	0.2	2
15	C	900	0.2	3
16	B	600	0.4	4
17	C	400	0.1	5
18	B	100	0.9	5
19	D	200	0.9	8
20	A	500	0.5	8

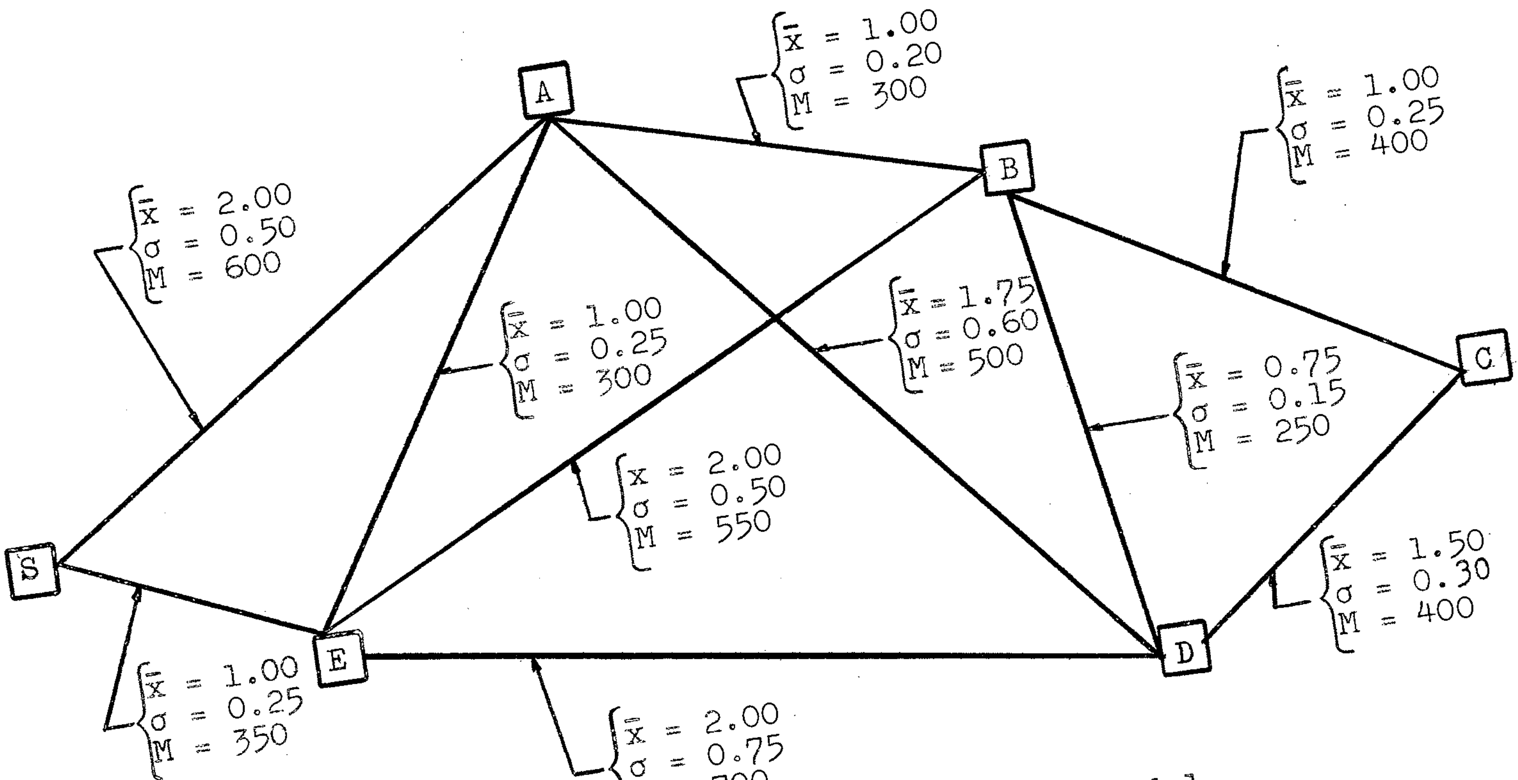

Figure 1. Schematic for the General Model

obtained a "near optimal" solution to the problem of minimizing the total mileage of a fleet of trucks required to service a large number of points from a central depot. Each service point has a demand of one or more specified products. Conway and Maxwell (18) simulated various arbitrary rules for establishing priorities in a system consisting of a network of queues. They arrived at the conclusion that, in general, the shortest-first operation discipline is "best."

Some of the more recent work in the area of delivery problems has been done by the following writers. Dantzig and Johnson (19) obtained iterative solutions for two air network problems. The two problems considered were the determination of the route of maximum payload per hour of flight time and the maximum payload flow through a network with base constraints. In their model, each arc of the air network is characterized on the basis of maximum payload (pounds) and flight time. Payload is considered a function of distance between refueling stops. In addition, all routes are terminated at the same end point. A major difference between this situation and the problem proposed in this dissertation is that Dantzig's and Johnson's model does not include the sequencing of orders nor does it consider any delivery time requirements. In addition, their model assumes deterministic travel times. Clarke and Wright (20) determined an iterative procedure which minimizes the total distance traveled by a fleet of trucks. Their model consists of a number of trucks of various capacities and a number of loads that must be delivered to several points from a central depot. Frank (21) considered the problem of a production-transportation system involving "K" units per unit time of a product requiring transportation to "N" different selling places. The times of transportation are assumed to be deterministic and the demand quantities for each selling place are known. The problem considers the effects of different orderings of servicing the selling points under different demand conditions. Neither Clarke's and Wright's model or Frank's model considered the problem of sequencing the loads under the conditions of due dates, limited transportation facilities, volume capacities, and probabilistic transportation times. These considerations are involved in the proposed model.

Balinski and Quandt (22) have obtained a solution to a delivery problem which has some characteristics that are similar to the proposed model. Their problem is concerned with the transportation, by carrier, to a number of clients at different destinations. The shipper's objective is to minimize the total cost of filling each client's orders. A given carrier could combine a number of orders to be delivered together provided that their destinations lie along one of a number of permissible geographical routes. For each possible destination, a rate schedule is specified relating weights to total costs. Thus, it became simply a distribution problem. This model does not include considerations of the best loads to put on a particular carrier under the conditions of different due dates, different penalty charges for each order, different volume requirements for each order, and the dependency of these items upon the probable transportation times between points in the system. In addition, the possibility of not being able to ship all the orders at one time (carrier facilities are not available at all times in the proposed model) is not included in their model.

These references on the sequencing and delivery problems indicate considerable interest in these two areas. Consequently, it is believed that the proposed model, which combines these two areas, will not only be of interest from an academic standpoint, but should have considerable industrial application. It is not implied that the proposed model is an answer to all sequencing-delivery problems. Rather, it is hoped that the solution to the proposed model and the methods employed to obtain the solution will prove beneficial in the analysis of other models.

In Chapter II, the analysis and solution of the proposed model is given. The approach that has been used in the analysis is to obtain solutions for models which are special cases of the proposed model. These solutions are then expanded and combined to obtain a generalized solution to the proposed model. Each of the special case models consider different aspects of the total problem. Model I is concerned with the sequencing problem and the restraint upon availability of vehicles. Model II includes the volume and/or weight restriction for a particular vehicle. Model III considers the case of the inability to ship partial orders. Model IV considers the problem of carrying orders to more than one destination. Model V is the proposed model (general model) which includes all these considerations. In addition, an approximation for the proposed model is given in Model V-A for cases where computation requirements are excessive. Numerical examples are used extensively in the presentation of each model in order to facilitate an understanding of the solutions. Also, a discussion is included at the end of the analysis of each model.

A summary of the conditions, assumptions, and solutions for each model is given in Chapter III.

The Conclusion, Chapter IV, gives the general conclusions and comments obtained as a result of the analysis and solution of the proposed model. In addition, recommended areas for future research are presented.

CHAPTER II

ANALYSIS

This chapter is concerned with the determination of decision rules for five models. The first four models are special cases of the general model (proposed model). The solutions to these models are utilized in obtaining a solution for the fifth model which is the general model proposed in Chapter I.

Model I: One Destination and One Order Per Vehicle

The conditions of this model follow:

1. There is only one destination (receiving point) and shipping point.
2. The time to reach the destination is distributed normally and the mean and standard deviation of the distribution are known.
3. The vehicle, used to transport an order, is limited to one order per trip and there is a vehicle available every "t" days. In this dissertation, $t = 1$ day is used for illustrative purposes.
4. The penalty for an order being late is assessed on a dollars per day basis.

A schematic for this model is shown in Figure 2 and the required data are given in Table II.

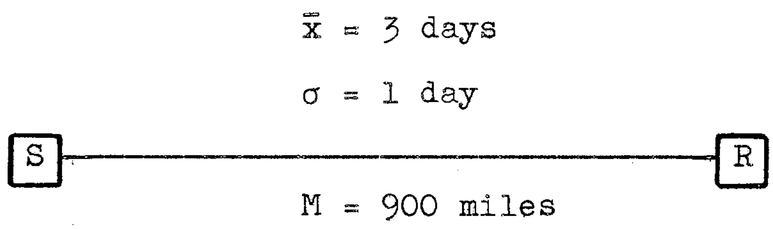

Figure 2. Schematic for Model I

TABLE II

DATA FOR MODEL I

i Order Number	T_i No. of Days to Due Date	P_i Penalty for Lateness $/Day
1	1	100
2	0	50
3	6	75
4	3	200
5	2	300
6	5	50
7	3	100
8	4	200

In arriving at a decision rule for this model and subsequent models, the practical situation of a trucking firm trying to decide the most effective shipping sequence should be considered (where effectiveness is measured by the costs incurred as a result of penalty charges). At a particular time, the decision must be made to ship a specific order immediately or to delay shipment and consider it for shipment as the next truck load. Consequently, the decision to delay shipment of an order results in a decision to delay shipment one day ($t = 1$). Therefore, the probability of being late is increased due to a decrease, by one day, of the number of days to the due date.

Mathematically, the expected cost, EC, due to being late for order "i" can be expressed as

$$EC_i = P_i \sum_{D=1}^{\infty} \alpha_D D \qquad (1)$$

where α_D is the probability of being exactly D days late. If the decision is made to delay shipment one day, the expected cost, EC'_i, is calculated using Equation (1) and decreasing the number of days to the due date by one day. In this manner, using the data given in Table II, the results shown in Table III are obtained. Sample calculations for the results shown in Table III follow.

TABLE III

EXPECTED COSTS DUE TO SHIPMENT BEING DELAYED

i Order Number	EC_i Expected Cost; Due to Order "i" Being Late $	EC'_i Expected Cost Due to Order "i" Being late (Shipment Delayed One Day) $
1	250.10	350.10
2	175.05	225.05
3	0.08	1.80
4	136.60	304.80
5	457.20	750.30
6	1.20	9.15
7	68.30	152.40
8	36.60	136.60

Order No. 1

The probability of being on time or earlier:

$$K = \frac{T_i - \bar{\bar{x}}}{\sigma} = \frac{1-3}{1} = -2.00; \quad \alpha = 0.023.$$

Probability of being one day late or less:

$$K = \frac{2-3}{1} = -1.00; \ \alpha = 0.159.$$

Probability of being two days late or less:

$$K = \frac{3-3}{1} = 0.00; \ \alpha = 0.500.$$

Probability of being three days late or less:

$$K = \frac{4-3}{1} = 1.00; \ \alpha = 0.841.$$

Probability of being four days late or less:

$$K = \frac{5-3}{1} = 2.00; \ \alpha = 0.977.$$

Probability of being five days late or less:

$$K = \frac{6-3}{1} = 3.00; \ \alpha = 0.999.$$

Probability of being six days late or less:

$$K = \frac{7-3}{1} = 4.00; \ \alpha = 1.000.$$

Using Equation (1), the expected cost of Order No. 1 is:

$$EC_1 = P_1 \Sigma \alpha_D D = 100[(.159 - .023)1 + (.500 - .159)2 + (.841 - .500)3 + (.977 - .841)4 + (.999 - .977)5 + (1.000 - .999)6]$$

$$EC_1 = \$250.10.$$

If the decision is made to delay shipment one day, then the expected cost can be calculated in a similar manner with the number of days to the due date decreased by one day as shown below:

Probability of being on time or earlier:

$$K = \frac{(T_i - t) - \bar{x}}{\sigma} = \frac{0-3}{1} = -3.00; \ \alpha = 0.001.$$

Probability of being one day late or less:

$$K = \frac{1-3}{1} = -2.00; \ \alpha = 0.023.$$

Probability of being two days late or less:

$$K = \frac{2-3}{1} = -1.00; \ \alpha = 0.159.$$

Probability of being three days late or less:

$$K = \frac{3-3}{1} = 0.000; \ \alpha = 0.500.$$

Probability of being four days late or less:

$$K = \frac{4-3}{1} = 1.000; \ \alpha = 0.841.$$

Probability of being five days late or less:

$$K = \frac{5-3}{1} = 2.00; \ \alpha = 0.977.$$

Probability of being six days late or less:

$$K = \frac{6-3}{1} = 3.00; \ \alpha = 0.999.$$

Probability of being seven days late or less:

$$K = \frac{7-3}{1} = 4.00; \ \alpha = 1.000.$$

Equation (1) gives:

$$\begin{aligned} EC'_1 = P_1 \Sigma \alpha_D D = {} & 100[(0.023 - .001)1 + (.159 - .023)2 + \\ & (.500 - .159)3 + (.841 - .500)4 + \\ & (.977 - .841)5 + (.999 - .977)6 + \\ & (1.000 - .999)7] \end{aligned}$$

$$EC'_1 = \$350.10.$$

The difference between EC_1 and EC'_1 is \$100.00 and represents an expected savings if Order No. 1 is shipped immediately and not postponed one day. Mathematically, this concept is shown by Equation (2):

$$\Delta EC_i = EC'_i - EC_i . \tag{2}$$

Similar calculations give the results shown in Table IV.

TABLE IV

ΔEC VALUES FOR MODEL I

i Order Number	ΔEC_i \$
1	100.00
2	50.00
3	1.72
4	168.20
5	293.10
6	7.95
7	84.10
8	100.00

In the previous calculations of the ΔEC values and throughout this dissertation, the probability distribution is terminated after a discrete number of days. Actually, the probability distribution assumed, the normal distribution, extends from $-\infty$ to $+\infty$. This simplification results in, essentially, no loss of accuracy since the calculations are accurate within three decimal places.

The optimum decision rule for determining the shipping sequence of orders is the order having the largest ΔEC value should be shipped first. Comparison of the ΔEC values given in Table IV shows that Order No. 5 should be shipped first and the optimum shipping sequence is 5-4-1-8-7-2-6-3. An arbitrary choice is made in determining the sequence of orders that have equal ΔEC values.

<u>Discussion of Model I</u>

In the case of limited vehicles for shipping, orders are shipped on the basis of the largest ΔEC. For example, if there were only three vehicles available at a particular time, Orders 5, 4, and 1 would be shipped. The remaining orders would be considered for shipment on the following day.

For a dynamic case where new orders are received each day, the ΔECs for the new orders should be considered along with the ΔECs for orders remaining from the previous day in determining a sequence.

In determining a decision rule for this model, the assumption has been made that mileage costs are the same for all orders due to there being only one receiving point. Consequently, mileage costs do not effect the sequencing rule. In a later model, this will not be the case.

The time between available shipping vehicles used in calculating EC'_i is taken as one day ($t = 1$) for illustrative purposes. It is possible to reflect a different situation by adjustment of "t" when calculating EC'_i. For example, if vehicles are available every second day, then the starting point for calculating EC'_i is $t = 2$.

Model II: One Destination and Several Orders Per Vehicle

This model has the same conditions as the general model described in Chapter I, except there is only one receiving point. One of the conditions of the general model is that each order has a certain space requirement. Consequently, the number of orders that can be carried by a vehicle is determined by the sum of the volumes required by each order and the total available volume of the vehicle. This condition may be expressed mathematically as:

$$\sum^{k} r_i \leq 1.0 \qquad (3)$$

where

$$r_i = \frac{\text{Total volume required by order "i"}}{\text{Total volume available in vehicle}}. \qquad (4)$$

If "x_i" is the proportion of Order No. i shipped, then it is possible to put this model in terms of a linear programming problem. The objective function is:

$$z = \sum_{i=1}^{n} (\Delta EC_i) x_i \qquad (5)$$

which is subject to the constraints:

$$\sum_{i=1}^{n} r_i x_i \leq 1.0 \qquad (6)$$

$$x_i \leq 1.0 \text{ where } i = 1, 2, \ldots, n. \qquad (7)$$

In Model I, it was shown that the term ΔEC_i represents a possible savings by not delaying the shipment of Order "i". Therefore, the objective function, Equation (5), must be maximized. Equation (6) represents the constraint due to the volume requirements for each order. Equation (7) represents the proportion of an order that should be shipped for an optimum solution.

The solution to this linear programming problem can be obtained by using the simplex method (23). The results of the simplex solution provide the decision rule for determining the optimum combination of orders for the load of a particular vehicle. This procedure is illustrated by using the data given in Table V which is an abridged Table IV with assumed values for "r_i".

TABLE V

DATA FOR MODEL II

i Order Number	ΔEC	r_i
1	100.00	.2
2	50.00	.4
3	1.72	.2
4	168.20	.1
5	293.10	.8

Using Equation (5), the objective function is:

$$z = \sum_{i=1}^{5} (\Delta EC_i)x_i = 100x_1 + 50x_2 + 1.72x_3 + 168.20x_4 + 293.10x_5 \quad (8)$$

and the constraints are:

$$.2x_1 + .4x_2 + .2x_3 + .1x_4 + .8x_5 \leq 1 \quad (9)$$

$$x_1 \leq 1; \; x_2 \leq 1; \; x_3 \leq 1; \; x_4 \leq 1; \; x_5 \leq 1. \quad (10)$$

Adding the slack variables to Equations (9) and (10) gives:

$$.2x_1 + .4x_2 + .2x_3 + .1x_4 + .8x_5 + U_1 = 1 \quad (11)$$

$$x_1 + U_2 = 1 \qquad x_4 + U_5 = 1$$

$$x_2 + U_3 = 1 \qquad x_5 + U_6 = 1. \quad (12)$$

$$x_3 + U_4 = 1$$

The initial tableau for these equations is shown in Figure 3. The iterations required to obtain an optimum solution are shown in Figures 4 through 7. The final tableau (Figure 7) indicates the following solution:

$$x_1 = 1.0$$
$$x_2 = 0.0$$
$$x_3 = 0.0$$
$$x_4 = 1.0$$
$$x_5 = 0.875$$
$$z = \$524.66.$$

The results indicate that all of Orders No. 1 and No. 4 should be shipped now and only 0.875 of Order No. 5. Orders No. 2, No. 3, and the remainder of Order No. 5 should be delayed one day and considered for shipment at that time. If this shipping sequence is followed, the advantage over shipping immediately and not delaying one day is an expected savings of $524.66. The expected savings can be checked by using Equation (5):

$$\begin{aligned} z = \sum_{i=1}^{n} (\Delta EC)_i x_i &= 100x_1 + 50x_2 + 1.75x_3 + 168.20x_4 + \\ &\quad 293.10x_5 \\ &= 100(1) + 50(0) + (1.72)(0) + (168.20)(1) \\ &\qquad + (243.10)(.875) \\ &= \$524.66 \end{aligned}$$

and the volume requirements by Equation (6):

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	b	θ
.2	.4	.2	.1	.8	1	0	0	0	0	0	1	1.25
1	0	0	0	0	0	1	0	0	0	0	1	
0	1	0	0	0	0	0	1	0	0	0	1	
0	0	1	0	0	0	0	0	1	0	0	1	
0	0	0	1	0	0	0	0	0	1	0	1	
0	0	0	0	1	0	0	0	0	0	1	1	1.00
-100	-50	-1.72	-168.20	-293.10	0	0	0	0	0	0	0	

Figure 3. Initial Tableau for Model II

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	b	θ
.2	.4	.2	.1	0	1	0	0	0	0	-.8	.2	2.00
1	0	0	0	0	0	1	0	0	0	0	1	
0	1	0	0	0	0	0	1	0	0	0	1	
0	0	1	0	0	0	0	0	1	0	0	1	
0	0	0	1	0	0	0	0	0	1	0	1	1.00
0	0	0	0	1	0	0	0	0	0	1	1	
-100	-50	-1.72	-168.20	0	0	0	0	0	0	293.10	293.10	

Figure 4. First Iteration for Model II

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	b	θ
.2	.4	.2	0	0	1	0	0	0	-.1	0.8	.1	0.50
1	0	0	0	0	0	1	0	0	0	0	1	1.00
0	1	0	0	0	0	0	1	0	0	0	1	
0	0	1	0	0	0	0	0	1	0	0	1	
0	0	0	1	0	0	0	0	0	1	0	1	
0	0	0	0	1	0	0	0	0	0	1	1	
-100	-50	-1.72	0	0	0	0	0	0	168.10	293.10	461.30	

Figure 5. Second Iteration for Model II

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	b	θ
1	2	1	0	0	5	0	0	0	-.5	-4	.5	
0	-2	-1	0	0	-5	1	0	0	.5	4	.5	0.125
0	1	0	0	0	0	0	1	0	0	0	1	
0	0	1	0	0	0	0	0	1	0	0	1	
0	0	0	1	0	0	0	0	0	1	0	1	
0	0	0	0	1	0	0	0	0	0	1	1	1.00
0	150	98.28	0	0	500	0	0	0	118.20	-106.90	511.30	

Figure 6. Third Iteration for Model II

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	b
1	0	0	0	0	0	1	0	0	0	0	1
0	-.5	-.25	0	0	-1.25	.25	0	0	.125	1	.125
0	1	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	0	0	0	1	0	0	1
0	0	0	1	0	0	0	0	0	1	0	1
0	.5	.25	0	1	1.25	-.25	0	0	-.125	0	.875
0	71.56	84.92	0	0	0	0	0	0	0	0	524.66

Figure 7. Final Tableau for Model II

$$\sum_{i=1}^{n} r_i x_i = .2x_1 + .4x_2 + .2x_3 + .1x_4 + .8x_5$$
$$= .2(1) + .4(0) + .2(0) + .1(1) + (.8)(.875)$$
$$= 1.00.$$

Both of these results agree with the simplex solution and the constraints of the model. Thus, the results of the simplex solution of Equations (5), (6), and (7) provide the optimum decision rule for Model II.

The proof that the preceding procedure provides an optimum solution lies in the inherent capability of the simplex method of solving a linear programming problem. References (23) and (24) state that the optimum solution to a linear programming problem is obtained when the simplex method is employed.

Discussion of Model II

The determination of the decision rule for this model is based on an evaluation of the difference in expected costs of shipping immediately and the delay of shipment by one day. Consequently, a simplex solution is required for each load (the orders placed in one vehicle constitute a load) and, in the case that an order is delayed one day, recalculation of ΔEC for that order. In the case of partial shipment of an order, the remaining part of an order can be considered as another order to be shipped the next day, provided the penalty and volume are adjusted in proportion to the amount of the order remaining. For example, the penalty for the remaining portion of Order No. 5 in the preceding example is:

$$P_5 = (1 - .875)(300) = \$37.50/\text{day}$$

and the volume ratio is:

$$r_5 = (1 - .875)(.8) = 0.1.$$

In this model, as in Model I, the mileage costs are not considered for the same reasons given in the Discussion of Model I.

In this model, a restriction due to the volume of an order was considered. This restriction could have been weight capacity. For example, the restriction

$$\sum_{i=1}^{n} w_i x_i \leq 1 \qquad (13)$$

where

$$w_i = \frac{\text{Weight of order "i"}}{\text{Weight capacity of a vehicle}} \qquad (14)$$

could replace Equation (6) for a case where weight considerations are of greater importance than volume requirements. For a case where weight and volume considerations are both important, it would be possible to impose, simultaneously, both restrictions.

Model III: Shipment of Partial Orders Not Allowed

In some instances, the shipment of partial orders is not possible or allowed because of physical requirements or penalty stipulations. Therefore, this model has the same conditions as Model II with the added condition that shipment of partial orders is not possible. Mathematically, this model is expressed by the objective equation:

$$z = \sum_{i=1}^{n} (\Delta EC)_i x_i \text{ (maximize)} \tag{15}$$

and by the constraints

$$\sum_{i=1}^{n} r_i x_i \leq 1.0 \tag{16}$$

$$x_i = 0 \text{ or } 1.0 \text{ for } i = 1, 2, \ldots, n. \tag{17}$$

The solution of Equations (15), (16), and (17) provides the optimum decision rule for this model. However, in order to obtain a solution to these equations, it is necessary to employ the concepts of an integer programming technique advanced by Gomory and Baumol (25). Their concepts and the procedure to obtain a decision rule for this model are illustrated using the data given in Table V (page 22).

The objective function which must be a maximum is:

$$z = \sum_{i=1}^{n} (\Delta EC)_i x_i = 100x_1 + 50x_2 + 1.72x_3 + 168.20x_4 + 293.10x_5 \qquad (18)$$

and the constraints are

$$0.2x_1 + 0.4x_2 + 0.2x_3 + 0.1x_4 + 0.8x_5 \leq 1.0 \qquad (19)$$

$$x_i = 0 \text{ or } 1.0 \text{ for } i = 1, 2, \ldots, 5. \qquad (20)$$

In obtaining a simplex solution to Equations (18), (19), and (20) that yields only integers for the x_i values, the constraints given in Equations (19) and (20) must be modified in order that this requirement is met. This can be accomplished by considering that if x_i can only have a value of 0 or 1, then Equation (19) can be written as

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 4.0, \qquad (21)$$

since the maximum number of orders that can be shipped is four (Orders No. 1, No. 2, No. 3, and No. 4). Otherwise, the sum of the volume ratios is greater than one and Equation (19) is not satisfied. In addition, and by the same reasoning, the constraint

$$x_2 + x_5 \leq 1.0 \qquad (22)$$

must also be satisfied.

In order to insure that an x_i value does not exceed

1.0, the constraints

$$x_i \leq 1.0 \quad \text{for} \quad i = 1, 2, \ldots, 5 \qquad (23)$$

must also be added.

The initial and final tableaus for a simplex solution of Equations (18), (21), (22), and (23) are shown in Figures 8 and 9. The final tableau indicated the solution

$$x_1 = 1 \qquad x_4 = 1$$
$$x_2 = 0 \qquad x_5 = 1$$
$$x_3 = 1 \qquad z = \$563.02.$$

Substituting these x_i values into Equation (19) results in $\Sigma r_i x_i = 1.3$. This result does not satisfy the constraint $\Sigma r_i x_i \leq 1.0$. Suggesting that the constraint

$$x_1 + x_3 + x_4 + x_5 \leq 3 \qquad (24)$$

must be included in the simplex solution. The initial and final tableaus for a simplex solution with Equation (24) included are shown in Figures 10 and 11. The final tableau indicates the solution

$$x_1 = 1 \qquad x_4 = 1$$
$$x_2 = 0 \qquad x_5 = 1$$
$$x_3 = 0 \qquad z = \$561.30.$$

Substituting these values into Equation (19) gives $\Sigma r_i x_i = 1.1$. This result does not satisfy the equation

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	U_7	b
1	1	1	1	1	1	0	0	0	0	0	0	4
0	1	0	0	1	0	1	0	0	0	0	0	1
1	0	0	0	0	0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	0	0	0	0	1	0	0	1
0	0	0	1	0	0	0	0	0	0	1	0	1
0	0	0	0	1	0	0	0	0	0	0	1	1
-100	-50	-1.72	-168.20	-293.10	0	0	0	0	0	0	0	0

Figure 8. Initial Tableau for Model III

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	U_7	b
0	0	0	0	0	1	-1	-1	0	-1	-1	0	0
0	1	0	0	1	0	1	0	0	0	0	0	1
1	0	0	0	0	0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	0	0	0	0	1	0	0	1
0	0	0	1	0	0	0	0	0	0	1	0	1
0	-1	0	0	0	0	-1	0	0	0	0	1	1
0	243.10	0	0	0	0	293.10	100	0	1.72	168.20	0	563.02

Figure 9. Final Tableau for Model III

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	b
1	1	1	1	1	1	0	0	0	0	0	0	0	4
0	1	0	0	1	0	1	0	0	0	0	0	0	1
1	0	1	1	1	0	0	1	0	0	0	0	0	3
1	0	0	0	0	0	0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	0	0	0	0	0	1	0	0	1
0	0	0	1	0	0	0	0	0	0	0	1	0	1
0	0	0	0	1	0	0	0	0	0	0	0	1	1
-100	-50	-1.72	-168.20	-293.10	0	0	0	0	0	0	0	0	0

Figure 10. Initial Tableau for Model III
(Constraint $x_1 + x_3 + x_4 + x_5 \leq 3$ Included)

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	b
0	1	0	0	0	1	0	-1	0	0	0	0	0	1
0	1	0	0	1	0	1	0	0	0	0	0	0	1
1	-1	1	0	0	0	-1	1	0	0	0	-1	0	1
0	1	-1	0	0	0	1	-1	1	0	0	-1	0	1
0	1	0	0	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	0	0	0	0	0	1	0	0	1
0	0	0	1	0	0	0	0	0	0	0	1	0	1
0	-1	0	0	0	0	0	-1	0	0	0	0	1	1
0	143.10	98.28	0	0	0	193.10	100	0	0	0	68.20	0	561.30

Figure 11. Final Tableau for Model III
(Constraint $x_1 + x_3 + x_4 + x_5 \leq 3$ Included)

$\Sigma r_i x_i \leq 1.0$ and suggests that the constraint

$$x_1 + x_4 + x_5 \leq 2 \qquad (25)$$

must be added. Adding this constraint gives the initial and final tableaus shown in Figures 12 and 13. The final tableau of this simplex solution indicates the solution

$x_1 = 0$	$x_4 = 1$
$x_2 = 0$	$x_5 = 1$
$x_3 = 1$	$z = \$463.02$

which also violates the constraint given by Equation (19). Therefore, the constraint

$$x_3 + x_4 + x_5 \leq 2 \qquad (26)$$

must also be added.

The initial and final tableaus for the simplex solution which includes Equation (26) are shown in Figures 14 and 15. The final tableau indicates the solution

$x_1 = 0$	$x_4 = 1$
$x_2 = 0$	$x_5 = 1$
$x_3 = 0$	$z = \$461.30$

which satisfies all the constraints and is the optimum solution.

The procedure shown in the previous calculations yields an optimum decision rule for this model. The proof

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	b
1	1	1	1	1	1	0	0	0	0	0	0	0	0	4
0	1	0	0	1	0	1	0	0	0	0	0	0	0	1
1	0	1	1	1	0	0	1	0	0	0	0	0	0	3
1	0	0	1	1	0	0	0	1	0	0	0	0	0	2
1	0	0	0	0	0	0	0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
0	0	0	1	0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	1	0	0	0	0	0	0	0	0	1	1
-100	-50	-1.72	-168.20	-293.10	0	0	0	0	0	0	0	0	0	0

Figure 12. Initial Tableau for Model III

(Constraint $x_1 + x_4 + x_5 \leq 2$ Included)

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	b
0	1	0	0	0	1	0	0	-1	0	0	-1	0	0	1
0	1	0	0	1	0	1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	1	-1	0	0	-1	0	0	1
1	-1	0	1	0	0	-1	0	1	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
-1	1	0	0	0	0	1	0	-1	0	0	0	1	0	0
0	-1	0	0	0	0	-1	0	0	0	0	0	0	1	0
68.20	143.10	0	0	0	0	124.90	0	168.20	0	0	1.72	0	0	463.02

Figure 13. Final Tableau for Model III

(Constraint $x_1 + x_4 + x_5 \leq 2$ Included)

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	b
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	4
0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	1
1	0	1	1	1	0	0	1	0	0	0	0	0	0	0	3
1	0	0	1	1	0	0	0	1	0	0	0	0	0	0	2
0	0	1	1	1	0	0	0	0	1	0	0	0	0	0	2
1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1
0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	1
0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1
-100	-50	-1.72	-168.20	-293.10	0	0	0	0	0	0	0	0	0	0	0

Figure 14. Initial Tableau for Model III

(Constraint $x_3 + x_4 + x_5 \leq 2$ Included)

x_1	x_2	x_3	x_4	x_5	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	b
1	1	0	0	0	1	0	0	0	-1	0	0	0	0	0	2
0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	1	0	-1	0	0	0	0	0	1
1	-1	0	1	0	0	-1	0	1	0	0	0	0	0	0	1
-1	0	1	0	0	0	0	0	-1	1	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0	1	0	0	0	1
1	0	0	0	0	0	0	0	1	-1	0	0	1	0	0	1
-1	1	0	0	0	0	1	0	-1	0	0	0	0	1	0	0
0	-1	0	0	0	0	-1	0	0	0	0	0	0	0	0	0
66.48	74.90	0	0	0	0	0	0	166.48	1.72	0	0	0	0	0	461.30

Figure 15. Final Tableau for Model III

(Constraint $x_3 + x_4 + x_5 \leq 2$ Included)

that the solution is optimum lies in the basic capability of a simplex solution. The simplex solution of a linear programming problem provides an optimum solution for a given objective function and set of constraints. The procedure used in this model is to obtain a simplex solution and check the results of this solution to insure that the constraints are not violated. Consequently, an optimum solution is obtained.

Discussion of Model III

The procedure used to obtain an optimum decision rule for this model employed a technique of integer programming which actually "generates" its own restrictions. In the example used to illustrate the procedure, many of the restrictions, determined after each simplex solution, could have been determined by inspection of Equation (19) and included in the initial simplex solution. This would have resulted in a reduction of the number of simplex solutions required to obtain the optimum solution. This was not done in the example in order to illustrate the procedure. From the standpoint of reducing the number of simplex solutions required to obtain an optimum solution, it is important that as many restrictions, of the types shown in Equations (22) through (26), are included in the initial solution. However, if pertinent restrictions are omitted, they will become evident when checking a solution to determine if it satisfies the constraints as shown in the illustrative example.

Determination of the initial constraints for use in the first simplex solution can be accomplished by inspection for a small number of orders. For the case of a large number of orders, the constraints can be determined by computer search techniques that determine combinations of the volume ratios which impose restrictions.

A weight constraint can also be applied to this model by the same method used for the volume constraint.

The remarks made in the Discussion of Model II concerning mileage considerations are also applicable to this model.

Model IV: A Predetermined Route

This model is the general case described in Chapter I with the modification that the vehicle must travel a predetermined route. This is the case, for example, of cargo being carried by a passenger bus or train.

The data given in Table VI (abridged data from Table I) is used for illustrative purposes and it is assumed that the prescribed route (Figure 1, page 7) is S-A-B-E-S.

The procedure used in obtaining a decision rule for this model is similar to the procedure used in Model II. That is, calculate the ΔECs for all orders and use the simplex method to obtain the orders which must be shipped for an optimum solution. The essential difference in this model and Model II is in the $\bar{x}$ and σ values used in the calculation of EC and EC′ for each order.

TABLE VI

DATA FOR MODEL IV

i Order Number	Destination	P_i Penalty $/Day	r_i Volume Ratio	T_i Available Days to Due Date
1	A	300	0.3	5
3	A	200	0.3	6
5	A	600	0.1	3
12	A	500	0.5	1
20	A	500	0.5	8
2	B	200	0.1	5
7	B	400	0.2	2
16	B	600	0.4	4
4	E	400	0.1	9
8	E	700	0.1	9
11	E	200	0.5	8
13	E	100	0.6	1

The values of $\bar{x}$ and σ used in calculation of EC and EC′ for each order are based on the following three statistical theorems (26):

1. The expected value of a linear combination of random variables is equal to the linear combination of the expected values.
2. The variance of a linear combination of independent random variables is equal to the sum of the variances.
3. The resulting distribution of a combination of normally distributed variables will be normal.

Mathematically, the first two theorems can be expressed as

$$\bar{x}_L = \sum_L \bar{x} \tag{27}$$

$$\sigma_L^2 = \sum_L \sigma^2 \tag{28}$$

where "L" denotes the receiving point for a particular route. Since the route has been established for this model, Equations (27) and (28) are used to calculate the $\bar{x}$ and σ values for all orders going to a particular point. The results of these calculations are summarized in Table VII.

TABLE VII

$\bar{x}$ AND σ VALUES FOR RECEIVING POINTS

Receiving Point	$\bar{x}_L$	σ_L
A	2.0	0.500
B	3.0	0.538
E	5.0	0.735

Sample calculations for $\bar{x}_B$ and σ_B are shown below:

$$\bar{x}_B = \Sigma\,\bar{x} = 2 + 1 = 3$$

$$\sigma_B^2 = \Sigma\sigma^2 = (0.5)^2 + (0.2)^2 = 0.29$$

$$\sigma = \sqrt{0.29} = 0.538.$$

The EC and EC′ values, shown in Table VIII, are calculated by the same method shown in Model I.

The optimum decision rule for this model is based on a simplex solution of the data given in Table VIII. Using Equation (5), the objective function is:

$$z = \sum_{i=1}^{n} (\Delta EC_i)x_i = 300x_5 + 500x_{12} + 6.80x_2$$

$$+ 400x_7 + 306x_{16} + 0.60x_{11} + 100x_{13} \qquad (29)$$

and the constraints, Equations (6) and (7), are:

TABLE VIII

ΔEC VALUES FOR MODEL IV

i Order Number	Destination	r_i Volume Ratio	EC	EC′	ΔEC
1	A	0.3	0	0	0
3	A	0.3	0	0	0
5	A	0.1	13.80	313.80	300.00
12	A	0.5	750.00	1250.00	500.00
20	A	0.5	0	0	0
2	B	0.1	0	6.80	6.80
7	B	0.2	600.00	1000.00	400.00
16	B	0.4	14.40	320.40	306.00
4	E	0.1	0	0	0
8	E	0.1	0	0	0
11	E	0.5	0	0.60	0.60
13	E	0.6	450.00	550.00	100.00

$$0.1x_5 + 0.5x_{12} + 0.1x_2 + 0.2x_7 + 0.4x_{16} + 0.5x_{11} + 0.6x_{13} \leq 1 \qquad (30)$$

$$\begin{array}{ll} x_5 \leq 1 & x_{16} \leq 1 \\ x_{12} \leq 1 & x_{11} \leq 1 \\ x_2 \leq 1 & x_{13} \leq 1 . \\ x_7 \leq 1 & \end{array} \qquad (31)$$

The simplex solution to these equations is shown in Figures 16 through 20 and indicate the solution

$$\begin{aligned} x_5 &= 1.00 \\ x_{12} &= 1.00 \\ x_7 &= 1.00 \\ x_{16} &= 0.50 \\ \Sigma\Delta EC &= \$1353.00. \end{aligned}$$

Therefore, the decision to ship all of Orders No. 5, No. 12, No. 7, and only 0.50 of Order No. 16 is an optimum decision. The remaining orders are considered for the next shipment with the penalty and the volume ratio of Order No. 16 reduced by 0.50.

Discussion of Model IV

The significant difference in this model and the models previously considered was the determination of the $\bar{\bar{x}}$ and σ values used in the calculation of ΔEC for each order; otherwise the calculations were the same as Model II.

The condition that shipment of partial orders is not allowed could be imposed on this model. The determination

x_5	x_{12}	x_2	x_7	x_{16}	x_{11}	x_{13}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	b	θ
.1	.5	.1	.2	.4	.5	.6	1	0	0	0	0	0	0	0	1	2.00
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1.00
0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1	
0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	
0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	
0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	
-300	-500	-6.80	-400	-306	-0.60	-100	0	0	0	0	0	0	0	0	0	

Figure 16. Initial Tableau for Model IV

x_5	x_{12}	x_2	x_7	x_{16}	x_{11}	x_{13}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	b	θ
.1	0	.1	.2	.4	.5	.6	1	0	-.5	0	0	0	0	0	.5	2.50
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	
0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1	
0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	1.00
0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	
0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	
-300	0	-6.80	-400	-306	-0.60	-100	0	0	500	0	0	0	0	0	500	

Figure 17. First Iteration for Model IV

x_5	x_{12}	x_2	x_7	x_{16}	x_{11}	x_{13}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	b	θ
.1	0	.1	0	.4	.5	.6	1	0	-.5	0	-.2	0	0	0	.3	0.75
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	
0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1	
0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	
0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	1.00
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	
0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	
-300	0	-6.80	0	-306	-0.60	-100	0	0	500	0	400	0	0	0	900	

Figure 18. Second Iteration for Model IV

x_5	x_{12}	x_2	x_7	x_{16}	x_{11}	x_{13}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	b	θ
.25	0	.25	0	1	1.25	1.50	2.5	0	-1.25	0	-.5	0	0	0	.75	3.00
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1.00
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	
0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1	
0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	
-.25	0	-.25	0	0	-1.25	-1.50	-2.5	0	1.25	0	.5	1	0	0	.25	
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	
0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	
-223.50	0	69.70	0	0	381.90	359	765	0	117.50	0	247	0	0	0	1129.50	

Figure 19. Third Iteration for Model IV

x_5	x_{12}	x_2	x_7	x_{16}	x_{11}	x_{13}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	b
0	0	.25	0	1	1.25	1.50	2.5	-.25	-1.25	0	-.5	0	0	0	.50
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1
0	0	-.25	0	0	-1.25	-1.50	-2.5	0	1.25	0	.5	1	0	0	.50
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1
0	0	69.70	0	0	381.90	359	765	223.50	117.50	0	247	0	0	0	1353

Figure 20. Final Tableau for Model IV

of a decision rule would be the same as described in Model III.

The remarks made in the Discussion of Model II concerning mileage costs and a weight restriction are also applicable to this model.

Model V: The General Model

The decision rules developed in Models I, II, III, and IV provide a basis for the determination of a decision rule for the general model described in Chapter I.

The facet of this model which makes it more complex than the preceding models is the dependency of the $\bar{x}$ and σ values, used in the calculation of ΔEC for each order, and mileage upon the route taken to reach different receiving points. This dependency of the $\bar{x}$ and σ values and the manner in which they were calculated for a particular route was shown in Model IV.

An optimum solution to the general case is obtained by evaluating all possible routes by the same methods shown in Model IV. Since different routes involve different mileage costs, a method must be included in the decision rule to account for mileage costs. The inclusion of mileage costs in the decision rule is accomplished by defining z_1 as the expected route profit for a particular route which is equal to the sum of the ΔECs for the orders that are to be shipped by a particular route minus the mileage cost. The route profit is defined by Equation (32):

$$z_1 = \sum^{k} \Delta EC_i - (C_m M) \qquad (32)$$

where C_m is the cost per mile and M is the shortest round trip mileage that connects the desired receiving points. The route having the largest value of "z_1" is the route chosen. Thus, Equation (32) becomes the decision rule for the general model since it determines the route that should be taken, the orders that should be shipped, and the expected route profit.

The decision rule for this model is demonstrated using the simple example shown in Figure 21 and the data given in Table IX. Considering Figure 21, it can be seen that possible routes are S-A-S, S-E-S, S-A-E-S, and S-E-A-S. The data that applies to these different routes follows.

Route S-A-S

Mileage = 1200

Destination	$\bar{x}$	σ
A	2.0	0.5

Order Number	Destination	EC	EC′	ΔEC
1	A	0	0	0
3	A	0	0	0
5	A	13.80	313.80	300.
12	A	750.	1250.	500.
20	A	0	0	0

Route S-E-S

Mileage = 700

Destination	$\bar{x}$	σ
E	1.0	0.25

Order Number	Destination	EC	EC′	ΔEC
4	E	0	0	0
8	E	0	0	0
11	E	0	0	0
13	E	50.	150.	100.

Route S-A-E-S

Mileage = 1250

Destination	$\bar{x}$	σ
A	2.00	0.50
E	3.00	0.56

Order Number	Destination	EC	EC′	ΔEC
1	A	0	0	0
3	A	0	0	0
4	E	0	0	0
5	A	13.80	313.80	300.
8	E	0	0	0
11	E	0	0	0
12	A	750.	1250.	500.
13	E	250.	350.	100.
20	A	0	0	0

Route S-E-A-S

Mileage = 1250

Destination	$\bar{x}$	σ
A	2.00	0.35
E	1.00	0.25

Order Number	Destination	EC	EC′	ΔEC
1	A	0	0	0
3	A	0	0	0
4	E	0	0	0
5	A	1.20	301.20	300.
8	E	0	0	0
11	E	0	0	0
12	A	900.	1500.	600.
13	E	50.	150.	100.
20	A	0	0	0

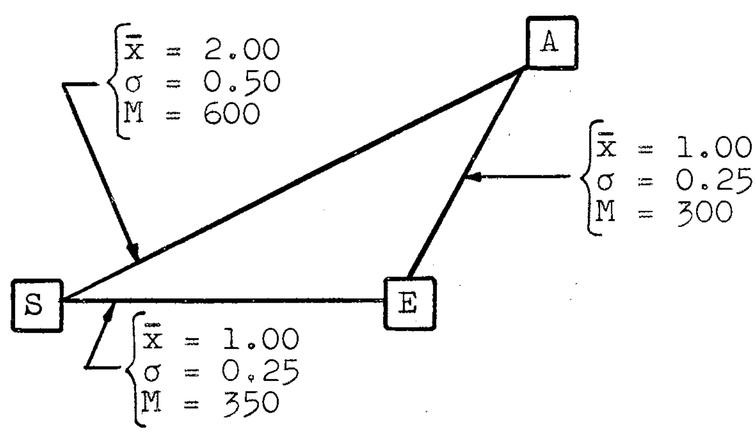

Figure 21. Schematic for Model V

TABLE IX

DATA FOR MODEL V

i Order Number	Destination	P_i Penalty \$/Day	r_i Volume Ratio	T_i Available Days to Due Date
1	A	300	0.3	5
3	A	200	0.3	6
4	E	400	0.1	9
5	A	600	0.1	3
8	E	700	0.1	9
11	E	200	0.5	8
12	A	500	0.5	1
13	E	100	0.6	1
20	A	500	0.5	8

Although it is possible to determine the optimum combination of orders to ship by inspection of the preceding equations, a simplex solution for each route is done in order to illustrate the procedure that is followed in a complex problem. The simplex solutions for the four possible routes are shown in Figures 22 through 29 and indicate the following solutions:

Route S-A-S

$x_5 = 1$

$x_{12} = 1$

$z = 800$

Route S-E-S

$x_{13} = 1$

$z = 100$

x_5	x_{12}	U_1	U_4	U_5	b
.1	.5	1	0	0	1
1	0	0	1	0	1
0	1	0	0	1	1
-300	-500	0	0	0	0

Figure 22. Initial Tableau for Route S-A-S

x_5	x_{12}	U_1	U_4	U_5	b
0	0	1	-.1	-.5	.4
1	0	0	1	0	1
0	1	0	0	1	1
0	0	0	300	500	800

Figure 23. Final Tableau for Route S-A-S

x_{13}	U_1	U_4	b
.6	1	0	1
1	0	1	1
-100	0	0	0

Figure 24. Initial Tableau for Route S-E-S

x_{13}	U_1	U_4	b
0	1	-.6	.4
1	0	1	1
0	0	100	100

Figure 25. Final Tableau for Route S-E-S

x_5	x_{12}	x_{13}	U_1	U_5	U_8	U_9	b
.1	.5	.6	1	0	0	0	1
1	0	0	0	1	0	0	1
0	1	0	0	0	1	0	1
0	0	1	0	0	0	1	1
-300	-500	-100	0	0	0	0	0

Figure 26. Initial Tableau for Route S-A-E-S

x_5	x_{12}	x_{13}	U_1	U_5	U_8	U_9	b
0	0	1	$\frac{10}{6}$	$-\frac{1}{6}$	$-\frac{5}{6}$	0	$\frac{4}{6}$
1	0	0	0	1	0	0	1
0	1	0	0	0	1	0	1
0	0	0	$-\frac{10}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	1	$\frac{2}{6}$
0	0	0	$\frac{1000}{6}$	$\frac{1700}{6}$	$\frac{2500}{6}$	0	$\frac{5200}{6}$

Figure 27. Final Tableau for Route S-A-E-S

x_5	x_{12}	x_{13}	U_1	U_5	U_7	U_8	b
.1	.5	.6	1	0	0	0	1
1	0	0	0	1	0	0	1
0	1	0	0	0	1	0	1
0	0	1	0	0	0	1	1
-300	-600	-100	0	0	0	0	0

Figure 28. Initial Tableau for Route S-E-A-S

x_5	x_{12}	x_{13}	U_1	U_5	U_7	U_8	b
0	0	1	$\frac{10}{6}$	$-\frac{1}{6}$	$-\frac{5}{6}$	0	$\frac{4}{6}$
1	0	0	0	1	0	0	1
0	1	0	0	0	1	0	1
0	0	0	$-\frac{10}{6}$	$\frac{1}{6}$	$\frac{5}{6}$	1	$\frac{2}{6}$
0	0	0	$\frac{1000}{6}$	$\frac{1700}{6}$	$\frac{3100}{6}$	0	$\frac{5800}{6}$

Figure 29. Final Tableau for Route S-E-A-S

Route S-A-E-S

$x_5 = 1$

$x_{12} = 1$

$x_{13} = 4/6$

$z = \frac{5200}{6}$

Route S-E-A-S

$x_5 = 1$

$x_{12} = 1$

$x_{13} = 4/6$

$z = \frac{5800}{6}$

Assuming a value (0.2) for C_m, the route profit for each route is:

Route S-A-S

$$z_1 = 800 - (0.2)(1200) = \$560.00$$

Route S-E-S

$$z_1 = 100 - (0.2)(700) = -\$40.00$$

Route S-A-E-S

$$z_1 = \frac{5200}{6} - (0.2)(1250) = \frac{3700}{6} = \$616.67$$

Route S-E-A-S

$$z_1 = \frac{5800}{6} - (0.2)(1250) = \frac{4300}{6} = \$716.67.$$

The results of the preceding calculations indicate that the optimum route is S-E-A-S, all of Orders No. 5 and No. 12 require shipment 4/6 of Order No. 13 requires shipment, and the expected route profit is $716.67.

Discussion of Model V

A decision rule has been obtained for the general case, which provides an optimum solution. The inclusion of different mileage costs, due to the different routes that might be taken, has been accomplished by defining a route profit (Equation (32)) which must be evaluated for all possible routes.

In this model, as in previous models, the added condition of not allowing partial shipment of orders could have been included. In addition, a weight restriction could have been included, if required.

Although the solution to this model involves considerable computations, it is particularly applicable for the condition of a fixed number of receiving points at fixed locations. In this case, all possible routes to the receiving points must be determined only once and, consequently, the amount of calculations is reduced. The $\bar{\bar{x}}$ and σ values for the different receiving points can be established for each route. The $\bar{x}$ and σ values can be adjusted, if desired, to reflect changes in traveling conditions (new roads, road repairs, weather, etc.).

In addition, the condition can be established to travel only predetermined routes to different combinations of receiving points. For example, it could be decided to travel only routes (Figure 1, page 7) S-E-A-S, S-E-D-B-A-S, and S-E-D-C-B-A-S alternately or assign one vehicle to each route. This condition decreases the number of calculations by reducing the number of routes that require evaluation.

In the case of changing locations and number of receiving points, all possible routes must be determined each time a solution is desired requiring considerably more calculations. Because of this disadvantage, an approximate solution for Model V is presented in Model V-A which decreases the number of calculations and provides a good decision rule.

Model V-A: An Approximation for the General Model

An approximate decision rule for the general case is presented in the following steps and uses Figure 1 (page 7) and the data given in Table X (selected data from Table I, page 6) for illustrative purposes.

<u>Step 1</u>: Determine the shortest route to each receiving point.

<u>Step 2</u>: Calculate the $\bar{x}$ and σ values for the different receiving points based on the shortest route to the particular receiving point. If the mileage is the same for two or more routes, use the route that has the smallest $\Sigma\bar{\bar{x}}$. If the mileage and $\Sigma\bar{\bar{x}}$ values are the same, use the route that has the smallest $\Sigma\sigma^2$. If all three are the same, make an arbitrary choice.

The results of Steps No. 1 and No. 2 are shown in Table XI.

<u>Step 3</u>: Calculate the ΔEC value for each order

TABLE X

DATA FOR MODEL V-A

i Order Number	Destination	P_i Penalty $/Day	r_i Volume Ratio	T_i Available Days to Due Date
5	A	600	0.1	3
6	D	400	0.1	1
7	B	400	0.2	2
12	A	500	0.5	1
13	E	100	0.6	1
15	C	900	0.2	3
16	B	600	0.4	4

TABLE XI

SUMMARY OF STEPS NO. 1 AND NO. 2 FOR MODEL V-A

Destination	Shortest Route to Destination	$\bar{\bar{x}}$ for Shortest Route	σ for Shortest Route	Shortest Round Trip Route	Shortest Round Trip Mileage
A	S-A	2.00	0.500	S-A-S	1200
B	S-A-B	3.00	0.538	S-A-B-A-S	1800
C	S-A-B-C	4.00	0.594	S-A-B-C-B-A-S	2600
D	S-E-D	3.00	0.790	S-E-D-ES	2100
E	S-E	1.00	0.250	S-E-S	700

using the $\bar{x}$ and σ values determined in Step No. 2. These results are given in Table XII.

Step 4: Establish the objective function and constraints for a simplex solution. The objective equation (Equation (5)) is:

$$z = 300x_5 + 400x_6 + 400x_7 + 500x_{12} + 100x_{13} + 900x_{15} + 306x_{16}$$

and the constraint equations (Equations (6) and (7)) are:

$$0.1x_5 + 0.1x_6 + 0.2x_7 + 0.5x_{12} + 0.6x_{13} + 0.2x_{15} + 0.4x_{16} \leq 1.00$$

$$x_5 \leq 1.00 \qquad x_{13} \leq 1.00$$

$$x_6 \leq 1.00 \qquad x_{15} \leq 1.00$$

$$x_7 \leq 1.00 \qquad x_{16} \leq 1.00.$$

$$x_{12} \leq 1.00$$

Step 5: Obtain a simplex solution to the equations determined in Step No. 4. The results of the simplex solution indicate the orders to be shipped and, consequently, the receiving points to be serviced. The route to service these receiving points is the shortest possible and the return route to the shipping point is the shortest possible. The initial and final tableaus for

TABLE XII

ΔEC VALUES FOR MODEL V-A

i Order Number	Destination	r_i Volume Ratio	EC	EC′	ΔEC
5	A	0.1	13.80	313.80	300
6	D	0.1	1000.00	1400.00	400
7	B	0.2	600.00	1000.00	400
12	A	0.5	750.00	1250.00	500
13	E	0.6	50.00	150.00	100
15	C	0.2	1350.00	2250.00	900
16	B	0.4	14.40	320.40	306

the simplex solution are shown in Figures 30 and 31 and give the following solution:

$$x_5 = 1 \qquad x_{15} = 1$$

$$x_6 = 1 \qquad x_{12} = 0.8$$

$$x_7 = 1 \qquad z = \$2400.00.$$

This solution implies that all receiving points must be serviced in order to deliver these orders. The shortest round trip route that serves all receiving points is S-A-B-C-D-E-S, which has a mileage of 2750.

Step 6: Eliminate orders designated for the farthest receiving point determined in Step No. 5 and repeat Steps No. 4 and No. 5. In this example, receiving point "C" is the farthest. Therefore, Order No. 15 is eliminated and the objective function is

$$z = 300x_5 + 400x_6 + 400x_7 + 500x_{12} + 100x_{13} + 306x_{16}.$$

The constraints are

$$0.1x_5 + 0.1x_6 + 0.2x_7 + 0.5x_{12} + 0.6x_{13} + 0.4x_{16} \leq 1.00$$

x_5	x_6	x_7	x_{12}	x_{13}	x_{15}	x_{16}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	b
.1	.1	.2	.5	.6	.2	.4	1	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1
0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1
0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1
-300	-400	-400	-500	-100	-900	-306	0	0	0	0	0	0	0	0	0

Figure 30. Initial Tableau for Model V-A

(All Destinations Considered)

x_5	x_6	x_7	x_{12}	x_{13}	x_{15}	x_{16}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	b
0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1
1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1
0	0	0	0	-1.2	0	-.8	-2	.2	0	.4	1	0	.4	0	.2
0	0	0	1	1.2	0	.8	2	-.2	0	-.4	0	0	-.4	0	.8
0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1
0	0	0	0	500	0	94	1000	200	0	200	0	0	700	0	2400

Figure 31. Final Tableau for Model V-A

(All Destinations Considered)

$$x_5 \leq 1.00 \qquad x_{12} \leq 1.00$$

$$x_6 \leq 1.00 \qquad x_{13} \leq 1.00$$

$$x_7 \leq 1.00 \qquad x_{16} \leq 1.00.$$

The initial and final tableaus for a simplex solution of these equations are shown in Figures 32 and 33, respectively. The final tableau gives the following solution:

$$x_5 = 1 \qquad x_{12} = 1$$

$$x_6 = 1 \qquad x_{13} = 1/6$$

$$x_7 = 1 \qquad z = \$1616.67$$

and indicates that receiving points A, B, D, and E must be serviced. The shortest route that connects these receiving points is S-A-B-D-E-S and has a mileage of 2200. The route used should correspond, as much as possible, to the route used in Step No. 5.

In the case that two or more receiving points have the same mileage and are the farthest points, arbitrarily eliminate one receiving point and the orders going to it. Set up the appropriate objective function, constraints, and simplex the resulting equations. Repeat this procedure for each of the receiving points having the same

x_5	x_6	x_7	x_{12}	x_{13}	x_{16}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	b
.1	.1	.2	.5	.6	.4	1	0	0	0	0	0	0	1
1	0	0	0	0	0	0	1	0	0	0	0	0	1
0	1	0	0	0	0	0	0	1	0	0	0	0	1
0	0	1	0	0	0	0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	0	0	0	1	0	0	1
0	0	0	0	1	0	0	0	0	0	0	1	0	1
0	0	0	0	0	1	0	0	0	0	0	0	1	1
-300	-400	-400	-500	-100	-306	0	0	0	0	0	0	0	0

Figure 32. Initial Tableau for Model V-A

(Destination "C" Omitted)

x_5	x_6	x_7	x_{12}	x_{13}	x_{16}	U_1	U_2	U_3	U_4	U_5	U_6	U_7	b
0	0	0	0	1	$\frac{4}{6}$	$\frac{10}{6}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{6}$	$-\frac{5}{6}$	0	0	$\frac{1}{6}$
1	0	0	0	0	0	0	1	0	0	0	0	0	1
0	1	0	0	0	0	0	0	1	0	0	0	0	1
0	0	1	0	0	0	0	0	0	1	0	0	0	1
0	0	0	1	0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	$-\frac{4}{6}$	$-\frac{10}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{5}{6}$	1	0	1
0	0	0	0	0	1	0	0	0	0	0	0	1	1
0	0	0	0	0	$\frac{400}{6}$	$\frac{1000}{6}$	$\frac{500}{6}$	$\frac{2300}{6}$	$\frac{2200}{6}$	$\frac{2500}{6}$	0	0	$\frac{9700}{6}$

Figure 33. Final Tableau for Model V-A

(Destination "C" Omitted)

mileage. Use the larger value of z for Step No. 7.

Step 7: Determine the route profits, (Equation (32)), and the maximum error term (Equation (39)), using the solutions obtained in Steps No. 5 and No. 6. These calculations are shown below where a value of $0.20 per mile is assumed for C_m.

$$z_1 = \Sigma(\Delta EC) - C_m M$$

$$2400 - 0.2(2750) = \$1850.00$$

$$e_m = \Sigma\, t\, P_i - \Sigma(\Delta EC)$$

$$= (1)[600 + 400 + 400 + 900] + (.8)(500) - 2400 = \$300.00$$

$$z_1 = \$1616.67 - 0.2(2200) = \$1167.67$$

$$e_m = 600 + 400 + 400 + 500 + \frac{100}{6} - 1667.67$$

$$= \$300.00.$$

Both maximum error terms are the same and both result from the same order (Order No. 5). Consideration of the routes to be taken show that the first receiving point that is reached is "A" and the ΔEC for Order No. 5 is the actual ΔEC. Therefore, the maximum error term can be disregarded. The interpretation of the error term is discussed in the next section. Since z_1 for Step No. 5 is the larger, the best route and choice of orders to be shipped have been obtained. If, after consideration of the maximum error term, z_1 for Step No. 5 is less than for Step No. 6, a better approximate solution exists. Therefore, repeat Steps No. 6 and No. 7 until the solution obtained in Step No. 7 is less than the preceding solution. In this numerical example, z_1 for Step No. 6 is greater than z_1 for Step No. 7. Therefore, this is the best approximate solution and the decision would be made to ship all of Orders No. 5, No. 6, No. 7, No. 15, and 0.8 of Order No. 12 by route S-A-B-C-D-E-S. For illustrative purposes, consider the case that z_1 for Step No. 6 was greater than z_1 for Step No. 5. In this case, orders designated for receiving point "D" would be eliminated and Step No. 6 would be repeated. A comparison would then be made between z_1 for receiving points A, B, D, E and z_1 for receiving points A, B, E.

Derivation and Interpretation of Error Term in Model V-A

It is the purpose of this section to show that the decision rules established for Model V-A provide a conservative approach and a good solution to the general model. This purpose is accomplished by the derivation of the error term given in Step No. 7 and by providing guide lines for the interpretation of the error term.

The ΔEC values are obtained using minimum $\bar{\bar{x}}$ and σ values which are based on the shortest route to their respective receiving stations. The ΔECs, calculated in this manner, are minimum values. Any deviations from the shortest route increase the $\bar{x}$ and σ values and the ΔEC values, correspondingly, increase. Since the choice of routes, to be compared (Step No. 7) are based on the results of the simplex solution of these minimum ΔEC values, a conservative approach is taken in the choice of routes that are to be compared. It is for these reasons that the rule regarding the use of minimum $\bar{x}$ and σ values for the ΔEC values has been established in Step No. 2.

If the route to different receiving points changes from the route used in determining the $\bar{x}$ and σ values, the ΔEC values will increase. Consequently, the route profits used in comparing two routes is less than the actual route profit. The route profit used in comparing two routes is:

$$z_1 = \Sigma(\Delta EC) - C_m M. \tag{33}$$

The actual route, z_a, profit is given by the equation:

$$z_a = \Sigma(\Delta EC)_a - C_m M_a \tag{34}$$

where the ΔEC values in the $\Sigma(\Delta EC)_a$ term are based on the actual $\bar{x}$ and σ values for the route taken.

The error, e, is the difference between Equations (33) and (34).

$$e = z_a - z_1 = \Sigma(\Delta EC)_a - C_m M_a - \Sigma(\Delta EC) + C_m M. \tag{35}$$

The actual mileage and the mileage used in Equation

(35) are the same. Consequently, Equation (35) reduces to:

$$e = \Sigma(\Delta EC)_a - \Sigma(\Delta EC). \tag{36}$$

Since the smallest values for $\bar{x}$ and σ have to be used in calculating $\Sigma(\Delta EC)$, Equation (36) cannot be less than zero.

As the value for $\bar{x}$ increases, for a particular order, the actual ΔEC value will approach a limit. Mathematically, this limit can be expressed as:

$$\lim_{\bar{x}\to\infty} (\Delta EC)_{ai} = t\,P_i, \tag{37}$$

or extending this result further

$$\lim_{\bar{x}\to\infty} \Sigma(\Delta EC)_{ai} = \Sigma t\,P_i. \tag{38}$$

Substituting Equation (28) into Equation (27), the maximum error, e_m, will be

$$e_m = \Sigma t\,P_i - \Sigma(\Delta EC) \tag{39}$$

for a particular route. Therefore, the route error is in the range of

$$0 \leq e \leq e_m. \tag{40}$$

Equations (39) and (40) can be used in the comparison of two routes to insure that the best choice is made. Once two routes have been decided, it is possible to calculate the actual route profits and compare the results. These results indicate the correct (greatest z_a) route choice. However, it is possible, in many cases, to eliminate the need to evaluate the actual route profits. This is possible, employing the following considerations:

1. Determine the maximum error for the route having the smallest route profit (z_1). Add this to the route profit. If this result is still less than the other route profit, without its maximum error, the route with the greatest z_1, is the best choice.

2. In comparing the effect of an increase in $\bar{\bar{x}}$ and σ, it should be remembered that if $-3.5 \leq K \leq 3.5$, then $(\Delta EC)_i = t\,P_i$. In addition, some of the ΔEC values will be correct values. This is determined by considering the route taken to the receiving point for a particular order. If the route taken is the same as the shortest, then the ΔEC value is correct. These two facts can be used to determine the bounds of the actual error.

3. If the difference between ΣP_i and $\Sigma\Delta EC$ results from the same orders and there is no change in the route to the receiving points for these orders, it is possible to neglect the maximum error term. The choice of route would then be based on the greater route profit. This is

the reason for duplicating routes, as near as possible, mentioned in Step No. 6.

Discussion of Model V-A

The major objectives in considering this model were to obtain a decision rule that would closely approximate the optimum and decrease the number of calculations necessary to arrive at the decision rule. These two objectives have been accomplished by the procedure outlined in this model. The closeness of the approximation has been obtained by consideration of an error term. The number of calculations has been reduced, since all possible routes do not have to be evaluated. These objectives have been attained without added conditions or restrictions to the generality of the model. Since no added condition has been made in this model, the remarks made in the Discussion of Model V concerning weight restriction and shipment of partial orders are also applicable to this model.

CHAPTER III

SUMMARY OF MODELS

In Chapter II, decision rules have been established for five models. Each model was described by different conditions and, in addition, certain assumptions were made. These conditions, assumptions, and the decision rules are emphasized in the following summary of each model.

Model I: One Destination and One Order Per Vehicle

Conditions:

1. The time required to travel between any two points in the system is distributed according to some known probability distribution.
2. The vehicle, used in transporting orders, is limited to one order per trip.
3. The penalty for an order being late is assessed on a dollars per day basis.
4. There is only one receiving point and one shipping point.

Assumptions:

1. The time required to travel between any two points in the system is distributed normally and the mean and standard deviation are known.
2. Mileage costs are the same for each order.

Solution and Decision Rules:

1. Determine the ΔEC for each order using Equations (1) and (2).
2. Sequence the orders to be shipped on the basis of the greatest ΔEC first.

Model II: One Destination and Several Orders Per Vehicle

Conditions:

1. The time required to travel between any two points in the system is distributed according to some known probability distribution.
2. The vehicle, used in transporting orders, is able to carry more than one order.
3. The penalty for an order being late is assessed on a dollars per day basis and varies directly with the proportion of the order not shipped.

4. There is only one receiving point and one shipping point.

Assumptions:

1. The time to travel between any two points in the system is distributed normally and the mean and standard deviation are known.
2. The volume ratio defined by Equation (4) is less than one for all orders and varies directly with the proportion of the order shipped.
3. Mileage costs are the same for each order.

Solution and Decision Rules:

1. Determine the ΔEC and r_i for each order using Equations (1), (2), and (4).
2. Set up the objective functions and constraints defined by Equations (5), (6), and (7) and solve these equations by the simplex method.
3. Sequence the orders and proportions of orders to be shipped on the basis of the simplex solution.

Model III: Shipment of Partial Orders Not Allowed

Conditions:

1. The time required to travel between any two points in the system is distributed according to some known probability distribution.
2. The vehicle, used in transporting orders, is able to carry more than one order.
3. The penalty for being late is assessed on a dollars per day basis.
4. There is only one receiving point and one shipping point.
5. Shipment of partial orders is not allowed.

Assumptions:

1. The time to travel between any two points in the system is distributed normally and the mean and standard deviation are known.
2. The volume ratio defined by Equation (4) is less than one for all orders and varies directly with the proportion of the order shipped.
3. Mileage costs are the same for each order.

Solution and Decision Rules:

1. Determine the ΔEC and r_i for each order using Equations (1), (2), and (4).
2. Set up the objective function and constraints defined by Equations (15), (16), and (17) and express the constraint equations in terms of integer values as shown by Equations (22) through (26).
3. Obtain a simplex solution and check the solution to determine if it satisfies the constraints.
4. The results of the simplex solution provide the decision rule.

Model IV: A Predetermined Route

Conditions:

1. The time required to travel between any two points in the system is distributed according to some known probability distribution.
2. The vehicle, used in transporting orders, is able to carry more than one order.
3. The penalty for an order being late is assessed on a dollars per day

basis and varies directly with the proportion of the order not shipped.

4. There is only one shipping point, but several receiving points.

5. The vehicle must travel a predetermined route.

Assumptions:

1. The time to travel between any two points in the system is distributed normally and the mean and standard deviations are known.

2. The volume ratio defined by Equation (4) is less than one for all orders and varies directly with the proportion of the orders shipped.

3. Mileage costs are the same for each order.

4. The time required to travel between receiving points is independent of the time to travel between other receiving points.

5. The time lost at a receiving point due to unloading orders is essentially zero.

Solution and Decision Rules:

1. Calculate the $\bar{\bar{x}}$ and σ values for

each receiving point on the route to be traveled using Equations (27) and (28).

2. Determine the ΔEC and r_i for each order using Equations (1), (2), and (4).

3. Set up the objective functions and constraints defined by Equations (5), (6), and (7) and simplex these equations.

4. Sequence the orders and proportion of orders to be shipped on the basis of the simplex solution.

<u>Model V: The General Case</u>

<u>Conditions</u>:

1. The time required to travel between any two points in the system is distributed according to some known probability distribution.

2. The vehicle, used in transporting orders, is able to carry more than one order.

3. The penalty for an order being late is assessed on a dollars per day basis and varies directly with the proportion of the order not shipped.

4. There is only one shipping point, but several receiving points.

Assumptions:

1. The time required to travel between any two points in the system is distributed normally and the mean and standard deviation are known.
2. The volume ratio defined by Equation (4) is less than one for all orders and varies directly with the proportion of the order shipped.
3. The time required to travel between receiving points is independent of the time to travel between other receiving points.
4. The time lost at a receiving point due to unloading orders is essentially zero.

Solution and Decision Rules:

1. Determine all possible routes to all possible combinations of receiving points and calculate the following items for each route:
   a. The $\bar{\bar{x}}$ and σ values for each receiving point on the route using Equations (27) and (28).

b. The ΔEC and r_i values for each order using Equations (1), (2), and (4).

3. Establish, for each route, an objective function and constraints using Equations (5), (6), and (7).
4. Obtain a simplex solution for each route.
5. Calculate the route profit for each route using Equation (32).
6. The route with the greatest route profit is chosen and the results of its simplex solution indicate the orders that should be shipped.

Model V-A: An Approximation for the General Case

The conditions and assumptions for this model are the same as those summarized in Model V. For this reason, they will not be repeated. In addition, it is felt that repeating the decision rules for this model is not warranted since they are presented in outline form in Chapter II.

CHAPTER IV

CONCLUSIONS

This chapter is divided into two sections. The first section consists of general remarks and conclusions about the models considered in Chapter II and the decision rules established for each. Specific conclusions and remarks are made in the discussions presented at the end of each model. Summaries of the conditions, assumptions, solutions, and decision rules are presented in Chapter III. The second section proposes possible, future investigations for the specific models considered in this dissertation and in the general area of sequencing-delivery models.

General Remarks and Conclusions

In this dissertation, two simple, but powerful, concepts are combined to provide optimum decision rules for the probabilistic sequencing-delivery models considered. Specifically, the two concepts employed are a comparison of expected cost of making a decision and the expected cost of delaying the decision for a specified period of time. This concept coupled with the simplex method for solving linear programming problems provides optimum decision rules.

Two desirable requirements of a model are that the model be practical and that the solution of the model be easily implemented. It is believed that these requirements have been met since the restrictions and conditions of the model are not considered to be so stringent as to make the models impractical for industrial applications; and, in addition, the decision rules can be readily understood and easily implemented. Although the calculations to arrive at a decision are not difficult, in some models they are laborious. Consequently, the use of a computer to perform the calculations for a large system is desirable.

Another requirement of a model is that the model and its solution be defined in such a way as to facilitate adjustments due to change and control of the model. It is believed that these requirements are met since the variables which define the models and the model parameters are established in such a way as to allow generality.

Proposals for Future Investigations

Investigations into the following two models would be interesting and worthwhile:

1. A model similar to the general model, considered in this dissertation, with the added condition that each point in the system may be a receiving point and/or a shipping point.

2. A model where the possibility exists of a vehicle waiting at a receiving station to be unloaded. This waiting time could be assumed to be distributed according to some probability distribution (i.e., Poisson Distribution).

As pointed out in the Introduction, relatively little progress has been made in the mathematical analysis of the sequencing problem. It is possible that applications of the concepts presented in this dissertation might result in the solution of, as yet, unsolved sequencing and delivery problems. It is recommended that this possibility be investigated.

A SELECTED BIBLIOGRAPHY

1. Churchman, C. W., R. L. Ackoff, and E. L. Arnoff. Introduction to Operations Research. New York: John Wiley and Sons, 1957, p. 450.

2. Starr, M. K. Production Management. Englewood Cliffs, N. J.: Prentice-Hall, 1964, p. 427.

3. Ackoff, R. L. (Editor). Progress in Operations Research. R. L. Sisson, Chapter 7, "Sequencing Theory." New York: John Wiley and Sons, Inc., 1961, pp. 295-326.

4. Johnson, S. M. "Optimal Two- and Three-Stage Production Schedules With Setup Times Included," Nav. Res. Log. Quat., 1, No. 1, Mar. 1954, pp. 61-68.

5. Akers, S. B., Jr., and J. Friedman. "A Non-Numerical Approach to Production Scheduling Problems," J. Oprs. Res. Soc. Am., Vol. 3, No. 4, Nov. 1955, pp. 429-442.

6. Mitten, L. G. "Sequencing 'n' Jobs on Two Machines With Arbitrary Time Lags," Management Sci., Vol. 5, Apr. 1958, pp. 293-298.

7. ____________. "A Schedule Problem," J. Industrial Eng., Vol. 10, Mar.-Apr. 1959, pp. 131-137.

8. Sisson, R. L. "Methods of Sequencing in Job Shops," J. Oprs. Res. Soc. Am., Vol. 7, 1959, pp. 10-29.

9. Giffler, B., and G. L. Thompson. "Algorithms for Solving Production Scheduling Problems," J. Oprs. Res. Soc. Am., Vol. 8, No. 2, 1960, pp. 487-503.

10. Thompson, G. L. "Recent Developments in Job-Shop Scheduling Problems," Nav. Res. Log. Quat., 7, No. 4, Dec. 1960, pp. 585-589.

11. Heller, J. "Some Numerical Experiments for an M x J Flow Shop and Its Decision Theoretic Aspects," J. Oprs. Res. Soc. Am., Vol. 8, No. 2, 1960, pp. 178-184.

12. __________, and G. Logeman. "Algorithm for Construction and Evaluation of Schedules," Management Sci., Vol. 8, Apr. 1961, pp. 169-183.

13. Rowe, A. J. "Toward a Theory of Scheduling," J. Industrial Eng., Vol. 11, Mar.-Apr. 1960, pp. 125-136.

14. Giffler, B. "Scheduling General Production Systems Using Schedule Algebra," Nav. Res. Log. Quat., 10, No. 3, Sept. 1963, pp. 237-255.

15. Dudek, R. A., and O. F. Teuton, Jr. "Development of M Stage Decision Rule for Scheduling n Jobs Through m Machines," J. Oprs. Res. Soc. Am., Vol. 12, 1964, pp. 471-497.

16. Ferguson, A. R., and G. B. Dantzig. "The Allocation of Aircraft to Routes - An Example of Linear Programming Under Uncertain Demand," Management Sci., Vol. 3, 1956, pp. 45-73.

17. Dantzig, G. B., and J. H. Ramser. "The Truck Dispatching Problem," Management Sci., Vol. 6, 1959, pp. 80-91.

18. Conway, R. W., and W. L. Maxwell. "Network Dispatching by the Shortest-Operation Discipline," Management Sci., Vol. 10, 1962, pp. 51-73.

19. Dantzig, G. B., and D. L. Johnson. "Maximum Payloads Per Unit Time Delivered Through an Air Network," J. Oprs. Res. Soc. Am., Vol. 12, Mar.-Apr. 1964, pp. 230-236.

20. Clarke, G., and J. W. Wright. "Scheduling of Vehicles From a Central Depot to a Number of Delivery Points," J. Oprs. Res. Soc. Am., Vol. 12, July-Aug. 1964, pp. 568-581.

21. Frank, O. "The Optimal Order to Serve in Certain Sequencing Problems," J. Oprs. Res. Soc. Am., Vol. 12, May-June 1964, pp. 433-440.

22. Balinski, M. L., and R. E. Quandt. "On an Integer Program for a Delivery Problem," J. Oprs. Res. Soc. Am., Vol. 12, 1964, pp. 300-304.

23. Loomba, N. P. Linear Programming. New York: McGraw-Hill Book Co., 1964.

24. Llewellyn, R. W. Linear Programming. New York: Holt, Rinehart, and Winston, 1964.

25. Gomory, R. A., and W. J. Baumol. "Integer Programming and Pricing," Econometrica, 28, 1960, pp. 521-550.

26. Bowker, A. H., and G. J. Lieberman. Engineering Statistics. Englewood Cliffs, N. J., 1961.

VITA

Gladstone Taylor Stevens, Jr.

Candidate for the Degree of

Doctor of Philosophy

Thesis: A PROBABILISTIC SEQUENCING-DELIVERY MODEL

Major Field: Engineering

Biographical:

Personal Data: Born December 16, 1930, in Brockton, Massachusetts, the son of Gladstone Taylor and Blanche Ruth Stevens, of Brockton, Massachusetts.

Education: Attended high school in Brockton, Massachusetts and graduated in 1948. Entered the University of Oklahoma at Norman, Oklahoma, in 1952; received the Bachelor of Science degree in Mechanical Engineering in 1956. Entered Case Institute of Technology in Cleveland, Ohio, in 1959 (part-time); received the Master of Science degree in Mechanical Engineering in 1962. Completed the requirements for the Degree of Doctor of Philosophy in January, 1966.

Professional Experience: Employed by E. I. duPont deNemours, Company as a Project Engineer from June, 1956 to August, 1959. Employed as a Research Engineer by Thompson-Ramo-Wooldridge, Inc., from June, 1960 to April, 1962. Served as an Assistant Professor in the School of Mechanical Engineering and in the School of Industrial Engineering, Lamar State College of Technology, from September, 1962 to present.

Professional Membership: American Society of Mechanical Engineers, American Institute of Industrial Engineers, American Society for Engineering Education, Registered Professional Engineer, Alpha Pi Mu, and Sigma Tau.