CONVEX KERNELS

By

CHARLES RICHARD SMITH

Bachelor of Science
Northwestern Louisiana State College
Natchitoches, Louisiana
1957

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1964

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the Oklahoma State University
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Thesis Approved:

E. K. McFadden
Thesis Adviser

Leonard Agnew

John E. Seesly

J. Leroy Folks

J. W. Bryan
Dean of the Graduate College

PREFACE

This dissertation deals with a certain class of subsets of real linear spaces. It is the purpose of this paper to give a status report, in an expository manner, on the work involving the convex kernel of star-shaped sets. The concept of the convex kernel has been involved in a recent flurry of activity by some of the leading students of convexity.

The desired audience for this paper is the student of convexity or functional analysis with a minimum background of the material in Parts I and II of Valentine's book [10]. Several concepts used freely throughout this exposition with which the reader should be familiar are: linear space, convex set, hyperplane of support, convex cone, the notion of one point seeing another via a set, topological properties of sets such as interior, boundary, open, closed, bounded, compact and connected, and sub-spaces and flats. The only notation which might be new to the reader is for the convex kernel of a set S, denoted by ckS.

Chapter I is concerned with the basic definitions of star-shapedness and of the convex kernel of a star-shaped set, a historical development of the progress concerning convex kernels, and a statement of the two basic problems of interest involving convex kernels.

In Chapter II, the first problem, that of finding and

characterizing the convex kernel of a given non-convex set, is investigated. All of the major results of attempts at a general solution of the problem are given and explained with various examples. Although most of the progress on this problem has concerned sets in finite-dimensional Euclidean space, some mention is made of sets in certain infinite-dimensional spaces.

The second problem, that of finding a non-convex set for which a given convex set is its convex kernel, is discussed in Chapter III. The work on a general solution of this problem has appeared in the literature only since 1964, so some comparative analysis is made of the published results. Work on this problem has had the setting of both finite and infinite-dimensional linear spaces, so examples are given in all cases.

Chapter IV concerns itself with some recent activity, some yet to be published, in the study of convex kernels and with some unsolved problems which should be of interest to most students of convexity.

It was the purpose of the author to delete various unnecessary details of the known results involving convex kernels. All results, as well as copied problems, are referenced and if a statement is not referenced, it is that of the author.

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Professor E. K. McLachlan, whose inspiration and encouragement has been invaluable throughout my graduate program; and the other members of my advisory committee, Professors Jeanne Agnew, Leroy Folks and John E. Susky, for their help in the completion of my program.

Without the cooperation and patience of my wife, Wanda, and daughter, Karla, this dissertation would have been almost impossible, so to them I want to express my deepest appreciation.

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CHAPTER I

INTRODUCTION

In the study of mathematics, many questions of interest arise when certain desirable conditions are relaxed. If such a relaxation proves feasible, the new result is sometimes called a generalization. Several attempts have been made to generalize the concept of convexity in a linear space. One such modification of convexity is that of star-shapedness. A set is star-shaped with respect to a point if the connecting segment of the given point and each point of the set belongs to the set. The set S in Figure 1, below, is a non-convex set which is star-shaped. This set is star-shaped with respect to each point of the rectangular region with vertices A, B, C and D.

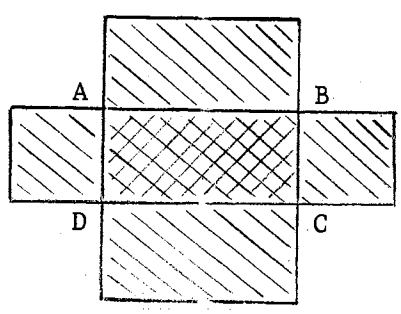


Figure 1

The collection of points of a set with respect to which a set is star-shaped is called the convex kernel of the set. The idea of the kernel, which is necessarily a convex set if it exists, or "Kerneigebeit" of a set was introduced by H. Brunn [1] in 1913.

This concept of the convex kernel of a set requires only the structure of a linear space, however, the more interesting questions arise and more solutions are available in the setting of a linear topological space. Up to the present, two types of problems are of interest concerning the convex kernel of a set: 1) given a non-convex set, find its convex kernel and 2) given a convex set, find a non-convex set for which the given set is its convex kernel. In the first problem, more than just show the existence of the convex kernel, it is desirable to characterize it, i.e. give a set theoretic description of the kernel. The second problem is one of extending the given set, such as the rectangular region ABCD in Figure 1, to a new set such that the original set plays the role of the convex kernel. Observation of Figure 1 shows that a given set may be the kernel of more than one non-convex set.

Concerning the first problem, Krasnosel'skii [6], in 1946, made the earliest major contribution in showing the existence of the convex kernel of certain sets in finite dimensional Minkowski space. An important tool in Krasnosel'skii's work was Helly's theorem [3] which was discovered in 1923. For several years after Krasnosel'skii's work, there was a lapse in the activity involving the convex kernel and only recently, since 1964, has the literature contained articles in the area of star-shapedness and the convex kernel.

Two writers, Valentine [10], [11], [12], and Robkin [8], have

recently characterized the convex kernel of certain classes of non-convex sets not applicable to Krasnosel'skii's original theorem. The extensions made by Valentine and Robkin are concerned with the properties of the sets and not extensions relative to the dimension of the space.

All the contributions of Krasnosel'skii, Valentine and Robkin have been focused on finding or proving the existence of convex kernels. This work has been contrasted by the activity involved with the second situation mentioned heretofore, the so-called extension problem. All of the literature concerning the extension problem is very recent. Klee [4] and Post [7], apparently similtaneously, have investigated the problems concerning the realization of a given convex set as the convex kernel of a non-convex set. Klee's results are more general and include much of Post's work as a special case. It is interesting to compare the papers of Klee and Post since they are similar in spirit, but quite different in detail. The setting for Klee's paper [4] is that of a separable Banach space, while Post [7], pursuing a problem posed by L. Fejes Toth, works in one of the more common Banach spaces, E_2. An intuitive approach to the extension problem would put Post's work first.

CHAPTER II

EXISTENCE OF CONVEX KERNELS

When confronted with the problem of finding a point of the convex kernel of a plane set, the student of convexity quite often exhibits such a point by inspection. Occasionally one can find the entire convex kernel and can give a simple geometric proof that his assertions are correct. In this section of the paper, a systematic approach, with respect to the properties of sets, will be made to show that the convex kernel of some sets exist.

In any linear space, if a set is convex then the set is star-shaped with respect to each point, hence the set and its convex kernel are identical. Similarly, if a point of a set can see every other point of the set via the given set, then the convex kernel is non-empty. However, any other general statements concerning the existence of the kernel are impossible without some knowledge of certain properties of the set being considered.

The first class of sets to be considered will be those sets in E_2 which are bounded and closed, that is, compact sets, whose boundary is a simple closed polygon. If such a set is not convex, then one of the vertices must be a point of <u>local non-convexity</u>, that is the intersection of every neighborhood of the vertex and the set produces a non-convex set. For an example, see the star-shaped set in Figure 2,

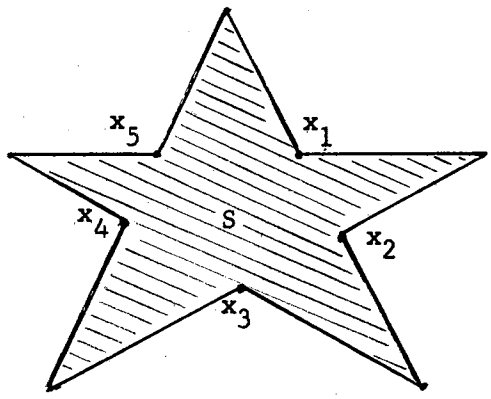


Figure 2

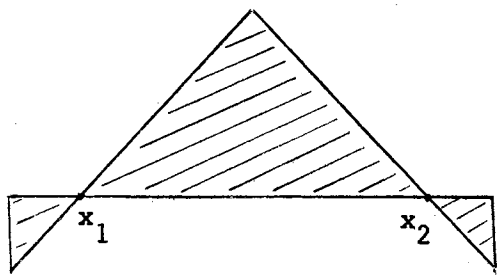


Figure 3

in which it is seen that the vertices x_1, x_2, x_3, x_4 and x_5 are points of local non-convexity of the set.

Another concept required before formulation of the known results concerning the kernel of such polygonal regions is that of an external ray of support. A closed ray is an <u>external ray of support to a set</u> if its end-point belongs to the boundary of the set and the ray does not intersect the set in any other point. An external ray of support to the interior of a set can intersect the boundary of the set. The closed ray with the same end-point as a given external ray of support and which is the reflection in the same line of the given ray is called the <u>complementary ray of the support ray</u>.

The best results concerning convex kernels of such sets are summarized by Valentine [12] with the following theorem:

<u>Theorem 1:</u> Let S be a compact set in E_2 whose boundary is a simple closed polygon. Suppose that for each three or fewer vertices which are also points of local non-convexity of S there exist corresponding external rays of support to the interior of S whose complementary rays are concurrent and meet in S. Then these conditions are both necessary and sufficient for the convex kernel of S to be non-empty.

Again, considering the set in Figure 2, it is not difficult to see that such rays of external support as required in Theorem 1 exist and that the set is star-shaped. In fact the set in Figure 2 has as its convex kernel the pentagonal region with vertices x_1, x_2, x_3, x_4 and x_5. The need for the boundary of the set being considered in Theorem 1 to be

a simple polygon is shown by the set in Figure 3. Since the only vertices which are not points of local convexity are x_1 and x_2, complementary rays of the required rays of external support meet along the segment x_1x_2 but the set is not star-shaped, i.e. the set has an empty convex kernel.

Theorem 1 can be proved as a corollary to a later result, however in a logical approach to a solution of the original problem, it seems natural to consider those sets with polygonal boundaries first. The obvious question would concern the extension of the first theorem to a space of dimension greater than two. Again, a partial solution exists in the so-called Krasnosel'skii type theorems (cf. Krasnosel'skii [6] and Robkin [8]).

One could think of several possibilities for extensions of Theorem 1 to E_n. Certainly the condition of external rays of support is a strong one as is the condition that the boundary of the region considered be a simple polygon, that is, the boundary is contained in a finite number of hyperplanes. It should be noted that the rays of support were demanded only at the points of local non-convexity. A rather strong theorem which retains the need for the existence of the external rays of support but requires only that the boundary of the set not isolate any regions of its complement, as does the set bounded by the annulus in E_2, is given by Robkin [8]. Actually, this theorem is also a corollary of a more general, but awkward, statement.

<u>Theorem 2:</u> Suppose that S is a compact set in E_n which is the closure of a non-empty open set. Further suppose that for every $n + 1$ or fewer boundary points of S there exist corresponding rays of external

support to S (not simply supporting the interior of S) whose complementary rays are concurrent and meet in S. Then the convex kernel of S is non-empty.

Obviously a set can have a non-empty kernel and not be compact. But, the converse of Theorem 2 concluding the existence and concurrency of the complementary rays of support is false. This fact is verified by an example in E_2 which is demonstrated by the set in Figure 4. This set is compact and is the closure of a non-empty open set. It is also star-shaped with respect to x_o, yet the complementary rays of any of the external rays of support to S at x_o do not intersect S. Hence any collection of three boundary points which includes x_o does not satisfy the hypothesis of Theorem 2.

To see the need for each member of the class of sets applicable to Theorem 2 to have a non-empty interior, the set in Figure 5 is considered. It is found that set S in Figure 5 is compact and possesses the property that for every three boundary points there exist external rays of support to the set whose complementary rays are concurrent and meet in the set. However, the set has an empty interior and is not star-shaped. This non-empty interior requirement is still not strong enough as there exist compact sets which have non-empty interiors and satisfy the conditions required of the external rays of support in the hypothesis of Theorem 2, but the sets are not star-shaped. Hence the need for each set to be the closure of a non-empty open set is seen. For such a set in E_2, see the set in Figure 6 which is composed of two circles that are tangent externally, a segment of their common tangent line and the region bounded by one of the circles. This set has the

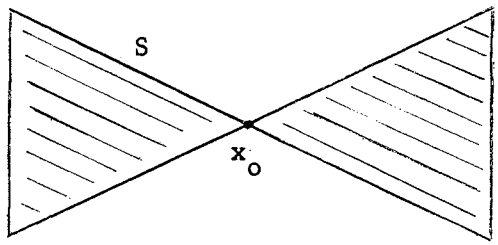


Figure 4

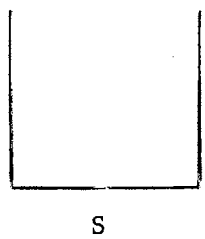


Figure 5

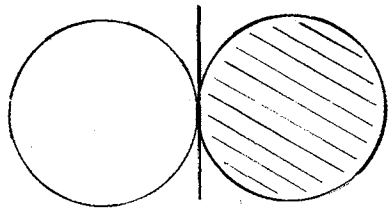


Figure 6

empty set for a convex kernel, yet it satisfies all of the conditions of Theorem 2 except it is not the closure of an open set.

The conditions in Theorem 2 are not necessary for the existence of a convex kernel as evidenced by the example in Figure 4, which is star-shaped and does not satisfy the hypothesis of the theorem. To obtain conditions which are both necessary and sufficient for star-shapedness, the classes of sets are somewhat restricted. The best result in the two dimensional case has been given by Valentine [12], while the best result in the n-dimensional case is still Krasnosel'skii's [6] original theorem of 1946.

Before investigating Valentine's results, it will be helpful to consider two additional concepts. One concept is that of the external cone of support to a set at a point of the set. <u>An external cone of support to a set at a point</u> is simply the union of all external rays of support to the set which have a common end-point belonging to the boundary of the set. The second concept is that of a one-sided point of external support to the interior of a set. If the cone of external support to the interior of a set exists and is contained in a half-space, the boundary point of the set, or the apex of the cone, is called a <u>one-sided point of external support to the interior of the set</u>.

It should be noted that in Theorem 2 the sets considered were not required to possess a connected interior. Obviously a star-shaped set must be connected, in fact, it is polygonally connected with at most two segments required to connect two points, but its interior may not have this property as evidenced by the set S in Figure 4. With the restriction added that the sets considered have a connected interior,

Valentine [12] produced the following which concerns a rather large class of sets in E_2:

Theorem 3: Suppose S is a bounded set in E_2 which is the closure of an open connected set. Then necessary and sufficient conditions that S be star-shaped (has a non-empty convex kernel) are:

1) Each point of local non-convexity has a non-empty cone of external support to the interior of S.
2) Given three points of local non-convexity of S which are also one-sided points of external support to the interior of S, there exist rays in the external cone at each point such that the corresponding complementary rays are concurrent and meet in S.

To see that both conditions of Theorem 3 are necessary, examples are readily available. In Figure 7, each point of local non-convexity has a non-empty cone of external support to the interior of S while in Figure 8 this is not the case, since a region of the complement of the set is isolated. If points of local non-convexity x_1, x_2 and x_3 in the set S of Figure 7 are considered, then condition (2) of the theorem is not satisfied while the set in Figure 8 satisfies condition (2) vacuously. To see that the set S in Figure 7 does not satisfy condition (2) of the theorem, the cones of external support to the set are illustrated by the restricting rays which are the broken lines.

As mentioned previously, Theorem 1 is a corollary of Theorem 3. The requirements of Theorem 1 included the feature of the sets having

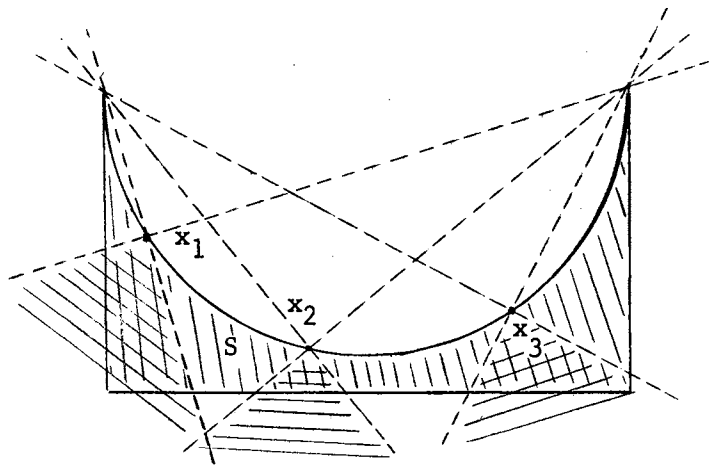


Figure 7

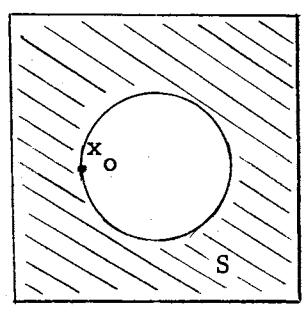


Figure 8

a simple polygon as its boundary. This fact allows for the existence of the cone of external support for each vertex of local non-convexity. Condition (2) of Theorem 3 is almost the same as the hypothesis of Theorem 1.

All of the results previously stated have been proved by using Helly's theorem [3] which is rather astounding to the beginning student. That is, it seems strange that the following (Helly's Theorem) has anything to do with star-shapedness: "Let F be a family of compact convex sets in a n-dimensional Minkowski space containing at least $n + 1$ members, then a necessary and sufficient condition that all members of F have a non-empty intersection is that every $n + 1$ members have a point in common."

The proof of Theorem 3 actually gives a method of finding the convex kernel of the set being considered. If a set satisfies the conditions in Theorem 3, then its convex kernel is the intersection of the convex hull of the set and the closed convex hull of the complementary cones of each point of local non-convexity. Although this intersection might appear otherwise, it is a subset of the set being considered. The need to intersect the convex hulls of the set and complementary cone of support is to satisfy the hypothesis of Helly's theorem. For example, in Figure 9 the points of local non-convexity of the set are x_1, x_2, x_3 and x_4 while the intersection of their closed complementary cones and the convex hull of the set has as its boundary the pentagon with vertices x_4, x_5, x_6, x_7 and x_8. Of course this procedure is relatively simple to use if the set of points of local non-convexity is finite and the cones of external support are

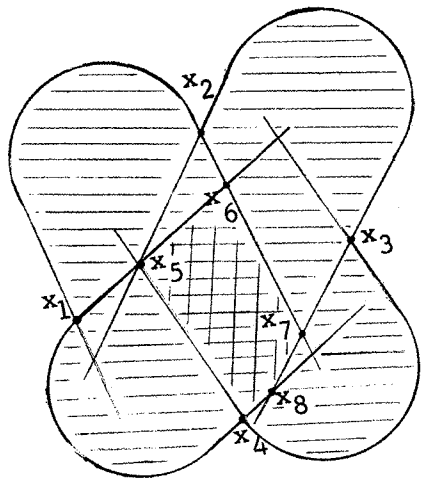


Figure 9

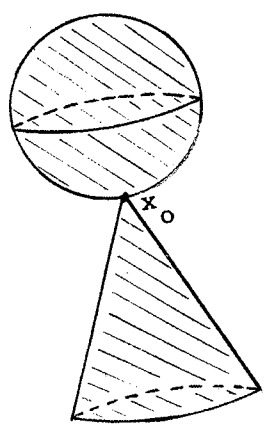


Figure 10

easily found.

In proceeding to E_n, $n \geq 2$, first to be considered is an existence theorem of Valentine [11]. This result is a by-product of some results involving a study of polygonally connected sets.

<u>Theorem 4:</u> Suppose S is a closed connected set in E_n, which has a unique point of local non-convexity. Then S is star-shaped with respect to this point.

Figure 10 exhibits a set in E_3 with the properties described in the hypothesis of Theorem 4. It happens that x_o is the only point of local non-convexity and the only point in the convex kernel. A question which might follow is, "Is this unique point of local non-convexity always the only point in the kernel?" The answer is negative. In Figure 11, the given set has a unique point of local non-convexity but the kernel is found to contain points other than the point p. It is easy to see that the converse of Theorem 4 does not hold, for the set in Figure 12 is star-shaped but both p and q are not points of local convexity.

As mentioned previously, any of the attempts to establish the existence of the convex kernel of a given set have utilized properties of the boundary of the set. A subset of the boundary which plays a very important role in Krasnosel'skii's theorem is the set of regular points of the set. Precisely, a boundary point p of a set S is a <u>regular point</u> of S if a hyperplane exists, which contains the point p and also which supports the subset of S which can see p via S.

In Figure 13, x_1 is a regular point of the given set with the

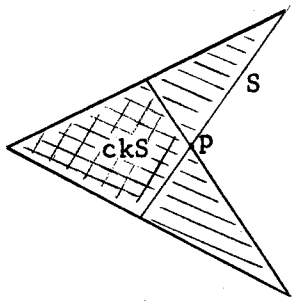


Figure 11

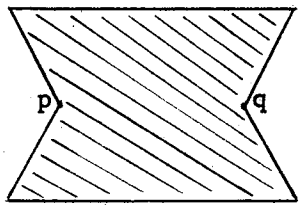


Figure 12

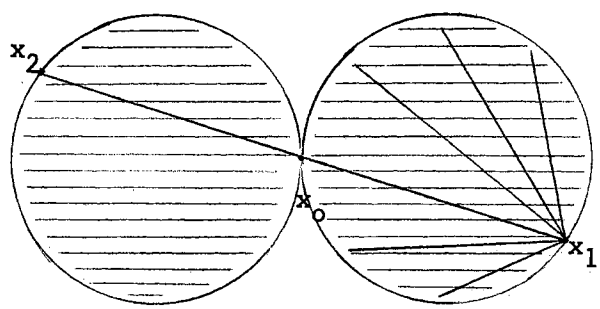


Figure 13

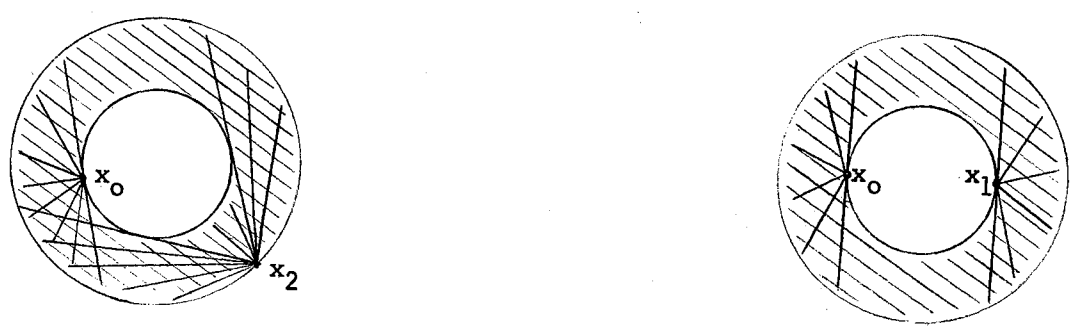


Figure 14

required hyperplane being the line tangent to the disk at x_1. The boundary point x_o is not a regular point of the set since every hyperplane passing through x_o strictly separates at least two points which can see x_o via the set.

With this background, Krasnosel'skii's theorem [6] can now be stated:

Theorem 5: Let S be a non-empty compact connected set in E_n. Suppose that each n + 1 regular points of S can see at least one point of S via S. Then the convex kernel of S is non-empty.

Each collection of three regular points of the set in Figure 13 can see x_o via the set and x_o belongs to the kernel of the set. In Figure 14, the set does not possess a non-empty kernel while 14a exhibits two regular points which can see some common points. The need for all collections of three or fewer regular points to see a common point is exemplified in 14b where the subsets which see x_o and x_1 via the set are disjoint.

The requirement in Theorem 5 for the subcollections to contain n + 1 points is necessitated by Helly's theorem. It is easily seen that a converse of Theorem 5 is true since every point of a star-shaped set can see the kernel via the set, however, as mentioned earlier, the set need not be compact.

If a set satisfies the hypotheses of Theorem 5, its convex kernel can actually be characterized by the following theorem which is deduced from Valentine's proof [10] of the theorem:

<u>Theorem 6:</u> If S is a non-empty compact connected set in E_n such that each collection of n + 1 regular points of S sees at least one point of S, then

$$\mathrm{ck}S = \bigcap_y (C_y \cap \mathrm{conv}S)$$

where y is a regular point of S and C_y denotes the intersection of all closed half-spaces which contain y and support those points of S which can see y via S.

Although it is often quite easy to find the kernel of two and three dimensional sets, the formulation of a procedure for arbitrary finite dimensional spaces is rather cumbersome. For any set which can be exhibited by a diagram, one often relies on intuition to find the convex kernel, however, Theorems 5 and 6 are actually being employed.

Occasionally in arbitrary linear topological spaces, it can be shown that a non-convex set is star-shaped. However, it appears that any astounding results, say an analogue to Krasnosel'skii's theorem, are not to be found. One slightly interesting generalization for closed sets is given by Valentine [10]:

<u>Theorem 7:</u> Let S be a closed set in a linear topological space and suppose K is a compact subset of S of dimension n. If each set of n + 1 boundary points of S can see at least one point of K via S, then S is star-shaped.

A general converse to Theorem 7 does not exist. If the set S of the theorem is star-shaped with respect to each point of a compact subset, then each boundary point can certainly see it via S. However, a star-shaped set need not be star-shaped with respect to each compact

subset.

Although the work heretofore in this chapter has been concerned with finite dimensional Euclidean spaces, it is sometimes possible to characterize the convex kernel of some sets in infinite dimensional spaces. The example which follows will demonstrate this idea.

<u>Example 1:</u> In the space of real sequences consider the following set:

$$E = \{ x = (c_1, c_2, \cdots) \mid c_i = 0,\ i \text{ odd; or } c_i = 0,\ i \text{ even; or } c_i = 0,\ i = 1, 2, \cdots \}.$$

Both $x = (1,0,0,\cdots)$ and $y = (0,1,0,0,\cdots)$ belong to E, yet $\frac{1}{2}x + \frac{1}{2}y = (\frac{1}{2},\frac{1}{2},0,0,\cdots)$ does not belong to E. Hence E is not convex. It happens that the convex kernel of E contains exactly one point, the origin, ϕ. The kernel contains ϕ since $\alpha\phi + (1-\alpha)x = (1-\alpha)x$ for all x in E and for $0 \le \alpha \le 1$, in fact for all real α. Also for every z in E, $z \neq \phi$, there exists a vector w in E such that w cannot see z via E. To demonstrate this fact, let $z = (z_1, z_2, \cdots)$, then for some coordinate, $z_i \neq 0$ and $z_{i+1} = 0$. Consider $w = (w_1, w_2, \cdots)$ in E such that $w_{i+1} = z_i \neq 0$. This choice of w_{i+1} forces w_i to be zero. Then $\frac{1}{2}z + \frac{1}{2}w = (\frac{1}{2}z_1 + \frac{1}{2}w_1, \frac{1}{2}z_2 + \frac{1}{2}w_2, \cdots, \frac{1}{2}w_{i-1}, \frac{1}{2}z_i, \frac{1}{2}w_{i+1}, \cdots)$ does not belong to E since $\frac{1}{2}w_{i+1} = \frac{1}{2}z_i \neq 0$, i.e. the point $\frac{1}{2}z + \frac{1}{2}w$ has two successive non-zero coordinates. Hence the set E is star-shaped with respect to the origin and not with respect to any other point.

At present, no general results are available concerning the characterization of the convex kernels of sets such as E in Example 1.

Below is an example in a function space of a non-convex set that has a convex kernel consisting of more than one point:

<u>Example 2:</u> In the space of real functions defined on $I = [0,1]$, let $S = \{ f \mid f(I) \subset I,\ f(0) = 0,\ f$ continuous except at $x_i \in I,\ 1 \le i \le N \}$. S is not convex since

$$f(x) = \begin{cases} x, & x \neq 1, \\ 0, & x = 1, \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0, & x \neq 1, \\ 1, & x = 1, \end{cases}$$

both belong to S, yet $(\frac{1}{2}f + \frac{1}{2}g)(x) = x/2$ for all $x \in I$, which implies $\frac{1}{2}f + \frac{1}{2}g \notin S$. The convex kernel of S,

$$\text{ckS} = \{ f \mid f \in S,\ \alpha f + (1-\alpha)g \in S \text{ for all } g \in S,\ 0 \le \alpha \le 1 \},$$

is characterized as follows:

> Given $f \in S$, a necessary and sufficient condition that $f \in \text{ckS}$ is that f is discontinuous at $x = 0$.

The sufficiency of this characterization follows from the definition of ckS and from the fact that for any $g \in S$, $g(0) = 0$. To verify the necessity of f being discontinuous at $x = 0$, a contrary assumption allows for a choice of a function in S, namely,

$$g(x) = \begin{cases} (1-f(x))(x/c), & 0 \le x \le c, \\ 1-f(x), & c \le x \le 1, \end{cases}$$

where f is continuous on $[0,c]$ and $f(x) \le \frac{1}{2}$ for all $0 \le x \le c$, such that $\frac{1}{2}f + \frac{1}{2}g \notin S$, a contradiction of $f \in \text{ckS}$.

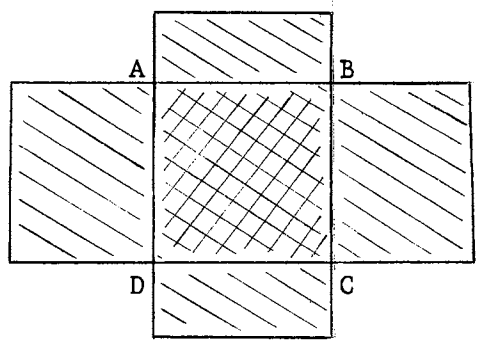
CHAPTER III

CONVEX SETS AS CONVEX KERNELS OF NON-CONVEX SETS

A situation of equal interest to that of characterizing the convex kernel of a given set concerns the realization of a given convex set as the kernel of a non-convex set. Problems of this type are the so-called extension problems.

If the convex kernel of a set exists, then clearly it is unique. However, the above extension of the given convex set may not be unique. For example, in Figure 15, both non-convex sets are extensions of the regions bounded by the squares ABCD and A'B'C'D'. Thus it is seen that in the case of the region with the square as its boundary there can exist several extensions.

All sets do not admit extensions. This fact is illustrated by a sequence of plane sets in Figure 16. Convex set K in 16a, which is an open triangular region and three points, x_1, x_2 and x_3 of the interior of the distinct sides of the triangle, cannot be realized as the convex kernel of a non-convex set. Under a contrary assumption, suppose $p_o \notin cl(K)$ sees K via some set S, then a segment of the interior of one side of the triangle, e.g. $x_o x_3$ in 16b, is contained in the convex kernel of S. This fact is verified in 16c since if p_1 sees y_1 via S, p_2 sees y_2 via S and p_3 sees x_2 via S, then $x_o x_3 \subset ckS$. If $p_o \in cl(K)$, as in the set in 16d, then p_o plays the role of x_o in 16b



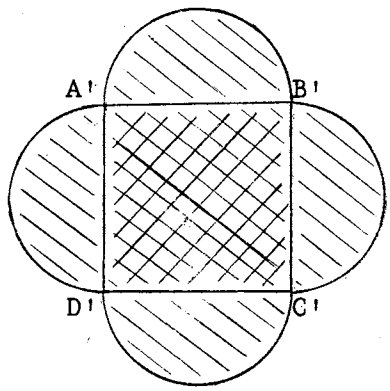


Figure 15

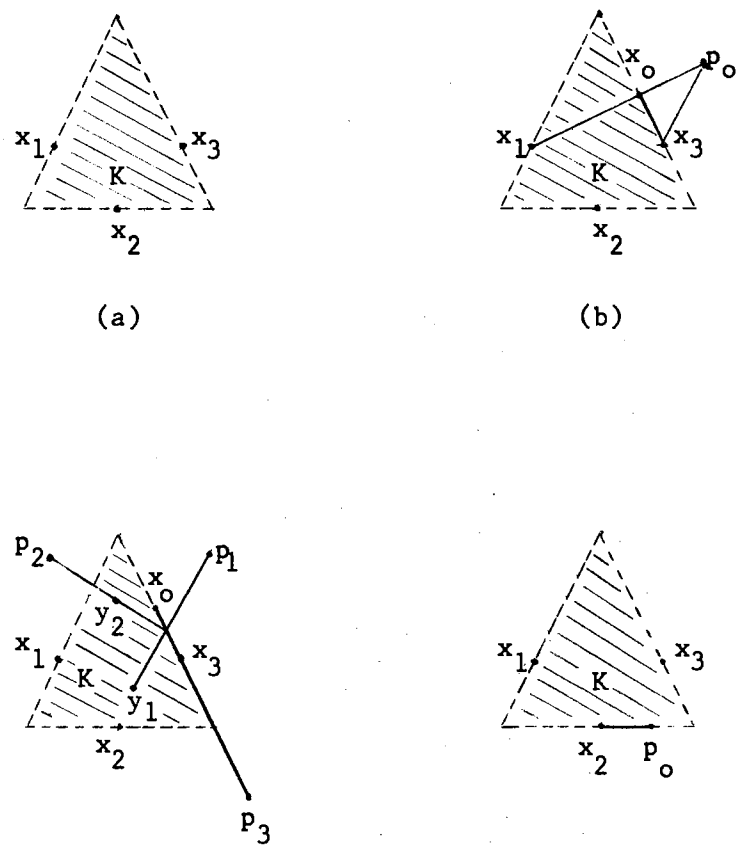

x1
x3
K
x2
(a)
xo
po
x1
x3
K
x2
(b)
p2
p1
xo
y2
x1
x3
K
y1
x2
p3
(c)
x1
x3
K
x2
po
(d)

Figure 16

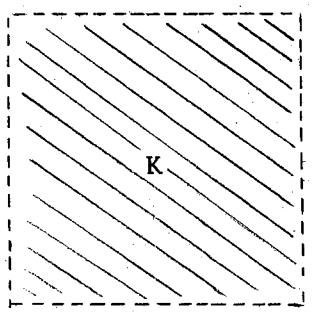
and 16c. Hence K is properly contained in the convex kernel of S, i.e. S is not an extension of K.

In Figure 15 the sets under consideration were closed while in Figure 16, K is not closed. In general, the results concerning the problems of extension are based on having a closed set to play the role of the kernel. However, other results are available for large classes of plane sets and they will be investigated first.

A concept required for discussion of the extension problems is that of the K-star of a convex set K. A set S, different from K, such that K = ckS, will be called a <u>K-star</u>; moreover, if cl(S) is different from cl(K), then S is a <u>proper</u> K-star. If $K \neq S$, K = ckS and cl(K) = cl(S), then S is called an <u>improper</u> K-star. Consequently any K-star of a closed set K is proper since cl(K) is exactly K. The region in Figure 15 admits a proper K-star which is illustrated in both cases. In Figure 17, the open-square region in 17a has a K-star, namely the set in 17b, which is the union of K and two of its boundary points, x_o and x_1, but this K-star is improper. This is true since the closures of S and K are identical.

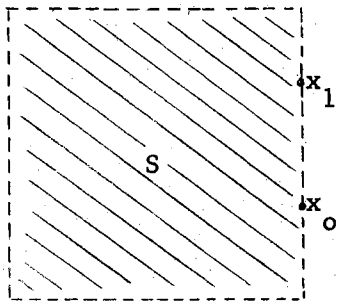
Possibly the simplest of all convex sets, subsets of straight lines, would be a good beginning point for any development of the theory of extensions. Post [7] produced the following, seemingly obvious, result:

<u>Theorem 8:</u> If K is a convex subset of a straight line L in E_2, then a proper K-star exists if and only if $K \neq L$.



Open Convex Set

(a)



Improper Extension

(b)

Figure 17

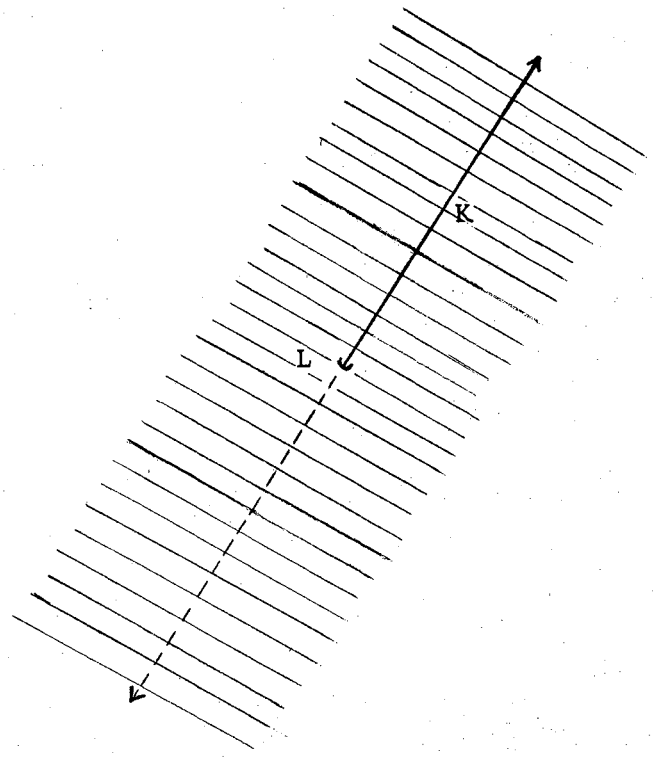


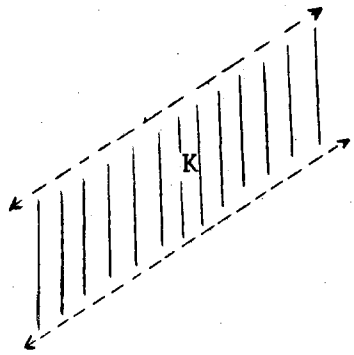
Figure 18

As seen in Figure 18, if $K \neq L$, then K is the convex kernel of the union of K and the complement of L. The closure of such a set is the whole plane while the closure of K is a closed ray. Therefore K is the kernel of a proper K-star.

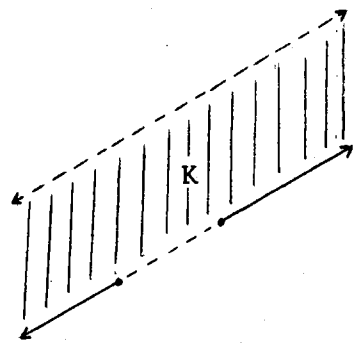
In developing a proof of Theorem 8 some unexpected results concerning infinite strips in the plane and half-spaces are observed. While verifying that the existence of a proper K-star is sufficient to show that K cannot be the line L, it is seen that a convex, two-way infinite strip between two parallel lines in E_2, although it is the kernel of a non-convex set, has no proper K-star.

The open two-way infinite strip K in Figure 19a admits many K-stars, one of which is the set in Figure 19b, which is the union of K and two rays of one bounding line. This set is not convex and each point of the set sees K via the set. However, the set in 19b is not a proper K-star since its closure is identical to the closure of K. In fact, as seen in Figure 19c, if a point which is not in the closure of K can see K via some set, then a boundary point of K must also be in the kernel of the set.

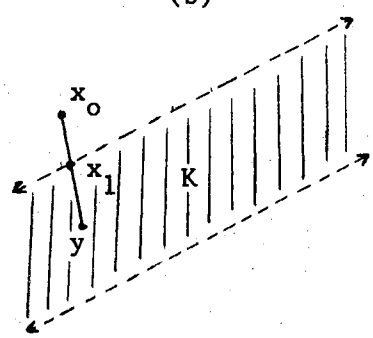
Another consequence of the development of Theorem 8 is that a convex half-plane has no proper star extension and a convex half-plane which is not open admits no star extension at all. For example, in Figure 20, if a point x_o can see each point of the neither open nor closed convex half-space H, then another point x_1 which sees H via some set S also sees x_o via S. Hence a new convex set, which is the convex hull of x_o and H and properly contains H, contains the kernel of S. Therefore, no star extension exists for H.



(a)



(b)



(c)

Figure 19

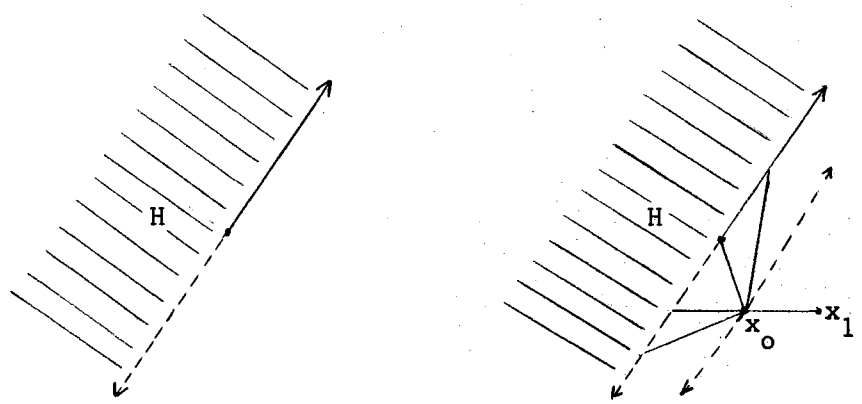
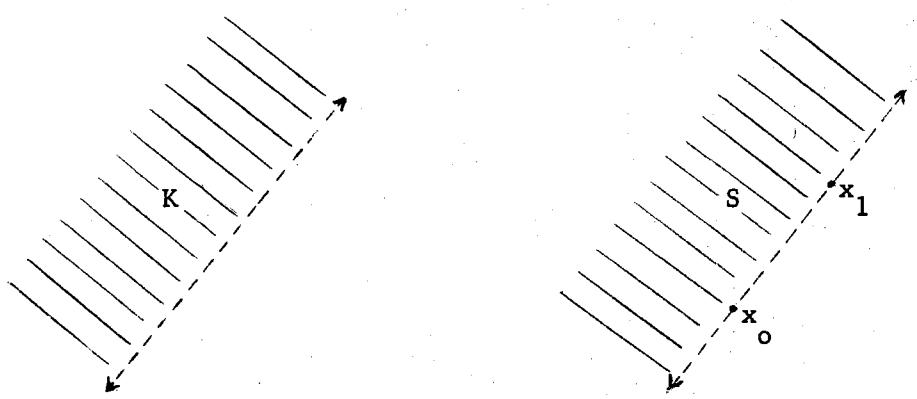


Figure 20



(a) (b)

Figure 21

An open half-plane can be extended, as seen in Figure 21. The set K in 21a is exactly the kernel of the set in 21b which is the union of K and the set consisting of the two points x_o and x_1, but the closure of K and the closure of the extension are the same. Hence the K-star is improper.

Therefore by consequences of Theorem 8, it can be verified as to whether or not those plane convex sets which are subsets of a line, two-way infinite strips or half-planes admit a star extension. Furthermore, it can be determined if the extension is proper or improper. It is necessary that the setting for Theorem 8 be a plane since a line in E_3 can be the convex kernel of a non-convex set. One such non-convex set is the union of two distinct planes which have the given line in common. To be discussed later in this chapter will be a theorem of Klee [4] which gives a similar result for certain Banach spaces that are more general than E_2.

When a student, not necessarily a student of convexity, is asked for an example of a convex set, the reply is usually that of a closed disk or a slight distortion of such a region. Sets of this type are called strictly convex sets. Precisely, a <u>strictly convex set</u> is one for which each hyperplane of support has exactly one point in common with the boundary of the set. In Figure 22, the set S in 22a is constructed to be strictly convex while in 22b any hyperplane of support at x_o does not have a one point intersection with the boundary of T.

It happens that any star extension of a strictly convex set K is a proper K-star. This is a lemma of Post [7] and follows from the fact

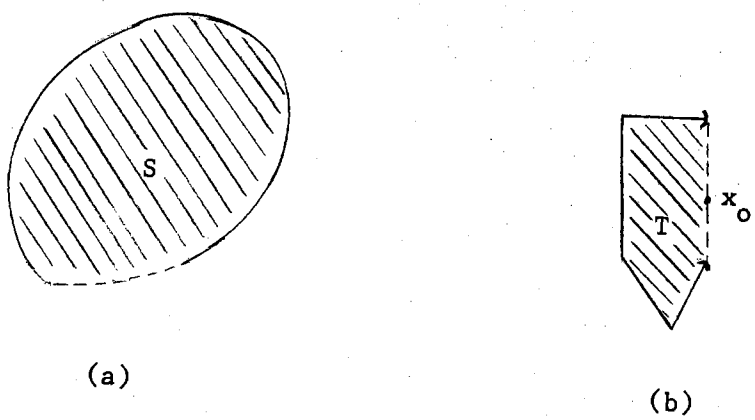


Figure 22

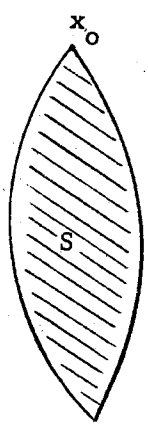


Figure 23

that any set S, satisfying $K \subset S \subset cl(K)$, is convex if K is strictly convex, hence an improper K-star would be convex, which is a contradiction of the definition of a K-star. For example, the extension of an open circular disk, if it exists, must include points not in the bounding circle, otherwise an extension would only be a new convex set which is contrary to the definition of a K-star.

A nice class of convex sets is that collection in which a hyperplane of support for a given set not only intersects the set in exactly one point but also the plane of support is unique for that point. Simply stated, this class is those strictly convex sets with each boundary point similar to the boundary points of a circular disk. A strictly convex set which is <u>not</u> in this class is given in Figure 23. Point x_o of the given convex set S does not have a unique line of support passing through it.

With the above class of sets, the following result was obtained by Post [7]:

<u>Theorem 9:</u> For the existence of a star extension of a strictly convex set K in E_2 such that each boundary point of the set admits only one line of support through it, it is necessary and sufficient that bd(K) contains an arc A such that $A \setminus K$ is at most countable.

As is often the case, the existence Theorem 9 does not produce the method for a solution. For example, to find a non-convex set for which a given closed disk in the plane is its kernel is not an easy task. Post [7] proves that such a set must exist, but a geometric formulation is lacking.

Some corollaries to Theorem 9 are 1) that a strictly convex closed set K with boundary as required in the theorem has a proper K-star while 2) such a set which is non-empty and open does not admit an extension. Simply stated, convex sets which are disk-like and contain an arc of their boundary can be properly extended while sets similar to the open disk in E_2 cannot be extended. It should be noted that these somewhat general results do not require the set which is being extended to be closed. However, there is no hope of a proper extension in the case of the open strictly convex set.

Mentioned earlier was the fact that any general theorems concerning the extension problem evolved around closed sets. Even though such a set can be extended, Figure 15 reveals that the extension may not be unique. Summarizing and making use of Theorem 9 and some of its corollaries, the most complete result for extension of a closed set in the plane is also given by Post [7]:

<u>Theorem 10:</u> A closed convex set in E_2 can be realized as the convex kernel of a non-convex set if and only if it is neither a half-plane nor a two-way infinite strip.

Although the extension may not be proper, some open sets can be extended. This is illustrated in Figure 17. Also, sets which are neither open nor closed might be extended. Figures 24 and 25 both exhibit such sets. The set in 24b is an improper extension of the set in 24a which consists of the open triangular region and two boundary points, x_1 and x_2. It happens that the set in Figure 24a has no proper extension. In Figure 25, the set in part (a) which is

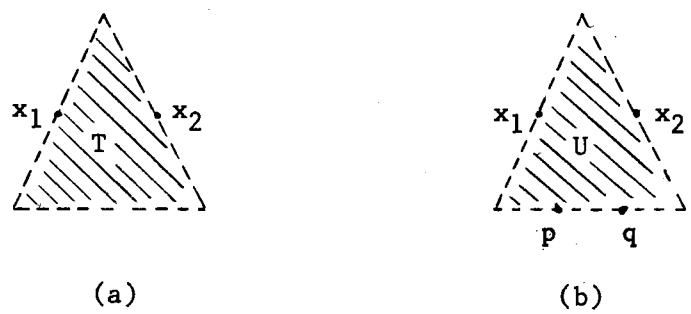


Figure 24

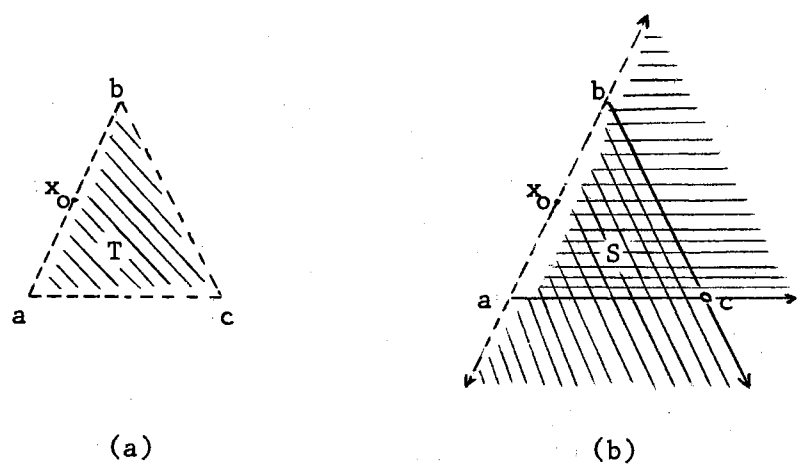


Figure 25

the union of the open triangular region and the boundary point x_o, can be properly extended. An extension is shown in 25b. This non-convex set has as its convex kernel, the set T in 25a. This fact may not be obvious at first but careful inspection of 25b and noting that point c does not belong to S produces the desired conclusion.

So far in this chapter, all of the accomplishments mentioned have been in E_2. Several of the examples given and some of the results stated could have had the setting of E_n, n larger than two. For example, a closed cube in E_3 can be realized as the kernel of a non-convex set. Such a set could be formed by constructing closed cubes on each face of the given cube. Since all of the spaces E_n, $n = 1,2,\cdots$, are complete normed linear spaces which possess a countable dense subset, i.e. separable, the available generalization of the extension problem is somewhat natural in development. Klee [4] gave such a generalization.

<u>Theorem 11:</u> If K is a closed convex set of a separable Banach space, the following three assertions are equivalent:

a) K contains no hyperplane;

b) K is the convex kernel of a non-convex set;

c) for all $0 < s < p$, K is the convex kernel of a closed non-convex set S, such that $K_s \subset S \subset K_p$, where K_s and K_p are open parallel bodies of K.

If the space of Theorem 11 is E_2, then parts (a) and (b) of the theorem are the hypothesis and conclusion of Theorem 10. This equivalence is realized since the closed set K could not be a half-

plane nor could it contain a two-way infinite strip without containing a hyperplane, which is a line in this case. Hence the statement that Theorem 11 is a generalization of Theorem 10 follows. This new result, Theorem 11, however, is not a generalization of Theorem 9 concerning strictly convex sets. To satisfy the hypothesis of Theorem 9, the set to be extended need not be closed, therefore Theorem 11 is not applicable.

Examples of sets in two and three dimensional Euclidean space to illustrate the corresponding forms of Theorem 11 have been given throughout this chapter. Examples to verify the theorem's assertions in infinite dimensional spaces can also be found. The example which follows gives an extension for a one-dimensional flat in an infinite dimensional space, however, the procedure of extension is the same for any finite-dimensional flat.

<u>Example 3:</u> In the space of real sequences, ℓ^2, which is a separable Banach space, consider the set

$$K = \{x = (c_1, c_2, \cdots) \mid c_n = 0,\ n = 2,\ 3,\ \cdots\}.$$

The set is convex and closed and contains no hyperplane since it has co-dimension greater than one. Hence by the authority of Theorem 11, there exists a non-convex set S such that $K = ckS$. One such set can be defined by

$$S = \{x \mid c_{2n} = 0 \text{ or } c_{2n+1} = 0,\ n = 1,\ 2,\ \cdots,\ \text{or } c_n = 0,\ n = 1,\ 2, \cdots\}.$$

This set is not convex since $x = (1,1,0,0,\cdots)$ and $y = (1,0,1,0,0,\cdots)$ belong to S, but $\frac{1}{2}x + \frac{1}{2}y = (1,\frac{1}{2},\frac{1}{2},0,0,\cdots)$ does not belong to S. To

see that K is a subset of ckS, consider $x = (c,0,0,\cdots)$ in K and $y = (d_1,d_2,\cdots)$ in S. Then $(1-\alpha)x + \alpha y = ((1-\alpha)c + \alpha d_1,\alpha d_2,\alpha d_3,\cdots)$ belongs to S for all α such that $0 \leq \alpha \leq 1$ since $d_i = 0$ implies $\alpha d_i = 0$ for any $i = 1, 2, \cdots$. To complete the verification that K = ckS, consider any $x = (c_1,c_2,\cdots)$ in $S\setminus K$. Since x does not belong to K, there exists an $i \geq 2$ such that $c_i \neq 0$. Choose $y = (d_1,d_2,\cdots)$ in S such that $d_{i+1} = c_i$, which forces $d_i = 0$, then $\frac{1}{2}x + \frac{1}{2}y$ has two successive non-zero coordinates, neither being the first coordinate. Hence $\frac{1}{2}x + \frac{1}{2}y \notin S$ and $x \notin ckS$. Therefore $S\setminus K$ and ckS are disjoint and K = ckS.

As mentioned prior to the example, the technique used in Example 3 for the extension of a one-dimensional flat can be used to find a K-star for any finite-dimensional flat. This fact is emphasized to contrast the works of Post [7] and Klee [4] concerning extensions of convex sets. In the setting of Post's paper, any finite-dimensional flat did not admit a star-extension, whereas the setting of Klee's paper does allow such an extension.

Although Klee's theorem, Theorem 11, is an existence-type theorem, it does give some information about the nature of the possible extensions. This partial description is in part (c) of the theorem and says that if an extension of a set exists, then there is at least one K-star between any two open parallel bodies of a set K. However, for certain norms, a parallel set is not always meaningfully characterized.

CHAPTER IV

RECENT ACTIVITY AND SOME INTERESTING UNSOLVED PROBLEMS

In Chapter I, it was stated that the basic problems of interest concerning convex kernels were 1) given a non-convex set, find its kernel and 2) given a convex set, find a non-convex set for which the given set is its kernel. Chapters II and III were devoted to various attempts to answer these questions in general. Throughout those chapters certain specific problems which might be of interest were mentioned in passing.

Although the basic problems remain in a general setting, several specific interesting situations have arisen while general solutions were being attempted. For example, in Chapter II several theorems were stated which involved the existence of a convex kernel for a given set. Suppose the restriction is added that the kernel have a certain dimension, then certainly a much stronger hypothesis could be expected if the theorems were to be true. Consequently, a more general problem is at hand.

All the results concerning the extension problem of Chapter III were in the setting of a separable Banach space. The fact that the space was separable was very important in the work of both Klee [4] and Post [7]. As of the present, a stronger result, deleting the condition of separability from the hypothesis, has not been published. Hence

another case of a particular problem which is an out-growth of one of the original questions, that of extension.

Since much of the activity involving the convex kernel has been relatively recent, a current collection of unsolved or recently solved problems is lacking. Valentine [10] provides some problems in this area, but his collection is dominated by questions concerning star-shapedness relative to spherical surfaces and by questions about Helly-type theorems.

Concerning the existence of the convex kernel of a given set the most general problem is given first:

Problem 1: Let S be a set in a linear space L (a linear topological space if a topology is required). Find necessary and sufficient conditions for the existence of $K = ckS$ and for the characterization of K.

Of course, the solution of Problem 1 is ideal and almost the entirety of Chapter II was devoted to partial solutions of this problem. All of the published partial solutions have had the setting of a finite-dimensional space. Various problems could be stated with the setting of a particular space, such as a Banach space.

A problem of Valentine [10] which has received considerable attention recently is:

Problem 2: Let S be a set in a linear space L. Determine necessary and sufficient conditions that the convex kernel of S has dimension k. In particular, determine this for $k = 0$ so that S has a one point kernel.

All of the known results concerning Problem 2 have been for the particular case where the kernel was to be a singleton set. One partial solution (which is incorrect) in the current literature was given by Hare and Kenelly [2]. Their result is summarized with the following definition and theorem: Definition: A set S in E_n, $n \geq 2$, has property k_n if and only if given the set, $\{x_1, x_2, \cdots, x_{k+1}\}$, of affinely independent points ($2 \leq k \leq n$), there exists a unique point $p \in S$ such that $x_i p \subset S$, $i = 1, 2, \cdots, k+1$. Theorem: A compact set $S \subset E_n$ has a one point kernel if and only if S has property k_n, $n \geq 2$.

The above theorem is false since it is not necessary for the set S to have property k_n in order for it to possess a one point kernel. For a counterexample, see the set in Figure 4. This set has a one point kernel yet it lacks property k_2 since many collections of three affinely independent points can see arbitrarily many points of the set. From the results of Tidmore [9], it is reasonable to believe that the sufficiency portion of the theorem of Hare and Kenelly is correct.

Tidmore [9] produced a more general sufficient condition for the existence of a one point kernel than was suggested by Valentine and supposedly given by Hare and Kenelly [2]. This result is as follows: Theorem: Consider a set S in a linear space such that the dimension of S is larger than one. If for each set of k affinely independent points ($k = 3, 4$), there exists a unique point x_o such that $x_o x_i \subset S$, $1 \leq i \leq k$, then S has a one point kernel.

From Figure 4, it is easy to see that the star-like condition in the hypothesis of Tidmore's result is not necessary since each set of

three affinely independent points can see more than one point via the set, but still the set has a one point convex kernel. Although Tidmore does not have necessary conditions for the convex kernel to have dimension zero, it is of particular importance that he required neither the condition of compactness nor connectedness on the set S in his sufficient condition.

A tool used in the study of convexity is that of decomposition of a set into a certain class of its subsets. One seemingly natural collection of subsets of a non-convex set which is often used is the collection of convex subsets of the given set. Much of this paper has concerned itself with characterizing the convex kernel of a star-shaped set by one of several methods. It might be possible to describe the convex kernel of a non-convex set in terms of the convex subsets of the non-convex set. The following problem, an adaptation of a Valentine [10] problem, is interesting.

Problem 3: Characterize the convex kernels of those star-shaped sets S in L, a linear topological space, in terms of the maximal convex subsets of S.

Concerning the problem of extension of convex sets to be convex kernels of non-convex sets, several interesting unsolved problems exist. Analogous to Problem 1, the following problem is the most general of the extension problems:

Problem 4: Let K be a closed convex set in a linear topological space L. Find minimal sufficient conditions such that K admits a K-star.

As stated in Chapter III, the most complete solution of Problem 4 is given in Theorem 11 by Klee [4]. By restricting the setting to a separable Banach space, Klee was able to give a necessary and sufficient condition that a closed set K admits a K-star.

Two special cases of Problem 4, which, if solved, might lead to a solution of the general problem, are also given by Klee [4]:

Problem 5: Find necessary and sufficient conditions that the unit ball K of a non-separable inner product space admits a K-star.

Problem 6: Find necessary and sufficient conditions for K, a closed and bounded convex set which has no supporting hyperplane, in a separable incomplete inner product space to admit a K-star.

Before Problem 6 could be attacked, it might be of interest to know whether or not a closed and bounded set with no supporting hyperplanes existed. Several years prior to proposing Problem 6, Klee [5] gave the following example of a set which satisfies the requirements of the set K in the problem:

Example 4: Consider the linear transformations T of ℓ^2 into itself such that if $x = (c_1, c_2, \cdots)$ then $T(x) = (d_1, d_2, \cdots)$ with $d_i = 2^{-i}c_i$, $i = 1, 2, \cdots$. Let $C = \{x \mid \|x\| \leq 1\}$ and $L = \{x \mid \sum_{i=1}^{\infty} (2^i c_i)^2 < \infty\}$. If M is a dense linear subspace of ℓ^2 such that $L \cup M$ spans the space, then $T(C \cap M)$ is a closed and bounded convex set which has no points of support. The verification of this example is rather involved and will not be given. The details are

supplied by Klee [5].

Certainly in function spaces as well as Euclidean spaces, the concept of the x-star, i.e. those points of a set which can see x via the set, has yet to be exploited. This concept seems to have a place in both the problem of existence of convex kernels and the problem of extension of convex sets.

It may be that more tools are needed before attacking the problems of generalization mentioned in this chapter. Possibly the equipment needed to solve these problems is available but the students of convexity are unaware of its existence. After all, it was more than twenty years after the publication of Helly's theorem before Krasnosel'skii published his results and almost another twenty years before the recent flurry of activity involving the convex kernel.

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VITA

Charles Richard Smith

Candidate for the Degree of

Doctor of Education

Thesis: CONVEX KERNELS

Major Field: Mathematics

Biographical:

Personal Data: Born in Campti, Louisiana, September 11, 1936, the son of Albert Jackson and Mary Julia Smith.

Education: Attended elementary and high school at Montgomery High School, Montgomery, Louisiana, and was graduated in 1954; received the Bachelor of Science degree from Northwestern State College, Natchitoches, Louisiana, with a major in mathematics, in August, 1957; attended Lamar State College, Beaumont, Texas, McNeese State College, Lake Charles, Louisiana, and the University of Southwestern Louisiana, Lafayette, Louisiana, part-time from 1959 until 1963; received the Master of Science degree in mathematics from Oklahoma State University, Stillwater, Oklahoma, in August, 1964; completed requirements for the Doctor of Education degree with emphasis in mathematics at Oklahoma State University, in July, 1966.

Professional Experience: Worked in operations with East Texas Pulp and Paper Company, Silsbee, Texas, from January, 1958, until September, 1959; high school Mathematics-Science teacher with Calcasieu Parish School Board, Lake Charles, Louisiana, from September, 1959 until June, 1963; instructor in the Night School, Chennault Air Force Base, Lake Charles, Louisiana, from 1959 until 1963; instructor in the Enrichment Program, Department of Special Education, McNeese State College, Summer, 1961; graduate assistant in the Department of Mathematics and Statistics at Oklahoma State University during the period 1964-1966.

Professional Organizations: Member of the Mathematical Association of America.