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# LORENTZ ENSEMBLES IN THE STOCHASTIC THEORY OF SPACE-TIME

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LORENTZ ENSEMBLES IN THE STOCHASTIC

THEORY OF SPACE-TIME

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# LORENTZ ENSEMBLES IN THE STOCHASTIC THEORY OF SPACE-TIME

## CHAPTER I

## INTRODUCTION

It seems unlikely that man is aware of more than an infinitesimal fraction of all that affects his existence. His progressive nature however, results in a continual investigation and consequent expansion of his perception of that within which he exists, i.e., his universe. As his awareness has increased, the means by which he tries to describe his universe has evolved from rather simple bodies of knowledge and speculation having limited ranges of validity to more comprehensive theories. It has often been the inability of current theories to cope with problems arising from man's awareness of new domains, e.g., smaller or greater distances, that has led to the development of new theories.

The first great formulation of laws describing what man calls his physical universe was the mechanics of Newton. His theory, developed around 1670, resulted from observations that were comparatively slight extensions of man's direct sensory perception. Newtonian mechanics was therefore developed to describe particles which are massive compared to, say, an electron and slowly moving compared to an electromagnetic photon. As man extended the range of phenomena he might observe, he became interested in problems associated with new domains. It was the recognition of the inability of Newtonian, or classical, mechanics to describe either particles moving near the speed of light or particles of atomic dimensions that led to the two most significant advances of physical theory in this century, the theory of relativity and quantum theory. Significantly, both theories yield the mechanics of Newton as a limiting case, i.e., as velocities become small and the particles become more massive. In fact, the extraordinary success of Newtonian mechanics within its proper domain seems to imply that it must be a special case of all more general formulations of physical theory. It is an interesting fact that the development of more general theories has always resulted in more precise specification of the domain of Newtonian mechanics.

The development of physical theory is again on the threshold of great advancement. With the advent of non-relativistic quantum theory it was recognized that a more generally applicable theory, a relativistic quantum theory, must follow. Although there have been many attempts to construct a quantum theory which is relativistically correct, a satisfactory formulation has not been realized to date. In addition to being expected on theoretical grounds, the existence of a more general theory is also necessary since once again man has found current theories inadequate to describe observed phenomena.

Neither special relativity nor quantum theory serves to describe phenomena in the sub-nuclear domain. This paper is concerned with the foundations upon which a more general formulation, i.e., relativistic quantum theory, may be constructed.

The possibility of mass-energy transmutation, i.e., the creation and annihilation of material particles, as predicted by the theory of relativity, was appreciated at the time of the first attempts at formulating a relativistically correct quantum theory. Although Dirac's electron theory[1] (1928,1931) did account for the existence of anti-particles, its formalism did not include the creation and annihilation phenomena. That particular difficulty was removed with the introduction of quantum field theory (hereafter abbreviated as QFT) by Heisenberg and Pauli[1] in 1929. Dirac's electron theory was later shown to be included in the quantum theory of the electron field.

The QFT of Heisenberg and Pauli scored many successes. At the very least it was far superior to its predecessors in giving qualitative descriptions of many natural phenomena. In addition to including in its formalism the creation and annihilation of particles, photons emerged in a natural way upon quantization of the electromagnetic field. The manner in which photons were described immediately suggested the possibility that other particles whose existence is observed are also related to force fields by the same quantization procedure. It was on this basis that Yukawa[1] predicted the existence of the pion from knowledge of the existence of nuclear forces.

Perhaps one of the most important contributions of the old QFT was that it heralded the development of a new philosophical approach to the formulation of physical theory. It was a natural extension of the considerations of Yukawa and others to associate with each kind of observed particle in nature a field $\phi(x)$ which satisfies an assumed wave equation. A particle interpretation is obtained upon quantization of the field. At least so far as formalism is concerned, this marked the first real departure from the pre-relativistic concept of a binary universe in which the relationships between matter and energy did not include transmutation. The primary intent of Dirac's theory was to describe particles, a feature it shared with Newtonian and non-relativistic quantum mechanics. By contrast, QFT is a theory of quantized fields in which increments of relativistic mass-energy characterized by rest mass, charge, spin, etc., are secondarily identified as particles. Significantly, the formalism of QFT regards the fields as more fundamental entities than the particles.

QFT has not, in general, been quantitatively successful. In fact, its only quantitative results which are in agreement with experiment are those concerned with the description of electrodynamics. Amazingly, the agreement of these results with experiment is extremely good. Its difficulties emphasized its unreliability but this single success was indicative of the presence of some degree of truth in the foundations of QFT. For this reason it was felt that rather than disregard the entire structure of QFT, a proper modification should proceed from extreme scrutiny of its postulatory basis. Theoretical investigations of this nature, begun in the early 1950's, resulted in what is now called axiomatic QFT. Essentially, an effort was made in the axiomatic approach to apply only what were felt to be universally satisfied physical constraints. In a word, although these efforts did yield a more rigorous formalism, few of the major difficulties of the old QFT were avoided.

As a result of his analysis (1936,1938,1939) of the applicability of QFT in connection with the self-fields of the elementary particles, Heisenberg[1] classfied interactions as those of the first and second kind. For the first kind, QFT shows that the density of particles of the self-field (photons) does not depend essentially on their energies whereas the density of self-field photons for the second kind show strong energy dependence. He concluded that the interactions of the first and second kind are, respectively, within and without the range of applicability of QFT. The electromagnetic self-field would be of the first kind.

Heisenberg also indicated how QFT must be modified to describe interactions of the second kind. He noted that in QFT one finds a probability of observing all photons of the self-field. They are found to be distributed like a cloud in a region immediately around their source particle. The source particle and this cloud are observed as a single particle. Besides reacting with their own source particle, the contact of the photon cloud of one elementary particle with that of a second particle gives rise to interactions between the elementary particles. In 1938 Wick[1] demonstrated that the photon cloud of a self-field of mass m extends mainly in a region of radius $\hbar/mc$ around the source particle. The density of high energy photons of the self-fields is large at points near the origin and gives rise to various kinds of infinities, e.g., the infinite mass. Heisenberg suggested that since the infinities are due to the high energy photons of the self-fields, their high energy contributions must be delimited in a new formulation of QFT. It appears necessary to introduce a fundamental constant $\lambda = \hbar c/E_o$ which discriminates photon energies in such a way that the high energy photons correspond to energies greater than $E_o$. The density of the high energy photons near the origin of the source particle would be moderated by some factor depending on the constant length $\lambda$. Just as Planck's constant h served to cut off the high energy radiation and thus avoid the ultra-violet catastrophe, $\lambda$ may play the role of cutting off the high energy region of the self-fields of elementary particles and thereby eleminate the ultra-violet divergences of QFT.

In the case of particles with Compton wave lengths much greater than $\lambda$ the effects of the self-fields may be expected to be contained only in constants, i.e., masses and charges. On the other hand, in phenomena concerning particles with wave lengths small compared to $\lambda$, the self-fields show dynamical effects. These considerations provide a means of estimating the magnitude of the fundamental length since QFT is known to work quite well for the electron-photon interaction but not for any other. $\lambda$ must therefore be less than the Compton wavelength of the electron ($10^{-11}$ cm) but greater than the Compton wavelength of a nucleon ($10^{-14}$ cm). Of course, the question then became one of how the fundamental length suggested by Heisenberg was to be incorporated into the formalism in a logical manner. A means by which this might be accomplished was suggested by R.L. Ingraham in his stochastic theory of space-time. This writer considers the work of Ingraham as a major contribution toward the development of relativistic quantum theory.

# CHAPTER II

## INGRAHAM'S THEORY OF STOCHASTIC SPACE-TIME*

Ingraham recognized that a possible explanation for the divergences of QFT had root in the structure of the index space of the quantum mechanical operators, i.e., space-time itself. The problem of infinite renormalization suggested the necessity for a high-momentum cut-off beyond which the density of high momentum states is drastically reduced, or equivalently, a fundamental length $\lambda = \hbar/mc$ expressing a lower bound to the fineness of measuring space-time coordinates. Moreover, he recognized that within this interpretation, the problem of divergences could be attributed to the insistence that all equivalent Lorentz observers describe a given physical field by the same mathematical object, e.g., a tensor or spinor. He noted that this requirement of form invariance need not be equated to the Principle of Relativity. Such considerations led him to the development of a stochastic theory of space-time and subsequently to a formulation of a finite QFT (as opposed to the older theory which may be styled "infinite" in every sense of the word!). The major features of his stochastic theory of space-time will be described in the following.

*This chapter is primarily a summary of references 2, 3, and 4.

Since space-time has an indefinite metric, it is not possible to define a notion of nearness in it without adding extra structure. This additional structure is included by asking for frame or observer dependence of physical field determinations. Ingraham's method of accomplishing this was to treat the index space of the quantum mechanical operators as a manifold of random, or stochastic, coordinates rather than perfectly sharp coordinates as is the procedure in infinite QFT. This is done in such a way that the Principle of Relativity, in its general sense of complete equivalence of all Lorentz frames, is satisfied. An immediate consequence is the violation of form invariance, i.e., different observers do not in general describe the same physical field by identical mathematical fields (functions which are tensor or spinor transforms of each other). In fact, it was shown by Ingraham that stochastic space-time theory precludes the possibility of general form invariance.

From another point of view, Ingraham's theory amounts to recognition in current sharp theories of space-time that a given Lorentz observer is capable of measuring the coordinates of an event E with perfect accuracy, i.e., his repeated measurements of the same event would always yield the same four numbers. It is also assumed in sharp theories that he can measure observable fields $\varphi(x)$ at E with perfect accuracy. Moreover, since the measurements of the coordinates and fields at E by equivalent Lorentz observers are connected in sharp theories via Lorentz transformations, a given physical field is mathematically described over all space-time by a single entity, i.e., current sharp theories are form invariant. The assumptions of sharp space-time were considered as "unphysical idealization" by Ingraham. Rather, he assumed that the process of measurement (whatever its details) of observable entities has an intrinsic dispersion characterized by a length $\lambda$ which is the same for all equivalent observers, independent of the field being measured. Thus, repeated measurements by observer L of the coordinates or any other observable field on a given space-time event do not always yield the same values, but values which are distributed about certain mean values.

It is significant to note that the proposed lack of sharpness in stochastic theory is not the same as the quantum mechanical indeterminacy which results from the non-commutativity of conjugate operators in Hilbert space. Whereas the lower bound imposed upon experimental conjugate pair observations by quantum mechanical indeterminacy is measured by Planck's constant h, the lower bound imposed by stochastic theory on space-time coordinate observations is measured by the fundamental length $\lambda$. Of course, this is not to say that the two indeterminacies are unrelated. The reason for the intrinsic dispersion in space-time must ultimately emerge from consideration of the measurement process itself. As noted by Ingraham,[3] measurement theory, having to deal with the interaction of physical particles, must concern itself with quantum theory due to the atomic nature of matter. An indication of such a relationship was demonstrated by Cohn[5] in which he was able to show that the Heisenberg uncertainty relationship between conjugate variables places a lower bound on the sharpness of space-time measurements.

In any case, Ingraham's theory does not require knowledge of the details of measurement theory but only the assumption that they are mathematically describable by the formalism of random variable theory. Thus, the operation of measuring the coordinates of an event E by Lorentz observer L defines a quadruple random or stochastic variable X. Let the probability distribution or frequency function of this stochastic variable $X^{\mu}$ be $f(\eta,E,L)$. The support $s(E,L)$ is some manifold of space-time having measure $d\mu(\eta,E,L)$ at $\eta = (\eta^1,\eta^2,\eta^3,\eta^4)$. The mean value of the field $\varphi$, interpreted as the average of the physical field measurements at E by L, is defined by

$$\varphi(x,L) = \int_{s(E,L)} \varphi(X)\, f(\eta,E,L)\, d\mu(\eta,E,L) \tag{1}$$

where $x = x(E,L)$, the mean value of the coordinate of E as measured by L, is defined by the appropriate moment of the distribution, as given below:

$$\int_{s(E,L)} f(\eta,E,L)\, d\mu(\eta,E,L) = 1 \tag{2}$$

$$\int_{s(E,L)} f(\eta,E,L)\, X(\eta,E,L)\, d\mu(\eta,E,L) = x(E,L) \tag{3}$$

$$\int_{s(E,L)} f(\eta,E,L)\, (X - x)^2\, d\mu(\eta,E,L) = 3\lambda^2(E,L) \tag{4}$$

Equations (1) through (4) apply to the measurements of a given observer as referred to his own frame.

Ingraham's first postulate is that for equivalent observers L and $\underline{L}$, connected by a Lorentz transformation or Poincare transformation P,

POSTULATE I: $\underline{x}(E,\underline{L}) = P\, x(E,L) = \Lambda x(E,L) + a$

Since the mean coordinates of a given event as measured by different observers will be connected by Lorentz transformations, the events E may be placed in 1-1 correspondence with their mean coordinates.

For a physical field at event E as measured by $\underline{L}$ and referred to the frame of L, the stochastic average is defined as

$$\varphi(x,\underline{L}) = \int_{s(E,\underline{L})} \varphi(X)\, f(\eta,E,\underline{L})\, d\mu(\eta,E,\underline{L}) \tag{5}$$

which is related to $\underline{\varphi}(\underline{x},\underline{L})$ by some Lorentz transformation

$$\varphi(x,\underline{L}) = U(P)\, \underline{\varphi}(\underline{x},\underline{L}) \tag{6}$$

Form invariance would require that for any two equivalent observers, $\varphi(x,L) = \varphi(x,\underline{L})$. This is not, however, the case in stochastic space-time.

POSTULATE II -- Principle of Relativity: Any two observers L and $\underline{L}$ are equivalent in that their stochastic formulations of physical laws must be subjectively identical.

Specifically, consider two corresponding or subjectively identical points Q and $\underline{Q}$ such that $x(Q,L) = \underline{x}(\underline{Q},\underline{L})$, i.e., Q stands in the same relation to the frame L as $\underline{Q}$ does to the frame $\underline{L}$. The Principle of Relativity imposes the following constraints:

(1) $X^{\mu}(\eta,x,L) = X^{\mu}(\underline{\eta},\underline{x},\underline{L})$ for $\underline{\eta} = \eta$ and $\underline{x} = x$.

(2) $s(x,L)$ and $s(x,\underline{L})$ are subjectively identical. In other words, the set of numerical values of $\eta$ allowed by $s(x,L)$ is identical with the set $\underline{\eta}$ allowed by $s(\underline{x},\underline{L})$ if $\underline{x} = x$. This allows one to choose the element of measure $d\mu$ of the support such that $d\mu(\eta,x,L) = d\mu(\underline{\eta},\underline{x},\underline{L})$ for $\underline{\eta} = \eta$ and $\underline{x} = x$. With the measure element so chosen, the next statement is made:

(3) $f(\eta,x,L) = f(\underline{\eta},\underline{x},\underline{L})$ for $\underline{\eta} = \eta$ and $\underline{x} = x$. This statement is necessary once the measure $d\mu$ is chosen to be observer independent since otherwise the probability that L finds a value of the coordinates in the measure element $d\mu(\eta,x,L)$ around $\eta$ when measuring the coordinates of a point Q to which he assigns the mean coordinates x, namely $f(\eta,x,L)\cdot d\mu(\eta,x,L)$, would in general differ from the probability that $\underline{L}$ find a value in the corresponding measure element $d\mu(\underline{\eta},\underline{x},\underline{L})$ with $\underline{\eta} = \eta$ and $\underline{x} = x$ around the corresponding value $\underline{\eta}$ when measuring the coordinates of $\underline{Q}$, to which he assigns the mean coordinates $\underline{x} = x$, namely $f(\eta,x,\underline{L})\cdot d\mu(\eta,x,\underline{L})$. But then one of L and $\underline{L}$ would be preferred in some way, which violates their complete equivalence.

This is all that is required by the Principle of Relativity. However, considering constraints imposed by the symmetry group, the proper, orthochronous Lorentz group, Ingraham made two additional statements:

POSTULATE III: Homogeneity of stochastic space-time.

The following are immediate consequences of this statement:

(1) $X^{\mu}(\eta+a,x+a) = X^{\mu}(\eta,x) + a$, for all a

(2) $\eta \in s(x,L) \Longrightarrow (\eta+a) \in s(x+a,L)$ for all a. The support for point (x+a) is therefore the translate of the support for point x by a. This means the measure element $d\mu$ may be further chosen such that $d\mu(\eta+a,x+a) = d\mu(\eta,x)$ for all a, i.e., $d\mu(\eta,x) = d\mu^{*}(\eta-x)$.

(3) The measure element so chosen requires that

$$f(\eta+a,x+a) = f(\eta,x) = f^{*}(\eta-x)$$

Concerning the rotations of the proper Lorentz group $L_{+}^{\uparrow}$, a postulate is necessary to assure that at least so long as one is confined to the support s(x,L), the probability of finding a value $X^{\mu}(\eta,x)$ in measuring the coordinates of the point having mean coordinates x(L) is independent of the direction of $\eta$ from x(L). This amounts to requiring a limited isotropy with respect to $L_{+}^{\uparrow}$:

POSTULATE IV: Limited isotropy of stochastic space-time.

In particular, there is postulated to exist a non-trivial group M $L_{+}^{\uparrow}$ such that if s(x,L) contains $\eta$, then $[m(\eta-x) + x] \in s(x,L)$ where $m \in M$ and x is taken as a fixed point or origin for M. Thus,

(1) $X[m(\eta-x) + x,x] = m[X - x] + x$, for all $m \in M$

(2) s(x,L), with x the fixed point of M, is invariant under M: $\eta \in s(x,L) \Longrightarrow [m(\eta-x) + x] \in s(x,L)$ for all $m \in M$. This means the measure element $d\mu$ can be further restricted by choice to have the property $d\mu[m(\eta-x)] = d\mu(\eta-x)$ for all $m \in M$.

(3) Having so chosen this property for the measure element it

is also required that $f[m(\eta-x)] = f(\eta-x)$ for all $m \in M$. This means that f must be a function of some group invariant, i.e., $(\eta-x)^2$.

Ingraham's next, and final, postulate recognizes that the measurement of one spatial coordinate should not depend upon measurement of the others:

POSTULATE V: Stochastic independence of spatial coordinates. If the spatial coordinates are stochastically independent, the frequency function must be of the form of a product of functions of each spatial variable, i.e.,

$$f(X,x) = \prod_{1}^{3} f_i(X^i, X^4, x) \tag{7}$$

This postulate places no restrictions on the stochastic time variable $X^4$. However there are only two possibilities, $X^4$ is either dependent on or independent of the spatial coordinates. Spatial dependence of $X^4$ means that either time is dispersionless or that time measurements can be reduced to coordinate measurements by some operational process using signals which have the same constant velocity for all observers, e.g., light signals.

Although he did not explicitly raise it to the status of a postulate, Ingraham did assume, as did Einstein, that the speed of light in vacuum is the same for all equivalent observers, independent of the motion of the source of the light.

Also regarding the last postulate, the objection has been raised that it allows for the time variable to be treated in a manner different from the spatial variables, in a non-covariant way. Of course, this is in conflict with the more common interpretation of the Principle of Relativity. It is not, however, in conflict with the more general interpretation used by Ingraham. This interpretation, which only asks for subjectively identical formalisms among equivalent observers, seems to have originated with Einstein.[6] Thus, it is only required that all observers find the same relationship between stochastic space-time coordinates; the Principle of Relativity places no restriction on the precise nature of such relationships.

Ingraham did not make a definite commitment as to the form of the distribution or its support. However, from physical arguments, he did suggest that the most likely situation is that in which time may be expressed as a function of the spatial coordinates. In fact, if it is assumed that all four coordinate variables are stochastically independent, the frequency function is fixed as gaussian. If one requires that the support be of infinite extent, the normalization integral

$$1 = \int_{s(E,L)} d^4X \; f(X-x) = A \int_{s(E,L)} d^4X \; [\exp \alpha(X-x)^2]$$

eleminates the possibility of a four dimensional support for the gaussian distribution since divergence occurs whatever the choice of $\alpha$. It follows that all four variables may not be independent. Two dimensional supports were reasonably rejected for their asymmetric nature, thereby narrowing the possibilities to the one and three dimensional cases. Ingraham did not choose between these last possibilities but strongly favored the three dimensional support. In the case of dispersionless time, the frequency function for the three dimensional case is gaussian. It does seem likely that for the more physically realistic case of dispersion in all four coordinates, the three dimensional frequency function is a modified gaussian.

It is most convenient to express the stochastic averages of the field $\varphi(X)$ in terms of its Fourier transform

$$\varphi[X(\eta,x)] = \int d^4k\ \varphi(k)\ [\exp\ ikX(\eta,x)] \tag{8}$$

With (8), equation (1) becomes

$$\varphi(x,L) = \int d^4k\ \varphi(k)\ g(k,L)\ [\exp\ ikx] \tag{9}$$

while (5) becomes

$$\varphi(x,\underline{L}) = \int d^4k\ \varphi(k)\ g(k,\underline{L})\ [\exp\ ikx] \tag{10}$$

where the form factor $g(k,L)$ for an arbitrary observer is defined by

$$g(k,L) = \int_{s(E,L)} d\mu\ f(X-x,L)\ [\exp\ ik(X-x)] \tag{11}$$

It is seen that any difference in (9) and (10), i.e., lack of form invariance, is due to the difference in the form factors associated with the observers:

$$\varphi(x,\underline{L}) - \varphi(x,L) = \int d^4k\ \varphi(k)\ [\exp\ ikx]\ [g(k,\underline{L}) - g(k,L)] \tag{12}$$

Several properties of the form factors can be gleaned from the foregoing. For one thing, the reality of the measure element and the frequency function implies that

$$g^*(k,L) = g(-k,L) \tag{13}$$

Moreover, $g(k,L)$ must be real. This is seen by expanding (11) as

$$g(k,L) = \int_{s(E,L)} d\mu\ f(X-x) \cos k(X-x) + i\int_{s(E,L)} d\mu\ f(X-x) \sin k(X-x)$$

As a function of the invariant $(X-x)^2$, $f(X-x)$ is an even function. Since the sine function is odd, the last integral will vanish over a support of infinite extent. $g(k,L)$ is therefore real. In light of (13), this means that $g(k,L) = g(-k,L)$ and consequently $g(k,L)$ must be a function of $k^2 = k_\sigma k^\sigma$, i.e., $g(k^2,L)$.

Limitation of the support to be either three or one dimensional and the requirement that the support be of infinite extent suggest that $s(E,L) = s(x,L)$ be either a three dimensional plane or a one dimensional line in four-space, respectively. In both cases, since k appears as $k_\sigma(X^\sigma - x^\sigma)$, in the inner product, only the projection of k on the plane or line has non-zero contribution to the integral defining $g(k,L)$. Let $n(L)$ reprsent a unit four vector which is normal to the planar support in the three dimensional case, or is parallel to the linear support in the one dimensional case. Then the contributing projection of k for the planar support is

$$k_\perp = k + [k_\sigma n^\sigma(L)]n(L) \tag{14}$$

while for the linear support it is

$$k_\perp = k_\sigma n^\sigma(L) \tag{15}$$

Note that in (14) if $n(L) = (0,0,0,1)$ then $k_\sigma n^\sigma(L) = -k_4$ and therefore $k_\perp = \vec{k}$. In any case, the functional dependence of the form factor on k must be $k_\perp^2$ for either support. In addition, it may be noted that the form factor has no dimensions and so the actual dependence is $g(k_{\perp}^2\lambda^2, L)$, where $\lambda$ is the length defined by equation (4).

For what class of observers will the form factor have the same value? By (14) and (15) it must be that class whose support is characterized by the same vector n, i.e., those for whom $n(L_1) = n(L_2) = \ldots = n(L)$. The observer dependence is therefore carried entirely by $k_{\perp}$ and so the functional dependence of the form factor may be expressed simply as

$$g(k,L) = g(k_{\perp}^2\lambda^2) \tag{16}$$

An attempt has been made to summarize the basic features of Ingraham's theory of stochastic space-time. As indicated previously, this theory was developed as a prelude to the formulation of a finite quantum field theory. Without entering into a comprehensive discussion of finite field theory, it is of interest to note that the theory of stochastic space-time does allow for a logical formulation of quantum theory in which many of the discrepancies of infinite quantum field theory are removed. For example, renormalization is finite, the ultraviolet divergences do not occur due to the high-momentum cutoff, ambiguities associated with local Lagrangians (products of fields at the same point) disappear, etc. It is also important to recognize that the implications of stochastic space-time extend throughout all physical theory, i.e., stochastic space-time derives from the most fundamental considerations of physics.

In the chapters to follow, the stochastic theory of space-time will be presented from a point of view that differs from that of Ingraham. It will be demonstrated that the formulation may be approached philosophically as the relaxation of a postulate normally made in sharp theories of space-time. This in effect introduces an additional degree of freedom, which will be characterized by the parameter $\lambda$. This approach is appealing since there can be no argument as to the correctness of such a more general formulation if only its derivation is rigorous. Any debate is settled by evaluation of $\lambda$. If the parameter is eventually proven to be zero, then space-time is sharp and solution of the problem of developing relativistic quantum theory must be found elsewhere.

# CHAPTER III

## STOCHASTIC RELATIONSHIPS OF A SET OF COMMUNICATING OBSERVERS

It is desired to consider relationships which may exist between observers or frames of reference in communication with each other. The details of the process of physical communication are not important, it is only required that a given observer $L_i$ can by some means convey the results of his measurements and determinations of physical phenomena to any other observer $L_j$.

Being in physical communication, the observers possess the capability for agreement. They may, for example, agree to some common scheme of observer designation, such as observer one, two, etc. They may also establish unique mappings between each pair of frames in the set. Thus, a well defined coordinate mapping between frames $L_i$ and $L_j$ may be established by all observers agreeing that the space-time point designated as P in the frame of $L_j$ should always be mapped into the point designated as $\underline{P}$ in the frame of $L_i$. Moreover, this may be accomplished for all points in the frames. The coordinate mapping so defined from $L_j$ to $L_i$ will be represented by the two index quantity $L_{ij}$.

It should be noted that the particular functional form of the mapping may be different for different observers. The development thus far only requires common agreement as to which point of $L_j$ is mapped into a given point of $L_i$.

In addition to the coordinate mappings, the observers may agree as to the manner by which mathematical functions will be mapped between observers. These mappings, designated as $U_{ij}$, may or may not be functions of the $L_{ij}$.

For some physical event E, let $L_i$ measure, by a process involving single acts of observation at each step, the coordinates and value of some physical field at those coordinates. $L_i$ may repeat this procedure for other events, determining coordinate and physical field values for each point in his space. Eventually $L_i$ may analyze his data and conclude that for coordinates $Y_i(E,L_i)$ of the event E, the physical field has the value given by the mathematical field $\varphi_i[Y_i(L_i)]$. The field evaluated for some particular event E will be denoted by $\varphi_i[E,Y_i(L_i)]$.

Suppose a second observer $L_j$, in a similar manner, observes the same events as $L_i$. $L_j$ will assign a given event the coordinates $Y_j(E,L_j)$ and describe the same physical field by a generally different mathematical field $\varphi_j[Y_j(L_j)]$. Each member of the set of observers will likewise determine the coordinates of various events and assign a mathematical field to describe the physical field of interest.

Now let each observer transform his results to a chosen common frame, say $L_0$, by transformations of the type

$$Y_i(L_j) = L_{ij}Y_j(L_j) \tag{17}$$

$$\varphi_i[Y_i(L_j)] = U_{ij}\varphi_j[Y_j(L_j)] \tag{18}$$

Note that there is no justification for assuming that

$$Y_i(E,L_j) = Y_i(E,L_i)$$

or that

$$\varphi_i[Y_i(L_j)] = \varphi_i[Y_i(L_i)]$$

Once he has transformed the results of his observations to the frame of $L_o$ by the indicated mappings, let each observer communicate these transformed quantities to every other observer. The observers may then construct identical tables of the experimental data, all transformed to a common frame of reference ($L_o$) by means of agreed upon mappings. For an arbitrary field it is therefore possible for each of the observers to construct the following table:

TABLE I

| determined by $L_p$ | transform to $L_o$ via (18) |
|---|---|
| $\varphi_o[Y_o(L_o)]$ | $\varphi_o[Y_o(L_o)]$ |
| $\varphi_1[Y_1(L_1)]$ | $U_{o1}\varphi_1[Y_1(L_1)]$ |
| $\vdots$ | $\vdots$ |
| $\varphi_k[Y_k(L_k)]$ | $U_{ok}\varphi_k[Y_k(L_k)]$ |
| $\vdots$ | $\vdots$ |

Consider first how individual observers would analyze the distribution of coordinate values:

TABLE II

| determined by $L_p$ | transform to $L_o$ via (17) |
|---|---|
| $Y_o(L_o)$ | $Y_o(L_o)$ |
| $Y_1(L_1)$ | $L_{o1}Y_1(L_1)$ |
| $\vdots$ | $\vdots$ |
| $Y_k(L_k)$ | $L_{ok}Y_k(L_k)$ |
| $\vdots$ | $\vdots$ |

Having identical tables of data, the observers will agree on the distribution and frequency of coordinate values found in the right column of Table II.

The mappings $L_{ij}$ and $U_{ij}$ have been completely arbitrary thus far. If it is desired, however, that any lack of predictability or randomness in the values of the left column of Table II be reflected in a similar lack of predictability in the values of the right column, one must require that the mappings be 1-1, i.e., $L_{ij} = L_{ji}^{-1}$. Such will be the case for the development to follow. Likewise, it will be required that $U_{ij} = U_{ji}^{-1}$.

With no constraint beyond the ability to communicate, and the realization that the number of such observers is infinite, the qualification that $L_{ij} = L_{ji}^{-1}$ allows one to reasonably assume that the distribution of values in the right column of Table II is random. Let observer $L_o$ assign a random variable $X_o$, defined in his space-time frame, which duplicates the values of the right column as it ranges over some manifold of points. To average this variable $L_o$ would seek a frequency of distribution function $F_o(X_o,x_o)$ such that with the measure element $d\mu_o(X_o,x_o)$ of his support, $F_o(X_o,x_o)\cdot d\mu_o(X_o,x_o)$ is the probability of a coordinate value occurring in the table (and also in his space-time frame) in the range $X_o$, $X_o + dX_o$ about the mean value $x_o$. The mean value of the random variable would then be given by

$$x_o(L_o) = \int_{s(E,L_o)} X_o \; F_o(X_o,x_o) \; d\mu_o(X_o,x_o) \qquad (19)$$

Of course the total probability of some value of the random variable being in the table is unity:

$$\int_{s(E,L_o)} F_o(X_o,x_o) \; d\mu_o(X_o,x_o) = 1 \qquad (20)$$

Moreover, since it is not assumed that the values in the right column of the table will be identical, $L_o$ may find a non-zero dispersion

$$\int_{s(E,L_o)} [X_o - x_o]^2 \; F_o(X_o,x_o) \; d\mu_o(X_o,x_o) = 3\lambda^2(E,L_o) \qquad (21)$$

Suppose now that each observer transforms the right column of Table II to the frame of $L_k$:

TABLE III

| determined by $L_p$ | transform to $L_o$ | transform to $L_k$ |
|---|---|---|
| $Y_o(L_o)$ | $Y_o(L_o)$ | $L_{ko}\begin{bmatrix} Y_o(L_o) \\ L_{o1}Y_1(L_1) \\ \vdots \\ L_{ok}Y_k(L_k) \\ \vdots \end{bmatrix}$ |
| $Y_1(L_1)$ | $L_{o1}Y_1(L_1)$ | |
| $\vdots$ | $\vdots$ | |
| $Y_k(L_k)$ | $L_{ok}Y_k(L_k)$ | |
| $\vdots$ | $\vdots$ | |

In a manner similar to that of $L_o$, $L_k$ will average the column referenced to him by assigning a random variable $X_k$ defined in his own frame, etc:

$$x_k(L_k) = \int_{s(E,L_k)} X_k \; F_k(X_k,x_k) \; d\mu_k(X_k,x_k) \tag{22}$$

where

$$\int_{s(E,L_k)} F_k(X_k,x_k) \; d\mu_k(X_k,x_k) = 1 \tag{23}$$

and

$$\int_{s(E,L_k)} [X_k - x_k]^2 \; F_k(X_k,x_k) \; d\mu_k(X_k,x_k) = 3\lambda^2(E,L_k) \tag{24}$$

Consider now how $L_o$ will regard the column referenced to $L_k$. He will simply see it as the column referenced to himself multiplied by a factor $L_{ko}$. His conclusion will be that the average of the column referenced to $L_k$ is the average of the column referenced to himself, multiplied by $L_{ko}$, i.e., $x_k(L_o) = L_{ko}x_o(L_o)$. Therefore,

$$x_k(L_o) = L_{ko}\int_{s(E,L_o)} X_o \; F_o(X_o,x_o) \; d\mu_o(X_o,x_o)$$

or since $L_{ko}$ contains no variables of integration,

$$x_k(L_o) = \int_{s(E,L_o)} \underline{X}_k \; F_o(X_o,x_o) \; d\mu_o(X_o,x_o) \tag{25}$$

where $\underline{X}_k$ is a random variable in the frame of $L_k$ such that

$$\underline{X}_k = L_{ko}X_o$$

Since $\underline{X}_k$ and $X_k$ are both random variables defined in the frame of $L_k$, both of which duplicate the values of the column referred to $L_k$, they are in fact the same random variable. This means that the region of $L_k$'s frame over which the variable $X_k = \underline{X}_k = L_{ko}X_o$ assumes its values is simply the coordinate transform of the region of $L_o$'s frame in which $X_o$ assumes its values. Since the manifold of points included in the values assumed by these random variables must include the respective supports, the supports over which the averaging takes place must be transforms of each other.

It follows by inspection of Table III that the probability of the value $X_k$ occurring in the data referenced to $L_k$ is the same as the probability of $X_o$ occurring in the data referenced to $L_o$. Therefore,

$$F_o(X_o,x_o)\ d\mu_o(X_o,x_o) = F_k(X_k,x_k)\ d\mu_k(X_k,x_k) \qquad (26)$$

Using (26) and the fact that the supports $s(E,L_o)$ and $s(E,L_k)$ are transforms of one another, the variables of integration in (25) may be changed as

$$x_k(L_o) = \int_{s(E,L_k)} X_k\ F_k(X_k,x_k)\ d\mu_k(X_k,x_k) \qquad (27)$$

Comparing (22) and (27) it follows that for a given event E,

$$x_k = x_k(L_k) = x_k(L_o) = L_{ko}x_o \qquad (28)$$

This means that an event may be placed in 1-1 correspondance with its mean coordinates as determined in a particular frame.

It is significant to note that if the allowed mappings $L_{ij}$ are metric automorphisms of the space so that $[X_k-x_k]^2$ is an invariant, (26) substituted into (24) shows that $\lambda(E,L_i) = \lambda(E,L_j) = \lambda(E)$, i.e., at least for a given event, $\lambda(E)$ has the same value for all observers.

Specification of the manner in which observers will average mathematical fields in general is somewhat more arbitrary. However, if one wishes stochastic theory to reduce to the sharp theory of space-time in the event of zero coordinate dispersion, the prescription for averaging field values is determined. The form invariance of sharp theories is retained in the limit of zero coordinate dispersion only if one requires in stochastic theory that if two or more observers agree on one mathematical field, they will agree on all others. That is, if they measure the same coordinates for an event, they also obtain the same value for a physical field at that event. In particular, consider Table I. The observers may assign to frame $L_o$ a funcrion $\Psi_o(X_o)$ which duplicates the field values of the right column for an event E as its argument, the random variable $X_o$ of $L_o$'s frame, ranges over its values. Since the mathematical field $\Psi_o(X_o)$ is to have the same weight in the averaging process as its argument $X_o$, it follows that $F_o(X_o,x_o)\,d\mu_o(X_o,x_o)$ is the probability of occurrence of the value $\Psi_o(X_o)$ in the table. Observer $L_o$ will therefore average the function as

$$\phi_o(x_o,L_o) = \int_{s(E,L_o)} \Psi_o(X_o)\, F_o(X_o,x_o)\, d\mu_o(X_o,x_o) \qquad (29)$$

Now let all observers multiply the right column of Table I by the appropriate factor to reference the data to the frame of $L_k$:

TABLE IV

$$\begin{array}{ccc}
\underline{\text{determined by } L_p} & \underline{\text{transform to } L_o} & \underline{\text{transform to } L_k} \\
\begin{array}{c}
\varphi_o[Y_o(L_o)] \\
\varphi_1[Y_1(L_1)] \\
\vdots \\
\varphi_k[Y_k(L_k)] \\
\vdots
\end{array}
&
\begin{array}{c}
\varphi_o[Y_o(L_o)] \\
U_{o1}\varphi_1[Y_1(L_1)] \\
\vdots \\
U_{ok}\varphi_k[Y_k(L_k)] \\
\vdots
\end{array}
&
U_{ko}\begin{bmatrix}
\varphi_o[Y_o(L_o)] \\
U_{o1}\varphi_1[Y_1(L_1)] \\
\vdots \\
U_{ok}\varphi_k[Y_k(L_k)] \\
\vdots
\end{bmatrix}
\end{array}$$

The function which will generate this new column is $\Psi_k(X_k) = U_{ko}\Psi_o(X_o)$. Observer $L_k$ will therefore find the average of the right column of Table IV to be

$$\phi_k(x_k, L_k) = \int_{s(E,L_k)} \Psi_k(X_k)\ F_k(X_k, x_k)\ d\mu_k(X_k, x_k) \tag{30}$$

Any other observer, say $L_o$, will claim that the averages of the two columns can differ at most by the constant factor $U_{ko}$:

$$\begin{aligned}
\phi_k(x_k, L_o) &= U_{ko}\phi_o(x_o, L_o) \\
&= U_{ko}\int_{s(E,L_o)} \Psi_o(X_o)\ F_o(X_o, x_o)\ d\mu_o(X_o, x_o)
\end{aligned}$$

or

$$\phi_k(x_k, L_o) = \int_{s(E,L_o)} \Psi_k(X_k)\ F_o(X_o, x_o)\ d\mu_o(X_o, x_o) \tag{31}$$

Summarizing briefly, the foregoing has sought to describe the manner in which a set of communicating observers may establish common reference with regard to the coordinates of events and appropriate mathematical fields to be associated with a given physical field. It has been found that all observers will agree that the mean coordinates assigned to the various individual observers for a given event will be simple coordinate mappings of each other. However, if an arbitrary observer $L_i$ describes a particular physical field by the mathematical field

$$\phi_p(x_p,L_i) = U_{pq}\phi_q(x_q,L_i) = U_{pq}[U_{qr}\phi_r(x_r,L_i)] = \ldots$$

where

$$\phi_r(x_r,L_i) = \int_{s(E,L_i)} \Psi_r(X_r)\ F_i(X_i,x_i)\ d\mu_i(X_i,x_i) \tag{32}$$

another observer $L_j$ will describe the same physical field by the mathematical field

$$\phi_p(x_p,L_j) = U_{pq}\phi_q(x_q,L_j) = U_{pq}[U_{qr}\phi_r(x_r,L_j)] = \ldots$$

where

$$\phi_r(x_r,L_j) = \int_{s(E,L_j)} \Psi_r(X_r)\ F_j(X_j,x_j)\ d\mu_j(X_j,x_j) \tag{33}$$

Being integrated over different supports, it is not expected in general that the results of (32) and (33) will be identical, i.e., $\phi_r(x_r,L_i) \neq \phi_r(x_r,L_j)$ in general.

Not imposing the constraint of zero dispersion in coordinate and field measurements among the observers resulted in a degree of freedom not permitted in sharp or non-dispersive theories. If the agreed upon mappings are metric automorphisms this additional degree of freedom is characterized by the parameter $\lambda(E)$, which is the same for all observers for a given event.

## CHAPTER IV

## THE STOCHASTIC FORMALISM OF INERTIAL OBSERVERS

It has been possible to formulate the mathematical structure of the preceding chapter in terms of very general observers. Attention will now be restricted to a special set of communicating observers--those whose relative motion is uniform translatory, i.e., inertial observers.

POSTULATE I: The mean speed of light measured by rods and clocks at rest in a given inertial frame is always c, independent of the motion of the source.

It should be specifically noted that it is not assumed that the speed of light is sharp.

An immediate consequence of the above postulate is the specification of the mappings $L_{ij}$. Thus, let all inertial observers regard the emission and progress of a pulse of light. $L_p$ will determine mean coordinates for observers $L_i$ and $L_j$ for the point at which the pulse was emitted and note mean coordinates for the wave front some time later. For $L_i$ he will find

$$\delta\vec{r}_i \cdot \delta\vec{r}_i - c^2(\delta t_i)^2 = 0$$

where c is the mean speed of light found for all observers. Likewise he will find for $L_j$

$$\delta\vec{r}_j\cdot\delta\vec{r}_j - c^2(\delta t_j)^2 = 0$$

If $L_p$ then seeks a linear transformation satisfying the above he will find it to be a Lorentz transformation. Thus,

$$\delta t_j = \gamma[\delta t_i - \frac{1}{c^2}\delta\vec{r}_i\cdot\vec{v}_{ji}] \qquad (34)$$

$$L_{ji}: \qquad \delta\vec{r}_j = \delta\vec{r}_i + \vec{v}_{ji}[\frac{\gamma-1}{v_{ji}^2}\delta\vec{r}_i\cdot\vec{v}_{ji} - \gamma\delta t_i] \qquad (35)$$

$$\gamma^{-1} = \sqrt{1 - \frac{1}{c^2}\vec{v}_{ji}\cdot\vec{v}_{ji}} \qquad (36)$$

where the parameter $\vec{v}_{ji} = -\vec{v}_{ij}$ is the mean velocity of $L_j$ with respect to $L_i$.

The metric tensor is likewise determined as

$$g^{11} = g^{22} = g^{33} = -g^{44} = 1$$

Note that since the mean coordinates merely designate another point in the space of a given observer, i.e., the averaged points and the sharp points are members of the same manifold, g must be regarded as the metric tensor of the space. Thus, if the metrical interval $(x-\underline{x})^2$ is invariant under the Lorentz transformations, the metrical interval $(X-x)^2$ must likewise be an invariant.

The Lorentz transformations form a ten-parameter automorphic group. In particular, the metrical interval is invariant under four-space translations, rotations, and reflections. It has been shown that if $X_k = L_{ko}X_o$, the supports $s(E,L_k)$ and $s(E,L_o)$ must be transforms of one another. With respect to the Lorentz translations, this means that if $X_o \epsilon\, s(E,L_o) = s(x_o)$, then $X_k \epsilon\, s(x_o+a)$ if $X_k = X_o + a$. Similar statements follow for Lorentz rotations and reflections. These properties allow one to choose the measure element $d\mu$ such that

$$d\mu_k(X_k,x_k) \doteq d\mu_o(X_o,x_o) \tag{37}$$

The dot was placed above the sign of equality to emphasize the particular nature of the relationship. Equation (37) does not refer to a single volume element. It asks only that the measure element used by $L_o$ at $X_o$ have the same magnitude as the measure element used by $L_k$ at $X_k = L_{ko}X_o$. No constraint imposed on the theory so far requires the measure element be chosen as in (37). However, such a choice imposes no specialization.

With the measure element chosen as in (37), the probability measure will be the same (see equation (26)) only if the frequency functions are likewise related:

$$F_k(X_k,x_k) \doteq F_o(X_o,x_o) \tag{38}$$

That is, the function $F_k$, evaluated at the point $X_k = L_{ko}X_o$ in the frame of $L_k$, must give the same value as the (perhaps different) function $F_o$, evaluated at $X_o$ by $L_o$.

The preceding discussion, from chapter III to this point, has considered the description of a given event by various observers. The following postulate serves to establish relationships between different events:

POSTULATE II--The Principle of Relativity: The laws by which any two observers describe physical events are subjectively identical.

An immediate, and very important, consequence of this last statement is that, at the very least, the length $\lambda$ has the same value for all events of a given class, i.e., space-like, time-like, or null. To obtain this result, consider subjectively identical points $x_i$ in the frame of $L_i$ and $\underline{x}_j$ in the frame of $L_j$, i.e., $x_i^\mu = \underline{x}_j^\mu$. Of course, if $x_i$ is space-like to $L_i$, $\underline{x}_j$ is space-like to $L_j$. The Principle of Relativity requires that the dispersion of the random variable $X_i$ about the point $x_i$ (as seen by $L_i$) be the same as the dispersion of $\underline{X}_j$ about the point $\underline{x}_j$ (as seen by $L_j$). It follows from (4) that

$$\lambda(E,L_i) = \lambda(\underline{E},L_j)$$

where $x_i$ is the mean coordinate of E in $L_i$ and $\underline{x}_j$ is the mean coordinate of $\underline{E}$ in $L_j$.

If the coordinate dispersion about subjectively identical points is to be the same, it follows that one must also require

$$F_i(X_i,x_i)\, d\mu_i(X_i,x_i) \doteq F_j(\underline{X}_j,\underline{x}_j)\, d\mu_j(\underline{X}_j,\underline{x}_j) \qquad (39)$$

with $x_i^\mu = \underline{x}_j^\mu$ and $X_i^\mu = \underline{X}_j^\mu$. The equality is in the same sense as (37) and (38). The Principle of Relativity also requires that the support used by $L_i$ for the event $x_i$ be identical to that used by $L_j$ for $\underline{x}_j$. One may, without specialization, choose

$$d\mu_i(X_i,x_i) \doteq d\mu_j(\underline{X}_j,\underline{x}_j) \qquad (40)$$

With this choice, equation (39) requires

$$F_i(X_i,x_i) \doteq F_j(\underline{X}_j,\underline{x}_j)$$

Since the Principle demands subjective identity in all aspects of the formalism, the last condition can be allowed only if it is required that the frequency functions be the same for subjectively identical points, i.e.,

$$F(X_i, x_i) \doteq F(\underline{X}_j, \underline{x}_j) \tag{41}$$

Of course, $L_i$ uses the same frequency function for all points in his space. Since each point is subjectively identical to a point is some other frame, the frequency function must be the same for all points and all observers, i.e., it is Lorentz invariant in the sense

$$F(X,x) = F(LX, Lx) \tag{42}$$

This means

$$F(X,x) = F[(X^{\mu}-x^{\mu})(X_{\mu}-x_{\mu})] = F(\eta_{\mu}\eta^{\mu}) \tag{43}$$

The Principle of Relativity also requires that the supports of subjectively identical points be subjectively identical. Since it may be argued that a given observer will use the same type of support, e.g., a three-extent hyperboloid, one-extent line, etc., for each point in his space, it follows from arguments similar to those leading to (42) that the supports are form invariant. That is to say, if one observer uses a three-extent plane for his support, all observers will use three-extent planes.

To proceed with the derivation of the explicit form of the frequency function and its support, it is necessary to first consider the stochastic nature of the coordinates in more detail. In particular, one must deside whether or not there is dispersion in all four coordinates. To answer this consider two identical atoms or particles separated by some mean distance r, where the location of each particle at its respective end-point of the interval r can only be specified to within a region of radius $\delta$. Suppose each atom is in some energy state above its ground state initially but eventually returns to ground state with the emission of a photon. It is desired to investigate the simultaneity of these two events. For this purpose assume that an ideal detector has been placed at the midpoint of the separating interval such that the simultaneous incidence of the photons on the detector may be determined. (Such a detector is not possible according to arguments given by Cohn.[5])

Let one atom emit a photon when it is furthest from the midpoint of the separating interval and the other atom emit a photon $2\delta/c$ later at its closest approach to the midpoint, both photons proceeding toward the midpoint. If it is assumed that both photons travel with the same speed, they would strike the detector simultaneously, even though they were not emitted simultaneously. Of course, one may obtain a similar result even if the photons are allowed to have different speeds. In any case, it appears that one may not have a dispersion in spatial determination without a dispersion in the determination of simultaneity, i.e., of time. Further, one may demonstrate that a dispersion in time implies dispersion in spatial determination (simply let the emitting atoms have sharp positions at the end points of the interval r while allowing a dispersion $2\delta/c$ in time measurement). It must be concluded therefore that, in general, dispersion in spatial coordinates implies dispersion in time, and conversely. There is dispersion in all four coordinates.

A statement must now be made concerning the stochastic independence of the coordinate fluctuations. Suppose it is assumed that all four variables are stochastically independent. Then

$$F(\eta_\mu\eta^\mu) = F_1(\eta^1)F_2(\eta^2)F_3(\eta^3)F_4(\eta^4)$$

from which it follows that

$$\ln F = \ln F_1 + \ln F_2 + \ln F_3 + \ln F_4$$

From this last statement one obtains

$$\frac{\partial[\ln F]}{\partial[\eta_\mu\eta^\mu]}\,\frac{\partial[\eta_\mu\eta^\mu]}{\partial\eta^\sigma} = \frac{\partial[\ln F_\sigma]}{\partial\eta^\sigma} \qquad [\sigma \text{ not summed}]$$

Since

$$\frac{\partial[\eta_\mu\eta^\mu]}{\partial\eta^1} = 2\eta^1, \;\ldots,\; \frac{\partial[\eta_\mu\eta^\mu]}{\partial\eta^4} = -2\eta^4$$

one obtains

$$2\eta^1\,\frac{d[\ln F]}{d[\eta_\mu\eta^\mu]} = \frac{d[\ln F_1]}{d\eta^1}\;,\ldots,\; -2\eta^4\,\frac{d[\ln F]}{d[\eta_\mu\eta^\mu]} = \frac{d[\ln F_4]}{d\eta^4}$$

From this last statement

$$\frac{1}{\eta^1}\,\frac{d[\ln F_1]}{d\eta^1} = \ldots = -\frac{1}{\eta^4}\,\frac{d[\ln F_4]}{d\eta^4} = 2\,\frac{d[\ln F]}{d[\eta_\mu\eta^\mu]} = \underline{\alpha} = \text{const.}$$

from which it follows

$$F(\eta_\mu\eta^\mu) = \beta\,[\exp \underline{\alpha}(\eta_\mu\eta^\mu)] \qquad (44)$$

Consider first the possibility of a four-dimensional support, where the measure element may be taken as the four-volume $d^4X = dX^1dX^2dX^3dX^4$. With the frequency function (44), the normalization integral is

$$\int [\exp \underline{\alpha}(\eta_\mu \eta^\mu)]\, d^4X = \int [\exp \underline{\alpha}(\vec{\eta}\cdot\vec{\eta} - (\eta^4)^2]\, d^4X$$

This integral diverges for any four-dimensional region of infinite extent, whatever the choice of $\underline{\alpha}$. Four-dimensional supports are therefore not possible. With dispersion in all four variables, a support of smaller dimensionality implies a lack of independence of the stochastic variables, in contradiction to the original assumption that they were independent. Thus, not more than three of the space-time coordinate random variables are stochastically independent.

Symmetry suggests that if three of the random variables are independent, they are the three spatial coordinates. If this is the case, then the most general frequency function still permitted is

$$F(\eta_\mu \eta^\mu) = F_1(X^1, X^4, x)F_2(X^2, X^4, x)F_3(X^3, X^4, x) \qquad (45)$$

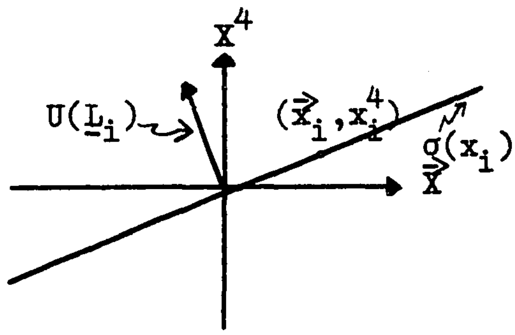
The metrical interval between the mean coordinates of an event and the origin must be in one of three classes: (i) space-like, (ii) time-like, (iii) null. In considering the possible supports for these cases, the surfaces $\eta_\mu \eta^\mu = K^2$, K a constant, may be eliminated since they imply $F(\eta_\mu \eta^\mu) = F(K^2)$, which can not be normalized for a surface of infinite extent. Whatever the nature of the support, however, it must depend upon the event to be described in some manner.

As one possibility, suppose observer $L_i$ used the family of space-like surfaces $\eta_\mu T^\mu(L_i) = K$ for his supports, where K depends on the event, and $T(L_i)$ is a unique time-like vector associated with the frame of $L_i$. The Principle of Relativity requires that the four-vector associated with a different observer $L_j$ have the same direction in four-space with respect to $L_j$ that $T(L_i)$ has with respect to $L_i$. This amounts to a preferred direction in space-time. The time axis defines an acceptable preferred direction but T(L) directed along this axis may be ruled out since it implies zero dispersion in time. Even if the notion of a preferred direction other than an axis is not in conflict with the most general interpretation of the Principle of Relativity, I feel that it must be rejected since it is equivalent to saying "Nature points to the left."

More generally, it is felt that any support requiring a preferred direction, frame, etc., in space-time must be rejected. The following development is consistent with this point of view.

<u>The mean coordinates are space-like</u>

If such is the case, observer $L_i$ may find a frame $\underline{L}_i$ in which the time component of the mean event is zero:




frame of $L_i$

$$0 = \gamma(\vec{\underline{v}}_i)[cx_i^4 - \vec{x}_i \cdot \vec{\underline{v}}_i]c^{-2}$$

where $\vec{\underline{v}}_i$ is the velocity of the frame $\underline{L}_i$ relative to $L_i$. The above condition is satisfied by

$$\underline{\vec{v}}_i = \frac{cx_i^4}{\vec{x}_i \cdot \vec{x}_i} \vec{x}_i \qquad (46)$$

One may now associate the four-velocity $U^\mu$ of $\underline{L}_i$ with a space-like plane through the origin,

$$X_\mu U^\mu = 0$$

or, with $\eta = X - x$,

$$\eta_\mu U^\mu = 0 \qquad (47)$$

All events on the plane defined by (47) would appear simultaneous to $\underline{L}_i$. It is suggested that this space-like plane is the support to be used by $L_i$ for the event having mean coordinates $x_i$. (Note that this is not the plane over which $\underline{L}_i$ would do his own averaging.)

In the frame $\underline{L}_i$, the condition (45) becomes

$$F(\underline{\eta}_\mu \underline{\eta}^\mu) = F_1(\underline{\eta}^1)F_2(\underline{\eta}^2)F_3(\underline{\eta}^3) \qquad (48)$$

By a derivation identical to that leading to (44), one finds that (48) is satisfied by the function

$$F(\underline{\eta}_\mu \underline{\eta}^\mu) = \beta\,[\exp -\alpha^2(\vec{\underline{\eta}} \cdot \vec{\underline{\eta}})] \qquad (49)$$

Equation (49) may be expressed in covariant form by defining the coordinates

$$\eta_\perp^\mu = \eta^\mu + \frac{1}{c^2}\,[\eta_\sigma U^\sigma]U^\mu \qquad (50)$$

It follows that

$$\eta_\perp \eta_\perp = \eta_\mu \eta^\mu + \frac{1}{c^2}\,[\eta_\sigma U^\sigma]^2 \qquad (51)$$

Note that in the proper frame $\underline{L}_i$, for which $U^\mu = (0,c)$, $\eta_\perp \eta_\perp = \vec{\underline{\eta}} \cdot \vec{\underline{\eta}}$. The frequency function may now be expressed as

$$F(\eta_\perp \eta_\perp) = \beta[\exp -\alpha^2(\eta_\perp \eta_\perp)] \qquad (52)$$

which is valid in the frame of $L_i$ as well as that of $\underline{L}_i$.

Calculations for $L_i$ may be performed in the frame of $\underline{L}_i$. That is, $L_i$ may express the relationship

$$\phi_r(x_r,L_i) = \int_{\sigma(x_i)} \Psi_r(X_r)\, F(\eta_\perp\eta_\perp)\, d\sigma \qquad (53)$$

simply as

$$\phi_r(x_r,L_i) = \beta\int \underline{\Psi}_r(\underline{X}_r)\, [\exp -\alpha^2\vec{\underline{\eta}}_i\cdot\vec{\underline{\eta}}_i]\, d^3\underline{\eta}_i \qquad (54)$$

where the integration in the frame $\underline{L}_i$ extends over all three space. To carry out the actual integration one must transform the function $\Psi_r(X_r)$ to the frame of $\underline{L}_i$:

$$\begin{aligned}\Psi_r(X_r) &= U_{ri}\Psi_i(X_i)\\ &= U_{ri}U(\vec{\underline{v}}_i)\underline{\Psi}_i(\underline{X}_i)\end{aligned} \qquad (55)$$

For example, the normalization integral may be evaluated in $\underline{L}_i$. Thus

$$\int F(\eta_\perp\eta_\perp)\, d\sigma = \beta\int [\exp -\alpha^2\vec{\underline{\eta}}_i\cdot\vec{\underline{\eta}}_i]\, d^3\underline{\eta}_i \qquad (56)$$

so

$$1 = \beta\left[\frac{\sqrt{\pi}}{\alpha}\right]^3 \qquad (57)$$

If (21) is likewise evaluated, it is found that

$$\lambda_s^2 = \frac{1}{2}\,\beta\alpha^{-5}[\sqrt{\pi}]^3 \qquad (58)$$

where $\lambda_s$ is a fundamental length associated with space-like coordinate dispersion. From (57) and (58) one finds

$$\begin{aligned}2\lambda_s^2 &= \alpha^{-2}\\ \beta &= [\lambda_s\sqrt{2\pi}]^{-3}\end{aligned} \qquad (59)$$

The temporal intervals $\eta^4$ may be expressed in terms of the spatial intervals $\vec{\eta}$. It must be true that

$$\vec{\underline{\eta}}\cdot\vec{\underline{\eta}} = \vec{\eta}\cdot\vec{\eta} - (\eta^4)^2 \tag{60}$$

Using (35) to transform $\vec{\underline{\eta}}\cdot\vec{\underline{\eta}}$ it is found that

$$\vec{\underline{\eta}}\cdot\vec{\underline{\eta}} = \vec{\eta}\cdot\vec{\eta} + c^{-2}\gamma^2(\vec{\eta}\cdot\vec{\underline{v}})^2 + (c^{-1}\gamma\underline{v})^2(\eta^4)^2 - 2c^{-1}\gamma^2\vec{\eta}\cdot\vec{\underline{v}}\eta^4 \tag{61}$$

Comparing (60) and (61) it is found that

$$(\eta^4)^2 - \left[\frac{2c\gamma^2\vec{\eta}\cdot\vec{\underline{v}}}{\gamma^2\underline{v}^2 + c^2}\right]\eta^4 + \left[\frac{\gamma^2(\vec{\eta}\cdot\vec{v})^2}{\gamma^2 v^2 + c^2}\right] = 0 \tag{62}$$

Equation (62) may be solved for $\eta^4$ to yield

$$\eta^4 = \vec{\eta}\cdot\frac{\vec{\underline{v}}}{c} \tag{63}$$

If one substitutes (46) into (63) it is found that

$$x_i^4 = \frac{x_i^4\vec{x}_i}{\vec{x}_i\cdot\vec{x}_i}\cdot\vec{x}_i \tag{64}$$

Substituting (63) into (51) it is found that for $L_i$,

$$\eta_\perp\eta_\perp = \vec{\eta}_i\cdot\vec{\eta}_i - \frac{1}{c^2}(\vec{\eta}_i\cdot\vec{\underline{v}}_i)^2 \tag{65}$$

Equation (52), in the frame of $L_i$, is therefore

$$F(\eta_\perp\eta_\perp) = \beta[\exp -\alpha^2\vec{\eta}_i\cdot\vec{\eta}_i][\exp \frac{\alpha^2}{c^2}(\vec{\eta}_i\cdot\vec{\underline{v}}_i)^2] \tag{66}$$

The form factor $g(k,L)$ defined in equation (11) may be evaluated in the frame of $\underline{L}_i$. Thus,

$$g(k,L_i) = \int_{\sigma(x_i)} F(\eta_\perp\eta_\perp)\,[\exp ik_\mu\eta^\mu]\,d\sigma$$

$$g(\underline{k},L_i) = \beta\int[\exp -\alpha^2\vec{\underline{\eta}}\cdot\vec{\underline{\eta}}][\exp i\vec{\underline{k}}\cdot\vec{\underline{\eta}}]\,d^3\underline{\eta}$$

so

$$g(\underline{k}_i\cdot\underline{k}_i) = [\exp - \tfrac{1}{2}\lambda_s^2\,\underline{k}_i\cdot\underline{k}_i] \tag{67}$$

Equation (67) may be put in invariant form by using

$$k_\perp^\mu = k^\mu + c^{-2}[k_\sigma U^\sigma]U^\mu \tag{68}$$

From (68) one obtains

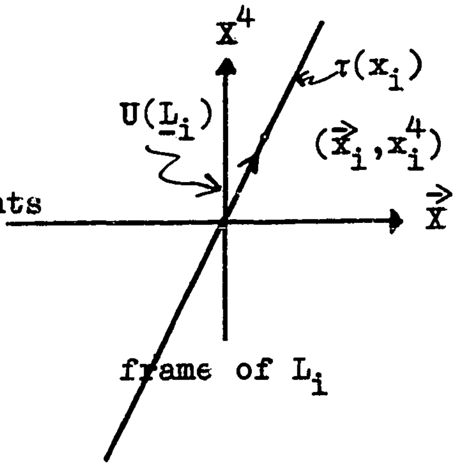
$$k_\perp k_\perp = k_\mu k^\mu + c^{-2}[k_\sigma U^\sigma]^2 \tag{69}$$

Therefore,

$$g(k_\perp k_\perp) = [\exp - \tfrac{1}{2}\lambda_s^2 k_\perp k_\perp] \tag{70}$$

Evaluated in the frame of $\underline{L}_i$, (70) yields,

$$g(k_i) = [\exp - \tfrac{1}{2}\lambda_s^2 \vec{k}_i \cdot \vec{k}_i][\exp -\frac{1}{2c^2}\lambda_s^2(\vec{k}_i \cdot \underline{\vec{v}}_i)^2][\exp \frac{1}{c}\lambda_s^2 \vec{k}_i \cdot \underline{\vec{v}}_i k_i^4] \tag{71}$$




The mean coordinates are time-like

In this situation, $L_i$ seeks a frame $\underline{L}_i$ in which the spatial components of the mean event are zero:

$$0 = \vec{x}_i + \underline{\vec{v}}_i\left[\frac{(\gamma-1)}{\underline{v}_i^2}\vec{x}_i \cdot \underline{\vec{v}}_i - \frac{\gamma}{c}x_i^4\right]$$

which may be satisfied for

$$\underline{\vec{v}}_i = \frac{c}{x_i^4}\vec{x}_i \tag{72}$$

One may now associate the four-velocity of $\underline{L}_i$ with a line $\tau(x_i)$ through the origin. All events on $\tau(x_i)$ will have zero spatial separation. It is suggested that this time-line is the support to be used by $L_i$ for the event having mean coordinates $x_i$.

In the frame of $\underline{L}_i$, (45) reduces to

$$F(\eta_\mu \eta^\mu) = F[(\eta^4)^2]$$

which is satisfied by

$$F(\underline{\eta}_{\mu}\underline{\eta}^{\mu}) \quad = \quad \beta\ [\exp -\alpha^2(\underline{\eta}_i^4)^2] \tag{73}$$

which in terms of the variable

$$\eta_{\parallel}^{\mu} \quad = \quad c^{-2}[\eta_{\varepsilon}U^{\varepsilon}]U^{\mu} \tag{74}$$

may be expressed in the covariant form

$$F(\eta_{\parallel}\eta_{\parallel}) \quad = \quad \beta\ [\exp -\alpha^2\eta_{\parallel}\eta_{\parallel}] \tag{75}$$

or

$$F(\eta_{\parallel}\eta_{\parallel}) \quad = \quad \beta\ [\exp -c^{-2}\alpha^2(\eta_{\mu}U^{\mu})^2] \tag{76}$$

which is valid in the frame of $L_i$ as well as that of $\underline{L}_i$.

For the average value of the field $\Psi_r(X_r)$, $L_i$ may write

$$\phi_r(x_r, L_i) \quad = \quad \int_{\tau(x_i)} \Psi_r(X_r)\ F(\eta_{\parallel}\eta_{\parallel})\ d\tau \tag{77}$$

In the frame of $\underline{L}_i$, (77) has the form

$$\phi_r(x_r, L_i) \quad = \quad \beta\int \underline{\Psi}_r(\underline{X}_r)\ [\exp -\alpha^2(\underline{\eta}_i^4)^2]\ d\underline{\eta}_i^4 \tag{78}$$

The normalization integral may be evaluated in the frame of $\underline{L}_i$ to yield

$$1 \quad = \quad \beta\left[\frac{\sqrt{\pi}}{\alpha}\right] \tag{79}$$

Since the evaluation of (21) must yield a negative number, define the constant

$$3\lambda_s^2 \quad = \quad -\lambda_\tau^2 \tag{80}$$

Evaluation of (21) in the frame of $\underline{L}_i$ then yields

$$\lambda_\tau^2 \quad = \quad \beta\ \frac{\sqrt{\pi}}{\alpha^3} \tag{81}$$

where $\lambda_\tau$ is a fundamental length associated with time-like coordinate dispersion. From (79) and (81) one finds

$$\begin{aligned} 2\lambda_\tau^2 \quad &= \quad \alpha^{-2} \\ \beta \quad &= \quad [\lambda_\tau\sqrt{2\pi}]^{-1} \end{aligned} \tag{82}$$

The spatial intervals $\vec{\eta}$ may be expressed in terms of the temporal intervals $\eta^4$. This is accomplished by noting that

$$-(\underline{\eta}^4)^2 = \vec{\eta}\cdot\vec{\eta} - (\eta^4)^2 \tag{83}$$

Transforming $\underline{\eta}^4$ by (34), it is found that

$$(\underline{\eta}^4)^2 = \gamma^2[(\eta^4)^2 - 2c^{-1}\vec{\eta}\cdot\vec{\underline{v}}\eta^4 + c^{-2}(\vec{\eta}\cdot\vec{\underline{v}})^2] \tag{84}$$

Comparing (83) and (84) one may solve to find

$$\vec{\eta} = \frac{\eta^4}{c\gamma^2}\vec{\underline{v}} \tag{85}$$

Using (85) to evaluate (76) in the frame of $L_i$, it is found that

$$F(\eta_{||}\eta_{||}) = \beta\,[\exp -\alpha^2(\eta_i^4)^2][\exp\,(\frac{2\gamma^2c^2 - v^2}{\gamma^2c^4})(\eta_i^4)^2] \tag{86}$$

The form factor $g(k,L)$ may be evaluated as

$$g(k_i) = [\exp - \tfrac{1}{2}\lambda_\tau(k_i^4)^2][\exp -\frac{1}{2c^2}\lambda_\tau^2(\vec{k}_i\cdot\vec{\underline{v}}_i)^2][\exp \tfrac{1}{c}\lambda_\tau\vec{k}_i\cdot\vec{\underline{v}}_i k_i^4] \tag{87}$$

<u>The mean coordinates are null</u>

At this point one may identify what has been implicit in the preceding discussion of space-like and time-like supports. With space-like dispersion confined to a space-like plane and time-like dispersion to a time-line, it follows that if one observer finds an event to be null, all observers differing by a Lorentz rotation will agree. The basic viewpoint is that, for a given observer, the coordinate dispersion of an event is confined to be exclusively space-like, time-like, or null.

The support for null events is the light cone. Also, all observers differing by Lorentz rotations will agree to this support. Equation (21) is not too informative since the metrical interval is zero for all observers, i.e., $(\eta^4)^2 = \vec{\eta}\cdot\vec{\eta}$. However, this same fact means that the dispersion in the spatial and the temporal coordinates must be identical. Since the spatial coordinates are assumed to be stochastically independent, one may guess that the appropriate distribution function for spatial coordinate dispersion is gaussian:

$$F = \beta\,[\exp -\alpha^2\vec{\eta}\cdot\vec{\eta}] \tag{88}$$

$$= \beta\,[\exp -\alpha^2(\eta^4)^2] \tag{89}$$

CHAPTER V

CONCLUSION

It has been shown that the basic structure of a stochastic theory of space-time may be derived from two postulates, i.e., the invariance of the mean speed of light and the Principle of Relativity. This derivation, while differing in approach, is found to parallel the development of Ingraham's theory. His theory was derived by considering the repeated observations of individual observers whereas this paper is based on the observations of an infinite number of observers. It isn't hard to show that the nexus of the two theories is the Principle of Relativity.

Some basic features of the theory are shown to be a natural consequence of dispersion in coordinate determinations, e.g., agreement on the mean coordinates of an event, lack of general form invariance under tensor transformations, etc. Significantly, one finds that the two postulates of the theory require a lower bound to measurement if physical determinations are observer dependent. It is shown, however, that the value of this lower bound may depend on whether or not an event is space-like, time-like, or null.

It was also shown that if one goes a step further and assumes that the coordinate dispersion for a given event is confined to intervals which are exclusively space-like, time-like, or null, the basic requirements of the theory may be satisfied by choosing the support of the space-like events to be certain space-like planes, the support of the time-like events to be certain time-lines, and the support of null events to be the light cone. With this choice, it was possible to derive the appropriate distribution functions and consequently, the appropriate formalism to be used for calculations. This procedure was not entirely arbitrary as it provided a means of avoiding specification of a preferred direction or frame of reference for all space-time. It is not suggested that the supports and distribution functions demonstrated constitute the only possible formalism. It is felt that they are plausible and consistent to the extent that this investigation has proceeded. Final conclusions can be drawn only after the theory has been developed further.

The assumption of confined dispersion has some important implications. For one thing, it indicates that observers would agree absolutely on the speed of light, not just its mean value. Also, with respect to microcausality, all observers will agree on the causal nature of two events since the dispersion in their coordinate determinations will not cross the light cone.

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