THE DISTRIBUTION OF CURRENT ON AN INFINITE ANTENNA IN AN ANISOTROPIC INCOMPRESSIBLE PLASMA

By

GARY LEE JOHNSON

Bachelor of Science
Kansas State University
Manhattan, Kansas
1961

Master of Science
Kansas State University
Manhattan, Kansas
1963

Submitted to the faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
July, 1966

THE DISTRIBUTION OF CURRENT ON AN INFINITE

ANTENNA IN AN ANISOTROPIC

INCOMPRESSIBLE PLASMA

Thesis Approved:

K. R. Cook

Thesis Adviser

Richard L. Cummins

J. Wayne Johnson

[illegible]

[illegible]

Dean of the Graduate College

PREFACE

The advent of artificial earth satellites has made the subject of satellite antennas a very important one. It is difficult to experimentally design such antennas because of the difficulty of duplicating the ionosphere in a laboratory. Therefore theoretical examinations of antenna paramaters such as the current distribution and impedance have been undertaken by many people. This thesis contains a theoretical treatment of the current distribution on an infinite antenna. It is found that the current decays fast enough that the distribution on the infinite antenna would be a good approximation to the distribution on a long but finite antenna.

I wish to express my gratitude to my thesis adviser, Dr. K. R. Cook, for his valuable assistance and guidance during my doctoral studies. I also wish to thank Mr. Loren S. Bearce of the Naval Research Laboratory for his advice and support in the form of a contract. Others who have helped me are Bruce Edgar and Robert Buchanan with many illuminating conversations, Dr. R. G. McIntyre with whom I learned something about functional analysis, and the other members of my advisory committee, Professors W. L. Hughes, R. L. Cummins, and L. W. Johnson, with advice and encouragement throughout my doctoral studies.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

1.1 Statement of the Problem. Within the past decade considerable interest has been generated in the use of cylindrical antenna structures for diagnostic techniques as well as communication systems for satellites orbiting in the ionosphere. The cylindrical antenna has proven to be an excellent tool for diagnostic experiments designed to provide experimental data describing certain parameters of the upper ionosphere. However, in order to make quantitative interpretations of such data, certain parameters of the antenna must be accurately known. In general, these critical parameters rely on one common bit of information, that being the current distribution existing on a cylindrical antenna when situated in an homogeneous anisotropic plasma. When used as a diagnostic or communication tool, accurate information must be available as to the input impedance of the antenna structure, which in turn depends on the current distribution. In particular it is desired to know how the impedance of a dipole antenna varies with length of the antenna and with the plasma parameters. This is a very difficult problem, especially when it is realized that no completely satisfactory solution has been found in free space when the dipole is longer than about one-half wave length. Therefore one is led to examine a simpler problem in the hope that it can be solved to some order of approximation, and that this solution can be extended to obtain an estimate of the dipole impedance.

It is basic to the solution of boundary value problems in antenna theory that the current distribution in the antenna and the electromagnetic fields external to the antenna are uniquely determined by each other. The more common approach is to assume a current distribution and find a solution to the inhomogeneous Helmholtz equation by the Green's function - Fourier Transform techniques. This is a powerful method and has yielded many satisfactory field solutions for particular problems. It should be kept in mind that this method is not exact insofar as the assumed current distribution may not be exact, and the integrations may involve approximations.

The method of first solving for the fields from the homogeneous Maxwell's equations and the boundary conditions at the antenna and at infinity, is more exact but also more limited. There are very few geometrical arrangements which admit exact solutions to the homogeneous Helmholtz equation, since most arrangements do not fit entire coordinate surfaces. One such entire coordinate surface is the infinite circular cylinder. The appropriate cylindrical coordinate system is shown in Figure 1.

Thus it appears that the infinite antenna is the preferred structure to consider. An infinite antenna is an infinitely long perfectly conducting cylinder with a prescribed electric field located in a finite slot, as in Figure 1. The model to be chosen for the ionosphere is an anisotropic, incompressible plasma, as discussed in Chapter II. Losses in the plasma do not alter the system of equations to be solved, so are included, except for certain explicit results which are more easily obtained if the losses are made zero. Because of the separability of the differential equations involved, the antenna is chosen aligned with the earth's magnetic field.

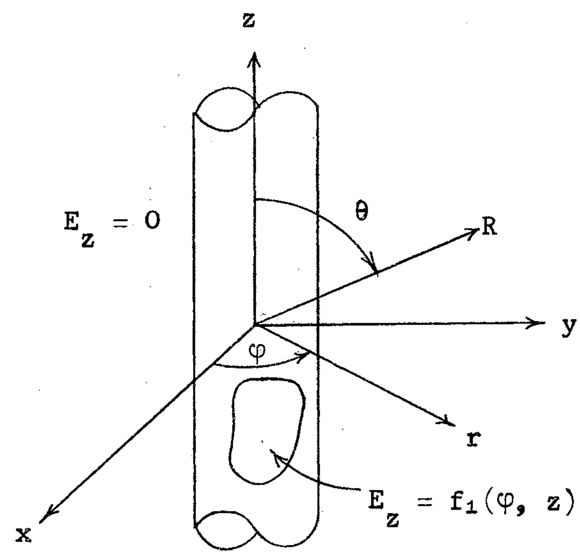

Figure 1. Infinite Antenna in Three Coordinate Systems.

The inhomogeneous wave equation for this medium is derived for sources composed of arbitrary electric and magnetic currents and charges in Chapter II. However, the electric field distribution in the finite slot is taken to be part of the boundary conditions, so that the wave equation for the infinite antenna problem becomes an homogeneous fourth-order differential equation. That is, all of the source terms which make the wave equation inhomogeneous are set equal to zero. The finite slot is further specialized in Chapter III to be a narrow circumferential slot with only a longitudinal electric field in the slot.

The problem considered in this research is that of finding the electromagnetic fields which satisfy the wave equation of the medium, boundary conditions on the antenna, and the boundary conditions at infinity. This simplified problem is subject to analytic treatment, and approximate expressions for the electromagnetic fields near the antenna but some distance from the source are derived in Chapter IV.

Finding the impedance from a known electromagnetic field distribution is really a separate problem from finding the fields; however, the basic approach which should be taken is outlined at the close of Chapter V.

1.2 Previous Work in the Area. The input impedance of an arbitrarily oriented, long cylindrical antenna in an anisotropic homogeneous ionosphere has been analyzed recently by several investigators (Katzin, 1961; Brandstatter, 1964; Ament, 1963, 1964). All of these papers have the disadvantage that extensive numerical integration is necessary to obtain fairly accurate results. Also, these papers have bypassed the difficulties of theoretically determining the current distribution by assuming a sinusoidal distribution with a particular wave number. Their results will therefore remain in question until it can be shown that the assumed distribution is close to the actual.

In Brandstatter's work it was assumed that, due to the anisotropy of the external medium there should be two sinusoidal current distributions, linearly superimposed, on a cylindrical antenna, and the wave numbers chosen for the distributions were those associated with the ordinary and extraordinary plane waves propagating in a direction parallel to the axis of the antenna. Ament originally chose to use an "average" wave number, assuming only one dominant mode for the current distribution on the antenna. However, in the 1963 paper, Ament found another wave number through a variational formulation of the impedance of the cylindrical dipole. These assumed wave numbers will be discussed in Chapter IV.

The possibly simpler problem of an arbitrarily oriented electrically short cylindrical antenna has received a great deal of attention. Only a few of the more recent papers are mentioned here.

Balmain (1963, 1964) and Blair (1964) independently solved the quasi-static differential equation for electromagnetic fields adjacent to the antenna, and obtained closed form solutions for the input impedance, valid for most frequencies below HF.

Storey (1963) discussed the design of a cylindrical antenna which was used on an Aerobee ionospheric probe. His optimum design was based on the effect the ionospheric particles would have on the antenna impedance.

There has been some concern shown about a certain "infinity catastrophe" in the impedance of a small dipole by Staras (1964). This and other questions about the validity of the formulation of the problem have been raised by Felsen (1965) and de Wolf (1965).

1.3 Outline of the Method of Solution. The primary problem under consideration is the determination of the exterior fields produced by a specified field distribution in a slot in the wall of a perfectly conducting circular cylinder, surrounded by an anisotropic plasma. The same infinite antenna problem, except for a medium of free space, has been considered by Papas (1949), Silver and Saunders (1950), and Northover (1958). The same general approach will be followed here, although the anisotropy of the medium forces some phases of the problem to be treated differently.

Basically, the approach is to construct a solution of Maxwell's equations that assumes the prescribed value of the tangential electric field over the cylinder and that satisfies a radiation condition at infinity. The radiation condition can not be expressed simply in terms of "outgoing waves" at infinity in the anisotropic case. This is quite in contrast to the case of isotropic media, and the discussion of this topic in Chapter III forms an important part of this paper.

The procedure is to find basic sets of cylindrical waves, or eigenfunctions of the homogeneous wave equation to be developed in Chapter II, which satisfy the radiation condition at infinity. These eigenfunctions generally do not also satisfy the boundary condition on the cylinder so a superposition is required. Both the tangential electric field over the cylinder and the field in space are synthesized in the form of a Fourier integral representation. The coefficients involved in this representation are known for the surface field. The coefficients of the representation for the space field are determined by the requirement that its tangential components at the surface of the cylinder shall reduce to the Fourier representation of the field prescribed over the surface.

Once the representation for the field in space is completely specified, the next step is to evaluate the integral. This type of integral is usually treated by asymptotic techniques, with results valid over certain restricted ranges of the parameters. One may attempt the saddle point method, used by Papas (1949) and Silver and Saunders (1950), or the branch cut integral method, used by Northover (1958). The branch cut method is used in this paper primarily because the saddle point is extremely difficult to determine, and even if found would be difficult to use in obtaining a solution.

CHAPTER II

DERIVATION OF THE WAVE EQUATION

2.1 The Permittivity Tensor. The equation relating the motion of electrons in a plasma to the impressed electric and magnetic fields can be written

$$\frac{d}{dt}\underline{v} = -\frac{1}{nm}\nabla P + \frac{q}{m}(\underline{E} + \underline{v} \times \underline{B}) - \nu\underline{v} \qquad (2.1.1)$$

where:

$\underline{v}$ = velocity of electrons

n = electron density

m = electron mass

q = charge on electron

ν = collision frequency

Equation 2.1.1 is called the momentum transfer equation and it and two similar equations for ions and neutral particles are necessary to fully describe the plasma. A number of approximations are usually made at this point to get a simplified version of Equation 2.1.1. For the particular case under consideration, these are as follows:

1. It is assumed that the effects of ions and neutral particles are negligible, so that only Equation 2.1.1 of the system of three momentum transfer equations need be considered,
2. Equation 2.1.1 is linearized by assuming that the static magnetic field is much larger than the impressed time varying

field. That is, $\underline{B}$ is approximately given by $\underline{B}_0$,

3. It is assumed that the ∇P term is negligible. This amounts to ignoring the compressibility of the plasma and is an useful approximation only if the plasma is not too dense or not too hot,

4. It is assumed that steady state has been reached, so that all the variables have an $\exp(i\omega t)$ time variation.

These are common assumptions, and are discussed in detail in a number of plasma physics books, including Uman (1964).

After applying these assumptions, Equation 2.1.1 becomes

$$i\omega m \underline{v} = q \underline{E} + q \underline{v} \times \underline{B}_0 - \nu\underline{v} \qquad (2.1.2)$$

Equation 2.1.2 may be solved for $\underline{v}$ in terms of $\underline{E}$.

$$\underline{v} = \frac{q}{m} \hat{\gamma} \cdot \underline{E} \qquad (2.1.3)$$

where

$$\hat{\gamma} = \begin{bmatrix} \frac{\nu + i\omega}{(\nu + i\omega)^2 + \omega_g^2} & \frac{\omega_g}{(\nu + i\omega)^2 + \omega_g^2} & 0 \\ \frac{-\omega_g}{(\nu + i\omega)^2 + \omega_g^2} & \frac{\nu + i\omega}{(\nu + i\omega)^2 + \omega_g^2} & 0 \\ 0 & 0 & \frac{1}{\nu + i\omega} \end{bmatrix} \qquad (2.1.4)$$

One of Maxwell's equations is

$$\nabla \times \underline{H} - i\omega \epsilon_0 \underline{E} = \underline{J} \qquad (2.1.5)$$

where $\underline{J}$ is due to the motion of the electrons in the plasma, and is related to the impressed electric field by

$$\underline{J} = nq\underline{v} = \frac{nq^2}{m} \hat{V} \cdot \underline{E} \tag{2.1.6}$$

Combining Equations 2.1.5 and 2.1.6 gives

$$\nabla \times \underline{H} - i\omega\, \epsilon_0 \left[\hat{I} + \frac{nq^2 \hat{V}}{i\omega\, \epsilon_0 m}\right] \cdot \underline{E} = 0 \tag{2.1.7}$$

The term in brackets in Equation 2.1.7 is called the relative permittivity tensor, and is usually written

$$\hat{K} = \begin{bmatrix} K_{11} & -iK_{12} & 0 \\ iK_{12} & K_{11} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \tag{2.1.8}$$

where for a lossless plasma

$$K_{11} = 1 - \frac{X}{1 - Y^2} \tag{2.1.9}$$

$$K_{12} = - \frac{XY}{1 - Y^2} \tag{2.1.10}$$

$$K_{33} = 1 - X \tag{2.1.11}$$

where $$X = \frac{\omega_p^2}{\omega^2} \quad \text{and} \quad Y = \frac{\omega_g}{\omega} \tag{2.1.12}$$

with $$\omega_p = \sqrt{\frac{nq^2}{m\epsilon_0}} \quad \text{(the electron plasma frequency)} \tag{2.1.13}$$

$$\omega_g = \frac{qB_0}{m} \quad \text{(the electron cyclotron frequency)} \tag{2.1.14}$$

The effects of collisional damping can be included by replacing

$$X \text{ by } \frac{X}{1 - iZ} \quad \text{and} \quad Y \text{ by } \frac{Y}{1 - iZ} \tag{2.1.15}$$

where

$$Z = \frac{\nu}{\omega} \tag{2.1.16}$$

It should be pointed out that a plasma always has some loss, and a valid model for a plasma should include these losses. This will get rid of some physically unrealistic mathematical phenomena that arise in problems concerning lossless plasmas. This means that the term ν should be carried along in all the analytical expressions, although it is permissible to let it be arbitrarily small in the final result. It is somewhat easier, however, to do as much algebra as possible for the lossless case and then use the change of variable in Equations 2.1.15 to introduce losses into the system. This will be the approach taken in this thesis.

2.2 Wave Equation for an Anisotropic Plasma with Sources.

Maxwell's equations in an anisotropic medium for arbitrary electric and magnetic currents and charges may be written.

$$\nabla \times \underline{H} = \underline{J}_e + i\omega\, \epsilon_0\, \hat{K} \cdot \underline{E} \tag{2.2.1}$$

$$\nabla \times \underline{E} = \underline{J}_m - i\omega\, \mu_0\, \underline{H} \tag{2.2.2}$$

$$\nabla \cdot \underline{H} = -\frac{\rho_m}{\mu_0} \tag{2.2.3}$$

$$\nabla \cdot (\hat{K} \cdot \underline{E}) = \frac{\rho_e}{\epsilon_0} \tag{2.2.4}$$

The common factor $\exp(i\omega t)$ has been suppressed in the above equations. The subscript e refers to electric sources while the subscript m refers to magnetic sources. In a temperate plasma with the static magnetic field aligned with the z axis, the relative permittivity tensor $\hat{K}$ is given by Equation 2.1.8. The procedure to be used in this chapter for separating the fields into their components and obtaining a wave equation satisfied by one field component will not work if the static magnetic field is not aligned with the z axis. It is always possible, of course, to choose the coordinate system such that the static magnetic field is aligned with the z axis, and so the derivations in this chapter are always valid. Solutions to the wave equation to be derived may be very difficult to obtain, however, unless the conducting surfaces and sources possess rather simple geometrical relationships to the axis defined by the static magnetic field. This point will become more evident later.

Maxwell's equations will now be manipulated to obtain the differential equation satisfied by one component of the electric or magnetic fields. Once this component is isolated, the other five field components may be obtained in terms of it without great difficulty.

It will be convenient to separate the vectors and vector operators into their transverse and longitudinal components.

$$\underline{E} = \underline{E}_t + E_z \underline{a}_z \tag{2.2.5}$$

$$\underline{H} = \underline{H}_t + H_z \underline{a}_z \tag{2.2.6}$$

$$\nabla = \nabla_t + \underline{a}_z \frac{\partial}{\partial z} \tag{2.2.7}$$

With this notation

$$\nabla \times \underline{E} = \nabla_t \times \underline{E}_t - \underline{a}_z \times (\nabla_t E_z - \frac{\partial \underline{E}_t}{\partial z}) \tag{2.2.8}$$

where $\nabla_t \times \underline{a}_z = 0$ because $\underline{a}_z$ is not a function of the transverse co-ordinates.

It can be shown that the elements of $\hat{K}$, Equation 2.1.8, are invarient under the transformation from cartesian coordinates to any orthogonal cylindrical coordinate system with the same z axis. A coordinate system is chosen such that one transverse axis is parallel to $\underline{E}_t$ and the other transverse axis perpendicular to $\underline{E}_t$. Then the term $\hat{K} \cdot \underline{E}$ can be written in matrix form as

$$\hat{K} \cdot \underline{E} = \begin{bmatrix} K_{11} & -iK_{12} & 0 \\ iK_{12} & K_{11} & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \begin{bmatrix} \underline{E}_t \\ 0 \\ E_z \end{bmatrix} = \begin{bmatrix} K_{11}\underline{E}_t \\ iK_{12}\underline{E}_t \\ K_{33}E_z \end{bmatrix} \tag{2.2.9}$$

and in vector form as

$$\hat{K} \cdot \underline{E} = K_{11}\underline{E}_t + iK_{12}\underline{a}_z \times \underline{E}_t + K_{33}E_z\underline{a}_z \tag{2.2.10}$$

The transverse portions of Equations 2.2.1 and 2.2.2 can now be written as

$$-\underline{a}_z \times \left(\nabla_t H_z - \frac{\partial \underline{H}_t}{\partial z}\right) = \underline{J}_{te} + i\omega\, \epsilon_0 K_{11}\underline{E}_t - \omega\, \epsilon_0 K_{12}\underline{a}_z \times \underline{E}_t \tag{2.2.11}$$

$$-\underline{a}_z \times \left(\nabla_t E_z - \frac{\partial \underline{E}_t}{\partial z}\right) = \underline{J}_{tm} - i\omega\, \mu_0 \underline{H}_t \tag{2.2.12}$$

while the longitudinal portions are

$$\nabla_t \times \underline{H}_t = \underline{a}_z\, [i\omega\, \epsilon_0 K_{33} E_z + J_{ze}] \tag{2.2.13}$$

$$\nabla_t \times \underline{E}_t = \underline{a}_z\, [-i\omega\, \mu_0 H_z + J_{zm}] \tag{2.2.14}$$

Equations 2.2.3 and 2.2.4 become, after some simplification

$$\nabla_t \cdot \underline{H}_t + \frac{\partial H_z}{\partial z} = - \frac{\rho_m}{\mu_0} \qquad (2.2.15)$$

$$K_{11}\nabla_t \cdot \underline{E}_t + iK_{12}\nabla_t \cdot \underline{a}_z \times \underline{E}_t + K_{33}\frac{\partial E_z}{\partial z} = \frac{\rho_e}{\epsilon_0} \qquad (2.2.16)$$

The next step is to eliminate $\underline{E}_t$ and $\underline{H}_t$ from these equations, leaving two equations in two unknowns, E_z, and H_z. This is not difficult but is rather tedious, so several steps will just be outlined. Taking the cross product of $\underline{a}_z$ and Equations 2.2.11 and 2.2.12 and then taking the transverse divergence of the result gives

$$\nabla_t^2 H_z - \nabla_t \cdot \frac{\partial \underline{H}_t}{\partial z} = \nabla_t \cdot \underline{a}_z \times \underline{J}_{te} + i\omega\,\epsilon_0 K_{11}\nabla_t \cdot \underline{a}_z \times \underline{E}_t$$

$$+ \omega\,\epsilon_0 K_{12}\nabla_t \cdot \underline{E}_t \qquad (2.2.17)$$

$$\nabla_t^2 E_z - \nabla_t \cdot \frac{\partial \underline{E}_t}{\partial z} = \nabla_t \cdot \underline{a}_z \times \underline{J}_{tm} - i\omega\,\mu_0\nabla_t \cdot \underline{a}_z \times \underline{H}_t \qquad (2.2.18)$$

The dot product of $\underline{a}_z$ with Equations 2.2.13 and 2.2.14 yields, after some simplification,

$$\underline{a}_z \cdot \nabla_t \times \underline{H}_t = i\omega\,\epsilon_0 K_{33} E_z + J_{ze} \qquad (2.2.19)$$

$$\underline{a}_z \cdot \nabla_t \times \underline{E}_t = - i\omega\,\mu_0 H_z + J_{zm} \qquad (2.2.20)$$

Equations 2.2.15, 2.2.16, 2.2.19 and 2.2.20 may be used in Equations 2.2.17 and 2.2.18 to eliminate the transverse fields. Then

$$(\nabla_t^2 + \frac{\partial^2}{\partial z^2} + k_0^2 [K_{11} - \frac{K_{12}^2}{K_{11}}]) H_z = - \frac{\omega \epsilon_0 K_{12} K_{33}}{K_{11}} \frac{\partial E_z}{\partial z}$$

$$+ \nabla_t \cdot \underline{a}_z \times \underline{J}_{te} - \frac{\partial}{\partial z} (\frac{\rho_m}{\mu_0}) - i\omega \epsilon_0 K_{11} J_{zm} + \omega \epsilon_0 K_{12} (\frac{\rho_e}{K_{11} \epsilon_0}$$

$$+ \frac{iK_{12}}{K_{11}} J_{zm}) \tag{2.2.21}$$

$$(\nabla_t^2 + \frac{K_{33}}{K_{11}} \frac{\partial^2}{\partial z^2} + k_0^2 K_{33}) E_z = \omega \mu_0 \frac{K_{12}}{K_{11}} \frac{\partial H_z}{\partial z} + \frac{\partial}{\partial z} (\frac{\rho_e}{K_{11} \epsilon_0})$$

$$+ \frac{iK_{12}}{K_{11}} \frac{\partial J_{zm}}{\partial z} + \nabla_t \cdot \underline{a}_z \times \underline{J}_{tm} + i\omega \mu_0 J_{ze} \tag{2.2.22}$$

It will be convenient to rewrite the above two equations using operator notation.

$$L_1 H_z = a_1 L_2 E_z + S_1 \tag{2.2.23}$$

$$L_3 E_z = a_2 L_2 H_z + S_2 \tag{2.2.24}$$

The constants are

$$a_1 = - \omega \epsilon_0 \frac{K_{12} K_{33}}{K_{11}}$$

$$a_2 = \omega \mu_0 \frac{K_{12}}{K_{11}}$$

and S_1 and S_2 are the terms involving sources in Equations 2.2.21 and 2.2.22 respectively. To eliminate H_z, operate on Equation 2.2.23 with L_1^{-1}. This gives

$$H_z = a_1 L_1^{-1} L_2 E_z + L_1^{-1} S_1 \tag{2.2.25}$$

Equations 2.2.24 and 2.2.25 are now combined to yield

$$L_3E_z = a_2L_2\,[a_1L_1^{-1}L_2E_z + L_1^{-1}S_1] + S_2 \qquad (2.2.26)$$

If the assumption is made that all operators commute, then the last equation can be operated on by L_1 and the wave equation satisfied by the longitudinal E field for this anisotropic medium is

$$[L_1L_3 - a_1a_2L_2^2]\,E_z = f_e = a_2L_2S_1 + L_1S_2 \qquad (2.2.27)$$

The longitudinal H field satisfies the same differential equation except that the source term is

$$f_h = a_1L_2S_2 + L_3S_1 \qquad (2.2.28)$$

2.3 The Green's Function Approach. The wave equation of Section 2.2 is in the form

$$LE_z = f_e \qquad (2.3.1)$$

There are two fundamental methods of solving this linear equation, where L is the linear operator, f_e a given vector or function, and E_z an unknown vector or function. One method is to construct the inverse operator L^{-1}; so that

$$E_z = L^{-1}f_e \qquad (2.3.2)$$

This is the Green's function approach. The other method is to use the spectral representation of the operator L, as discussed in Chapter I. The second method is the one used in this thesis and perhaps it should be mentioned briefly why the Green's function approach was not used.

As is well known, Green's function $G(\underline{r}, \underline{r}')$ must satisfy both the inhomogeneous wave equation

$$LG(\underline{r}, \underline{r}') = - \delta(\underline{r} - \underline{r}') \qquad (2.3.3)$$

and the boundary conditions imposed on E_z. In Equation 2.3.3, the vectors $\underline{r}$ and $\underline{r}'$ extend from the origin to the point of observation and the source point respectively, and $\delta(\underline{r} - \underline{r}')$ is the three dimensional impulse function. Friedman (1956) shows that if L can be separated into the sum of three commutative operators, one in each coordinate variable, the Green's function and final spectral representation can be constructed for L from a knowledge of the spectral representation of each of the operators into which L was separated. Unfortunately, the operator L has coupled terms such as $\nabla_t^2 \partial^2/\partial z^2$, so that this method of separation of variables is not directly applicable. One could get around this by defining a change of variable such that L is decoupled. However, in this case the boundary surfaces must also undergo this coordinate transformation. It thus appears that whatever gains were made originally by choosing a simple cylindrical boundary aligned with the z axis would be lost at this point. It should be evident from this discussion that finding the Green's function for this medium and geometry by this general approach is very difficult indeed.

A slightly different method of finding the Green's function, or at least using it to solve equations like Equation 2.3.2 is presented by Wait (1959). This technique depends on the use of Green's theorem which Wait wrote in the special form

$$\int [G(\underline{r}, \underline{r}') \nabla^2 E_z(\underline{r}') - E_z(\underline{r}') \nabla^2 G(\underline{r}, \underline{r}')] \, dv'$$

$$= \int_{S'} [G(\underline{r}, \underline{r}') \frac{\partial E_z(\underline{r}')}{\partial n} - E_z(\underline{r}') \frac{\partial G(\underline{r}, \underline{r}')}{\partial n}] \, ds' \qquad (2.3.4)$$

where n is the outward directed normal to the surface surrounding the radiating structure. By using the boundary conditions on the perfectly conducting cylinder and those at infinity, Wait is able to write the explicit form of Equation 2.3.2 rather easily. However, this technique works only if the differential operator describing the medium is ∇^2, so that Green's theorem in the form of Equation 2.3.4 can be used to advantage. This is not the case in an anisotropic medium. There should be an analog to Equation 2.3.4 for the operator L, but even if this analog could be found, there is no guarantee that the explicit form of Equation 2.3.2 would follow easily.

It thus appears to this author, after an examination of both methods, that the technique of superposition of elementary eigenfunctions is easier to apply to this particular problem than the Green's function approach because of some of the difficulties outlined above.

2.4 The Homogeneous Wave Equation. For the particular problem outlined in the Introduction, the general wave equation for this medium, Equation 2.2.27, can be considerably simplified. First, it is assumed that the external fields are produced entirely by a specified electric field distribution in a slot in the surface of a perfectly conducting infinite cylinder. Since there are no source currents or charges anywhere, the source terms of Equation 2.2.27 go to zero, making it a homogeneous wave equation. The problem is now to solve a homogeneous differential equation with non-homogeneous boundary conditions. That is,

$$LE_z = 0 \tag{2.4.1}$$

with the boundary conditions

$$E_z = f_1(\varphi, z) \quad \text{inside the slot} \tag{2.4.2}$$

$$= 0 \quad \text{outside the slot} \tag{2.4.3}$$

If it were desired to solve the other type of problem, an inhomogeneous differential equation with homogeneous boundary conditions, then a source of specified currents would have to be placed exterior to the unslotted cylinder. This other type of problem will not be considered in this thesis.

The second assumption to be made is that all fields and currents have an $\exp(i\omega t - ikz)$ variation. This special time and z variation does not really restrict the allowable spatial and time variations of the fields, since they can always be represented by a Fourier series or a Fourier integral, as long as they are piece-wise continuous.

Under these assumptions, Equations 2.2.21 and 2.2.22 become

$$\nabla_t^2 E_z + aE_z = bH_z \tag{2.4.4}$$

$$\nabla_t^2 H_z + cH_z = dE_z \tag{2.4.5}$$

where

$$a = \frac{K_{33}}{K_{11}} (- k^2 + k_0^2 K_{11}) \tag{2.4.6}$$

$$b = - \frac{ikK_{12}\omega \mu_0}{K_{11}} \tag{2.4.7}$$

$$c = - k^2 + k_0^2(K_{11} - \frac{K_{12}^2}{K_{11}}) \tag{2.4.8}$$

$$d = ik\omega \epsilon_0 \frac{K_{12}K_{33}}{K_{11}} \tag{2.4.9}$$

Eliminating H_z between Equations 2.4.4 and 2.4.5 gives the uncoupled fourth-order homogeneous wave equation,

$$\nabla_t^4 E_z + (a + c)\, \nabla_t^2 E_z + (ac - bd)\, E_z = 0 \qquad (2.4.10)$$

Since the transverse Laplacian ∇_t^2 is a commutative operator, the wave equation may be factored into the form

$$(\nabla_t^2 + \lambda_1^2)(\nabla_t^2 + \lambda_2^2)\, E_z = 0 \qquad (2.4.11)$$

Comparing the last two equations shows that $\lambda_1^2 + \lambda_2^2 = a + c$ and $\lambda_1^2 \lambda_2^2 = ac - bd$. These two relationships yield the following equation which is quadradic in $\lambda_{1,2}^2$.

$$\lambda_{1,2}^4 - (a + c)\, \lambda_{1,2}^2 + (ac - bd) = 0 \qquad (2.4.12)$$

This equation is called the dispersion relation (Allis, 1963). Solving for $\lambda_{1,2}^2$ by the quadradic formula gives

$$\lambda_{1,2}^2 = \frac{1}{2}(a + c) \pm \frac{1}{2} \sqrt{(a + c)^2 - 4(ac - bd)}$$

$$= \frac{1}{2}(a + c) \pm \frac{1}{2} \sqrt{(a - c)^2 + 4bd} \qquad (2.4.13)$$

After substituting from Equations 2.4.6 through 2.4.9, this equation becomes

$$\lambda_{1,2}^2 = \frac{1}{2} \left[-k^2 \left(\frac{K_{33}}{K_{11}} + 1 \right) + k_0^2 \left(K_{33} + K_{11} - \frac{K_{12}^2}{K_{11}} \right) \right]$$

$$\pm \frac{1}{2} \sqrt{ \left[-k^2 \left(\frac{K_{33}}{K_{11}} - 1 \right) + k_0^2 \left(K_{33} + \frac{K_{12}^2}{K_{11}} - K_{11} \right) \right]^2 + 4k^2 k_0^2 \frac{K_{33} K_{12}^2}{K_{11}^2} }$$

(2.4.14)

This equation expresses the transverse wave numbers λ_1 and λ_2 in terms of the longitudinal wave number k. The real and imaginary parts of these wave numbers determine the propagation and attenuation of waves in this medium. It may be noted that for each longitudinal wave number k_i there are two transverse wave numbers $\lambda_{1,2}(k_i)$. Also, it is possible for $\lambda_{1,2}$ to be complex when k is real, and conversely, for $\lambda_{1,2}$ to be real when k is complex.

As mentioned earlier, the main object of this boundary value problem is to find those specific wave numbers k_i which correspond to components of the current distribution. The last equation will be useful later when examining various possible waves to make sure they are physically plausible.

<u>2.5 The Transverse Field Components.</u> Once a solution E_z of the wave equation has been obtained for a particular set of boundary conditions, the other five field components can be found in terms of E_z. This is a case where the fields cannot be separated into independent TE and TM modes. There are degenerate cases, however, where the two coupled second-order wave equations, Equations 2.4.4 and 2.4.5 become uncoupled and the possibility of independent TE and TM modes arises. This would be when b = d = 0. From the expressions for b and d, and the plasma parameters discussed in Section 2.1, it is seen that there are three conditions for which b = d = 0.

1. $Y = 0$ (isotropic plasma)
2. $Y = \infty$ (birefringent plasma) (2.5.1)
3. $X = 0$ (free space)

E_z and H_z can be determined in these limiting cases from the now uncoupled second-order wave equations. The equations in Section 2.2 can then be used to determine the transverse fields.

If b and d are not zero, then H_z and the four transverse fields can be specified in terms of E_z. This has been done by Allis (1963), Likuski (1964), and Mushiake (1965), so a rather brief development will be given here.

It will be convenient to construct a matrix equation which can be solved for the transverse components in terms of the longitudinal components. Two equations which are used for this are obtained by taking the cross product of $\underline{a}_z$ with Equations 2.2.11 and 2.2.12 and realizing that $\underline{a}_z \times \underline{a}_z \times \underline{A} = - \underline{A}$.

$$\nabla_t H_z + ik\underline{H}_t = i\omega\, \epsilon_0 K_{11}\, \underline{a}_z \times \underline{E}_t + \omega\, \epsilon_0 K_{12} \underline{E}_t \qquad (2.5.2)$$

$$\nabla_t E_z + ik\underline{E}_t = - i\omega\, \mu_0 \underline{a}_z \times \underline{H}_t \qquad (2.5.3)$$

The source current terms have been dropped from these equations for simplicity; however, they cause no particular theoretical difficulty if carried along. The last two equations and Equations 2.2.11 and 2.2.12 may now be put into matrix form.

$$\begin{bmatrix} -ik & 0 & 0 & -i\omega\,\mu_0 \\ \omega\,\epsilon_0 K_{12} & -ik & i\omega\,\epsilon_0 K_{11} & 0 \\ 0 & i\omega\,\mu_0 & -ik & 0 \\ -i\omega\,\epsilon_0 K_{11} & 0 & \omega\,\epsilon_0 K_{12} & -ik \end{bmatrix} \begin{bmatrix} \underline{E}_t \\ \underline{H}_t \\ \underline{a}_z \times \underline{E}_t \\ \underline{a}_z \times \underline{H}_t \end{bmatrix} = \begin{bmatrix} \nabla_t E_z \\ \nabla_t H_z \\ \underline{a}_z \times \nabla_t E_z \\ \underline{a}_z \times \nabla_t H_z \end{bmatrix} \qquad (2.5.4)$$

By inverting this matrix equation a solution for the transverse fields is obtained in terms of the longitudinal fields.

The cofactor matrix is

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \tag{2.5.5}$$

where

$$A_{11} = A_{22} = A_{33} = A_{44} = -\ ik(-k^2 + k_0^2 K_{11})$$

$$A_{12} = A_{34} \qquad = k^2 \omega\ \epsilon_0 K_{12}$$

$$A_{13} = A_{24} = -A_{31}\ -A_{42} = kk_0^2 K_{12}$$

$$A_{14} = -A_{32} \qquad = -i\omega\ \epsilon_0 [k_0^2 (K_{12}^2 - K_{11}^2) + k^2 K_{11}]$$

$$A_{23} = -A_{41} \qquad = i\omega\ \mu_0 (k^2 - k_0^2 K_{11})$$

$$A_{21} = A_{43} \qquad = -k_0^2 \omega\ \mu_0 K_{12}$$

and the determinant is

$$D = (k^2 - k_0^2 K_{11})^2 - k_0^4 K_{12}^2 \tag{2.5.6}$$

where

$$k_0^2 = \omega^2\ \mu_0 \epsilon_0$$

The inverse of a matrix is the transpose of the cofactor matrix divided by the determinant. Thus the transverse fields can be written

$$\begin{bmatrix} \underline{E}_t \\ \underline{H}_t \\ \underline{a}_z \times \underline{E}_t \\ \underline{a}_z \times \underline{H}_t \end{bmatrix} = \frac{1}{D}[A]^T \begin{bmatrix} \nabla_t E_z \\ \nabla_t H_z \\ \underline{a}_z \times \nabla_t E_z \\ \underline{a}_z \times \nabla_t H_z \end{bmatrix} \tag{2.5.7}$$

It will be convenient to split E_z, the solution to the wave equation discussed earlier, into two parts, one part for each transverse wave number.

$$E_z = E_{z1} + E_{z2} \tag{2.5.8}$$

From this equation and Equation 2.4.4 and 2.4.5 it can be seen that

$$H_z = h_1 E_{z1} + h_2 E_{z2} \tag{2.5.9}$$

where

$$h_i = \frac{a - \lambda_i^2}{b} = \frac{d}{c - \lambda_i^2} \qquad (i = 1,\ 2) \tag{2.5.10}$$

Then from Equation 2.5.7, the transverse electric and magnetic fields for one of the transverse wave numbers can be written

$$\underline{E}_{ti} = \frac{1}{D}\left[A_{11} + A_{21}\left(\frac{a-\lambda_i^2}{b}\right)\right]\nabla_t E_{zi} + \frac{1}{D}\left[A_{31} + A_{41}\left(\frac{a-\lambda_i^2}{b}\right)\right]\underline{a}_z \times \nabla_t E_{zi} \tag{2.5.11}$$

$$\underline{H}_{ti} = \frac{1}{D}\left[A_{12} + A_{22}\left(\frac{a-\lambda_i^2}{b}\right)\right]\nabla_t E_{zi} + \frac{1}{D}\left[A_{32} + A_{42}\left(\frac{a-\lambda_i^2}{b}\right)\right]\underline{a}_z \times \nabla_t E_{zi} \tag{2.5.12}$$

From Equation 2.4.13 it can be shown that

$$a - \lambda_1^2 = - c + \lambda_2^2 \qquad (2.5.13)$$

The coefficients of Equations 2.5.11 and 2.5.12 may now be written as

$$\beta_i = \frac{1}{D}\left[A_{11} + A_{21}\left(\frac{- c + \lambda_i^2}{b}\right)\right] = \frac{i}{k}\left[1 - \frac{k_0^2 K_{11}\lambda_i^2}{(k^2 - k_0^2 K_{11})^2 - k_0^4 K_{12}^2}\right] \qquad (2.5.14)$$

$$\gamma_i = \frac{1}{D}\left[A_{31} + A_{41}\left(\frac{- c + \lambda_i^2}{b}\right)\right] = \frac{K_{11}}{kK_{12}}\left[1 + \frac{k_0^2 K_{11}\lambda_i^2}{(k^2 - k_0^2 K_{11})^2 - k_0^4 K_{12}^2}\right] \qquad (2.5.15)$$

$$\alpha_i = \frac{1}{D}\left[A_{12} + A_{22}\left(\frac{- c + \lambda_i^2}{b}\right)\right] = - \frac{K_{11}}{K_{12}\omega\,\mu_0}\left[1 + \frac{(k^2 - k_0^2 K_{11})\lambda_i^2}{(k^2 - k_0^2 K_{11})^2 - k_0^4 K_{12}^2}\right] \qquad (2.5.16)$$

$$\delta_i = \frac{1}{D}\left[A_{32} + A_{42}\left(\frac{- c + \lambda_i^2}{b}\right)\right] = - \frac{i\omega\,\epsilon_0 K_{11}\lambda_i^2}{(k^2 - k_0^2 K_{11})^2 - k_0^4 K_{12}^2} \qquad (2.5.17)$$

The transverse gradient can be split into its component vectors in cylindrical coordinates.

$$\nabla_t E_z = \underline{a}_r \frac{\partial E_z}{\partial r} + \frac{\underline{a}_\varphi}{r}\frac{\partial E_z}{\partial \varphi} \qquad (2.5.18)$$

Using the vector relationships $\underline{a}_z \times \underline{a}_r = \underline{a}_\varphi$ and $\underline{a}_z \times \underline{a}_\varphi = - \underline{a}_r$ it is now possible to write the components of the transverse fields as follows:

$$E_\varphi = \frac{\beta_2}{r}\frac{\partial E_{z1}}{\partial \varphi} + \frac{\beta_1}{r}\frac{\partial E_{z2}}{\partial \varphi} + \gamma_2 \frac{\partial E_{z1}}{\partial r} + \gamma_1 \frac{\partial E_{z2}}{\partial r} \qquad (2.5.19)$$

$$E_r = \beta_2 \frac{\partial E_{z1}}{\partial r} + \beta_1 \frac{\partial E_{z2}}{\partial r} - \frac{\gamma_2}{r}\frac{\partial E_{z1}}{\partial \varphi} - \frac{\gamma_1}{r}\frac{\partial E_{z2}}{\partial \varphi} \qquad (2.5.20)$$

$$H_\varphi = \frac{\alpha_2}{r}\frac{\partial E_{z1}}{\partial \varphi} + \frac{\alpha_1}{r}\frac{\partial E_{z2}}{\partial \varphi} + \delta_2 \frac{\partial E_{z1}}{\partial r} + \delta_1 \frac{\partial E_{z2}}{\partial r} \qquad (2.5.21)$$

$$H_r = \alpha_2 \frac{\partial E_{z1}}{\partial r} + \alpha_1 \frac{\partial E_{z2}}{\partial r} - \frac{\delta_2}{r}\frac{\partial E_{z1}}{\partial \varphi} - \frac{\delta_1}{r}\frac{\partial E_{z2}}{\partial \varphi} \qquad (2.5.22)$$

In these equations, each component of the transverse electric and magnetic fields is given explicity in terms of the longitudinal electric field with an $\exp(i\omega t - ikz)$ variation.

CHAPTER III

DOMAIN OF THE WAVE EQUATION

3.1 Definition of Domain. Basically, the domain of a differential operator L is that set of functions u which satisfy certain boundary conditions, and are such that both u and Lu satisfy certain continuity or integrability conditions. A complete definition of L and its domain thus requires not only that Lu = 0, for example, but also the conditions satisfied by u. For precise notation a different symbol for the operator should be used each time the conditions on u are changed. Usually, however, the same letter is used for the differential operator under all conditions, but the boundary conditions satisfied by u are always specified in addition. Friedman (1956) has an excellent discussion of this material.

The operator equation and boundary conditions for the infinite antenna may be written as follows:

$$LE_z = (\nabla_t^2 + \lambda_1^2)(\nabla_t^2 + \lambda_2^2)E_z = 0 \qquad (3.1.1)$$

where

$$E_z(r = a) = f_1(\varphi, z) \quad \text{inside the slot} \qquad (3.1.2)$$

$$E_z(r = a) = 0 \quad \text{outside the slot} \qquad (3.1.3)$$

$$E_z(r, z) \quad \text{satisfies the radiation condition} \qquad (3.1.4)$$

There are also, of course, the related requirements that λ_1^2 and λ_2^2 satisfy the dispersion relation, Equation 2.4.12, and that $E_\varphi(r = a) = 0$.

It is evident that finding the domain of L is equivalent to solving the infinite antenna problem.

<u>3.2 Eigenfunctions and Elementary Wave Functions of the Wave Equation.</u> It will be convenient to define the eigenfunctions e_{zj} as those functions which satisfy Equation 3.1.1, but which may not individually satisfy any of the boundary conditions. Actually the eigenfunctions of primary interest will be those which satisfy the radiation conditions to be discussed later in this chapter. The total E_z field is then built up of these eigenfunctions by summation or integration techniques such that the boundary conditions on the antenna are satisfied.

There are four linearly independent eigenfunction solutions to the fourth-order equation $Le_z = 0$. One set of these eigenfunctions are the Hankel functions of order n and of the first and second kinds.

$$\begin{aligned}
e_{z1} &= H_n^{(1)}(\lambda_1 r)\, e^{i(\omega t - kz + n\varphi)} \\
e_{z2} &= H_n^{(2)}(\lambda_1 r)\, e^{i(\omega t - kz + n\varphi)} \\
e_{z3} &= H_n^{(1)}(\lambda_2 r)\, e^{i(\omega t - kz + n\varphi)} \\
e_{z4} &= H_n^{(2)}(\lambda_2 r)\, e^{i(\omega t - kz + n\varphi)}
\end{aligned} \qquad (3.2.1)$$

These are the only Bessel functions which represent wave motion and will be used here for that reason.

The special case of azimuthal symmetry is given by n = 0 in the above expression. It is convenient to choose a slot in the antenna such that an azimuthally symmetric electric field exists in it. The infinite antenna problem with this circumferential slot will now be solved for that portion of the fields which are azimuthally symmetric. In the isotropic case (Stratton, 1941) this n = 0 portion of the total field is the dominant part, although no similar analysis has been performed in the anisotropic case. This would be a good area for future investigation.

The slotted antenna now appears as in Figure 2, with its orientation to various coordinate systems.

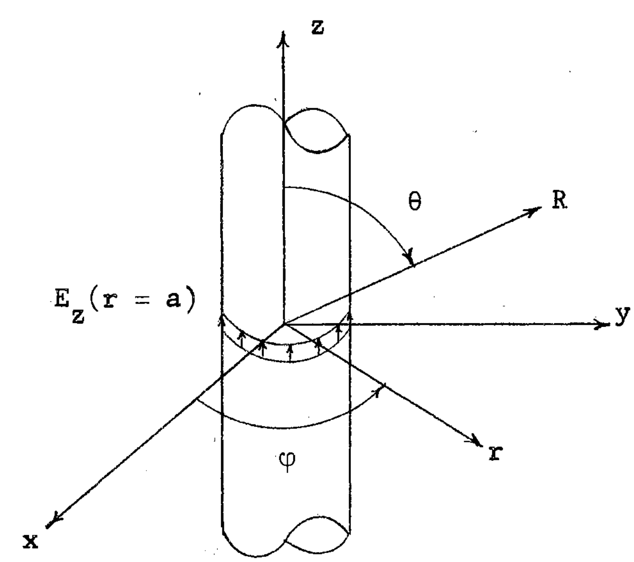

Figure 2. Infinite Antenna and Cylindrical and Spherical Coordinate Systems.

The eigenfunctions of $Le_z = 0$ may now be written as

$$\begin{aligned} e_{z1} &= H_0^{(1)}(\lambda_1 r)\, e^{i(\omega t - kz)} \\ e_{z2} &= H_0^{(2)}(\lambda_1 r)\, e^{i(\omega t - kz)} \\ e_{z3} &= H_0^{(1)}(\lambda_2 r)\, e^{i(\omega t - kz)} \\ e_{z4} &= H_0^{(2)}(\lambda_2 r)\, e^{i(\omega t - kz)} \end{aligned} \qquad (3.2.2)$$

Somewhat different results occur for the special case of $\lambda_1^2 = \lambda_2^2 = \lambda^2$. Equation 3.1.1 becomes

$$(\nabla_t^2 + \lambda^2)^2 E_z = 0 \qquad (3.2.3)$$

This is still a fourth-order differential equation, but now only two of the eigenfunctions in Equation 3.2.2 are independent; those containing $H_0^{(1)}(\lambda r)$ and $H_0^{(2)}(\lambda r)$. Two new eigenfunctions of the above equations are needed to span the space or to form a complete set of solutions of the fourth-order operator. It may be shown by direct substitution that these eigenfunctions are $rH_1^{(1)}(\lambda r)$ and $rH_1^{(2)}(\lambda r)$. These new functions are the generalized eigenfunctions of the operator L (Friedman, 1956). Generalized eigenfunctions (of rank 2) of an operator M are those functions u' where $Mu' \neq 0$ but $M^2u' = 0$. Strictly speaking, $rH_1^{(1)}(\lambda r)$ is a generalized eigenfunction of the operator L for $\lambda_1^2 \neq \lambda_2^2$ and an eigenfunction for $\lambda_1^2 = \lambda_2^2$, but this need not be included in the terminology. The set of eigenfunctions and generalized eigenfunctions for Equation 3.2.3, denoted by primes, are therefore given by

$$
\begin{aligned}
e^{1}_{z_1} &= H_0^{(1)}(\lambda r)\, e^{i(\omega t - kz)} \\
e^{1}_{z_2} &= H_0^{(2)}(\lambda r)\, e^{i(\omega t - kz)} \\
e^{1}_{z_3} &= rH_1^{(1)}(\lambda r)\, e^{i(\omega t - kz)} \\
e^{1}_{z_4} &= rH_1^{(2)}(\lambda r)\, e^{i(\omega t - kz)}
\end{aligned}
\qquad (3.2.4)
$$

The term "elementary wave function" will now be defined as a linear sum of eigenfunctions. For $\lambda_1^2 \neq \lambda_2^2$, the elementary wave function is

$$e_z = AH_0^{(1)}(\lambda_1 r) + BH_0^{(2)}(\lambda_1 r) + CH_0^{(1)}(\lambda_2 r) + DH_0^{(2)}(\lambda_2 r) \qquad (3.2.5)$$

while for $\lambda_1^2 = \lambda_2^2$ the elementary wave function is

$$e'_z = A'H_0^{(1)}(\lambda r) + B'H_0^{(2)}(\lambda r) + C'rH_1^{(1)}(\lambda r) + D'rH_1^{(2)}(\lambda r) \qquad (3.2.6)$$

The $\exp(i\omega t - ikz)$ variation is to be understood in all field descriptions where it does not explicitly appear.

Both of the elementary wave functions e_z and e'_z have four undetermined coefficients. The specified values of E_z and E_φ on the antenna can be used to yield two relations between these four coefficients. This problem thus has the common characteristic with other problems in wave motion that in an infinite medium the wave equation and the boundary conditions on any conducting structures do not by themselves determine the solution uniquely. However, any well-formulated mathematical problem which has a physical basis should possess a unique solution. Sommerfeld (1912) appears to be the first to recognize this difficulty, and has proposed additional restrictions on the behavior of the waves at infinity. In isotropic problems there are only two eigenfunctions in the elementary wave function, and when the Sommerfeld radiation condition is applied, one of the two eigenfunctions will be discarded. After a suitable radiation condition is defined for anisotropic media later in the chapter, two of the four eigenfunctions in the elementary wave function will be rejected, resulting in an unique solution.

Two points on terminology should be mentioned here. First, the total field E_z, satisfying all boundary conditions, will be built up of elementary wave functions satisfying the boundary conditions on the antenna. Second, in the literature (Harrington, 1961) the terms "eigenfunctions" and "elementary wave functions" are used interchangeably. This is satisfactory in isotropic problems, because after applying the radiation condition, the elementary wave function reduces to one eigenfunction. However, in the anisotropic case it will be convenient to examine each eigenfunction individually for the radiation condition, but the total field will be formed from an elementary wave function or two eigenfunctions.

3.3 The Radiation Condition in Isotropic Media. Most wave motion problems considered in the literature have dealt with isotropic media. A number of mathematicians have examined the question of the proper boundary condition at infinity, the so-called radiation condition. It is only from this extensive groundwork laid for the isotropic case that one can hope to define a suitable radiation condition for anisotropic media. This is not to imply that no more work needs to be done concerning the radiation condition for isotropic media. For example, Dolph (1956) outlines a number of difficulties which mathematicians have with the Sommerfeld radiation condition, particularly with normal mode theory and backward scattering by the Schwinger variational principle. He then discusses a "Dirichlet" principle for the wave equation as a possible way of avoiding the Sommerfeld radiation condition entirely. Of course, this then requires a complete knowledge of initial conditions, something at least as difficult to work with as the radiation condition.

For the purposes of this thesis, the original statement of Sommerfeld (1912) concerning uniqueness may be formulated as follows:

The function u is uniquely specified throughout all space if

(i) u is specified at all points on the closed surface S,

(ii) u has continuous second derivatives on and outside S,

(iii) $\nabla^2 u + k^2 u = 0$ (the scalar Helmholtz equation) on and outside S,

(iv) Ru remains bounded as $R \to \infty$, uniformly in all directions,

(v) $R(iku + \frac{\partial u}{\partial R}) \to 0$ as $R \to \infty$, also uniformly in all directions.

In the last two conditions R is the distance from any fixed point, which is taken to be the origin of a spherical coordinate system. Condition (iv) is Sommerfeld's "condition of finiteness" (Endlichkeitsbedingung); condition (v) is his "radiation condition" (Ausstrahlungsbedingung). This ensures that at great distances from the source the field represents a divergent traveling wave. The algebraic sign betwen the two terms is for an exp(+ iω t) time variation. The function u is called the scalar radiation function. Usually conditions (iv) and (v) are taken together as the Sommerfeld radiation condition. Atkinson (1949) appears to be the first to offer a rigorous proof for this statement. It seems that Sommerfeld argued the validity of his radiation condition for physical or somewhat non-rigorous reasons. It does seem quite obvious from a physical view-point that waves should be of order $O(1/R)$ for large R, and that the waves should exhibit outward wave motion. This intuitive approach was probably the reason for the delay between Sommerfeld's original statement and its proof.

There are other, essentially equivalent, ways of expressing conditions (iv) and (v). Atkinson, for example, proves that both conditions (iv) and (v) are given by either of the statements

$$R_e^{-ikR}\left[\left(ik - \frac{1}{R}\right)u + \frac{\partial u}{\partial R}\right] \longrightarrow 0 \quad \text{as } R \rightarrow \infty \qquad (3.3.1)$$

$$R^3 e^{ikR}\left[\left(ik - \frac{1}{R}\right)u + \frac{\partial u}{\partial R}\right] \quad \text{bounded as } R \rightarrow \infty \qquad (3.3.2)$$

The radiation condition can also be formed from vector functions. The vector Helmholtz equations for $\underline{E}$ and $\underline{H}$ are

$$\nabla \times \nabla \times \underline{E} - k^2 \underline{E} = 0 \qquad (3.3.3)$$

$$\nabla \times \nabla \times \underline{H} - k^2 \underline{H} = 0 \qquad (3.3.4)$$

Silver (1947) starts with these equations (actually, the vector Helmholtz equations including electric and magnetic sources) and uses a Green's function technique to obtain the following vector radiation condition.

(1) $R\underline{E}$, $R\underline{H}$ remain bounded as $R \to \infty$

(2) $R\left[(\underline{a}_R \times \underline{H}) + \sqrt{\frac{\epsilon}{\mu}}\,\underline{E}\right]$, $R\left[\sqrt{\frac{\epsilon}{\mu}}\,(\underline{a}_R \times \underline{E}) - \underline{H}\right] \to 0$ as $R \to \infty$

Again, condition (1) is regularity at infinity, while condition (2) ensures that radiation consists of divergent waves.

Wilcox (1956) was able to prove that regularity at infinity is a consequence of the divergent waves requirement so that only a single statement is needed. This he called the Silver-Müller radiation condition and wrote as

$$R\left[\underline{a}_R \times (\nabla \times \underline{U}) + ik\underline{U}\right] \longrightarrow 0 \quad \text{as } R \to \infty$$

uniformly in all directions. (3.3.5)

where

$$\underline{U} = \sqrt{-\,i(\omega\,\epsilon + i\sigma)}\,\underline{E} \quad \text{or} \quad \sqrt{i\omega\,\mu}\,\underline{H} \qquad (3.3.6)$$

By analogy with Sommerfeld's formulation, $\underline{U}$ is called a vector radiation function.

In addition to a number of other important results, Wilcox (1956) proves the following expansion theorem.

Let $\underline{U}(\underline{R})$ be a vector radiation function for a region $R > c$ where (R, θ, φ) are spherical coordinates. Then $\underline{U}(\underline{R})$ has an (unique) expansion

$$\underline{U}(\underline{R}) = \frac{e^{ikR}}{R} \sum_{n=0}^{\infty} \frac{\underline{U}_n(\theta, \varphi)}{R^n} \qquad (3.3.7)$$

which is valid for $R > c$, and $\text{Im}\, k \geq 0$, and which converges absolutely and uniformly in the parameters R, θ, and φ in any region $R \geq c + \epsilon > c$. The series can be differentiated term by term with respect to R, θ, and φ any number of times and the resulting series all converge absolutely and uniformly. The proof of this theorem is not given here, but the result will be useful later when examining yet another form of the radiation condition.

It might be pointed out that the Cartesian components of $\underline{U}$ satisfy the scalar Helmholtz equation so that problems involving vector radiation functions may be reduced to problems involving scalar wave functions. However, the additional constraint of $\nabla \cdot \underline{U} = 0$ must be satisfied before any arbitrary scalar wave functions u_x, u_y, and u_z can be combined to form a vector wave function.

There is one subtle but important point in the hypothesis of all this work on the three dimensional radiation condition. That is the requirement that all sources and scattering surfaces be contained within a sphere of finite radius. This excludes such special but non-trivial cases as the infinite cylindrical antenna and the infinite biconical antenna. This point seems to have been ignored by several workers in the area (Papas, 1949; Silver and Saunders, 1950; Northover, 1958), as they have explicitly or implicitly used the radiation condition as outlined above on the infinite cylindrical antenna. From the physical viewpoint this is probably all right since it is still physically plausable that waves travel outward. However, from the mathematics viewpoint, the relaxation of this part of the hypothesis may make it difficult or impossible to prove any theorems on uniqueness using the same general approach.

Thus one is led to reexamine the whole question of uniqueness, in the hope that a form of the radiation condition can be found which does not depend on the requirement that all sources and scattering surfaces be contained within a bounded sphere. Historically, the radiation condition has been defined by properties of the individual electric or magnetic fields. This has resulted primarily from physical reasoning about how these fields ought to behave, and was later put into precise mathematical form which follows closely the original reasoning process. It appears to this author that this line of thinking has been carried about as far as is practicable and will not yield a suitable radiation condition for either certain special cases in isotropic media or for more general cases in anisotropic media. The alternate physical approach which will be discussed next is that of energy stored in the fields. This is evidently at least as basic a consideration as that of wave motion. The radiation condition so deduced does not depend on the form of Maxwell's equations for homogeneous, isotropic media. Therefore it will be written for a more general media and specialized when necessary to the isotropic case.

3.4 The Radiation Condition in Anisotropic Media. The time average complex power flow across a closed surface S is given by the integral of the complex Poynting vector over that surface (Harrington, 1961).

$$P = \oint_S \underline{E} \times \underline{H}^* \cdot d\underline{s} \qquad (3.4.1)$$

The asterisk signifies the complex conjugate value. The real time average power flow is given by the real part of P while the imaginary part represents energy stored in the electric and magnetic fields. Applying the divergence theorem to Equation 3.4.1 gives

$$\oint_S \underline{E} \times \underline{H}^* \cdot d\underline{s} = \int_V \nabla \cdot \underline{E} \times \underline{H}^* \, dV \tag{3.4.2}$$

where
$$\nabla \cdot \underline{E} \times \underline{H}^* = \underline{H}^* \cdot \nabla \times \underline{E} - \underline{E} \cdot \nabla \times \underline{H}^* \tag{3.4.3}$$

The curl of $\underline{E}$ and $\underline{H}^*$ may be replaced by $- i\omega \underline{B}$ and $(i\omega \underline{D})^* + \underline{J}^*$ from Maxwell's equations. Thus, Equation 3.4.2. becomes

$$\oint_S \underline{E} \times \underline{H}^* \cdot d\underline{s} = \int_V [i\omega(- \underline{B} \cdot \underline{H}^* + \underline{E} \cdot \underline{D}^*)] \, dV - \int_V \underline{E} \cdot \underline{J}^* \, dV \tag{3.4.4}$$

From this equation, the time-average Poynting theorem, the identification is usually made that the time-average amount of energy stored in the electric field is (Harrington, 1961)

$$W_e = \frac{1}{2} \operatorname{Re} \int_V \underline{E} \cdot \underline{D}^* \, dV \tag{3.4.5}$$

The time-average energy stored in the magnetic field is then

$$W_m = \frac{1}{2} \operatorname{Re} \int_V \underline{B} \cdot \underline{H}^* \, dV \tag{3.4.6}$$

With these definitions, Equation 3.4.4 contains a factor $2i\omega(W_e - W_m)$.

These equations are valid in a region where there are impressed current sources $\underline{J}$. Any current induced because of a finite conductivity of the medium is included in $\underline{D}^*$. In an anisotropic media $\underline{D} = \epsilon_0 \hat{K} \cdot \underline{E}$ where $\hat{K}$ is given by Equation 2.1.8. It can be seen by a careful inspection of Equations 2.1.8 and 2.1.4 that the elements of $\hat{K}$, written in matrix form as $[K_{ij}]$, can be written as the sum of an hermitian matrix $[h_{ij}]$ plus Z times an antihermitian matrix $[a_{ij}]$, where $Z = \nu/\omega$.

$$[K_{ij}] = [h_{ij}] + Z[a_{ij}] \qquad (3.4.7)$$

The expressions for h_{ij} and a_{ij} have been worked out and are tabulated in the literature (Kelso, 1964). It may be noted that if $[h_{ij}]$ is hermitian, then $h_{ij} = h^*_{ji}$, which also implies that h_{11}, h_{22}, and h_{33} must be pure real. Similarly, if $[a_{ij}]$ is antihermitian, then $a_{ij} = - a^*_{ji}$ and a_{11}, a_{22}, and a_{33} are pure imaginary.

The integrand of Equation 3.4.5 can now be written as

$$\underline{E} \cdot \underline{D}^* = \epsilon_0 \, \underline{E} \cdot \hat{K}^* \cdot \underline{E}^* = \epsilon_0 \sum_{i,j} K^*_{ij} E_i E^*_j \qquad (3.4.8)$$

With the introduction of Equation 3.4.7, this becomes

$$\underline{E} \cdot \underline{D}^* = \epsilon_0 \sum_{i,j} h^*_{ij} E_i E^*_j + Z \, \epsilon_0 \sum_{i,j} a^*_{ij} E_i E^*_j \qquad (3.4.9)$$

Since $[h_{ij}]$ is an hermitian matrix and $[a_{ij}]$ is an antihermitian matrix, some conclusions can be drawn about the summations of the above equation. First, the terms $h^*_{ij} E_i E^*_j$ are considered. When $i = j$ the terms are all real, since the diagonal elements h_{ij} of an hermitian matrix are always real, and since the product $E_i E^*_j$ equals the square of the magnitude of E_j and is therefore real. For $i \neq j$, the terms $h^*_{ij} E_i E^*_j$ can be paired off with $h^*_{ji} E_j E^*_i$. Since $h_{ij} = h^*_{ji}$, it can be seen that

$$h^*_{ij} E_i E^*_j + h^*_{ji} E^*_i E_j = 2 \, \mathrm{Re}(h^*_{ij} E_i E^*_j) \qquad (3.4.10)$$

Therefore, it may be concluded that the first summation in Equation 3.4.9 is pure real and hence contributes to the integral in Equation 3.4.5.

Similarly, the properties of an antihermitian matrix (diagonal elements are imaginary; $a_{ij} = - a^*_{ji}$) may be used to show that the summation involving a_{ij} in Equation 3.4.9 is pure imaginary.

For the sake of symmetry in the field equations, the permeability will also be written as the sum of an hermitian matrix $[\mu_{ij}]$ plus Z' times an antihermitian matrix $[a'_{ij}]$, where $Z' = \nu'/\omega$.

$$[\mu'_{ij}] = [\mu_{ij}] + Z'[a'_{ij}] \tag{3.4.11}$$

The real part of the Poynting theorem, Equation 3.4.4, can be written as

$$- \text{Re} \int_V \underline{E} \cdot \underline{J}^* dV = \text{Re} \oint_S (\underline{E} \times \underline{H}^*) \cdot dS - \text{Re}\; i\omega\, \epsilon_0 Z \int_V \sum_{i,j} a_{ij} E^*_i E_j \; dV + \text{Re}\; i\omega\, Z' \int_V \sum_{i,j} a'_{ij} H^*_i H_j \; dV \tag{3.4.12}$$

Since the summations involving a_{ij} and a'_{ij} are pure imaginary, i times these summations will be real, so that the Re notation in front of the last two integrals is redundant. This equation is interpreted to mean that the real power supplied by the sources within a region is equal to the real power leaving the region plus the power dissipated within the region. It can be seen that the only contribution to the absorption comes from the antihermitian parts of $[K_{ij}]$ and $[\mu'_{ij}]$. Equation 3.4.12 also shows that for $Z = Z' = 0$

(i.e. $\nu = \nu' = 0$), the fields do not suffer absorption.

The sum of the time-averaged energies stored in the electric and magnetic fields is

$$W_e + W_m = \frac{1}{2} \mathrm{Re} \int_V [h_{ij} E_i E_j^* + \mu_{ij} H_i H_j^*] \, dV \qquad (3.4.13)$$

As shown earlier, this integral is strictly real, so the condition that the real part be taken is unnecessary.

Suppose now the medium is isotropic with a non-zero conductivity. Then from Equation 3.3.7, and Im k > 0, it can be seen that the vector radiation functions $\underline{E}$ and $\underline{H}$ have an exponential decay with increasing radial distance. It is not hard to show from elementary calculus that for such field variations

$$\int_V [h_{ij} E_i E_j^* + \mu_{ij} H_i H_j^*] \, dV < \infty \qquad (3.4.14)$$

where the integration is over the entire three-dimensional space except for a finite region containing the sources. This equation essentially states that when steady state conditions are reached, the energy supplied by the source is offset by the energy dissipated, and the total energy stored in the electric and magnetic fields is finite. It can be shown that this finite energy condition is equivalent to the Sommerfeld radiation condition in isotropic media. However, it can also be shown that this statement results in unique field representations for anisotropic media, so only the more general proof will be given. It will be necessary to invoke the powerful methods of functional analysis to handle such a problem, and a statement of the Uniqueness Theorem can be made after a suitable terminology has been defined. This Uniqueness Theorem is similar to one proposed by Wilcox (1963), as well as the nature of its proof. Its extension to lossless media has not been considered elsewhere to this author's knowledge.

An antenna may be considered as an obstacle O (not necessarily bounded) in the unbounded Euclidean space R^3 so that the region Ω is the set of all points such that $\Omega = R^3 - O$. Maxwell's equations for time harmonic fields are

$$\left.\begin{aligned} \nabla \times \underline{H} - i\omega\, \epsilon_0 \hat{K} \cdot \underline{E} &= \underline{J}_e \\ \nabla \times \underline{E} + i\omega\, \mu_0 \underline{H} &= \underline{J}_m \end{aligned}\right\} \quad \text{in } \Omega \qquad (3.4.15)$$

These equations define a linear operator L, such as the bracketed term in Equation 2.2.27, on a linear vector space. A linear vector space S has the properties that: (1) Any two elements x and y in S may be added, and the result is an element z in S; (2) The operation of addition is commutative and associative; (3) S contains a unique element called the null or zero element; (4) For any x in S, there exists an element (- x) such that x + (- x) = 0. Abstractly, an operator L is a mapping which assigns to a vector x in a linear vector space S another vector in S which we denote by Lx. The set of vectors x for which the mapping is defined is called the domain of L. The set of vectors y which are equal to Lx for some x in the domain is called the range of the operator. Equations 3.4.15 may be used to define in S a complex scalar product. The complex vectors $\underline{E}$ and $\underline{H}$ which satisfy such a complex scalar product are said to be Lebesgue square integrable on Ω, (now the volume of integration), and denoted by $L_2(\Omega)$. Such a linear vector space with a complex scalar product is called a Hilbert space.

The problem is now to find the domain of L (the vector $\underline{F} = \underline{E}$ or $\underline{H}$) where

$$L\underline{E} = \underline{f}_e(\underline{J}_e, \underline{J}_m) \tag{3.4.16}$$

$$L\underline{H} = \underline{f}_h(\underline{J}_e, \underline{J}_m) \tag{3.4.17}$$

$$\underline{n} \times \underline{E} = 0 \quad \text{on } \partial\Omega \tag{3.4.18}$$

and

$$\int_\Omega [h_{ij}E_iE_j^* + \mu_{ij}H_iH_j^*]\, dV < \infty \tag{3.4.19}$$

In Equation 3.4.18, $\partial\Omega$ denotes the boundary of Ω on which the tangential electric field goes to zero. Equation 3.4.19 will be called the Wilcox radiation condition to distinguish it from the wave motion formulation of Sommerfeld.

The precise statement of the Uniqueness Theorem is as follows.

Theorem 1. If h_{ij} and μ_{ij} are bounded, Lebesgue measurable functions of position, and if $\omega \neq 0$ and real, then the steady state solution $\underline{F}$ of Equations 3.4.16-19 exists and is unique for every source field $\underline{f}(\underline{J}_e, \underline{J}_m)$ in $L_2(\Omega)$.

The proof of this theorem is found in Appendix A.

Thus it is seen that the Wilcox radiation condition causes the fields to be uniquely specified for rather general media, sources, and conducting surfaces, but with non-zero losses. The question of uniqueness as the losses go to zero will be considered next.

3.5 Extension of the Wilcox Radiation Condition to Lossless Media. To see why such an extension is necessary, consider the eigenfunctions of the infinite antenna problem, Equation 3.2.2. The longitudinal wave number k and the transverse wave numbers λ_1 and λ_2 are related by Equation 2.4.14 where they occur only in the squared form. For a lossy medium both k^2 and $\lambda_{1,2}^2$ will be complex, in general. Therefore a choice of square root has to be made. It is evident that the fields near the antenna have the proper decay if the square root of k^2 is chosen such that

$$\begin{aligned} \text{Im } k &< 0, \quad z > 0 \\ \text{Im } k &> 0, \quad z < 0 \end{aligned} \tag{3.5.1}$$

Next the radial variation has to be considered. The large argument formulas for the Hankel functions are

$$H_n^{(1)}(\lambda r) \xrightarrow[\lambda r \to \infty]{} \sqrt{\frac{2}{i\pi\lambda r}}\,(i)^{-n}\,e^{i\lambda r} \tag{3.5.2}$$

$$H_n^{(2)}(\lambda r) \xrightarrow[\lambda r \to \infty]{} \sqrt{\frac{2i}{\pi\lambda r}}\,(i)^{n}\,e^{-i\lambda r} \tag{3.5.3}$$

If, for the sake of illustration, the square roots of $\lambda_{1,2}^2$ are chosen such that

$$\text{Im } \lambda_1 \leq 0, \quad \text{Im } \lambda_2 \leq 0 \tag{3.5.4}$$

then the Hankel functions of the first kind, $H_0^{(1)}(\lambda_1 r)$ and $H_0^{(1)}(\lambda_2 r)$ become unbounded for large r. Such unbounded field variation is obviously not Lebesgue square integrable and so is rejected from the set of possible solutions. Meanwhile, the other Hankel functions, $H_0^{(2)}(\lambda_1 r)$ and $H_0^{(2)}(\lambda_2 r)$, experience an exponential decay in the radial direction. This radial decay plus the longitudinal decay due to

Im k $\neq$ 0 yields eigenfunctions which are in $L_2(\Omega)$. Thus two of the four eigenfunctions remain, and these, together with the two boundary conditions $E_z = E_\varphi = 0$ at $r = a$ will yield a unique solution.

Now suppose the losses go to zero. A simple analysis will show that if k is real, then none of the four eigenfunctions of Equation 3.2.2 are in $L_2(\Omega)$. The same is true of any eigenfunction for which the transverse wave number is real. Such eigenfunctions are bounded at infinity, but still correspond to states of infinite energy. Three general types of waves which fall into this category are plane waves, spherical waves, and cylindrical waves. These types of waves are admittedly idealized, but have proved quite useful in solving a broad class of fields problems. Therefore, one wants to keep such waves as possible solutions to the wave equation, and modify the Wilcox radiation condition as necessary to do this.

First, a rule by which such bounded waves not in $L_2(\Omega)$ may be included in the possible domain of L will be given. Then the mathematical properties of such waves will be discussed.

Friedman (1956) gives a rule for uniquely defining the Green's function G of a second order differential operator over a semi-infinite or infinite interval. He uses the "outgoing waves" concept to choose the proper square root of the squared eigenvalue when it is complex, and then uses the principle of analytic continuation to define G for a real eigenvalue. A similar rule for eigenfunctions may be stated in the terminology of this chapter as follows.

<u>Rule.</u> Suppose that L is a differential operator which is in the limit-point case at infinity; then the eigenfunctions e_{zj} for L are defined by implicitly requiring that e_{zj} vanish for large values

of r if $\lambda^2_{1,2}$ are complex and Im $\lambda_1 < 0$, Im $\lambda_2 < 0$. For real positive values of $\lambda^2_{1,2}$, the eigenfunctions are defined as the limit of e_{zj} for complex values of $\lambda_{1,2}$ as $\lambda_{1,2}$ becomes real.

A differential operator L is said to be in the limit-point case at infinity if the differential equation without boundary conditions, represented by $Lu = 0$, has at least one solution which is not of integrable square in an interval containing infinity.

Even for the case of equal wave numbers, $\lambda^2_1 = \lambda^2_2$, it may be seen that this rule eliminates exactly two eigenfunctions of the four in Equation 3.2.4 without any confusion. For λ^2 complex, the exponential decay of the Hankel functions will dominate any multiplying factor of r. Therefore the two eigenfunctions satisfying the rule (and the Wilcox radiation condition) are $H_0^{(2)}(\lambda r)$ and $rH_1^{(2)}(\lambda r)$. The same two will be the eigenfunctions for real positive λ^2 even though neither are then in $L_2(\Omega)$.

The choice of square root of $\lambda^2_{1,2}$ made in the rule appears rather arbitrary, so some explanation is in order. The choice was made such that the Hankel function of the second kind, Equation 3.5.3, should have an exponential decay for large r. If at the same time Re $\lambda \geq 0$, then this Hankel function represents outward phase travel for an exp(+ $i\omega$ t) time variation. In isotropic media the two roots of the squared wave number, proportional to the first radical in Equation 3.3.6, lie in the second and fourth quadrants. Thus it is always possible in such media to have a wave number with a positive real part and a negative imaginary part. Therefore in Friedman's rule, the requirement that the imaginary part of an eigenvalue be negative is identical to saying the real part is positive, or that the eigenfunction behaves like outgoing waves at infinity. (Friedman chose an $\exp(-i\omega t)$ time variation so his inequality signs are actually opposite to those given here). Therefore as an anisotropic medium becomes isotropic the rule stated here reduces to the rule stated by Friedman, as it should. However, in anisotropic media, choosing the sign of the imaginary part of $\lambda_{1,2}$ does not automatically fix the sign of the real part. The real part may be positive or negative depending on the frequency and the plasma parameters, as can be seen from a numerical investigation of Equation 2.4.14. This implies that the phase variation of each eigenfunction will be either inward or outward, depending on the parameters. This is an interesting change from the isotropic case. However, the question is whether this change is important or not, or if the direction of phase travel in anisotropic media has the same meaning as in isotropic media. The answer is no, the phase behavior of these eigenfunctions has little or no correlation with such things as direction of power flow in dissipative anisotropic media. This may be argued from several viewpoints.

First, it should be emphasized that the individual eigenfunctions have little physical meaning by themselves. Only the total field $\underline{E}$ may be reasonably expected to show an outward flow of energy. Second, the direction of energy flow and the direction of phase travel are not necessarily the same in anisotropic media. Hines (1951a, b, c, d, 1952) has made an exhaustive study of energy flow in various types of media. He concludes that the Poynting theorem (non time averaged) is not as representative of the expected flow of energy in some media as the Macdonald theorem (Macdonald, 1902). The use of the Poynting vector in dissipative anisotropic media leads to a rather dismaying picture in which different bits of energy in the wave may move in different directions. The motion may be very disorderly and complicated. Even under the best conditions, the energy in a bit of a wave oscillates through a cycle, energy being absorbed at one time and regenerated at a later time.

The Macdonald vector gives an alternative expression for the energy of the wave. One example where it differs from Poynting's vector is that when a constant current flows in a wire, a slight modification of Macdonald's expression leads to the conclusion that the energy is flowing in the wire, whereas the Poynting vector arguments indicate that energy must flow into the wire from the surrounding medium. In the notation of this section, Macdonald's theorem gives, for the energy of an electromagnetic wave,

$$\oint_S [(\underline{E} \times \underline{H}) + \frac{1}{2}\frac{\partial}{\partial t}(\underline{A} \times \underline{H})] \cdot d\underline{s} = \int_V [\underline{E} \cdot \frac{\partial}{\partial t}\underline{D} + \frac{1}{2}\left(\frac{\partial \underline{A}}{\partial t}\right)\cdot\left(\frac{\partial \underline{D}}{\partial t}\right)]\, dV \tag{3.5.5}$$

where $\underline{A}$ is the vector potential, given by $\underline{B} = \nabla \times \underline{A}$. The bracketed expression in the integral on the left-hand side is called the Macdonald vector.

There is a third technique for determining the direction of energy flow, called the packet method (Kelso, 1964). This approximate method gives a direction different from the Poynting vector or Macdonald vector method. For some special cases it appears that the packet method is more realistic than the two energy theorems.

A complete discussion of the packet method and Macdonald's theorem is beyond the scope of this thesis. The main reason for introducing them is to suggest that in a dissipative anisotropic medium the exact character of the movement of a bit of energy is not known, nor can it be known from a purely mathematical approach.

Therefore, with Wilcox (1963), one concludes that phase information is irrelevant in defining the domain of L in an anisotropic medium, and that only the finite energy condition has undisputed meaning. The only real requirement on the rule for choosing eigenfunctions is that it agrees with the isotropic case when the anisotropy goes to zero, and as mentioned earlier, this has been accomplished.

Now that the rule has been established which produces an extended domain of L, the type of wave introduced by this extension can be discussed. The domain of L includes only those functions belonging to $L_2(\Omega)$ or having finite energy. These are called eigenfunctions and have their corresponding eigenvalues k_i and $\lambda_{1,2}(k_i)$. The extended domain of L contains the domain of L plus a restricted class of functions not belonging to $L_2(\Omega)$. These new functions are called improper eigenfunctions corresponding to a set of improper eigenvalues. As has been pointed out, the only time the rule for choosing an unique set of functions is satisfied by functions not in $L_2(\Omega)$ is when there are no losses. This implies that only real positive eigenvalues are improper eigenvalues and conversely. According to Friedman (1956) the eigenfunctions form the discrete spectrum of L while the improper eigenfunctions form the continuous spectrum.

The physical significance of the eigenfunctions and improper eigenfunctions will become more evident after the infinite antenna problem is solved in the next chapter. However, from Friedman (1956) the statement may be made that in general, the eigenfunctions produce surface waves which are exponentially damped in the direction transverse to the direction in which the wave is moving. The improper eigenfunctions produce space waves; that is, waves which cause energy to be transported away from any radiating sources without being confined to the near vicinity of some surface.

CHAPTER IV

INTEGRAL SOLUTION FOR THE DOMAIN OF THE WAVE EQUATION

4.1 Formulation of the Integral. The material given in the previous two chapters may now be used to find integral expressions for the fields external to an infinitely long antenna aligned with a static magnetic field in a plasma, and excited by a narrow circumferential slot. The general procedure is similar to that used by Silver and Saunders (1950) and Northover (1958). The wave equation of the medium, the differential operator L, is given by

$$Le_z(r, k) = (\nabla_t^2 + \lambda_1^2)(\nabla_t^2 + \lambda_2^2)e_z(r, k) = 0 \qquad (4.1.1)$$

Here, $e_z(r, k)$ is the elementary wave function from which $E_z(r, z)$ is to be constructed, ∇_t^2 is the transverse Laplacian, and λ_1 and λ_2 are the transverse wave numbers obtained from the dispersion relation and are given by Equation 2.4.14.

The slot may be considered to be arbitrarily narrow with a voltage V_0 across it. The longitudinal electric field is assumed zero on the surface of the antenna except for the slot. The transverse electric field component E_φ is assumed to be zero on the surface of the antenna and also within the slot. Thus the boundary conditions on the antenna are

$$E_z(r = a, z) = V_0\delta(z) \qquad (4.1.2)$$

$$E_\varphi(r = a,\ z) = 0 \qquad (4.1.3)$$

$E_\varphi(r, z)$, as well as the other field components, H_z, H_φ, H_r and E_r are given in terms of $E_z(r, z)$ by Equations 2.5.9, 2.5.19, 2.5.20, 2.5.21, and 2.5.22. Both the total electric and magnetic fields satisfy the Wilcox radiation condition, or the finite energy condition, Equation 3.4.19.

$$\int_\Omega [h_{ij} E_i E_j^* + \mu_{ij} H_i H_j^*] dV < \infty \qquad (4.1.4)$$

In this equation, Ω is the entire three dimensional space except for the region occupied by the antenna.

According to the arguments presented in Section 3.5 the square roots of $\lambda_{1,2}^2$ may be chosen such that

$$\text{Im}\ \lambda_1 \leq 0,\ \text{Im}\ \lambda_2 \leq 0 \qquad (4.1.5)$$

Only two of the four eigenfunctions of Equation 4.1.1 satisfy Equation 4.1.4 so that the elementary wave function of Equation 3.2.5 becomes, after renaming the coefficients,

$$e_z(r, k) = AH_0^{(2)}(\lambda_1 r) + B\ H_0^{(2)}(\lambda_2 r) \qquad (4.1.6)$$

From Equation 2.5.19 with no φ variation one finds an expression for e_φ.

$$e_\varphi(r, k) = -A\gamma_2\lambda_1 H_1^{(2)}(\lambda_1 r) - B\gamma_1\lambda_2 H_1^{(2)}(\lambda_2 r) \qquad (4.1.7)$$

In these equations A, B, $\gamma_{1,2}$, and $\lambda_{1,2}$ are all functions of the longitudinal wave number k. The exact functional form of A and B is yet to be determined.

An arbitrary z dependence may now be built up by superposition of the elementary wave functions over various values of k. Regarding k as a complex quantity, different representations can be obtained according to the path in the complex k plane along which k is allowed to vary. In the present case where the field at r = a is to be fitted to a Fourier integral along the surface of the cylinder, the path along the real k axis is indicated.

That is, since

$$E_z(a, z) = V_0\delta(z) = \frac{V_0}{2\pi}\int_{-\infty}^{\infty} e^{-ikz}dk \qquad (4.1.8)$$

it is desirable to write for each field component integrals of the form

$$E_z(r, z) = \int_{-\infty}^{\infty} e_z(r, z, k)dk = \int_{-\infty}^{\infty} e_z(r, k)e^{-ikz}dk \qquad (4.1.9)$$

Since Equation 4.1.9 is just a direct Fourier transform, the inverse transform can be written

$$e_z(r, k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_z(r, z)e^{ikz}dz \qquad (4.1.10)$$

Therefore, on the antenna,

$$e_z(a, k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_0\delta(z)e^{ikz}dz = \frac{V_0}{2\pi} \qquad (4.1.11)$$

and similarly,

$$e_\varphi(a, k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (0)e^{ikz}dz = 0 \qquad (4.1.12)$$

Comparing the last two equations with Equations 4.1.6 and 4.1.7 evaluated at r = a yields the following pair of equations.

$$\begin{aligned} A(k)H_0^{(2)}(\lambda_1 a) + B(k)H_0^{(2)}(\lambda_2 a) &= \frac{V_0}{2\pi} \\ -A(k)\gamma_2\lambda_1 H_1^{(2)}(\lambda_1 a) - B(k)\gamma_1\lambda_2 H_1^{(2)}(\lambda_2 a) &= 0 \end{aligned} \tag{4.1.13}$$

Solving these equations for A(k) and B(k) yields

$$A(k) = \frac{\dfrac{V_0}{2\pi}\,\dfrac{\gamma_1}{\lambda_1 H_1^{(2)}(\lambda_1 a)}}{\left[\dfrac{\gamma_1 H_0^{(2)}(\lambda_1 a)}{\lambda_1 H_1^{(2)}(\lambda_1 a)} - \dfrac{\gamma_2 H_0^{(2)}(\lambda_2 a)}{\lambda_2 H_1^{(2)}(\lambda_2 a)}\right]} \tag{4.1.14}$$

$$B(k) = \frac{\dfrac{V_0}{2\pi}\,\dfrac{\gamma_2}{\lambda_2 H_1^{(2)}(\lambda_2 a)}}{\left[\dfrac{\gamma_1 H_0^{(2)}(\lambda_1 a)}{\lambda_1 H_1^{(2)}(\lambda_1 a)} - \dfrac{\gamma_2 H_0^{(2)}(\lambda_2 a)}{\lambda_2 H_1^{(2)}(\lambda_2 a)}\right]} \tag{4.1.15}$$

After factoring out and canceling a term common to both γ_1 and γ_2, which are given by Equation 2.5.15, the total longitudinal electric field of Equation 4.1.9 can be written as

$$E_z(r, z) = \frac{V_0}{2\pi}\int_{-\infty}^{\infty} \frac{\left[\dfrac{H_0^{(2)}(\lambda_1 r)}{H_1^{(2)}(\lambda_1 a)}\,\dfrac{[\lambda_1^2+f(k)]}{\lambda_1} - \dfrac{H_0^{(2)}(\lambda_2 r)}{H_1^{(2)}(\lambda_2 a)}\,\dfrac{[\lambda_2^2+f(k)]}{\lambda_2}\right] e^{-ikz}dk}{\left[\dfrac{H_0^{(2)}(\lambda_1 a)}{H_1^{(2)}(\lambda_1 a)}\,\dfrac{[\lambda_1^2+f(k)]}{\lambda_1} - \dfrac{H_0^{(2)}(\lambda_2 a)}{H_1^{(2)}(\lambda_2 a)}\,\dfrac{[\lambda_2^2+f(k)]}{\lambda_2}\right]} \tag{4.1.16}$$

where

$$f(k) = k^2 - k_0^2 + \frac{k_0^4 K_{12}^2}{-k^2 + k_0^2 K_{11}} \tag{4.1.17}$$

As a quick check on boundary conditions it may be noted that, when $r = a$, the integral in Equation 4.1.16 reduces to the integral in Equation 4.1.8, as it must.

It might be pointed out that the $E_z(r, z)$ found by evaluating Equation 4.1.16 is guaranteed to identically satisfy the original wave equation of the medium, Equation 2.2.27, but will not necessarily satisfy the finite energy condition. This may be shown by considering what is meant by the spectrum of L (Friedman, 1956). A number k is said to belong to the spectrum of the operator L if L(k) does not have a bounded inverse. If k belongs to the spectrum of L, there are the two following possibilities: either there exists a non-zero function u in the space over which L is defined such that $Lu = 0$, or no such function exists. In the first case, k is an eigenvalue in the discrete spectrum of L and u is an eigenfunction or is in the domain of L. In the second case, k is in the continuous spectrum of L, and u is an improper eigenfunction or is in the extended domain of L.

An example of a continuous spectrum can be obtained from the isotropic infinite antenna problem. Northover (1958) evaluates the isotropic equivalent of Equation 4.1.16 and finds that an asymptotic expression for the total longitudinal electric field for z large and r moderate is

$$E_z(r, z)_N = -\frac{4\pi}{z} e^{-ikz} \log\left(\frac{a}{r}\right) \frac{1}{\left[\log\left(\frac{z}{2ka^2}\right)\right]^2}$$

where $k = (\omega^2\mu\epsilon + i\mu\sigma\omega)^{\frac{1}{2}}$. The number k is in the continuous spectrum and there is no discrete spectrum. This function satisfies the boundary condition on the antenna because at r = a, $E_z(r, z)_N = 0$. However, $E_z(r, z)_N$ does not satisfy the scalar Helmholtz equation except in the limiting case as z goes to infinity. This is a result of the asymptotic techniques used in evaluating the integral. It could be shown by going back to the original integral that $E_z(r, z)_N$ does not satisfy the finite energy condition, for no losses. Therefore, $E_z(r, z)_N$ is an improper eigenfunction.

In going from the isotropic to the anisotropic case, one might therefore expect to also find a continuous spectrum. It is conceivable, of course, that the anisotropy might also introduce elements into the discrete spectrum. This point will be considered in Section 4.4. The exact nature of the continuous spectrum is one of the main difficulties in the theory of differential operators. It is difficult to tell if a particular differential equation plus boundary conditions will have a continuous spectrum or not, without actually solving for the domain of the operator. One clue to the nature of the spectrum is given by Friedman (1956) in that, at least for a rather broad class of problems, poles of the integral solution to this differential equation plus boundary conditions give rise to the discrete spectrum while branch line integrations yield the continuous spectrum.

The usual method of handling integrals such as Equation 4.1.16 is to deform the path of integration into the complex k plane and apply the Cauchy integral theorem. Poles in the integrand are evaluated by residues and the integrals along any branch cuts are evaluated by asymptotic techniques. This is a valid approach as long as the integrand is an analytic function of k over the entire half plane into which the path of integration is deformed except at a finite number of poles. One difficulty which appears in this integral expression for E_z and which does not appear in the corresponding isotropic problem (Northover, 1958), is that the integrand is not analytic everywhere. For some complex k, $\lambda_1 = \lambda_2$, and the integrand of Equation 4.1.16 assumes the indeterminate form 0/0. This exceptional case is the result of the eigenfunctions not spanning the space for this value of k.

Because the dispersion relation, Equation 2.4.12, is quadradic in k^2 and λ^2, there will be two pairs of values of k for which $\lambda_1 = \lambda_2$. These will be called $\pm k_1$ and $\pm k_3$, and their properties will be considered later. Thus the integrand of Equation 4.1.16 becomes indeterminate at two points in the complex half plane where it is desired to apply the Cauchy integral theorem. By the definition of analytic, that an analytic function is defined and differentiable at all points of some region, it is seen that the integrand is not analytic for $k = \pm k_1$ or $\pm k_3$. Therefore it is seen that the Cauchy integral theorem can not be applied around the half plane. It can be applied, of course, in a region excluding these undefined points but an integral around such a region will not necessarily give the desired value of $E_z(r, z)$.

At this point, then, it seems necessary to go to some other formulation which does not have this non-analytic problem. This particular situation has not arisen in any previously considered problem in the literature, to this author's knowledge. Therefore some discussion of the modification of the integral solution is in order. As mentioned earlier, the difficulty is caused by the eigenfunctions of rank one not spanning the space. Therefore the proper way around this difficulty is to use a set of eigenfunctions which do span the space. This requires the use of the generalized eigenfunctions discussed in Section 3.2. From Equations 3.2.6, 4.1.4 and 4.1.5 it may be argued that the proper elementary wave function satisfying Equation 4.1.1 is

$$e_z'(r, k) = A' H_0^{(2)}(\lambda_1 r) + B' r H_1^{(2)}(\lambda_2 r) \qquad (4.1.18)$$

This wave function is composed of a linear sum of one eigenfunction and one generalized eigenfunction. Such a sum spans the space, by which is meant that any function satisfying the differential equation and the associated boundary conditions may be constructed from such an elementary wave function. Now for $k \neq \pm k_{1,3}$, both wave functions, Equations 4.1.6 and 4.1.18, span the space. It seems reasonable that the solution for the total field $\underline{E}$ exterior to the antenna should be the same, no matter which wave function was used to construct it, with the possible exception of a contribution from a neighborhood of $\pm k_{1,3}$. This difference would be because one wave function spans the space there and the other does not.

Starting with Equation 4.1.18 and going through the same procedure as follows Equation 4.1.6 the following integral expression for E_z is found.

$$E_z(r, z) = \frac{V_0}{2\pi}\int_{-\infty}^{\infty} \frac{\left[\frac{H_0^{(2)}(\lambda_1 r)}{H_1^{(2)}(\lambda_1 a)}\,\frac{[\lambda_1^2+f(k)]}{\lambda_1} + \frac{r}{a}\,\frac{H_1^{(2)}(\lambda_2 r)}{H_0^{(2)}(\lambda_2 a)}\,\frac{[\lambda_2^2+f(k)]}{\lambda_2}\right] e^{-ikz}\,dk}{\left[\frac{H_0^{(2)}(\lambda_1 a)}{H_1^{(2)}(\lambda_1 a)}\,\frac{[\lambda_1^2+f(k)]}{\lambda_1} + \frac{H_1^{(2)}(\lambda_2 a)}{H_0^{(2)}(\lambda_2 a)}\,\frac{[\lambda_2^2+f(k)]}{\lambda_2}\right]}$$

(4.1.19)

It may be seen that when $r = a$ the bracketed terms in the integrand become equal so that this integral reduces to the integral in Equation 4.1.8. Thus the longitudinal E field takes on its prescribed values on the surface of the antenna. The tangential E_φ field will be forced to be zero at $r = a$ because of the relation which has been used, similar to the second of Equations 4.1.13. The wave function satisfies the Wilcox radiation condition so that the total field found by a summation of this wave function will also satisfy this condition. The wave function is a solution, in a generalized sense, of the wave equation, and an integral over this solution to find E_z is guaranteed to also be a solution. It may be noted that for $r \neq a$, the integrand of Equation 4.1.19 can not possibly be indeterminate.

This integral formulation for E_z thus satisfies the wave equation, the boundary conditions, and has the desired analytic properties. Unless the use of generalized eigenfunctions introduces some new requirement not prominently discussed in the literature, this is all that can be required of a solution to the infinite antenna problem or to any other problem. This would seem to be a fertile area of investigation, a mathematical description of when generalized eigenfunctions are necessary, and an orderly procedure of using them to solve the class of problems of which this infinite antenna problem seems to be a member. Such far ranging questions are beyond the scope of this thesis. Therefore, it will be considered that Equation 4.1.19 represents the desired integral solution for E_z, and the remainder of this chapter will be devoted to obtaining asymptotic approximations to the functional form of E_z.

4.2 The Riemann Surface in the Complex k Plane. As has been mentioned, the integrand of Equation 4.1.19 is analytic everywhere. However, it is not single valued in the k plane. This means that a multisheeted Riemann surface has to be defined such that the integrand will be analytic and single valued on each sheet. The original path of integration in Equation 4.1.19 is taken along the real k axis from $-\infty$ to $+\infty$. Cauchy's integral theorem can be applied to the region between this path and any other path starting and ending at the same points as long as both paths lie entirely on the same sheet. In order for the original integral to be convergent, a particular sheet must be chosen so that the integrand vanishes exponentially as $z \to \infty$. The exponential decay is handled by requiring Equation 3.5.1 to hold. For the sake of explicitness only the half space $z > 0$ will be considered so that $\text{Im } k < 0$ is always the proper inequality. This requirement, taken with Equation 4.1.5, will guarantee the convergence of the integral when the path of integration is deformed into the lower half k plane.

In common with the isotropic problem (Northover, 1958), branch points of the Riemann surface occur for those values of k, called k_I and k_{II}, for which $\lambda_1 = 0$ and $\lambda_2 = 0$. This is easily seen by considering the value of λ_1^2 as k describes a small circle about k_I in the k plane.

It will be shown that $\pm k_1$ and $\pm k_3$, yielding $\lambda_1^2 = \lambda_2^2$, are also branch points. To do this, suppose Equation 2.4.14 is written as

$$\begin{aligned} \lambda_1^2 &= C + \sqrt{D} \\ \lambda_2^2 &= C - \sqrt{D} \end{aligned} \qquad (4.2.1)$$

Since the radicand D is a fourth order polynomial in k, it has four zeros and can be written

$$D = D_1 \sqrt{(k - k_1)(k + k_1)(k - k_3)(k + k_3)} \qquad (4.2.2)$$

Letting k trace a small circle about $\pm k_1$ or $\pm k_3$ will produce the result

$$\sqrt{D} \rightarrow -\sqrt{D} \qquad (4.2.3)$$

This is equivalent to

$$\lambda_1^2 \rightarrow \lambda_2^2, \qquad \lambda_2^2 \rightarrow \lambda_1^2 \qquad (4.2.4)$$

It is evident that the integrand of Equation 4.1.19 assumes a new value when λ_1 and λ_2 are interchanged. This means that the small circle must have passed through a branch cut and arrived back at the same value of k on a different sheet. The original choice of which transverse wave number went with which eigenfunction was arbitrary but must remain fixed for the integrand of Equation 4.1.19 to be single valued.

Before these branch points are actually located in the k plane, their functional form should be known. Thus a small digression will be made here to examine $k_{1,3}$ and then $k_{I,II}$.

The transverse wave numbers being equal implies that the radical in Equation 2.4.13 is zero, or that

$$(a - c)^2 = -4bd \qquad (4.2.5)$$

Taking the square root gives

$$(a - c) = \pm 2\sqrt{-bd} = \pm 2\sqrt{-k^2 k_0^2 \frac{K_{12}^2}{K_{11}^2} K_{33}} \qquad (4.2.6)$$

or

$$(a - c) = \pm\ 2kk_0 \frac{K_{12}}{K_{11}} \sqrt{-K_{33}} \qquad (4.2.7)$$

From Equations 2.4.6, 2.4.8, and 4.2.7 there is obtained the following relationship.

$$k^2\left(1 - \frac{K_{33}}{K_{11}}\right) + k_0^2\left(K_{33} - K_{11} + \frac{K_{12}^2}{K_{11}}\right) = \pm\ 2kk_0 \frac{K_{12}}{K_{11}} \sqrt{-K_{33}} \quad (4.2.8)$$

Introducing the normalized plasma- and gyro-frequencies, Equation 2.1.12, the plasma parameters, Equations 2.1.9-11, and the effects of collisional damping, Equation 2.1.15, into the above equation and using the quadradic formula gives

$$k_{1,3} = \pm\frac{k_0}{|Y|}\left[\sqrt{Y^2 + X - 1 + Z^2 + iZ(2 - X)} \pm \sqrt{X - 1 + Z^2 + iZ(2 - X)}\right] \qquad (4.2.9)$$

Here, $+ k_{1,3}$ are chosen with negative imaginary parts, and $-k_{1,3}$ are chosen with positive imaginary parts. It may be seen from this equation that for $Z = 0$, k_1 and k_3 are real for $X > 1$, complex conjugate for $X < 1$ and $X + Y^2 > 1$, and purely imaginary for $X + Y^2 < 1$. This sign information is shown in Figure 3. For the sake of explicitness, Z will be considered to be small so that the inclusion of non-zero losses will only slightly perturb k_1 and k_3. That is, for $Z \neq 0$ and $X > 1$, k_1 and k_3 will have a relatively large real part and a relatively small imaginary part.

It will be shown later in the chapter that the total longitudinal field $E_z(r, z)$ contains terms with factors such as $\exp(-ik_1z)$ multiplying other functions of z and r. If $k_{1,3}$ are purely imaginary (for $Z = 0$), the $\exp(-ik_{1,3}z)$ variation causes $E_z(r, z)$ to be cutoff or evanescent in the z direction. Such a field would not be expected

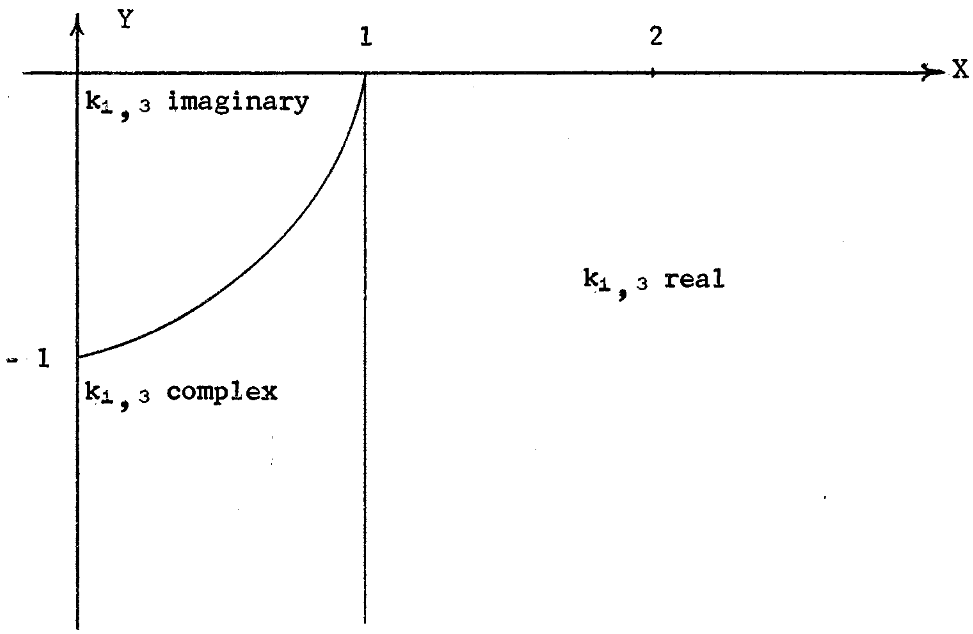

Figure 3. Signs of the Wave Numbers $k_{1,3}$ on the XY Plane for no Losses.

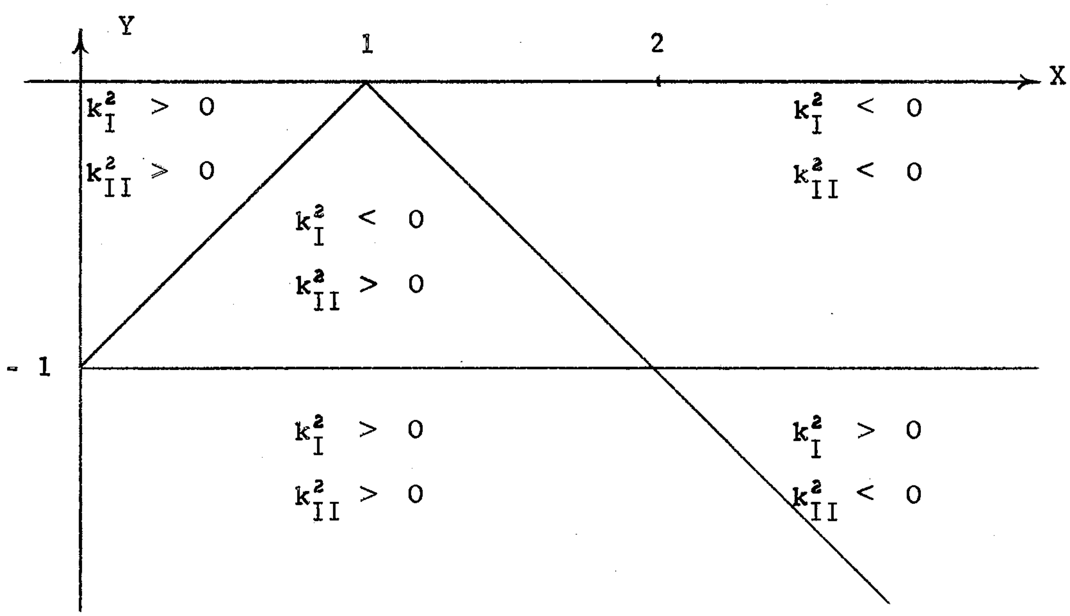

Figure 4. Signs of the Wave Numbers $k^2_{I,II}$ on the XY Plane for no Losses.

to induce a significant amount of current in the antenna some appreciable distance from the source. Therefore this part of $E_z(r, z)$ might be neglected if this is the only desired result. If $k_{1,3}$ are complex for Z = 0, then the associated field component carries no real power (Allis, 1963). A more complete discussion along this line will appear after $E_z(r, z)$ has actually been evaluated.

The other set of wave numbers which form branch points are found from the zeros of $\lambda_{1,2}^2$. In Equation 2.4.12, if (ac - bd) = 0, then one of the roots of the equation is $\lambda^2 = 0$. Setting (ac - bd) = 0 gives the following equation in k.

$$k^4 - 2K_{11}k_0^2k^2 + k_0^4(K_{11}^2 - K_{12}^2) = 0 \qquad (4.2.10)$$

Solving this equation for k^2 gives

$$k_{I,II}^2 = (K_{11} \pm K_{12})k_0^2 \qquad (4.2.11)$$

This is the Appleton-Hartree equation with $\theta = 0$. It specifies the wave numbers of the medium's two possible plane waves when these waves are propagating parallel to the static magnetic field. These wave numbers can be expressed in terms of X, Y and Z as follows.

$$k_{I,II}^2 = k_0^2\left[1 - \frac{X(1 - Y^2 + Z^2)(1 \pm Y)}{(1 - Y^2 - Z^2)^2 + 4Z^2} - \frac{iXZ(1 + Y^2 + Z^2 \pm 2Y)}{(1 - Y^2 - Z^2)^2 + 4Z^2}\right] \qquad (4.2.12)$$

For Z = 0, this reduces to the simpler expression,

$$k_{I,II}^2 = k_0^2\left[\frac{1 - X - Y^2 \pm XY}{1 - Y^2}\right]_{Z=0} \qquad (4.2.13)$$

It may be noted that $k_{I,II}$ are either real or pure imaginary for no losses. There is no possibility of them being complex, as there is for $k_{1,3}$ for certain ranges of the plasma parameters.

The sign information in Equation 4.2.13 is plotted in Figure 4.

The branch points $\pm k_I$, $\pm k_{II}$, $\pm k_1$, $\pm k_3$ are shown in Figure 5 for small losses and approximately for the plasma parameters bounded by $X = 1$ and $1 - X - Y^2 - XY = 0$. Within this region the wave numbers of Equations 4.2.9 and 4.2.12 are nearly real for small Z. As the plasma- and gyro-frequencies vary, these wave numbers will assume different values, of course. This does not affect the evaluation of the integral since the information about a wave number being nearly real or nearly imaginary is used only after the asymptotic formulas are obtained.

Figure 5 also shows the branch cuts which have been chosen to separate the sheets of a Riemann surface. As far as the single valuedness requirement is concerned, the locations of these branch cuts are almost completely arbitrary. They can lie along curves between k_1 and k_3, and between k_I and k_{II}, or they can pass from these branch points to the point at infinity along rather arbitrary curves. The locations of these cuts are more limited by the desire to evaluate the integral in Equation 4.1.19 without undue difficulty. Since the original integration is over the real axis, no branch cuts should cross the axis. If a cut did cross the real k axis, the integrand would not be single valued along its original path of integration. Since each pair of branch points is symmetrical about the origin, this means that all the branch cuts must extend from their branch points to the point at infinity. The exact nature of these

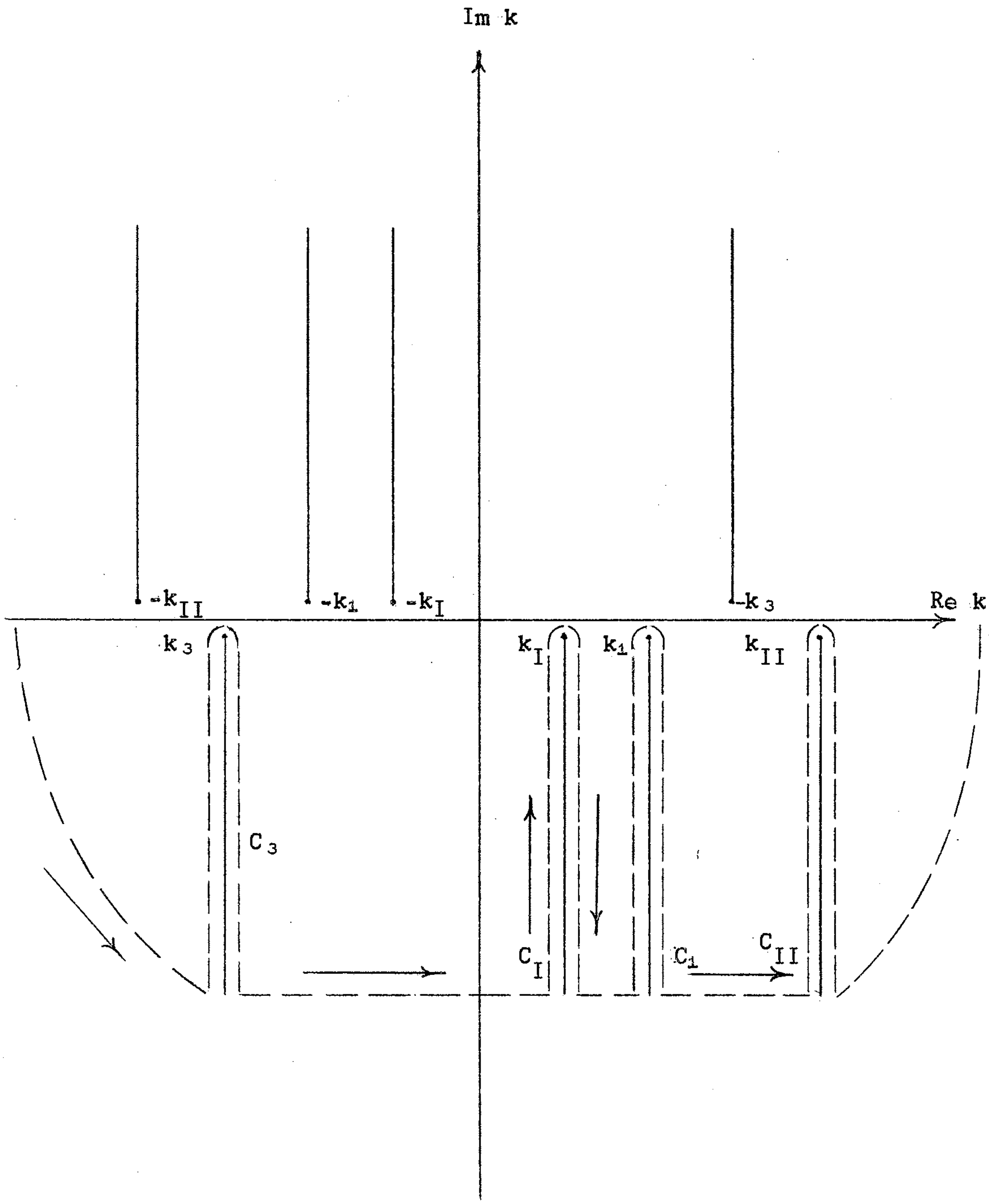

Figure 5. The k Plane with Branch Cuts and the Deformed Path of Integration.

cuts might be determined by requiring that the integral in Equation 4.1.19 converge for all values of k on the original path of integration and on the corresponding sheet of the Riemann surface. It may be assumed that k has an arbitrarily small negative imaginary part along the original path of integration, in which case Im $\lambda_1 < 0$, Im $\lambda_2 < 0$, and convergence is guaranteed.

A rather detailed study of branch cuts and integrations on a four sheeted Riemann surface has been done by Banos and Wesley (Part I, 1953; Part II, 1954). In Part I they require, in the notation of this section, that Im $\lambda_1 < 0$ and Im $\lambda < 0$ at every point in the half plane into which the path of integration is deformed. This gave them a parametric curve for each branch cut. It turned out that the integral was difficult to evaluate along these cuts and that the solutions did not possess certain desired analytic properties, or at least such properties could not be proved. Therefore they reexamined the problem to see if other branch cuts could be used. In Part II they point out that as long as r and z are finite the integral will converge for more arbitrary branch cuts, including the ones shown in Figure 5 as half-lines parallel to the imaginary axis. They made such a choice of cut and this made evaluation of their integral much easier.

Of course, the integral solved by Banos and Wesley (1954) is not the same as Equation 4.1.19, but the argument for the choice of branch cuts in each case is identical. That is, the cuts are specified by the conditions

$$\mathrm{Re}(k) = \mathrm{Re}(\pm k_1, \pm k_3, \pm k_I, \pm k_{II}) \qquad (4.2.14)$$

The original path of integration may now be deformed as shown in Figure 5. Starting on the real axis at $k \to -\infty$, the proposed path follows, first, the semi-circle at infinity in the third quadrant, next the contour C_3 completely around the lower branch cut for k_3, then the contours C_I, C_1, and C_{II} and, finally the semi-circle at infinity in the fourth quadrant terminating on the real axis at $k \to +\infty$. By Cauchy's theorem, the proposed path is completely equivalent to the original path along the real axis except for the residues of any pole type singularities lying between the two paths. Furthermore, it can be readily shown (Banos and Wesley, 1954) that the contribution over the semi-circle at infinity in the lower half-plane vanishes, with the result that the original integral can be expressed as a sum of several branch cut integrals plus the sum of residues at any poles present.

$$E_z(r, z) = I_1 + I_3 + I_I + I_{II} + 2\pi i \sum \text{Residues} \qquad (4.2.15)$$

Determining the location of any poles present and evaluating the residues at these poles can be done independently of the branch cut calculations, and therefore will be considered later.

All of the branch points are determined by the behavior of $\lambda_{1,2}$, which come from a fourth order dispersion relation. This implies that the Riemann surface consists of four sheets, which may be arbitrarily numbered as shown in Table I.

The interconnection diagram for the four sheets of the Riemann surface is shown in Figure 6. It may be seen from Equation 4.2.4 that when the cuts associated with k_1 or k_3 are crossed, both λ_1

TABLE I

ARGUMENTS OF λ_1 AND λ_2 ON THE VARIOUS SHEETS

OF THE RIEMANN SURFACE

Sheet	Im λ_1	Im λ_2
1	-	-
2	+	-
3	-	+
4	+	+

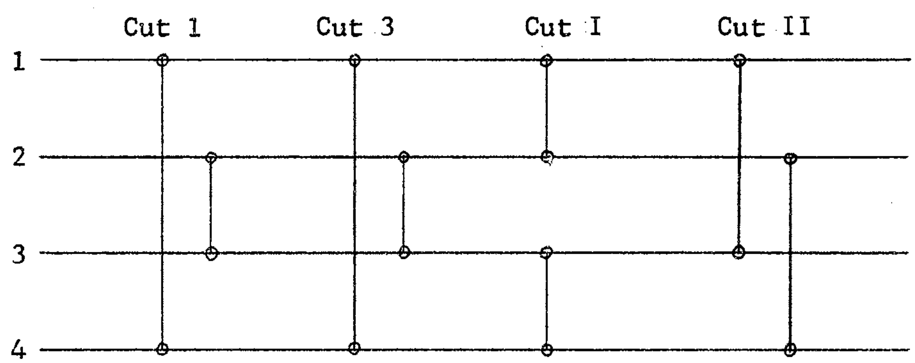

Figure 6. The Riemann Surface of Four Sheets in the k Plane.

and λ_2 change sign. Therefore, one can pass from sheet 1 to sheet 4 and back as well as from sheet 2 to sheet 3 since in these transfers the sign of the imaginary parts of both λ_1 and λ_2 change sign. However, in going across the cut associated with k_I only λ_1 changes sign, so one can move between sheets 1 and 2 and between sheets 3 and 4. A similar situation holds for k_{II}.

4.3 Evaluation of the Branch Line Integrals. The procedure outlined in the preceding section for splitting the original integral for $E_z(r, z)$ into the sum of several integrals along the branch cuts plus possibly some residues does not yield integrals which can be evaluated exactly in closed form. It does yield integrals for which asymptotic approximations exist in certain regions of space; for example, near the antenna but far from the source. Obtaining these asymptotic forms for various regions of space and for varying system parameters such as plasma- and gyro-frequencies is a long and tedious process, so the following presentation will not be complete. It would seem that a computer solution would be the only way to obtain a complete solution for all possible parameters. However, integrals such as Equation 4.1.19 do not lend themselves to computer solutions, and besides, it is not clear that such a computer solution would be as useful as the asymptotic forms. This is because the asymptotic expressions clearly show the most important characteristics of the solution in an analytic form, characteristics which may be obscured in a group of curves.

Only one branch line integral will be evaluated in detail, that associated with k_I, and the other expressions will be inferred from this one. The technique to be used is basically the same as used by Northover (1958). The integral of Equation 4.1.19 is to be evaluated along both parts of the contour C_I in the directions indicated. Using the change of variable,

$$k = k_I - \frac{i\beta}{z} \tag{4.3.1}$$

the integral I_I can be written as

$$I_I = E_{zI}(r, z) = -\frac{iV_0}{2\pi z} e^{-ik_I z}\left[\int_\infty^0 \frac{N_L}{D_L} e^{-\beta} d\beta + \int_0^\infty \frac{N_R}{D_R} e^{-\beta} d\beta\right]$$

$$= -\frac{iV_0}{2\pi z} e^{-ik_I z}\left[\int_0^\infty \frac{N_R}{D_R} - \frac{N_L}{D_L} e^{-\beta} d\beta\right] \tag{4.3.2}$$

where

$$N_L = \frac{H_0^{(2)}(\lambda_{1L} r)}{H_1^{(2)}(\lambda_{1L} a)} \frac{[\lambda_{1L}^2 + f(k)]}{\lambda_{1L}} + \frac{r}{a} \frac{H_1^{(2)}(\lambda_2 r)}{H_0^{(2)}(\lambda_2 a)} \frac{[\lambda_2^2 + f(k)]}{\lambda_2} \tag{4.3.3}$$

$$D_L = \frac{H_0^{(2)}(\lambda_{1L} a)}{H_1^{(1)}(\lambda_{1L} a)} \frac{[\lambda_{1L}^2 + f(k)]}{\lambda_{1L}} + \frac{H_1^{(2)}(\lambda_2 a)}{H_0^{(2)}(\lambda_2 a)} \frac{[\lambda_2^2 + f(k)]}{\lambda_2} \tag{4.3.4}$$

These are simply the bracketed portions of the numerator and denominator of Equation 4.1.19 evaluated on the left-hand side of C_I. Similar expressions hold for the right-hand side. Since k_I is not a branch point for λ_2, λ_2 does not change value in crossing the cut and so does not need a distinctive subscript to indicate which side of the branch line is being considered.

The asymptotic evaluation of Equation 4.3.2 is a rather tedious process and is carried out in some detail in Appendix B. An outline of the procedure and the results will be given here.

It is desired to find the fields close to the antenna, so that the current distribution can be computed from the relationship

$$I(z) = 2\pi a[H_\varphi(r, z)]_{r = a} \tag{4.3.5}$$

This current distribution can then theoretically be used to find the input impedance of the infinite antenna as well as a starting point for estimating the current distribution on a finite dipole.

The assumption has already been made that there is no azimuthal variation of E_z, which is a reasonable assumption only for antennas with radii small compared to a wavelength. This asymptotic evaluation will thus be made for the antenna radius a arbitrarily small. This assures that $D_L \doteq D_R$, which eases the evaluation of Equation 4.3.2 considerably.

The integration of I_I is split into two ranges, $0 \leq \beta < K$ and $K \leq \beta \leq \infty$ where K has to be taken large enough to make

$$\left| \int_K^\infty \right| \ll \left| \int_0^K \right| \tag{4.3.6}$$

Over the first range, the range of primary contribution to the integral, it is further assumed that z is large enough to make $\beta/z \ll 1$. Using these assumptions, it is shown in Appendix B that I_I is approximately given as

$$I_I \doteq - \frac{V_0 a^2}{2z} e^{-ik_I z} \log\left(\frac{\gamma\lambda_2' a}{2}\right) J_0(\lambda_1' r)\left[F_1(k_I) + \frac{ik_I}{z} F_2(k_I)\right] \tag{4.3.7}$$

The primed transverse wave numbers are "average" values chosen to represent the actual wave numbers over the entire path of integration, C_I. The values used by Northover (1958) and which are also used here are

$$\lambda'_{1,2}(k_I) = \lambda_{1,2}(k = k_I - \frac{i}{z}) \qquad (4.3.8)$$

That is, the arguments of the log and Bessel functions, both slowly varying functions for small arguments, are evaluated for $\beta = 1$ and then factored out of the integration as constants.

The quantities F_1 and F_2 are rather involved functions of the plasma parameters X, Y, and Z, whose functional form depends on the branch point being considered, hence $F_{1,2}(k_I)$. They can be determined by working backwards through Equations B.47, B.40, B.37, B.35, B.34, B.31, and B.30.

Because of the symmetry between k_I and k_{II} it may be argued that the integral I_{II} will have the same form as I_I, with simply an interchange of k_I and k_{II}, and λ_1 and λ_2. Therefore,

$$I_{II} \doteq - \frac{V_0 a^2}{2z} e^{-ik_{II}z} \log(\frac{\gamma\lambda'_1 a}{2}) J_0(\lambda'_2 r)[F_1(k_{II}) + \frac{ik_{II}}{z} F_2(k_{II})] \qquad (4.3.9)$$

In this equation, $\lambda'_{1,2}$ are given by Equation 4.3.8 except that k_I is replaced by k_{II}.

Appendix B also contains a solution for the line integral around the branch point k_1. From Equation B.55 the integral I_1 is approximately given by

$$I_1 \doteq -\frac{V_0}{2z} e^{-ik_1 z} [i\lambda_2' r J_1(\lambda_2' r) +$$

$$+ a^2 \log(\frac{\gamma\lambda_2' a}{2}) J_0(\lambda_1' r)[F_1(k_1) + \frac{ik_1}{z} F_2(k_1)]] \qquad (4.3.10)$$

The integral involving the contour C_3 around k_3 will be identical to the above equation except for the interchange of k_1 and k_3.

$$I_3 \doteq -\frac{V_0}{2z} e^{-ik_3 z} [i\lambda_2' r J_1(\lambda_2' r) +$$

$$+ a^2 \log(\frac{\gamma\lambda_2' a}{2}) J_0(\lambda_1' r)[F_1(k_3) + \frac{ik_3}{z} F_2(k_3)]] \qquad (4.3.11)$$

The total integral around all the branch cuts is now found by summing I_I, I_{II}, I_1, and I_3. The residue portion of the original integral, Equation 4.1.19, will be considered in the next section.

Now that the asymptotic solutions to the various branch line integrals have been obtained, these solutions should be checked in the differential equation, Equation 4.1.1. Because of the nature of asymptotics, it would not be expected that every solution would exactly satisfy the differential equation, but all asymptotic solutions should at least approximately satisfy it. Checking first Equation 4.3.7, it is seen that the only radial variation is $J_0(\lambda_1' r)$. The transverse wave numbers $\lambda_{1,2}$ in the differential equation have to be evaluated for the integration, or according to Equation 4.3.8. The differential equation operating on I_I is thus

$$LI_I = M_I(\nabla_t^2 + \lambda_1'^2)(\nabla_t^2 + \lambda_2'^2) J_0(\lambda_1' r) \qquad (4.3.12)$$

M_I contains all the factors of Equation 4.3.7 which do not vary with r. It happens for this solution that, after commuting operators, the term $(\nabla_t^2 + \lambda_1'^2)\, J_0(\lambda_1' r)$ is identically zero, in which case Equation 4.1.1 is identically satisfied. A quick examination will show that the same argument holds for I_{II} in Equation 4.3.9.

Checking now the solutions I_1 and I_3, it is seen that they contain a $J_0(\lambda_1' r)$ term which satisfies the differential equation exactly, and a $rJ_1(\lambda_2' r)$ term. The second order operator $(\nabla_t^2 + \lambda_2'^2)$ operating on $rJ_1(\lambda_2' r)$ does not give a zero result. Therefore, a closer look must be taken at this proposed solution. Note that the transverse wave numbers are evaluated according to

$$\lambda_{1,2}'(k_1) = \lambda_{1,2}(k = k_1 - \frac{i}{z}) \qquad (4.3.13)$$

and also with k_1 replaced by k_3. The solution I_1 is asymptotic as z gets very large. In this asymptotic limit, the above expression becomes

$$\lambda_{1,2}'(k_1) \doteq \lambda_{1,2}(k = k_1) \qquad (4.3.14)$$

But, by definition,

$$\lambda_1(k_1) = \lambda_2(k_1) \qquad (4.3.15)$$

Now Equation 4.1.1 can be written as

$$LI_1 \doteq M_1(\nabla_t^2 + \lambda_2'^2)(\nabla_t^2 + \lambda_2'^2) rJ_1(\lambda_2' r) = 0 \qquad (4.3.16)$$

It is not hard to show that the operator $(\nabla_t^2 + \lambda_2'^2)$ operating twice on $rJ_1(\lambda_2' r)$ does give a zero result. This implies that the portion of the solution involving $rJ_1(\lambda_2'r)$ is truly asymptotic in the sense that it satisfies the differential equation to a better and better approximation as the distance from the source is increased more and more.

4.4 The Discrete Spectrum. The branch line integrals of the previous section constitute the continuous spectrum of the operator L given by Equations 4.1.1-4. The discrete spectrum of L is found by evaluating Equation 4.1.19 at the poles of its integrand. In the context of this type of problem, the elements of the continuous spectrum are called space waves and those of the discrete spectrum are called surface waves (Friedman, 1956). Generally speaking, surface waves are tied to a surface in some fashion while space waves are not. Historically, the space wave portion of the solutions to many boundary problems is the portion which has been experimentally verified and about which there exists little disagreement. On the other hand, the nature of and even the existance of so called surface waves have been topics on which disagreement has existed for at least the past sixty five years. Stratton (1941) discusses some of the points in question. What amounts to a debate on the meaning of the term "surface wave" between such people as Wait, Barlow, Cullen, and Zucker appears in Silver (1963). The debate ended without agreement being reached. With this kind of background, a rather cautious approach is taken toward the subject of surface waves in this chapter.

The definition proposed by Zucker (Silver, 1963) for surface waves is as follows: A surface wave is a source-free solution of Maxwell's equations over an interface. It satisfies the radiation condition at infinity and boundary conditions at the interface.

When the source is removed from the infinite antenna, it becomes an infinite cylinder. This would suggest that, to obtain surface waves, an alternate procedure to solving the integral expression by residues would simply be to solve for the fields exterior to a cylinder. The solution should be the same, at least for large distances from the slot on the antenna.

The people who investigated the infinite antenna in an isotropic medium (Northover, 1958, etc.) found only a space wave contribution. According to Sommerfeld (1947) and Goubau (1950), a surface wave will exist on a long cylinder only if the surface of the cylinder is modified or if the cylinder has a finite conductivity. They argue that a surface wave cannot exist over a perfectly conducting cylinder because such a wave, in the language of Chapter III, would be an improper eigenfunction. As discussed in Chapter III, this is not always a valid reason for discarding such a wave. This is a point of academic interest only, of course, because such a model could never be built and tested. However, it appears to this author that the reason for surface waves not appearing in the isotropic infinite antenna problem is more probably that a circumferential slot does not launch surface waves than that such a wave could not exist even if launched.

Moving now to the anisotropic case, at least two papers (Johnson and Cook, 1965; Mushiake, 1965) have considered the infinite cylinder problem. The Johnson and Cook paper suggested that the longitudinal wave numbers k_1 and k_3 were associated with surface waves. This analysis was questioned by Seshadri (1965), and the alternate infinite antenna approach performed earlier in this chapter shows that k_1 and k_3 are actually associated with space waves. The reason this misunderstanding arose can be seen from Equation 4.1.16. A necessary condition for $k_{1,3}$ to belong to surface wave structures is that the denominator of the integrand be zero for these values of k. This is not a sufficient condition because the numerator of the integrand may be zero at the same points as the denominator, in which case there is no pole. This is what happens for k_1 and k_3.

Another wave number, $k_2 = k_0\sqrt{K_{11}}$, has been suggested by both Johnson and Cook (1965) and Mushiake (1965) to belong to a surface wave. This same wave number was used by Ament (1964) in his impedance calculations for a dipole antenna, although he arrived at it in a somewhat different fashion. This wave number has the advantage that it reduces to k_0 as the plasma becomes free space. Of course, k_I and k_{II} have the same advantage. It is not obvious from Equation 4.2.9 that either k_1 or k_3 has such a property, but this is not important because the analysis from which they come is strictly valid only for X, $Y \not\equiv 0$ anyway.

Unfortunately, both papers made the same error mentioned earlier in assuming that a necessary condition was also a sufficient one. To show this consider that either the integral representation of Equation 4.1.16 or Equation 4.1.19 is valid away from the branch cuts. If k_2 is a pole then both integrands should show this, and likewise if k_2 is not a pole. It would not be expected, of course, that both integrands would have the same value at nonsingular points. From the infinite cylinder formulation it appears that k_2 may be a pole asymptotically as the radius goes to zero, so this will be the limiting case to be considered.

The integrand of Equation 4.1.19 will now be examined to see if $k = k_2$ is a pole, or if the integrand is unbounded for $k = k_2$. It may be noted from Equation 4.1.17 that k_2 is a pole of $f(k)$. For k near k_2, $f(k) \gg \lambda_1^2$ or λ_2^2, and since $f(k)$ is then a common factor, it can be divided out. The integrand is then

$$\frac{N}{D} = \frac{\dfrac{H_0^{(2)}(\lambda_1 r)}{\lambda_1 H_1^{(2)}(\lambda_1 a)} + \dfrac{r}{a}\dfrac{H_1^{(2)}(\lambda_2 r)}{\lambda_2 H_0^{(2)}(\lambda_2 a)}}{\dfrac{H_0^{(2)}(\lambda_1 a)}{\lambda_1 H_1^{(2)}(\lambda_1 a)} + \dfrac{H_1^{(2)}(\lambda_2 a)}{\lambda_2 H_0^{(2)}(\lambda_2 a)}} \tag{4.4.1}$$

$$\xrightarrow[a \to 0]{} \quad \frac{r}{a}\frac{H_1^{(2)}(\lambda_2 r)}{H_1^{(2)}(\lambda_2 a)} \doteq \frac{r H_1^{(2)}(\lambda_2 r)}{(2i/\pi\lambda_2)} \tag{4.4.2}$$

This equation has been obtained by using Equations B.13 and B.14 and neglecting the terms which go to zero as the radius a goes to zero. It is evident that Equation 4.4.2 is not unbounded even for the radius identically zero. This would imply that k_2 is not a pole of the integrand and hence there is no guided wave with this longitudinal wave number.

Making the same small argument approximations, the integrand of Equation 4.1.16 becomes

$$\frac{N}{D} \doteq \frac{(\pi a/2i)H_0^{(2)}(\lambda_1 r) - (\pi a/2i)H_0^{(2)}(\lambda_2 r)}{-a[\log\frac{\gamma\lambda_1 a}{2} + \frac{i\pi}{2}] + a[\log\frac{\gamma\lambda_2 a}{2} + \frac{i\pi}{2}]}$$

$$= \frac{(\pi a/2i)[H_0^{(2)}(\lambda_1 r) - H_0^{(2)}(\lambda_2 r)]}{a\log(\frac{\lambda_2}{\lambda_1})} \tag{4.4.3}$$

The denominator goes to zero with the radius a, but the numerator does too, with the result that the integrand is not unbounded at $k = k_2$. Thus from both integral representations it is seen that k_2 is not a part of the spectrum.

The next question to consider is the possibility of other values of k being members of the discrete spectrum. This is a difficult question to answer because of the transcendental nature of the functions involved. However, after a considerable amount of analytic and computer work, it appears to this author that there are no real values of k in the discrete spectrum for a lossless medium. This implies that any members k of the descrete spectrum of L are either complex or purely imaginary. Allis (1963) shows that waves with a complex propagation constant in a lossless medium carry no real power. It is hard to imagine much use for this type of wave. The purely imaginary longitudinal wave numbers would correspond to waves cutoff or evanescent in the z direction. This type of wave is useful in matching boundary conditions at some discontinuity, but would not contribute to the current distribution some distance from the source.

The conclusion is that the discrete spectrum of L is null in the sense that there are no surface waves propagating along the cylinder with little or no attenuation in the direction of travel.

Therefore, from Equation 4.2.15 it is seen that the total electric field is written as the sum of the four integrals I_I, I_{II}, I_1, and I_3. The main characteristics of $E_z(r, z)$ are the exponential and the 1/z factors. This is more clearly illustrated by writing Equation 4.2.15 in the abbreviated form

$$E_z(r, z) = - \frac{V_0}{2z} [G_I \ e^{-ik_I z} + G_{II} \ e^{-ik_{II} z} + G_1 \ e^{-ik_1 z} + G_3 \ e^{-ik_3 z}] \quad (4.4.4)$$

where the G terms are the functions of r, z, X, Y, and Z given in Equations 4.3.7, 4.3.9, 4.3.10, and 4.3.11. The current distribution will be obtained from this expression in the next chapter.

CHAPTER V

SUMMARY AND RECOMMENDATIONS

5.1 Summary. Chapter II starts with basic concepts of plasma physics and contains the derivation of the wave equation for an anisotropic plasma. A short discussion is given on why Green's functions were not used to solve this wave equation. The wave equation is then put into a special form suitable for solving by Fourier transform techniques. Most of this material is in the literature, and this chapter forms the basis for the more original work in the following chapters.

Chapter III forms an important part of this thesis by setting forth a rigorous discussion of the boundary condition at infinity. Most of these concepts have been presented elsewhere by various people, but this chapter represents the first relatively complete discussion of radiation conditions in both isotropic and anisotropic media. In reading a good deal of literature relating to the solution of Maxwell's equations in an exterior region, it appears that most people do not bother to state the particular form of the radiation condition they are using, nor do they explain how it applies to their particular problem. This could easily lead to misunderstandings by the uninitiated. One interesting example of this is in the problem of propagation along an infinite circular cylinder in an isotropic medium (Stratton, 1941; Goubau, 1950). Both Stratton and Goubau make the same choice of square root, which requires them to use Hankel functions of the first kind, $H_n^{(1)}(\lambda r)$, to ensure the "proper behavior" of the fields at infinity. However, Stratton uses an $\exp(-i\omega t)$ time convention while Goubau uses an $\exp(i\omega t)$ convention. This means that Stratton has waves traveling away from the cylinder while Goubau has waves going toward the cylinder. This difference does not automatically imply that one of the developments is wrong, because both men have satisfied the Sommerfeld radiation condition. The source is located at $z = -\infty$, so that "outward phase travel" means only that the phase travel is in the + z direction. The Sommerfeld radiation condition has nothing to say about a radial component of phase travel either toward or away from the cylinder. Goubau is apparently using the Poynting vector argument that power should flow inward to supply the losses in the cylinder, and therefore appears to be on firmer ground philosophically than is Stratton. The results are formally the same, of course, because the time variation does not appear in their determinantal equations. It is unfortunate that neither researcher explained exactly what condition at infinity should be required, and how their particular solution satisfied this condition.

It was pointed out in Chapter III that the usual statement of Sommerfeld's radiation condition involving boundedness and outward traveling phase is not necessarily adequate in anisotropic media. Reasons for this as well as statements by several authors to this effect were cited. A finite energy condition, named the Wilcox radiation condition after the man who suggested it, was shown to be a suitable statement of the boundary condition at infinity for both isotropic and anisotropic media. A rule for extending this Wilcox radiation condition to lossless media was adapted from the concepts contained in Friedman (1956).

The work of Chow (1962) might be mentioned here to show that complete agreement does not yet exist on these points. Chow claimed to have justified the validity of Sommerfeld's radiation condition in anisotropic media. Unfortunately, he does not state anywhere just what he means by the Sommerfeld radiation condition, so that one cannot say he is wrong by requiring outward phase travel. The point is that one needs to be careful in stating the radiation condition used in a particular situation so that there can be no confusion on the part of the reader.

Chapter IV contains a formulation of the integral solution to the infinite antenna problem, as well as an asymptotic evaluation of this integral. Both the formulation and the evaluation are important contributions to the theory of wave propagation in plasmas. The most important part of the formulation is the recognition of the fact that generalized eigenfunctions must be used to span the space of the differential operator. There are several ways to argue this point, but perhaps the simplest is the following. In general, a nth-order differential equation without boundary conditions has to have n independent solutions before a solution to the differential equation with boundary conditions can be found. These boundary conditions have to give exactly n relations between the n independent solutions for the solution to the differential equation with boundary conditions to be unique. For $k \neq \pm k_{1,3}$, the four expressions in Equations 3.2.1 are four independent solutions to $Le_z = 0$. However, for $k = \pm k_{1,3}$, $\lambda_1 = \lambda_2$, and these four expressions reduce to only two independent solutions of $Le_z = 0$.

This means that two more independent solutions must be found, and these turn out to be the generalized eigenfunctions.

It should be mentioned that the term "solution" has been used to mean two different things in the preceding paragraph, as has the word "eigenfunction" earlier. The solution (eigenfunctions) of the differential equation only is obviously not the same as the solution (eigenfunctions) of the differential equation plus boundary conditions. It should be clear from the context in each case which meaning is intended.

The asymptotic expression for the longitudinal electric field close to the antenna but far from the source was given in Equation 4.4.4, and is repeated here for convenience.

$$E_z(r, z) = -\frac{V_0}{2z}\left[G_I e^{-ik_I z} + G_{II} e^{-ik_{II} z} + G_1 e^{-ik_1 z} + G_3 e^{-ik_3 z}\right] \tag{5.1.1}$$

This field has a continuous spectrum only or is composed entirely of space waves. The most interesting part of this result is that every term of the total longitudinal electric field has a basic variation of 1/z times an exp(-ikz) factor. The G functions also contain a z variation, an example of which is G_I of Equation 4.3.7.

$$G_I = a^2 \log\left(\frac{\gamma\lambda_2' a}{2}\right) J_0(\lambda_1' r)\left[F_I(k_I) + \frac{ik_I}{z} F_2(k_I)\right] \tag{5.1.2}$$

Here, $\lambda_{1,2}'(k_I)$ are functions of z, Equation 4.3.8, while $F_{1,2}(k_I)$ are not functions of z or r.

The next step in finding the current distribution is to find $H_\varphi(r, z)$, which from Equation 2.5.21 for no φ variation is

$$H_\varphi(r, z) = \delta_2 \frac{\partial E_{z1}}{\partial r} + \delta_1 \frac{\partial E_{z2}}{\partial r} \tag{5.1.3}$$

The k_I portion of $E_z(r, z)$ has a single factor involving r; $J_0(\lambda_1' r)$. The λ_1' in the argument of the Bessel function means that this k_I portion is part of $E_{z1}(r, z)$. Now the k_I portion of $H_\varphi(r, z)$ can be found as

$$H_{\varphi I}(r, z) = (\delta_2)_I \frac{\partial (E_{z1})_I}{\partial r}$$

$$= \left[\frac{-i\omega\, \epsilon_0 K_{11} \lambda_2'^2}{(k^2 - k_0^2 K_{11})^2 - k_0^4 K_{12}^2}\right]\left[-\frac{V_0}{2z}\right] e^{-ik_I z}\, a^2 \log\left(\frac{\gamma \lambda_2' a}{2}\right)[-\lambda_1' J_1(\lambda_1' r)] \cdot$$

$$[F_1(k_I) + \frac{ik_I}{z} F_2(k_I)] \qquad (5.1.4)$$

It may be noted that $H_\varphi(r, z)$ has approximately the same z variation as $E_z(r, z)$ for large z. The only difference is that $H_{\varphi I}(r, z)$ has a $\lambda_2'^2 \lambda_1' J_1(\lambda_1' r)$ factor while $E_{zI}(r, z)$ has a $J_0(\lambda_1' r)$ term, and the z variation of both functions becomes negligible for large z.

Performing the appropriate differentiation for each part of $E_z(r, z)$ to obtain $H_\varphi(r, z)$, the current distribution I(z) can be written from Equation 4.3.5 as

$$I(z) = 2\pi a\, [H_\varphi(r, z)]_{r = a}$$

$$= 2\pi a\left[-\frac{V_0}{2z}\right][G_I' e^{-ik_I z} + G_{II}' e^{-ik_{II} z} + G_1' e^{-ik_1 z} + G_3' e^{-ik_3 z}]$$

$$(5.1.5)$$

where, from Equation 5.1.4

$$G'_I = \frac{-i\omega\, \epsilon_0 K_{11} \lambda_2'^2}{(k^2 - k_0^2 K_{11})^2 - k_0^4 K_{12}^2} a^2 \log\left(\frac{\gamma\lambda_2' a}{2}\right)[-\lambda_1' J_1(\lambda_1' a)]\cdot[F_1(k_I) + \frac{ik_I}{z} F_2(k_I)] \tag{5.1.6}$$

Similar expressions for G'_{II}, G'_1, and G'_3 are easily obtained by the same procedure.

The current distribution I(z) has a basic 1/z times exp(-ikz) variation plus terms which decay as $1/z^2$, $1/z^3$, etc., for z large. The exponential term carries the phase variation while the factors of 1/z simply indicate that the antenna is radiating, which requires the magnitude of the current to decrease away from the source.

Only the wave numbers which are real for zero losses can make a significant contribution to the current distribution some distance from the source. Depending on the plasma- and gyro-frequencies, one to three of the four wave numbers may be complex or imaginary and therefore can be neglected for large z. The wave numbers k_1 and k_3 can be neglected for $X < 1$ for this reason. This line of reasoning, together with possibly a computer evaluation of the G functions of Equation 5.1.1, will give the current distribution for each value of X and Y.

It is interesting to note that the real part of k_3 is opposite in sign to the real part of k_1. This implies that the phase variation of one component of the wave is away from the slot source while the other is toward it. In isotropic media, a wave with phase travel toward the source could be discarded because there is no source at infinity to produce such a wave. This same argument may or may not be valid for this anisotropic problem. It is conceivable that because of the interrelated nature of k_1 and k_3, one wave traveling outward may actually produce another wave traveling inward. One example where something like this does happen is in radial waveguides (Harrington, 1961).

The wave numbers k_I and k_{II} are those of plane waves in this medium, and would therefore be expected to appear in the solution to an antenna problem even for arbitrary orientation of the antenna with respect to the magnetic field. The appearance of the wave numbers k_1 and k_3, on the other hand, is somewhat unexpected. Although they do come from the mathematical formulation in a natural fashion, some caution in their use is in order. Their portion of the total field may be vanishingly small as compared with the $k_{I,II}$ portions, or such waves may exist only for the antenna within a very small cone about $\theta = 0$. In either case they would not be of much use in computing the impedance of an antenna. It will be interesting to examine the experimental current distribution of antennas in the ionosphere when such experiments are performed to see if there is evidence of more than the two plane wave wave numbers. This, of course, will be the final test in determining the usefulness of k_1 and k_3. This whole analysis is invalid for $Y = 0$, so examining $k_{1,3}$ for this limiting case can not give an answer to the question of their existance.

Lo and Lee (1966) have also considered this same problem of an infinite antenna aligned with the static magnetic field in an anisotropic incompressible plasma. At the time of this writing, this author had only a brief correspondence item at hand, which, of course, is not enough on which to base a full comparison of results. A few preliminary remarks can be made, however. It appears from their results that they may have used a rotating coordinate system. This could easily change many of the details involved in obtaining a solution. That is, the elementary eigenfunctions may look different, as well as the integral solution. The final form of their solution and the one presented here should be the same when referred to either the rotating or ordinary cylindrical coordinate systems, at least to within the limits of the asymptotic approximations made in evaluating the integrals. The z variation of the fields should be the same in both coordinate systems.

They say that their branch cut integrals yield two current waves with propagation coefficients $K_r k_0$ and $K_\ell k_0$, where

$$K_r = K_{11} + K_{12} \qquad K_\ell = K_{11} - K_{12} \tag{5.1.7}$$

From Equation 4.2.11, however, it is seen that

$$k_I = \sqrt{K_r}\, k_0 \qquad k_{II} = \sqrt{K_\ell}\, k_0 \tag{5.1.8}$$

This implies that one of two things are true. Either Lo and Lee (1966) found wave numbers which are different from those of plane waves in the medium or else their paper contains typographical errors. This author would suppose the latter to be more probable.

They also comment that the integrand of their integral solution changes its basic form with variation of plasma parameters. They say that part of the time the integrand consists of two double-valued functions, and the rest of the time of one double-valued function with four branch points plus two single-valued functions. This statement also implies a different formulation from the one used by this author because the form of the integrand of Equation 4.1.19 does not have to be changed for varying plasma parameters.

The fact that other people are working on the infinite antenna problem would imply that this is not a problem without interest. Some comment is therefore in order on the particular merits of this thesis. It seems that the main contribution of this thesis has been the language and techniques applied to the infinite antenna problem with a secondary contribution of a partial solution to this problem. This partial solution is the part which is mathematically interesting; the remainder is primarily writing and running a computer program and plotting the data obtained, not a trivial task in itself. The powerful tools of functional analysis have only been used in the solution of field problems for perhaps the last ten years. As the class of problems under consideration becomes more complex, the advantages of functional analysis become more pronounced. A careful investigation of the infinite antenna problem has required as much or more functional analysis than this author has seen applied to any other physical problem. The use of the continuous spectrum is not really uncommon but the use of generalized eigenfunctions may prove to be a first, at least for this type of problem. Therefore, it is felt that the particular solution obtained is of secondary importance to the rather general methods used in obtaining this solution.

5.2 Recommendation for Further Study. The model chosen for the ionosphere is the simplest possible that still includes anisotropy. The medium in the near vicinity of a dipole on a satellite will probably be much more complex than indicated by this model. There will be a sheath around the antenna, for example. There may also be acoustic waves (Cook and Edgar, 1965). These factors, as well as possible nonlinear effects and ion modes, may change the current distribution and the impedance drastically. The infinite antenna problem should be examined in an anisotropic compressible plasma, first without a sheath and then with one. It will probably be some time in the future before effects other than sheaths and compressibility can be satisfactorily considered.

Once a model is chosen, a problem can be posed as a differential equation and a set of boundary conditions. A solution can be obtained by the powerful methods of functional analysis, as in this research. This area of mathematics is very much a research area, and the topics of higher order differential equations and generalized eigenfunctions need a great deal more examination.

Perhaps the most obvious next step is to find the impedance of the infinite antenna. There are at least two distinct approaches to finding the impedance of an antenna. One is to integrate the radial component of Poynting's vector over the surface of a large sphere. This integration yields the radiated power, and division by $V^2/2$ gives the radiation conductance. This approach was used by Papas (1949) in the isotropic infinite antenna problem. Northover (1958) looked at this approach again and was able to verify some of the assumptions which Papas made. This requires solving the integral expression for the E field for large r and/or large z and converting the solution to spherical coordinates.

Using this Poynting vector method for the infinite antenna in an anisotropic medium has some disadvantages. As mentioned in Chapter III, there is some question about Poynting's vector actually representing true power flow. Also, the field solutions obtained earlier in this chapter are only valid near the antenna. It would require a good deal of work to obtain the asymptotic form of $E_z(r, z)$ suitable for large r and z, convert to spherical coordinates, and integrate this asymptotic expression over a sphere.

The second approach uses the Lorentz reciprocity theorem or its modernized form, the reaction concept (Richmond, 1961). The Lorentz reciprocity theorem, neglecting magnetic currents, is given by Collin (1960) as

$$\oint_S (\underline{E}_1 \times \underline{H}_2 - \underline{E}_2 \times \underline{H}_1) \cdot d\underline{s} = \int_V (\underline{E}_2 \cdot \underline{J}_1 - \underline{E}_1 \cdot \underline{J}_2) dV \qquad (5.2.1)$$

Collin actually derived this result for symmetric permittivity and permeability tensors, but the same form applies for the antisymmetric permittivity tensor of this problem if one set of solutions, say $\underline{E}_2$ and $\underline{H}_2$, are found in the transposed medium. That is, $\underline{E}_1$ and $\underline{H}_1$ are solutions to Maxwell's equations when the static magnetic field is in the positive z direction while $\underline{E}_2$ and $\underline{H}_2$ are solutions for the magnetic field reversed. The volume V is bounded by the closed surface S.

Richmond (1961) uses this theorem to obtain the following formula for the self impedance of an antenna.

$$Z_{in} = -\frac{1}{I^2} \int_{\partial\Omega} \underline{J} \cdot \underline{E}_2 \, ds \qquad (5.2.2)$$

In this formula, $\partial\Omega$ is the surface of the antenna and $\underline{J}$ is the current density along the antenna. This current density is easily found from the current distribution, for which Equation 5.1.5 is suitable for large z. The current I is the input current at the antenna terminals which gives rise to $\underline{J}$. The field $\underline{E}_2$ is the field produced by $\underline{J}$ when the magnetic field is reversed and the perfectly conducting surface of the antenna removed. This means, of course, that $\underline{E}_2(a, z)$ is not identically zero for any z.

From this discussion it may be seen that finding the impedance of the infinite antenna is not a trivial task, and may be of the same order of difficulty as finding the current distribution. This would certainly make a good research topic.

A SELECTED BIBLIOGRAPHY

Allis, W. P., S. J. Buchsbaum, and A. Bers. (1963). Waves in Anisotropic Plasmas. Cambridge: M.I.T. Press, 1963.

Ament, W. S. (1963). "The Impedance of a Cylindrical Dipole in a Homogeneous Anisotropic Medium, II." Scientific Report No. 3 on Contract NAS5-585, Electromagnetic Research Corporation, College Park, Md. 1963.

Ament, W. S., J. C. Katzin, M. Katzin, and B. Y. C. Koo. (1964). "Impedance of a Cylindrical Dipole having a Sinusoidal Current Distribution in a Homogeneous Anisotropic Ionosphere." Radio Science, J. of Research, NBS/USNC-URSI, Vol. 68D, April, 1964, 379-405.

Arbel, E. and L. B. Felsen. (1962). "On the Theory of Radiation from Sources in Plane Stratified Anisotropic Plasma Media." Report PIBMRI-1069-62, Dept. of Electrophysics, Polytechnic Inst. of Brooklyn, New York, September, 1962.

_______ "Radiation from a Dipole in an Infinite, Homogeneous, Anisotropic Plasma." Report PIBMRI-1070-62, ibid. These reports have been published as Theory of Radiation from Sources in Anisotropic Media, I-II, in Electromagnetic Theory and Antennas, E. C. Jordon, Ed. New York: Pergamon, 1963, 391.

Atkinson, F. V. (1949). "On Sommerfeld's 'Radiation Condition'." Philosophical Magazine, Vol. 40, 1949, 645-651.

Balmain, K. G. (1963). "The Impedance of a Short Dipole Antenna in a Magnetoplasma." Ph.D. dissertation, University of Illinois, Urbana, Illinois. 1963.

_______ "The Impedance of a Short Dipole Antenna in a Magnetoplasma." IEEE Transactions on Antennas and Propagation, Vol. AP-12, September, 1964, 605-617.

Banos, A., Jr. and J. P. Wesley. (1953, 1954). "The Horizontal Electric Dipole in a Conducting Half-Space." Parts I and II. University of California, Los Angeles. Contract NObsr-43356, NE-120221-5, SIO References 53-33, 54-31. AD nos. 18344, 47854. 1953, 1954.

Barlow, H. M. and A. E. Karbowiak. (1953). "An Investigation of the Characteristics of Cylindrical Surface Waves." Proceedings of the IEE, Vol. 100, Part III, No. 68, November, 1953, 321-328.

Barlow, H. M. and A. L. Cullen. (1953). "Surface Waves." Proceedings of the IEE, Vol. 100, Part III, No. 68, Nov. 1953, 329-347.

Blair, W. E. (1964). "The Driving-Point Impedance of an Electrically Short Cylindrical Antenna in the Ionosphere." Technical Report EE-109, Engineering Experiment Station, University of New Mexico, Albuquerque, June, 1964.

Brandstatter, J. J., and A. J. Penico. (1964). "A Study of the Impedance of a Cylindrical Dipole in an Anisotropic Plasma." Final Report, SRI Project PMU-3417, Stanford Research Institute, Menlo Park, California. 1964.

Chow, Yutze. (1962). "Note on Radiation in a Gyro-Electric-Magnetic Medium-An Extension of Bunkin's Calculation." IRE Transactions on Antennas and Propagation, Vol. AP-10, July, 1962, 464-469.

Clemmow, P. C. (1963). "The Theory of Electromagnetic Waves in a Simple Anisotropic Medium." Proceedings IEE, Vol. 110, January, 1963, 101-106.

Collin, R. E. (1960). Field Theory of Guided Waves. New York: McGraw-Hill, 1960.

Cook, K. R. and B. C. Edgar. (1966). "Current Distribution and Impedance of a Cylindrical Antenna in an Isotropic Compressible Plasma." Radio Science, Vol. 1, January, 1966, 13-19.

deWolf, D. A. (1965). "Impedance of a Small VLF-Dipole in the Ionosphere." Presented at the 1965 USNC-URSI Fall Meeting, Hanover, New Hampshire.

Dolph, C. L. (1956). "The Mathematician Grapples with Linear Problems Associated with the Radiation Condition." IRE Transactions on Antennas and Propagation, Vol. 4, July, 1956, 302-311.

Epstein, P. S. (1954). "On the Possibility of Electromagnetic Surface Waves." Proceedings of the National Academy of Sciences, U.S.A., Vol. 40, December, 1954, 1158-1165.

Felsen, L. B. and H. Staras. (1965). "Comment on 'The Impedance of an Electric Dipole in a Magneto-Ionic Medium'." IEEE Transactions on Antennas and Propagation, Vol. AP-13, July, 1965, 637-638.

Friedman, B. (1956). Principles and Techniques of Applied Mathematics. New York: John Wiley, 1956.

Goubau, Georg. (1950). "Surface Waves and Their Application to Transmission Lines." Journal of Applied Physics. Vol. 21, November, 1950, 1119-1128.

Goubau, Georg. (1952). "On the Excitation of Surface Waves." Proceedings of the IRE, Vol. 40, July, 1952, 865-868.

Harrington, R. F. (1961). Time-Harmonic Electromagnetic Fields. New York: McGraw-Hill. 1961.

Hines, C. O. (1951a). "Wave Packets, the Poynting Vector, and Energy Flow: Part I-Non-dissipative (Anisotropic) Homogeneous Media." Journal of Geophys. Res., Vol. 56, March, 1951, 63-72.

Hines, C. O. (1951b). "Wave Packets, the Poynting Vector, and Energy Flow: Part II-Group Propagation Through Dissipative Isotropic Media." Journal of Geophys. Res., Vol. 56, June, 1951, 197-206.

Hines, C. O. (1951c). "Wave Packets, the Poynting Vector, and Energy Flow: Part III-Packet Propagation Through Dissipative Anisotropic Media." Journal of Geophys. Res., Vol. 56, June, 1951, 207-220.

Hines, C. O. (1951d). "Wave Packets, the Poynting Vector, and Energy Flow: Part IV-Poynting and and MacDonald Velocities in Dissipative Anisotropic Media (Conclusion)." Journal of Geophys. Res., Vol. 56, December, 1951, 535-544.

Hines, C. O. (1952). "Electromagnetic Energy Density and Flux." Canad. Journal of Phys., Vol. 30, March, 1952, 123-129.

Jahnke, E. and F. Emde. (1945). Tables of Functions. New York: Dover Publications, 1945.

Johnson, G. L. and K. R. Cook. (1965). "Current Distribution for a Cylindrical Dipole in an Homogeneous Anisotropic Ionosphere." IEEE Transactions on Antennas and Propagation, Vol. AP-13, May, 1965, 471-472.

Katzin, J. C. and M. Katzin. (1961). "The Impedance of a Cylindrical Dipole in a Homogeneous Anisotropic Ionosphere." Electromagnetic Research Corporation, Report No. NAS585-2, September, 1961.

Kelso, J. M. (1964). Radio Ray Propagation in the Ionosphere. New York: McGraw-Hill, 1964.

Kogelnik, H. (1960). "On Electromagnetic Radiation in Magneto-ionic Media." Journal of Research NBS, Vol. 64D, September-October, 1960, 515-523.

Kuehl, H. H. (1962). "Electromagnetic Radiation from an Electric Dipole in a Cold Anisotropic Plasma." Phys. of Fluids, Vol. 5, September, 1962, 1095-1103.

Likuski, R. (1964). Free and Driven Modes in Anisotropic Plasma Guides and Resonators. Technical Documentary Report No. AL-TDR-64-157, Engineering Experiment Station, University of Illinois, Urbana, Illinois, July, 1964.

Langer, R. E., Editor. (1962). Proceedings of the Symposium of Electromagnetic Waves. University of Wisconsin Press, 1962, 243-290.

Lo, Y. T. and S. W. Lee. (1966). "Current Distribution and Input Impedance of an Infinite Cylindrical Antenna in Anisotropic Plasma." Electronic Letters. Vol. 2, 1966.

Macdonald, H. M. (1902). Electric Waves. Cambridge University Press, 1902.

Marcuvitz, N. (1956). "On Field Representations in Terms of Leaky Modes or Eigenmodes." IRE Transactions on Antennas and Propagation, Vol. AP-4, July, 1956, 192-194.

Mushiake, Y. (1965). "Electromagnetic Waves Along an Infinitely Long and Thin Conducting Wire in a Magneto-Ionic Medium." Radio Science, Journal of Res. NBS/USNC-URSI, Vol. 69D, No. 4, April, 1965, 503-510.

Northover, F. H. (1958). "Radiation and Surface Currents From a Slot on an Infinite Conducting Cylinder." Canad. Journal of Phys., Vol. 36, 1958, 206-217.

Papas, C. H. (1949). "On the Infinitely Long Cylindrical Antenna." Journal of Applied Phys., Vol. 20, May, 1949, 437-440.

Richmond, J. H. (1961). "A Reaction Theorem and its Application to Antenna Impedance Calculations." IRE Transactions on Antennas and Propagation, Vol. AP-9, November, 1961, 515-520.

Roberts, T. E., Jr. (1953). "Theory of the Single-Wire Transmission Line." Journal of Applied Phys., Vol. 24, January, 1953, 57-60.

Saunders, W. K. (1952). "On Solutions of Maxwell's Equations in an Exterior Region." Proceedings of the National Academy of Sciences, Vol. 38, 1952, 342-348.

Saxton, J. A., Editor. (1964). Advances in Radio Research. New York: Academic, 1964, 157-217.

Seshadri, S. R., G. L. Johnson and K. R. Cook. (1965). "Comment on 'Current Distribution for a Cylindrical Dipole in an Homogeneous Anisotropic Ionosphere'." IEEE Transactions on Antennas and Propagation, Vol. AP-13, November, 1965, 971-973.

Silver, S., Editor. (1949). Microwave Antenna Theory and Design. M.I.T. Radiation Laboratory Series, Vol. 12, New York: McGraw-Hill, 1949.

Silver, S. and W. K. Saunders. (1950). "The External Field Produced by a Slot in an Infinite Circular Cylinder." Journal of Applied Physics, Vol. 21, February, 1950, 153-158.

Silver, S., Editor. (1963). Radio Waves and Circuits. Amsterdam: Elsevier, 1963, 86-95.

Sommerfeld, A. (1912). Jahresbericht der D. M. V., Vol. 21, 1912, 309-353. See also A. Sommerfeld, Die Differential-und Integral-Gleichungen der Mechanik und Physik. Vol. II. Chapter 19, Edited by P. Frank and R. von Mises, New York: Mary S. Rosenberg, 1943.

Sommerfeld. A. N. (1949). Partial Differential Equations. New York: Academic Press, 1949.

Staras, H. (1964). "The Impedance of an Electric Dipole in a Magneto-Ionic Medium." IEEE Transactions on Antennas and Propagation, Vol. AP-12, November, 1964, 695-702.

Storey, L. R. O. (1963). "The Design of an Electric Dipole Antenna for V. L. F. Reception within the Ionosphere." Centre National D'etudes des Telecommunications, Technical Report No. 308TC, 1963.

Stratton, J. A. and L. J. Chu. (1941). "Steady-State Solutions of Electromagnetic Field Problems. I. Forced Oscillations of a Cylindrical Conductor." Journal of Applied Phys., Vol. 12, March, 1941, 230-235.

Stratton, J. A. (1941). Electromagnetic Theory. New York: McGraw-Hill, 1941.

Tamir, T. and A. A. Oliner. (1962). "The Influence of Complex Waves on the Radiation Field of a Slot-excited Plasma Layer." IRE Transactions on Antennas and Propagation, Vol. AP-10, January, 1962, 55-65.

Tamir, T. and A. A. Oliner. (1963a). "The Spectrum of Electromagnetic Waves Guided by a Plasma Layer." Proceedings of the IEEE, Vol. 51, February, 1963, 317-332.

Tamir, T. and A. A. Oliner. (1963b). "Guided Complex Waves: Part I. Fields at an Interface, Part II. Relation to Radiation Patterns." Proceedings of the IEE, Vol. 110, February, 1963, 310-334.

Uman, M. A. (1964). Introduction to Plasma Physics. New York: McGraw-Hill, 1964.

Wait, J. R. (1959). Electromagnetic Radiation From Cylindrical Structures. New York: Pergamon Press, 1959.

Wait, J. R. (1965). "A Note on Surface Waves and Ground Waves." IEEE Transactions on Antennas and Propagation, Vol. AP-13, November, 1965, 996-997.

Wilcox, C. H. (1956). "An Expansion Theorem for Electromagnetic Fields." Comm. Pure Appl. Math., Vol. 9, 1956, 115-134.

Wilcox, C. H. (1963). "The Steady-State Diffraction of Electromagnetic Radiation by an Obstacle in an Inhomogeneous Anisotropic Conducting Medium." Arch. Rational Mech. Anal., Vol. 14, 1963, 326-336.

APPENDIX A

PROOF OF UNIQUENESS THEOREM

The uniqueness theorem, Theorem 1, is repeated here for convenience.

Theorem 1. If h_{ij} and μ_{ij} are bounded, Lebesgue measurable functions of position, and if $\omega \neq 0$ and real, then the steady state solution $\underline{F}$ of Equations 3.4.17-20 exists and is unique for every source field $\underline{f}(\underline{J}_e, \underline{J}_m)$ in $L_2(\Omega)$.

The proof given here is essentially the same as that presented by Wilcox (1963) except that an attempt has been made to include more explanatory material than was presented by Wilcox.

There are two distinct questions to be answered by this proof. First, does a solution exist and, second, is it unique? The system of Equations 3.4.16 may be given as one operator equation, $Lx = a$. Friedman (1956) proves that the solution to $Lx = a$ is unique only if the solution to $Lx = 0$ is trivial. This will be one part of the proof, showing that there are no nonzero solutions to $Lx = 0$.

Friedman then goes on to prove that the solution to $Lx = a$ exists if the range of L is closed, and if and only if a is orthogonal to every solution of the adjoint homogeneous equation $L^a z = 0$. The adjoint system is obtained by replacing all anisotropic materials by materials for which $\hat{K}^a$ is the transposed form of $\hat{K}$. Showing that these requirements on the range of L and on a hold will be the other part of the proof.

Wilcox (1963) assumes that the components of the permittivity tensor are real-valued. This ensures that the quadratic form for the energy density is positive definite. However, the energy density was shown in Chapter III to also be positive definite for a hermittian permittivity. Since the positive definite energy density appears to be the factor actually used in the proof, this author has left the real-valuedness requirement out of the statement of the theorem.

Several definitions must be made at this time.

$L_2(\Omega)$ = The set of all vector fields $\underline{F}$ such that $\underline{F}$ is Legesque measurable on Ω and $\int_\Omega \underline{F}(\underline{r})^2 \, dV < \infty$.

$L_2(\nabla \times; \Omega)$ = The set of vector fields $\underline{F}$ such that $\underline{F}$ and $\nabla \times \underline{F}$ are in $L_2(\Omega)$.

$L_2^0(\nabla \times; \Omega)$ = The set of vector fields which belong to both $L_2(\nabla \times; \Omega)$ and the set for which $\int_\Omega \underline{F} \cdot \nabla \times \underline{G} \, dV = \int_\Omega \underline{G} \cdot \nabla \times \underline{F} \, dV$, for all $\underline{G}$ in $L_2(\nabla \times; \Omega)$.

The last definition is motivated by the integral identity

$$\int_\Omega \underline{F} \cdot \nabla \times \underline{\varphi} \, dV - \int_\Omega \underline{\varphi} \cdot \nabla \times \underline{F} \, dV = \int_{\partial\Omega} \underline{n} \times \underline{F} \cdot \underline{\varphi} \, ds \qquad \text{(A.1)}$$

where

$$\int_{\partial\Omega} \underline{n} \times \underline{F} \cdot \underline{\varphi} \, ds = 0 \qquad \text{(A.2)}$$

generalizes the boundary condition "$\underline{n} \times \underline{F} = 0$ on $\partial\Omega$". If $\underline{F}$ and $\nabla \times \underline{F}$ are continuous in the closure of Ω (the closure is the smallest closed set containing Ω) and if $\partial\Omega$ is sufficiently smooth, then

$\underline{F} \in L_2^0(\nabla\, x;\ \Omega)$ implies

$$\int_{\partial\Omega} \underline{n} \times \underline{F} \cdot \underline{\varphi}\, ds = 0$$

for all $\underline{\varphi}$ which are continuously differentiable in the closure of Ω, and it follows that $\underline{n} \times \underline{F} = 0$ on $\partial\Omega$. The integral boundary condition is more flexible than the pointwise one and is useful in approximation methods.

The currents $\underline{J}_e$ and $\underline{J}_m$ are in $L_2(\Omega)$ for every $\underline{E}$ in $L_2^0(\nabla\, x;\ \Omega)$ and $\underline{H}$ in $L_2(\nabla\, x;\ \Omega)$. Since the operator L of Equations 3.4.17, 18 involves two functions which are each in $L_2(\Omega)$, it is said that L is defined on the Hilbert space $L_2(\Omega) \times L_2(\Omega) = HS$ which is formed from the Cartesian product of the Hilbert space $L_2(\Omega)$ with itself. However, the domain of this linear operator ($\underline{E}$ and $\underline{H}$) is not all of $L_2(\Omega) \times L_2(\Omega)$ but is

$$D(L) = L_2^0(\nabla\, x;\ \Omega) \times L_2(\nabla\, x;\ \Omega) \tag{A.3}$$

The same ideas are expressed in the following definition of strict solution.

Definition. The fields $\underline{E}$ and $\underline{H}$ define a strict solution of the steady state problem for the region Ω and source fields $\underline{J}_e \in L_2(\Omega)$ and $\underline{J}_m \in L_2(\Omega)$ provided that $\underline{E} \in L_2^0(\nabla\, x;\ \Omega)$, $\underline{H} \in L_2(\nabla\, x;\ \Omega)$ and Maxwell's Equations 2.2.1-4 hold almost everywhere in Ω.

Wilcox was able to prove that for $\underline{E}$ and $\underline{H}$ strict solutions the following energy inequality theorem holds.

$$\int_{\Omega} [h_{ij} E_i E_j^* + \mu_{ij} H_i H_j^*] dV \leq C \int_{\Omega} (|\underline{J}_e|^2 + |\underline{J}_m|^2) dV < \infty \tag{A.4}$$

where C is a determined constant depending on the medium parameters.

It is also required that h_{ij}, μ_{ij}, and ω satisfy the hypothesis of Theorem 1. This inequality has been called the Wilcox radiation condition in this thesis. It may be considered equivalent to the statements $\underline{E} \in L_2^0(\nabla \times; \Omega)$ and $\underline{H} \in L_2(\nabla \times; \Omega)$. This energy inequality theorem will not be proved here, but the result will be useful in proving the uniqueness theorem.

From the energy inequality theorem, it may be shown that

$$\int_\Omega (|\underline{E}|^2 + |\underline{H}|^2) dV \leq C_1 \int_\Omega (|\underline{J}_e|^2 + |\underline{J}_m|^2) dV \tag{A.5}$$

where C_1 is another determined constant depending on the medium parameters.

Since $L_2(\Omega)$ is a Hilbert space, the norm in this space may be defined as

$$||\underline{f}||^2 = \int_\Omega |\underline{f}|^2 dV \tag{A.6}$$

Uniqueness may now be shown. Let $\underline{E}$, $\underline{H}$ and $\underline{E}'$, $\underline{H}'$ be strict solutions of the problem corresponding to source fields $\underline{J}_e$, $\underline{J}_m$ and $\underline{J}_e'$, $\underline{J}_m'$, respectively. Then the differences $\underline{E} - \underline{E}'$, $\underline{H} - \underline{H}'$ define a strict solution with source fields $\underline{J}_e - \underline{J}_e'$, $\underline{J}_m - \underline{J}_m'$, because Maxwell's equations are linear. Therefore, Equation A.5 can be written

$$||\underline{E} - \underline{E}'||^2 + ||\underline{H} - \underline{H}'||^2 \leq C_1 (||\underline{J}_e - \underline{J}_e'||^2 + ||\underline{J}_m - \underline{J}_m'||)^2 \tag{A.7}$$

If $\underline{E}$, $\underline{H}$ and $\underline{E}'$, $\underline{H}'$ are strict solutions with the same data $\underline{J}_e$, $\underline{J}_m$ then Equation A.7 with $\underline{J}_e = \underline{J}_e'$, $\underline{J}_m = \underline{J}_m'$ implies $\underline{E}' = \underline{E}$, $\underline{H} = \underline{H}'$. This means that the solutions $\underline{E}$ and $\underline{H}$ are unique.

To prove the existence of a solution the range of the operator, R(L), must be shown to be the entire Hilbert space $L_2(\Omega) \times L_2(\Omega)$. The range is by definition the set of all elements obtained when L acts on all of D, the domain of L. Now using the hypothesis of Theorem 1 it is desired to show that the closure of the range, $\overline{R(L)} = L_2(\Omega) \times L_2(\Omega)$. But $\overline{R(L)}$ is a closed linear subspace so that

$$L_2(\Omega) \times L_2(\Omega) = \overline{R(L)} \oplus [\overline{R(L)}]^{\perp} \tag{A.8}$$

where $[\overline{R(L)}]^{\perp}$ is the orthogonal complement of $\overline{R(L)}$ and $\oplus$ denotes the orthogonal sum of the two closed (also complete) linear subspaces. Let $\underline{E}'$, $\underline{H}'$ be orthogonal to R(L) in $L_2(\Omega) \times L_2(\Omega)$, which means that

$$\int_{\Omega} [\underline{E}' \cdot \underline{J}_e + \underline{H}' \cdot \underline{J}_m] dV = 0 \quad \text{for all fields } \underline{E} \text{ in } L_2^0(\nabla \times; \Omega), \ \underline{H} \text{ in } L_2(\nabla \times; \Omega). \tag{A.9}$$

Write Maxwell's equations in the form

$$J_{ei} = (\nabla \times \underline{H})_i - i\omega\, \epsilon_0 K_{ij} E_j \tag{A.10}$$

$$J_{mi} = (\nabla \times \underline{E})_i + i\omega\, \mu_{ij} H_j \tag{A.11}$$

Take the two cases

$$\underline{E} = \underline{P} \text{ with } \underline{P} \in L_2^0(\nabla \times; \Omega), \ \underline{H} = 0$$

and $$\underline{E} = 0, \ \underline{H} = \underline{Q} \text{ with } \underline{Q} \in L_2(\nabla \times; \Omega)$$

Combining these two cases with Maxwell's equations and Equation A.9 gives the pair of identities

$$\int_{\Omega} [E_i'(i\omega\, \epsilon_0 K_{ij}) P_j + H_i'(\nabla \times \underline{P})_i] dV = 0$$

$$\int_{\Omega} [E_i^{\prime}(\nabla \times \underline{Q})_i - i\omega\, \mu_{ij} H_i^{\prime} Q_j] dV = 0 \qquad (A.12)$$

for all $\underline{P} \in L_2^0(\nabla \times; \Omega)$, $\underline{Q} \in L_2(\nabla \times; \Omega)$

It should be evident from the formulation that the above identities express exactly the same thing as was given in the definition on strict solution although this is called the definition for a "finite energy solution." Wilcox proves the equivalence in a Lemma, but the proof is short and elementary, and therefore will not be included here.

Equations A.12 say that $\underline{E}'$, $\underline{H}'$ is a finite energy solution of an adjoint problem with $-\omega$, $\hat{K}^a$, $\hat{\mu}^a$ (the transposes) replacing ω, $\hat{K}$, $\hat{\mu}$ but with source fields $\underline{J}_e$, $\underline{J}_m = 0$. But the conditions for the energy inequality to hold are the same for the adjoint problem so that $\underline{E}'$, $\underline{H}'$ is a strict solution of the adjoint problem. Since $\underline{J}_e$, $\underline{J}_m = 0$, then by Equation A.4, $\underline{E}' = \underline{H}' = 0$. This proves that $[\overline{R(L)}]^{\perp}$ is the null space and hence $\overline{R(L)} = L_2(\Omega) \times L_2(\Omega)$. A set S is dense in a Hilbert space HS if the closure of S coincides with the whole space HS, which means that R(L) is dense in $L_2(\Omega) \times L_2(\Omega)$. It was said earlier that the solution to Lx = a exists if the range of L is closed, and if and only if a is orthogonal to every solution of the adjoint homogeneous equation $L^a x = 0$. The second part of this has just been proved, so now the closure of the range must be considered.

Let $\underline{J}_e$, $\underline{J}_m$ be in $L_2(\Omega)$. From the preceding remarks there exist sequences of fields $[\underline{E}_n]$ in $L_2^0(\nabla \times; \Omega)$, $[\underline{H}_n]$ in $L_2(\nabla \times; \Omega)$ such that their sources $[\underline{J}_{e,n}]$, $[\underline{J}_{m,n}]$ converge in $L_2(\Omega) \times L_2(\Omega)$. These sequences are therefore Cauchy sequences, hence, by setting $\underline{E} - \underline{E}' = \underline{E}_n - \underline{E}_j$, and similarly for the other quantities in Equation

A.7, there results

$$||\underline{E}_n - \underline{E}_j||^2 = ||\underline{H}_n - \underline{H}_j||^2 \leq C_1(||\underline{J}_{e,n} - \underline{J}_{e,j}||^2 + ||\underline{J}_{m,n} - \underline{J}_{m,j}||^2) \tag{A.13}$$

From this it follows that the sequences $[\underline{E}_n]$, $[\underline{H}_n]$ are Cauchy sequences in $L_2(\Omega)$ since the right side of Equation A.13 tends to zero as $n,j \to \infty$. But since $L_2(\Omega)$ is a complete set the above sequences of fields converge to a limit field which lies in $L_2(\Omega)$, namely

$$\underline{E}_n \to \underline{E}, \quad \underline{H}_n \to \underline{H} \quad \text{as} \quad n \to \infty$$

Each $\underline{E}_n$, $\underline{H}_n$ is a strict solution with source fields $\underline{J}_{e,n}$, $\underline{J}_{m,n}$ and are therefore also solutions with finite energy which means that Equations A.12 hold for each $\underline{E}_n$, $\underline{H}_n$ and their corresponding source fields. But since all fields $\underline{E}_n$, $\underline{H}_n$ converge in $L_2(\Omega)$ then Equations A.12 hold as $n \to \infty$ for the limit fields and their sources. Therefore $\underline{E}$, $\underline{H}$ is a solution with finite energy and thus also a strict solution with source fields $\underline{J}_e$, $\underline{J}_m$ and is unique. The above argument shows that $R(L)$ is a closed linear subspace and is therefore equal to its closure. That is, $R(L) = \overline{R(L)}$ in the space $L_2(\Omega) \times L_2(\Omega)$. Therefore

$$R(L) = L_2(\Omega) \times L_2(\Omega)$$

so that the range is both closed and complete. This concludes the proof of Theorem 1.

APPENDIX B

BESSEL FUNCTIONS

Bessel's equation of order n is

$$z \frac{d}{dz}\left(z \frac{dy}{dz}\right) + (z^2 - n^2)y = 0 \qquad (B.1)$$

Two independent solutions to this equation for n = 0 and n = 1 are

$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{z}{2}\right)^{2m} \qquad (B.2)$$

$$N_0(z) = \frac{2}{\pi} \log \frac{\gamma z}{2} J_0(z) + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(m!)^2} \left(\frac{z}{2}\right)^{2m} \varphi(m) \qquad (B.3)$$

$$J_1(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{z}{2}\right)^{2m+1} \qquad (B.4)$$

$$N_1(z) = \frac{2}{\pi} \log \left(\frac{\gamma z}{2}\right) J_1(z) - \frac{2}{\pi z} - \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{z}{2}\right)^{2m+1} [\varphi(m)+\varphi(m+1)] \qquad (B.5)$$

where $\gamma = 1.781$

$\log \gamma = 0.5772$

$$\varphi(m) = \sum_{n=1}^{m} \frac{1}{n} \qquad (B.6)$$

For the expression of wave phenomena, it is convenient to define linear combinations of the Bessel functions,

$$H_n^{(1)}(z) = J_n(z) + iN_n(z)$$

$$H_n^{(2)}(z) = J_n(z) - iN_n(z) \tag{B.7}$$

called Hankel functions of the first and second kinds. Only the Hankel functions have the possibility of remaining bounded for arbitrarily large complex argument z. Both the Bessel and Neumann functions $J_n(z)$ and $N_n(z)$ will become unbounded as the complex z goes to infinity.

Ratios of Hankel functions are often encountered in integral solutions for the fields around an antenna. One such ratio is

$$\frac{H_0^{(2)}(z)}{H_1^{(2)}(z)} = \frac{J_0(z)J_1(z) + N_0(z)N_1(z)}{J_1^2(z) + N_1^2(z)} - \frac{\frac{2i}{\pi z}}{J_1^2(z) + N_1^2(z)} \tag{B.8}$$

where use has been made of the Wronskian of Bessel's equation, which is

$$J_n(z)N_n'(z) - N_n(z)J_n'(z) = \frac{2}{\pi z} \tag{B.9}$$

The inverse ratio also is used.

$$\frac{H_1^{(2)}(z)}{H_0^{(2)}(z)} = \frac{J_0(z)J_1(z) + N_0(z)N_1(z)}{J_0^2(z) + N_0^2(z)} + \frac{\frac{2i}{\pi z}}{J_0^2(z) + N_0^2(z)} \tag{B.10}$$

The small argument forms of the Bessel and Neumann functions are

$$J_0(z) \xrightarrow[z \to 0]{} 1$$

$$N_0(z) \xrightarrow[z \to 0]{} \frac{2}{\pi} \log\left(\frac{\gamma z}{2}\right) \tag{B.11}$$

$$J_1(z) \xrightarrow[z \to 0]{} \frac{z}{2}$$

$$N_1(z) \xrightarrow[z \to 0]{} -\frac{2}{\pi z} \qquad (B.12)$$

Using Equations B.11 and B.12, the Hankel function ratios become

$$\frac{H_0^{(2)}(z)}{H_1^{(2)}(z)} \xrightarrow[z \to 0]{} -z[\log(\frac{\gamma z}{2}) + i\,\frac{\pi}{2}] \qquad (B.13)$$

$$\frac{H_1^{(2)}(z)}{H_0^{(2)}(z)} \xrightarrow[z \to 0]{} -\frac{1}{z\,\log(\frac{\gamma z}{2})} + \frac{i}{\frac{2}{\pi}\,z[\log(\frac{\gamma z}{2})]^2} \qquad (B.14)$$

Comparing the exact values of these ratios with the approximate values given in Equations B.13 and B.14 (Jahnke and Emde, 1945) shows that these approximate values are accurate to within about 3 per cent for $z \leq 0.1$. Usually, in using such ratios, the imaginary part is neglected in making the small argument approximation. This should be done only when necessary because it introduces at least a 3 per cent error in magnitude unless $|z| \leq 0.001$.

It will also be convenient to determine these ratios for a negative argument $(-z)$. Equation B.2 does not change when z is replaced by $-z$. The only part of Equation B.3 which changes is the log term. That is,

$$\log(-\frac{\gamma z}{2}) = \log(\frac{\gamma z}{2}) + i\pi = \log(\frac{\gamma |z|}{2}) + i\pi + i\arg(z) \qquad (B.15)$$

The choice of plus sign on the $i\pi$ term is arbitrary. It is motivated, however, by the fact that $z = \lambda a$ has an argument $-\pi \leq \theta \leq 0$, at least for k near the real axis. The principal value of the logarithm is defined for $-\pi \leq \theta \leq \pi$, and adding π to the argument of z falls within this range while subtracting π from the argument of z does not. Therefore,

$$N_0(-z) = N_0(z) + i2J_0(z) \tag{B.16}$$

A similar examination of Equations B.4 and B.5 shows that

$$J_1(-z) = - J_1(z) \tag{B.17}$$

$$N_1(-z) = - N_1(z) - i2J_1(z) \tag{B.18}$$

The ratio of the Hankel functions of negative argument can now be written.

$$\frac{H_0^{(2)}(-z)}{H_1^{(2)}(-z)} = \frac{J_0(z) - i(N_0(z) + i2J_0(z))}{-J_1(z) - i(-N_1(z) - i2J_1(z))}$$

$$= \frac{-9J_0(z)J_1(z) - N_0(z)N_1(z)}{9J_1^2(z) + N_1^2(z)} + \frac{\frac{i6}{\pi z}}{9J_1^2(z) + N_1^2(z)} \tag{B.19}$$

$$\xrightarrow[z \to 0]{} z\log(\gamma z/2) + i3\pi z/2 \tag{B.20}$$

$$\frac{H_1^{(2)}(-z)}{H_0^{(2)}(-z)} = \frac{[-3J_1(z) + iN_1(z)]\,[3J_0(z) + iN_0(z)]}{[\ 3J_0(z) - iN_0(z)]\,[3J_0(z) + iN_0(z)]}$$

$$= \frac{-9J_0(z)J_1(z) - N_0(z)N_1(z)}{9J_0^2(z) + N_0^2(z)} - \frac{\frac{i6}{\pi z}}{9J_0^2(z) + N_0^2(z)} \tag{B.21}$$

$$\xrightarrow[z \to 0]{} \frac{1}{z\log(\frac{\gamma z}{2})} - \frac{3i}{\frac{2}{\pi}\, z(\log(\frac{\gamma z}{2}))^2} \tag{B.22}$$

The integrand of Equation 4.3.2 may now be investigated. The transverse wave numbers on the two sides of the branch cut are related as follows:

$$\lambda_{1L} = - \lambda_{1R}, \quad \lambda_{2L} = \lambda_{2R} \tag{B.23}$$

As the radius of the antenna gets very small, the first term in the denominator, Equation 4.3.4 goes to zero while the second term goes to infinity. This may be seen by comparing Equations B.13 and B.14 as z goes to zero. This second term does not involve λ_1, and since λ_2 does not change across this cut, $D_L \doteq D_R$. Both numerator terms of Equation 4.3.2 have a common member containing λ_2, so when the indicated subtraction is performed, this λ_2 term disappears. Thus the integrand of Equation 4.3.2 becomes, in the small radius limit,

$$\frac{N_R}{D_R} - \frac{N_L}{D_L} \doteq \frac{[\lambda_1^2 + f(k)]}{[\lambda_2^2 + f(k)]} \; \frac{\dfrac{J_0(\lambda_1 r) - iN_0(\lambda_1 r)}{(i2/\pi a)} - \dfrac{3J_0(\lambda_1 r) - iN_0(\lambda_1 r)}{(i2/\pi a)}}{-\dfrac{1}{\lambda_2^2 a \log(\frac{\gamma\lambda_2 a}{2})} + \dfrac{i}{\frac{2}{\pi}\lambda_2^2 a[\log(\frac{\gamma\lambda_2 a}{2})]^2}}$$

$$\doteq - i\frac{\pi}{2} (\lambda_2 a)^2 \log(\frac{\gamma\lambda_2 a}{2}) \frac{[\lambda_1^2 + f(k)]}{[\lambda_2^2 + f(k)]} 2J_0(\lambda_1 r) \tag{B.24}$$

where

$$f(k) = k^2 - k_0^2 + \frac{k_0^4 \; K_{12}^2}{-k^2 + k_0^2 K_{11}} \tag{B.25}$$

The first of Equations B.24 contains a small imaginary term in the denominator which is neglected in the second equation. The most rapidly varying factor of this equation, the integrand of Equation

4.1.19, is λ_2^2. The term k^2 which appears in λ_2^2 may be written

$$k^2 = k_I^2 - 2ik_I \frac{\beta}{z} - \frac{\beta^2}{z^2}$$

$$= k_0^2(K_{11} + K_{12}) - 2ik_I \frac{\beta}{z} - \frac{\beta^2}{z^2} \qquad \text{(B.26)}$$

The first bracketed term of λ_2^2, Equation 2.4.14, becomes

$$-k_0^2(K_{11} + K_{12})(\frac{K_{33}}{K_{11}} + 1) + k_0^2(K_{33} + K_{11} - \frac{K_{12}^2}{K_{11}})$$

$$+ (2ik_I\frac{\beta}{z} + \frac{\beta^2}{z^2})(\frac{K_{33}}{K_{11}} + 1)$$

$$= -k_0^2K_{12}(\frac{K_{33}}{K_{11}} + \frac{K_{12}}{K_{11}} + 1) + (2ik_I \frac{\beta}{z} + \frac{\beta^2}{z^2})(\frac{K_{33}}{K_{11}} + 1) \qquad \text{(B.27)}$$

The bracketed term under the radical of Equation 2.4.14 becomes

$$-k_0^2(K_{11} + K_{12})(\frac{K_{33}}{K_{11}} - 1) + k_0^2(K_{33} + \frac{K_{12}^2}{K_{11}} - K_{11}) + (2ik_I \frac{\beta}{z} + \frac{\beta^2}{z^2})(\frac{K_{33}}{K_{11}} - 1)$$

$$= -k_0^2K_{12}(\frac{K_{33}}{K_{11}} + \frac{K_{12}}{K_{11}} - 1) + (2ik_I \frac{\beta}{z} + \frac{\beta^2}{z^2})(\frac{K_{33}}{K_{11}} - 1) \qquad \text{(B.28)}$$

For sufficiently large z the β^2/z^2 term may be ignored, so that λ_2^2 can be written as

$$\lambda_2^2 \doteq -k_0^2K_{12}(\frac{K_{33}}{K_{11}} + \frac{K_{12}}{K_{11}} + 1) + 2ik_I \frac{\beta}{z}(\frac{K_{33}}{K_{11}} + 1) - \sqrt{b_1 - ik_I \frac{\beta}{z}b_2} \qquad \text{(B.29)}$$

where

$$b_1 = k_0^4K_{12}^2[(\frac{K_{33}}{K_{11}} + \frac{K_{12}}{K_{11}} - 1)^2 + 4(K_{11} + K_{12})\frac{K_{33}K_{12}^2}{K_{11}^2}] \qquad \text{(B.30)}$$

$$b_2 = 4k_0^2K_{12}[(\frac{K_{33}}{K_{11}} - 1)(\frac{K_{33}}{K_{11}} + \frac{K_{12}}{K_{11}} - 1) + \frac{2K_{33}K_{12}}{K_{11}^2}] \tag{B.31}$$

The radical may be approximated, for b_1 on the same order of magnitude as b_2, and $\beta \ll z$, as

$$\sqrt{b_1 - ik_I \frac{\beta}{z} b_2} \doteq \sqrt{b_1} - i \frac{k_I}{z} \frac{\beta}{z} \frac{b_2}{\sqrt{b_1}} \tag{B.32}$$

Then Equation B.29 becomes

$$\lambda_2^2 \doteq b_3 + ik_I \frac{\beta}{z} b_4 \tag{B.33}$$

where

$$b_3 = -k_0^2K_{12}(\frac{K_{33}}{K_{11}} + \frac{K_{12}}{K_{11}} + 1) - \sqrt{b_1} \tag{B.34}$$

$$b_4 = 2(\frac{K_{33}}{K_{11}} + 1) + \frac{1}{2} \frac{b_2}{\sqrt{b_1}} \tag{B.35}$$

The other transverse wave number squared, λ_1^2, can be found by this same procedure. However, $\lambda_1^2 = 0$ for $k = k_I$ by definition, so the equivalent to Equation B.33 is

$$\lambda_1^2 \doteq ik_I \frac{\beta}{z} b_5 \tag{B.36}$$

where

$$b_5 = 2(\frac{K_{33}}{K_{11}} + 1) - \frac{1}{2} \frac{b_2}{\sqrt{b_1}} \tag{B.37}$$

Turning now to the other terms in Equation B.24, it may be noted that

$$f(k) \doteq k_0^2(K_{11} - 1) - 2ik_I \frac{\beta}{z} \tag{B.38}$$

Therefore the bracketed ratio in Equation B.24 is

$$\frac{\lambda_1^2 + f(k)}{\lambda_2^2 + f(k)} \doteq \frac{ik_I \frac{\beta}{z} b_5 + k_0^2(K_{11} - 1) - 2ik_I \frac{\beta}{z}}{b_3 + ik_I b_4 \frac{\beta}{z} + k_0^2(K_{11} - 1) - 2ik_I \frac{\beta}{z}} \quad (B.39)$$

Rationalizing and neglecting terms in β^2/z^2, this equation becomes

$$\frac{\lambda_1^2 + f(k)}{\lambda_2^2 + f(k)} \doteq \frac{[k_0^2(K_{11} - 1) + ik_I \frac{\beta}{z}(b_5 - 2)][b_3 + k_0^2(K_{33} - 1) - ik_I \frac{\beta}{z}(b_4 - 2)]}{[b_3 + k_0^2(K_{11} - 1)]^2}$$

$$\doteq \frac{k_0^2(K_{11}-1)[b_3+k_0^2(K_{33}-1)]+ik_I \frac{\beta}{z}[(b_5-2)(b_3+k_0^2(K_{33}-1))-(b_4-2)k_0^2(K_{11}-1)]}{[b_3 + k_0^2(K_{11} - 1)]^2}$$

$$= b_6 + ik_I \frac{\beta}{z} b_7 \quad (B.40)$$

The integral of Equation 4.3.2 may now be written as

$$I_I \doteq - \frac{i}{z} \frac{V_0}{2\pi} e^{-ik_I z} \int_0^\infty [(-i \tfrac{\pi}{2}) a^2 (b_3 + ik_I \frac{\beta}{z} b_4)(b_6 + ik_I \frac{\beta}{z} b_7) . \log(\frac{\gamma\lambda_2 a}{2}) 2J_0(\lambda_1 r)] e^{-\beta}\, d\beta$$

$$\doteq - \frac{V_0}{2z} e^{-k_I z} \int_0^\infty a^2 (b_3 b_6 + ik_I \frac{\beta}{z}[b_4 b_6 + b_3 b_7]) \log(\frac{\gamma\lambda_2 a}{2}) J_0(\lambda_1 r) e^{-\beta} d\beta \quad (B.41)$$

In this integral, $\log(\gamma\lambda_2 a/2)$ and $J_0(\lambda_1 r)$ are slowly varying functions of β compared with the β and $\exp(-\beta)$ factors, at least for the small arguments which have been assumed. Therefore it is possible to find some value of β between zero and infinity for which these functions take on an "average" value. This "average" value can then be used to represent these functions over the range of β from which most of the contribution to the integral arises. That is, the range of integration can be split into two ranges, $0 \leq \beta < K$ and $K \leq \beta \leq \infty$ where K has to be taken large enough to make

$$\left| \int_K^\infty \right| \ll \left| \int_0^K \right| \tag{B.42}$$

Some β in the range $0 \leq \beta < K$ will then give a suitable average value for $\log(\gamma\lambda_2 a/2)$ and $J_0(\lambda_1 r)$. To be able to compare the results for this anisotropic formulation to the isotropic antenna problem (Northover, 1958) the same choice of $\beta = 1$ will be made. This is a reasonable choice because it evaluates these slowly varying functions where the multiplying factor is approximately at its 1/e value, and it would appear that approximately half the value of the integral would come from $0 \leq \beta < 1$ and the other half from $\beta > 1$. For more rigorous arguments, one is referred to the original paper (Northover, 1958). Equation B.41 thus becomes

$$I_I \doteq - \frac{V_0 a^2}{2z} e^{-ik_I z} \log\left(\frac{\gamma\lambda_2' a}{2}\right) J_0(\lambda_1' r) \int_0^K \left[b_3 b_6 + ik_I \frac{\beta}{z}(b_4 b_6 + b_3 b_7)\right] e^{-\beta} d\beta \tag{B.43}$$

where

$$\lambda_2' = \sqrt{b_3 + ik_I \frac{b_4}{z}} \tag{B.44}$$

$$\lambda_1' = \sqrt{ik_I \frac{b_5}{z}} \tag{B.45}$$

Thus

$$I_I \doteq - \frac{V_0 a^2}{2z} e^{-ik_I z} \log\left(\frac{\gamma\lambda_2' a}{2}\right) J_0(\lambda_1' r) \left[-b_3 b_6 - i \frac{k_I}{z}(b_4 b_5 + b_3 b_7)(\beta + 1)\right] e^{-\beta} \Bigg|_0^K \tag{B.46}$$

Since K is not small, this may be further approximated as (Northover, 1958)

$$I_I \doteq - \frac{V_0 a^2}{2z} e^{-ik_I z} \log(\frac{\gamma\lambda_2' a}{2}) J_0(\lambda_1' r)[b_3 b_6 + \frac{ik_I}{z}(b_4 b_6 + b_3 b_7)]$$

(B.47)

To evaluate I_1, the integral around the contour C_1, the same procedure is followed except that the transverse wave numbers on the two sides of the branch cut are now related as

$$\lambda_{1L} = - \lambda_{1R}, \quad \lambda_{2L} = - \lambda_{2R} \qquad (B.48)$$

The denominators of the integral Equation 4.3.2 are approximately given, when the radius a is small enough that the Hankel function ratio involving λ_1 is much smaller than the inverse ratio involving λ_2, as

$$D_R \doteq \left[- \frac{1}{\lambda_2 a \log(\frac{\gamma\lambda_2 a}{2})} + \frac{i}{\frac{2}{\pi} \lambda_2 a [\log(\frac{\gamma\lambda_2 a}{2})]^2} \right] \frac{[\lambda_2^2 + f(k)]}{\lambda_2} \qquad (B.49)$$

$$D_L \doteq \left[\frac{1}{\lambda_2 a \log(\frac{\gamma\lambda_2 a}{2})} - \frac{3i}{\frac{2}{\pi} \lambda_2 a [\log(\frac{\gamma\lambda_2 a}{2})]^2} \right] \frac{[\lambda_2^2 + f(k)]}{-\lambda_2} \qquad (B.50)$$

For a quite small, $D_R \doteq D_L$, as was the case in the previous integral. The numerator terms involving λ_1 may be subtracted one from the other.

$$N_{R1} - N_{L1} = \left[\frac{H_0^{(2)}(\lambda_1 r)}{H_1^{(2)}(\lambda_1 a)} + \frac{H_0^{(2)}(-\lambda_1 r)}{H_1^{(2)}(-\lambda_1 a)} \right] \frac{[\lambda_1^2 + f(k)]}{\lambda_1}$$

$$\doteq \left[\frac{J_0(\lambda_1 r) - iN_0(\lambda_1 r) - J_0(\lambda_1 r) + iN_0(\lambda_1 r) - 2J_0(\lambda_1 r)}{\frac{2i}{\pi\lambda_1 a}} \right] \frac{[\lambda_1^2 + f(k)]}{\lambda_1}$$

$$= i\pi a J_0(\lambda_1 r)[\lambda_1^2 + f(k)] \qquad (B.51)$$

The numerator term involving λ_2 becomes

$$N_{R2} - N_{L2} = \frac{r}{a}\left[\frac{H_1^{(2)}(\lambda_2 r)}{H_0^{(2)}(\lambda_2 a)} + \frac{H_1^{(2)}(-\lambda_2 r)}{H_0^{(2)}(-\lambda_2 a)}\right]\frac{[\lambda_2^2 + f(k)]}{\lambda_2}$$

$$\doteq \frac{r}{a}\,\frac{1}{\frac{2}{\pi}\log(\frac{\gamma\lambda_2 a}{2})}[J_1(\lambda_2 r)-iN_1(\lambda_2 r)-J_1(\lambda_2 r)+iN_1(\lambda_2 r)-2J_1(\lambda_2 r)]\frac{[\lambda_2^2+f(k)]}{\lambda_2}$$

$$= -\frac{\pi r J_1(\lambda_2 r)}{a\log(\frac{\gamma\lambda_2 a}{2})}\,\frac{[\lambda_2^2 + f\ (k)]}{\lambda_2} \tag{B.52}$$

Then,

$$\frac{N_R}{D_R} - \frac{N_L}{D_L} \doteq \lambda_2\pi r J_1(\lambda_2 r) - i\pi a^2\lambda_2^2\log(\frac{\gamma\lambda_2 a}{2})J_0(\lambda_1 r)\,\frac{[\lambda_1^2 + f(k)]}{[\lambda_2^2 + f(k)]} \tag{B.53}$$

Evaluating Equation 4.3.2 for the first term of Equation B.53 for small $\lambda_2 r$ gives

$$I_1(\text{first term}) = -\frac{iV_0}{2\pi z}\,e^{-ik_1 z}\int_0^{\infty} \pi r\lambda_2 J_1(\lambda_2 r)e^{-\beta}d\beta$$

$$\doteq -\frac{iV_0}{2z}\,e^{-ik_1 z}\,\lambda_2'^2 r J_1(\lambda_2' r) \tag{B.54}$$

where λ_2' indicates that λ_2 has been evaluated for $k = k_1 - i(1/z)$. The other part of Equation B.53 is identical in form to Equation B.24 so the portion of the integral I_1 due to this part will be the same as I_I in Equation B.47 except that k_I is replaced by k_1 everywhere.

Therefore,

$$I_1 \doteq -\frac{V_0}{2z} e^{-ik_1 z} [i\lambda_2' r J_1(\lambda_2' r)/2 + g_1 a^2 J_0(\lambda_1 r)] \qquad \text{(B.55)}$$

where g_1 is the appropriate function of a, z, k_1, and the plasma parameters.

VITA

Gary Lee Johnson

Candidate for the Degree of

Doctor of Philosophy

Thesis: THE DISTRIBUTION OF CURRENT ON AN INFINITE ANTENNA IN AN ANISOTROPIC INCOMPRESSIBLE PLASMA

Major Field: Engineering

Biographical:

Personal Data: Born near Osage City, Kansas, November 20, 1938, the son of Elmer L. and Lelia P. Johnson.

Education: Attended grade school in Osage City, Kansas; graduated from Osage City High School in 1957; received the Bachelor of Science degree in Electrical Engineering from Kansas State University in August, 1961; received the Master of Science degree in Electrical Engineering from Kansas State University in June, 1963; attended the University of Washington, 1963-1964; completed requirements for the Doctor of Philosophy degree in July, 1966.

Professional experience: Employed by Boeing Airplane Company, Wichita, Kansas, during the summer of 1960. Employed by Boeing Company, Seattle, Washington, during the summer of 1963. Employed by the School of Electrical Engineering of Oklahoma State University since April, 1964, as a graduate assistant.

Professional and honorary organizations: Member of the Institute of Electrical and Electronic Engineers, Phi Kappa Phi, Pi Mu Epsilon, Eta Kappa Nu, and Phi Eta Sigma.