A STUDY OF INVERSE ARC MAPS AND AN EXAMPLE OF A SPECIAL DECOMPOSITION OF THE 2-CELL INTO NONLOCALLY CONNECTED CONTINUA

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PREFACE

This paper will be concerned with results related to inverse arc maps and a decomposition of continua that contain a topological 2-cell. Chapter I is an introductory chapter giving the definition of the above mentioned maps and definitions of related terms. The material of Chapter II is concerned with revealing some fundamental properties of an inverse arc map. This chapter presents some sufficient conditions on a space X for a map f defined on X to be an inverse arc map. A well known factorization theorem by G. T. Whyburn is extended for the inverse arc map. It is shown that some properties of X and subspaces of X are determined if f is an inverse arc map defined on X. Chapter III extends the notion of an inverse arc map and shows some properties of indecomposable continua and their relation to continuous and monotonic maps.

In Chapter IV a decomposition of the closed 2-cell into nonlocally connected continua is established. In conjunction with this a continuous and monotonic inverse arc map f is determined with domain the closed 2-cell, M, and range an arc, L, such that if $y \in L$ then $f^{-1}(y)$ is nonlocally connected. Also, Chapter IV reveals a characterization of an inverse arc map. As further results to Chapter IV, Chapter V establishes more general results for decompositions of general spaces. This chapter shows that every 2-manifold, M, can be represented as the union of uncountably many mutually exclusive nonlocally connected continua and there exists a continuous and monotonic inverse arc map, f, defined from M onto any arc L such that if $y \in L$, then $f^{-1}(y)$ is a nonlocally connected subcontinuum of M. The summary of all results is given in Chapter VI.

Numbers in brackets refer to the bibliography at the end of the paper. For example, [5-p, 127] refers to bibliography reference number five, page 127.

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CHAPTER I

INTRODUCTION

This paper will be devoted to certain results in connection with inverse arc maps and decompositions of continua that contain topological 2-cells. The following definition of an inverse arc map is stated.

Definition 1.1: If X is a space and f is a map such that $f(X) = Y$, then f is an inverse arc map if and only if for each arc L in Y there exists an arc L_1 in X such that $f(L_1) = L$.

All spaces in this paper will be assumed to be Moore spaces satisfying Axiom 0 and Axiom 1. If A is a subset of the space S then the notation F(A) will be used to represent the set of boundary points of the set A in S. If L is an arc from the point a to the boint b then L is denoted by the arc (ab). The results in this paper rely heavily upon material in Whyburn [5] and in Moore [3].

In Chapter II some fundamental properties of an inverse arc map are given. Among these are some sufficient conditions for a continuous and monotonic map to be an inverse arc map. It is well known that a continuous function f can be factored into a composite map $f = f_2f_1$ where both factors are continuous, f_1 is monotone, and f_2 is light, [5-p, 141]. A theorem in Chapter II is proved showing that if f is a continuous inverse arc map then f_2 in the above composite is also an inverse arc map.

Chapter III assumes X and Y are spaces and that a function f such that f(X) = Y is a continuous and monotonic map. Attention is given to studying the indecomposability of subsets of X if Y is indecomposable, and the indecomposability of Y if X is indecomposable. Some theorems consider the case Y = I where I is the unit interval. It is shown that if f(M) = I and M is indecomposable then there are at most two points of I, x_1, x_2, such that $f^{-1}(x_1)$ and $f^{-1}(x_2)$ are subcontinuua of M. Some extensions to the factorization theorem, [5-p, 141], mentioned in Chapter II are derived. In connection with P. M. Swingle's definition of finished sum of continua, [4], some theorems are proved concerning finished sums of indecomposable continua.

Chapter IV is devoted to showing a decomposition of the closed 2-cell into the union of uncountably many nonlocally connected continua. Once this decomposition is obtained then a map f is defined from the closed 2-cell onto I such that f is a continuous and monotonic inverse arc map. Thus, [5-p, 127] implies that this decomposition of the closed 2-cell is an upper semi-continuous decomposition of the closed 2-cell into nonlocally connected continua. At the conclusion of Chapter IV a characterization of an inverse arc map is given.

The contents of Chapter V are directed, mainly, toward showing some general consequences of Chapter IV. In particular it is shown that if M is a 2-manifold, as defined in [6-p, 95], then M can be represented as the union of uncountably many mutually exclusive nonlocally connected continua. For this decomposition of M there exists a continuous and monotonic inverse arc map, f, defined from M onto any arc E such that if $y \in E$ then $f^{-1}(y)$ is one of these nonlocally connected subcontinua of M.

The definitions which are pertinent to this paper are as follows:

Definition 1.2: [3-p, 379] A point set M is said to be aposyndetic at the point p if and only if p belongs to M and for each point x of M distinct from p there exists a domain with respect to M which contains p and is a subset of a connected subset of M - x which is closed relatively to M.

Definition 1.3: A continuum M is a hereditarily indecomposable continuum if and only if every subcontinuum of M is an indecomposable continuum.

Definition 1.4: A point set M is hereditarily locally connected if and only if every subcontinuum of M is locally connected.

Definition 1.5: A continuum M is said to be unicoherent if and only if it is true that if it is the sum of two continua their common part is a continuum. A continuum M is said to be hereditarily unicoherent if and only if every subcontinuum of it is unicoherent.

Suppose G is an upper semi-continuous collection of mutually exclusive closed and compact point sets filling up a space S. If the elements of G are called "points" and every region with respect to G is called a "region", then [3-p, 280] implies that with this definition of point and region, Axioms 0 and 1 of Moore are satisfied.

Definition 1.6: Let the space referred to in the preceeding paragraph be called S' and referred to as the hyperspace of S associated with the upper semi-continuous collection G.

Definition 1.7: The point set M is locally peripherally connected at the point p if and only if for every region R relative to M containing p there exists a region R_1 relative to M such that $p \in R_1 \subseteq \bar{R}_1 \subseteq R$ and $F(R_1)$ is connected.

CHAPTER II

SOME PROPERTIES OF INVERSE ARC MAPS

Let f be a continuous and monotonic map from a compact space X onto a space Y. This chapter will be concerned with:

(1) conditions that can be placed on the space X so that f is an inverse arc map,

(2) properties induced on $f^{-1}(L)$ if L is an arc in Y, when f is an inverse arc map, and

(3) a factorization theorem related to a factorization theorem by G. T. Whyburn.

An important comment is that since X is compact and since all spaces considered are assumed to satisfy Moore's Axioms 0 and 1 unless otherwise stated, it then follows that a continuous map, f, from X into Y is necessarily closed. This fact is used repeatedly throughout this thesis.

It is natural to expect that some type of local connectedness on the space X would be sufficient for f to be an inverse arc map. However, an effort was made to study the problem under weaker conditions than local connectedness by assuming X to be locally compact, locally peripherally connected, and connected. Theorem 2.1 shows that these conditions on X imply that X is locally connected and therefore nothing is gained by assuming them.

Theorem 2.1: If S is a locally compact, locally peripherally connected, and connected space, then S is locally connected.

Proof: Suppose there exists a point $p \in S$ such that S is not locally connected at p. Let U be any domain containing p. By Axiom 1 of Moore and the hypothesis of the theorem there exists a domain D such that $p \in D \subseteq \bar{D} \subseteq U$ such that D is compact while F(D) is connected. Let C be a component of D. The reference [3-p, 18], implies that F(D) contains a limit point of C.

Let C_p be the component of D that contains p. The point set C_p is not a domain since S is not locally connected at p. Therefore there exists a sequence of distinct points $\{p_n\}$ converging sequentially to p where $p_n \in C_n$, $n = 1,2,\ldots$, and $\{C_n\}$ is a collection of mutually exclusive components of D. Again by Axiom 1 of Moore and the hypothesis of the theorem we know there exists a domain D_1 such that $p \in D_1 \subseteq \bar{D}_1 \subset D$ and $\bar{D}_1$ is compact while $F(D_1)$ is connected. Since $\{p_n\}$ converges sequentially to p, there exists an integer N such that for every $n > N$, $p_n \in D_1$. Let C^1_p be the component of D_1 containing p and $C^1_{p_n}$ the component of D_1 containing p_n for $n > N$. Note that $C^1_p \subseteq C_p$. From the preceeding paragraph it is known that $C^1_{p_n}$ has a limit point in $F(D_1)$. Now consider the set $X = F(D_1) \cup [\bigcup_{n>N} C^1_{p_n}] \cup C^1_p$. The point set X is a connected point set since it is the union of a collection of connected point sets having a point in common. Therefore $X \subseteq C_p$ which contradicts the assumption that $p_n \notin C_p$, $n > N$. Thus the theorem is proved.

An interesting question in connection with the previous theorem is the following: If a space S is connected and locally peripherally connected must it also be locally compact? The following example shows the answer to this question to be in the negative.

Example 1: Consider the Moore space, X, satisfying Moore's Axioms 0 and 1, with the sequence of coverings of regions, $\{G_n\}$, where for each positive integer, n, $G_n = \{S(\overline{x},\epsilon): \epsilon < 1/n, \overline{x} \in E^2\}$. Now let $K = \{\overline{x}_1, \overline{x}_2, \dots\}$ be a sequence of all points of E^2 whose coordinates are both rational. Let x consist of the set of points in E^2.

Let $Y = \{\overline{x}: \overline{x} \in E^2$ and at least one of the coordinates of $\overline{x}$ is irrational$\}$. Now define the sequence of coverings of Y, $\{G_n'\}$, where for each positive integer, n,

$$G_n' = \left\{S(\overline{x},\epsilon): S(\overline{x},\epsilon) \in G_n \text{ and } \left[\overline{S(\overline{x},\epsilon)} \cap \{\bigcup_{i=1}^{n} \overline{x}_i\}\right] = \emptyset\right\}.$$

The importance of defining the sequence of coverings, $\{G_n'\}$, as above, is in showing that the space Y will satisfy Moore's Axioms 0 and 1, especially Axiom 1 part 4. The space Y is connected since in E^2 between any two points there exists an arc such that every point on this arc has at least one coordinate which is irrational. Thus, this arc lies in Y and Y is therefore arcwise connected which implies that Y is connected.

The space Y is locally peripherally connected since inside of every sphere in E^2 lies a rectangle such that any point on this rectangle has at least one irrational coordinate. This rectangle is therefore in Y and it follows that Y is locally peripherally connected.

The space Y is not locally compact since given any $S(\overline{y},\epsilon)$ in Y, there exists $\overline{x} \in K \ni \overline{x}$ will be a limit point of $S(\overline{y},\epsilon)$ in X. Thus, $S(\overline{y},\epsilon)$ in Y will contain an infinite subset with no limit point in Y and therefore, the space Y is not locally compact.

Theorem 2.2 gives a sufficient condition that a function be an inverse arc map.

Theorem 2.2: If X is a hereditarily locally connected space and f is a continuous and monotonic map such that $f(X) = Y$, then f is an inverse arc map.

Proof: Let L be any arc from a' to b' in Y and consider $f^{-1}(L)$. The point set $f^{-1}(L)$ is a subcontinuum of X since f is monotonic. Let $a \in f^{-1}(a')$ and $b \in f^{-1}(b')$. The subcontinuum, $f^{-1}(L)$, is locally connected since S is hereditarily locally connected. Therefore, [3-p, 84] implies $f^{-1}(L)$ is arcwise connected. Let L_1 be an arc from a to b in $f^{-1}(L)$. The point set $f(L_1)$ contains a' and b' since a, $b \in L_1$ and $f(L_1) \subseteq L$ since $L_1 \subseteq f^{-1}(L)$. The point set $f(L_1)$ is a subcontinuum of L since f is a closed continuous map. Therefore, $f(L_1) = L$ since L is irreducible with respect to being a continuum containing both a' and b'. Thus the theorem is proved.

Theorem 2.3: If X is a hereditarily locally connected space and f is a continuous and monotonic map such that $f(X) = Y$, then if J is a simple closed curve in Y there exists a simple closed curve $J_1 \subseteq X$ such that $f(J_1) = J$.

Proof: Let J be a simple closed curve in Y and a', b' $\in$ J such that $a' \neq b'$. The reference, [3-p, 44], implies $J = L_1' \cup L_2'$ where L_i', i = 1, 2, is an arc from a' to b' such that $L_1' \cap L_2' = \{a', b'\}$. Theorem 2.2 implies there exists arcs $L_1, L_2 \subseteq X$ from the point a to the point b such that:

(1) $f(L_1) = L_1'$ and $f(L_2) = L_2'$,

(2) $f(a) = a'$ and $f(b) = b'$, and

(3) $L_1 \cap L_2 \subseteq f^{-1}(a') \cap f^{-1}(b')$.

Let $c_1 \in L_1 - f^{-1}(a') \cup f^{-1}(b')$ and $c_2 \in L_2 - f^{-1}(a') \cup f^{-1}(b')$. Consider the subarc (c_1a) of L_1 and the subarc (c_2a) of L_2. Since (c_1a) and (ac_2) are intersecting arcs there exists a subarc $(c_1c_2)_a$ of $(c_1a) \cup (ac_2)$ from c_1 to c_2. The point set $f^{-1}(a') \cap (c_1c_2)_a \neq \emptyset$ since $(c_1c_2)_a$ is connected. Similarly there exists an arc $(c_1c_2)_b$ such that $f(b') \cap (c_1c_2)_b \neq \emptyset$. The point set $(c_1c_2)_a \cup (c_1c_2)_b = J_1$ is a simple closed curve since J_1 is the union of two arcs having only their end points in common. The reference, [5-p, 165], implies that simple closed curves are invariant under monotone maps. Therefore $f(J_1) = J$.

The following example is given to show that the hypothesis of Theorem 2.2 is not necessary for the map f to be an inverse arc map. In this example f is an inverse arc map, continuous and monotonic, but the space is not hereditarily locally connected.

Example 2: Let the space X be a subspace of E^2 (Figure 2.1) such that X is composed of the union of the points in the closure of $\{(x,y): y = \sin 1/x, 0 < x < 1/\pi\}$ and the points in the closed interval $K = \{(x,0): 1/\pi \leq x \leq 2\}$.

Define $f: X \to X$ such that:

$$f((x,y)) = \begin{cases} (1/\pi, 0) \text{ if } (x,y) \in X - K \\ \\ (x,0) \text{ if } (x,y) \in K \end{cases}$$

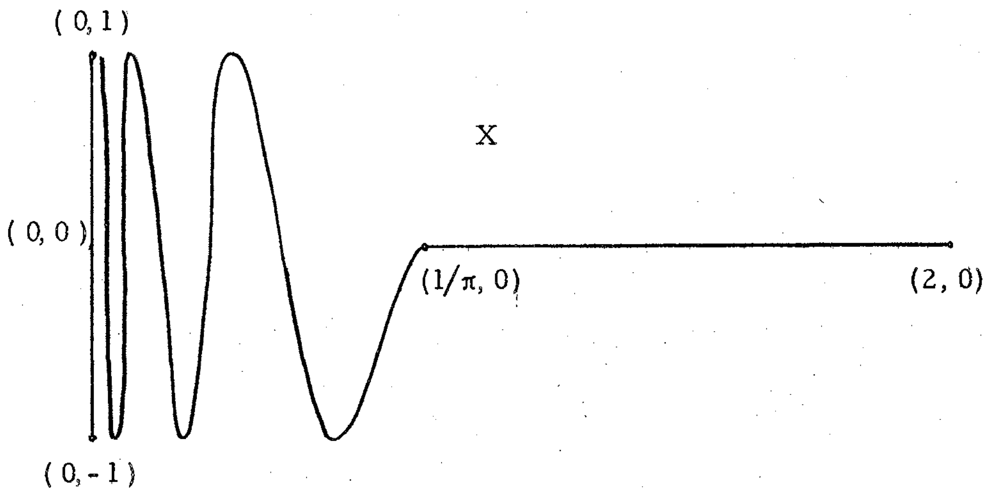

Figure 2.1

It can easily be seen that the map, as defined, is a continuous and monotonic map from X into X which is an inverse arc map but X is not hereditarily locally connected.

The example below points out that there exist spaces X and Y and a continuous and monotonic map f such that $f(X) = Y$, and that f is not an inverse arc map.

Example 3: Let the space X be the same as the space X in Example 2. Let $Y = \{(x,0) : 0 \leq x \leq 2\}$. Let f be the map of X onto Y such that $f((x,y)) = (x,0)$. It is easily seen that f is a continuous and monotonic map of X onto Y but one observes that for the arc $L = \{(x,0) : 0 \leq x \leq 1\}$, X contains no arc L_1 such that $f(L_1) = L$. Therefore f is not an inverse arc map.

A trivial result following directly from the definition of a function is now stated. If f is a monotone continuous map from a compact space X onto a space Y and if L is an arc in Y and there exists an arc L_1 in X such that $f(L_1) = L$, it will follow that for every $y \in L$, $f^{-1}(y) \cap L_1 \neq \emptyset$.

Theorems 2.4, 2.5, 2.6, 2.7, and 2.8 that follow show results of properties imposed on subspaces of the space X when f is a continuous monotonic inverse arc map.

<u>Theorem 2.4</u>: If X is a compact space, f is a continuous and monotonic map such that $f(X) = Y$, and L is an arc in Y such that there exists an arc L_1 in X where $f(L_1) = L$ then for every $y \in L$, there exists $p \in f^{-1}(y) \cap L_1$ such that if R is any region containing p there will exist a point $y_1 \in L$, $y_1 \neq y$, such that $R \cap f^{-1}(x) \neq \emptyset$ for every x in the arc (yy_1).

<u>Proof</u>: Let L be an arc in Y from a' to b' and y any point of L and L_1 an arc in X from a to b where $a \in f^{-1}(a')$ and $b \in f^{-1}(b')$. Consider $f^{-1}(y)$. Then $L_1 \cap f^{-1}(y) \neq \emptyset$. Let p be the last point of intersection of $f^{-1}(y)$ and L_1 on L_1 in the order from a to b. Consider any region R containing p. Now select a region D such that $p \in \overline{D} \subseteq R$ where D does not contain b. One should now focus his attention on the subarc (pb) of the arc (ab).

Suppose F(D) does not intersect the arc (pb). This implies that $(pb) = \{(pb) \cap D\} \cup \{(pb) - \overline{D}\}$ separate which contradicts (pb) being connected. Therefore, let p_1 be the first point of (pb) in the intersection of F(D) and (pb). Then the subarc (pp_1) of L_1 is contained entirely in R since $p \in \overline{D} \subseteq R$.

Select $y_1 \in L$ such that $f(p_1) = y_1$. The reference [5-p, 165], implies that arcs are invariant under monotonic maps. Therefore $f(pp_1) = (yy_1)$. Then, if $x \in (yy_1)$ then $f^{-1}(x) \cap (pp_1) \neq \emptyset$ and therefore, since $(pp_1) \subseteq R$, it is true that $f^{-1}(x) \cap R \neq \emptyset$.

Theorem 2.5: If X is a compact space, f is a continuous and monotonic map such that $f(X) = Y$, L is an arc in Y such that there exists an arc, L_1, in X such that $f(L_1) = L$, and $L - y = A \cup B$ separate for each $y \in L$ then the set, H, of all limit points of $f^{-1}(A)$ in $f^{-1}(y)$ is a continuum.

Proof: Let L be an arc in Y from a to b and $y \in L$. Theorem 2.4 implies that there exists a point $p \in f^{-1}(y)$ such that for each region R containing p there exists a point $y_1 \in L \cap A$ such that $R \cap f^{-1}(x) \neq \emptyset$ for every x in the arc (yy_1). Let $h \in H$ and consider a sequence of regions, $\{R_n\}$, closing down on h. Therefore, there exists a point $y_1 \in L \cap A$ such that $R_1 \cap f^{-1}(x) \neq \emptyset$ for every x in the arc (yy_1). Pick a point $a_1 \in f^{-1}(y_1)$. Since $f^{-1}(y_1)$ is closed there exists a positive integer n_2 such that $R_{n_2} \cap f^{-1}(y_1) = \emptyset$. Pick a point $a_2 \in R_{n_2}$. In general, if a_{i-1} has been defined such that $a_{i-1} \in f^{-1}(x)$ for some $x \in (yy_1)$, then there will exist a positive integer n_i such that $R_{n_i} \cap f^{-1}(x) = \emptyset$. Pick a point $a_i \in R_{n_i}$. In this way a sequence of distinct points, $\{a_n\}$, is obtained which is converging sequentially to h. With each a_n is associated $f^{-1}(x) = f^{-1}(f(a_n)) = M_n$. For each positive integer, n, M_n is a compact continuum since f is monotonic. Also for each positive integer, n, the point $a_n \in M_n$. The reference, [3-p, 23], implies that the limiting set M_h of the sequence sets $\{M_n\}$, is a continuum.

Let R be a region containing p and suppose $R \cap \{M_n\}^* = \emptyset$. Utilizing Theorem 2.4 as mentioned in the preceeding paragraph it is known that $R \cap f^{-1}(x) \neq \emptyset$ for every $x \in (yy_1)$, where $y_1 \in A$. Thus for every positive integer, n, it follows that $M_n = f^{-1}(x)$ for some $x \in (y_1a)$. Since $y \neq y_1$, there exists a region D containing y such that $(y_1a) \cap D = \emptyset$. Now $f^{-1}(D)$ is open in X and $h \in f^{-1}(y) \subseteq f^{-1}(D)$ while $f^{-1}(D) \cap \{M_n\}^* = \emptyset$ since $D \cap (y_1a) = \emptyset$. This contradicts h being an element of the limit set of $\{M_n\}$ and thus $R \cap \{M_n\}^* \neq \emptyset$ and $p \in M_n$.

The set, $\bigcup_{h \in H} M_h$ is a union of continua each of which contains p and therefore $\bigcup_{h \in H} M_h$ is connected. Since $H = \bigcup_{h \in H} M_h$ it is true that H is connected. Any point of $f^{-1}(y)$ that is a limit point of H must necessarily be a point of H and therefore H is also closed. Thus H is a continuum.

Theorem 2.6: If X is a compact space and f is a continuous and monotonic map such that:

(1) $f(X) = Y$,

(2) L is an arc in Y from a' to b',

(3) there exists an arc L_1 from a to b in X such that $f(L_1) = L$,

(4) every subcollection of $\{f^{-1}(y)\}$, as y varies over L, is semi-closed in $f^{-1}(L)$,

(5) p is the last point of intersection of L_1 from a to b with $f^{-1}(y)$ for a given $y \in L$, and

(6) $L - y = A \cup B$ separate,

then $p \cup f^{-1}(B)$ is locally connected at p.

Proof: It is understood that $f(a) = a'$ and $f(b) = b'$. Consider the point p and the set $p \cup f^{-1}(B)$. Let D be any domain relative to $p \cup f^{-1}(B)$ containing p and then pick a region R_1 relative to $p \cup f^{-1}(B)$ such that $p \in \overline{R}_1 \subseteq D$. Let p_1 be the first point of intersection of the arc (pb) with $F(R_1)$ from p to b. Consider $f(p_1)$ and the separation of L such that $L - f(p_1) = A'_y \cup B'$ separate, $y \in A'_y$. It follows that $H = f^{-1}(A'_y) \cap (p \cup f^{-1}(B))$ is open relative to $p \cup f^{-1}(B)$ since f is continuous. The half open arc $\{(pp_1) - p_1\}$ is contained in H. The set $R = R_1 \cap H$ is a region relative to $p \cup f^{-1}(B)$ such that $p \in \overline{R} \subseteq D$ and $R \subseteq f^{-1}\{(yf(p_1)) - f(p_1)\}$. This implies that $R \cap f^{-1}(x) \neq \emptyset$ for each $x \in$ arc $\{(yf(p_1)) - f(p_1)\}$.

If there exists no $x \in$ arc $\{(yf(p_1)) - f(p_1)\}$ such that $f^{-1}(x) - R \neq \emptyset$ then $R = f^{-1}\{(yf(p_1)) - f(p_1)\} \cap \{p \cup f^{-1}(B)\}$ which is connected since f is a monotonic map, and therefore $p \cup f^{-1}(B)$ is locally connected at p.

On the other hand if one supposes that there exists a point $x \in \{(yf(p_1)) - f(p_1)\}$ such that $f^{-1}(x) - R \neq \emptyset$ it naturally follows that there is a finite or an infinite number of such x. If there is only a finite number of such x then let x_1 be the first one from y to $f(p_1)$ on the arc $(yf(p_1))$. It is seen that $\{(yf(p_1)) - x_1\} = A'_y \cup B''$ separate, $y \in A'_y$, and $H_1 = f^{-1}(A'_y) \cap \{p \cup f^{-1}(B)\}$ is a region relative to $p \cup f^{-1}(B)$ which is connected since f is a monotone map and $p \in H_1 \subseteq R \subseteq D$. From this it follows that $p \cup f^{-1}(B)$ is locally connected at p.

Otherwise, if there is an infinite number of $x \in \{(yf(p_1)) - f(p_1)\}$ such that $f^{-1}(x) - R \neq \emptyset$ then let $\{a_n\}$ be a sequence of points of L,

$\{a_n\}^* \subseteq B$, converging sequentially to y. Theorem 2.4 implies there exists a positive integer, N, such that for every $n > N$ it follows that $f^{-1}(a_n) \cap R \neq \emptyset$. If there is an $x \in$ arc $(a_N y)$ such that $f^{-1}(x) - R \neq \emptyset$ pick one such x and name it b_1. Let a_{N+r_1} be the first point of $\left[\{a_n\}^* \cap (b_1 y)\right]$ from b_1 to y. Again if there is an $x \in (a_{N+r_1} y)$ such that $f^{-1}(x) - R \neq \emptyset$ pick one such x and name it b_2. In general if b_n has been defined let a_{N+r_n} be the first point of $\left[\{a_n\}^* \cap (b_n y)\right]$ from b_n to y. If there is an $x \in (a_{N+r_n} y)$ such that $f^{-1}(x) - R \neq \emptyset$ pick one such x and name it b_{n+1}. By this process we get a sequence of distinct points $\{b_n\}$ which also converges sequentially to y.

It is important to note that if at any time in the above construction there exists no such x, say at the a_{N+r_i} point, then $L - a_{N+r_{i+1}} = M_y \cup N$ separate, $y \in M_y$. This implies that $f^{-1}(M_y) \cap (p \cup f^{-1}(B)) = H_2$ is a region relative to $p \cup f^{-1}(B)$ and $p \in H_2 \subseteq R \subseteq D$. The region H_2 is connected since f is monotonic and therefore the theorem is true.

The sequence of points, $\{b_n\}$, is now reconsidered in conjunction with the sequence of sets, $\left\{f^{-1}(b_n)\right\}$. The reference [5 - p, 11] states that there exists a subsequence, $\left\{f^{-1}(b_{n_i})\right\}$, of $\left\{f^{-1}(b_n)\right\}$ which is convergent. Let $x_i \in \left\{f^{-1}(b_{n_i}) - R\right\}$. Since $f^{-1}(L)$ is compact it is clear that some subsequence of $\{x_i\}$, say $\{x_{ij}\}$, converges sequentially to some point x. Obviously this point, x, is in $f^{-1}(y)$ and $x \notin R$, so $x \neq p$.

Consider the convergent sequence of sets $\left\{f^{-1}(b_{n_{ij}})\right\}$. The hypothesis implies that the collection of sets that make up the sequence, $\left\{f^{-1}(b_{n_{ij}})\right\}$, is semi-closed. The limit set of the sequence of sets, $\left\{f^{-1}(b_{n_{ij}})\right\}$, contains at least the points x and p. Since neither x nor p are elements of $\left\{f^{-1}(b_{n_{ij}})\right\}^*$, the semi-closed property of this collection of sets is contradicted. Therefore, there does not exist an infinite number x in $\left\{(yf(p_1)) - f(p_1)\right\}$ such that $f^{-1}(x) - R \neq \emptyset$. Since all other possible cases have already yielded $p \cup f^{-1}(B)$ locally connected at p the theorem is therefore proved.

Theorem 2.7: If X is a compact space and f is a continuous and monotonic map such that:

(1) $f(X) = Y$

(2) L is an arc from a' to b' in Y,

(3) L_1 is an arc from a to b in X,

(4) $f(a) = a'$, $f(b) = b'$,

(5) $f(L_1) = L$,

(6) y and y_1 are points of L such that $L - y = A \cup B$ separate and y_1 is between y and b', and

(7) p is the last point of L_1 from a to b in $f^{-1}(y)$,

then the point set $p \cup f^{-1}\left\{(yy_1) - y\right\}$ is aposyndetic at p.

Proof: Let $x \in p \cup f^{-1}\left\{(yy_1) - y\right\}$, $x \neq p$. Theorem 2.4 implies that p is a limit point of $f^{-1}\left\{(yy_1) - y\right\}$. Therefore $p \cup f^{-1}\left\{(yy_1) - y\right\}$ is connected since f is monotonic and p is a limit point of $f^{-1}\left\{(yy_1) - y\right\}$.

Let $y_2 \in \left\{(yy_1) - y\right\}$ such that $f(x) = y_2$. If $y_2 = y_1$ then $f^{-1}\left\{(yy_1) - y_1\right\} \cap \left\{p \cup f^{-1}\{(yy_1) - y\}\right\} = R$ is a connected region relative to $p \cup f^{-1}\left\{(yy_1) - y\right\}$ containing p, since f is monotonic and continuous. However, $x \notin R$. Thus $p \cup f^{-1}\left\{(yy_1) - y\right\}$ is aposyndetic at p.

Similarly, if $y_2 \neq y_1$ then $f(x) = y_2$ where $y_2 \in \left\{(yy_1) - y\right\}$. The point set $\left\{(yy_1) - y_2\right\} = A_y \cup B_{y1}$ separate, $y \in A_y$, $y_1 \in B_{y1}$. It follows that $f^{-1}(A_y) \cap \{p \cup f^{-1}\{(yy_1) - y\}\}$ is a connected region relative to $p \cup f^{-1}\{(yy_1) - y\}$ containing p and not containing x, which again implies $p \cup f^{-1}\{(yy_1) - y\}$ is aposyndetic at p.

Definition 2.1: If M is a subset of a space S, then M is locally arcwise separate at a point $p \in M$ if and only if for each region R relative to M containing p then there exists a region R_1, $p \in R_1 \subseteq R$, and an arc L in S such that $R_1 - L = A \cup B$ separate. If M is locally arcwise separate at each of its points then M is said to be locally arcwise separate.

Theorem 2.8: If X is a compact space and f is a continuous and monotonic map such that:

(1) $f(X) = Y$,

(2) L is an arc in Y from a' to b',

(3) L_1 is an arc in X from a to b such that $f(L_1) = L$, and

(4) $f(a) = a'$, $f(b) = b'$,

then $f^{-1}(L)$ is locally connected or locally arcwise separate at every point of L_1.

Proof: Let $p \in L_1$ such that $p \neq a$, $p \neq b$. Suppose $f^{-1}(L)$ is not locally connected at p. Let D be any domain relative to $f^{-1}(L)$ containing p. Let R be a region such that $p \in R \subseteq \overline{R} \subseteq D$ and $a, b \notin R$. The point set $F(R)$ is not empty since $f^{-1}(L)$ is connected. Let p_1 be the last point of intersection of $F(R)$ on the subarc (ap) of the arc L_1 from a to p. Let p_2 be the first point of intersection of $F(R)$ on the subarc (pb) of the arc L_1 from p to b. Thus the arc (p_1p_2) is obtained such that the open arc $\{(p_1p_2) - p_1 - p_2\} = H \subseteq R$. Since

$f^{-1}(l)$ is not locally connected at p then $R = A \cup B$ separate where, say $H \subseteq A$. The point set $A - H \neq \emptyset$ for if $A - H = \emptyset$, H is a connected region containing p such that $p \in H \subseteq D$ which would contradict $f^{-1}(L)$ being not locally connected at p. Therefore $R - \overline{H} = (A - H) \cup B$ separate and it follows that $f^{-1}(L)$ is locally arcwise separate at p.

If p = a or p = b the argument is similar except the subarc H intersects F(R) in one and only one point.

The following theorem is motivated by a factorization theorem concerning continuous functions proved by G. T. Whyburn, [5-p, 141].

Theorem 2.9: If A is compact, f is a continuous inverse arc map such that $f(A) = B$, and $f(x) = f_2 f_1(x)$ is a factorization of f where f_1 and f_2 are both continuous, then f_2 is an inverse arc map.

Proof: Let L be an arc in B from $\bar{a}$ to $\bar{b}$. Since f is an inverse arc map, there exists an arc $L_1 \subseteq A$ such that $f(L_1) = L$ and $a, b \in L_1$ such that $f(a) = \bar{a}$, $f(b) = \bar{b}$. Now consider $f_1(L_1) \subseteq f_1(A)$ as a space and f_1 restricted to L_1 is a continuous closed map from the locally connected space L_1 onto the closed connected space $f_1(L_1)$. Thus, [1-p, 200] implies $f_1(L_1)$ is locally connected. Therefore, [3-p, 84] implies there exists an arc $L_1' \subseteq f_1(L_1)$ from $a' = f_1(a)$ to $f_1(b) = b'$. Now $f_2(L_1')$ is a subcontinuum of L since f_2 is a continuous closed map, but since L is irreducible with respect to being connected and containing both $\bar{a}$ and $\bar{b}$, it follows that $f_2(L_1') = L$. Therefore, the existence of subarc L_1' of $f_1(A)$ shows that f_2 is an inverse arc map.

The following theorem is an extension of a theorem by G. T. Whyburn.

Theorem 2.10: If A is compact and f is a continuous inverse arc map such that $f(A) = B$, then there exists a unique factorization

$$f(x) = f_2 f_1(x)$$

such that

$$f_1(A) = A'$$

where f_1 is continuous and monotonic and

$$f_2(A') = B$$

where f_2 is a light continuous inverse arc map.

Proof: The reference, [5-p, 141], proves the above stated theorem with the exception of showing f_2 is an inverse arc map. Theorem 2.9 proves that f_2 is an inverse arc map.

CHAPTER III

SOME PROPERTIES OF INDECOMPOSABLE CONTINUA AND THEIR RELATION TO CONTINUOUS AND MONOTONIC MAPS

Let X and Y be topological spaces and f a continuous and monotonic map such that $f(X) = Y$. This chapter will pay considerable attention to the study of two general questions connected with the map f.

(1) If Y is indecomposable what can be said about X?

(2) If X is indecomposable what can be said about Y?

In addition, attention will be given to P. M. Swingle's, [4], definition of a finished sum of a finite set of indecomposable continua.

The first theorem is a simple result concerning a pseudo arc as defined by E. E. Moise, [2].

Theorem 3.1: If L is a pseudo arc constructed from a point a to a point b then the only subcontinuum of L containing both a and b is L.

Proof: Let $Y_1, Y_2, \ldots, Y_i, \ldots$ be the sequence of chains used in the construction of L. Suppose there exists a proper subcontinuum, M, of L such that M contains both a and b. Let $x \in (L - M)$ and $S(x,\delta)$ be a sphere about x such that $S(x,\delta) \cap M = \emptyset$. The definition of the construction of the pseudo arc implies there exists a positive integer, i, such that the diameter of the links of the chain Y_i is $r < 1/i < \delta$. Let y be a link of Y_i such that $x \in y^* \subseteq S(x,\delta)$. Since $Y_i^* - y^* = A \cup B$

separate where $a \in A$, $b \in B$, it follows that $M \subseteq A \cup B$ separate and $M \cap A \neq \emptyset \neq M \cap B$. Therefore, $M = (M \cap A) \cup (M \cap B)$ separate which contradicts M being connected. Thus $M = L$ and the theorem is proved.

Theorem 3.2: If X is compact and f is a continuous and monotonic map such that $f(X) = Y$ where L is an idecomposable continuum which is irreducible about the points $\bar{a}$ and $\bar{b}$, and if $a \in f^{-1}(\bar{a})$, $b \in f^{-1}(\bar{b})$ then there exists an irreducible subcontinuum, L_1, of X with respect to containing both a and b such that $f(L_1) = L$.

Proof: The space $f^{-1}(L) = X$ is a compact continuum since f is continuous and monotonic. The reference, [3-p, 16], implies there exists an irreducible subcontinuum of X with respect to containing both a and b. Since X is compact f is a closed map. Therefore, $f(L_1) = L$ since L is the only subcontinuum of L containing both $\bar{a}$ and $\bar{b}$.

Corollary 3.1: If the same hypotheses as in Theorem 3.2 are assumed except that L is assumed to be a pseudo arc constructed from a point $\bar{a}$ to a point $\bar{b}$ then there exists an irreducible subcontinuum, L_1, of X with respect to being connected and containing both a and b such that $f(L_1) = L$.

Theorem 3.3: If X is compact and f is a continuous and monotonic map such that $f(X) = Y$ where L is an indecomposable continuum, then there exists an indecomposable subcontinuum, L_1, of X such that $f(L_1) = L$.

Proof: Again, the definition of the spaces implies that the map is a closed continuous map. The reference, [3-p, 59], implies that there exists two points, a and b in L, such that L is irreducible from a to b in L. Since f is continuous the sets $f^{-1}(a)$ and $f^{-1}(b)$ are disjoint closed subsets of X. It is noted that X is a continuum since f is monotonic. The reference, [3-p, 15], implies that X contains an irreducible continuum, L_1, from $f^{-1}(a)$ to $f^{-1}(b)$. Since $L_1 \cap f^{-1}(a) \neq \emptyset$, $L_1 \cap f^{-1}(b) \neq \emptyset$, $f(L_1)$ is a continuum, and L is irreducible about a and b it follows that $f(L_1) = L$.

Now suppose $L_1 = A \cup B$ where both A and B are proper subcontinua of L_1. Since L_1 is irreducible from $f^{-1}(a)$ to $f^{-1}(b)$ then without loss of generality it is assumed that $(L_1 \cap f^{-1}(a)) \subseteq A$ and $(L_1 \cap f^{-1}(b)) \subseteq B$. Therefore, $f(L_1) = L = f(A) \cup f(B)$ where f(A) and f(B) are both proper subcontinua of L since $b \notin f(A)$ and $a \notin f(B)$. This contradicts L being indecomposable and the theorem is proved.

Corollary 3.2: If the same hypotheses as in Theorem 3.3 are assumed except that L is assumed to be a pseudo arc then there exists an indecomposable subcontinuum, L_1, of X such that $f(L_1) = L$.

Theorem 3.4: If M is a compact space and f is a continuous and monotonic map such that f(X) = N and N has no cut points, then if x is a cut point of M then $f(M - x) = N$.

Proof: The hyperspace M' of M whose elements are the elements of the collection $\{f^{-1}(y)\}$, as y varies over N, is homeomorphic to N. Therefore, if $y \in N$ then $M' - y'$ is connected in M' since y does not separate N. Let $x \in y'^{*}$ and consider M - x. Suppose $M - x = A \cup B$ separate. Each element of $M' - y'$ is contained entirely in A or in B since each element of $M' - y'$ is a subcontinuum of M. From the definition of a region in M' it is implied that $A' = (A - y'^{*})'$ and $B' = (B - y'^{*})'$ are each regions of M' and $M' - y' = A' \cup B'$ separate in M'. Now since y' does not separate M' in M' it can be assumed that A' is empty and $A \subseteq y'^{*}$. Therefore, $f(M - x) = f(A) \cup f(B) = N$.

Corollary 3.3: If M is a compact space and f is a continuous and monotonic map such that $f(M) = N$ and N is an indecomposable continuum then if x is a cut point of M then $f(M - x) = N$.

The following three theorems give results related to composants of continua.

Theorem 3.5: If X is compact and f is a continuous and monotonic map such that $f(X) = N$ and N is an indecomposable continuum, then there exists an indecomposable subcontinuum $L_1 \subseteq X$ such that if L is a composant of L_1 containing a point p then $f(L)$ is a subset of the composant of N containing $f(p)$.

Proof: Let L_1 be the indecomposable subcontinuum of X implied to exist by Theorem 3.3. In the proof of Theorem 3.3 points a and b are points of N about which N is irreducible and L_1 is an irreducible subcontinuum of X from $f^{-1}(a)$ to $f^{-1}(b)$.

Let L be the composant of L_1 to which the point p belongs. Let $y \in L$ and L_y a proper subcontinuum of L_1 containing both p and y. Since f is a closed continuous map then $f(L_y)$ is a subcontinuum of N. Now L_y does not intersect both $f^{-1}(a)$ and $f^{-1}(b)$ since L_1 is irreducible from $f^{-1}(a)$ to $f^{-1}(b)$. Therefore, $f(L_y)$ is a proper subcontinuum of N containing both $f(y)$ and $f(p)$. By definition of a composant, for every $y \in L$, $f(y)$ and $f(p)$ are elements of the same composant of N. Therefore,

$f(L)$ is a subset of the composant of N to which $f(p)$ belongs.

Theorem 3.6: If L is a composant of a compact continuum, M, containing a point p, M contains more than one composant, and H is a proper subcontinuum of M such that $H \subseteq L$; then every component of L - H is nondegenerate.

Proof: Let $x \in (L - H)$ and $y \in (M - L)$. Let R_x and R_y be two regions such that $\overline{R}_x \cap \overline{R}_y = \emptyset$ and $x \in R_x$, $y \in R_y$. Also the region R_x is restricted such that $\overline{R}_x \cap H = \emptyset$. This can be done since H is closed.

Let N be the component of $\overline{R}_x$ containing x. The reference, [3-p, 18], implies that N is nondegenerate. Since $x \in L$ there exists a proper subcontinuum N_1 of M such that $x \in N_1 \subseteq L$. Now, $N_1 \cup N$ is a proper subcontinuum of M since $y \notin N_1 \cup N$ and since $N_1 \cup N$ is the union of two continua with $x \in N_1 \cap N$. Therefore, by the definition of composant and the region $\overline{R}_x$, it follows that $N \subseteq (L - H)$.

The component T of L - H containing x must then contain N and therefore, T is nondegenerate.

Theorem 3.7: If L is a composant of an indecomposable continuum, M, and H is a proper subcontinuum of M such that $H \subseteq L$ then L - H is a nondegenerate connected set.

Proof: Theorem 3.6 implies that L - H is nondegenerate. Suppose $L - H = (A \cup B)$ separate. The reference, [3-p, 25], implies that $(H \cup A)$ and $(H \cup B)$ are connected. Also the reference, [3-p, 58], implies every point of M - L is a limit point of either A or B. Therefore,

$M = (H \cup \bar{A}) \cup (H \cup \bar{B})$ where both $(H \cup \bar{A})$ and $(H \cup \bar{B})$ are proper subcontinua of M and this contradicts M being indecomposable.

The next three theorems give results obtained in considering the continuous mapping of certain spaces onto the unit interval, I.

Theorem 3.8: If L is a pseudo arc from a point $\bar{a}$ to a point $\bar{b}$ and L is mapped continuously onto the unit interval I such that

$$f(\bar{x}) = \frac{\rho(\bar{a},\bar{x})}{\rho(\bar{a},\bar{x}) + \rho(\bar{b},\bar{x})}$$

then $f^{-1}(\bar{y})$, $\bar{y} \in I$, is totally disconnected.

Proof: Suppose there exists $\bar{c} \in I$ such that $f^{-1}(\bar{c})$ contains a nondegenerate component, H. The pseudo arc, L, is considered imbedded in E^2. This can be done without loss of generality since [2] implies that all pseudo arcs are topologically equivalent. Let $\bar{a} = (a,0)$, $\bar{b} = (b,0)$ and the pseudo arc is constructed from $\bar{a}$ to $\bar{b}$.

$$\text{Let } S_c = \left\{(x_1,x_2) = \bar{x} : \frac{\rho(\bar{a},\bar{x})}{\rho(\bar{a},\bar{x}) + \rho(\bar{b},\bar{x})} = \bar{c} \in I\right\}$$

$$S_c = \left\{(x_1,x_2) = \bar{x} : \frac{\rho(\bar{a},\bar{x})}{\rho(\bar{b},\bar{x})} = \frac{\bar{c}}{1 - \bar{c}} = K \in \text{Reals}\right\}$$

$$S_c = \left\{(x_1,x_2) = \bar{x} : (x_1 - a)^2 + x_2^2 = k^2\left[(x_1 - b)^2 + x_2^2\right], \; k \in \text{Reals}\right\}$$

Therefore, S_c is a conic. Reference [2] implies H is a pseudo arc itself. However, H is defined such that H is a subcontinuum of S_c which contradicts H being a pseudo arc since every subcontinuum of S_c is locally connected. The theorem is proved.

Theorem 3.9: If M is a compact connected nondegenerate metric space then there exists a continuous map f such that $f(M) = I$ where I is the unit interval and $f^{-1}(0)$ and $f^{-1}(1)$ are each nondegenerate subcontinuum of M.

Proof: Let H_1 and H_2 be nondegenerate subcontinua of M such that $(H_1 \cap H_2) = \emptyset$. Let G be the collection of subsets of M made up of H_1, H_2, and single points of $M - (H_1 \cup H_2)$. The reference, [5-p, 122], implies G is an upper semicontinuous decomposition of M. Let M' be the hyperspace of the decomposition, G.

Let $f(M) = M'$ be the continuous map such that if

$$x \in H_i,\ i = 1,2, \text{ then } f(x) = H_i' \text{ or if } x \in \{M - (H_1 \cup H_2)\}$$

then $f(x) = x'$. Since M is connected we know that M' is connected. Therefore, [5-p, 34] implies that g, $g(M') = I$, is a continuous map where if H_1' and H_2' are considered as fixed points of M' and $x' \in M'$ then

$$g(x) = \frac{\rho(H_1', x')}{\rho(H_1',x') + \rho(H_2',x')}$$

Now, consider the composite continuous map gf, $(gf)(M) = I$. Then $f^{-1}(0) = H_1$ and $f^{-1}(1) = H_2$ which suffices to prove the theorem.

Corollary 3.4: If M is a pseudo arc then there exists a continuous map f such that $f(M) = I$ where I is the unit interval and $f^{-1}(0)$ and $f^{-1}(1)$ are each nondegenerate subcontinuum of M.

Theorem 3.10: If M is an indecomposable continuum and f is a continuous map onto the unit interval I such that $f(M) = I$, then there are at most two points of I, x_1, x_2, such that $f^{-1}(x_1)$ and $f^{-1}(x_2)$ are subcontinua of M.

Proof: Suppose there exists three points of I such that the inverse image of each is a subcontinuum of M. Let c be one of these points such that $c \neq 0$, $c \neq 1$. Since $I - c = (A \cup B)$ separate it follows that $M - f^{-1}(c) = \{f^{-1}(A) \cup f^{-1}(B)\}$ separate. The reference [3-p, 25], implies $\{f^{-1}(A) \cup f^{-1}(c)\}$ and $\{f^{-1}(B) \cup f^{-1}(c)\}$ are each proper subcontinuum of M and $M = \{f^{-1}(A) \cup f^{-1}(c)\} \cup \{f^{-1}(B) \cup f^{-1}(c)\}$ which contradicts M being indecomposable.

Theorems 3.11, 3.12, and 3.13 give fundamental results related to continuous and monotonic maps. In particular Theorems 3.11 and 3.13 give results that will be used to prove later theorems in this chapter. The results of these three theorems are obtained by putting further restrictions on the space X.

Theorem 3.11: If X is a compact indecomposable continuum and f is a continuous and monotonic map such that $f(X) = Y$, then Y is a compact indecomposable continuum.

Proof: Since f is a closed continuous map, Y is a compact continuum. Suppose Y is decomposable into proper subcontinua A and B. Then $X = \{f^{-1}(A) \cup f^{-1}(B)\}$ where $f^{-1}(A)$ and $f^{-1}(B)$ are both proper subcontinua of X which contradicts X being indecomposable.

Corollary 3.5: If X is a hereditarily indecomposable continuum and f is a continuous and monotonic map such that $f(X) = Y$, then every non-degenerate subcontinua of Y is indecomposable, that is, Y is hereditarily indecomposable.

Theorem 3.12: If X is a hereditarily unicoherent continuum, f is a continuous and monotonic map such that $f(X) = Y$ and H is a subcontinuum of X then f restricted to H is both continuous and monotonic.

Proof: Restricted to H, f is trivially continuous. Let $p \in f(H) \subseteq Y$. Since f is monotonic on X then $f^{-1}(p)$ is a subcontinuum of X. The point set $f^{-1}(p) \cap H \neq \emptyset$ implies that $f^{-1}(p) \cup H$ is a subcontinuum of X. Since X is hereditarily unicoherent it then is true that $f^{-1}(p) \cap H$ is a subcontinuum of H. Therefore, f restricted to H is both continuous and monotonic.

Theorem 3.13: If X is a compact and hereditarily unicoherent continuum and f is a continuous and monotonic map such that $f(X) = L$ where L is an indecomposable continuum, then there exists an indecomposable hereditarily unicoherent subcontinuum, L_1 of X such that $f(L_1) = L$ and f restricted to L_1 is both continuous and monotonic.

Proof: This result follows directly from Theorem 3.3 and Theorem 3.12.

The next two theorems give extensions of Theorem 2.10.

Theorem 3.14: If X is compact and f is a continuous and monotonic map such that $f(X) = L$ where L is an indecomposable continuum and $f(x) = f_2 f_1(x)$ is the factorization mentioned in Theorem 2.10 then there exist indecomposable subcontinua $L_1 \subseteq X$ and $L_2 \subseteq A'$ such that $f(L_1) = f_2(L_2) = L$.

Proof: Theorem 3.3 implies the existence of L_1 and if $L_2 = f_1(L_1)$ then Theorem 3.11 implies L_2 is an indecomposable subcontinuum of A'.

Therefore, $f_2(L_2) = f_2f_1(L_1) = f(L_1) = L$.

Theorem 3.15: If X is a compact and hereditarily unicoherent continuum, f is a continuous and monotonic map such that $f(X) = L$ where L is an indecomposable continuum, and $f(x) = f_2f_1(x)$ is the factorization mentioned in Theorem 2.10, then there exist compact indecomposable subcontinua $L_1 \subseteq X$ and $L_2 \subseteq A'$ such that $f(L_1) = f_2(L_2) = L$ where f restricted to L_1 is both continuous and monotonic and f_2 restricted to L_2 is continuous and light.

Proof: Theorem 3.14 gives the candidates for the desired L_1 and L_2. Theorem 3.13 implies that f meets the desired requirements restricted to L_1. Since [5-p, 141] has proved that f_2 is both continuous and light then f_2 meets the desired requirements restricted to L_2. Thus L_1 and L_2 satisfy the requirements of this theorem.

P. M. Swingle, [4], gave the following definition.

Definition 3.1: The set M is the k-finished sum of a set of subcontinua, $\{M_1, M_2, \ldots, M_k\}$, if and only if $M = \bigcup_{i=1}^{k} M_i$ and $M_j - M_i \neq \emptyset$ for each fixed j, $1 \leq j \leq k$, as i varies over the set,

$$\{1, 2, \ldots, j-1, j+1, \ldots, k\}.$$

The following three theorems involve the above definition.

Theorem 3.16: If M is the 2-finished sum of hereditarily indecomposable continua, M_1 and M_2, such that $M_1 \cap M_2 \neq \emptyset$, then there exists at least one point in $M_1 \cap M_2$ which is a limit point of both $M_1 - M_2$ and $M_2 - M_1$.

Proof: Suppose that no point of $M_1 \cap M_2$ is a limit point of both $M_1 - M_2 = H$ and $M_2 - M_1 = K$. Thus $\overline{H} \cap \overline{K} = \emptyset$. Therefore, $D = (M_1 \cup M_2) - (\overline{H} \cup \overline{K})$ is a domain relative to $M_1 \cup M_2$ such that $D \subseteq M_1 \cap M_2$ and $D \neq \emptyset$ since $M_1 \cup M_2$ is connected. The reference, [3-p, 58], implies domains D and H relative to M_1 both intersect every composant of M_1. Therefore, let $m_1 \in H$ and $x \in D$ such that m_1 and x both belong to the same composant of M_1. Let N_1 be a proper subcontinuum of M_1 such that $(x \cup m_1) \subseteq N_1 \subsetneq M_1$. Similarly consider the points $m_2 \in K$ and $x \in D$ lying in the same composant in M_2 and N_2 a proper subcontinuum of M_2 such that $(m_2 \cup x) \subseteq N_2 \subseteq M_2$.

The supposition implies that if $y \in F(D)$ then $y \in F(H)$ or $y \in F(K)$ but $y \notin \{F(H) \cap F(K)\}$ since $\overline{H} \cap \overline{K} = \emptyset$. Let I_1 be the component of $D \cap N_1$ that contains x and similarly define I_2. The reference, [3-p, 18], implies $\{F(I_1) \cap F(H)\} \neq \emptyset \neq \{F(I_2) \cap F(K)\}$. Thus $\overline{I}_1 - \overline{I}_2 \neq \emptyset \neq \overline{I}_2 - \overline{I}_1$. Since $x \in (I_1 \cap I_2)$ then $x \in (\overline{I}_1 \cap \overline{I}_2)$ and $\overline{I}_1 \cup \overline{I}_2$ is therefore a subcontinuum of $M_1 \cap M_2$. Since $\overline{I}_1 \cup \overline{I}_2$ contains no point of either H or K then $\overline{I}_1 \cup \overline{I}_2 \subseteq M_1 \cap M_2 \subseteq M_1$. The fact that $\overline{I}_1 - \overline{I}_2 \neq \emptyset \neq \overline{I}_2 - \overline{I}_1$ implies $\overline{I}_1 \cup \overline{I}_2$ is a decomposable subcontinuum of M_1. This contradicts M_1 being hereditarily indecomposable and thus the theorem is proved.

Theorem 3.17: If M is the k-finished sum of indecomposable continua, $M = \bigcup_{i=1}^{k} M_i$, M_i is an indecomposable continua, $1 \leq i \leq k$, and $\bigcap_{i=1}^{k} M_i \neq \emptyset$ then M has at most one cut point which is necessarily an element of $\bigcap_{i=1}^{k} M_i$.

Proof: Let j be an integer, $1 \leq j \leq k$, and $x \in M_j - \bigcap_{i=1}^{k} M_i$. Therefore, $M - x = (M_1 \cup \ldots \cup M_{j-1} \cup M_{j+1} \cup \ldots \cup M_k) \cup (M_j - x)$. Since M_j

is an indecomposable continuum then $M_j - x$ is connected and therefore $M - x$ is a union of connected point sets all of which have a point in common since $\bigcap_{i=1}^{k} M_i \neq \emptyset$. Thus $M - x$ is connected and M contains no cut point in $M - \bigcap_{i=1}^{k} M_i$.

If M has no cut points then the theorem is proved. Now suppose $p \in \bigcap_{i=1}^{k} M_i$ is a cut point of M. For each integer j, $1 \leq j \leq k$, let $H_j = M_j - p$. The point set H_j, $1 \leq j \leq k$, is a connected subset of M since no single point separates an indecomposable continuum. The point set $M - p = A \cup B$ separate since p is a cut point of M. The point set H_j, $1 \leq j \leq k$, lies entirely in A or in B since H_j is connected. In addition suppose p_1, $p_1 \neq p$, is a cut point of M. Since $p_1 \in M - p$ then $p_1 \in A$ or $p_1 \in B$. Without loss of generality assume that $p_1 \in A$. The reference, [3-p, 25], implies $p \cup B$ is a continuum containing p and $p \cup B \subseteq M - p_1$. For every j, $1 \leq j \leq k$, such that $H_j \subseteq A$, $H_j - p_1$ is connected since [3-p, 60], states that if T is the sum of countably many proper subcontinua of a compact indecomposable continuum M_j, then $M_j - T$ is connected. In this case $T = p \cup p_1$ and $M_j - T = H_j - p_1$. Therefore, $A - p_1$ is the union of a finite number of connected point sets. Since p is a limit point of H_j, $1 \leq j \leq k$, such that $H_j \subseteq A$ then p is a limit point of all such $H_j - p_1$. Therefore $M - p_1 = (A - p_1) \cup (p \cup B)$ is connected since $M - p_1$ is a finite union of connected sets each one of which has a common limit point p. This contradicts the point p_1 being a cut point of M and the theorem is proved.

<u>Theorem 3.18</u>: If $M = \bigcup_{i=1}^{k} M_i$ is the k-finished sum of indecomposable continua such that M_i, $1 \leq i \leq k$, is an indecomposable continuum and M has a cut point, p, then $p = \bigcap_{i=1}^{k} M_i$.

Proof: Theorem 3.17 implies $p \in \bigcap_{i=1}^{k} M_i$. Suppose there exists another point p_1, $p_1 \neq p$, such that $p_1 \in \bigcap_{i=1}^{k} M_i$. Let $H_j = M_j - p$, $1 \leq j \leq k$. The point set H_j is connected for each j, $1 \leq j \leq k$, since no point of the indecomposable continuum M_j separates M_j. The point set $M - p = \bigcup_{j=1}^{k} (M_j - p) = \bigcup_{j=1}^{k} H_j$ and $p_1 \in H_j$ for each j, $1 \leq j \leq k$, since $p \neq p_1$ and $p_1 \in \bigcap_{i=1}^{k} M_i$. Thus $M - p$ is connected since $M - p$ is the union of connected sets having the point p_1 in common. This contradicts p being a cut point of M and the theorem is proved.

Theorem 3.19: If M is the 2-finished sum of the indecomposable continua M_1 and M_2 and $M_1 \cap M_2 \neq \emptyset$ then either there exists no two points between which M is an irreducible continuum or $M_1 \cap M_2$ contains no domain relative to M.

Proof: Since $M_1 \cap M_2 \neq \emptyset$ it follows that M is a continuum. Suppose there exists two points a and b such that M is irreducible about $\{a,b\}$ and there exists a domain, D, relative to M such that $D \subseteq M_1 \cap M_2$. The points a and b cannot both belong to either M_1 or M_2 for if so M would not be an irreducible continuum about $\{a,b\}$. Therefore, without loss of generality, say $a \in (M_1 - M_2)$ and $b \in (M_2 - M_1)$. The point sets M_1 and M_2 are proper subcontinua of M containing a and b respectively since M is the finished sum of M_1 and M_2. The reference, [3-p, 60], states the following theorem. "If a and b are two points, M is a continuum which is irreducible from a to b, and T is a proper subcontinuum of M containing b, then M - T is connected." Therefore, since M_2 is a proper subcontinuum of M containing b, this theorem implies that $M_1 - M_2$ is a connected subset of M. In addition $M_1 - M_2$

is a connected subset of M_1.

Since $M_1 - M_2$ is a domain relative to M_1 then $M_1 - M_2$ intersects every composant of the indecomposable continuum M_1. Let x_1, x_2 be two points of $M_1 - M_2$ such that x_1 and x_2 are in different composants of M_1. Let $x \in D$ and consider the subcontinuum $(\overline{M_1 - M_2}) \subseteq M_1$. Since $D \subseteq M_1 \cap M_2$ then $x \notin (\overline{M_1 - M_2})$ since no point of D is a limit point of $M_1 - M_2$. Therefore, $(\overline{M_1 - M_2})$ is a proper subcontinuum of M_1 containing x_1 and x_2. This contradicts x_1 and x_2 being in different composants of M_1 and therefore, either there exists no two points of M such that M is irreducible from one to the other or $M_1 \cap M_2$ contains no domain relative to M.

CHAPTER IV

THE DECOMPOSITION OF THE CLOSED 2-CELL INTO NONLOCALLY CONNECTED CONTINUA

This chapter will be devoted to answering the following questions.

(1) Can a closed 2-cell be decomposed into the union of an uncountable number of mutually exclusive nonlocally connected compact continua?

(2) Does there exist a continuous and monotonic inverse arc map which maps the closed 2-cell onto an arc?

(3) What other characterizations of the inverse arc map can be given?

These three questions will be answered in the material that follows. As motivation, the following example is cited.

Example 4: Consider the subspace S of Euclidean three space made up of the cross product of the closure of $\left\{(x,y): y = \sin 1/x,\ 0 < x \leq 1\right\}$ and the closed interval $[0,1]$ on the z axis. Let $\{x_\alpha\}$, $\alpha \in \Upsilon$, be a well ordering of the reals on the closed interval $[0,1]$. Let

$$A_\alpha = \{\bar{x} : \bar{x} \in S \text{ and } x_\alpha \text{ is the third component of } \bar{x}\} .$$

From this it is seen that $S = \bigcup_{\alpha\in\Upsilon} A_\alpha$ where the index set, Υ, is uncountable. To make the above more meaningful one needs to observe that this example shows that the compact continuum S can be expressed as the union of an uncountable number of mutually exclusive compact continua no one of which is locally connected.

Examples of the above type are not hard to construct but, are more of a problem in Euclidean two space. This question will be answered later in the chapter. At the moment the above example will be extended one step further. Let f be a map such that $f(S) = I$, where I is the closed real interval $[0,1]$, be a map defined such that $f(A_\alpha) = x_\alpha$. It is apparent that this map is both continuous and monotonic. However, one can also notice, intuitively, just how bad this map is, since the preimage of every point, even though a continuum, has uncountable many points at which it is not locally connected. One might expect that if f is a continuous and monotonic map such that $f(X) = Y$ and L is an arc in Y then there would have to exist at least one point $y \in L$ such that $f^{-1}(y)$ is locally connected. Example 4 shows that this is not the case.

In the previous paragraph it was mentioned that a similar example would be exhibited in Euclidean two space. This example will be given in the form of a theorem. In this theorem the space M will be a closed 2-cell.

Theorem 4.1: If M is a closed 2-cell, $M = I \times I$, then M is the union of uncountably many mutually exclusive nonlocally connected continua.

Proof: Let $M = A \cup B \cup C$ where $A = \{I \times [0,1/4]\}$, $B = \{I \times (1/4, 3/4)\}$, and $C = \{I \times [3/4, 1]\}$. Before a complete description of the decomposition is given the following list of definitions is presented pertaining to M and its partitions already described.

Definition 4.1: If $(x,0) \in I$ then the point $(x,1)$ is called the

<u>associated</u> <u>point</u> of the point $(x,0)$.

<u>Definition 4.2</u>: An <u>S-arc</u> is an arc $L = L_1 \cup L_2 \cup L_3$ such that:

(1) L_1 is an arc from a point p in A to a point q in $C - \{I \times \{1\}\}$,

(2) L_2 is an arc from q to a point r in $A - \{I \times \{0\}\}$,

(3) L_3 is an arc from r to a point s in C,

(4) $L_i \cap \{I \times \{y\}\}$ is a single point for each y, $1/4 \leq y \leq 3/4$, $i = 1,2,3$,

(5) If $L_1 \cap \{I \times \{3/4\}\} = (x_1, 3/4)$ then $L_2 \cap \{I \times \{3/4\}\} = (x_2, 3/4)$ such that $x_1 < x_2$,

(6) If $L_2 \cap \{I \times \{1/4\}\} = (x_1, 1/4)$ then $L_3 \cap \{I \times \{1/4\}\} = (x_2, 1/4)$ such that $x_1 < x_2$,

(7) $L_i \cap \{I \times \{0\}\} = \emptyset$, $L_i \cap \{I \times \{1\}\} = \emptyset$, $i = 1,2,3$, unless otherwise stated.

<u>Definition 4.3</u>: <u>A</u> <u>semi-S</u> <u>arc</u> is an arc $L = L_1 \cup L_2$ such that:

(1) L_1 is an arc from a point p in A to a point q in $C - \{I \times \{1\}\}$,

(2) L_2 is an arc from q to a point r in $A - \{I \times \{0\}\}$,

(3) $L_i \cap \{I \times \{y\}\}$ is a single point for each y such that $1/4 \leq y \leq 3/4$, $i = 1,2$,

(4) if $L_1 \cap \{I \times \{3/4\}\} = (x_1, 3/4)$ then $L_2 \cap \{I \times \{3/4\}\} = (x_2, 3/4)$ such that $x_1 < x_2$,

(5) $L_i \cap \{I \times \{0\}\} = \emptyset$, $L_i \cap \{I \times \{1\}\} = \emptyset$, $i = 1,2$, unless otherwise stated.

If L is a semi-S arc and if $(x_1, 1/4) = p$ and $(x_2, 1/4) = q$ are the two points of intersection of L and $\{I \times \{1/4\}\}$ then the subarc (pq) of L along with the subarc (pq) of the arc $\{I \times \{1/4\}\}$ forms a simple closed curve. With each semi-S arc there is associated such a unique simple closed curve. With this in mind the following definition is made.

Definition 4.4: If L_1 and L_2 are semi-S arcs then it is said that L_2 is inside of L_1 if and only if the associated simple closed curve of L_2 is a subset of the closed region bounded by the simple closed curve associated with L_1.

Definition 4.5: If L_1 and L_2 are semi-S arcs such that L_2 is inside of L_1 then the closure of the region bounded by the associated simple closed curve of L_1 minus the open connected set bounded by the associated simple closed curve of L_2 is called the U-set formed by the semi-S arcs L_1 and L_2.

Definition 4.6: If C is the U-set formed by semi-S arcs L_1 and L_2 then $C \cap \left\{I \times \{1/2\}\right\}$ is the union of two disjoint arcs each of which we call a U-bar formed by semi-S arcs L_1 and L_2.

Definition 4.7: If L_1 is a semi-S arc then the closure of the region bounded by the associated simple closed curve intersected with $\left\{I \times \{1/2\}\right\}$ is an arc called the semi-S arc bar formed by the semi-S arc L_1.

This completes the list of definitions to be used to define the desired decomposition. In the definition of this decomposition no two defined arcs will be allowed to intersect and no defined arc will be allowed to intersect $\left\{\{1\} \times I\right\}$.

First, an S-arc is constructed from the point (0,0) to its associated point (0,1). Let the closed region bounded by this arc and the arc $\{[0\} \times I\}$ be named T. Now, $M = T \cup (M - T)$. The remaining definition of the construction will be done entirely in $(\overline{M - T})$. The closed two cell, $(\overline{M - T})$, will be decomposed in such a way as to induce the desired decomposition on M.

Next, from the point (1/2, 0) a semi-S arc is constructed inside the semi-S arc already constructed and then from the end point, which is in $\{A - \{I \times \{0\}\}\}$, of this semi-S arc an S-arc is constructed to the associated point of (1/2, 0) such that the simple closed curve associated with this S-arc does not intersect the region bounded by simple closed curves which are associated with any of the existing semi-S arcs. The construction of the S-arc mentioned last in the preceeding sentence will be referred to as constructing an S-arc to the right. It is important to note at this time that the compact continuum bounded by the S-arc from the point (0,0) to the point (0,1), the arc from the point (1/2, 0) to the point (1/2,1), the arc $\{[0, 1/2] \times \{1\}\}$, and the arc $\{[0, 1/2] \times \{0\}\}$ is homeomorphic to the closed 2-cell. So also is the closure of the complement of this continuum relative to $(\overline{M - T})$.

Attention is focused on the points (1/4, 0) and (3/4, 0). From the point (1/4, 0) a semi-S arc is constructed inside the semi-S arc beginning at the point (0,0) and then it is extended with the construction to the right of an S-arc to the associated point of the point (1/4, 0). From the point (3/4, 0) an arc is constructed to the associated point of (3/4, 0) such that this arc is the union of two semi-S arcs and an S-arc. Each of the two semi-S arcs will be inside distinct semi-S arcs of the arc beginning at the point (1/2, 0) and then these will be extended by the construction to the right of an S-arc to the associated point of the point $(3/4, 0)$.

In general, suppose the construction has been defined successfully at the $(n - 1)$ level, i.e. the constructions have been made beginning at points $(k/2^{n-1}, 0)$, $0 < k \leq 2^{n-1}$. The problem now is to describe the construction at the n level. At the point $(1/2^n, 0)$ construct an arc to the associated point of $(1/2^n, 0)$ which is the union of exactly one semi-S arc inside the single semi-S arc beginning at the point $(0,0)$ and an S-arc to the point $(1/2^n, 1)$ constructed to the right. For $k = 2$ the construction is complete. Therefore, for $k = 3$ an arc is constructed beginning at the point $(3/2^n, 0)$ which is the union of exactly the same number of semi-S arcs that begin at the point $(2/2^n, 0)$ each one of which is inside a distinct semi-S arc beginning at the point $(2/2^n, 0)$ and an S-arc constructed to the right to the point $(3/2^n, 1)$. It follows that from the point $(k/2^n, 0)$ where k is odd an arc is constructed to the point $(k/2^n, 1)$ which is the union of exactly the same number of semi-S arcs that begin at the point $(k-1/2^n, 0)$ each one of which is inside a distinct semi-S arc beginning at the point $(k-1/2^n, 0)$ and an S-arc constructed to the right to the point $(k/2^n, 1)$. This inductively defines the foundation of the decomposition to be discussed. Figures 4.1, 4.2, and 4.3 should help to clarify the preceeding definition.

Even though the basic portion of the foundation for the desired decomposition has been described, one further restriction must be placed on the construction of some of the arcs already described. If it is assumed that the S-arc beginning at the point $(0,0)$ is to be as shown in Figure 4.1 then adjustments will be as stated below.

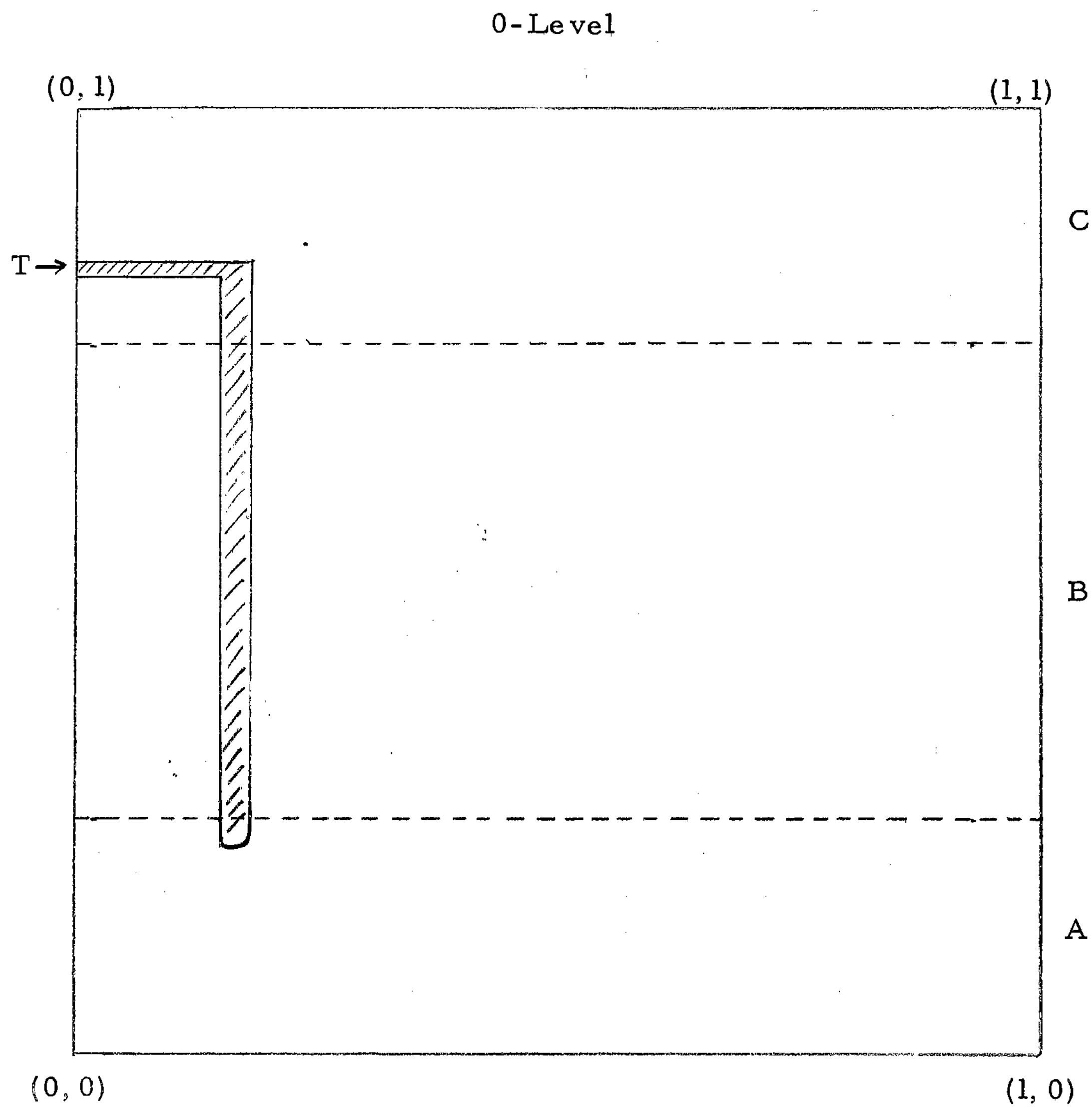

Figure 4.1

1-Level

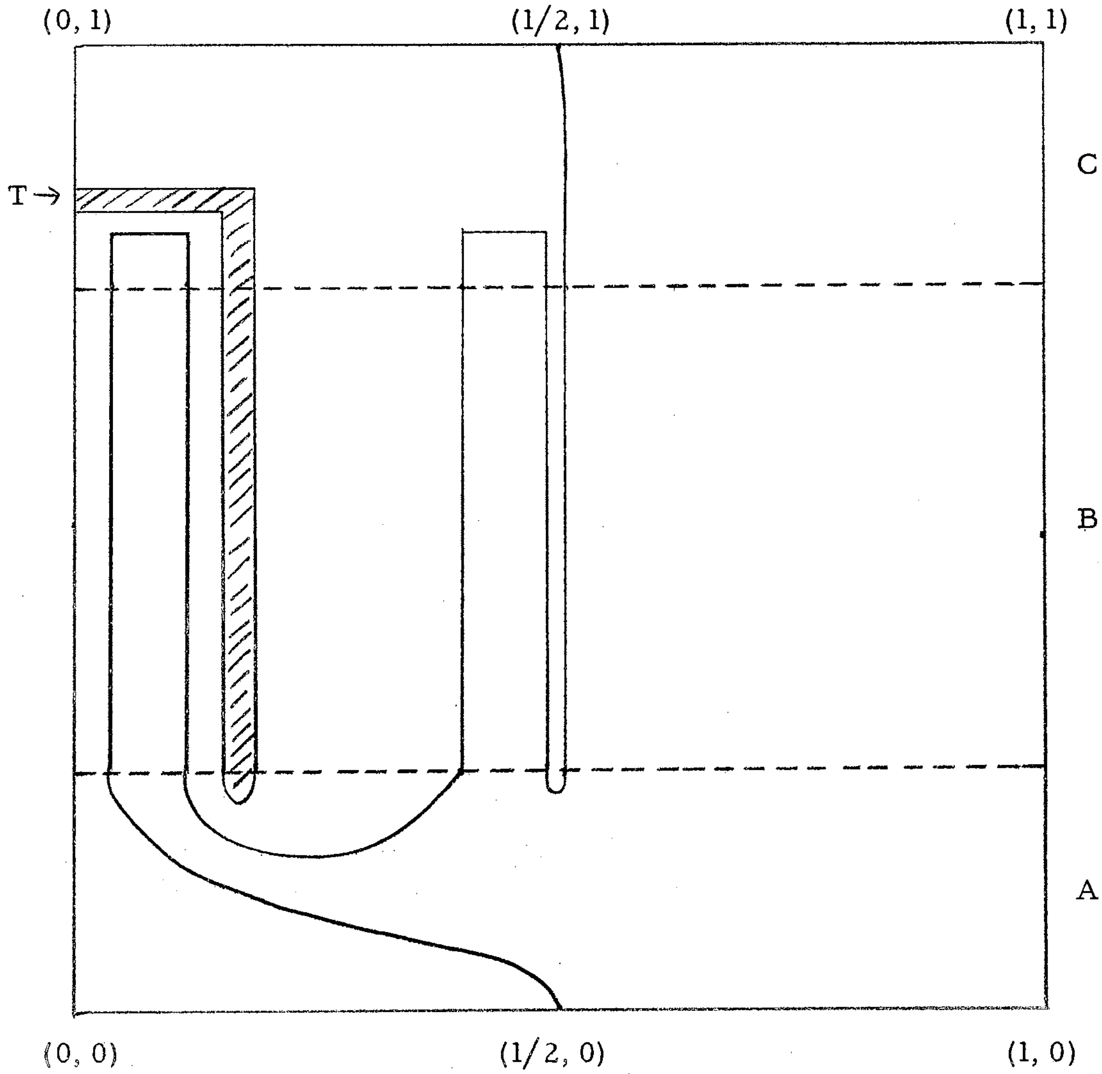

Figure 4.2

2-Level

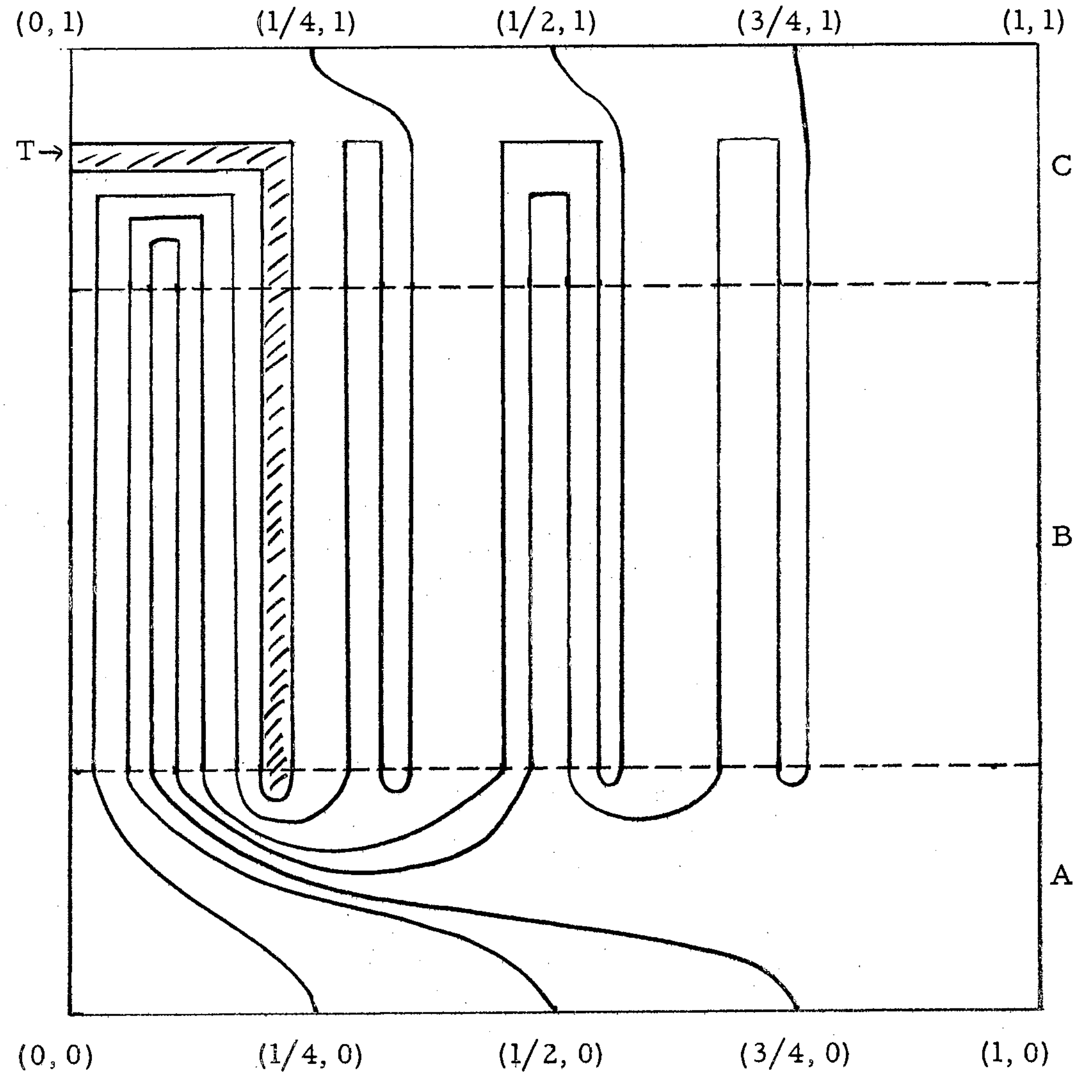

Figure 4.3

Consider the compact continuum consisting of the points of the closure of $\left\{(x,y): y = 1/16 \sin 1/x,\ 0 < x \leq 1/4\pi\right\}$. Let f be a translation defined on this compact continuum such that $f(x,y) = (x, y + 1/8$. Let $D \subseteq A$ be the image of the translation, f. The point set D is not locally connected since $f^{-1}(D)$ is not locally connected. The point set D does not separate E^2 since $f^{-1}(D)$ does not separate E^2. It is a well known theorem that there exists a monotonic decreasing sequence of closed topological 2-cells $\{D_n'\}$, in E^2 such that $\bigcap_{n=1}^{\infty} D_n' = D$, $D_1' \cap (\overline{M - T})$ is a subset of the interior of $A \cup \left\{(a,y) : x = 0,\ 0 < y < 1/4\right\}$, and D_n' is contained entirely in the interior of D_{n-1}', $n = 2,3,\ldots$. Thus it easily follows that there exists a monotonically decreasing sequence of closed topological 2-cells, $\{D_n\}$, such that $\bigcap_{n=1}^{\infty} D_n = D$, $D_1 \subseteq \left\{A - F(A)\right\} \cup \left\{(x,y) : x = 0,\ 0 < y < 1/4\right\}$, and D_n is contained entirely in the interior of D_{n-1}, $n = 2,3,\ldots$, relative to A.

From the definition of D one knows that there exists no point $(x,y) \in D$ such that $y < 1/16$ and there exists no point $(x,y) \in D$ such that $y > 3/16$. Consider the sequences of points of A such that

$$\{c_n\} = \left\{\left(\frac{2}{(2n+1)\pi}, \frac{1}{16} - \frac{1}{2n}\right) : n = 5,7,9,\ldots\right\} \text{ and}$$

$$\{d_n\} = \left\{\left(\frac{2}{(2n+1)\pi}, \frac{3}{16} + \frac{1}{2n}\right) : n = 4,6,8,\ldots\right\}.$$

The sequence $\{c_n\}$ converges to the point $(0, 1/16)$ and the sequence $\{d_n\}$ converges to the point $(0, 3/16)$ since the sequence $\left\{\left(\frac{2}{(2n+1)\pi}, \frac{1}{16}\right)\right\}$ converges to the point $(0, 1/16)$ and the sequence $\left\{\left(\frac{2}{(2n+1)\pi}, \frac{3}{16}\right)\right\}$ converges to $(0, 3/16)$. Therefore, there exists D_{n1} such that $(c_5 \cup d_4) \cap D_{n1} = \emptyset$. Again, there will exist a $D_{n2} \subseteq D_{n1}$ such that

$(c_7 \cup d_6) \cap D_{n2} = \emptyset$. In general if D_{ni} has been defined for the points c_n and d_{n-1} there will exist $D_{n(i+1)} \subseteq D_{ni}$ such that $(c_{n+2} \cup d_{n+1}) \cap D_{n(i+1)} = \emptyset$. It also follows that $\bigcap_{i=1}^{\infty} D_{ni} = D$.

Now, the previous construction of the arcs in $(\overline{M - T})$ will be considered. At this time the construction of the arcs beginning at the points $(1/2^n,0)$, $n = 1,2,\ldots$, will be altered. All that need be done is as follows:

(1) In place of the subarc from the point $(1/2^n,0)$ to the point $\bar{x}_n$, where $\bar{x}_n$ is the point of intersection of the previously constructed arc beginning at the point $(1/2^n,0)$ and the arc $\{I \times \{1/4\}\}$, the arc L_n, from the point $(1/2^n,0)$ to the point $\bar{x}_n$, is substituted where $L_n = L_{n1} \cup L_{n2} \cup L_{n3}$;

(2) L_{n1} is an arc from the point $(1/2^n,0)$ to the point $\bar{e}_i$ where $\bar{e}_i \in D_{ni} \cap \left\{(\frac{2}{(4i+7)\pi},y) : 0 \le y \le 1/16\right\}$;

(3) L_{n2} is an arc from the point $\bar{e}_i$ to the point $\bar{f}_i$ where $\bar{f}_i \in D_{ni} \cap \left\{(\frac{2}{(4i+5)\pi},y) : 3/16 \le y \le 1/4\right\}$ and the arc $(\bar{e}_i\bar{f}_i)$ is a subarc of $F(D_{ni})$ which lies in the interior of A;

(4) L_{n3} is an arc from the point $\bar{f}_i$ to the point $\bar{x}_n$.

Figure 4.4 will be a guide as to what adjustments are being made.

With this adjustment of the arcs beginning at the points $(1/2^n,0)$ one now proceeds to prove Theorem 4.1. This objective will be accomplished successfully obtaining the following four results:

I. Associating with each point of $\{I \times \{0\}\}$ a compact subcontinuum of M;

II. Showing that the collection of subcontinua of M acquired in I is a collection of disjoint continua;

III. Showing that the set theoretic union of all subcontinua acquired in M is exactly equal to M;

IV. Showing that each subcontinuum acquired in M is nonlocally connected at some point.

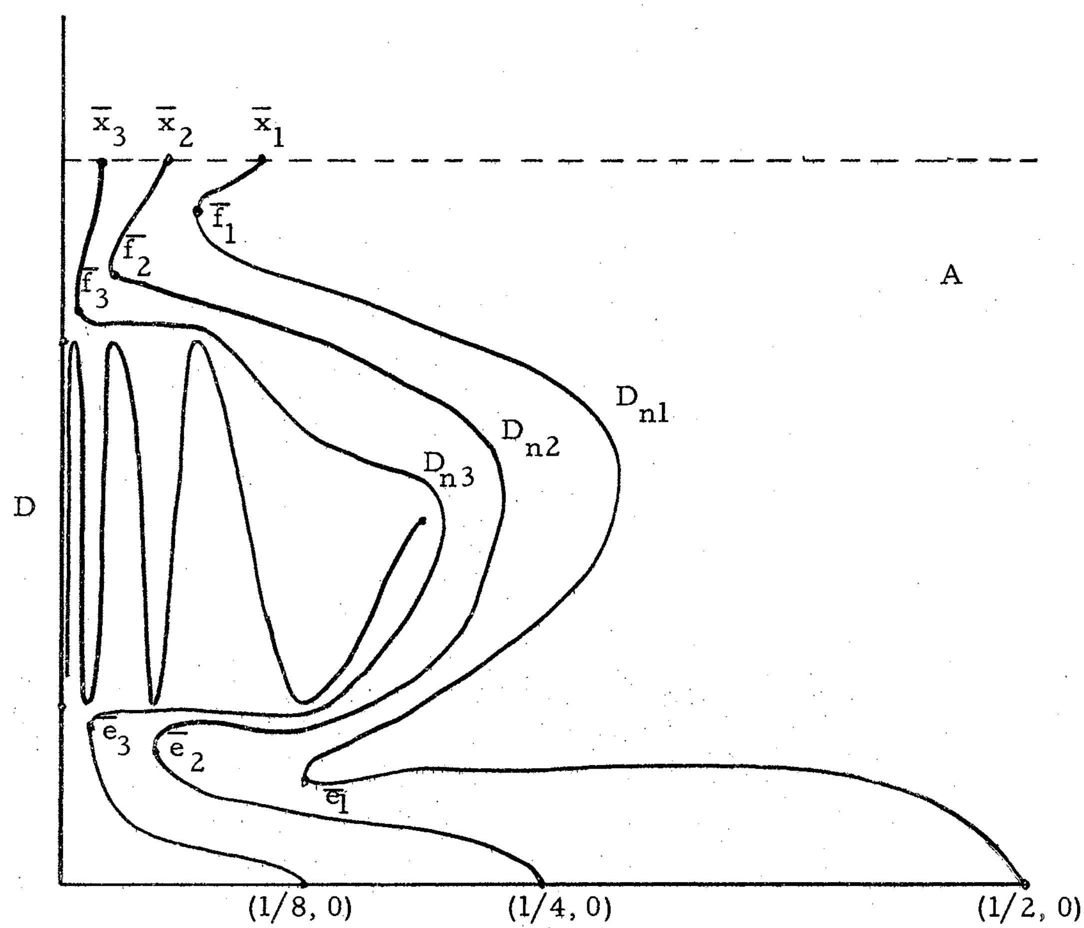

Figure 4.4

I. Let X_n, $n = 1,2,\ldots$, be the compact continuum bounded by the constructed arc beginning at the point $(0,0)$, the subarc of $\{I \times \{0\}\}$ from the point $(0,0)$ to the point $(1/2^n,0)$, the constructed arc beginning at the point $(1/2^n,0)$, and the subarc from the point $(0,1)$ to the point $(1/2^n,1)$ of the arc $\{I \times \{1\}\}$. The description of the construction of the arcs already established implies that the sequence, $\{X_n\}$, is monotonic decreasing. Reference [3-p, 14] implies that the point set $\bigcap_{n=1}^{\infty} X_n = X$ is a compact continuum containing the point $(0,0)$ and the point $(0,1)$. Also by the construction of the arcs beginning at points $(1/2^n,0)$, $n = 1,2,\ldots$, it follows that $D \subseteq X$ and that X is not locally connected at points $(0,y)$ where $1/16 \leq y \leq 3/16$. Since the subcontinua T and X have points in common then the point set $T \cup X$ is a continuum containing the point $(0,0)$. At this time the subcontinuum $T \cup X$ is associated with the point $(0,0)$ for future reference.

It is of importance to recognize that at level n the closed region bounded by the arc beginning at points $(k/2^n,0)$ and $(k-1/2^n,0)$ along with the arcs $\{(x,1) : k-1/2^n \leq x \leq k/2^n\}$ and $\{(x,0) : k-1/2^n \leq x \leq k/2^n\}$ is homeomorphic to a closed 2-cell. For each k, $1 \leq k \leq 2^n$, let this compact continuum be named Q_n^k. Now, with each point $(x,0) \in \{I \times \{0\}\}$ a compact continuum is associated in the following manner. If $(x,0) \in \{I \times \{0\}\}$ is a point such that $x \neq k/2^n$ for any k or n then there exists a unique k for each $n = 1,2,\ldots$, such that $(x,0) \in Q_n^k$. The description of the construction of the arcs already established implies that the sequence, $\{Q_n^k\}$, is monotonic decreasing. Let $A_x = \bigcap_{n=1}^{\infty} Q_n^k$. Note the fact that for each n and a fixed point x there is a unique positive integer k dependent upon both x and n. Reference

[3-p, 14] implies that for each such $(x,0) \in \{I \times \{0\}\}$ the point set A_x is a compact continuum containing points $(x,0)$ and $(x,1)$. Therefore, with each such point mentioned in this paragraph the subcontinuum A_x of M is associated.

Now, let the point $(x,0) \in \{I \times \{0\}\}$ be such that for some n and some k, $x = k/2^n$. Then, as in the above paragraph, a compact subcontinuum P_n^k of M can be defined.

Let P_n^k be the closed region bounded by the arcs beginning at the points $(k+1/2^n,0)$ and $(k-1/2^n,0)$ respectively along with the arcs $\{(x,1) : k-1/2^n \leq x \leq k+1/2^n\}$ and $\{(x,0) : k-1/2^n \leq x \leq k+1/2^n\}$. The point set P_n^k is also homeomorphic to a closed 2-cell.

If $x = k/2^N$ then at the N level P_N^k is defined as well as for all larger values of n. For all points $(x,0) \in \{I \times \{0\}\}$ such that there exists some positive integers N and k, where $x = k/2^N$, let the point set $B_x = \bigcap_{n=N}^{\infty} P_n^k$ be defined. Again, [3-p, 14] implies that for each $x \in I$ of the type considered in this paragraph the point set B_x is a compact continuum containing the point $(x,0)$ and its associated point, $(x,1)$. With each such point, $(x,0) \in \{I \times \{0\}\}$ the compact continuum, B_x, is associated.

So far a compact continuum has been associated with every point of the arc $\{I \times \{0\}\}$ except the point $(1,0)$. Let S_n be the closed topological 2-cell bounded by the arc beginning at the point $(2^n-1/2^n,0)$, the arc $\{\{1\} \times I\}$, the arc $\{(x,0) : 2^n-1/2^n \leq x \leq 1\}$, and the arc $\{(x,1) : 2^n-1/2^n \leq x \leq 1\}$. Let $Y = \bigcap_{n=1}^{\infty} S_n$. Again [3-p, 14] implies that Y is a compact subcontinuum of M containing the point $(1,0)$ and its associated point $(1,1)$. With the point $(1,0)$ let the compact subcontinuum Y be associated.

Now with every point $(x,0)$ of the arc $\{I \times \{0\}\}$ a compact subcontinuum of M has been associated which contains the points $(x,0)$ and $(x,1)$. The next objective is to argue that this collection of subcontinua of M is a mutually exclusive collection.

II. Let R_x, R_y be any two of the above mentioned compact subcontinua of M where the points $(x,0)$ and $(y,0)$ are associated respectively with R_x and R_y and $x < y$. The description of the construction of the arcs implies that $R_x = \bigcap_{n=1}^{\infty} E_n^k$, and $R_y = \bigcap_{n=1}^{\infty} F_n^k$ where the point sets E_n^k, F_n^k, $n = 1,2,\ldots,$ are the compact continua described in the construction above. Since the point set

$$K = \{k/2^n : n \text{ is a positive integer}, 0 \le k \le 2^n\}$$

is dense in I there exist positive integers k_1, k_2, n_1, and n_2 such that $x < k_1/n_1 < k_2/n_2 < y$ where, say, $n_2 > n_1$. From this the description of the construction implies that $E_{n2}^k \cap F_{n2}^k = \emptyset$. Therefore $R_x \cap R_y = \emptyset$. Thus it has been shown that the collection of subcontinua determined in I is a mutually exclusive collection. The question now is whether the set theoretic union of this collection is exactly equal to M.

III. Let $\{M_x\}$, $x \in I$, be the collection of subcontinua determined in I. The point set $\bigcup_{x \in I} M_x \subseteq M$ since $M_x \subseteq M$ for each $x \in I$. Let $\bar{x} \in M$. The description of the construction implies that there exists a sequence $\{X_n\}$, or a sequence $\{Q_n^k\}$, or a sequence $\{P_n^k\}$, or a sequence $\{S_n\}$ such that $\bar{x} \in \bigcap_{n=1}^{\infty} X_n$ or $\bar{x} \in \bigcap_{n=1}^{\infty} P_n^k$ or $\bar{x} \in \bigcap_{n=1}^{\infty} Q_n^k$ or $\bar{x} \in \bigcap_{n=1}^{\infty} S_n$ or $\bar{x} \in T$. In any case there would exist some $x \in I$ such that $\bar{x} \in M_x$. Thus $M \subseteq \bigcup_{x \in I} M_x$ and therefore, $M = \bigcup_{x \in I} M_x$.

The point set M has now been represented as the union of an uncountable number of mutually exclusive compact continua, $M = \bigcup_{x \in I} M_x$, indexed by the real numbers of the unit interval I. However, the most important aspect of this decomposition is to show that for any $x \in I$, M_x is not locally connected.

IV. Since the point set X is not locally connected, the description of the construction of the decomposition implies that the point set $T \cup X$ is nonlocally connected. Therefore, the set, $T \cup X$, which is associated with the point (0,0) is nonlocally connected.

Before proceeding some preliminary observations are needed. Let $x \in I$ such that $x \neq k/2^n$ for any k or n. Since the set K is dense in I there will exist an odd integer k and an integer n such that Q_n^k contains the point (x,0). Consider the first pair of U-bars looking from left to right from $\{\{0\} \times I\}$ formed by the first two semi-S arcs beginning at the points $(k/2^n,0)$ and $(k-1/2^n,0)$. It is important to note that when Q_{n+1}^k is selected it follows that the first pair of U-bars formed by the first two semi-S arcs beginning at the points $(k/2^{n+1},0)$ and $(k-1/2^{n+1},0)$ will each be subarcs of distinct U-bars formed by the first semi-S arcs beginning at the points $(k/2^n,0)$ and $(k-1/2^n,0)$. This similarly is the case for any pair of U-bars formed at the n-1 level relative to the n level. What this means is that if Q_n^k has a collection of t mutually exclusive U-bars then Q_{n+1}^k has a collection of at least t mutually exclusive U-bars each one of which is a subarc of one of the U-bars in Q_n^k. Therefore, consider the point set $M_x = \bigcap_{n=1}^{\infty} Q_n^k$. The first t U-bars at each level form t monotonic descending sequences of arcs from which [3-p, 14] implies there exist t disjoint nonempty intersections lying in M_x.

The very important observation to be made is that because the set K is dense in I and because of the description of the decomposition there will exist a Q^k_{n+p} such that the point $(x,0) \in Q^k_{n+p} \subseteq Q^k_n$ and Q^k_{n+p} will have t + 2 mutually exclusive nonempty intersections lying in M_x. Because of these observations the following choices of points in M_x can be made.

In relation to x there exists a closed 2-cell, Q^k_{n1}, that contains at least one U-bar. From this U-bar pick a point $\overline{p}_1 \in M_x$. This can be done since the previous paragraph points out that every U-bar contains at least one point of M_x. There also exists an integer n_2, $n_2 > n_1$, such that Q^k_{n2} has more than one U-bar. In Q^k_{n2} pick a point $\overline{p}_2 \in M_x$ where $\overline{p}_2$ is a point in a U-bar of Q^k_{n2} which is not a subset of the U-bar from which $\overline{p}_1$ was selected. In general if the point $\overline{p}_m$ has been chosen there will exist a Q^k_{ni} such that Q^k_{ni} has more than m U-bars. In Q^k_{ni} pick a point $\overline{p}_{m+1} \in M_x$ where $\overline{p}_{m+1}$ is a point in a U-bar of Q^k_{ni} which is not a subset of a U-bar from which $\overline{p}_r$, $r = 1,2,\ldots,m$, was selected. In this way a sequence of distinct points, $\{\overline{p}_n\}$, of M_x is obtained. Since M_x is compact and $\{\overline{p}_n\}^* \subseteq M_x$ then there exists a point $\overline{p}$ such that $\overline{p}$ is a limit point of $\{\overline{p}_n\}^*$. Also $\overline{p} \in M_x$ since M_x is closed.

Now, consider the closed 2-cell, Q^k_n, and suppose that Q^k_n contains at least two U-sets, C_1 and C_2. The description of the decomposition implies that

$$Q^k_n \cap \Big\{I \times [1/4,3/4]\Big\} = \Big\{C_1 \cap \{I \times [1/4,3/4]\}\Big\} \cup \Big\{(Q^k_n - C_2) \cap \{I \times [1/4,3/4]\}\Big\}$$

separate. To simplify this expresion let $W = \Big\{C_1 \cap \{I \times [1/4,3/4]\}\Big\}$ and $V = \Big\{(Q^k_n - C_2) \cap \{I \times [1/4,3/4]\}\Big\}$ which leads to

$Q_n^k \cap \{I \times [1/4,3/4]\} = V \cup W$ separate.

Let R be any sphere containing the point $\bar{p}$ with diameter less than 1/8. Since $\bar{p}$ is a limit point of $\{\bar{p}_n\}^*$ it follows that $R \cap \{\bar{p}_n\}^* \neq \emptyset$. Let $\bar{x}_1$, $\bar{x}_2 \in R \cap \{\bar{p}_n\}^*$. In the selection of $\bar{x}_1$ and $\bar{x}_2$ one should make sure that $\bar{x}_1$ and $\bar{x}_2$ do not belong to the same U-set of any Q_n^k. This can be done since $R \cap \{\bar{p}_n\}^*$ is infinite and therefore, $R \cap \{\bar{p}_n\}^*$ intersects an infinite number of U-sets. Let n and k be positive integers such that Q_n^k contains U-sets C_1 and C_2 and $\bar{x}_1 \in C_1$ and $\bar{x}_2 \in C_2$. Already it has been noted that $Q_n^k \cap \{I \times [1/4,3/4]\} = V \cup W$ separate where $C_1 \subseteq V$ and $C_2 \subseteq W$. Because of the selection of the radius of R and since $M_x \subseteq Q_n^k$ it follows that $R \cap M_x = \{(R \cap M_x) \cap V\} \cup \{(R \cap M_x) \cap W\}$ separate. Therefore, M_x is not locally connected at $\bar{p}$.

Thus the point sets M_x, where $x \neq k/2^n$ for any k or n, have been shown to be nonlocally connected. Now consider the point set M_x, where $x = k/2^N$ for some positive integers k and N. Also in this consideration the point set M_1 is excluded. It has previously been defined that $M_x = \bigcap_{n=N}^{\infty} P_n^k$. The description of the decomposition implies that for all $n > N$, $P_n^k = F_n^k \cup G_n^k$, where F_n^k is the compact continuum bounded by the arc beginning at the point $(k-1/2^n,0)$, the arc beginning at the point $(k/2^n,0)$, the arc $\{(x,0) : k-1/2^n \leq x \leq k/2^n\}$, and the arc $\{(x,1) : k-1/2^n \leq x \leq k/2^n\}$. The point set G_n^k is the compact continuum bounded by the arc beginning at the point $(k+1/2^n,0)$, the arc beginning at the point $(k/2^n,0)$, the arc $\{(x,0) : k/2^n \leq x \leq k+1/2^n\}$, and the arc $\{(x,1) : k/2^n \leq x \leq k+1/2^n\}$. It is clear that $F_n^k \cap G_n^k$ is exactly the arc initiating from the point $(k/2^N,0)$.

The last sentence in the preceeding paragraph implies

$M_x = \bigcap_{n=N}^{\infty} P_n^k = \left[\bigcap_{n=N}^{\infty} F_n^k\right] \cup \left[\bigcap_{n=N}^{\infty} G_n^k\right]$. The objective now is to argue that M_x is not locally connected. Also because of the last sentence of the preceeding paragraph it suffices, in this case, to show that $\bigcap_{n=N}^{\infty} F_n^k$ is not locally connected.

The description of the decomposition implies that F_{N+2}^k has at least one semi-S bar. Let this semi-S arc bar be B_1^2. Also the description of the decomposition implies that F_{N+3}^k contains a semi-S arc bar, B_2^2, such that $B_2^2 \subseteq B_1^2$. In general, F_{N+t}^k, $t = 2,3,\ldots$, contains a semi-S arc bar, B_{t-1}^2, such that $B_{t-1}^2 \subseteq B_{t-2}^2 \subseteq \cdots \subseteq B_2^2 \subseteq B_1^2$. The reference, [3-p, 3], implies that $\bigcap_{t=1}^{\infty} B_t^2 \neq \emptyset$ and $\bigcap_{t=1}^{\infty} B_t^2 \subseteq \bigcap_{n=N}^{\infty} F_n^k \subseteq M_x$. Therefore, from $B_1^2 \subseteq F_{N+2}^k$ pick a point $\bar{p}_1 \in \bigcap_{t=1}^{\infty} B_t^2 \subseteq \bigcap_{n=N}^{\infty} F_n^k \subseteq M_x$. The description of the decomposition implies F_{N+3}^k contains a semi-S arc bar, B_1^3, distinct from any F_{N+2}^k. In a similar way as described above pick a point $\bar{p}_2 \in \bigcap_{s=1}^{\infty} B_s^3 \subseteq M_x$. In general F_{N+t+1}^k, $t = 2,3,\ldots$, contains a semi-S arc bar B_1^{t+1}, distinct from any in F_{N+t}, $F_{N+t-1}, \ldots, F_{N+2}$. In a similar way as described above pick a point $\bar{p}_t \in \bigcap_{s=1}^{\infty} B_s^{t+1} \subseteq M_x$. In doing this a sequence of distinct points $\{\bar{p}_r\}$ is obtained such that $\{\bar{p}_r\}^* \subseteq \bigcap_{n=1}^{\infty} F_n^k \subseteq M_x$. Since $\bigcap_{n=1}^{\infty} F_n^k$ is closed and compact there will exist a limit point $\bar{p} \in \bigcap_{n=1}^{\infty} F_n^k$ of $\{\bar{p}_r\}^*$.

An argument can now be given, as was given when $x \neq k/2^n$ for any k or n, to show that $\bigcap_{n=1}^{\infty} F_n^k$ and therefore M_x is not locally connected at the point $\bar{p}$. This same type of consideration will suffice in the case of showing the previously defined compact continuum, Y, to be nonlocally connected. The continuum, Y, is associated with the point (1,0).

Thus all four results, I, II, III, and IV, have been accomplished and therefore the conclusion of the theorem follows.

The following lemma is proved using the notation of Theorem 4.1.

Lemma 4.1: If $y \in I$, $y \neq 0$, $y \neq 1$, then $M - M_y = A \cup B$ separate, where $A = \bigcup_{x<y} M_x$ and $B = \bigcup_{y<x} M_x$.

Proof: Theorem 4.1 implies that $A \cap B = \emptyset$. Also A and B are each connected subsets of M since each is the union of a collection of connected sets and an interval. Without loss of generality suppose that $\bar{p} \in A$. Let M_t be the nonlocally connected continuum of the collection defined in Theorem 4.1 such that $\bar{p} \in M_t$. The description of the decomposition in Theorem 4.1 implies $M_t = \bigcap_{k=1}^{\infty} N_k$ and $M_y = \bigcap_{k=1}^{\infty} E_k$ and that there exist integers r and s such that $N_s \cap E_r = \emptyset$. The decomposition of Theorem 4.1 implies that $\bar{p}$ is an interior point of N_s and that N_s intersects only points of sets M_x where $x < y$. This implies that $\bar{p}$ is not a limit point of B. Similarly no point of B is a limit point of A. Thus $M - M_y = A \cup B$ separate.

Theorem 4.2: There exists a continuous and monotonic inverse arc map, f, such that $f(M) = I$ and if $y \in I$ then $f^{-1}(y)$ is nonlocally connected.

Proof: If $\bar{p} \in M$ then define $f(\bar{p}) = x$, $x \in I$, if and only if $\bar{p} \in M_x$ in the decomposition of Theorem 4.1. Obviously this defines a monotonic map such that $f(M) = I$ and if $y \in I$ the $f^{-1}(y)$ is nonlocally connected. Let (ab) be a subarc of I from the point a to the point b. The map f maps the arc $\{(x,0) : a \leq x \leq b\}$ onto the subarc (ab) and therefore f is an inverse arc map. Thus it remains only to prove that f is continuous.

In order to prove that f is continuous it is sufficient to show that the inverse image under f of any open subinterval of I is a domain in M. Let (ab) be any open subinterval of I, $a \neq 0 \neq b$, $a \neq 1 \neq b$. The definition of f implies that $f^{-1}\{(ab)\} = \bigcup_{a<x<b} M_x$. Lemma 4.1 implies $M - M_a = A \cup B$ separate and $M - M_b = C \cup D$ separate. Without loss of generality suppose that $f^{-1}\{(ab)\} \subseteq A$ and $f^{-1}\{(ab)\} \subseteq D$. The point set $A \cap D$ is a domain relative to M since Lemma 4.1 implies that both A and D are domains relative to M. Therefore $f^{-1}\{(ab)\}$ is a domain relative to M since $f^{-1}\{(ab)\} = A \cap D$.

If $a = 0$ and $b = 1$ then $f^{-1}\{(ab)\} = M$ which is trivially a domain relative to M. If $a = 0$, $b \neq 0$, and $b \neq 1$ then Lemma 4.1 implies that $M - M_b = A \cup B$ separate where A is exactly $f^{-1}\{(ab)\}$ and therefore, $f^{-1}\{(ab)\}$ is a domain relative to M. Similar argument if $b = 1$, $a \neq 0$, and $a \neq 1$. Thus f is continuous.

The following definition is given in order to aid progress toward giving a characterization of the inverse arc map.

Definition 4.1: Let X and Y be spaces. If f is a map such that $f(X) = Y$, then Y is said to have property Z relative to the map f if and only if for every arc L in Y and every $y \in L$ there exists a region, U_y, relative to L such that $y \in U_y$ and such that y_1, $y_2 \in U_y$ there will exist $p_1 \in f^{-1}(y_1)$ and $p_2 \in f^{-1}(y_2)$ such that there exists an arc $(p_1p_2) \subseteq f^{-1}(L)$.

The results of Theorem 4.2 are considered in proving the following theorem.

Theorem 4.3: Let X and Y be spaces. If f is a continuous and monotonic map such that $f(X) = Y$ and f has the property that for each arc, L, in Y, there exists at least one point of L such that its inverse image is locally connected, then f is an inverse arc map if and only if Y has property Z relative to the map f.

Proof: Suppose f is an inverse arc map. Let L be any arc in Y and let L_1 be an arc in X such that $f(L_1) = L$. Let $y \in L$. Let U_y be any region relative to L containing y. Suppose $y_1, y_2 \in U_y$ and consider $f^{-1}(y_1)$ and $f^{-1}(y_2)$. Since $f(L_1) = L$ it then follows that $L_1 \cap f^{-1}(y_1) \neq \emptyset \neq L_1 \cap f^{-1}(y_2)$. Let $p_1 \in L_1 \cap f^{-1}(y_1)$ and $p_2 \in L_1 \cap f^{-1}(y_2)$. The subarc (p_1p_2) of L_1 fulfills the requirements of the definition for Y to have property Z relative to f.

On the other hand suppose Y has property Z and let L be any arc in Y from a' to b'. Let the points of L be well ordered, $\{x_\alpha\}$, $\alpha \in \Upsilon$. Since Y has property Z then for each x_α there exists a region, G_α, such that if $y_1, y_2 \in G_\alpha$ then there exists $p_1 \in f^{-1}(y_1)$, $p_2 \in f^{-1}(y_2)$, and an arc $(p_1p_2) \subseteq f^{-1}(L)$. The collection, $\{G_\alpha\}$, is an open covering of the point set L. Since L is connected there exists a finite chain of these sets from a' to b', say $H_1, H_2, \ldots, H_n$, where $a' \in H_1$, $b' \in H_n$, and $H_i \cap H_j \neq \emptyset$ if and only if $j = i + 1$.

For each $H_i \cap H_{i+1}$, $i = 1,2,\ldots, n-1$, there exists a subarc of L, C_i, such that $C_i \subseteq H_i \cap H_{i+1}$. The hypothesis implies that for each C_i there exists a point $c_i \in C_i \subseteq H_i \cap H_{i+1}$ such that $f^{-1}(c_i)$ is locally connected, $i = 1,2,\ldots,n-1$. Attention is now focused on the $n-1$ points, $\{c_1, c_2, \ldots, c_{n-1}\}$.

Property Z implies there exist points $a \in f^{-1}(a')$ and $p_{11} \in f^{-1}(c_1)$ such that there exists an arc $(ap_{11}) \subseteq f^{-1}(L)$. Also for the same reason there exist points $p_{12} \in f^{-1}(c_1)$ and $p_{21} \in f^{-1}(c_2)$ such that there exists an arc $(p_{12}p_{21}) \subseteq f^{-1}(L)$. If $(ap_{11}) \cap (p_{12}p_{21}) \neq \emptyset$ then by picking the first point of intersection of $(p_{12}p_{21})$ with (ap_{11}) from a to p_{11} and calling it x then the resulting arc (axp_{21}) is an arc from a to p_{21}. The plan is to continue this tying-up process for n times assuming of course that with every step there is a nonempty intersection. If this intersection is always nonempty the result will be an arc (ab) lying entirely in $f^{-1}(L)$ such that $f(a) = a'$ and $f(b) = b'$. Let $L_1 = (ab)$. The point set $f(L_1)$ is a subcontinuum of L containing both a' and b' since f is continuous and closed and a, b $\in L_1$. Therefore, $f(L_1) = L$ since [3-p, 40] implies that L is irreducible with respect to being connected and containing both a' and b'. Thus f is an inverse arc map.

However, suppose that (ap_{i1}) is an arc from a to p_{i1}, for some $i = 1,2,\ldots, n-1$. Since $f(p_{i1}) = c_i$ the hypothesis implies that $f^{-1}(c_i)$ is a locally connected continuum. The reference [3-p, 84], implies that $f^{-1}(c_i)$ is arcwise connected. The hypothesis implies that there exist $p_{i2} \in f^{-1}(c_i)$ and $p_{(i+1)1} \in f^{-1}(c_{i+1})$ such that there exists an arc $(p_{i2}p_{(i+1)1})$ lying entirely in $f^{-1}(L)$. Also since $f^{-1}(c_i)$ is arcwise connected there exists an arc $(p_{i1}p_{i2})$ in $f^{-1}(c_i)$. Let x_1 be the first point of intersection of $(p_{i1}p_{i2})$ with (ap_{i1}) from a to p_{i1}. Because of the preceeding sentence an arc (ax_1p_{i2}) exists in $f^{-1}(L)$. Now let x_2 be the first point of intersection of $(p_{i2}p_{(i+1)1})$ with (ax_1p_{i2}) from a to p_{i2}. Then the arc (ap_{i2}) exists in $f^{-1}(L)$. Therefore by a finite number of steps, whether a nonempty intersection is obtained or not, an arc (ab) is obtained lying entirely in $f^{-1}(L)$. Thus, let $(ab) = L_1$ and as shown in the preceeding paragraph, $f(L_1) = L$. Therefore, f is an inverse arc map.

CHAPTER V

DECOMPOSITIONS OF GENERAL SPACES

The contents of Chapter V are directed, mainly, toward showing some general consequences of Chapter IV. The decomposition of Theorem 4.1 will here after be referred to as, Decomposition β. Even more briefly in the theorems to follow, as β.

Since β shows that a closed topological 2-cell can be decomposed into the union of uncountably many mutually exclusive nonlocally connected continua, the following theorem follows by an induction argument.

<u>Theorem 5.1</u>: Every closed n-cell, $n \geq 2$, can be represented as a union of uncountably many mutually disjoint nonlocally connected continua.

Theorems 5.2 and 5.3 are theorems fundamental in the proof of Theorem 5.4. Theorem 5.4 is the first result of this chapter concerning a nonlocally connected decomposition of a general continuum.

<u>Theorem 5.2</u>: If M is a space, C_n is a closed topological n-cell ($n \geq 2$), and R_{n-1} is the spherical boundary of C_n such that $M - R_{n-1} = A \cup B$ separate where $A \cup R_{n-1} = C_n$ then no point of $\left\{R_{n-1} - F(M - C_n)\right\}$ is isolated relative to $\left\{R_{n-1} - F(M - C_n)\right\}$.

Proof: First, it is noted that the hypothesis implies that $B = (M - C_n)$. If $\{R_{n-1} - F(B)\} = \emptyset$ then the theorem is true. Otherwise, consider $\{R_{n-1} - F(B)\} \neq \emptyset$ and let $x \in \{R_{n-1} - F(B)\}$. Suppose that x is isolated relative to $\{R_{n-1} - F(B)\}$. Since R_{n-1} is connected it follows that x is a limit point of R_{n-1}. Because x is isolated relative to $\{R_{n-1} - F(B)\}$ there exists $S(x,\epsilon)$ such that $S(x,\epsilon) \cap \{R_{n-1} - F(B)\} = \emptyset$. But also since x is a limit point of R_{n-1} there must exist a point $y \in R_{n-1}$ such that $y \neq x$ and $y \in S(x,\epsilon)$. This implies that $y \in F(B)$. This will be true for every $\epsilon > 0$ and therefore, x is a limit point of $F(B)$. Since $F(B)$ is closed it follows that $x \in F(B)$. This contradicts the supposition that $x \in \{R_{n-1} - F(B)\}$. From this the theorem follows.

Theorem 5.3: If M is a space, C_n is a closed topological n-cell $(n \geq 2)$, and R_{n-1} is the spherical boundary of C_n such that:

(1) $M - R_{n-1} = A \cup B$ separate where $A \cup R_{n-1} = C_n$ and

(2) $\{R_{n-1} - F(B)\} \neq \emptyset$

then there exists a topological (n-1)-cell, C_{n-1}, such that $C_{n-1} \subseteq \{R_{n-1} - F(B)\}$.

Proof: Let $x \in \{R_{n-1} - F(B)\}$. Suppose there does not exist any (n-1)-cell, C_{n-1}, such that $x \in C_{n-1} \subseteq \{R_{n-1} - F(B)\}$. Theorem 5.2 implies that x is not an isolated point of $\{R_{n-1} - F(B)\}$ relative to $\{R_{n-1} - F(B)\}$. Therefore, x is a limit point of $\{R_{n-1} - F(B)\}$. Let R be any region in the space M such that $x \in R$. Since R_{n-1} is an (n-1) sphere there exists a topological (n-1)-cell, C_{n-1}, such that $x \in C_{n-1} \subseteq R \cap R_{n-1}$. The supposition implies that $C_{n-1} \not\subseteq R \cap \{R_{n-1} - F(B)\}$. Therefore, there must exist a point $y \in C_{n-1} \cap F(B)$

which implies that x is a limit point of F(B). Since F(B) is closed it then follows that $x \in F(B)$ which contradicts the fact that $x \in \{R_{n-1} - F(B)\}$. This implies the theorem is true.

<u>Theorem 5.4</u>: If M is a continuum, C_2 is a closed topological 2-cell, and R_1 is the spherical boundary of C_2 such that:

(1) $C_2 \subseteq M$,

(2) $M - R_1 = A \cup B$ separate where $A \cup R_1 = C_2$, and

(3) $\{R_1 - F(B)\} \neq \emptyset$

then (a) M can be represented as the union of uncountably many mutually exclusive nonlocally connected continua and (b) there exists a continuous and monotonic inverse arc map, f, defined from M onto any arc E such that if $y \in E$ then $f^{-1}(y)$ is a nonlocally connected continuum.

<u>Proof</u>: Theorem 5.3 implies that there exists an arc (ab) such that $(ab) \subseteq \{R_1 - F(B)\}$. On this arc pick a point a_1 between a and b and then a point b_1 between a_1 and b. Let L be an arc from the point a to the point b such that $L \cap R_1 = \{a, b\}$ and $L - R_1$ is a subset of the interior of C_2. It is clear that $L \cup (ab)$ along with the complementary domain it bounds is a closed 2-cell itself. Call this closed 2-cell T, and if the arc L is associated with the arc $\{\{0\} \times I\}$, the arc (aa_1) with the arc $\{I \times \{1\}\}$, the arc (a_1b_1) with the arc $\{\{1\} \times I\}$, and the arc (bb_1) with the arc $\{I \times \{0\}\}$ in the closed 2-cell, $I \times I$, then, by the method in Decomposition β, the closed 2-cell T can be decomposed into the union of uncountably many mutually exclusive nonlocally connected continua. One notes that the nonlocally connected subcontinuum of T that will be associated with the points a and b will also contain L.

For clarity let $T = \bigcup A_x$, $x \in (bb_1)$, where A_x is the nonlocally connected continuum associated with the point x. The point set $C_2 - T$ is connected since the arc L separated C_2 into two connected sets and $C_2 - T$ is one of these sets. Since the arc (ab) contains no limit points of $M - C_2$ it follows that $(C_2 - T) \cup (M - C_2) = M - T$ is a connected set. It is true that $M - T$ is not closed since $L \subseteq T$ and L contains limit points of $C_2 - T$. The point set $\{(M - T) \cup A_b\}$ is a continuum since $(\overline{M - T}) = (M - T) \cup L$ and $\{(M - T) \cup A_b\} = \{(\overline{M - T}) \cup A_b\}$ since A_b contains L. This shows that $\{(M - T) \cup A_b\}$ is the union of two closed connected sets. Also the point set $\{(M - T) \cup A_b\}$ is nonlocally connected since A_b is constructed as a nonlocally connected subcontinuum of the closed 2-cell T.

Now, let M be represented in the following manner.

$$M = \{(M - T) \cup A_b\} \cup \{\bigcup A_x : x \in \{(bb_1) - b\}\}$$

Considering the continuum $\{(M - T) \cup A_b\}$ as a single nonlocally connected subcontinuum of M, the desired decomposition of M in conclusion (a) is obtained.

To obtain conclusion (b) the map f, $f(M) = (bb_1)$, is defined such that $f\{(M - T) \cup A_b\} = b$ and $f(A_x) = x$ for each $x \in \{(bb_1) - b\}$. With this definition of the map f it follows, as in Lemma 4.1 and Theorem 4.2, that f is a continuous and monotonic inverse arc map such that if $y \in (bb_1)$ then $f^{-1}(y)$ is a nonlocally connected subcontinuum of M. Let E be any arc. Let h be a homeomorphism such that $h\{(bb_1)\} = E$. Therefore, $hf(M) = E$ and hf is the continuous and monotonic function called for by conclusion (b).

Some of the theorems to follow demand another decomposition of the closed 2-cell besides the one given in Theorem 4.1. A discussion will now be given to show that the closed 2-cell can be decomposed into uncountably many mutually exclusive nonlocally connected continua in a different way from that of Theorem 4.1. The only explanation given of this decomposition is to exhibit a figure, Figure 5.1, and from the information already given in Theorem 4.1 it will be clear that this decomposition will give the desired results.

In reference to Figure 5.1, each nonlocally connected continuum will be determined in the same way as the nonlocally connected continua were determined in Theorem 4.1. That is, each of these continua will be associated with a unique point of the unit interval except for the continuum associated with the point (1,0). The only difference lies in the fact that instead of constructing an arc beginning at each point $(k/2^n, 0)$, a simple closed curve is constructed through each point $(k/2^n, 0)$ in the manner described in Figure 5.1. The continuum associated with the point (1,0) will be the union of the one obtained in the manner of Theorem 4.1 and the disk A. Let this continuum be D. It follows that D is also a nonlocally connected continuum.

Notice, the continuum associated with the point (0,0) again is not locally connected and name this continuum H. The point set H is not locally connected because of the manner in which the point like nonlocally connected continuum T has been imbedded in H. As mentioned, this construction will determine the desired decomposition of the closed 2-cell in the same manner as the construction in Theorem 4.1 determined the desired decomposition. Let the decomposition obtained here be called Decomposition β_1.

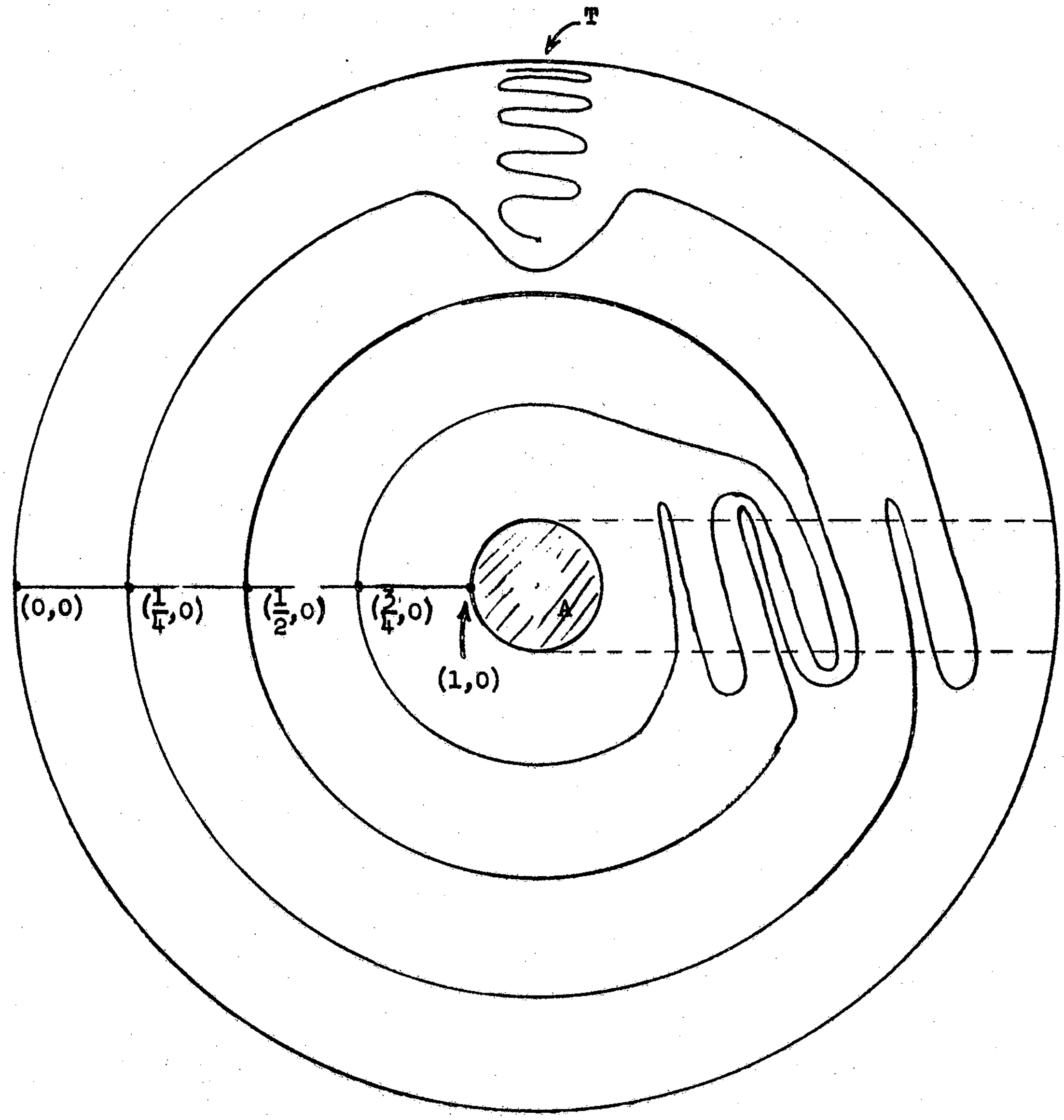

Figure 5.1a

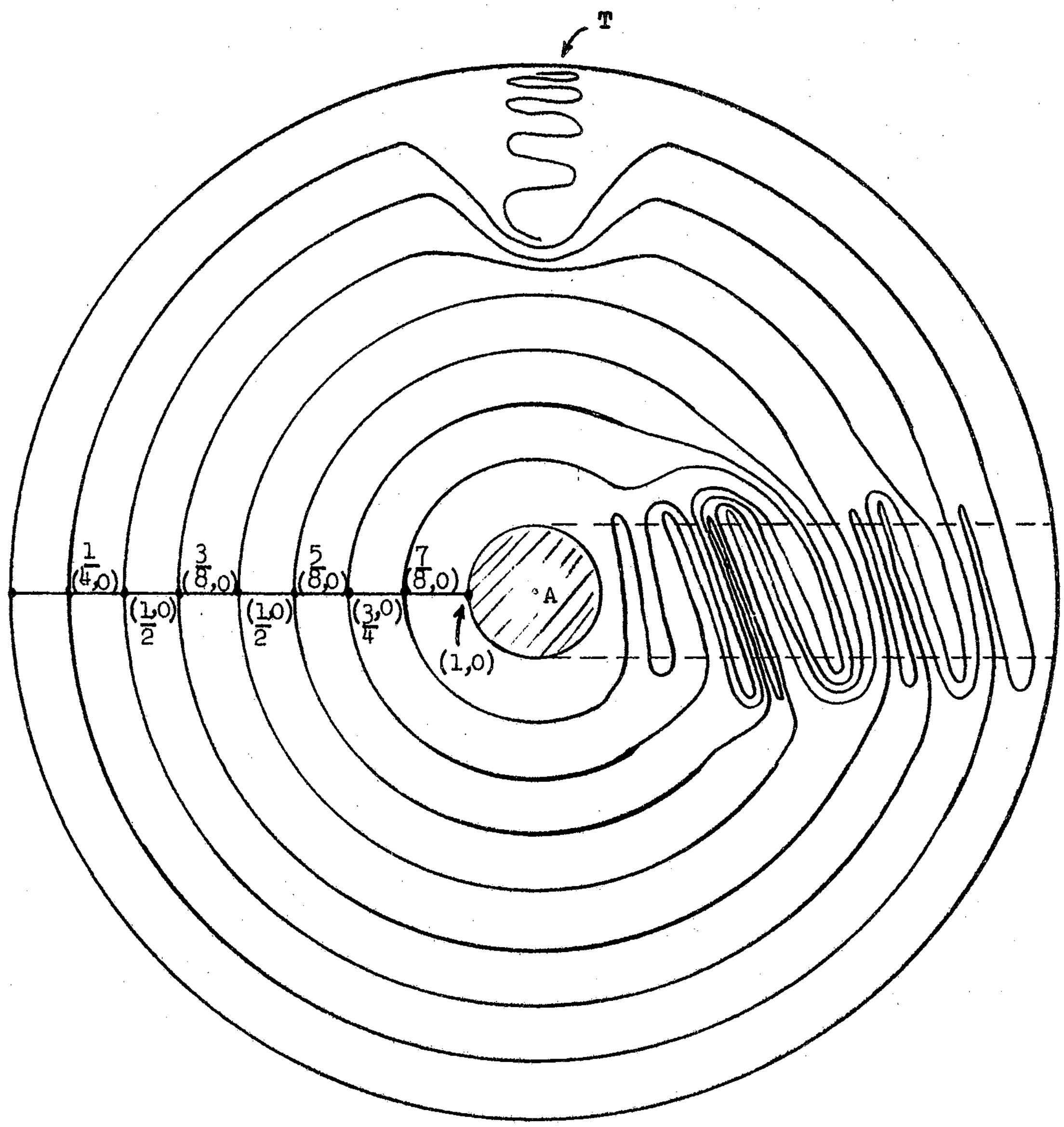

Figure 5.1b

Decomposition β_1 allows one to state and prove the following theorem.

Theorem 5.5: If M is a continuum, C_2 is a closed topological 2-cell, and R_1 is the spherical boundary of C_2 such that:

(1) $C_2 \subseteq M$,

(2) $M - R_1 = A \cup B$ separate where $A \cup R_1 = C_2$, and

(3) $\{R_1 - F(B)\} = \emptyset$

then (a) M can be represented as the union of uncountably many mutually exclusive nonlocally connected continua and (b) there exists a continuous and monotonic inverse arc map, f, defined from M onto any arc E such that if $y \in E$ then $f^{-1}(y)$ is a nonlocally connected continuum.

Proof: Let C_2' be a closed 2-cell such that $C_2' \subseteq C_2^o$ where C_2^o is the interior of C_2. Now consider a Decomposition β_1 of C_2'. Let $N = (M - C_2') \cup H'$ where H' is the homeomorphic image of the subcontinuum H in Decomposition β_1. It follows that N is a nonlocally connected continuum since H' is a nonlocally connected subcontinuum of C_2'. Now, let M be represented in the following manner.

$$M = N \cup \left\{ \cup A_x' : x \in (0,1) \right\} \cup D'$$

The point set A_x' is the homeomorphic image of the nonlocally connected continuum associated in Decomposition β_1 with the point $(x,0)$, $0 < x < 1$. The point set D' is a homeomorphic image of the D subcontinuum in Decomposition β_1. Therefore, this representation of M satisfies conclusion (a).

Let f be a map defined on M such that $f(M) = I$, $f(N) = (0,0)$, $f(A_x) = (x,0)$, and $f(D) = (1,0)$. With this definition of the map f it follows, as in Lemma 4.1 and Theorem 4.2, that f is a continuous and monotonic inverse arc map such that if $y \in I$ then $f^{-1}(y)$ is a nonlocally connected subcontinuum of M. Let E be any arc. Let h be a homeomorphism such that $h(I) = E$. Therefore, hf, $hf(M) = E$, is the desired continuous and monotonic function needed to obtain conclusion (b).

Theorem 5.6: If M is a continuum, C_2 is a closed topological 2-cell, and R_1 is the spherical boundary of C_2 such that:

(1) $C_2 \subseteq M$ and

(2) $M - R_1 = A \cup B$ separate where $A \cup R_1 = C_2$,

then (a) M can be represented as the union of uncountably many mutually exclusive nonlocally connected continua and (b) there exists a continuous and monotonic inverse arc map, f, defined from M onto any arc E such that if $y \in E$ then $f^{-1}(y)$ is a nonlocally connected continuum.

Proof: This is a direct result of Theorem 5.4 and Theorem 5.5.

A note of interest is that Theorem 5.6 implies that every 2-manifold, M, as defined in [6], can be represented as the union of uncountably many mutually exclusive nonlocally connected continua and there exists a continuous and monotonic inverse arc map, f, defined from M onto any arc E such that if $y \in E$ then $f^{-1}(y)$ is a nonlocally connected subcontinuum of M.

The following theorem gives a decomposition of E^2 into nonlocally connected subcontinua of E^2.

Theorem 5.7: Euclidean two space can be decomposed into uncountably many mutually exclusive nonlocally connected continua. Also there exists a continuous and monotonic inverse arc map, f, from E^2 onto E^1

such that the preimage of each point of E^1 is a nonlocally connected continuum.

Proof: In each closed 2-cell, $\{[n,n+1] \times I\}$, $n = 0, \pm 1, \pm 2, \ldots$, construct a Decomposition β. For each point c in the closed interval from n to n + 1, $n = 0, \pm 1, \pm 2, \ldots$, let A_c^n be the nonlocally connected continuum associated with the point (c,0) in Decomposition β. If c = n for some integer n then let the continuum $L_c = A_c^{n-1} \cup A_c^n$. The continuum L_c is a nonlocally connected continuum since A_c^n is a nonlocally connected subcontinuum of the closed 2-cell $\{[n,n+1] \times I\}$. In this way with each point (c,0) in E^2 a unique nonlocally connected subcontinuum of E^2 is associated.

Now, if c is a point of the open interval from n to n + 1 for some integer n then let $G_c = A_c^n \cup \{(c,x): x > 1 \text{ or } x < 0\}$. It follows that G_c is a nonlocally connected subcontinuum of E^2 since A_c^n is a nonlocally connected subcontinuum of the closed 2-cell $\{[n,n+1] \times I\}$. If c = n for some integer n then let $H_c = L_c \cup \{(c,x): x > 1 \text{ or } x < 0\}$. Again, it follows that H_c is a nonlocally connected subcontinuum of E^2 since L_c is a nonlocally connected subcontinuum of the closed 2-cell $\{[n-1, n+1] \times I\}$. Obviously, $E^2 = \{\cup H_c : c \text{ is an integer}\}^* \cup \{\cup G_c : c \text{ is real but } c \text{ is not an integer}\}^*$. Therefore, the collection of nonlocally connected subcontinua of E^2, $\{\{H_c\} \cup \{G_c\}\}$, is a decomposition of E^2 into nonlocally connected subcontinua of E^2. Figure 5.2 illustrates this decomposition.

Let f, $f(E^2) = E^1$, be a map defined such that $f(H_c) = c$ and $f(G_c) = c$. With this definition of the map f it follows, as in Lemma 4.1 and Theorem 4.2, that f is a continuous and monotonic inverse arc map such that if $c \in E^1$ then

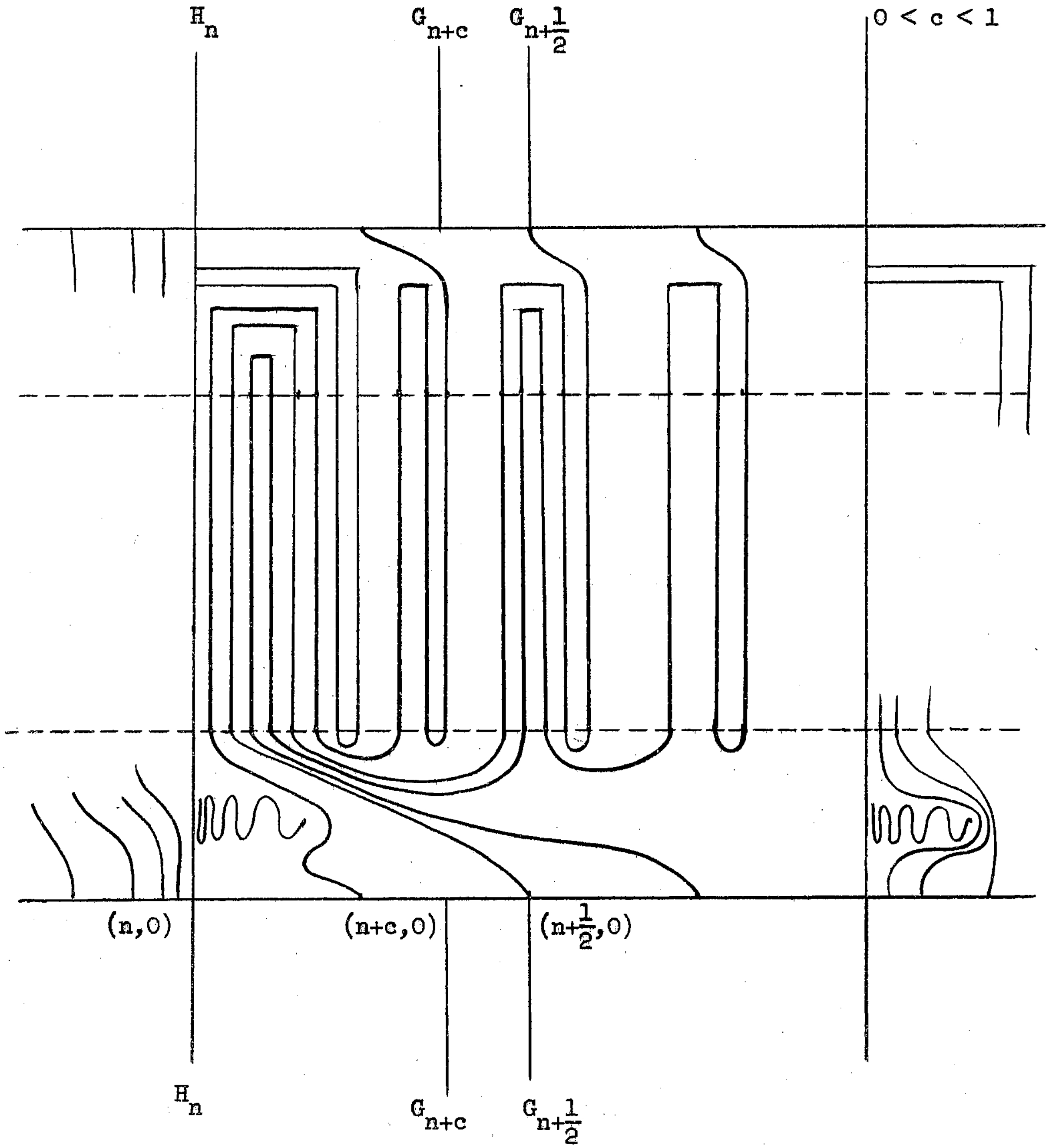

Figure 5.2

$f^{-1}(c)$ is a nonlocally connected subcontinuum of E^2. Therefore, the proof is complete.

<u>Theorem 5.8</u>: Euclidean n-space, $n \geq 2$, can be decomposed into the union of uncountably many mutually exclusive nonlocally connected continua.

<u>Proof</u>: Theorem 5.7 implies that the theorem is true for E^2. Assume the theorem is true for E^{k-1} where k is some positive integer, $k \geq 3$. Consider the subspace $K_c = \{(x_1, x_2, \ldots, x_{k-1}, c) : x_i$ is real, c is a fixed real$\}$, as a subspace of E^k for each real number c. The point set K_c is homeomorphic with E^{k-1} and therefore, the induction hypothesis implies that K_c can be represented as the union of uncountably many mutually exclusive nonlocally connected subcontinua of E^{k-1}. Since E^{k-1} is closed in E^k then the subcontinua mentioned in the previous sentence are also nonlocally connected subcontinua of E^k. Therefore, since $E^k = \bigcup K_c$ as c varies over the reals the theorem is then proved.

Whyburn, [5-p, 125], proves the following theorem concerning the hyperspace M' of an upper semi-continuous decomposition B of a space M. "If M is locally connected, so also is M'". As a consequence of this theorem the following two theorems can be stated and proved.

<u>Theorem 5.9</u>: If M is a space and G an upper semi-continuous decomposition of M, where M' is the hyperspace of M associated with G, then if M' can be decomposed into uncountably many mutually exclusive nonlocally connected subcontinua of M', so also can M.

<u>Proof</u>: Let $T' = \{T'_\alpha\}$, $\alpha \in \Upsilon$, be the decomposition of M' mentioned in the hypothesis. For each α, [5-p, 125] implies that T_α is a subcontinuum of M. The points of T' are elements of G (i.e. the collection

$\{g'_{\alpha_\eta}\}$, $\eta \in \psi$, is the set of points of T'_α but for each η, $g_{\alpha_\eta} \subseteq M$). Since any subcollection of an upper semi-continuous collection is itself upper semi-continuous then the points of T' form an upper semi-continuous decomposition of T_α. Thus, in reference to Whyburn's results mentioned above it follows that since T'_α is nonlocally connected in M' then T_α is not locally connected in M. Therefore, the decomposition $T = \{T_\alpha\}$, $\alpha \in \Upsilon$, is one of the type desired for the space M.

Theorem 5.10: If M is a space and G an upper semi-continuous decomposition of M, where M' is the hyperspace of M associated with G, and there exists a continuous map f' defined on M' with range an arc L such that the preimage of each point of L is a nonlocally connected subcontinuum of M' then there exists a continuous map f defined on M with range L such that the preimage of each point of L is a nonlocally connected subcontinuum of M.

Proof: The continuous map f' decomposes M' into the collection $\{T'_y\}$, where $y \in L$, $f^{-1}(y) = T'_y$, such that for each $y \in L$, T'_y is a nonlocally connected subcontinuum of M'. It follows by Whyburn's theorem that T_y is a nonlocally connected subcontinuum of M. Therefore, let h, $h(M) = M'$, be a map defined such that $h(x) = g' \in M'$ if and only if $x \in g$ in M. The reference, [5-p, 125], implies that h is a continuous map. Let f, $f(M) = (f'h)(M) = L$, be the continuous composite map from M onto L. Let $y \in L$ and consider $f^{-1}(y)$.

$$f^{-1}(y) = (f'h)^{-1}(y) = h^{-1}\left[(f')^{-1}(y)\right] = h^{-1}(T'_y) = T_y$$

Again, since T'_y is a nonlocally connected subcontinuum of M', then [5-p,125] implies that T_y is a nonlocally connected subcontinuum of M. Thus the theorem is proved.

CHAPTER VI

SUMMARY

This paper is primarily concerned with two objectives, namely those of a study of some fundamental properties of a continuous and monotonic inverse arc map and the decomposition of the closed 2-cell into the union of uncountably many mutually exclusive nonlocally connected continua.

The inverse arc map is defined and then in Chapter II some of the fundamental properties pertaining to this map are proved. In Chapter III the investigation of the notion of an inverse arc map is not pursued in detail, but the general theme of the inverse arc map is maintained. If f is a continuous and monotonic map such that $f(X) = Y$, then an investigation is made into the question of the effect on X if Y is an indecomposable continuum; and conversely into the question of the effect on Y if X is an indecomposable continuum.

One of the principal results of this paper is in Chapter IV. If M is a closed 2-cell, that is, $M = I \times I$, then M can be decomposed into the union of uncountably many mutually exclusive nonlocally connected continua. As a consequence of this result, there exists a continuous and monotonic inverse arc map, f, such that $f(M) = I$ and such that if $y \in I$ then $f^{-1}(y)$ is nonlocally connected. Therefore, [5-p, 125] implies the collection $\left\{f^{-1}(y)\right\}$, $y \in I$, is an upper semi-continuous decomposition of M into uncountably many nonlocally connected continua.

Chapter V shows that if M is a 2-manifold then M can be decomposed into the union of uncountably many mutually exclusive nonlocally connected continua. Also, there exists a continuous and monotonic inverse arc map, f, such that $f(M) = I$ and such that if $y \in I$ then $f^{-1}(y)$ is nonlocally connected.

Some questions for further study are the following. What are some other characterizations of an inverse arc map? Can an indecomposable continuum be decomposed into uncountably many mutually exclusive nondegenerate subcontinua? If M_1 and M_2 are indecomposable continua and $M_1 \cap M_2 \neq \emptyset$ then can $M_1 \cap M_2$ contain a domain relative to $M_1 \cup M_2$? Can E^2 be decomposed into a collection of compact nondegenerate indecomposable subcontinua?

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