ANALYSIS OF PLANAR FRAMES LOADED NORMALLY
BY COMPLEMENTARY POTENTIAL ENERGY

By

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PREFACE

The method of complementary energy for the analysis of planar frames loaded normal to the plane is presented in this thesis. A cantilever basic structure is adopted and the cross-sectional vectors of the near end are chosen as the redundants. These are then defined in terms of applied loads and redundants of the whole system. The complementary energy is expressed in a matrix form. Minimization with respect to each of the redundants yields a sufficient number of simultaneous algebraic equations for the solution.

This research is the result of ideas expressed by Professor Jan J. Tuma during the writer's consultations with him before the undertaking of this project.

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NOMENCLATURE

a (subscript)	A member.
b	Action influence coefficient.
A	Area of cross-section.
E	Modulus of elasticity.
E (superscript)	Refers to the element reference system.
f_{iars}	rs^{th} element of flexibility matrix of member a at end i.
f'_{iars}	rs^{th} element of deformation influence coefficient matrix.
G	Modulus of rigidity.
GJ	Equivalent torsional rigidity.
$\left\{ H_i \right\}$	Column vector of reactive actions at end i.
i	Near end of member a.
I	Moment of inertia of cross-section.
j	Far end of member a.
k	Shear constant.
M	Reactive moment.
M (superscript)	Refers to member reference system.
N	Reactive force.
O (superscript)	Refers to basic reference system.
p	Applied distributed load.
P	Applied concentrated load.
q	Position of a general section on a span.
Q	Applied moment in plane of member.

$[r]$	Transmission matrix.
R	Redundant.
s	Interval of integration.
SM	Static equivalent moment.
SN	Static equivalent force.
$\{SW\}$	Static equivalent vector of actions.
t	Location of generalized force on basic structure.
T	Twisting moment.
T (superscript)	Refers to the transpose of a matrix.
U	Strain energy.
U*	Complementary strain energy.
v	Volume.
V*	Complementary potential energy of prescribed displacements.
w	Generalized displacement.
W	Generalized load.
X,Y,Z	Coordinate axes.
α	Influence coefficient.
β	Angle.
γ	Constant.
γ (subscripted)	Unit shear strain.
δ	Variation.
$\{\Delta\}$	Column vector of end displacements.
ϵ	End deflection.
ε	Unit normal strain.
η	Deflection Load function.
θ	Angle.

μ	Constant, EI/GJ.
$\{\nu\}$	Vector of load functions.
Π^*	Total complementary energy.
ρ	Constant, kEI/GA.
σ	Unit normal stress.
τ	Angular load function.
τ	Unit shear stress (when subscripts are coordinate axes).
ϕ	End rotation of member.
ψ, Ψ	Influence coefficients
$[\omega]$	Rotation matrix.
$\{\ \}$	Column matrix.
$[\ ]$	Rectangular matrix.

CHAPTER I

INTRODUCTION

1-1. Statement of the Problem

A structure lying in one plane and loaded normal to it is defined as an Order II structure. The analysis of frames of this type by means of complementary potential energy is the subject of this study. The investigation is restricted to frames consisting of straight, bent, or curved members having either constant or variable cross-sections. The causes of deformations may be forces, moments, displacements of supports, temperature variations, or changes in volume of the material.

1-2. Limitation of the Problem

The analysis is based on the following assumptions:

1. One of the principal planes of each member coincides with the plane of the structure.
2. The material of the structure is homogeneous, isotropic and continuous.
3. All deformations are small and elastic.
4. The material obeys Hooke's Law.
5. The moduli of elasticity and rigidity are known numbers.

1-3. Historical Review

The principle of complementary energy, first stated by Engesser in 1889 is a generalization of Castigliano's theorems on least work presented in 1873 (1). A rigorous presentation of the principle as applied in the theory of elasticity has been given by Sokolnikoff (2). Formulation, proofs, and applications of the theorem of minimum complementary energy for statically loaded structures with minute deformations may be found in the works of Argyris and Kelsey (3), Brown (4), Hoff (1), Langhaar (5), Pippard (6), and others. Westergaard (7) used this principle in a generalized form in the solution of buckling and vibration problems. Charlton (8) approached the problem from a new perspective, employing the law of conservation of energy as a starting point. Libove (9) extended the theory to include structures with finite deformations. Berman (10) applied the complementary energy method in matrix form to planar structures loaded in the plane, and Li (11) analyzed a truss as a rigid frame by the same method. The extension of the method of complementary energy to the analysis of frames of Order II is developed herein.

1-4. Notation

The symbols adopted are defined where they first appear and are arranged alphabetically under Nomenclature.

CHAPTER II

BASIC STRUCTURE

2-1. General

The frame to be analyzed is made "statically determinate" by temporarily releasing an adequate number of actions.[1] The structure thus formed is called a "Basic Frame" (Fig. 2-1). If a frame has n degrees of redundancy, n releases will be required. The suppression of each internal action at a given cross-section of a frame corresponds to one release. The solution of a statically indeterminate frame will be considered achieved when the actions at the chosen releases (redundants) are found.

2-2. Coordinate Systems

Three coordinate systems are introduced: the reference system, the member system, and the element system (Fig. 2-2). Each one consists of a right-handed set of orthogonal axes. The first is oriented so that the X and Y axes are in the plane of the structure with the origin arbitrarily located. This system is referred to as the "O" system and all terms associated with it are labeled with an O superscript.

The Z axis of the second is parallel to the Z axis of the O system while its X axis is a straight line originating at end i of the member and

[1] In this dissertation "action" will indicate a generalized force and "displacement" will refer to a deflection or a rotation.

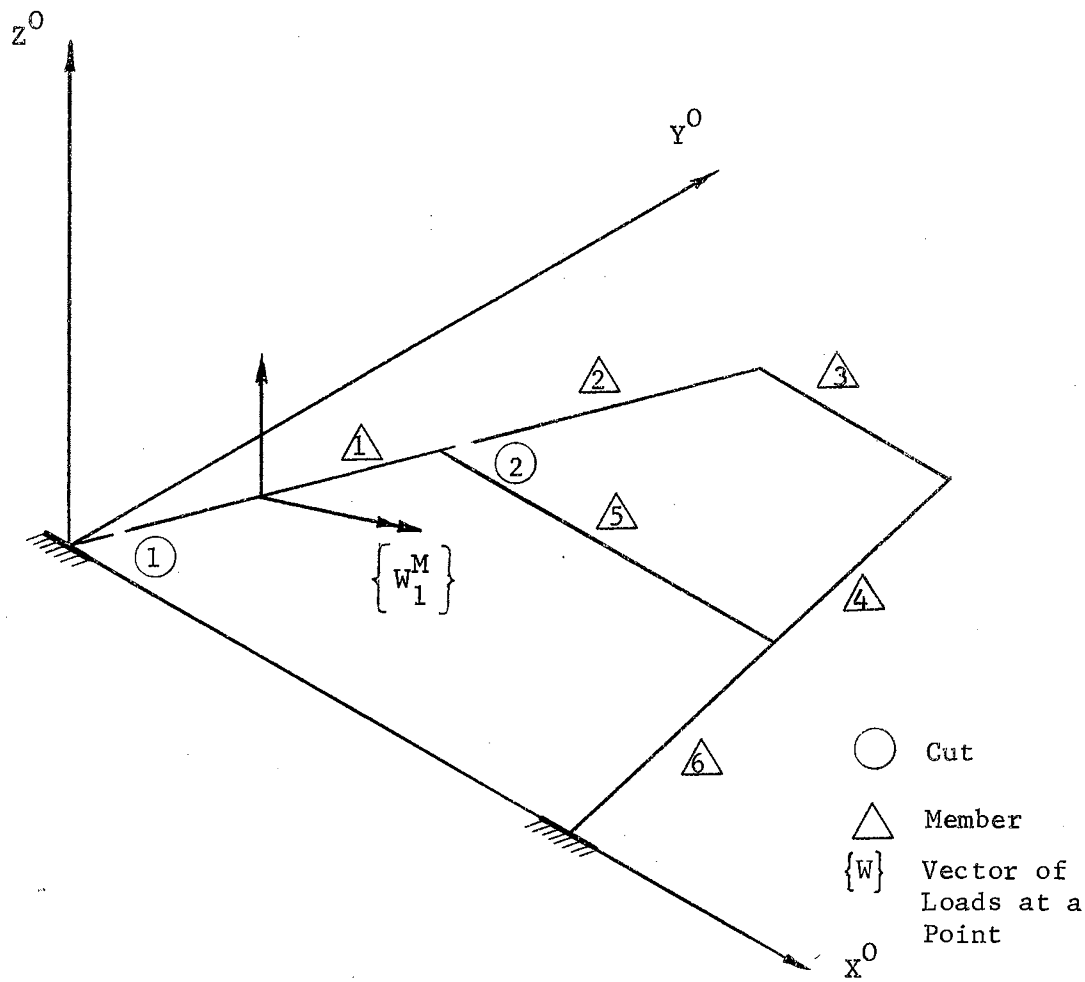

Figure 2-1. Basic Frame

directed through the other end j. All quantities in this system are distinguished by the superscript M. The angle measured from the X axis of the 0 system to the X axis of the M system according to the right hand rule is designated by θ (Fig. 2-2).

The third system is required only in the derivation of the flexibilities of a curved member. Its Y-Z plane is the plane of the cross-section

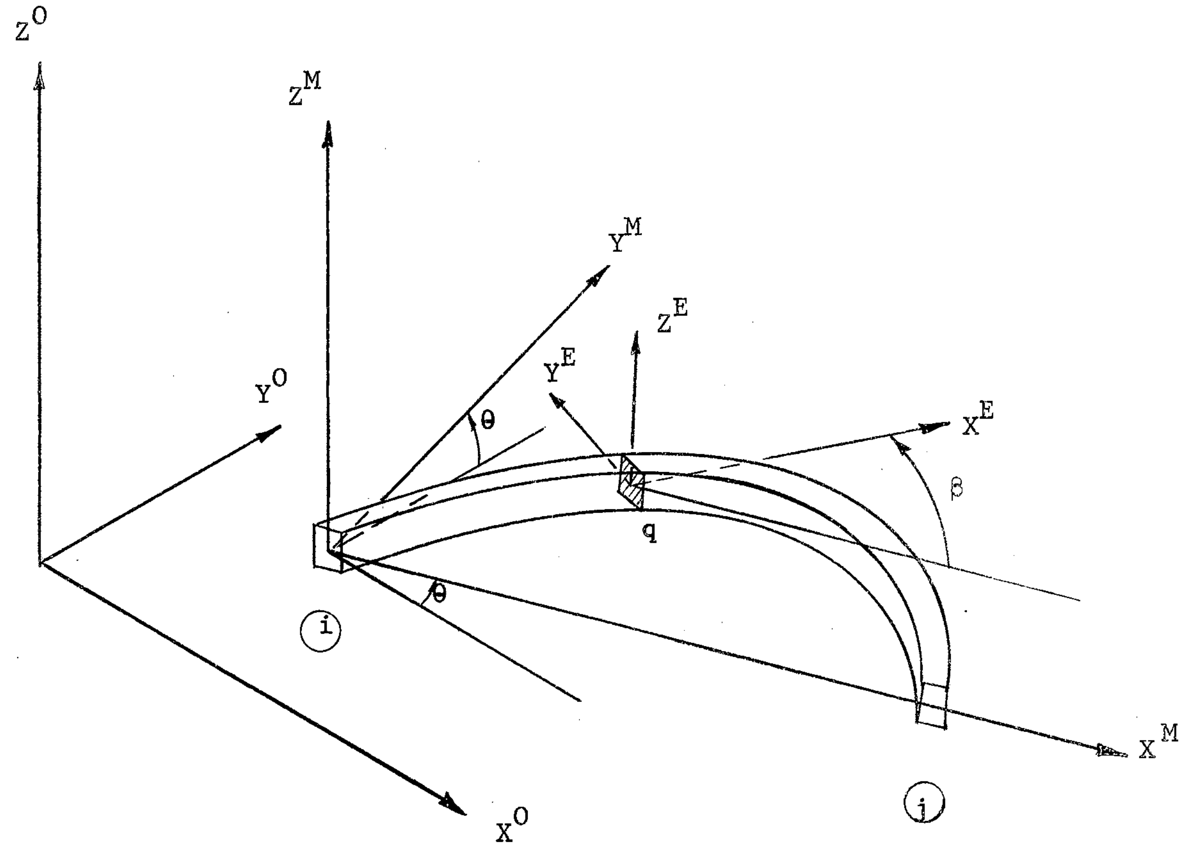

Figure 2-2. Coordinate Systems

at any point along the member, and its X axis is normal to the cross-section at that point and passes through its centroid. Related terms are characterized by the superscript E. β is the angle measured in the X-Y plane from the M system to the element system (Fig. 2-2).

It is assumed that the shear center of any cross-section coincides with its centroid.

2-3. Cantilever Basic Structure

The cantilever basic structure (Fig. 2-3) is fixed at end j and free at end i. $P_1^M, \ldots, P_t^M$ are applied forces normal to the plane, and $Q_1^M, \ldots, Q_s^M$ are applied moment vectors in the plane of the member. Moments M_{iax}^M and M_{iay}^M, and force N_{iaz}^M are the member redundants and are treated as arbitrary loads. Collectively they are designated by the column matrix $\left\{H_{ia}^M\right\}$. The moments and force at j (M_{jax}^M, M_{jay}^M and N_{jax}^M) can be expressed as a function of the applied loads and member redundants utilizing the static equilibrium of the cantilever.

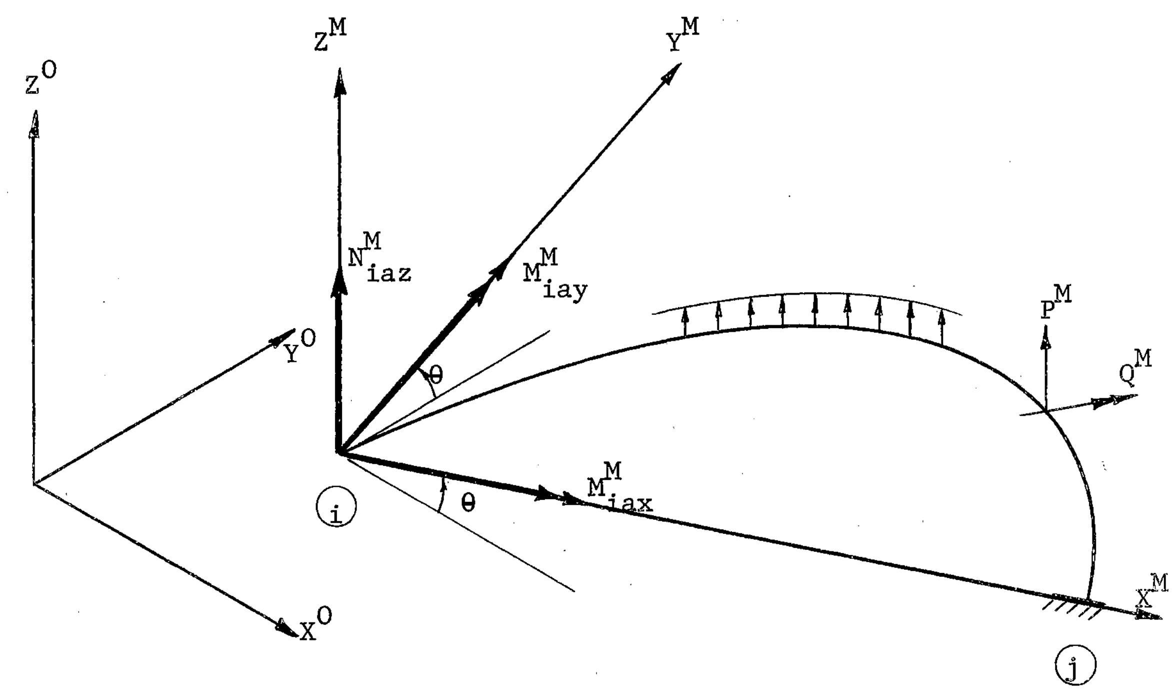

Figure 2-3. Cantilever Basic Structure

Force and moment vectors are positive if acting in the positive sense of the appropriate reference system.

2-4. Cross-Sectional Elements

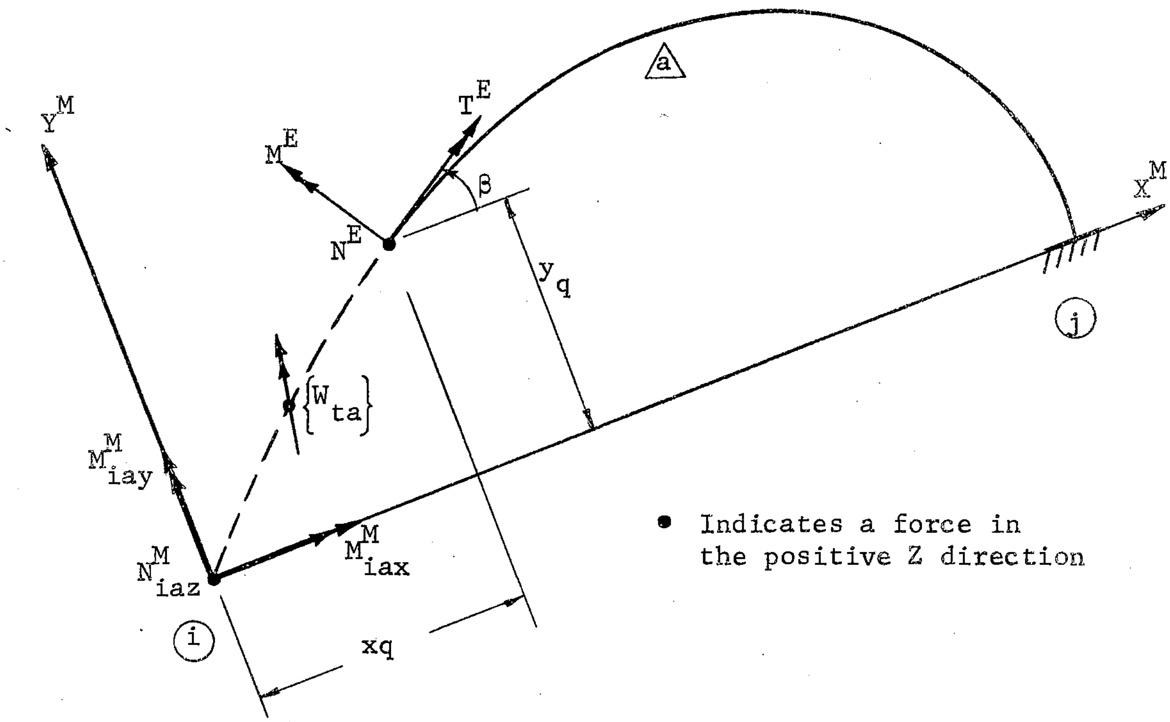

Figure 2-4. Cross-sectional Elements at Point q

The torsional moment, bending moment, and shearing force at any section q of member a due to end actions $\left\{H^M_{ia}\right\}$, are given by

$$\begin{bmatrix} T^{E(H)}_{qa} \\ M^{E(H)}_{qa} \\ N^{E(H)}_{qa} \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -y^M_{qi} \\ 0 & 1 & x^M_{qi} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M^M_{iax} \\ M^M_{iay} \\ N^M_{iaz} \end{bmatrix} \tag{2-1}$$

where x^M_{qi} and y^M_{qi} are the coordinates of the shear center at point q with respect to end i in the member system. Symbolically, Eq. (2-1) may be written as

$$\left\{W^{E(H)}_{qa}\right\} = \left[\omega^{EM}_q\right] \left[r^M_{qi}\right] \left\{H^M_{ia}\right\} \tag{2-1a}$$

where $\left[r^M_{qi}\right]$ is the transmission matrix that transfers the end actions in the M system to point q from point i, and $\left[\omega^{EM}_q\right]$ is the rotation matrix at point q that rotates these actions into the element coordinate system. Similarly, the cross-sectional elements at q due to applied loads on the cantilever are

$$\left\{W^{E(L)}_{qa}\right\} = \left[\omega^{EM}_q\right] \left[r^M_{qt}\right] \left\{W^M_{ta}\right\} \tag{2-2}$$

in which the summation is taken only over the segment to the left of q (Fig. 2-4). By superposition the total actions at q due to end actions and applied loads are:

$$\left\{W^E_{qa}\right\} = \left\{W^{E(H)}_{qa}\right\} + \left\{W^{E(L)}_{qa}\right\} \tag{2-3}$$

2-5. Deformations of Basic Structure

For an elastic member the total displacement at the end i is the sum of the partial displacements caused by successive applications of the loads and of the member redundants. These may be found by employing Castigliano's first theorem, which gives

$$\phi^M_{iax} = \frac{\partial U_a}{\partial M^M_{iax}}$$

$$\phi^M_{iay} = \frac{\partial U_a}{\partial M^M_{iay}}$$

$$\epsilon^M_{iaz} = \frac{\partial U_a}{\partial N^M_{iaz}}$$

where

ϕ^M_{iax} denotes the total rotation at i of member a about the X axis of the M system,

ϕ^M_{iay} denotes the total rotation at i of member a about the Y axis of the M system, and

ϵ^M_{iaz} denotes the total deflection at i of member a in the Z direction of the M system.

The strain energy U_a stored in member a is given by

$$U_a = U_{torsion} + U_{bending} + U_{shear}$$

or

$$U_a = \frac{1}{2}\left[\int_s (T^E_{qa})^2 \frac{ds}{GJ} + \int_s (M^E_{qa})^2 \frac{ds}{EI} + k \int_s (N^E_{qa})^2 \frac{ds}{GA}\right] \qquad (2\text{-}4)$$

where G and E are the moduli of rigidity and elasticity respectively, GJ is the equivalent torsional rigidity, A and I are the area and moment of inertia of the cross-section, and k is the shear constant for the section.

Substituting from Eq. (2-1) into Eq. (2-4) the displacements at i due to end actions $\left\{H^M_{ia}\right\}$ become

$$\phi^{M(H)}_{iax} = \frac{\partial U_a}{\partial M^M_{iax}} = M^M_{iax}\, f^M_{iaxx} + M^M_{iay}\, f^M_{iaxy} + N^M_{iaz}\, f^M_{iaxz}$$

$$\phi^{M(H)}_{iay} = \frac{\partial U_a}{\partial M^M_{iay}} = M^M_{iax}\, f^M_{iayx} + M^M_{iay}\, f^M_{iayy} + N^M_{iaz}\, f^M_{iayz} \qquad (2\text{-}5)$$

$$\epsilon^{M(H)}_{iaz} = \frac{\partial U_a}{\partial N^M_{iaz}} = M^M_{iax}\, f^M_{iazx} + M^M_{iay}\, f^M_{iazy} + N^M_{iaz}\, f^M_{iazz}$$

or in matrix form

$$\begin{bmatrix} \phi^{M(H)}_{iax} \\ \phi^{M(H)}_{iay} \\ \epsilon^{M(H)}_{iaz} \end{bmatrix} = \begin{bmatrix} f^M_{iaxx} & f^M_{iaxy} & f^M_{iaxz} \\ f^M_{iayx} & f^M_{iayy} & f^M_{iayz} \\ f^M_{iazx} & f^M_{iazy} & f^M_{iazz} \end{bmatrix} \begin{bmatrix} M^M_{iax} \\ M^M_{iay} \\ N^M_{iaz} \end{bmatrix} \qquad (2\text{-}6)$$

which may be written as

$$\left\{\Delta^{M(H)}_{ia}\right\} = \left[f^M_{ia}\right] \left\{H^M_{ia}\right\} \qquad (2\text{-}6a)$$

where the coefficient matrix is symmetrical and its terms are as given in Table 2-1. A typical coefficient f^M_{iars}, defined as an end flexibility, denotes the displacement of i of member a in the direction of the r axis of the M system due to a unit cause at i in the direction of the s axis of the M system, all other causes being zero.

The displacements at i due to a load vector $\left\{W^M_{ta}\right\}$ at point t (Fig. 2-4) may be found in a like manner if the static equivalent of $\left\{W^M_{ta}\right\}$ is placed at point i and the partial derivatives are taken with respect to these equivalent end forces. The interval of integration for the energy expression is tj.

TABLE 2-1. ALGEBRAIC EXPRESSIONS FOR FLEXIBILITY COEFFICIENTS

$$f^M_{iaxx} = \int_s \cos^2 \beta \frac{ds}{GJ} + \int_s \sin^2 \beta \frac{ds}{EI}$$

$$f^M_{iaxy} = \int_s \sin \beta \cos \beta \frac{ds}{GJ} - \int_s \sin \beta \cos \beta \frac{ds}{EI}$$

$$f^M_{iaxz} = \int_s (x \sin \beta \cos \beta - y \cos^2 \beta)\frac{ds}{GJ} - \int_s (x \sin\beta \cos\beta + y \sin^2 \beta)\frac{ds}{EI}$$

$$f^M_{iayy} = \int_s \sin^2 \beta \frac{ds}{GJ} + \int_s \cos^2 \beta \frac{ds}{EI}$$

$$f^M_{iayz} = \int_s (x \sin^2\beta - y \sin \beta \cos \beta) \frac{ds}{GJ} + \int_s (x \cos^2\beta + y \sin\beta \cos\beta)\frac{ds}{EI}$$

$$f^M_{iazz} = \int_s (x \sin \beta - y \cos \beta)^2 \frac{ds}{GJ} + \int_s (x \cos\beta + y \sin\beta)^2 \frac{ds}{EI} + k \int_s \frac{ds}{GA}$$

The static equivalent of $\left\{W^M_{ta}\right\}$ at point i is

$$\left\{ SW^M_{ita} \right\} = \left[r^M_{it} \right] \left\{ W^M_{ta} \right\} \tag{2-7}$$

that is

$$\begin{bmatrix} SM^M_{itax} \\ SM^M_{itay} \\ SN^M_{itaz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -y^M_{it} \\ 0 & 1 & x^M_{it} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q^M_{tax} \\ Q^M_{tay} \\ P^M_{taz} \end{bmatrix}$$

Thus

$$\phi^{M(L)}_{iax} = \frac{\partial U'_a}{\partial SM^M_{itax}} = \tau^M_{itax}$$

$$\phi^{M(L)}_{iay} = \frac{\partial U'_a}{\partial SM^M_{itay}} = \tau^M_{itay}$$

$$\epsilon^{M(L)}_{iaz} = \frac{\partial U'_a}{\partial SN^M_{itaz}} = \eta^M_{itaz}$$

where U'_a is the energy stored in member a from t to j, due to the loads at point t, and where τ^M_{itax}, τ^M_{itay}, and η^M_{itaz} are the load functions at point i resulting from the same cause. This gives

$$\begin{bmatrix} \tau^M_{itax} \\ \tau^M_{itay} \\ \eta^M_{itaz} \end{bmatrix} = \begin{bmatrix} f^{M'}_{iaxx} & f^{M'}_{iaxy} & f^{M'}_{iaxz} \\ f^{M'}_{iayx} & f^{M'}_{iayy} & f^{M'}_{iayz} \\ f^{M'}_{iazx} & f^{M'}_{iazy} & f^{M'}_{iazz} \end{bmatrix} \begin{bmatrix} SM^M_{itax} \\ SM^M_{itay} \\ SN^M_{itaz} \end{bmatrix} \tag{2-8}$$

or

$$\left\{ \nu^{M}_{ita} \right\} = \left[f^{M'}_{ia} \right] \left\{ SW^{M}_{ita} \right\} \qquad (2\text{-}8a)$$

where the coefficients $f^{M'}_{ia}$ are the same algebraic expressions as those of f^{M}_{ia} shown in Table 2-1, except that in this case all the integrations are performed over the interval tj. If several loads are applied on the span, the total load functions are obtained by superposition. Therefore,

$$\left\{ \nu^{M}_{ia} \right\} = \sum_{t} \left\{ \nu^{M}_{ita} \right\} \qquad (2\text{-}9)$$

and the total displacements at end i of member a are

$$\begin{bmatrix} \phi^{M}_{iax} \\ \phi^{M}_{iay} \\ \epsilon^{M}_{iaz} \end{bmatrix} = \begin{bmatrix} \tau^{M}_{iax} \\ \tau^{M}_{iay} \\ \eta^{M}_{iaz} \end{bmatrix} + \begin{bmatrix} f^{M}_{iaxx} & f^{M}_{iaxy} & f^{M}_{iayz} \\ f^{M}_{iayx} & f^{M}_{iayy} & f^{M}_{iayz} \\ f^{M}_{iazx} & f^{M}_{iazy} & f^{M}_{iazz} \end{bmatrix} \begin{bmatrix} M^{M}_{iax} \\ M^{M}_{iay} \\ N^{M}_{iaz} \end{bmatrix} \qquad (2\text{-}10)$$

or

$$\left\{ \Delta^{M}_{ia} \right\} = \left\{ \nu^{M}_{ia} \right\} + \left[f^{M}_{ia} \right] \left\{ H^{M}_{ia} \right\} \qquad (2\text{-}10a)$$

The derivation of flexibilities and load functions for a parabolic bar is given in Appendix A, and values are tabulated for the bar configuration used in the example problem.

CHAPTER III

TOTAL COMPLEMENTARY POTENTIAL ENERGY

3-1. General

The total complementary potential energy is defined as the sum of the complementary strain energy U* and the complementary potential energy of the prescribed displacements V*. In equation form this may be written as

$$\pi^* = U^* + V^* \qquad (3\text{-}1)$$

Here U* represents the area above the stress strain curve (Fig. 3-1), and is given by

$$U^* = \int_v \int_o^\sigma \varepsilon(\sigma)\ d\sigma\ dv$$

for a uniaxial state of stress.

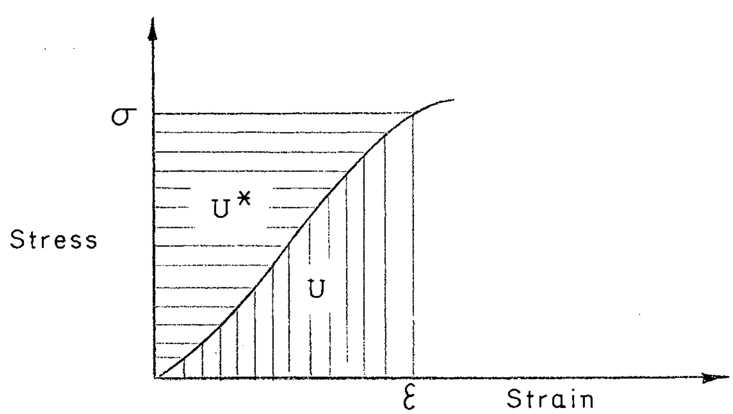

Figure 3-1. Stress-Strain Curve

When a three-dimensional state of stress exists in an elastic body, this equation must be replaced by

$$U^* = \Sigma \int_v \left(\int_o^{\sigma_x} \varepsilon_x \, d\sigma_x + \int_o^{\sigma_y} \varepsilon_y \, d\sigma_y + \int_o^{\sigma_z} \varepsilon_z \, d\sigma_z \right.$$

$$\left. + \int_o^{\tau_{xy}} \gamma_{xy} \, d\tau_{xy} + \int_o^{\tau_{yz}} \gamma_{yz} \, d\tau_{yz} + \int_o^{\tau_{zx}} \gamma_{zx} \, d\tau_{zx} \right) dv$$

In Eq. (3-1)

$$V^* = - \left\{W_k\right\}^T \left\{w_k\right\} \tag{3-2}$$

where w_k are specified displacements and W_k are the forces required to maintain these displacements.

A structure is in a true state of stress when not only the equilibrium conditions are satisfied but also the requirement of compatibility of deformations is fulfilled. For variations in stress when the system is in its true state of equilibrium, the total complementary potential energy π^* has a stationary value. Because the stationary value is a minimum when the system is in stable equilibrium, the complementary energy principle may be stated as follows: The total complementary potential energy is a minimum with respect to variations in stress when the system is in a state of stable equilibrium (1). That is

$$\delta \; (U^* + V^*) = 0 \tag{3-3}$$

thus

$$\frac{\partial U^*}{\partial W_k} \delta W_k + \frac{\partial V^*}{\partial W_k} \delta W_k = 0 \tag{3-4}$$

Since the δW_k are not all zero and are independent of each other

$$\frac{\partial U^*}{\partial W_k} + \frac{\partial V^*}{\partial W_k} = 0 \tag{3-4a}$$

or substituting from Eq. (3-2)

$$\frac{\partial U^*}{\partial W_k} = w_k \tag{3-5}$$

Thus the partial derivative of the complementary strain energy of an elastic system with respect to a force at which there is a prescribed displacement is equal to that displacement.

3-2. Compatibility

For a given redundant structure the equations of equilibrium may be established first. The complementary strain energy U* and the complementary potential energy of the prescribed displacements V* can both be expressed as functions of loads and redundants only, since the other forces may be eliminated by using the relationships in the equations of equilibrium. The compatibility conditions, which are the same in number as the redundants, are then obtained by minimizing the total complementary potential energy with respect to each of the redundants. Thus

$$\frac{\partial U^*}{\partial R_k} = r_k \tag{3-6}$$

where r_k are the specified displacements at the redundants and will either be zero or have some known values.

3-3. Redundants

The vector of member redundants $\left\{ H^M_{ia} \right\}$ is a function of loads on the frame and some of the structure redundants. At end i of member a, this vector represents the static equivalent of the applied loads and structure redundants that affect the member and therefore may be written as

$$\left\{H^{M}_{ia}\right\} = \left[\omega^{MO}\right] \left[r^{O}_{a}\right] \left\{\begin{matrix} SW^{O}_{j} \\ R^{O}_{k} \end{matrix}\right\} \qquad (3\text{-}7)$$

where

$\left[\omega^{MO}\right]$ is the rotation matrix from the "O" system to the "M" system,

$\left[r^{O}_{a}\right]$ is the linear transmission matrix for member a,

$\left\{SW^{O}_{j}\right\}$ are the load vectors at ends j of the loaded members, statically equivalent to the applied loads on the members, and

$\left\{R^{O}_{k}\right\}$ is the vector of structure redundants.

For example, the frame in Fig. (2-1) has two structure redundant vectors $\left\{R^{O}_{1}\right\}$ and $\left\{R^{O}_{2}\right\}$ at cuts ① and ② and member ⚠1 only is loaded. The member redundant vectors $\left\{H^{1}_{i1}\right\}$ and $\left\{H^{5}_{i5}\right\}$ are given by

$$\left\{H^{1}_{i1}\right\} = \left\{\begin{matrix} M^{1}_{i1x} \\ M^{1}_{i1y} \\ N^{1}_{i1z} \end{matrix}\right\} = \left[\omega^{10}\right] \left[0_{3} \,\middle|\, I_{3} \,\middle|\, 0_{3}\right] \left\{\begin{matrix} SW^{O}_{j1} \\ R^{O}_{1} \\ R^{O}_{2} \end{matrix}\right\}$$

$$\left\{H^{5}_{i5}\right\} = \left\{\begin{matrix} M^{5}_{i5x} \\ M^{5}_{i5y} \\ N^{5}_{i5z} \end{matrix}\right\} = \left[\omega^{50}\right] \left[I_{3} \,\middle|\, r^{O}_{21} \,\middle|\, -I_{3}\right] \left\{\begin{matrix} SW^{O}_{j1} \\ R^{O}_{1} \\ R^{O}_{2} \end{matrix}\right\}$$

where I_3 is a 3 x 3 unit matrix, and O_3 is a 3 x 3 null matrix.

3-4. Complementary Strain Energy

Since for linearly elastic structures the complementary strain energy is equal to the internal strain energy, then by Clapeyron's theorem (13), the complementary strain energy of member a is given by

$$U^*_a = \frac{1}{2}\left\{H^M_{ia}\right\}^T\left\{\Delta^M_{ia}\right\} + \frac{1}{2}\left\{W^M_{ta}\right\}^T\left\{w^M_{ta}\right\} \qquad (3\text{-}8)$$

in which W^M_{ta} are the externally applied loads on member a and w^M_{ta} are the corresponding displacements. Substituting Eq. (2-10a) into Eq. (3-8),

$$U^*_a = \frac{1}{2}\left\{H^M_{ia}\right\}^T\left[\left\{\nu^M_{ia}\right\} + \left[f^M_{ia}\right]\left\{H^M_{ia}\right\}\right] + \frac{1}{2}\left\{W^M_{ta}\right\}^T\left\{w^M_{ta}\right\} \qquad (3\text{-}9)$$

and Eq. (3-7) substituted into Eq. (3-9) yields

$$U^*_a = \frac{1}{2}\left\{H^O_{ia}\right\}^T\left[\omega^{MO}\right]^T\left\{\nu^M_{ia}\right\}$$

$$+ \frac{1}{2}\left\{H^O_{ia}\right\}^T\left[\omega^{MO}\right]^T\left[f^M_{ia}\right]\left[\omega^{MO}\right]\left\{H^O_{ia}\right\}$$

$$+ \frac{1}{2}\left\{W^M_{ta}\right\}^T\left\{w^M_{ta}\right\} \qquad (3\text{-}10)$$

Since

$$\left[\omega^{MO}\right]^T = \left[\omega^{OM}\right]$$

then

$$\left[\omega^{MO}\right]^T\left\{\nu^M_{ia}\right\} = \left\{\nu^O_{ia}\right\}$$

and

$$\left[\omega^{MO}\right]^T\left[f^M_{ia}\right]\left[\omega^{MO}\right] = \left[f^O_{ia}\right]$$

Therefore

$$U_a^* = \frac{1}{2}\left\{W_{ta}^M\right\}^T\left\{w_{ta}^M\right\} + \frac{1}{2}\left\{H_{ia}^0\right\}^T\left\{\nu_{ia}^0\right\}$$

$$+ \frac{1}{2}\left\{H_{ia}^0\right\}^T\left[f_{ia}^0\right]\left\{H_{ia}^0\right\} \qquad (3\text{-}11)$$

that is

$$U_a^* = \frac{1}{2}\left\{W_{ta}^M\right\}^T\left\{w_{ta}^M\right\}$$

$$+ \frac{1}{2}\left\{H_{ia}^0\right\}^T\left[\nu_{ia}^0 \;\middle|\; f_{ia}^0\right]\left\{\begin{array}{c}1\\ \hline H_{ia}^0\end{array}\right\} \qquad (3\text{-}11a)$$

or

$$U_a^* = \frac{1}{2}\left\{W_{ta}^M\right\}^T\left\{w_{ta}^M\right\}$$

$$+ \frac{1}{2}\left\{\begin{array}{c}1\\ \hline H_{ia}^0\end{array}\right\}^T\left[\begin{array}{c|c}0 & 0\\ \hline \nu_{ia}^0 & f_{ia}^0\end{array}\right]\left\{\begin{array}{c}1\\ \hline H_{ia}^0\end{array}\right\} \qquad (3\text{-}11b)$$

Substitution of the value of H_{ia}^0 from Eq. (3-7) yields

$$U_a^* = \frac{1}{2}\left\{W_{ta}^M\right\}^T\left\{w_{ta}^M\right\}$$

$$+ \frac{1}{2}\left\{\begin{array}{c}1\\ \hline SW_j^0\\ \\ R_k^0\end{array}\right\}^T\left[\begin{array}{c|c}1 & 0\\ \hline 0 & r_a^0\end{array}\right]^T\left[\begin{array}{c|c}0 & 0\\ \hline \nu_{ia}^0 & f_{ia}^0\end{array}\right]\left[\begin{array}{c|c}1 & 0\\ \hline 0 & r_a^0\end{array}\right]\left\{\begin{array}{c}1\\ \hline SW_j^0\\ \\ R_k^0\end{array}\right\}$$

or

$$U^*_a = \frac{1}{2}\left\{W^M_{ta}\right\}^T\left\{w^M_{ta}\right\} + \frac{1}{2}\begin{Bmatrix} 1 \\ \hline SW^0_j \\ R^0_k \end{Bmatrix}^T \left[\begin{array}{c|c} 0 & 0 \\ \hline \left[r^0_a\right]^T\left\{\nu^0_{ia}\right\} & \left[r^0_a\right]^T\left[f^0_{ia}\right]\left[r^0_a\right] \end{array}\right] \begin{Bmatrix} 1 \\ \hline SW^0_j \\ R^0_k \end{Bmatrix} \tag{3-12}$$

Denoting

$$\left[r^0_a\right]^T\left\{\nu^0_{ia}\right\} = \begin{bmatrix} \alpha^0_{aS\nu} \\ \alpha^0_{aR\nu} \end{bmatrix}$$

and

$$\left[r^0_a\right]^T\left[f^0_{ia}\right]\left[r^0_a\right] = \left[\begin{array}{c|c} \alpha^0_{aSS} & \alpha^0_{aSR} \\ \hline \alpha^0_{aRS} & \alpha^0_{aRR} \end{array}\right]$$

Eq. (3-12) may be expressed as

$$U^*_a = \frac{1}{2}\left\{W^M_{ta}\right\}^T\left\{w^M_{ta}\right\} + \frac{1}{2}\begin{Bmatrix} 1 \\ \hline SW^0_j \\ \hline R^0_k \end{Bmatrix}^T \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \alpha^0_{aS\nu} & \alpha^0_{aSS} & \alpha^0_{aSR} \\ \hline \alpha^0_{aR\nu} & \alpha^0_{aRS} & \alpha^0_{aRR} \end{array}\right] \begin{Bmatrix} 1 \\ \hline SW^0_j \\ \hline R^0_k \end{Bmatrix} \tag{3-13}$$

The total complementary strain energy of the entire structure is then the summation of the complementary energies of the individual members. Thus

$$U^* = \frac{1}{2}\sum_a \left\{W^M_{ta}\right\}^T\left\{w^M_{ta}\right\} + \frac{1}{2}\begin{Bmatrix} 1 \\ \hline SW^0_j \\ \hline R^0_k \end{Bmatrix}^T \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \alpha_{S\nu} & \alpha_{SS} & \alpha_{SR} \\ \hline \alpha_{R\nu} & \alpha_{RS} & \alpha_{RR} \end{array}\right] \begin{Bmatrix} 1 \\ \hline SW^0_j \\ \hline R^0_k \end{Bmatrix} \tag{3-14}$$

where $\left[\alpha\right] = \sum_a \left[\alpha^0_a\right]$

The redundants $\left\{R_k^O\right\}$ are determined by the use of Eq. (3-6). However, before the partial differentiation $\frac{\partial U^*}{\partial R_k^O}$ can be carried out, $\left\{w_t\right\}$ must be expressed in terms of $\left\{R_k^O\right\}$. This operation is possible but becomes intractable. By using the equation for the deflection curve for the "basic cantilever" the displacements $\left\{w_{ta}\right\}$ at the load points can be stated in terms of $\left\{W_{ta}\right\}$ and the member redundants $\left\{H_{ia}^M\right\}$, but since $\left\{H_{ia}^M\right\}$ are functions of the structure redundants $\left\{R_k^O\right\}$ the expression for $\sum_a \left\{W_{ta}\right\}^T \left\{w_{ta}\right\}$ quickly reaches forbidding proportions. The necessity for differentiating this matrix product in the energy equation is circumvented by developing an alternate matrix formulation. This derivation is carried out in the next chapter, and comparison of the new expression with Eq. (3-14) shows that $\frac{\partial U^*}{\partial R_k^O}$ can be evaluated without explicitly expressing $\left\{W_{ta}\right\}^T \left\{w_{ta}\right\}$ in terms of $\left\{R_k^O\right\}$.

CHAPTER IV

ALTERNATIVE DERIVATION OF COMPLEMENTARY STRAIN ENERGY

4-1. General

Instead of grouping the static equivalents of the on-span loads at the "free" end of the bar, their effects could be treated separately in the positions at which they occur on the span. At every point of loading there will be a displacement caused by each of the loads. The energy expression therefore would require the generation of a full influence coefficient matrix. The general procedure for dealing with concentrated loads on a span is to introduce an imaginary node at each load point thereby creating members with no loads between the end points (10). The increase in the number of members to be dealt with is an obvious drawback of this method. A further disadvantage occurs if the span is subjected to distributed loads, for in this instance, these loads must be approximated by concentrated loads before the node points are assigned. However, by the use of generalized forces, Meek (12) showed that the definition of influence coefficients arising from such loadings can be extended to cases where loads occur between selected node points of a structure.

4-2. Generalized Forces

A generalized force is any group of statically interdependent forces that can be completely defined by one symbol (13). The corresponding displacement must be taken in such a way that the result of the product of the generalized force and the increment of the corresponding generalized displacement will be the work. This means that in the case of a distributed load, if the load intensity is taken as the generalized force, the corresponding generalized displacement will be the area between the original position of the beam and its deflected position (Fig. 4-1) and is found by taking the partial derivative of the strain energy with respect to the load intensity.

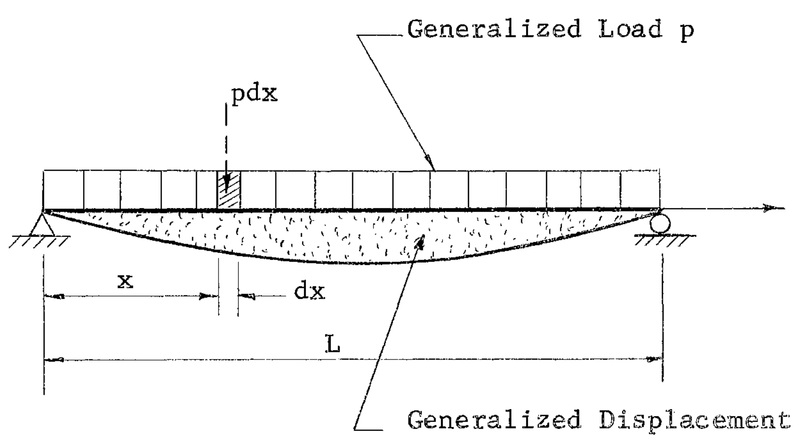

Figure 4-1. Generalized Displacement Due to Uniform Load

4-3. Generalized Displacements of Basic Structure

The basic structure (Fig. 2-3) is acted upon by the applied loads

$$\left\{ W^M_{ta} \right\} = \left\{ Q^M_{tax} \quad Q^M_{tay} \quad P^M_{taz} \right\} \qquad (t = 1, 2, 3, \ldots)$$

and the member redundants

$$\left\{ H^M_{ia} \right\} = \left\{ M^M_{iax} \quad M^M_{iay} \quad N^M_{iaz} \right\}$$

The torsional moment, bending moment, and shearing force at any section q of member a are given by

$$T^E_{qa} = b^E_{qixx} \, M^M_{iax} + b^E_{qixy} \, M^M_{iay} + b^E_{qixz} \, N^M_{iaz}$$

$$+ \sum_t b^E_{qtxx} \, Q^M_{tax} + \sum_t b^E_{qtxy} \, Q^M_{tay} + \sum_t b^E_{qtxz} \, P^M_{taz}$$

$$M^E_{qa} = b^E_{qiyx} \, M^M_{iax} + b^E_{qiyy} \, M^M_{iay} + b^E_{qiyz} \, N^M_{iaz} \qquad (4\text{-}1)$$

$$+ \sum_t b^E_{qtyx} \, Q^M_{tax} + \sum_t b^E_{qtyy} \, Q^M_{tay} + \sum_t b^E_{qtyz} \, P^M_{taz}$$

$$N^E_{qa} = b^E_{qizx} \, M^M_{iax} + b^E_{qizy} \, M^M_{iay} + b^E_{qizz} \, N^M_{iaz}$$

$$+ \sum_t b^E_{qtzx} \, Q^M_{tax} + \sum_t b^E_{qtzy} \, Q^M_{tay} + \sum_t b^E_{qtzz} \, P^M_{taz}$$

A typical influence coefficient b^E_{qtyx} is the action at q in the Y direction of the element system due to a unit cause at t in the X direction of the member system; b^E_{qixy} is the action at q in the X direction of the element system due to a unit cause at i in the Y direction of the member system.

The summation $\sum_t$ in determining the cross-sectional elements at q on the basic cantilever is taken over the interval i q.

The total complementary strain energy of member a is given by

$$U^*_a = U^*_{torsion} + U^*_{bending} + U^*_{shear}$$

that is

$$U^*_a = \frac{1}{2} \int_s \left[\left(T^E_{qa} \right)^2 \lambda^E_{qx} + \left(M^E_{qa} \right)^2 \lambda^E_{qy} + \left(N^E_{qa} \right)^2 \lambda^E_{qz} \right] \qquad (4\text{-}2)$$

where

$\lambda^E_{qx} = \frac{ds}{GJ}$ = angular deformation of the differential element ds in the X^E direction,

$\lambda^E_{qy} = \frac{ds}{EI}$ = angular deformation of the differential element ds in the Y^E direction,

$\lambda^E_{qz} = \frac{kds}{GA}$ = linear deformation of the differential element ds in the Z^E direction.

The generalized displacements corresponding to the generalized forces on the member are

$w^M_{tax} = \frac{\partial U^*_a}{\partial Q^M_{tax}}$ = rotation at t of member a about the X axis of the member system,

$w^M_{tay} = \frac{\partial U^*_a}{\partial Q^M_{tay}}$ = rotation at t of member a about the Y axis of the member system,

$w^M_{taz} = \frac{\partial U^*_a}{\partial P^M_{taz}}$ = deflection at t of member a in the Z direction,

$w^M_{iax} = \frac{\partial U^*_a}{\partial M^M_{iax}}$ = rotation of i of member a about the X axis of the member system,

$$w^M_{iay} = \frac{\partial U^*_a}{\partial M^M_{iay}} = \text{rotation at } i \text{ of member } a \text{ about the Y axis of the member system, and}$$

$$w^M_{iaz} = \frac{\partial U^*_a}{\partial N^M_{iaz}} = \text{deflection at } i \text{ of member } a \text{ in the Z direction}$$

that is

$$w^M_{tax} = \int_s \left[\left(T^E_{qa} \frac{\partial T^E_{qa}}{\partial Q_{tax}} \lambda^E_{qx} \right) + \left(M^E_{qa} \frac{\partial M^E_{qa}}{\partial Q_{tax}} \lambda^E_{qy} \right) + \left(N^E_{qa} \frac{\partial N^E_{qa}}{\partial Q_{tax}} \lambda^E_{qz} \right) \right]$$

$$w^M_{tay} = \int_s \left[\left(T^E_{qa} \frac{\partial T^E_{qa}}{\partial Q^M_{tay}} \lambda^E_{qx} \right) + \left(M^E_{qa} \frac{\partial M^E_{qa}}{\partial Q^M_{tay}} \lambda^E_{qy} \right) + \left(N^E_{qa} \frac{\partial N^E_{qa}}{\partial Q^M_{tay}} \lambda^E_{qz} \right) \right]$$

$$w^M_{taz} = \int_s \left[\left(T^E_{qa} \frac{\partial T^E_{qa}}{\partial P^M_{taz}} \lambda^E_{qx} \right) + \left(M^E_{qa} \frac{\partial M^E_{qa}}{\partial P^M_{taz}} \lambda^E_{qy} \right) + \left(N^E_{qa} \frac{\partial N^E_{qa}}{\partial P^M_{taz}} \lambda^E_{qz} \right) \right]$$

$$w^M_{iax} = \int_s \left[\left(T^E_{qa} \frac{\partial T^E_{qa}}{\partial M^M_{iax}} \lambda^E_{qx} \right) + \left(M^E_{qa} \frac{\partial M^E_{qa}}{\partial M^M_{iax}} \lambda^E_{qy} \right) + \left(N^E_{qa} \frac{\partial N^E_{qa}}{\partial M^M_{iax}} \lambda^E_{qz} \right) \right]$$

$$w^M_{iay} = \int_s \left[\left(T^E_{qa} \frac{\partial T^E_{qa}}{\partial M^M_{iay}} \lambda^E_{qx} \right) + \left(M^E_{qa} \frac{\partial M^E_{qa}}{\partial M^M_{iay}} \lambda^E_{qy} \right) + \left(N^E_{qa} \frac{\partial N^E_{qa}}{\partial M^M_{iay}} \lambda^E_{qz} \right) \right]$$

$$w^M_{iaz} = \int_s \left[\left(T^E_{qa} \frac{\partial T^E_{qa}}{\partial N^M_{iaz}} \lambda^E_{qx} \right) + \left(M^E_{qa} \frac{\partial M^E_{qa}}{\partial N^M_{iaz}} \lambda^E_{qy} \right) + \left(N^E_{qa} \frac{\partial N^E_{qa}}{\partial N^M_{iaz}} \lambda^E_{qz} \right) \right]$$

Thus

$$w^M_{tax} = \int_s \left[\left(T^E_{qa} \, b^E_{qtxx} \, \lambda^E_{qx} \right) + \left(M^E_{qa} \, b^E_{qtyx} \, \lambda^E_{qy} \right) + \left(N^E_{qa} \, b^E_{qtzx} \, \lambda^E_{qz} \right) \right]$$

$$(t = 1, 2, 3, \ldots)$$

$$w^{M}_{tay} = \int_s \left[\left(T^{E}_{qa} \; b^{E}_{qtxy} \; \lambda^{E}_{qx}\right) + \left(M^{E}_{qa} \; b^{E}_{qtyy} \; \lambda^{E}_{qy}\right) + \left(N^{E}_{qa} \; b^{E}_{qtzy} \; \lambda^{E}_{qz}\right)\right]$$

$$(t = 1, 2, 3, \ldots)$$

$$w^{M}_{taz} = \int_s \left[\left(T^{E}_{qa} \; b^{E}_{qtxz} \; \lambda^{E}_{qx}\right) + \left(M^{E}_{qa} \; b^{E}_{qtyz} \; \lambda^{E}_{qy}\right) + \left(N^{E}_{qa} \; b^{E}_{qtzz} \; \lambda^{E}_{qz}\right)\right]$$

$$(t = 1, 2, 3, \ldots)$$

$$w^{M}_{iax} = \int_s \left[\left(T^{E}_{qa} \; b^{E}_{qixx} \; \lambda^{E}_{qx}\right) + \left(M^{E}_{qa} \; b^{E}_{qiyx} \; \lambda^{E}_{qy}\right) + \left(N^{E}_{qa} \; b^{E}_{qizx} \; \lambda^{E}_{qz}\right)\right]$$

$$w^{M}_{iay} = \int_s \left[\left(T^{E}_{qa} \; b^{E}_{qixy} \; \lambda^{E}_{qx}\right) + \left(M^{E}_{qa} \; b^{E}_{qiyy} \; \lambda^{E}_{qy}\right) + \left(N^{E}_{qa} \; b^{E}_{qizy} \; \lambda^{E}_{qz}\right)\right]$$

$$w^{M}_{iaz} = \int_s \left[\left(T^{E}_{qa} \; b^{E}_{qixz} \; \lambda^{E}_{qx}\right) + \left(M^{E}_{qa} \; b^{E}_{qiyz} \; \lambda^{E}_{qy}\right) + \left(N^{E}_{qa} \; b^{E}_{qizz} \; \lambda^{E}_{qz}\right)\right]$$

Substituting the values from Eq. (4-1) into the above, and omitting the superscripts for the sake of brevity, the displacements become

$$\begin{aligned}
w_{tax} = \int_s \Big[& b_{qtxx} \Big(b_{qixx} M_{iax} + b_{qixy} M_{iay} + b_{qixz} N_{iaz} \\
& + \sum_t b_{qtxx} Q_{tax} + \sum_t b_{qtxy} Q_{tay} + \sum_t b_{qtxz} P_{taz} \Big) \lambda_{qx} \\
& + b_{qtyx} \Big(b_{qiyx} M_{iax} + b_{qiyy} M_{iay} + b_{qiyz} N_{iaz} \\
& + \sum_t b_{qtyx} Q_{tax} + \sum_t b_{qtyy} Q_{tay} + \sum_t b_{qtyz} P_{taz} \Big) \lambda_{qy} \\
& + b_{qtzx} \Big(b_{qizx} M_{iax} + b_{qizy} M_{iay} + b_{qizz} N_{iaz} \Big) \\
& + \sum_t b_{qtzx} Q_{tax} + \sum_t b_{qtzy} Q_{tay} + \sum_t b_{qtzz} P_{taz} \Big) \lambda_{qz} \Big]
\end{aligned}$$

$$\begin{aligned}
w_{tay} = \int_s \Big[& b_{qtxy} \Big(b_{qixx} M_{iax} + b_{qixy} M_{iay} + b_{qixz} N_{iaz} \\
& + \sum_t b_{qtxx} Q_{tax} + \sum_t b_{qtxy} Q_{tay} + \sum_t b_{qtxz} P_{taz} \Big) \lambda_{qx}
\end{aligned}$$

$$+ b_{qtyy}\Big(b_{qiyx} M_{iax} + b_{qiyy} M_{iay} + b_{qiyz} N_{iaz}$$

$$+ \sum_t b_{qtyx} Q_{tax} + \sum_t b_{qtyy} Q_{tay} + \sum_t b_{qtyz} P_{taz}\Big) \lambda_{qy}$$

$$+ b_{qtzy}\Big(b_{qizx} M_{iax} + b_{qizy} M_{iay} + b_{qizz} N_{iaz}$$

$$+ \sum_t b_{qtzx} Q_{tax} + \sum_t b_{qtzy} Q_{tay} + \sum_t b_{qtzz} P_{taz}\Big) \lambda_{qz}\Big]$$

$$w_{taz} = \int_s \Big[b_{qtxz} \Big(b_{qixx} M_{iax} + b_{qixy} M_{iay} + b_{qixz} N_{iaz}$$

$$+ \sum_t b_{qtxx} Q_{tax} + \sum_t b_{qtxy} Q_{tay} + \sum_t b_{qtxz} P_{taz}\Big) \lambda_{qx}$$

$$+ b_{qtyz}\Big(b_{qiyx} M_{iax} + b_{qiyy} M_{iay} + b_{qiyz} N_{iaz}$$

$$+ \sum_t b_{qtyx} Q_{tax} + \sum_t b_{qtyy} Q_{tay} + \sum_t b_{qtyz} P_{taz}\Big) \lambda_{qy}$$

$$+ b_{qtzz}\Big(b_{qizx} M_{iax} + b_{qizy} M_{iay} + b_{qizz} N_{iaz}$$

$$+ \sum_t b_{qtzx} Q_{tax} + \sum_t b_{qtzy} Q_{tay} + \sum_t b_{qtzz} P_{taz}\Big) \lambda_{qz}\Big]$$

The displacements at the free end of member a w_{iax}, w_{iay}, and w_{iaz} are given by similar equations, except that the subscript t in each coefficient in front of the parentheses is replaced by the subscript i. These displacements may be expressed in matrix form as

$$\begin{Bmatrix} \{W_{1a}\} \\ \cdot \\ \cdot \\ \cdot \\ \{W_{ta}\} \\ \cdot \\ \cdot \\ \cdot \\ \hline \{W_{ia}\} \end{Bmatrix} = \int_s \left[\begin{array}{c|c} [b_{q1a}]^T [\lambda_q] \left[[b_{q1a}] \cdots [b_{qta}] \cdots \right] & [b_{q1a}]^T [\lambda_q][b_{qia}] \\ & \cdot \\ & \cdot \\ & \cdot \\ [b_{qta}]^T [\lambda_q] \left[[b_{q1a}] \cdots [b_{qta}] \cdots \right] & [b_{qta}]^T [\lambda_q][b_{qia}] \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \hline [b_{qia}]^T [\lambda_q] \left[[b_{q1a}] \cdots [b_{qta}] \cdots \right] & [b_{qia}]^T [\lambda_q][b_{qia}] \end{array} \right] \begin{Bmatrix} \{W_{1a}\} \\ \cdot \\ \cdot \\ \cdot \\ \{W_{ta}\} \\ \cdot \\ \cdot \\ \cdot \\ \hline \{H_{ia}\} \end{Bmatrix} \qquad (4\text{-}3)$$

in which typical submatrices are of the form

$$[b_{qta}]^T [\lambda_q] [b_{qja}] \qquad (j = 1, 2, 3, \ldots)$$

where

$$[b_{qta}] = \begin{bmatrix} b_{qtxx} & b_{qtxy} & b_{qtxz} \\ b_{qtyx} & b_{qtyy} & b_{qtyz} \\ b_{qtzx} & b_{qtzy} & b_{qtzz} \end{bmatrix}$$

and

$$[\lambda_q] = \begin{bmatrix} \lambda_{qx} & 0 & 0 \\ 0 & \lambda_{qy} & 0 \\ 0 & 0 & \lambda_{qz} \end{bmatrix}$$

Eq. (4-3) may be written in abbreviated form as

$$\left\{\begin{array}{c} w_{ta} \\ \hline w_{ia} \end{array}\right\} = \left[\begin{array}{c|c} \psi_{tat} & \psi_{tai} \\ \hline \psi_{iat} & \psi_{iai} \end{array}\right] \left\{\begin{array}{c} W_{ta} \\ \hline H_{ia} \end{array}\right\} \tag{4-3a}$$

where

$\left[\psi_{tat}\right]$ is the influence coefficient matrix of displacements at points t due to generalized forces acting at the points t of bar a,

$\left[\psi_{tai}\right]$ is the influence coefficient matrix of displacements at the points t due to member redundants of bar a,

$\left[\psi_{iat}\right]$ is the influence coefficient matrix of displacements at i due to generalized forces acting at the points t of bar a, and

$\left[\psi_{iai}\right]$ is the influence coefficient matrix of displacements at i due to member redundants of bar a.

4-4. Complementary Strain Energy

Since the structure is assumed to be elastic the complementary strain energy is equal to the strain energy stored in member a. Therefore

$$U^*_a = \frac{1}{2} \left\{\begin{array}{c} W_{ta} \\ \hline H_{ia} \end{array}\right\}^T \left\{\begin{array}{c} w_{ta} \\ \hline w_{ia} \end{array}\right\} \tag{4-4}$$

Substitution of Eq. (4-3a) into Eq. (4-4) gives

$$U^*_a = \frac{1}{2}\begin{Bmatrix} W_{ta} \\ \hline H_{ia} \end{Bmatrix}^T \left[\begin{array}{c|c} \psi_{tat} & \psi_{tai} \\ \hline \psi_{iat} & \psi_{iai} \end{array}\right] \begin{Bmatrix} W_{ta} \\ \hline H_{ia} \end{Bmatrix} \qquad (4\text{-}5)$$

Carrying out the indicated matrix multiplication,

$$2U^*_a = \{W_{ta}\}^T [\psi_{tat}] \{W_{ta}\} + \{H_{ia}\}^T [\psi_{iat}] \{W_{ta}\}$$

$$+\{W_{ta}\}^T [\psi_{tai}] \{H_{ia}\} + \{H_{ia}\}^T [\psi_{iai}] \{H_{ia}\}$$

and substituting for $\{H_{ia}\}$ from Eq. (3-7)

$$2U^*_a = \{W_{ta}\}^T [\psi_{tat}] \{W_{ta}\} + \begin{Bmatrix} SW^O_j \\ R^O_k \end{Bmatrix}^T [r^O_a] [\omega^{MO}]^T [\psi_{iat}] \{W_{ta}\}$$

$$+\{W_{ta}\}^T [\psi_{tai}] [\omega^{MO}] [r^O_a] \begin{Bmatrix} SW^O_j \\ R^O_k \end{Bmatrix}$$

$$+\begin{Bmatrix} SW^O_j \\ R^O_k \end{Bmatrix}^T [r^O_a]^T [\omega^{MO}]^T [\psi_{iai}] [\omega^{MO}] [r^O_a] \begin{Bmatrix} SW^O_j \\ R^O_k \end{Bmatrix} \qquad (4\text{-}6)$$

or

$$2U^*_a = \{W_{ta}\}^T [\Psi_{tat}] \{W_{ta}\} + \begin{Bmatrix} SW^O_j \\ R^O_k \end{Bmatrix}^T [\Psi_{iat}] \{W_{ta}\}$$

$$+\left\{W_{ta}\right\}^T \left[\bar{\Psi}_{tai}\right] \begin{Bmatrix} SW_j^O \\ R_k^O \end{Bmatrix} + \begin{Bmatrix} SW_j^O \\ R_k^O \end{Bmatrix}^T \left[\bar{\Psi}_{iai}\right] \begin{Bmatrix} SW_j^O \\ R_k^O \end{Bmatrix} \qquad (4\text{-}7)$$

where

$$\left[\bar{\Psi}_{tat}\right] = \left[\psi_{tat}\right]$$

$$\left[\bar{\Psi}_{iat}\right] = \left[r_a^O\right]^T \left[\omega^{MO}\right]^T \left[\psi_{iat}\right]$$

$$\left[\bar{\Psi}_{tai}\right] = \left[\psi_{tai}\right] \left[\omega^{MO}\right] \left[r_a^O\right]$$

$$\left[\bar{\Psi}_{iai}\right] = \left[r_a^O\right]^T \left[\omega^{MO}\right]^T \left[\psi_{iai}\right] \left[\omega^{MO}\right] \left[r_a^O\right]$$

Combining the matrices of Eq. (4-7)

$$U_a^* = \frac{1}{2} \begin{Bmatrix} W_{ta} \\ \hline SW_j^O \\ R_k^O \end{Bmatrix}^T \left[\begin{array}{c|c} \bar{\Psi}_{tat} & \bar{\Psi}_{tai} \\ (t \times t) & (t \times p) \\ \hline \bar{\Psi}_{iat} & \bar{\Psi}_{iai} \\ (p \times t) & (p \times p) \end{array}\right] \begin{Bmatrix} W_{ta} \\ (t \times 1) \\ \hline SW_j^O \\ R_k^O \\ (p \times 1) \end{Bmatrix} \qquad (4\text{-}8)$$

where p equals three times the sum of the SW_j^O vectors and R_k^O vectors on the structure. Examination of Eq. (4-7) shows that this new $\left[\bar{\Psi}\right]$ matrix is also symmetrical.

If the load vector contained all applied loads on the structure and there were n such loads, then Eq. (4-8) would become

$$U_a^* = \frac{1}{2} \begin{Bmatrix} W_t \\ \hline SW_j^0 \\ R_k^0 \end{Bmatrix}^T \left[\begin{array}{c|c} \Psi_{tat} & \Psi_{tai} \\ (n \times n) & (n \times p) \\ \hline \Psi_{iat} & \Psi_{iai} \\ (p \times n) & (p \times p) \end{array}\right] \begin{Bmatrix} W_t \\ (n \times 1) \\ \hline SW_j^0 \\ R_k^0 \\ (p \times 1) \end{Bmatrix} \qquad (4\text{-}9)$$

in which in the appropriate places,

$\left[\Psi_{tat}\right]$ now has n minus t rows and columns of zeros

$\left[\Psi_{tai}\right]$ has n minus t rows of zeros, and

$\left[\Psi_{iat}\right]$ has n minus t columns of zeros.

The total complementary energy stored in the entire structure is then

$$U^* = \sum_a U_a^* = \frac{1}{2} \begin{Bmatrix} W_t \\ \hline SW_j^0 \\ R_k^0 \end{Bmatrix}^T \left[\begin{array}{c|c} \Psi_{tt} & \Psi_{ti} \\ \hline \Psi_{it} & \Psi_{ii} \end{array}\right] \begin{Bmatrix} W_t \\ \hline SW_j^0 \\ R_k^0 \end{Bmatrix} \qquad (4\text{-}10)$$

where

$$\left[\begin{array}{c|c} \Psi_{tt} & \Psi_{ti} \\ \hline \Psi_{it} & \Psi_{ii} \end{array}\right] = \sum_a \left[\begin{array}{c|c} \Psi_{tat} & \Psi_{tai} \\ \hline \Psi_{iat} & \Psi_{iai} \end{array}\right] \qquad (4\text{-}11)$$

The lower elements of the influence coefficient matrix in Eq. (4-10) can be partitioned further so that

$$\left[\begin{array}{c|c} \Psi_{it} & \Psi_{ii} \end{array}\right]$$

may be replaced by

$$\left[\begin{array}{c|c|c} \Psi_{St} & \Psi_{SS} & \Psi_{SR} \\ \hline \Psi_{Rt} & \Psi_{RS} & \Psi_{RR} \end{array}\right]$$

Hence

$$U^* = \frac{1}{2} \left\{\begin{array}{c} W_t \\ \hline SW_j^O \\ \hline R_k^O \end{array}\right\}^T \left[\begin{array}{c|c|c} \Psi_{tt} & \Psi_{tS} & \Psi_{tR} \\ \hline \Psi_{St} & \Psi_{SS} & \Psi_{SR} \\ \hline \Psi_{Rt} & \Psi_{RS} & \Psi_{RR} \end{array}\right] \left\{\begin{array}{c} W_t \\ \hline SW_j^O \\ \hline R_k^O \end{array}\right\} \qquad (4\text{-}12)$$

The subscript S is used to identify those terms associated with $\left\{SW_j^O\right\}$.

CHAPTER V

SOLUTION OF STRUCTURE REDUNDANTS

5-1. Differentiation of the Complementary Strain Energy

The triple multiplication indicated in Eq. (4-12) yields

$$U^* = \frac{1}{2}\left\{W_t\right\}^T\left[\Psi_{tt}\right]\left\{W_t\right\}+\left\{SW_j^0\right\}^T\left[\Psi_{St}\right]\left\{W_t\right\} + \left\{R_k^0\right\}^T\left[\Psi_{Rt}\right]\left\{W_t\right\}$$

$$+\left\{W_t\right\}^T\left[\Psi_{St}\right]\left\{SW_j^0\right\} + \left\{SW_j^0\right\}^T\left[\Psi_{SS}\right]\left\{SW_j^0\right\}+\left\{R_k^0\right\}^T\left[\Psi_{RS}\right]\left\{SW_j^0\right\}$$

$$+\left\{W_t\right\}^T\left[\Psi_{tR}\right]\left\{R_k^0\right\}+\left\{SW_j^0\right\}^T\left[\Psi_{SR}\right]\left\{R_k^0\right\} + \left\{R_k^0\right\}^T\left[\Psi_{RR}\right]\left\{R_k^0\right\}$$

and differentiation with respect to one of the redundants gives

$$\frac{\partial U^*}{\partial R_k^0} = \frac{1}{2}\Bigg[\left[\Psi_{Rt}\right]\left\{W_t\right\} + \left[\Psi_{tR}\right]^T\left\{W_t\right\} + \left[\Psi_{RS}\right]\left\{SW_j^0\right\}$$

$$+\left[\Psi_{SR}\right]^T\left\{SW_j^0\right\} + 2\left[\Psi_{RR}\right]\left\{R_k^0\right\}\Bigg] \qquad (5\text{-}1)$$

Since the $\left[\Psi\right]$ matrix is symmetrical,

$$\frac{\partial U^*}{\partial R_k^0} = \left[\Psi_{Rt}\right]\left\{W_t\right\} + \left[\Psi_{RS}\right]\left\{SW_j^0\right\} + \left[\Psi_{RR}\right]\left\{R_k^0\right\} \qquad (5\text{-}2)$$

which can be combined into compact form to give

$$\frac{\partial U^*}{\partial R_k^0} = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \Psi_{Rt} & \Psi_{RS} & \Psi_{RR} \end{array}\right] \left\{\begin{array}{c} W_t \\ \hline SW_j^0 \\ \hline R_k^0 \end{array}\right\} \tag{5-2a}$$

5-2. Comparison of the Energy Expressions

Since Eqs. (3-14) and (4-12) both represent the total complementary strain energy of the structure,

$$\left\{\begin{array}{c} W_t \\ \hline SW_j^0 \\ \hline R_k^0 \end{array}\right\}^T \left[\begin{array}{c|c|c} \Psi_{tt} & \Psi_{tS} & \Psi_{tR} \\ \hline \Psi_{St} & \Psi_{SS} & \Psi_{SR} \\ \hline \Psi_{Rt} & \Psi_{RS} & \Psi_{RR} \end{array}\right] \left\{\begin{array}{c} W_t \\ \hline SW_j^0 \\ \hline R_k^0 \end{array}\right\}$$

$$= \sum_a \left\{W_{ta}^M\right\}^T \left\{w_{ta}^M\right\} + \left\{\begin{array}{c} 1 \\ \hline SW_j^0 \\ \hline R_k^0 \end{array}\right\}^T \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \alpha_S & \alpha_{SS} & \alpha_{SR} \\ \hline \alpha_R & \alpha_{RS} & \alpha_{RR} \end{array}\right] \left\{\begin{array}{c} 1 \\ \hline SW_j^0 \\ \hline R_k^0 \end{array}\right\} \tag{5-3}$$

Examination of the last three of Eqs. (4-3) and Eqs. (2-10) reveals that they are equivalent, thus

$$\left[\Psi_{it}\right] \left\{W_t\right\} \equiv \left\{\nu_{ia}\right\} \tag{5-4}$$

and

$$\left[\Psi_{ii}\right] \equiv \left[f_{ia}\right] \tag{5-5}$$

In Eq. (4-12)

$$\begin{bmatrix} \Psi_{St} \\ \Psi_{Rt} \end{bmatrix} = \sum_a \left[\left[r_a^O \right]^T \left[\omega^{MO} \right]^T \left\{ \Psi_{iat} \right\} \right]$$

$\left\{ W_t \right\}$ is the vector of all externally applied loads, and

$$\left[\Psi_{ii} \right] = \left[\begin{array}{c|c} \Psi_{SS} & \Psi_{SR} \\ \hline \Psi_{RS} & \Psi_{RR} \end{array} \right] = \sum_a \left[\left[r_a^O \right]^T \left[\omega^{MO} \right]^T \left[\Psi_{iai} \right] \left[\omega^{MO} \right] \left[r_a^O \right] \right]$$

while in Eq. (3-14)

$$\begin{Bmatrix} \alpha_{S\nu} \\ \alpha_{R\nu} \end{Bmatrix} = \left[r^O \right]^T \left\{ \nu^O \right\} = \sum_a \left[\left[r_a^O \right]^T \left[\omega^{MO} \right]^T \left\{ \nu_a^M \right\} \right]$$

and

$$\left[\begin{array}{c|c} \alpha_{SS} & \alpha_{SR} \\ \hline \alpha_{RS} & \alpha_{RR} \end{array} \right] = \left[r^O \right]^T \left[f^O \right] \left[r^O \right]$$

$$= \sum_a \left[\left[r_a^O \right]^T \left[\omega^{MO} \right]^T \left[f_a^M \right] \left[\omega^{MO} \right] \left[r_a^O \right] \right]$$

Therefore

$$\begin{bmatrix} \Psi_{St} \\ \Psi_{Rt} \end{bmatrix} \left\{ W_t \right\} \equiv \begin{Bmatrix} \alpha_{S\nu} \\ \alpha_{R\nu} \end{Bmatrix} \left\{ 1 \right\} \tag{5-6}$$

and

$$\left[\begin{array}{c|c} \Psi_{SS} & \Psi_{SR} \\ \hline \Psi_{RS} & \Psi_{RR} \end{array} \right] \equiv \left[\begin{array}{c|c} \alpha_{SS} & \alpha_{SR} \\ \hline \alpha_{RS} & \alpha_{RR} \end{array} \right] \tag{5-7}$$

These can be arranged in the following form:

$$\begin{bmatrix} 0 & 0 & 0 \\ \Psi_{St} & 0 & 0 \\ \Psi_{Rt} & \Psi_{RS} & \Psi_{RR} \end{bmatrix} \begin{Bmatrix} W_t \\ SW_j^0 \\ R_k^0 \end{Bmatrix} \equiv \begin{bmatrix} 0 & 0 & 0 \\ \alpha_{S\nu} & 0 & 0 \\ \alpha_{R\nu} & \alpha_{RS} & \alpha_{RR} \end{bmatrix} \begin{Bmatrix} 1 \\ SW_j^0 \\ R_k^0 \end{Bmatrix} \qquad (5\text{-}8)$$

From Eq. (5-2a) the left side of Eq. (5-8) is equal to

$$\frac{\partial U^*}{\partial R_k^0}$$

therefore by Eq. (5-8)

$$\frac{\partial U^*}{\partial R_k^0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_{R\nu} & \alpha_{RS} & \alpha_{RR} \end{bmatrix} \begin{Bmatrix} 1 \\ SW_j^0 \\ R_k^0 \end{Bmatrix} \qquad (5\text{-}9)$$

5-3. Compatibility Equations

By minimizing the total complementary potential energy with respect to each of the redundants, it was shown that the partial derivative of the complementary strain energy is equal to the prescribed displacement at and in the direction of the associated redundant. When R_k^0 is an internal redundant, compatibility requires that no relative displacements occur at the imaginary cut, therefore $r_k = 0$. Similarly, when R_k^0 is a redundant at an unyielding support, it is obvious that $r_k = 0$.

In the case of a redundant at a support having a known initial deflection or rotation, r_k is that displacement.

By permitting k to assume in turn all the index values of the redundants, Eq. (5-9) yields

$$\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \alpha_{R\nu} & \alpha_{RS} & \alpha_{RR} \end{array}\right] \left\{\begin{array}{c} 1 \\ \hline SW_j^0 \\ \hline R_k^0 \end{array}\right\} = \left\{ r_k \right\} \tag{5-10}$$

which may be solved to give the redundants of the entire structure. Thus

$$\left[\begin{array}{c|c} \alpha_{R\nu} & \alpha_{RS} \end{array}\right] \left\{\begin{array}{c} 1 \\ \hline SW_j^0 \end{array}\right\} + \left[\alpha_{RR}\right] \left\{ R_k^0 \right\} = \left\{ r_k \right\} \tag{5-11}$$

or

$$\left\{ R_k^0 \right\} = \left[\alpha_{RR}\right]^{-1} \left\{ r_k \right\} - \left[\alpha_{RR}\right]^{-1} \left[\begin{array}{c|c} \alpha_{R\nu} & \alpha_{RS} \end{array}\right] \left\{\begin{array}{c} 1 \\ \hline SW_j^0 \end{array}\right\} \tag{5-12}$$

Once the structure redundants are known, the actions at i of any member can be determined readily by the use of Eq. (3-7). The reactions at j can then be calculated by statics.

CHAPTER VI

APPLICATION

Planar Frame Loaded Normally

The two-bay planar frame loaded normally (Fig. 6-1) has been solved on the IBM 1620 computer. All members have equal EI values of 150,000 k-ft^2 and EI/GJ = unity. The shear effects in the calculations of the flexibilities are excluded by setting kEI/GA = 0. The curved members are parabolic in shape.

Three support conditions are considered in the analysis of the example problem:

1) fixed supports at A, B, and C,
2) pinned support at B and fixed supports at A and C,
3) supports A and C fixed and displacements

$$\Delta_z^0 = 0.02 \text{ ft.}, \qquad \theta_x^0 = 0.05 \text{ rad.}, \qquad \theta_y^0 = 0 \text{ at B.}$$

The resulting values of the structure redundants and member redundants are listed in Tables 6-1 and 6-9, respectively.

Tables 6-2 and 6-9 illustrate the step by step formulation of the problem (condition (1)). The procedure for analysis on the computer is given in Appendix B.

Eq. (5-11) is used to solve for the redundants, condition (1). Since all supports are fixed

$$\left\{ r_k^0 \right\} = \left\{ 0 \right\}$$

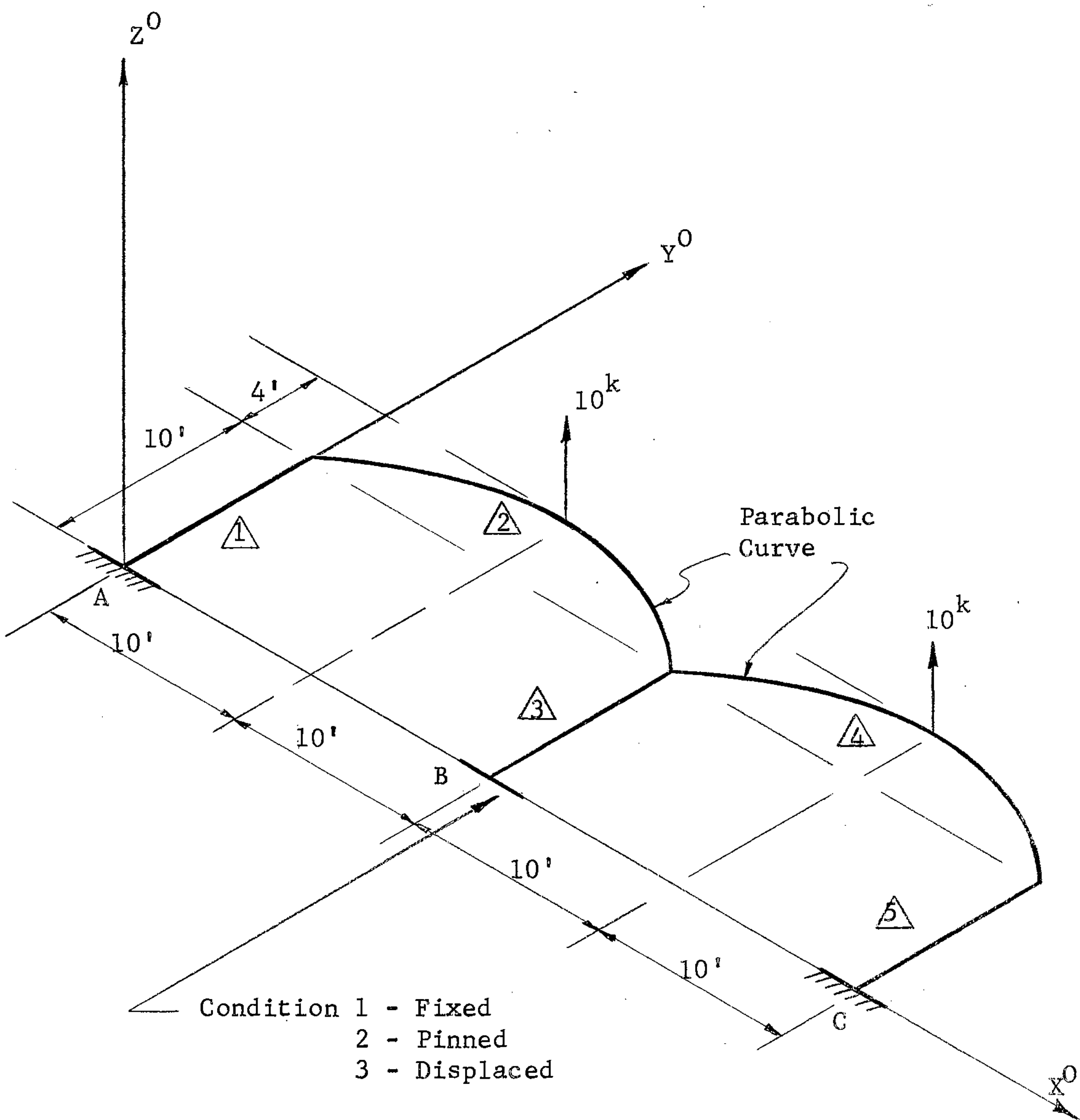

Figure 6-1. Plane Frame Loaded Normally

TABLE 6-1

STRUCTURE REDUNDANTS

Condition	Location of Redundant	Redundant		
		M^0_{ix} k-ft	M^0_{iy} k-ft	N^0_{iz} kips
1	Base of Member 1	-79.37	17.02	-5.09
	Base of Member 5	-79.37	-17.02	-5.09
2	Base of Member 3	0	0	-1.18
	Base of Member 1	-140.00	46.74	-9.41
3	Base of Member 3	594.09	0	42.20
	Base of Member 1	-437.04	195.77	-31.10

TABLE 6-2

DEVELOPMENT OF H^0_{ia} MATRIX

$$\left\{ SW^0_{j2} \right\} = \left\{ SW^0_{j4} \right\} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 40 \\ 100 \\ 10 \end{bmatrix}$$

$$\left\{ H^0_{ia} \right\} = \left\{ 40 \quad 100 \quad 10 \;\vdots\; 40 \quad 100 \quad 10 \;\vdots\; M^0_{i1x} \quad M^0_{i1y} \quad N^0_{i1z} \;\vdots\; M^0_{i2x} \quad M^0_{i2y} \quad N^0_{i2z} \right\}$$

Therefore

$$\begin{bmatrix} \alpha_{Rv} & | & \alpha_{RS} \end{bmatrix} \begin{Bmatrix} 1 \\ --- \\ SW_j^0 \end{Bmatrix} + \begin{bmatrix} \alpha_{RR} \end{bmatrix} \begin{Bmatrix} R_k^0 \end{Bmatrix} = \begin{Bmatrix} 0 \end{Bmatrix}$$

Cuts 1 and 2 are introduced at the base of members 1 and 5 respectively, thereby providing the six releases necessary to render the structure "statically determinate". This furnishes the vector of redundants.

$$\begin{Bmatrix} R_k^0 \end{Bmatrix} = \begin{Bmatrix} M_{iix}^0 & M_{iiy}^0 & N_{iiz}^0 & M_{i2x}^0 & M_{i2y}^0 & N_{i2z}^0 \end{Bmatrix}$$

$\begin{Bmatrix} SW_j^0 \end{Bmatrix}$ is the column matrix of load vectors at ends j of all the members subjected to external loads and is statically equivalent to the applied loads on the members. This vector is obtained by premultiplying the actual load vector on each member by the transmission matrix necessary to transfer the static effects of the loads to end j of that member.

The construction of

$$\begin{Bmatrix} H_{ia}^0 \end{Bmatrix} = \begin{Bmatrix} SW_j^0 \\ --- \\ R_k^0 \end{Bmatrix}$$

which is used in setting up the computer solution is demonstrated in Table 6-2.

The flexibilities and load functions for the various members in their own systems are next calculated and listed in Table 6-3. The rotation matrices shown in Table 6-4 are required in transforming these values from the member system to the reference system. The reverse of this operation can be accomplished with the transpose of the tabulated matrices.

TABLE 6-3

FLEXIBILITIES AND LOAD FUNCTIONS IN "M" SYSTEM

Member	$E\ I \left[f^{M}_{ia} \right]$	$E\ I \left[\nu^{M}_{ia} \right]$
1, 3, 5	$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 50 \\ 0 & 50 & 333.33333 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
2, 4	$\begin{bmatrix} 21.964602 & 0 & -56.546076 \\ 0 & 21.964602 & 219.64602 \\ -56.546076 & 219.64602 & 3157.3710 \end{bmatrix}$	$\begin{bmatrix} 156.56164 \\ 573.03370 \\ 9403.9710 \end{bmatrix}$

TABLE 6-4

ROTATION MATRICES

Member	1	2, 4	3, 5
$\left[\omega^{MO} \right]$	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In order to formulate the $\left[\alpha^{0}\right]$ matrix the flexibilities and load functions in the "O" system are calculated from the equations

$$\left[f^{O}_{ia}\right] = \left[\omega^{MO}\right]^{T} \left[f_{ia}^{M}\right] \left[\omega^{MO}\right]$$

and

$$\left\{\nu^{O}_{ia}\right\} = \left[\omega^{MO}\right]^{T} \left\{\nu^{M}_{ia}\right\}$$

The resulting values are shown in Table 6-5.

TABLE 6-5

FLEXIBILITIES AND LOAD FUNCTIONS IN "O" SYSTEM

Member	$EI\left[f^{O}_{ia}\right]$			$EI\left[\nu^{O}_{ia}\right]$
1	10	0	-50	0
	0	10	0	0
	-50	0	333.33333	0
3, 5	10	0	50	0
	0	10	0	0
	50	0	333.33333	0
2, 4	21.964602	0	-56.546076	156.56164
	0	21.964602	219.64602	573.03370
	-56.546076	219.64602	3157.3710	9403.9710

These matrices are expanded to the form

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline \nu^0_{ia} & f^0_{ia} \end{array}\right]$$

A linear transmission matrix for each member is established next. This matrix transfers to the end i of a member all effects that influence the composition of the particular member redundant. Since $\left\{H^0_{ia}\right\}$ is premultiplied by $\left[r^0_a\right]$, and $\left\{H^0_{ia}\right\}$ includes all the $\left\{SW^0_j\right\}$ and $\left\{R^0_k\right\}$ vectors of the structure, the transmission matrix $\left[r^0_{ia}\right]$ (Table 6-6) contains zeros in appropriate locations wherever that portion of the $\left\{H^0_{ia}\right\}$ matrix does not influence the member redundant.

The transmission matrices are expanded to the form

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & r^0_a \end{array}\right]$$

and are used to calculate $\left[\alpha^0_a\right]$ from Eq. (3-14). Table 6-7 demonstrates the computation of $\left[\alpha^0_2\right]$. All other $\left[\alpha^0_a\right]$ matrices in this problem are calculated in a similar manner.

Summation over the entire structure gives

$$\left[\alpha^0\right] = \sum_a \left[\alpha^0_a\right]$$

from which

$\left[\alpha_{Rv} \mid \alpha_{RS}\right]$ and $\left[\alpha_{RR}\right]$ are easily extracted.

Finally,

$$\left[\alpha_{Rv} \mid \alpha_{RS}\right] \left[\begin{array}{c} 1 \\ \hline SW^0_j \end{array}\right] + \left[\alpha_{RR}\right] \left\{R^0_k\right\} = \left\{0\right\}$$

TABLE 6-6

LINEAR TRANSMISSION MATRICES

$$\left[r^0_{i1} \right] = \left[\begin{array}{ccc|ccc|ccc|ccc} & & & & & & 1 & 0 & 0 & & & \\ & 0 & & & 0 & & 0 & 1 & 0 & & 0 & \\ & & & & & & 0 & 0 & 1 & & & \end{array}\right]$$

$$\left[r^0_{i2} \right] = \left[\begin{array}{ccc|ccc|ccc|ccc} & & & & & & 1 & 0 & -10 & & & \\ & 0 & & & 0 & & 0 & 1 & 0 & & 0 & \\ & & & & & & 0 & 0 & 1 & & & \end{array}\right]$$

$$\left[r^0_{i3} \right] = \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -10 & 1 & 0 & -10 \\ 0 & 1 & 0 & 0 & 1 & -20 & 0 & 1 & 20 & 0 & 1 & -20 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[r^0_{i4} \right] = \left[\begin{array}{ccc|ccc|ccc|ccc} & & & -1 & 0 & 0 & & & & -1 & 0 & 10 \\ & 0 & & 0 & -1 & 20 & & 0 & & 0 & -1 & 20 \\ & & & 0 & 0 & -1 & & & & 0 & 0 & -1 \end{array}\right]$$

$$\left[r^0_{i5} \right] = \left[\begin{array}{ccc|ccc|ccc|ccc} & & & & & & & & & -1 & 0 & 10 \\ & 0 & & & 0 & & & 0 & & 0 & -1 & 0 \\ & & & & & & & & & 0 & 0 & -1 \end{array}\right]$$

TABLE 6-7

DEVELOPMENT OF $\left[\alpha_2^0\right]$ MATRIX

$$
\left[\alpha_2^0\right] = \left[\begin{array}{c|ccc}
1 & 0 & 0 & 0 \\ \hline
0 & & & \\
0 & & \mathbf{0} & \\
0 & & & \\ \hline
0 & & & \\
0 & & \mathbf{0} & \\
0 & & & \\ \hline
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -10 & 0 & 1 \\ \hline
0 & & & \\
0 & & \mathbf{0} & \\
0 & & &
\end{array}\right]
\left[\begin{array}{c|c|c|c}
0 & 0 & 0 & 0 \\ \hline
156.56164 & 21.964602 & 0 & -56.546076 \\
573.03370 & 0 & 21.964602 & 219.64602 \\
9403.9710 & -56.546076 & 219.64602 & 3157.3710
\end{array}\right]
\left[\begin{array}{c|ccc|ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & & & & & & & 1 & 0 & -10 & & & \\
0 & & \mathbf{0} & & & \mathbf{0} & & 0 & 1 & 0 & & \mathbf{0} & \\
0 & & & & & & & 0 & 0 & 1 & & &
\end{array}\right]
$$

$$
= \left[\begin{array}{c|ccc|ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & & & & & & & & & & & & \\
0 & & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & \\
0 & & & & & & & & & & & & \\ \hline
156.56164 & & & & & & & 21.964602 & 0 & -276.19210 & & & \\
573.03370 & & \mathbf{0} & & & \mathbf{0} & & 0 & 21.964602 & 219.64602 & & \mathbf{0} & \\
7838.3546 & & & & & & & -276.19210 & 219.64602 & 6484.7528 & & & \\ \hline
0 & & & & & & & & & & & & \\
0 & & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & \\
0 & & & & & & & & & & & & \\ \hline
0 & & & & & & & & & & & & \\
0 & & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & \\
0 & & & & & & & & & & & &
\end{array}\right]
$$

becomes

$$\begin{Bmatrix} 1956.5616 \\ 573.0337 \\ 505.0210 \\ 1956.5617 \\ -573.0337 \\ 505.0170 \end{Bmatrix} + \begin{bmatrix} 41.9646 & 0 & -376.1921 & 10.0000 & 0 & -50,0000 \\ & 41.9646 & 419.6460 & 0 & 10.0000 & -200.0000 \\ & & 11151.419 & -50.0000 & 200.0000 & -3666.6667 \\ & & & 41.9646 & 0 & -376.1921 \\ & \text{Symmetrical} & & & 41.9646 & -419.6460 \\ & & & & & 11151.419 \end{bmatrix} \begin{bmatrix} M^O_{i1x} \\ M^O_{i1y} \\ N^O_{i1z} \\ M^O_{i5x} \\ M^O_{i5y} \\ N^O_{i5z} \end{bmatrix} = \{0\},$$

This set of equations is solved to give the structure redundants listed in Table 6-1.

The member redundants are next found by substituting the values of the structure redundants R^O_k in Eq. (3-7).

$$\left\{H^M_{ia}\right\} = \left[\omega^{MO}\right]\left[r^O_a\right]\begin{Bmatrix} SW^O_j \\ \hline R^O_k \end{Bmatrix}$$

Sample calculations for member 2 are shown in Table 6-8 and the values of the member redundants are listed in Table 6-9. The member redundants for the other two support conditions are shown in this table also.

TABLE 6-8

CALCULATION OF REDUNDANTS FOR MEMBER 2

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\left[\begin{array}{c|c|ccc|c} 0 & 0 & 1 & 0 & -10 & 0 \\ & & 0 & 1 & 0 & \\ & & 0 & 0 & 1 & \end{array}\right]
\begin{Bmatrix} 40.0000 \\ 100.0000 \\ 10.0000 \\ 40.0000 \\ 100.0000 \\ 10.0000 \\ -79.3685 \\ 17.0244 \\ -5.0864 \\ -79.3685 \\ -5.0864 \end{Bmatrix}
=
\begin{Bmatrix} -28.5043 \\ 17.0244 \\ -5.0864 \end{Bmatrix}
$$

TABLE 6-9

MEMBER REDUNDANTS

Condition	Member	Redundant		
		M_{ix}^{M} k-ft	M_{iy}^{M} k-ft	N_{iz}^{M} kips
(1)	1	17.02	79.36	-5.09
	2	-28.50	17.02	-5.09
	3	0	22.99	9.83
	4	-11.50	15.30	-4.91
	5	-17.02	28.50	5.09
(2)	1	46.74	140.00	-9.41
	2	-45.89	46.74	-9.41
	3	0	-11.78	1.18
	4	5.89	-41.48	-0.59
	5	-46.74	45.89	9.41
(3)	1	195.77	437.04	-31.10
	2	-126.05	195.77	-31.10
	3	0	-172.10	-42.20
	4	86.05	-326.21	21.10
	5	-195.77	126.05	31.10

CHAPTER VII

SUMMARY AND CONCLUSIONS

7-1. Summary

The analysis of planar frames loaded normal to their plane, by the use of the complementary energy principle, is presented in this study.

Each member is treated separately as a basic cantilever. The actions at i, the "free" end of the cantilever, resulting from interaction with adjoining members are treated as member redundants. These are expressed in terms of all applied loads and structure redundants. Summation of the complementary strain energy of each member gives the complementary strain energy for the entire frame. When this is added to the complementary potential energy of all prescribed displacements of the structure the total complementary potential energy π^* is obtained. The compatibility conditions are obtained by minimization of this function with respect to each of the redundants.

The complementary strain energy is represented in two forms:

$$U^* = \frac{1}{2} \sum_a \left\{ W^M_{ta} \right\}^T \left\{ w_{ta} \right\} + \frac{1}{2} \left\{ \begin{array}{c} 1 \\ \hline SW^0_j \\ \hline R^0_k \end{array} \right\}^T \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline \alpha_{Sv} & \alpha_{SS} & \alpha_{SR} \\ \hline \alpha_{Rv} & \alpha_{RS} & \alpha_{RR} \end{array} \right] \left\{ \begin{array}{c} 1 \\ \hline SW^0_j \\ \hline R^0_k \end{array} \right\} \qquad (3\text{-}14)$$

and

$$U^* = \frac{1}{2} \begin{Bmatrix} W_t \\ \hline SW_j^0 \\ \hline R_k^0 \end{Bmatrix}^T \left[\begin{array}{c|c|c} \Psi_{tt} & \Psi_{tS} & \Psi_{tR} \\ \hline \Psi_{St} & \Psi_{SS} & \Psi_{SR} \\ \hline \Psi_{Rt} & \Psi_{RS} & \Psi_{RR} \end{array} \right] \begin{Bmatrix} W_t \\ \hline SW_j^0 \\ \hline R_k^0 \end{Bmatrix} \qquad (4\text{-}12)$$

In the first equation, the flexibility portion of the $[\alpha]$ matrix remains constant for a given structure and the ν values can be taken from tables, if available, or can be calculated separately. Although the first term in the expression is not readily differentiable, for reasons explained in Chapter III, it was shown in Chapter V that it is not necessary to perform this differentiation when solving for redundants. Thus a structure can be efficiently analyzed for various loading conditions by Eq. (5-9) derived from this formulation.

In the second form, Eq. (4-12), differentiation of the $[\Psi]$ matrix is straightforward. In a practical problem, however, this procedure would entail calculating a different influence coefficient matrix for each member under each type of loading, hence would be inefficient for analysis of a structure for several conditions of loading.

General expressions for end flexibilities and influence coefficients for load functions of a planar curved member are given in Chapter II. These expressions are evaluated for the case of a parabolic bar and the derivations are given in Appendix A. A numerical example is included to demonstrate the theory presented.

7-2. Conclusions

The formulation of the energy expression, Eq. (3-14) offers two major advantages:

1. It presents the load function matrix in a form suitable for the systematic solution of a given frame under various load conditions without undue algebraic or numerical work. Hence, it is not necessary to start calculations from the beginning if the load system on the structure is changed.

2. By the use of the concept of generalized forces, the load function matrix is prepared without the necessity of creating fictitious nodes at points of application of each load.

The second formulation, Eq. (4-12), allows the calculation of the work expression without the requirement of evaluating the displacements at load points or expressing these deformations as functions of the redundants.

Combination of these two formulations in one method greatly simplifies the construction of the compatibility matrix equation.

7-3 Extensions

The theory developed in this research could be extended to:

1. The solution of space frame problems.

2. The investigation of planar frames subjected to vibrations normal to the plane.

3. The investigation of structures composed of materials with nonlinear behavior.

4. The solution of plate problems by finite element techniques.

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APPENDIX A

FLEXIBILITIES AND LOAD FUNCTIONS

A-1. Derivation of Deformation Influence Coefficients

Flexibilities and load functions for a planar parabolic member loaded normally are derived in this section.

The end flexibility matrix of a member is one which when post-multiplied by a unit vector gives the displacements at the near end due to unit values of the near-end actions. The algebraic expressions for the coefficients for a general planar member are given in Table 2-1 of Chapter II. It is to be noted that the integration is over the complete length of the member.

A load function influence coefficient matrix $\left[f'\right]$ can be developed. When this is post-multiplied by the near-end vector which is the static equivalent of a unit load vector at a given point, the product will be the end displacements due to the unit load vector. The coefficients of this matrix have the same algebraic expressions as those shown in Table 2-1, except that the integration is now over only the segment extending from the point of loading to the far end of the member. Thus the two matrices are identical if the unit load vector is applied at the near end.

In this section the expressions given in Table 2-1 are integrated for the particular case of a parabolic arc Fig. A-1 over the interval from the point of loading to the far end. The coefficients thus obtained may be used either as the end flexibilities or the load function influence coefficients by substitution of the correct limits. A separate matrix must be generated for each concentrated load vector.

For the case of a uniformly distributed load, the load functions for a unit load must be integrated over the interval of loading and the results multiplied by the load intensity. In a numerical problem this is easily accomplished by using Simpson's one-third rule, which is precise up to third degree curves.

The algebraic derivation is given in A-2 of the appendix. All the coefficients are in the member system; therefore the superscript M is omitted.

A-2. Algebraic Derivation

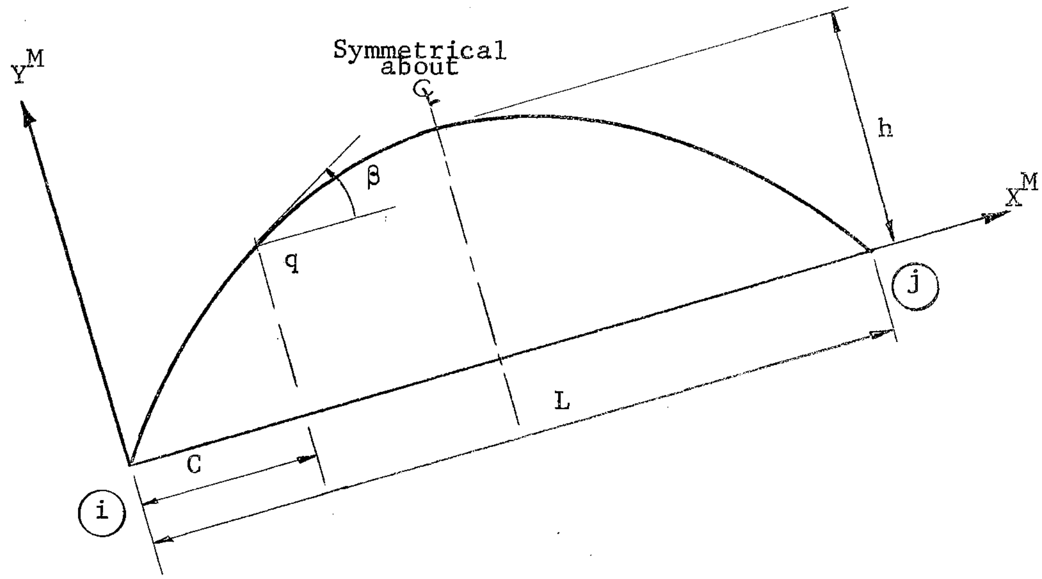


Figure A-1. Parabolic Arc

The equation of the parabola shown in Fig. A-1 is

$$y = \frac{4hx}{L^2} (L - x) \qquad \text{(A-2.1)}$$

This may be represented in parametric form by

$$x = \frac{Lt}{2} \qquad \text{(A-2.2)}$$

$$y = ht\ (2 - t)$$

For any point q along the curve, the limits of integration become

$$t = \frac{2C}{L} \quad \text{to} \quad t = 2 \qquad \text{(A-2.3)}$$

and

$$dx = \frac{L}{2}\ dt \qquad \text{(A-2.4)}$$

Denoting the angle between the X axis and the tangent at q by β,

$$\tan \beta = \frac{dy}{dx} = \frac{4h}{L} (1 - t) \tag{A-2.5}$$

$$\cos \beta = \frac{L}{4h \sqrt{(1 - t)^2 + a^2}}$$

$$\sin \beta = \frac{1 - t}{\sqrt{(1 - t)^2 + a^2}}$$

where

$$a = \frac{L}{4h}$$

$$ds = \frac{dx}{\cos \beta}$$

therefore

$$ds = 2h \sqrt{(1 - t)^2 + a^2} \; dt$$

Changing the variable of integration from t to u by letting

$$1 - t = u \tag{A-2.6}$$

gives

$$\cos \beta = \frac{L}{4h \sqrt{u^2 + a^2}} \tag{A-2.7}$$

$$\sin \beta = \frac{u}{\sqrt{u^2 + a^2}} \tag{A-2.8}$$

and

$$dt = -du$$

therefore

$$ds = -2h \sqrt{u^2 + a^2} \; du \tag{A-2.9}$$

The new limits of integration are now $(1 - \frac{2C}{L})$ to (-1) thus

$$\int_{C}^{L} f(x)\, dx = \int_{\frac{2C}{L}}^{2} g(t)\, dt = \int_{1-\frac{2C}{L}}^{-1} G(u)\,(-du)$$

$$= \int_{-1}^{1-\frac{2C}{L}} G(u)\, du$$

All integrals will therefore be of the form

$$\int_{-1}^{\gamma} G(u)\, du$$

where

$$\gamma = 1 - \frac{2C}{L} \qquad (-1 \le \gamma \le 1) \qquad \text{(A-2.10)}$$

The integrals necessary for evaluation of the coefficients f'_{ia} are given in Table A-2.1. The symbols listed in the third column were used in preparing the computer program and are helpful in shortening the expressions for f'_{ia}, while at the same time preventing the meaning of the equations from being obscured.

For members with constant EI, GJ and GA, if the ratios EI/GJ and kEI/GA are denoted by μ and ρ respectively, the influence coefficients are reduced to the values given in Table A-2.2.

TABLE A-2.1

INTEGRALS FOR EVALUATION OF f'_{ia}

ξ	$\int_{-1}^{\gamma} \frac{\xi \, du}{\sqrt{u^2 + a^2}}$	Symbol
1	$\log \left(\frac{\gamma + \sqrt{\gamma^2 + a^2}}{-1 + \sqrt{1 + a^2}} \right)$	DU
u	$\sqrt{\gamma^2 + a^2} - \sqrt{1^2 + a^2}$	UDU
u^2	$\frac{1}{2} \left[\gamma \sqrt{\gamma^2 + a^2} + \sqrt{1 + a^2} - a^2 \text{ (DU)} \right]$	U2DU
u^3	$\frac{1}{3} \left[\left(\sqrt{\gamma^2 + a^2} \right)^3 - \left(\sqrt{1 + a^2} \right)^3 \right] - a^2 \left[\sqrt{\gamma^2 + a^2} - \sqrt{1 + a^2} \right]$	U3DU
u^4	$\frac{1}{4} \left(\gamma^3 \sqrt{\gamma^2 + a^2} + \sqrt{1 + a^2} \right) - \frac{3a^2}{8} \left(\gamma \sqrt{\gamma^2 + a^2} + \sqrt{1 + a^2} \right) + \frac{3a^4}{8} \text{ (DU)}$	U4DU
u^6	$\frac{1}{6} \left(\gamma^5 \sqrt{\gamma^2 + a^2} + \sqrt{1 + a^2} \right) - \frac{5a^2}{24} \left(\gamma^3 \sqrt{\gamma^2 + a^2} + \sqrt{1 + a^2} \right) + \frac{5a^4}{16} \left(\gamma \sqrt{\gamma^2 + a^2} + \sqrt{1 + a^2} \right) - \frac{5a^6}{16} \text{ (DU)}$	U6DU
$u^2 + a^2$	$\frac{1}{2} \left[\gamma \sqrt{\gamma^2 + a^2} + \sqrt{1 + a^2} + a^2 \text{ (DU)} \right]$	DUIN

TABLE A-2.2

DEFORMATION INFLUENCE COEFFICIENTS FOR A CANTILEVER PARABOLIC BAR	
EI x Coefficient	Values in terms of Symbols in Table A-2.1
f'_{iaxx}	$\frac{\mu L^2}{8h}$ (DU) + 2h(U2DU)
f'_{iaxy}	$\frac{L}{2}$ $(\mu - 1)$ (UDU)
f'_{iaxy}	$\frac{\mu L^2}{8}\left[2(\mathrm{UDU}) - (\mathrm{DU}) - (\mathrm{U2DU})\right] + \frac{L^2}{4}\left[(\mathrm{U2DU}) - (\mathrm{UDU})\right] - 2h^2\left[(\mathrm{U2DU}) - (\mathrm{U4DU})\right]$
f'_{iayy}	$2\mu h$ (U2DU) + $\frac{L^2}{8h}$ (DU)
f'_{iayz}	$\mu Lh\left[(\mathrm{U2DU}) - \mathrm{U3DU}) - \frac{1}{2}\left\{(\mathrm{UDU}) - (\mathrm{U3DU})\right\}\right] + \frac{L^3}{16h}\left[(\mathrm{DU}) - (\mathrm{UDU})\right] + \frac{Lh}{2}\left[(\mathrm{UDU}) - (\mathrm{U3DU})\right]$
f'_{iazz}	$\frac{L^2 h}{8}\Big\{\mu\left[(\mathrm{DU}) - 4(\mathrm{UDU}) + 6(\mathrm{U2DU}) - 4(\mathrm{U3DU}) + (\mathrm{U4DU})\right] + \frac{L^2}{4h}\left[(\mathrm{DU}) - 2(\mathrm{UDU}) + (\mathrm{U2DU})\right]$ $+ 4\left[(\mathrm{UDU}) - (\mathrm{U2DU}) - (\mathrm{U3DU}) + (\mathrm{U4DU})\right] + \frac{16h^2}{L^2}\left[(\mathrm{U2DU}) - 2(\mathrm{U4DU}) + (\mathrm{U6DU})\right]$ $+ \frac{16\rho}{L^2}$ (DUIN) $\Big\}$

A-3. Flexibility Data for Parabolic Cantilever

Influence coefficients for end flexibilities and load functions for a parabolic cantilever beam of constant cross-section are presented in Table A-3.1 This beam configuration (Fig. A-2) is used in the numerical example. The coefficients were evaluated from the equations given in Table (A-2.2).

Deformations are shown by arrows with a slash. Double-headed arrows indicate rotations and the single-headed arrow denotes a deflection. The positive directions are as shown.

Table A-3.1 DEFORMATION INFLUENCE COEFFICIENTS PARABOLIC CANTILEVER, CONSTANT SECTION

Data: L = 20 ft. h = 4 ft. $\frac{EI}{GJ} = 1$ $\frac{kEI}{GA} = 0$

End Flexibilities:

$f_{ixx} = 21.9646/EI$	$f_{iyy} = 21.9646/EI$
$f_{ixy} = 0.0000$	$f_{iyz} = 219.6460/EI$
$f_{ixz} = -56.5461/EI$	$f_{izz} = 3157.3710/EI$

Location CL	Load Function		
	$EI\ \tau_{ix}$	$EI\ \tau_{iy}$	$EI\ \eta_{iz}$
0	-56.5461	219.6460	3157.3710
.1	-26.6490	178.2136	2639.1469
.2	- 6.0480	141.5352	2151.8574
.3	6.9035	109.3002	1703.5174
.4	13.6851	81.2722	1298.7262
.5	15.6561	57.3034	940.3971
.6	14.1129	37.3430	631.2004
.7	10.3515	21.4418	374.8180
.8	5.7313	9.7477	177.1248
.9	1.7319	2.4967	47.4409
1.0	0	0	0

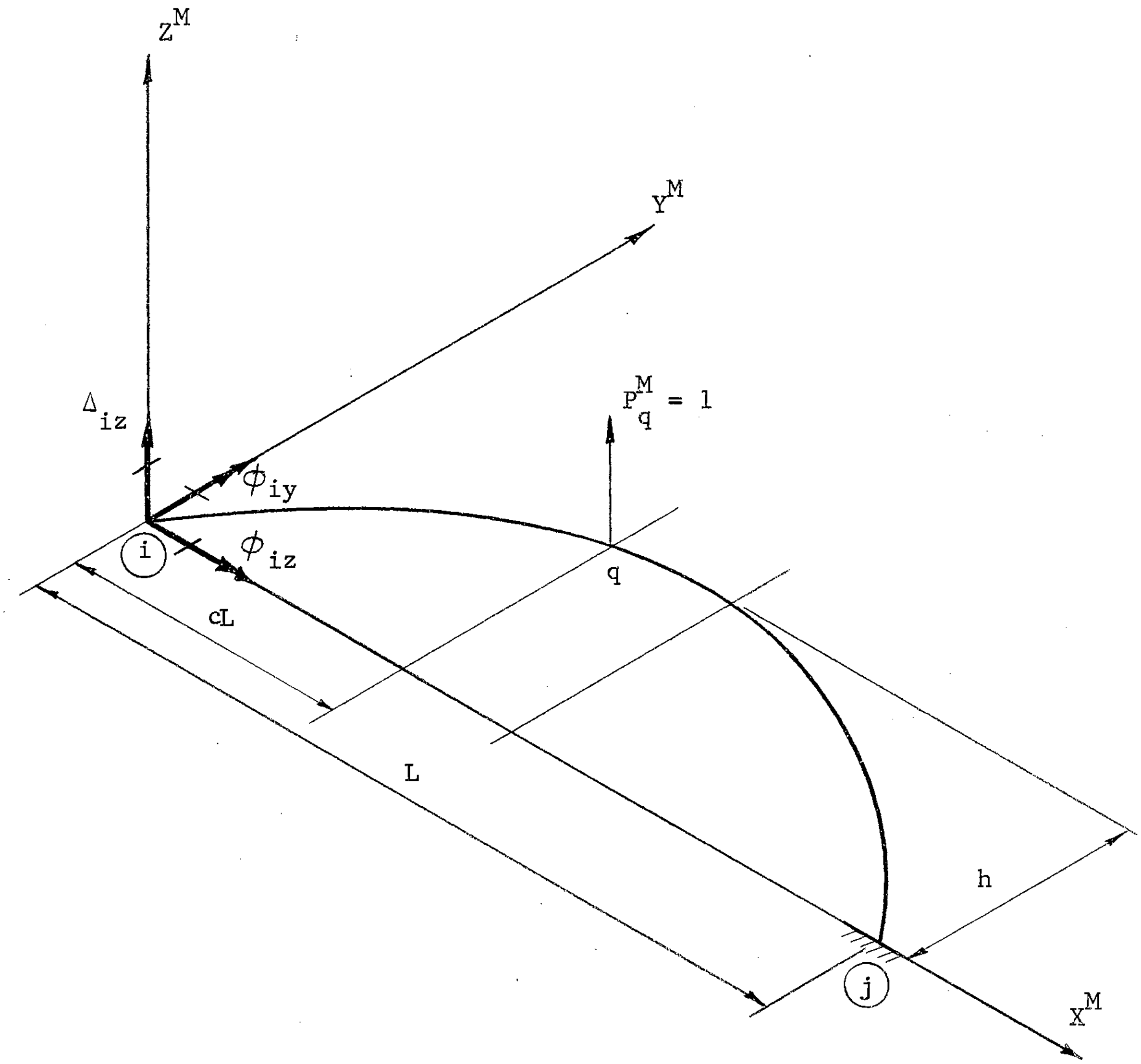


Figure A-2. Geometry and Definition Sketch Parabolic Cantilever

APPENDIX B

COMPUTER ANALYSIS

A computer program for the analysis of the class of frames investigated in this thesis was written for the IBM 1620I. Since the storage capacity of the computer is only 20,000 binary digits, the program was subdivided into six phases.

A macro flow diagram (which follows) illustrates the basic logic of the process used in the solution of the problems. Input data required for each phase are indicated below. The member redundants constitute the output of the final phase.

INPUT DATA - PHASE I

Information Needed in Phase II

M, N, MA, NA, MB, NB, AM, BM

M, N   Rows and columns of expanded matrix

The next four numbers indicate the number of rows and columns that matrices A and B are to be shifted. The last two are the multipliers of A and B.

Member Number and Shape, Number of Loads, and Coordinates in Reference System

MM, MS, NL, XI, YI, XJ, YJ	
MM	member number
MS	member shape
NL	number of loads
The last four numbers indicate the coordinates of the i and j ends of the member.	

Member Properties

EMU, RO, H, ETA	
EMU	the ratio EI/GJ
RO	the ratio kEI/GA
H	the y ordinate at mid-point of the member in its own system
ETA	the ratio EI/EI_0

Location and Values of Applied Loads in the Member System

XL, QX, QY, P	
XL	x ordinate of the load
The other three numbers are the values of the applied loads.	

INPUT DATA - PHASE II

Output of Phase I

M, N, MA, NA, MB, NB, AM, BM (Same as first card of Phase I)

I, J, FO(I, J)	
I, J	location of matrix element
FO(I, J)	flexibility coefficient in reference system

I, ENUO(I)	
I	location of matrix element
ENUO(I)	load function in reference system

INPUT DATA - PHASE III

Coordinates of Member and Number of Influences for Transmission Matrix

XA, YA, NI	
XA, YA	coordinates of end i of member a
NI	number of redundant vectors plus the number of statically equivalent applied load vectors at ends j of members

Influence Constant and Coordinates

C, XB, YB	
C	0, 1, or -1 depending on how member a is affected by the influence
XB, YB	coordinates of the influence

INPUT DATA - PHASE IV

M, N	number of rows and columns in output from Phase II

Output of Phases II and III

I, J, B(I, J)	
I, J	location of matrix element
B(I, J)	load function or flexibility coefficient

I, J, A(I, J)	
I, J	location of matrix element
A(I, J)	element of expanded transmission matrix

INPUT DATA - PHASE V

Matrix Selection Details

M, N, MA, NA	
M, N	number of rows and columns needed from the structure flexibility matrix
MA	number of rows ignored
NA	number of elements in load vector

Structure Flexibility Matrices

I, J, A(I, J)	
I, J	location of matrix element
A(I, J)	element of structure flexibility matrix

Prescribed Displacements

I, W(I)

Load Vector

```
I, S(I)
```

Details for Solution of Simultaneous Equations

```
M2, N2, MA2, NA2

M2        number of redundants

N2        number of columns of augmented matrix

MA2, NA2  number of rows and columns omitted from structure

          flexibility matrix
```

INPUT DATA - PHASE VI

```
M, N      number of rows and columns in transmission matrix
```

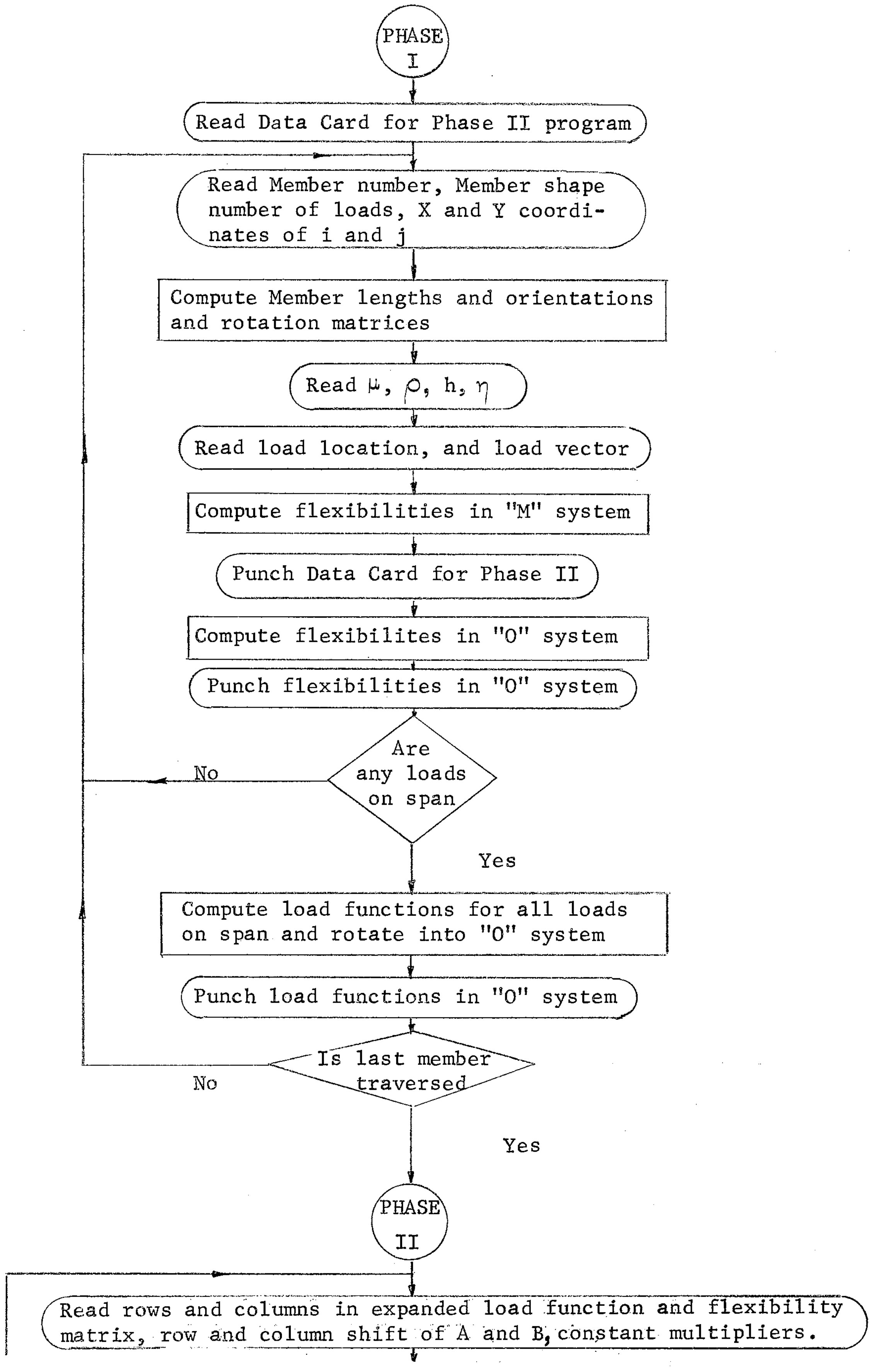
Vector of Loads and Structure Redundants

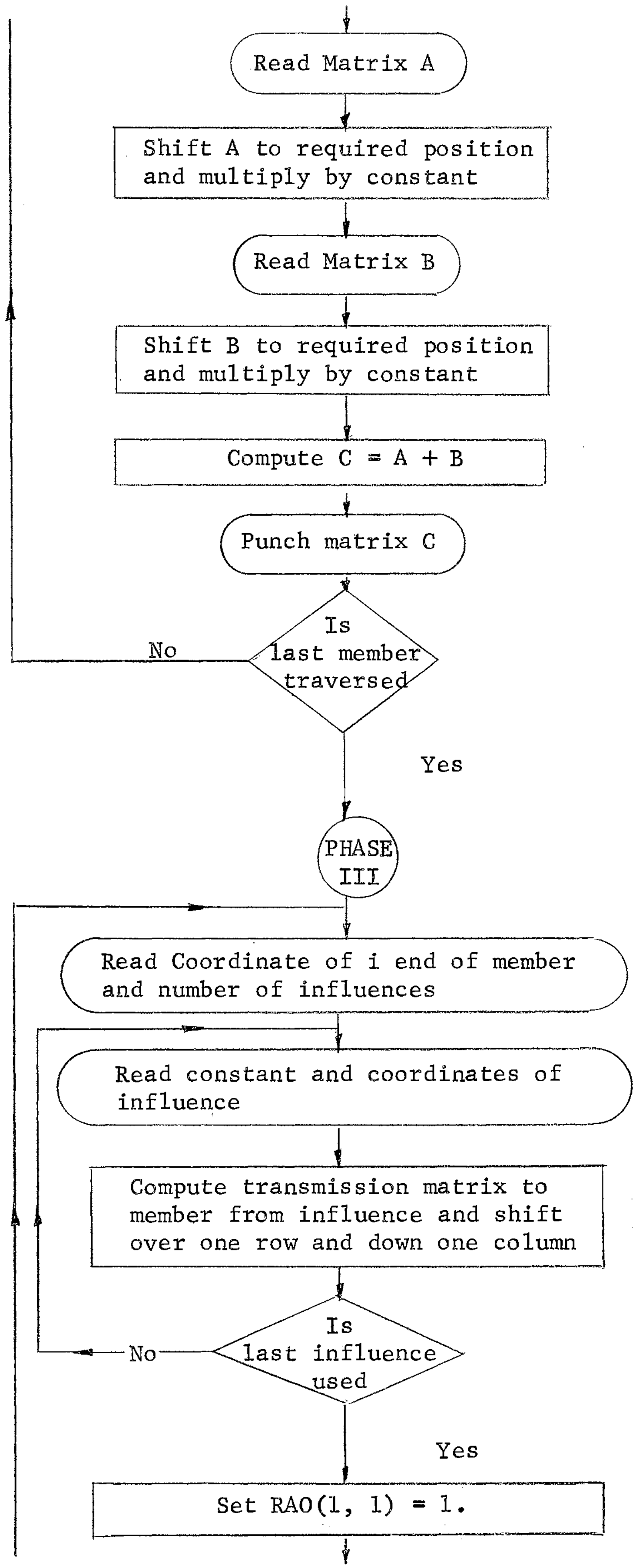
```
I, SRO(I)

   structure redundants are output of Phase V
```

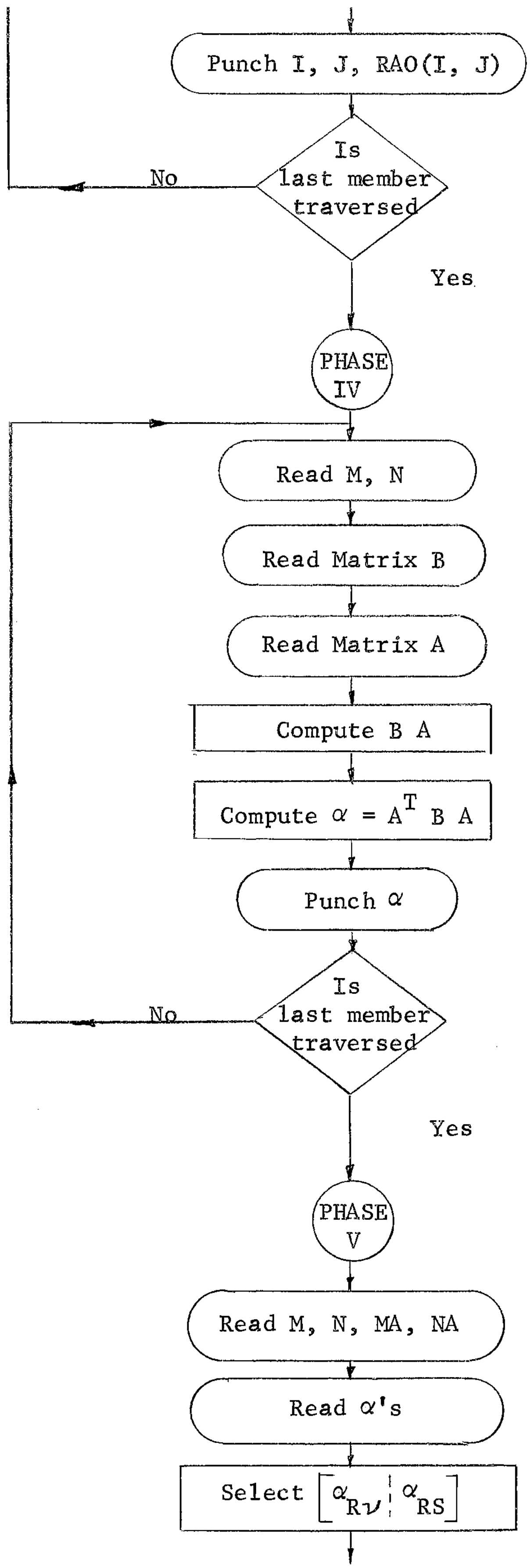
Transmission Matrices

```
I, J, RAO(I, J)

I, J      location of matrix element

RAO(I, J) element of transmission matrix
```

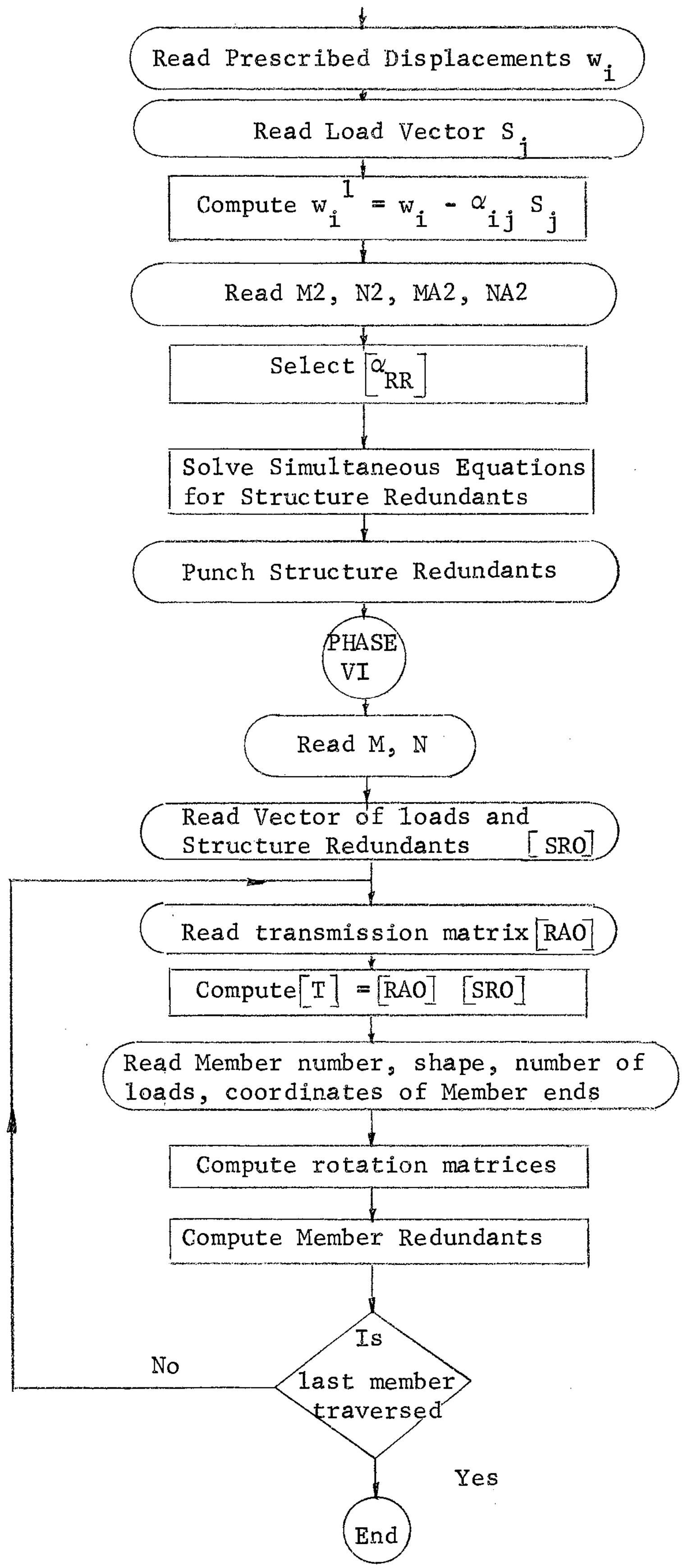
```
MM, MS, NL, XI, YI, XJ, YJ

(same as second card of Phase I)
```


PHASE I
Read Data Card for Phase II program
Read Member number, Member shape number of loads, X and Y coordinates of i and j
Compute Member lengths and orientations and rotation matrices
Read μ, ρ, h, η
Read load location, and load vector
Compute flexibilities in "M" system
Punch Data Card for Phase II
Compute flexibilites in "O" system
Punch flexibilities in "O" system
Are any loads on span
No
Yes
Compute load functions for all loads on span and rotate into "O" system
Punch load functions in "O" system
Is last member traversed
No
Yes
PHASE II
Read rows and columns in expanded load function and flexibility matrix, row and column shift of A and B, constant multipliers.


Read Matrix A
Shift A to required position and multiply by constant
Read Matrix B
Shift B to required position and multiply by constant
Compute C = A + B
Punch matrix C
Is last member traversed
No
Yes
PHASE III
Read Coordinate of i end of member and number of influences
Read constant and coordinates of influence
Compute transmission matrix to member from influence and shift over one row and down one column
Is last influence used
No
Yes
Set RAO(1, 1) = 1.



Punch I, J, RAO(I, J)
Is last member traversed
No
Yes
PHASE IV
Read M, N
Read Matrix B
Read Matrix A
Compute B A
Compute $\alpha = A^T B A$
Punch α
Is last member traversed
No
Yes
PHASE V
Read M, N, MA, NA
Read α's
Select $[\alpha_{R\nu} \mid \alpha_{RS}]$


Read Prescribed Displacements w_i
Read Load Vector S_j
Compute $w_i^1 = w_i - \alpha_{ij} S_j$
Read M2, N2, MA2, NA2
Select $[\alpha_{RR}]$
Solve Simultaneous Equations for Structure Redundants
Punch Structure Redundants
PHASE VI
Read M, N
Read Vector of loads and Structure Redundants [SRO]
Read transmission matrix [RAO]
Compute [T] = [RAO] [SRO]
Read Member number, shape, number of loads, coordinates of Member ends
Compute rotation matrices
Compute Member Redundants
Is last member traversed
No
Yes
End

VITA

Matthew McCartney Douglass

Candidate for the Degree of

Doctor of Philosophy

Thesis: ANALYSIS OF PLANAR FRAMES LOADED NORMALLY BY COMPLEMENTARY POTENTIAL ENERGY

Major Field: Engineering

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