STATE-SPACE SYNTHESIS OF PASSIVE ONE-PORT NETWORKS

By

MARVIN EMMETT DANIEL

Bachelor of Science in Electrical Engineering
Kansas State University
Manhattan, Kansas
1961

Master of Science in Electrical Engineering
University of New Mexico
Albuquerque, New Mexico
1963

Submitted to the faculty of the Graduate College of
the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
May, 1966

STATE-SPACE SYNTHESIS OF PASSIVE

ONE-PORT NETWORKS

Thesis Approved:

Thesis Adviser

Dean of the Graduate College

ACKNOWLEDGMENTS

I wish to express my gratitude to Professor W. A. Blackwell, Chairman of my advisory committee, for his assistance and guidance during my doctoral studies. A great deal of the background in state-space techniques necessary for my research was obtained in a course in Advanced Control Theory taught by Professor Blackwell.

I am especially indebted to Professor L. L. Grigsby for the original idea which led to this thesis and for his encouragement and aid during its preparation.

I also wish to thank the other members of my advisory committee, Professors Jeanne L. Agnew, R. L. Cummins, and A. M. Breipohl, for their advice and encouragement throughout my doctoral studies.

TABLE OF CONTENTS

Chapter Page

LIST OF FIGURES

CHAPTER I

INTRODUCTION

<u>1.1 Statement of the Problem.</u> With the advent of high-speed computers, a new approach to network analysis and synthesis has evolved. The new approach is based on the dynamical description of a physical system, generally referred to as the state model or state-space description of the system. The state model is composed of a set of differential equations explicit in the first derivatives of the state variables and a set of algebraic equations.

$$\frac{d}{dt}\psi_i = F_i\,(\psi_1, \ldots, \psi_n; r_1, \ldots, r_m)$$

$$c_j = G_j\,(\psi_1, \ldots, \psi_n; r_1, \ldots, r_m) \qquad (1.1.1)$$

$$i = 1, 2, \ldots, n$$

$$j = 1, 2, \ldots, p$$

$\psi_i(t)$ = i-th component of the vector of state variables

$r_i(t)$ = i-th component of the vector of forcing functions (drivers)

$c_i(t)$ = i-th component of the vector of remaining system variables

The state model most commonly considered is the linear time-invariant case. Equations 1.1.1 become matrix differential equations whose coefficient matrices have constant elements.

$$\frac{d}{dt}\underline{\psi}(t) = \underline{A}\,\underline{\psi}(t) + \underline{B}\,\underline{r}(t)$$

$$\underline{c}(t) = \underline{C}\,\underline{\psi}(t) + \underline{D}\,\underline{r}(t) \qquad (1.1.2)$$

where:

$\underline{x}(t)$ = n by 1 vector of state variables

$\underline{r}(t)$ = m by 1 vector of forcing functions (drivers)

$\underline{c}(t)$ = p by 1 vector of remaining system variables (outputs)

$\underline{A}$ = n by n coefficient matrix

$\underline{B}$ = n by m coefficient matrix

$\underline{C}$ = p by n coefficient matrix

$\underline{D}$ = p by m coefficient matrix

A good treatment of the characterization of a system by the state-space method is given in Chapter 3 of Tou (1).

There are several books which are devoted entirely to the subject of the state space. Probably the most comprehensive and rigorous treatment is presented by Zadeh and Desoer (2).

Although the state-space approach is relatively new in engineering, there are many papers which discuss the subject. Almost all of the current literature is, however, concerned with the problem of analysis rather than synthesis. The classical network theory treats both the analysis and synthesis of networks. Similarly, it would be desirable to find a synthesis procedure for the state-space model to complement the methods of analysis.

In classical network synthesis, the problem is to find a physical network which is described by a given function of the Laplace transform variable, s. The procedures for testing a Laplace function for physical realizability are well developed for one-port networks. Van Valkenburg (3) and Weinberg (4) give conditions that a function be positive real and conditions that the network realization of the function be composed of certain combinations of resistors, capacitors, and inductors.

In this thesis the author has considered the state model, Equation 1.1.2, with $\underline{r}(t)$ and $\underline{c}(t)$ being scalar functions of time. (m, p = 1)

$$\frac{d}{dt}\,\underline{\psi}(t) = \underline{A}\,\underline{\psi}(t) + \underline{B}\,r(t)$$
$$c(t) = \underline{C}\,\underline{\psi}(t) + D\,r(t) \qquad (1.1.3)$$

The problems investigated are the following:

1. Is it possible to find a one-port passive network having r(t) as the input voltage and c(t) as the input current (or r(t) as the input current and c(t) as the input voltage) which has a state model of the form of Equation 1.1.3?

$$\overset{\circ}{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{d}\,v(t)$$
$$i(t) = \underline{h}\,\underline{x}(t) + G\,v(t) \qquad (1.1.4)$$

or

$$\overset{\circ}{\underline{x}}(t) = \underline{B}\,\underline{x}(t) + \underline{e}\,i(t)$$
$$v(t) = \underline{k}\,\underline{x}(t) + R\,i(t) \qquad (1.1.5)$$

2. If such a network realization is possible, will it be an LC, RC, RL, RLC, or RLCT network?

In the theorems and proofs which follow, it is assumed that the reader has an understanding of classical one-port network analysis and synthesis. An elementary knowledge of linear-graph theory is also assumed.

1.2 Previous Work in This Area. The references mentioned in Section 1.1, Tou (1) and Zadeh and Desoer (2), provide a good summary of the current state-of-the-art of state-space analysis. A straightforward and systematic procedure for formulating the state model of a network is given by Blackwell and Grigsby (5).

The previous work which has considered the synthesis of the state model has been confined primarily to technical papers. Dervisoglu (6) has developed a method for realizing the A-matrix (as defined by Bashkow (7)) for a special class of RLC networks. He has found necessary and sufficient conditions that a given matrix be realizable as the A-matrix of an RLC network, but the conditions are quite restrictive.

The paper by Morgan (8) treats the problem of state variable synthesis. However, he assumes the realizability of the network and develops a synthesis procedure which uses state-variable feedback. Brockett (9) discusses the effect of state-variable feedback on the poles and zeros of the transfer function of a control system. The effect of state-variable feedback on the pole-zero configuration and the proper feedback for optimal control are examined.

Kalman (10, 11) derives some rather theoretical conditions that a system of state equations describe an N-port network having an N by N impedance matrix which is non-negative real. These are general conditions which guarantee that no eigenvalue of the coefficient matrix of his state model can have a positive real part; and hence the state equations can be realized. An example of his realization procedures as applied to LC driving-point synthesis is provided.

The theory of matrix transformations and canonical matrices is summarized in a paper by Browne (12) and Chapters 15 and 17 of (13) by the same author. A comprehensive and thoroughly understandable treatment of canonical matrices is given by Gantmacher (14). Proofs of the existence of transformations between canonical forms of a matrix are done by Turnbull and Aitken (15).

Holmes (16) has done some work on applications of the rational canonical form of the state equations to control systems. He derives a method, which is used by the author in this thesis, for obtaining the transfer function from the state equations.

1.3 Outline of the Method of Solution. The author has answered the two questions posed in Section 1.1 by finding conditions on the coefficient matrices ($\underline{A}$ or $\underline{B}$) which correspond to realizability conditions for driving-point functions in classical network synthesis. The conditions are found by performing a linear transformation of the state equations and then inspecting the coefficient matrices of the transformed equations.

Linear nonsingular transformations are found such that $\underline{A}$ and $\underline{B}$ are each transformed to the rational canonical form. If $\underline{x} = \underline{P}\,\underline{y}$ and $\underline{x} = \underline{Q}\,\underline{y}$ in Equations 1.1.4 and 1.1.5, respectively, then

$$\dot{\underline{y}} = \underline{P}^{-1}\,\underline{A}\,\underline{P}\,\underline{y} + \underline{P}^{-1}\,\underline{d}\,v = \underline{C}_1\,\underline{y} + \underline{D}\,v \qquad (1.3.1)$$

$$i = \underline{h}\,\underline{P}\,\underline{y} + G\,v = \underline{H}\,\underline{y} + G\,v$$

$$\overset{\circ}{\underline{y}} = \underline{Q}^{-1}\ \underline{B}\ \underline{Q}\ \underline{y} + \underline{Q}^{-1}\ \underline{e}\ i = \underline{C}_2\ \underline{y} + \underline{E}\ i$$

$$v = \underline{k}\ \underline{Q}\ \underline{y} + R\ i = \underline{K}\ \underline{y} + R\ i \qquad (1.3.2)$$

where

$$\underline{C}_1 = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & -a_n \\ 1 & 0 & & & \cdot & -a_{n-1} \\ 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 0 & -a_2 \\ 0 & \cdot & \cdot & 0 & 1 & -a_1 \end{bmatrix}$$

$$\underline{C}_2 = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & -b_n \\ 1 & 0 & & & \cdot & -b_{n-1} \\ 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 0 & -b_2 \\ 0 & \cdot & \cdot & 0 & 1 & -b_1 \end{bmatrix}$$

Inspection of Equations 1.3.1 and 1.3.2 gives some necessary conditions for the state model to be realizable as a one-port network.

To find necessary and sufficient conditions for realizability, another nonsingular transformation, $\underline{y} = \underline{T}\ \underline{z}$, takes the rational canonical form to the Jordan canonical form.

$$\overset{\circ}{\underline{z}} = \underline{T}^{-1}\ \underline{C}_1\ \underline{T}\ \underline{z} + \underline{T}^{-1}\ \underline{D}\ v = \underline{J}_1\ \underline{z} + \underline{T}^{-1}\ \underline{D}\ v$$

$$i = \underline{H}\ \underline{T}\ \underline{z} + G\ v = \underline{H}\ \underline{T}\ \underline{z} + G\ v \qquad (1.3.3)$$

$$\dot{\underline{z}} = \underline{U}^{-1} \underline{C}_2 \underline{U} \underline{z} + \underline{U}^{-1} \underline{E} i = \underline{J}_2 \underline{z} + \underline{U}^{-1} \underline{E} i$$

$$v = \underline{K} \underline{U} \underline{z} + R i = \underline{K} \underline{U} \underline{z} + R i \qquad (1.3.4)$$

where:

$$\underline{J}_2 = \begin{bmatrix} \underline{J}_{21} & 0 & \cdot & \cdot & 0 \\ 0 & \underline{J}_{22} & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \underline{J}_{2k} \end{bmatrix}$$

$$\underline{J}_{2i} = \begin{bmatrix} \beta_i & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \beta_i & 1 & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & 0 \\ \cdot & & & & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & \beta_i \end{bmatrix}$$

Next form a matrix $\underline{S}$:

$$\underline{S} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & -\frac{c_n}{c_o} \\ 1 & 0 & & & \cdot & -\frac{c_{n-1}}{c_o} \\ 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 0 & -\frac{c_2}{c_o} \\ 0 & \cdot & \cdot & 0 & 1 & -\frac{c_1}{c_o} \end{bmatrix} \qquad (1.3.5)$$

where the c_i's are specified combinations of the a_i's and b_i's. The matrix $\underline{S}$ is then transformed to its Jordan canonical form, $\underline{J}_3$.

An inspection of the diagonal blocks of $\underline{J}_2$ and $\underline{J}_3$ provide the necessary and sufficient conditions for realizability.

A discussion of the procedures for obtaining the canonical forms of matrices is given in Appendices A and B.

Various tests have been devised by the author for determining the type of network which can be realized. Inspection of the original state model, inspection of $\underline{C}_1$ and $\underline{C}_2$, and a short computation using the coefficients of $\underline{J}_2$ provide necessary and sufficient conditions for an LC realization.

To test for an RC (or RL) realization, an inspection of $\underline{J}_2$ and a short computation involving the coefficients of $\underline{J}_1$ and $\underline{J}_2$ are necessary and sufficient.

If neither the LC, RC, nor RL conditions are satisfied, then the realization must be of the RLC or RLCT type.

No attempt is made to find the values of the components of the realization network, since the problem has now been reduced to the well-defined procedures of classical synthesis.

CHAPTER II

DERIVATION OF THE CANONICAL STATE EQUATIONS

2.1 Theorems Relating the State Equations to Classical Network Functions. The author has formulated and proved several theorems which facilitate the correlation between the abstract mathematical equations and physical systems. These theorems are also used in some later proofs. First, some terms will be defined.

Definition 2.1.1. When G is equal to zero in Equation 1.1.4, we have i explicit in the state variables only. This set of state equations will be called the I-equations.

$$\begin{aligned} \underline{\mathring{x}} &= \underline{A}\,\underline{x} + \underline{d}\,v \\ i &= \underline{h}\,\underline{x} \end{aligned} \tag{2.1.1}$$

Definition 2.1.2. When R is equal to zero in Equation 1.1.5, then v is a function of state variables only. These equations are designated the V-equations.

$$\begin{aligned} \underline{\mathring{x}} &= \underline{B}\,\underline{x} + \underline{e}\,i \\ v &= \underline{k}\,\underline{x} \end{aligned} \tag{2.2.2}$$

Definition 2.1.3. A passive one-port network is said to contain a driving-point circuit of capacitors if, when an ideal voltage source is connected to the input terminals, there exists a circuit consisting of the voltage source and one or more capacitors.

Definition 2.1.4. A passive one-port network contains a driving-point cutset of inductors if, when an ideal current source is connected to the input terminals, there exists a cutset formed by the current source and one or more inductors.

The passive one-port networks discussed throughout this section are assumed to contain no circuits of capacitors or cutsets of inductors.

Definition 2.1.5. Let θ_i be the voltage across capacitor C_i. Let ϕ_j be the current through inductor L_j. θ_i and ϕ_j will be referred to as the i-th and j-th state variables of the network.

To write the state model for a system of n state variables, we have

$$\begin{bmatrix} C_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & & & & & & \cdot \\ \cdot & & \cdot & & & & & \cdot \\ \cdot & & & C_k & & & & \cdot \\ \cdot & & & & L_{k+1} & & & \cdot \\ \cdot & & & & & \cdot & & \cdot \\ \cdot & & & & & & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & L_n \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \cdot \\ \cdot \\ \dot{\theta}_k \\ \dot{\phi}_{k+1} \\ \cdot \\ \cdot \\ \dot{\phi}_n \end{bmatrix} = \begin{bmatrix} i_1 \\ \cdot \\ \cdot \\ i_k \\ v_{k+1} \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \tag{2.1.3}$$

where:

i_j is the current through C_j, ($j = 1, 2, \ldots, k$), and

v_ℓ is the voltage across L_ℓ, ($\ell = k + 1, \ldots, n$).

In writing this state model, it is assumed that the system contains a minimum number of reactive elements. For example, if there are two capacitors in parallel, they are first combined into one equivalent capacitor, etc. A state model of a system is said to be of order n when the system contains exactly n reactive elements and vice versa.

To obtain the I-equations, it is necessary to express the capacitor currents ($i_1, \ldots, i_k$) and the inductor voltages ($v_{k+1}, \ldots, v_n$) as functions of the input voltage, v, and the state variables ($\theta_1, \ldots, \theta_k, \phi_{k+1}, \ldots, \phi_n$). The input current, i, must be expressed as a function of state variables only.

To obtain the V-equations, the requirement is that the capacitor currents and the inductor voltages be expressed as functions of i and the state variables. v must be expressed as a function of state variables only.

<u>Theorem 2.1.1.</u> The V-equations can be written for a one-port network if and only if the network contains a driving-point circuit of capacitors.

Proof:

1. Assume the network contains a driving-point circuit of capacitors.

2. Then $v = \sum_i \theta_i$, where capacitor C_i is in the driving-point

circuit. Therefore v is a function of state variables only.

3. Every capacitor current and inductor voltage ($i_1, \ldots, v_n$) can be written as a function of i, v, and the state variables.

$$i_j = f(\underline{\theta}, \underline{\phi}, i, v), \quad j = 1, 2, \ldots, k$$

$$v_\ell = g(\underline{\theta}, \underline{\phi}, i, v), \quad \ell = k + 1, \ldots, n$$

4. By statement 2, $v = v(\underline{\theta})$, so we can write

$$i_j = f(\underline{\theta}, \underline{\phi}, i), \quad j = 1, 2, \ldots, k$$

$$v_\ell = g(\underline{\theta}, \underline{\phi}, i), \quad \ell = k + 1, \ldots, n$$

5. Therefore, the V-equations can be written for the network.

6. Assume the network contains no driving-point circuit of capacitors.

7. Apply an ideal voltage source v to the input. It is possible to choose a tree consisting of v, all of the capacitors, and a portion of the resistors. The cotree then is composed of the inductors and the remaining resistors.

8. The currents through the resistors in the cotree can be written as

$$\underline{i}_{RC} = f(\underline{\theta}, \underline{\phi}, v)$$

The voltages across the resistors in the tree can be written as

$$\underline{v}_{RT} = g(\underline{\theta}, \underline{\phi}, v)$$

The current through v (which is i) can be written as

$$i = h\,(\underline{i}_{RC}, \underline{\phi}) = h\,(\underline{\theta}, \underline{\phi}, v)$$

9. Therefore, in general, v is a function of i; and hence the V-equations cannot be written. (For a more complete discussion and proofs of statements 7-8, see Blackwell and Grigsby (5).)

10. By logic, if $\overline{A}$ implies $\overline{B}$, then B implies A. Therefore, if the V-equations can be written, the network contains a driving-point circuit of capacitors.

Theorem 2.1.2. The V-equations can be written for a one-port network if and only if the input impedance has a zero at infinity.

Proof:

A well-known fact from classical network analysis is that the input impedance of a one-port network has a zero at infinity if and only if the network contains a driving-point circuit of capacitors.

The theorem follows immediately from this fact and Theorem 2.1.1.

Theorem 2.1.3. (dual to Theorem 2.1.1) The I-equations can be written for a one-port network if and only if the network contains a driving-point cutset of inductors.

Proof:

1. Assume the network contains a driving-point cutset of inductors.

2. Then $i = \sum_j \phi_j$, where inductor L_j is in the driving-point cutset. Therefore, i is a function of state variables only.

3. Every capacitor current and inductor voltage can then be written as

$$i_j = f(\underline{\theta}, \underline{\phi}, i, v) = f(\underline{\theta}, \underline{\phi}, v), \quad j = 1, 2, \ldots, k$$

$$v_\ell = g(\underline{\theta}, \underline{\phi}, i, v) = g(\underline{\theta}, \underline{\phi}, v), \quad \ell = k + 1, \ldots, n$$

4. Therefore, the I-equations can be written.

5. Assume the network does not contain a driving-point cutset of inductors.

6. If an ideal current source is applied to the input, it is possible to choose a tree such that all of the inductors, part of the resistors, and the current source are in the cotree. The tree then consists of the remaining resistors and all of the capacitors.

7. The currents through the resistors in the cotree are

$$\underline{i}_{RC} = f(\underline{\theta}, \underline{\phi}, i)$$

The voltages across the resistors in the tree are

$$\underline{v}_{RT} = g(\underline{\theta}, \underline{\phi}, i)$$

The voltage across the current source is v and can be written as

$$v = h(\underline{v}_{RT}, \underline{\theta}) = h(\underline{\theta}, \underline{\phi}, i)$$

8. Therefore, in general, i is a function of v, and the I-equations cannot be written.

9. Therefore, if the I-equations can be written, the network contains a driving-point cutset of inductors.

Theorem 2.1.4. The I-equations can be written for a one-port network if and only if the input impedance has a pole at infinity.

Proof:

The input impedance of a one-port network has a pole at infinity if and only if the network contains a driving-point cutset of inductors. Using Theorem 2.1.3, the theorem follows immediately.

Theorem 2.1.5. The I-equations and the V-equations cannot both be written for a given one-port network.

Proof: (by contradiction)

Assume both the I-equations and the V-equations can be written. Then by Theorem 2.1.2, the input impedance has a zero at infinity. By Theorem 2.1.4, the input impedance also has a pole at infinity, an impossibility.

If the network does not contain either a driving-point circuit of capacitors or a driving-point cutset of inductors, then $G \neq 0$ and $R \neq 0$ in Equations 1.1.4 and 1.1.5, respectively, and neither the I-equations nor the V-equations can be written.

2.2 Reduction of the Coefficient Matrices to Rational Canonical Form. A nonsingular linear transformation, $\underline{x} = \underline{P}\,\underline{y}$, is found such that $\underline{P}^{-1}\,\underline{A}\,\underline{P} = \underline{C}_1$, the rational canonical form of $\underline{A}$. (See Appendix A for a discussion of the rational form and the method for finding $\underline{P}$.) In general

$$\underline{C}_1 = \begin{bmatrix} \underline{C}_{11} & 0 & \cdot & \cdot & 0 \\ 0 & \underline{C}_{12} & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \underline{C}_{1k} \end{bmatrix} \qquad (2.2.1)$$

where each of the submatrices has the form

$$\underline{C}_{1i} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & -a_{\zeta_i} \\ 1 & 0 & & & \cdot & -a_{\zeta_i - 1} \\ 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 0 & -a_2 \\ 0 & \cdot & \cdot & 0 & 1 & -a_1 \end{bmatrix} \tag{2.2.2}$$

$$0 \leq \zeta_i \leq n, \quad \sum_{i=1}^{k} \zeta_i = n$$

The reduced characteristic function of $\underline{A}$ is

$$\phi_1(\lambda) = (-1)^r \, [\lambda^r + a_1 \lambda^{r-1} + \ldots + a_{r-1} \lambda + a_r] \tag{2.2.3}$$

$$r = \max [\zeta_i], \; i = 1, 2, \ldots, k$$

For the present time, assume that $\zeta_1 = r = n$. Then $\underline{C}_1$ contains only one block of the form $\underline{C}_{1i}$. The case where $\underline{C}_1$ has more than one block will be considered in the next section.

Similarly, find another nonsingular linear transformation, $\underline{x} = \underline{Q}\,\underline{y}$, such that $\underline{Q}^{-1}\,\underline{B}\,\underline{Q} = \underline{C}_2$, the rational canonical form of $\underline{B}$.

$$\underline{C}_2 = \begin{bmatrix} \underline{C}_{21} & 0 & \cdot & \cdot & 0 \\ 0 & \underline{C}_{22} & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \underline{C}_{2\ell} \end{bmatrix} \tag{2.2.4}$$

where each of the submatrices has the form

$$\underline{C}_{2i} = \begin{bmatrix} 0 & 0 & \circ & \circ & 0 & -b_{\xi_i} \\ 1 & 0 & & & \circ & -b_{\xi_i - 1} \\ 0 & 1 & \circ & & \circ & \cdot \\ \circ & & \circ & \circ & \circ & \circ \\ \circ & & & \circ & 0 & -b_2 \\ 0 & \circ & \circ & 0 & 1 & -b_1 \end{bmatrix} \qquad (2.2.5)$$

$$0 \leq \xi_i \leq n, \quad \sum_{i=1}^{\ell} \xi_i = n$$

The reduced characteristic function of $\underline{B}$ is

$$\phi_2(\lambda) = (-1)^s \, [\lambda^s + b_1 \, \lambda^{s-1} + \ldots + b_{s-1} \, \lambda + b_s] \qquad (2.2.6)$$

$$s = \max \, [\xi_i], \; i = 1, 2, \ldots, \ell$$

It is also assumed that $\underline{C}_2$ contains only one block of the form $\underline{C}_{2i}$.

2.3 Definition of the Canonical State Equations. When the state variable $\underline{x}$ is transformed by a linear, nonsingular, n by n coefficient matrix $\underline{P}$, the state model is also transformed. Replacing $\underline{x}$ by $\underline{P}\,\underline{y}$ in Equation 1.1.4 gives

$$\underline{P}\,\mathring{\underline{y}} = \underline{A}\,\underline{P}\,\underline{y} + \underline{d}\,v$$

$$i = \underline{h}\,\underline{P}\,\underline{y} + G\,v \qquad (2.3.1)$$

Multiplying the top equation by $\underline{P}^{-1}$(since $\underline{P}$ is nonsingular) produces

$$\mathring{\underline{y}} = \underline{P}^{-1}\,\underline{A}\,\underline{P}\,\underline{y} + \underline{P}^{-1}\,\underline{d}\,v$$

$$i = \underline{h}\,\underline{P}\,\underline{y} + G\,v \qquad (2.3.2)$$

where $\underline{y}$ is an n by 1 vector which defines the state variables of the transformed state model.

Define the new coefficient matrices

$$\underline{P}^{-1}\ \underline{A}\ \underline{P} = \underline{C}_1 = \text{n by n coefficient matrix}$$

$$\underline{P}^{-1}\ \underline{d} = \underline{D} = \text{n by 1 coefficient matrix}$$

$$\underline{h}\ \underline{P} = \underline{H} = \text{1 by n coefficient matrix}$$

Then

$$\begin{aligned} \mathring{\underline{y}} &= \underline{C}_1\ \underline{y} + \underline{D}\ v \\ i &= \underline{H}\ \underline{y} + G\ v \end{aligned} \qquad (2.3.3)$$

The other state model is similarly transformed by the linear, nonsingular matrix $\underline{Q}$, an n by n coefficient matrix. Replacing $\underline{x}$ by $\underline{Q}\ \underline{y}$ in Equation 1.1.5 gives

$$\begin{aligned} \underline{Q}\ \mathring{\underline{y}} &= \underline{B}\ \underline{Q}\ \underline{y} + \underline{e}\ i \\ v &= \underline{k}\ \underline{Q}\ \underline{y} + R\ i \end{aligned} \qquad (2.3.4)$$

Multiply the first equation by $\underline{Q}^{-1}$ and define the new coefficient matrices

$$\underline{Q}^{-1}\ \underline{B}\ \underline{Q} = \underline{C}_2 = \text{n by n coefficient matrix}$$

$$\underline{Q}^{-1}\ \underline{e} = \underline{E} = \text{n by 1 coefficient matrix}$$

$$\underline{k}\ \underline{Q} = \underline{K} = \text{1 by n coefficient matrix}$$

Then

$$\mathring{\underline{y}} = \underline{C}_2 \underline{y} + \underline{E} i$$

$$v = \underline{K} \underline{y} + R i \qquad (2.3.5)$$

Equations 2.3.3 and 2.3.5 with $\underline{y}$ as the vector of state variables will hereinafter be referred to as the canonical form of the state equations.

Now consider the cases where $\underline{C}_1$ and $\underline{C}_2$ contain more than one block of the form of $\underline{C}_{1i}$ and $\underline{C}_{2i}$, respectively.

Assume that the n by n matrix $\underline{C}_1$ can be partitioned into submatrices as in Equation 2.2.1. The submatrices $\underline{C}_{11}, \ldots, \underline{C}_{1k}$ have dimensions ζ_1 by $\zeta_1, \ldots, \zeta_k$ by ζ_k, respectively, where

$$\sum_{i=1}^{k} \zeta_i = n .$$

The n-dimensional state vector $\underline{y}$ can be partitioned into k subvectors, each with dimension ζ_i.

$$\underline{y} = \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \cdot \\ \cdot \\ \underline{y}_k \end{bmatrix} \qquad (2.3.6)$$

Partition $\underline{D}$ and $\underline{H}$ such that they are conformable to $\underline{C}_1$. Then Equation 2.3.3 becomes

$$\begin{bmatrix} \dot{\underline{y}}_1 \\ \dot{\underline{y}}_2 \\ \cdot \\ \cdot \\ \dot{\underline{y}}_k \end{bmatrix} = \begin{bmatrix} \underline{C}_{11} & 0 & \cdot & \cdot & 0 \\ 0 & \underline{C}_{12} & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \underline{C}_{1k} \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \cdot \\ \cdot \\ \underline{y}_k \end{bmatrix} + \begin{bmatrix} \underline{D}_1 \\ \underline{D}_2 \\ \cdot \\ \cdot \\ \underline{D}_k \end{bmatrix} v$$

$$i = \begin{bmatrix} \underline{H}_1 & \underline{H}_2 & \cdot & \cdot & \underline{H}_k \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \cdot \\ \cdot \\ \underline{y}_k \end{bmatrix} + (G_1 + G_2 + \ldots + G_k)\, v \qquad (2.3.7)$$

where $(G_1 + G_2 + \ldots + G_k) = G$

Equation 2.3.7 has a k by k diagonal coefficient matrix $\underline{C}_1$, so it can be divided into k equations, each having the form of Equation 2.3.3.

$$\dot{\underline{y}}_j = \underline{C}_{1j}\, \underline{y}_j + \underline{D}_j\, v$$

$$i_j = \underline{H}_j\, \underline{y}_j + G_j\, v \qquad (2.3.8)$$

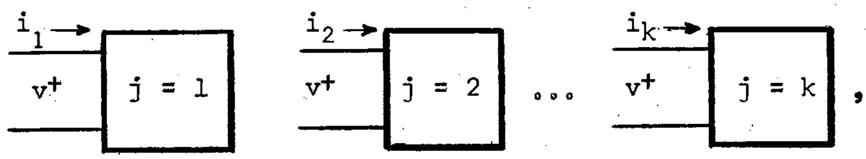
$$j = 1, 2, \ldots, k$$

The procedure for physically realizing the k sets of equations is now stated in the form of a theorem.

Theorem 2.3.1. If each set of equations is realizable (j = 1, 2, ..., k), then Equation 2.3.3 is realizable as k one-port networks connected in parallel.

Proof:

Assume that each set of equations is realizable as a one-port network.



, $\sum_{j=1}^{k} i_j = i$

Figure 2.3.1. One-Port Realizations for Equations 2.3.8

Now Equation 2.3.3 is realized by connecting the k one-ports of Figure 2.3.1 in parallel.

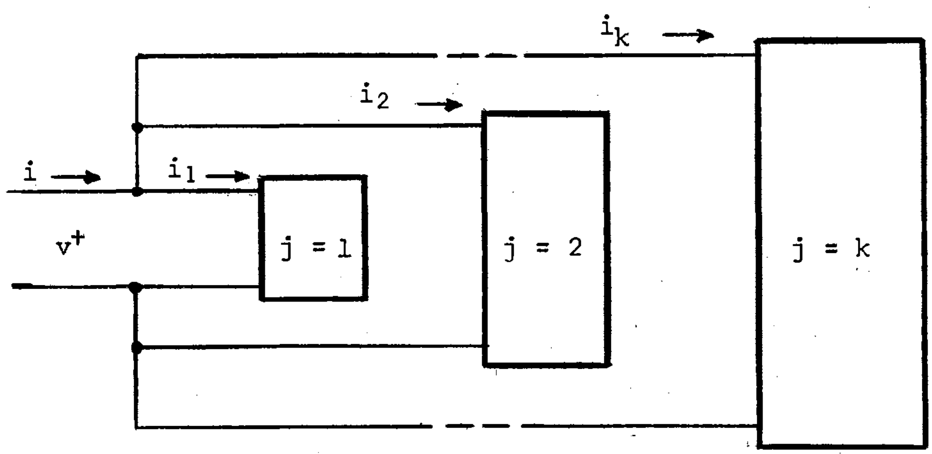


Figure 2.3.2. One-Port Realization When $\underline{C}_1$ Contains More Than One Block

Similarly, the n by n matrix $\underline{C}_2$ can be partitioned such that all non-zero submatrices are square and on the main diagonal. Partition $\underline{y}$, $\underline{E}$, and $\underline{K}$ such that they are each composed of ℓ subvectors, each of dimension ξ_j ($j = 1, 2, \ldots, \ell$). Then Equation 2.3.5 can be written as

$$\begin{bmatrix} \mathring{\underline{y}}_1 \\ \mathring{\underline{y}}_2 \\ \cdot \\ \cdot \\ \mathring{\underline{y}}_\ell \end{bmatrix} = \begin{bmatrix} \underline{C}_{21} & 0 & \cdot & \cdot & 0 \\ 0 & \underline{C}_{22} & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \underline{C}_{2\ell} \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \cdot \\ \cdot \\ \underline{y}_\ell \end{bmatrix} + \begin{bmatrix} \underline{E}_1 \\ \underline{E}_2 \\ \cdot \\ \cdot \\ \underline{E}_\ell \end{bmatrix} i$$

$$v = \begin{bmatrix} \underline{K}_1 & \underline{K}_2 & \cdot & \cdot & \underline{K}_\ell \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \cdot \\ \cdot \\ \underline{y}_\ell \end{bmatrix} + (R_1 + R_2 + \ldots + R_\ell)\, i \qquad (2.3.9)$$

where $(R_1 + R_2 + \ldots + R_\ell) = R$ and $\underline{C}_{2j}$ is a square matrix of order ξ_j $(j = 1, 2, \ldots, \ell)$.

Divide Equation 2.3.9 into ℓ sets of equations, each having the form of Equation 2.3.5.

$$\mathring{\underline{y}}_j = \underline{C}_{2j}\, \underline{y}_j + \underline{E}_j\, i$$

$$v_j = \underline{K}_j\, \underline{y}_j + R_j\, i \qquad (2.3.10)$$

$$j = 1, 2, \ldots, \ell$$

<u>Theorem 2.3.2.</u> If each set of equations is realizable ($j = 1, 2, \ldots, \ell$), then Equation 2.3.5 is realizable as ℓ one-port networks connected in series.

Proof:

Assume each set of equations can be realized as a one-port network.

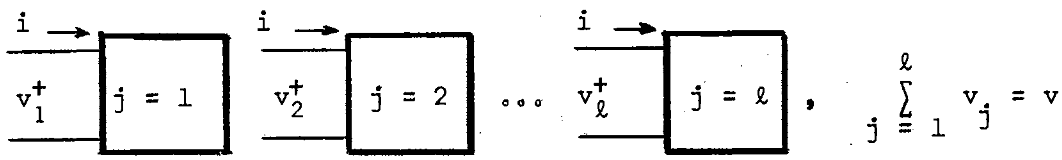


Figure 2.3.3. One-Port Realizations for Equations 2.3.10

Now Equation 2.3.5 is realized by connecting the ℓ one-port networks of Figure 2.3.3 in series.

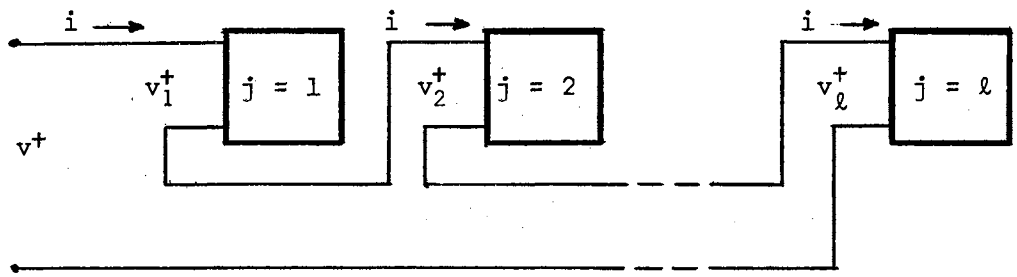


Figure 2.3.4. One-Port Realization When $\underline{C}_2$ Contains More Than One Block

2.4 Relation of the Canonical State Equations to Classical Network Functions. Define the polynomials $p(\lambda)$ and $q(\lambda)$ such that they are polynomials with the highest degree term having unity coefficient.

$$p(\lambda) = (-1)^n \phi_1(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n \qquad (2.4.1)$$

$$q(\lambda) = (-1)^m \phi_2(\lambda) = \lambda^m + b_1 \lambda^{m-1} + \ldots + b_{m-1} \lambda + b_m \qquad (2.4.2)$$

In all the preceding discussions m and n have been equal. However, as will be shown, the degrees of $p(\lambda)$ and $q(\lambda)$ may differ by one.

Assume $Z(\lambda) = p(\lambda)/q(\lambda)$. The correlation between classical networks and the state model is now stated in the form of a theorem.

Theorem 2.4.1. If Z(s) is a positive real function, then it is realizable as the classical input impedance of a one-port network. The resulting network has Equations 2.3.3 and 2.3.5 as canonical state equations.

Proof:

$p(\lambda)$ and $q(\lambda)$ are the reduced characteristic functions (within a factor of -1) of Equations 1.1.4 and 1.1.5, respectively. They are invariant under the transformations $\underline{x} = \underline{P}\,\underline{y}$ and $\underline{x} = \underline{Q}\,\underline{y}$. There can be any number of state-space models with coefficient matrices which have the same reduced characteristic functions. All of these state equations can be reduced to the same canonical state equations.

There are many one-port networks which have the same classical input impedance, Z(s); but for some given one-port, the input impedance is unique.

Therefore, if a network realization is found which has input impedance $Z(s) = p(s)/q(s)$, then the network is a realization of the canonical state equations. Hence, as far as the classical theory, which considers only external characteristics of the network, is concerned, the network is also a realization of Equations 1.1.4 and 1.1.5.

Figure 2.4.1 shows diagrammatically the relationship of the state model and classical networks.

The previous discussion has implicitly assumed that G and R are non-zero. However, in general, this is not the case. If $\underline{A}$ and $\underline{B}$ are

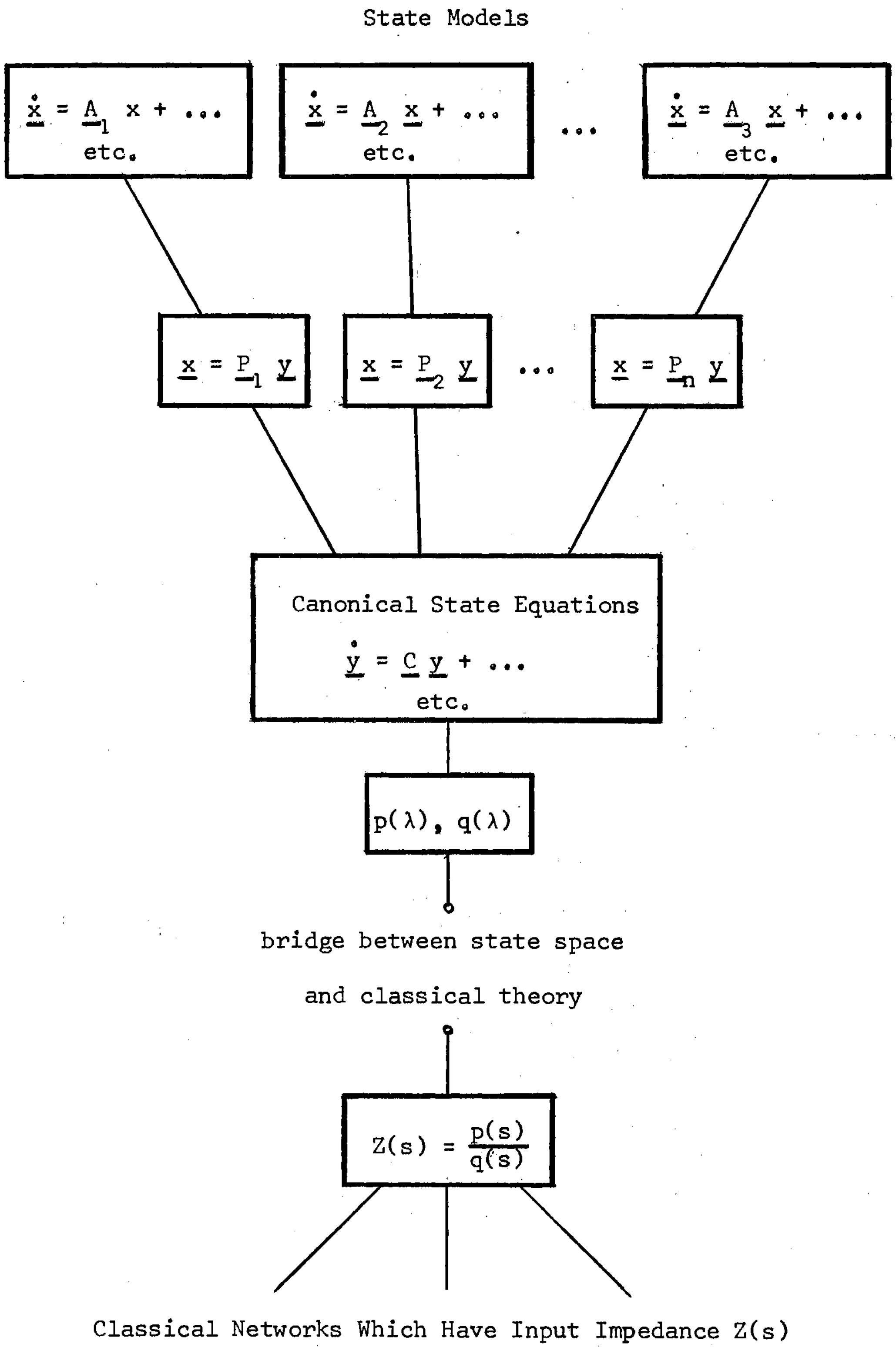


Figure 2.4.1. Block Diagram Showing the Relation Between Classical Networks and the State Model

not both available, then another method for obtaining $p(\lambda)$ and $q(\lambda)$ must be found. The procedure is stated in the form of a theorem and proof.

Theorem 2.4.2. The rational canonical matrices $\underline{C}_1$ and $\underline{C}_2$--or actually $p(\lambda)$ and $q(\lambda)$--can always be found from the given state model.

Proof:

Case I. If $G \neq 0$ in Equation 1.1.4 and $R \neq 0$ in Equation 1.1.5, then $p(\lambda)$ and $q(\lambda)$ can be found by reducing $\underline{A}$ and $\underline{B}$, respectively, to rational form.

Case II. Assume that Equation 1.1.4 is given with $G = 0$. Then we have the I-equations as defined in Section 2.1.

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{d}\,v$$
$$i = \underline{h}\,\underline{x} \qquad (2.1.1)$$

Only $\underline{C}_1$ can be found by canonical reduction of $\underline{A}$. Therefore, $p(\lambda)$ is available by inspection. The canonical form of Equation 2.1.1 is

$$\dot{\underline{y}} = \underline{C}\,\underline{y} + \underline{D}\,v$$
$$i = \underline{H}\,\underline{y} \qquad (2.4.3)$$

Let the canonical state model be solved by Laplace transform theory for I(s) in terms of V(s). In classical theory, the input impedance is $Z(s) = V(s)/I(s)$; and the input admittance is $1/Z(s) = I(s)/V(s)$.

$$s\,\underline{Y}(s) - \underline{y}(0^+) = \underline{C}_1\,\underline{Y}(s) + \underline{D}\,V(s)$$

$$I(s) = \underline{H}\,\underline{Y}(s) \qquad (2.4.4)$$

As is customary, assume initial conditions are zero, $\underline{y}(0^+) = 0$. Solving the first equation for $\underline{Y}(s)$ and substituting into the second equation gives

$$I(s) = \underline{H}\,(s\,\underline{U} - \underline{C}_1)^{-1}\,\underline{D}\,V(s) \qquad (2.4.5)$$

$$I(s)/V(s) = \underline{H}\,(s\,\underline{U} - \underline{C}_1)^{-1}\,\underline{D} \qquad (2.4.6)$$

From elementary matrix theory, the inverse of a matrix is equal to the adjoint of the matrix divided by the determinant of the matrix.

$$\frac{I(s)}{V(s)} = \frac{\underline{H}\,\mathrm{adj}\,[s\,\underline{U} - \underline{C}_1]\,\underline{D}}{\det\,[s\,\underline{U} - \underline{C}_1]} \qquad (2.4.7)$$

It is proved by Holmes (16) that $\det\,[s\,\underline{U} - \underline{C}_1]$ is equal to $p(s)$, where $p(s)$ is as defined in Equation 2.4.1. Assume that $V(s) = p(s)$. Then

$$q(s) = I(s) = \underline{H}\,\mathrm{adj}\,[s\,\underline{U} - \underline{C}_1]\,\underline{D} \qquad (2.4.8)$$

If $\underline{C}_1$ is of order n, the elements of $\mathrm{adj}\,[s\,\underline{U} - \underline{C}_1]$ are polynomials of degree $n - 1$. Therefore if $p(s)$ is of degree n, then $q(s)$ will have degree $m = n - 1$.

The polynomial $q(s)$ can be found by ordinary methods of Laplace transformation and matrix inversion. However, Holmes has developed an algorithm for evaluating the coefficients of $q(s)$ which involves only multiplication of constant matrices.

It is already known that m must be greater than n, from Theorem 2.1.4. The I-equations are given so $Z(\lambda) = p(\lambda)/q(\lambda)$ has a pole at infinity; implying the degree of $p(\lambda)$ is greater than the degree of $q(\lambda)$.

Case III. Assume that Equation 1.1.5 is given with R = 0. Then $q(\lambda)$ can be found by canonical reduction of $\underline{B}$. The canonical state equations are

$$\mathring{\underline{y}} = \underline{C}_2\ \underline{y} + \underline{E}\ i$$
$$v = \underline{K}\ \underline{y} \qquad (2.4.9)$$

Solving as before by methods of Laplace transforms gives

$$Z(s) = \frac{V(s)}{I(s)} = \underline{K}\ [s\ \underline{U} - \underline{C}_2]^{-1}\ \underline{E} = \frac{\underline{K}\ \mathrm{adj}\ [s\ \underline{U} - \underline{C}_2]\ \underline{E}}{\det\ [s\ \underline{U} - \underline{C}_2]} \qquad (2.4.10)$$

Then, we have

$$q(s) = I(s) = \det\ [s\ \underline{U} - \underline{C}_2] \qquad (2.4.11)$$

and

$$p(s) = V(s) = \underline{K}\ \mathrm{adj}\ [s\ \underline{U} - \underline{C}_2]\ \underline{E} \qquad (2.4.12)$$

By Theorem 2.1.2, Z(s) has a zero at infinity. Therefore, p(s) is of lower degree than q(s); and Holmes' algorithm can be applied to find p(s).

Therefore, both $p(\lambda)$ and $q(\lambda)$ can always be found for a given state model; and the degrees of $p(\lambda)$ and $q(\lambda)$ differ by one at most.

CHAPTER III

DEVELOPMENT OF A REALIZABILITY TEST

The author has developed an algorithm which, after a transformation of the canonical state equations, gives necessary and sufficient conditions that a set of state equations be realizable as a one-port network.

3.1 Necessary Conditions for Realizability. There are some necessary conditions which can be checked by inspection of $\underline{C}_1$ and $\underline{C}_2$ of Equations 2.3.3 and 2.3.5, respectively. If the state model is realizable, then

1. All of the coefficients, a_i's and b_i's, are real and positive.
2. m and n differ by one at most.
3. The lowest powers of $p(\lambda)$ and $q(\lambda)$ differ by one at most.
4. There are no missing terms in $p(\lambda)$ and $q(\lambda)$ between the highest and lowest degree terms unless all even or all odd terms are missing.

The state model is realizable as a one-port network if and only if $Z(s) = p(s)/q(s)$ is a positive real function. In this statement it is assumed that p(s) and q(s) have no common factor. If they have a common factor, then define $Z(s) = p'(s)/q'(s)$, where p'(s) and q'(s) have no common factor.

The properties of a positive real function F(s), which is a quotient of rational polynomials, are tabulated on page 106 of Van Valkenberg (3).

If F(s) is positive real, all polynomial coefficients are real and positive. For the state model, the a_i's and b_i's are the polynomial coefficients. Therefore, the last column of both $\underline{C}_1$ and $\underline{C}_2$ must have all nonpositive elements.

The highest and lowest power of the numerator and denominator polynomials differ by one at most. This statement is equivalent to conditions 2 and 3 above on the polynomials $p(\lambda)$ and $q(\lambda)$. The order of matrices $\underline{C}_1$ and $\underline{C}_2$ must differ by no more than one. The number of successive zeros at the top of the last column of $\underline{C}_1$ and $\underline{C}_2$ must differ by no more than one.

If F(s) is positive real, then there must be no missing terms in numerator and denominator polynomials unless all even or all odd terms are missing. This condition is obvious in $\underline{C}_1$ and $\underline{C}_2$. The last column of both matrices may contain zeros only in the top rows of the last column <u>or</u> all alternate rows of the last column must be zero, starting with a zero in the last row.

<u>3.2 Jordan Canonical Form for the Coefficient Matrices.</u> From the rational canonical form, the reduced characteristic function and the elementary divisors of a matrix are found by inspection. The coefficients of the rational form are always real. The main diagonal elements of the Jordan canonical form of a matrix are the eigenvalues of the matrix, which are, in general, complex. The Jordan form is not always a diagonal matrix, but for the purposes of this thesis a diagonal form is not necessary.

We would like to find a nonsingular transformation matrix $\underline{T}$ such that $\underline{T}^{-1}\ \underline{C}_1\ \underline{T} = \underline{J}_1$, the Jordan canonical form of $\underline{C}_1$. (See Appendix B for a discussion of the Jordan form and some methods for finding $\underline{T}$.)

When $\underline{C}_1$ has only one block of the form of Equation 2.2.2, we have

$$\underline{J}_1 = \begin{bmatrix} \underline{J}_{11} & \cdot & \cdot & \cdot & 0 \\ 0 & \underline{J}_{12} & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \underline{J}_{1k} \end{bmatrix} \qquad (3.2.1)$$

where

$$\underline{J}_{1i} = \begin{bmatrix} \alpha_i & 1 & 0 & \cdot & \cdot & 0 \\ 0 & \alpha_i & 1 & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & 0 \\ \cdot & & & & \cdot & 1 \\ 0 & \cdot & \cdot & \cdot & 0 & \alpha_i \end{bmatrix} \qquad (3.2.2)$$

and is of order ν_i.

$$\begin{aligned} p(\lambda) &= \lambda^n + a_1 \lambda^{n-1} + \ldots + a_{n-1} \lambda + a_n \\ &= (\lambda - \alpha_1)^{\nu_1} (\lambda - \alpha_2)^{\nu_2} \ldots (\lambda - \alpha_k)^{\nu_k}, \quad \sum_{i=1}^{k} \nu_i = n \end{aligned} \qquad (3.2.3)$$

The α_i's are distinct eigenvalues of $\underline{C}_1$ (i.e., $\alpha_i \neq \alpha_j$ if $i \neq j$), where α_i has multiplicity ν_i.

Similarly, find a nonsingular transformation matrix $\underline{U}$ such that, for $\underline{C}_2$ having only one block, we have

$$\underline{U}^{-1}\ \underline{C}_2\ \underline{U} = \underline{J}_2 = \begin{bmatrix} \underline{J}_{21} & 0 & \cdot & \cdot & 0 \\ 0 & \underline{J}_{22} & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \underline{J}_{2j} \end{bmatrix} \tag{3.2.4}$$

where

$$\underline{J}_{2i} = \begin{bmatrix} \beta_i & 1 & 0 & \cdot & 0 \\ 0 & \beta_i & 1 & & \cdot \\ \cdot & & \cdot & \cdot & 0 \\ \cdot & & & \cdot & 1 \\ 0 & \cdot & \cdot & 0 & \beta_i \end{bmatrix} \tag{3.2.5}$$

and is of order μ_i.

$$\begin{aligned} q(\lambda) &= \lambda^m + b_1 \lambda^{m-1} + \ldots + b_{m-1} \lambda + b_m \\ &= (\lambda - \beta_1)^{\mu_1} (\lambda - \beta_2)^{\mu_2} \ldots (\lambda - \beta_j)^{\mu_j}, \quad \sum_{i=1}^{j} \mu_i = m \end{aligned} \tag{3.2.6}$$

The β_i's are distinct eigenvalues of $\underline{C}_2$, where β_i has multiplicity μ_i.

Next, form the matrix $\underline{S}$, where

$$\underline{S} = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & -\frac{c_n}{c_o} \\ 1 & 0 & & & \cdot & -\frac{c_{n-1}}{c_o} \\ 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 0 & -\frac{c_2}{c_o} \\ 0 & \cdot & \cdot & 0 & 1 & -\frac{c_1}{c_o} \end{bmatrix} \tag{3.2.7}$$

where

$$\left. \begin{aligned} c_i &= (-1)^i \sum_{k=0}^{i} \left(a_{2k+1}\, b_{2i-2k} - a_{2k}\, b_{2i-2k+1}\right) \\ i &= 0, 1, 2, \ldots, m \end{aligned} \right\} n = m + 1 \quad (3.2.8)$$

$$\left. \begin{aligned} c_i &= (-1)^{i+1} \left(\sum_{k=0}^{i} a_{2k+1}\, b_{2i-2k-1} - \sum_{k=0}^{i+1} a_{2k}\, b_{2i-2k} \right) \\ i &= 0, 1, 2, \ldots, n \end{aligned} \right\} m = n$$

(3.2.9)

$$\left. \begin{aligned} c_i &= (-1)^i \sum_{k=0}^{i} \left(b_{2k+1}\, a_{2i-2k} - b_{2k}\, a_{2i-2k+1}\right) \\ i &= 0, 1, 2, \ldots, n \end{aligned} \right\} m = n + 1$$

(3.2.10)

in Equations 3.2.8 through 3.2.10,

$$a_o = b_o = 1$$

$$a_\ell = 0,\ n < \ell < 0$$

$$b_\ell = 0,\ m < \ell < 0$$

Find the nonsingular transformation matrix $\underline{W}$ such that

$$\underline{W}^{-1}\,\underline{S}\,\underline{W} = \underline{J}_3 = \begin{bmatrix} \underline{J}_{31} & 0 & \cdot & \cdot & 0 \\ 0 & \underline{J}_{32} & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \underline{J}_{3\ell} \end{bmatrix} \tag{3.2.11}$$

where

$$\underline{J}_{3i} = \begin{bmatrix} \gamma_i & 1 & 0 & \cdot & 0 \\ 0 & \gamma_i & 1 & & \cdot \\ \cdot & & \cdot & \cdot & 0 \\ \cdot & & & \cdot & 1 \\ 0 & \cdot & \cdot & 0 & \gamma_i \end{bmatrix} \tag{3.2.12}$$

and is of order n_i.

<u>3.3 Necessary and Sufficient Conditions for Realizability.</u> The necessary and sufficient conditions that the state model be realizable as a passive one-port network are now stated in the form of a theorem.

<u>Theorem 3.3.1.</u> If the necessary conditions of Section 3.1 are satisfied, then the state model is realizable as a one-port network if and only if

1. The real part of every β_i in $\underline{J}_2$ is nonpositive.
2. If any β_i is pure imaginary (real part of $\beta_i = 0$), then μ_i is unity; and the following condition is satisfied:

$$(\lambda - \beta_i)\,\frac{p(\lambda)}{q(\lambda)}\bigg|_{\lambda = \beta_i} > 0 \text{ and real} \tag{3.3.1}$$

3. If any diagonal element, γ_i, of $\underline{J}_3$ is positive and real, it must appear in a submatrix of even order (i.e., n_i is even if γ_i is real and positive).

$$\begin{bmatrix} \gamma_i & 1 \\ 0 & \gamma_i \end{bmatrix}, \quad \begin{bmatrix} \gamma_i & 1 & 0 & 0 \\ 0 & \gamma_i & 1 & 0 \\ 0 & 0 & \gamma_i & 1 \\ 0 & 0 & 0 & \gamma_i \end{bmatrix}, \text{ etc.} \qquad (3.3.2)$$

Note: A sufficient condition that Statement 3 be satisfied is

$$c_i \geq 0, \; (i = 0, 1, \ldots, n)$$

Proof:

The conditions of Theorem 3.3.1 are equivalent to the necessary and sufficient conditions for positive real character of a function which is the quotient of two rational polynomials. The positive real conditions due to Van Valkenberg (3) are summarized below.

<u>Definition 3.3.1.</u> $F(s) = f(s)/g(s)$ is a positive real function if and only if

A. $F(s)$ is real when s is real.

B. $F(s)$ has no poles in the right-half plane.

C. Imaginary axis poles of $F(s)$ are simple; residues evaluated at these poles are real and positive.

D. $\text{Re } F(j\omega) \geq 0, \; 0 \leq \omega \leq \infty$.

If the necessary conditions of Section 3.1 are satisfied, then (A) of Definition 3.3.1 is satisfied. If the coefficients of $f(s)$ and g(s) are real, then F(s) is real when s is real.

The poles of $Z(s) = p(s)/q(s)$ are the values of s for which q(s) is zero. Writing Equation 3.2.6 with λ replaced by s gives

$$q(s) = (s - \beta_1)^{\mu_1} (s - \beta_2)^{\mu_2} \dots (s - \beta_j)^{\mu_j}, \quad \sum_{i=1}^{j} \mu_i = m \qquad (3.3.3)$$

The zeros of Equation 3.3.3 are $\beta_1, \beta_2, \dots, \beta_j$. If the real part of β_i $(i = 1, 2, \dots, j)$ is nonpositive, then there are no poles of Z(s) in the right-half plane. Therefore, (1) of Theorem 3.3.1 is equivalent to (B) of Definition 3.3.1.

If the real part of some β_i $(1 \leq i \leq j)$ is zero, then the pole of Z(s) at β_i is pure imaginary. The pole at β_i is simple if μ_i is unity. If μ_i is unity, then the residue evaluated at β_i is given by

$$(s - \beta_i) \frac{p(s)}{q(s)} \Bigg|_{s = \beta_i} \qquad (3.3.4)$$

Therefore, (2) of Theorem 3.3.1 is equivalent to (C) of Definition 3.3.1.

Let

$$F(s) = \frac{f(s)}{g(s)} = \frac{m_1(s) + n_1(s)}{m_2(s) + n_2(s)} \qquad (3.3.5)$$

where $m_1(s)$ = even part of f(s)

$n_1(s)$ = odd part of f(s)

$m_2(s)$ = even part of g(s)

$n_2(s)$ = odd part of g(s)

In order that Re $F(j\omega) \geq 0$ for all ω, it is necessary and sufficient that

$$A(\omega^2) = m_1(s)\, m_2(s) - n_1(s)\, n_2(s) \Big|_{s = j\omega}$$

have no real positive roots of odd multiplicity.

$$A(\omega^2) = c_o\, \omega^{2p} + c_1\, (\omega^2)^{p-1} + \ldots + c_{p-1}\, \omega^2 + c_p \qquad (3.3.6)$$

where $p = \max [m, n]$

The coefficients defined by Equations 3.2.8 through 3.2.10 are identically the c_i's of Equation 3.3.6. The ordinary methods for finding the roots of $A(\omega^2)$ are by factoring or by using Sturm's theorem (see Guillemin (17)).

If the c_i's are put into a matrix $\underline{S}$, as defined by Equation 3.2.7, then the Jordan form of $\underline{S}$ has the roots of $A(\omega^2)$ displayed on the main diagonal. The eigenvalues of $\underline{S}$ are the roots of

$$A(\omega^2) = (\omega^2 - \gamma_1)^{\eta_1}\, (\omega^2 - \gamma_2)^{\eta_2} \ldots (\omega^2 - \gamma_\ell)^{\eta_\ell} \qquad (3.3.7)$$

If any of the γ_i's are real and positive, this fact is found by inspection of $\underline{J}_3$. The γ_i's are distinct (because $\underline{S}$ is a rational form with only one block), and the multiplicity of each γ_i is η_i. Each block, $\underline{J}_{3i}$ $(i = 1, 2, \ldots, \ell)$, of $\underline{J}_3$ is of order η_i; so the multiplicity of each root of $A(\omega^2)$ is immediately obvious from $\underline{J}_3$. Therefore, (3) of Theorem 3.3.1 is equivalent to condition (D) of Definition 3.3.1.

An equivalent condition to (B) and (C) of Definition 3.3.1 is the following:

B'. If $F(s) = f(s)/g(s)$, then $f(s) + g(s)$ must be a Hurwitz polynomial.

The author has developed a test on the coefficients of $p(\lambda)$ and $q(\lambda)$

which produces the same information as (1) and (2) of Theorem 3.3.1, which is, in turn, equivalent to (B'). Define

$$r(\lambda) = \lambda^{p} + d_1 \lambda^{p-1} + \dots + d_{p-1} \lambda + d_p \qquad (3.3.8)$$

$$r(\lambda) = p(\lambda) + q(\lambda) \text{ if } m \neq n$$

$$= \frac{p(\lambda) + q(\lambda)}{2} \text{ if } m = n$$

where $p = \max [m, n]$, and each d_i is the sum of one a_i and one b_i.

Let $\psi(s) = m(s)/n(s)$, for p even (or $n(s)/m(s)$, for p odd) where

$$m(s) = \text{even part of } r(s)$$

$$n(s) = \text{odd part of } r(s)$$

Then r(s) is a Hurwitz polynomial if the continued fraction expansion below has every δ_i $(i = 1, 2, \dots, p)$ positive and real.

$$\psi(s) = \delta_1 s + \cfrac{1}{\delta_2 s + \cfrac{1}{\ddots + \cfrac{1}{\delta_p s}}} \qquad (3.3.9)$$

A complete proof of the conditions on the d_i's for the general p-th order polynomial to be Hurwitz is quite laborious. The author has done a weak induction proof, one step of which is produced below. After doing the solution up through p = 6, a pattern has formed such that the general condition can be stated.

An equivalent statement to (1) and (2) of Theorem 3.3.1 is the following:

1'. For $p \geq 3$ and odd:

$$d_1 \, d_{2k} > d_{2k+1}, \; k = 1, 2, \ldots, \frac{p-1}{2}$$

for $p \geq 4$ and even:

$$d_1 \, d_{2k} > d_{2k+1}, \; k = 1, 2, \ldots, \frac{p}{2} - 1$$

$$d_1 \, d_2 \, d_{p-1} > d_3 \, d_{p-1} + d_1^2 \, d_p$$

for $p \leq 2$, the necessary conditions are also sufficient.

Proof: (for p = 6)

$$\psi(s) = \frac{1}{d_1} s + \cfrac{1}{\frac{d_1^2}{[I]} s + \cfrac{1}{\frac{[I]^2}{d_1[III]} s + \cfrac{1}{\frac{d_1[III]^2}{[I]\,[V]} s + \cfrac{1}{\frac{[I]\,[V]^2}{d_1[III]\,[VI]} s + \cfrac{1}{\frac{[VI]\,s}{d_6[I]\,[IV]}}}}}}$$

where: $[I] = d_1 \, d_2 - d_3$

$[II] = d_1 \, d_4 - d_5$

$[III] = d_3 \, [I] - d_1 \, [II]$

$[IV] = d_5 [I] - d_1^2 d_6$

$[V] = d_1 [II] [III] - [I] [IV]$

$[VI] = [I] [IV] - d_1 d_6 [III]^2$

(1) $\delta_1 > 0 \Rightarrow \frac{1}{d_1} > 0$, satisfied because $d_i \geq 0$ $(i = 1, 2, \ldots, 6)$

(2) $\delta_2 > 0 \Rightarrow \frac{d_1^2}{[I]} > 0 \Rightarrow [I] > 0 \Rightarrow d_1 d_2 > d_3$

(3) $\delta_3 > 0 \Rightarrow \frac{[I]^2}{d_1 [III]} > 0 \Rightarrow [III] > 0 \Rightarrow d_3[I] > d_1[II]$

(4) $\delta_4 > 0 \Rightarrow \frac{d_1[III]^2}{[I] [V]} > 0 \Rightarrow [V] > 0 \Rightarrow d_1[II] [III] > [I] [IV]$

(5) $\delta_5 > 0 \Rightarrow \frac{[I] [V]^2}{d_1[III] [VI]} > 0 \Rightarrow [VI] > 0 \Rightarrow [I] [IV] > d_1 d_6[III]^2$

$\Rightarrow [IV] > 0$

$\Rightarrow d_5[I] > d_1^2 d_6$

$\Rightarrow d_1 d_2 d_5 > d_3 d_5 + d_1^2 d_6$

(6) $\delta_6 > 0 \Rightarrow \frac{[VI]}{d_6 [I] [IV]} > 0$, satisfied if δ_2 and $\delta_5 > 0$

(7) Substitute the results of (5) into (4)

$[IV] > 0 \Rightarrow [II] > 0$

$\Rightarrow d_1 d_4 > d_5$

(8) Go back to (3)

$$d_1 d_2 d_3 + d_1 d_5 > d_3^2 + d_1^2 d_4$$

divide by d_3

$$d_1 \; d_2 + \frac{d_1 \; d_5}{d_3} > d_3 + \frac{d_1^{\,2} \; d_4}{d_3}$$

use results of (7)

$$\Rightarrow d_1 \; d_2 + \frac{d_1 \; d_5}{d_3} > d_3 + \frac{d_1 \; d_5}{d_3}$$

$$\Rightarrow d_1 \; d_2 > d_3, \text{ satisfied in (2)}$$

Therefore, conditions on the d_i's come from (2), (5), and (7).

$$d_1 \; d_2 > d_3$$

$$d_1 \; d_4 > d_5$$

$$d_1 \; d_2 \; d_5 > d_3 \; d_5 + d_1^{\,2} \; d_6$$

With respect to $\underline{C}_1$, $\underline{C}_2$, $\underline{J}_2$, $\underline{S}$, and $\underline{J}_3$, the necessary and sufficient conditions that the state model be realizable are summarized below (assuming $p(\lambda)$ and $q(\lambda)$ have no common factor):

Necessary Conditions:

1. The last column of $\underline{C}_1$ and the last column of $\underline{C}_2$ have all nonpositive elements.
2. The orders of $\underline{C}_1$ and $\underline{C}_2$ differ by one at most.
3. The number of successive zeros at the top of the last column of $\underline{C}_1$ and the top of the last column of $\underline{C}_2$ differs by no more than one.
4. In both $\underline{C}_1$ and $\underline{C}_2$, the last column contains zeros only in the top rows, or alternate rows of the last column are zero, beginning with a zero in the bottom row.

Necessary and Sufficient Conditions:

1. If any diagonal element, γ_i, of $\underline{J}_3$ is positive and real, it must appear in a submatrix of even order.

$$\begin{bmatrix} \gamma_i & 1 \\ 0 & \gamma_i \end{bmatrix}, \quad \begin{bmatrix} \gamma_i & 1 & 0 & 0 \\ 0 & \gamma_i & 1 & 0 \\ 0 & 0 & \gamma_i & 1 \\ 0 & 0 & 0 & \gamma_i \end{bmatrix}, \text{ etc.}$$

A sufficient condition for this to be true is that $c_i \geq 0$, $(i = 0, 1, 2, \ldots, n)$ in $\underline{S}$.

This statement and <u>either one</u> of the following two statements must be satisfied.

2. The d_i's defined in Equation 3.3.8 must satisfy

$$d_1\, d_{2k} > d_{2k+1},\ k = 1, 2, \ldots, \frac{p-1}{2}, \text{ for } p \geq 3 \text{ and odd}$$

$$\left.\begin{aligned} & d_1\, d_{2k} > d_{2k+1},\ k = 1, 2, \ldots, \frac{p}{2} - 1 \\ & d_1\, d_2\, d_{p-1} > d_3\, d_{p-1} + d_1^{\,2}\, d_p \end{aligned}\right\} \text{for } p \geq 4 \text{ and even}$$

for $p \leq 2$, the necessary conditions $(d_i \geq 0)$ are also sufficient.

3. The real part of every β_i in $\underline{J}_2$ is nonpositive. If any β_i in $\underline{J}_2$ is pure imaginary (Re $\beta_i = 0$), then μ_i is unity and

$$(\lambda - \beta_i) \frac{p(\lambda)}{q(\lambda)} \bigg|_{\lambda = \beta_i} > 0 \quad \text{and real.}$$

CHAPTER IV

TYPES OF REALIZATIONS

After the state model is found to be realizable, it is desired to find whether the realization network will be LC, RL, etc. The author has formulated a set of tests for determining the type of network which can be synthesized. The tests are applied in the order in which they are given.

4.1 Test for LC Realization. A non-constant, rational function F(s) is realizable as a one-port LC network if and only if it is an F_{LC} function. An F_{LC} function is defined as follows (due to Weinberg (4), page 209):

Definition 4.4.1. F(s) is an F_{LC} function if and only if

1. Its poles and zeros are simple and occur only on the imaginary axis in the complex plane.
2. Its poles and zeros alternate on the imaginary axis (separation property).
3. The constant multiplier is positive.

Properties of F_{LC} functions:

1. $\text{Re } F(j\omega) \equiv 0$
2. $F_{LC}(s) = \dfrac{M_1(s)}{N_2(s)} \left(\text{or } \dfrac{N_1(s)}{M_2(s)}\right)$

M(s) is an even polynomial

N(s) is an odd polynomial

3. The point at infinity is either a pole or a zero.

The above conditions for an F_{LC} function will now be translated into state-space notation.

Theorem 4.4.1. If either Equation 1.1.4 or Equation 1.1.5, or both, is given (R, G $\neq$ 0), there cannot be an LC realization for $Z(\lambda)$.

Proof: (by contradiction)

Assume that the state model has an LC realization. Then $Z(\lambda)$ has either a pole or a zero at infinity.

If $Z(\lambda)$ has a pole at infinity, then, by Theorem 2.1.4, the I-equations can be written. This implies G = 0, a contradiction.

Similarly, if $Z(\lambda)$ has a zero at infinity, then, by Theorem 2.1.2, the V-equations can be written; implying R = 0, a contradiction.

Theorem 4.1.2. The state model can be realized as an LC network if and only if

1. The realizability conditions of Chapter III are satisfied.
2. The last columns of both $\underline{C}_1$ and $\underline{C}_2$ have zeros in alternate rows, starting with a zero in the last row.
3. $$(\lambda^2 + \beta_i^2) \left.\frac{p(\lambda)}{\lambda\, q(\lambda)}\right|_{\lambda^2 = -\beta_i^2} > 0$$

$$\left.\frac{p(\lambda)}{q(\lambda)}\right|_{\lambda = 0} > 0$$ (if $q(\lambda)$ has a zero at the origin, i.e., if $q(\lambda)$ is an odd polynomial).

Proof:

If the realizability conditions are satisfied for $Z(\lambda)$, then both $p(\lambda)$ and $q(\lambda)$ are Hurwitz polynomials. By Theorem 4.1.1, an LC realization is possible only if R, G = 0. The degrees of $p(\lambda)$ and $q(\lambda)$ differ by one, as was pointed out in the proof of Theorem 2.4.2. To satisfy statement 2, if $a_n \neq 0$, then $b_m = 0$ and vice versa. This says that if $p(\lambda)$ is an even polynomial, then $q(\lambda)$ is an odd polynomial and vice versa.

If $p(\lambda)$ is even ($a_n \neq 0$) and is Hurwitz, then its zeros are simple and pure imaginary. If $p(\lambda)$ is odd ($a_n = 0$) and Hurwitz, then $p(\lambda)/\lambda$ is an even polynomial as in the previous statement. The zeros are still simple and pure imaginary (the origin being a trivial case). See Van Valkenberg (3), page 122. The same statements can be made for $q(\lambda)$ when it is an even (or odd) polynomial.

Therefore, the poles and zeros of $Z(\lambda)$ are simple and occur only on the imaginary axis.

$$Z(\lambda) = \frac{(\lambda^2 + \alpha_1^2)\,(\lambda^2 + \alpha_2^2)\,\ldots\,(\lambda^2 + \alpha_k^2)}{(\lambda^2 + \beta_1^2)\,(\lambda^2 + \beta_2^2)\,\ldots\,(\lambda^2 + \beta_j^2)} \tag{4.1.1}$$

$$= \lambda + \frac{k_o}{\lambda} + \frac{2\,k_2\lambda}{(\lambda^2 + \beta_1^2)} + \ldots + \frac{2\,k_j\lambda}{(\lambda^2 + \beta_j^2)} \tag{4.1.2}$$

The λ term is present if $Z(\lambda)$ has a pole at infinity.

The k_o/λ term is present if there is a pole at the origin.

$$k_o = \left.\frac{p(\lambda)}{q(\lambda)}\right|_{\lambda = 0} \tag{4.1.3}$$

$$k_{2i} = \left.\frac{(\lambda^2 + \beta i^2)}{2\lambda}\,\frac{p(\lambda)}{q(\lambda)}\right|_{\lambda^2 = -\beta_i^2}, \quad i = 1, 2, \ldots, \frac{j}{2} \tag{4.1.4}$$

The constants, k_{2i} (i = 0, 1, 2, ..., j/2), are positive if and only if the poles and zeros of Z(λ) alternate along the imaginary axis.

The constant multiplier of Z(λ) is always unity because of the way p(λ) and q(λ) are defined.

Therefore, Theorem 4.1.2 is equivalent to the F_{LC} conditions of Definition 4.1.1.

4.2 Test for RC (or RL) Realization. The state model is realizable as an RC (or RL) network if and only if Z(λ) is an F_{RCZ} function or an F_{RCY} function. The following definitions of F_{RCZ} and F_{RCY} functions are due to Weinberg (4):

Definition 4.2.1. Z(λ) is an F_{RCZ} function if and only if

1. All the zeros and poles of Z(λ) are simple and lie on the negative real axis or at the origin of the complex plane.
2. The poles and zeros alternate on the nonpositive real axis.
3. The lowest critical frequency, i.e., the one at or nearest the origin, is a pole. The highest critical frequency, which may be at infinity, is a zero.
4. The constant multiplier is positive.

Definition 4.2.2. Z(λ) is an F_{RCY} function if and only if

1. All the zeros and poles are simple and lie on the negative real axis or at the origin.
2. The poles and zeros alternate on the nonpositive real axis.
3. The lowest critical frequency, which may be at the origin, is a zero. The highest critical frequency, which may be at infinity, is a pole.

4. The constant multiplier is positive.

The partial fraction expansions for F_{RCZ} and F_{RCY} functions are

$$F_{RCZ} = k_o + \frac{k_1}{\lambda + \lambda_1} + \frac{k_3}{\lambda + \lambda_3} + \dots + \frac{k_n}{\lambda + \lambda_n} \tag{4.2.1}$$

$$\frac{F_{RCY}}{\lambda} = k_\infty + \frac{k_o}{\lambda} + \frac{k_2}{\lambda + \lambda_2} + \dots + \frac{k_n}{\lambda + \lambda_n} \tag{4.2.2}$$

$\left(\frac{F_{RCY}}{\lambda} \text{ is an } F_{RCZ} \text{ function.} \right)$

An alternative and equivalent statement for Definitions 4.2.1 and 4.2.2 is stated by Weinberg (4) in terms of the partial-fraction expansions.

Definition 4.2.3. $Z(\lambda)$ is an F_{RCZ} function or an F_{RCY} function if and only if Equation 4.2.1 or Equation 4.2.2, respectively, satisfies the following conditions:

1. All the poles are simple and lie on the negative real axis or at the origin of the complex plane.
2. All the residues (k_i's) and the constant term (k_o in Equation 4.2.1 or k_∞ in Equation 4.2.2) are real and non-negative.
3. No pole at infinity is present.

The conditions of Definition 4.2.3 are now translated into conditions on the state model and on the canonical matrices $\underline{C}_1$, $\underline{C}_2$, and $\underline{J}_2$.

Theorem 4.2.1. $Z(\lambda)$ cannot be an F_{RCZ} function if only Equation 1.1.4 is given, with $G = 0$.

Proof:

The I-equations are given. By Theorem 2.1.4, there is a pole at infinity.

By Definition 4.2.1, $Z(\lambda)$ cannot be an F_{RCZ} function since it has a pole at infinity.

Theorem 4.2.4. $Z(\lambda)$ is an F_{RCZ} function if and only if

1. $\underline{J}_2$ is a diagonal matrix having diagonal terms which are nonpositive, real, and distinct; i.e.,

$$\underline{J}_2 = \begin{bmatrix} \beta_1 & 0 & \cdot & \cdot & 0 \\ 0 & \beta_2 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \beta_m \end{bmatrix}, \quad \beta_i \neq \beta_j \text{ for } i \neq j \qquad (4.2.3)$$

$$\beta_i \leq 0, \; i = 1, \ldots, m$$

2. $$(\lambda - \beta_i) \frac{p(\lambda)}{q(\lambda)} \Bigg|_{\lambda = \beta_i} > 0 \text{ and real, } i = 1, 2, \ldots, m \qquad (4.2.4)$$

3. $n \leq m$

Proof:

Show that these are equivalent to the conditions of Definition 4.2.3 for an F_{RCZ} function.

The β_i's of Equation 4.2.3 are identical to the λ_i's of Equation 4.2.1 and are the zeros of $q(\lambda)$. If the β_i's are distinct, then the poles of $Z(\lambda)$ are simple. If every β_i is nonpositive, all poles lie on the negative real axis or at the origin.

When the β_i's are distinct, Equation 4.2.4 is the formula for the residues evaluated at the poles of $Z(\lambda)$. Equation 4.2.4 has real answers when all the zeros of $p(\lambda)$ and $q(\lambda)$ are real. The constant term is non-negative (zero or unity) because of the way $p(\lambda)$ and $q(\lambda)$ are defined.

If $n \leq m$, the degree of $p(\lambda)$ is less than or equal to the degree of $q(\lambda)$. Therefore, $Z(\infty)$ is zero or unity; and hence there is no pole at infinity.

<u>Theorem 4.2.3.</u> $Z(\lambda)$ cannot be an F_{RCY} function if only Equation 1.1.5 is given, with $R = 0$.

Proof:

The V-equations are given. By Theorem 2.1.2, $Z(\lambda)$ has a zero at infinity.

By Definition 4.2.2, $Z(\lambda)$ cannot be an F_{RCY} function since it has a zero at infinity.

<u>Theorem 4.2.4.</u> $Z(\lambda)$ is an F_{RCY} function if and only if

1. The matrix

$$\underline{J}_4 = \begin{bmatrix} 0 & \underline{0} \\ \underline{0} & \underline{J}_2 \end{bmatrix}$$

is diagonal with diagonal terms which are all real, non-positive, and distinct; i.e.,

$$\underline{J}_4 = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 \\ 0 & \beta_1 & & & \cdot \\ \cdot & & \beta_2 & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \beta_m \end{bmatrix} \qquad (4.2.5)$$

$$\beta_i \neq \beta_j \text{ if } i \neq j$$

$$\beta_i < 0 \text{ for } i = 1, 2, \ldots, m$$

2. $$\left. \frac{(\lambda - \beta_i)}{\lambda} \frac{p(\lambda)}{q(\lambda)} \right|_{\lambda = \beta_i} > 0 \text{ and real} \qquad (4.2.6)$$

$i = 1, 2, \ldots, m$

$$\left. \frac{p(\lambda)}{q(\lambda)} \right|_{\lambda = 0} > 0 \text{ and real} \qquad (4.2.7)$$

3. $n \leq m + 1$

Proof:

The poles of $Z(\lambda)$ are at the origin and at the zeros of $q(\lambda)$ and so are the poles of Equation 4.2.2. If the β_i's are distinct and non-zero, then the poles are simple. If every β_i is negative, all poles lie at the origin and on the negative real axis.

Equation 4.2.6 gives the residues evaluated at the internal poles of $Z(\lambda)/\lambda$ since the β_i's are distinct and non-zero. The residue evaluated at the pole of $Z(\lambda)/\lambda$ at the origin is given by Equation 4.2.7.

$Z(\lambda)/\lambda$ has no pole at infinity if the degree of $p(\lambda)$ is less than or equal to one plus the degree of $q(\lambda)$, i.e., if $n \leq m + 1$. This condition is automatically satisfied if the state model is realizable.

4.3 Type of Realization if Not Realizable as an LC or RC Network.

If the state model satisfies the realizability conditions of Chapter III but does not satisfy Theorems 4.1.2, 4.2.2, or 4.2.4, then the realization network must be of the RLC or RLCT type.

If $Z(\lambda)$ is a minimum function, then the methods of Brune ideal transformers, Bott-Duffin, or Darlington can be applied to find the realization network. In terms of the canonical matrices, $Z(\lambda)$ is a minimum function if

1. The realizability conditions are satisfied. ($Z(\lambda)$ is a positive real function.)
2. Re $\alpha_i \neq 0$ for $i = 1, 2, \ldots, k$ in Equation 3.2.3.
   Re $\beta_i \neq 0$ for $i = 1, 2, \ldots, j$ in Equation 3.2.6.
3. $m = n$ and a_n, $b_m \neq 0$.
4. At least one of the γ_i's of Equations 3.2.11 and 3.2.12 is zero.

These requirements are equivalent to the following classical conditions for a minimum function:

1. $F(s)$ is positive real.
2. $F(s)$ has no poles or zeros on the imaginary axis.
3. $F(s)$ is real, finite, and positive for $s = 0$ and $s = \infty$.
4. $F(j\omega) = 0$ for at least one finite real frequency, ω_1.

CHAPTER V

ILLUSTRATIVE EXAMPLES

Some examples are given to illustrate the procedure for testing a state model for realizability and for finding the type of network realization which is possible. The examples are chosen to illustrate systems of various order and type.

<u>Example 5.1.</u> Second-Order System With i a Function of v

The state model is given by

$$\underline{\mathring{x}} = \begin{bmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} v$$

$$i = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v \qquad (5.1.1)$$

$$\underline{\mathring{x}} = \begin{bmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} i$$

$$v = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + i \qquad (5.1.2)$$

For this first example, the steps will be carried out in considerable detail, although many of the operations are almost trivial.

$$\underline{A} = \begin{bmatrix} -5 & 1 \\ 3 & -3 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$$

To find $\underline{P}$ such that $\underline{P}^{-1}\ \underline{A}\ \underline{P} = \underline{C}_1$, let

$$\underline{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{A}\ \underline{p}_1 = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$\underline{P} = [\underline{p}_1, \underline{A}\ \underline{p}_1] = \begin{bmatrix} 1 & -5 \\ 0 & 3 \end{bmatrix}$$

$$\underline{P}^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}$$

$$\underline{P}^{-1}\ \underline{A}\ \underline{P} = \underline{C}_1 = \begin{bmatrix} 0 & -12 \\ 1 & -8 \end{bmatrix}$$

Therefore, $p(\lambda) = \lambda^2 + 8\lambda + 12$.

Also,

$$\underline{A}^2\ \underline{p}_1 = \begin{bmatrix} 28 \\ -24 \end{bmatrix} = -8\ \underline{A}\ \underline{p}_1 - 12\ \underline{p}_1$$

$$(\underline{A}^2 + 8\ \underline{A} + 12\ \underline{U})\ \underline{p}_1 = \underline{0} \text{ and } p(\lambda) = \lambda^2 + 8\lambda + 12$$

Similarly, find $\underline{Q}$ such that $\underline{Q}^{-1}\ \underline{B}\ \underline{Q} = \underline{C}_2$.

$$\underline{Q} = [\underline{q}_1, \underline{B}\ \underline{q}_1] = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$$

$$\underline{Q}^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\underline{Q}^{-1}\ \underline{B}\ \underline{Q} = \underline{C}_2 = \begin{bmatrix} 0 & 0 \\ 1 & -4 \end{bmatrix}$$

Therefore, $q(\lambda) = \lambda^2 + 4\lambda$.

The canonical state equations are

$$\dot{\underline{y}} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & -12 \\ 1 & -8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} v$$

$$i = \begin{bmatrix} -1 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + v \qquad (5.1.3)$$

$$\dot{\underline{y}} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} i$$

$$v = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + i \qquad (5.1.4)$$

Next, find a matrix $\underline{T}$ such that $\underline{T}^{-1}\ \underline{C}_2\ \underline{T} = \underline{J}_2$. The roots of $q(\lambda)$ are $\lambda_1 = 0$ and $\lambda_2 = -4$.

$$[\underline{C}_2 + 4\ \underline{U}]\ \underline{T}_1 = \underline{0}\ ,$$

$$\underline{C}_2\ \underline{T}_2 = \underline{0}$$

$$\underline{T} = [\underline{T}_1, \underline{T}_2] = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\underline{T}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$$

$$\underline{T}^{-1}\,\underline{C}_2\,\underline{T} = \underline{J}_2 = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$$

Form the matrix $\underline{S}$, where (using Equation 3.2.9) we have $c_o = 1$, $c_1 = 20$, and $c_2 = 0$.

$$\underline{S} = \begin{bmatrix} 0 & 0 \\ 1 & -20 \end{bmatrix}$$

Check the Realizability Conditions:

Necessary Conditions:

1. The last column of $\underline{C}_1$ and the last column of $\underline{C}_2$ have all nonpositive elements.
2. The order of $\underline{C}_1$ equals the order of $\underline{C}_2$.
3. There is one zero at the top of the last column of $\underline{C}_2$. There are no zeros at the top of the last column of $\underline{C}_1$.
4. There are no zeros in the last columns of $\underline{C}_1$ and $\underline{C}_2$ except in the top row of $\underline{C}_2$.

Necessary and Sufficient Conditions:

1. All of the c_i's are non-negative.
2. $\beta_1 = 0$, $\beta_2 = -4$

$$\frac{p(\lambda)}{q(\lambda)} = \frac{\lambda^2 + 8\lambda + 12}{\lambda^2 + 4\lambda} = \frac{(\lambda + 2)\,(\lambda + 6)}{\lambda\,(\lambda + 4)}$$

$$\lambda \left.\frac{p(\lambda)}{q(\lambda)}\right|_{\lambda = 0} = 3$$

$$(\lambda + 4) \left.\frac{p(\lambda)}{q(\lambda)}\right|_{\lambda = -4} = 1$$

Therefore, the state model is realizable.

Check for the Type of Realization:

1. Cannot be LC since i is a function of v.

2. $\underline{J}_2$ is a diagonal matrix; and the β_i's are real, distinct, and nonpositive. The residues evaluated at 0 and -4 are positive and real, and m = n.

Therefore $Z(\lambda)$ is an F_{RCZ} function, and the realization network is RC.

Example 5.2. Third-Order System With i a Function of v

The state model is

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} v$$

$$i = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + v \qquad (5.2.1)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} i$$

$$v = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + i \qquad (5.2.2)$$

$$\underline{A} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

Find $\underline{P}$ to reduce $\underline{A}$ to rational form. Let

$$\underline{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \underline{A}\ \underline{p}_1 = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}, \ \underline{A}^2\ \underline{p}_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \ \underline{A}^3\ \underline{p}_1 = \begin{bmatrix} -2 \\ -8 \\ -6 \end{bmatrix}$$

$$[\underline{A}^3 + 3\ \underline{A}^2 + 3\ \underline{A} + 2\ \underline{U}]\ \underline{p}_1 = \underline{0}$$

$$\underline{P} = [\underline{p}_1, \underline{A}\ \underline{p}_1, \underline{A}^2\ \underline{p}_1] = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 4 \\ 0 & -2 & 4 \end{bmatrix}$$

$$\underline{P}^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\underline{P}^{-1}\ \underline{A}\ \underline{P} = \underline{C}_1 = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

$$p(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 2 = (\lambda^2 + \lambda + 1)\ (\lambda + 2)$$

$$\underline{B} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix}$$

Let

$$\underline{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{Q} = [\underline{q}_1, \underline{B}\,\underline{q}_1, \underline{B}^2\,\underline{q}_1] = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\underline{Q}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\underline{Q}^{-1}\,\underline{B}\,\underline{Q} = \underline{C}_2 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

$$q(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$$

Now find $\underline{J}_2$ from $q(\lambda)$, the reduced characteristic function of $\underline{C}_2$. Method 2 of Appendix B will be used.

$$\psi(\lambda, \mu) = \frac{q(\mu) - q(\lambda)}{\mu - \lambda} = \mu^2 + \mu(\lambda + 3) + (\lambda^2 + 3\lambda + 3)$$

$$\underline{C}\,(\lambda) = \psi(\lambda\underline{U}, \underline{C}_2) = \underline{C}_2^{\,2} + (\lambda + 3)\,\underline{C}_2 + (\lambda^2 + 3\lambda + 3)\,\underline{U}$$

$$\underline{C}(\lambda) = \begin{bmatrix} 0 & -1 & 3 \\ 0 & -3 & 8 \\ 1 & -3 & 6 \end{bmatrix} + (\lambda + 3) \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

$$+ (\lambda^2 + 3\lambda + 3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{C}'(\lambda) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} + (2\lambda + 3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{C}''(\lambda) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underline{C}(-1) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\underline{C}'(-1) = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\underline{C}''(-1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Form the transformation matrix from the first columns of the three above matrices.

$$\underline{T} = [C_1(-1),\ C_1'(-1),\ C_1''(-1)] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\underline{T}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\underline{T}^{-1}\ \underline{C}_2\ \underline{T} = \underline{J}_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Form the matrix $\underline{S}$, where $c_o = 1$, $c_1 = 3$, $c_2 = 0$, and $c_3 = 2$.

$$\underline{S} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

Check for Realizability:

The last columns of $\underline{C}_1$ and $\underline{C}_2$ have all elements negative, and they are of the same order. Therefore, the necessary conditions are satisfied.

The sufficient condition on $\underline{S}$ is satisfied; all c_i's are non-negative.

$$p(\lambda) + q(\lambda) = 2\lambda^3 + 6\lambda^2 + 6\lambda + 3$$

$$d_1 = 3,\ d_2 = 3,\ d_3 = \frac{3}{2}$$

$$d_1 \; d_2 = 9 > \frac{3}{2} = d_3$$

as required. Therefore, the state model is realizable.

Type of Realization:

1. Cannot be LC because i is a function of v.
2. Cannot be RC because $\underline{J}_2$ is not a diagonal matrix.
3. Therefore, $Z(\lambda)$ can be realized as an RLC or RLCT network.

Example 5.3. Fourth-Order System With i Not a Function of v

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{5}{12} & \frac{-5}{12} \\ 6 & -6 & 0 & 0 \\ 0 & \frac{18}{5} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v$$

$$i = x_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad (5.3.1)$$

Choose

$$\underline{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Then

$$\underline{P} = [\underline{p}_1, \underline{A}\,\underline{p}_1, \underline{A}^2\,\underline{p}_1, \underline{A}^3\,\underline{p}_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & \frac{-3}{2} & 0 \\ 0 & 0 & 0 & 9 \\ 0 & \frac{18}{5} & 0 & \frac{-27}{5} \end{bmatrix}$$

$$\underline{P}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{5}{18} \\ \frac{2}{3} & \frac{-2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & 0 \end{bmatrix}$$

$$\underline{P}^{-1}\,\underline{A}\,\underline{P} = \underline{C}_1 = \begin{bmatrix} 0 & 0 & 0 & -9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$p(\lambda) = \lambda^4 + 10\lambda^2 + 9 = (\lambda^2 + 1)(\lambda^2 + 9)$$

$$\underline{D} = \underline{P}^{-1}\,\underline{d} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \\ 0 \end{bmatrix}$$

$$\underline{H} = \underline{h}\,\underline{P} = [H_1 \quad H_2 \quad H_3 \quad H_4] = [1 \quad 0 \quad 0 \quad 0]$$

The canonical state equations are

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \\ 0 \end{bmatrix} v$$

$$i = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad (5.3.2)$$

The algorithm given by Holmes (16) is used to find $q(\lambda)$, which in turn gives $\underline{C}_2$.

$$q(\lambda) = b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4$$

$$b_1 = D_1H_1 + D_2H_2 + D_3H_3 + D_4H_4 = 1$$

$$b_2 = D_1H_2 + D_2H_3 + D_3H_4 + a_1D_1H_1 + a_1D_2H_2 + a_1D_3H_3 - a_4D_4H_1$$
$$- a_3D_4H_2 - a_2D_4H_3 = 0$$

$$b_3 = D_1H_3 + D_2H_4 + a_1D_1H_2 + a_1D_2H_3 + a_2D_1H_1 + a_2D_2H_2$$
$$- a_4D_4H_2 - a_3D_4H_3 - a_4D_3H_1 - a_3D_3H_2 = 4$$

$$b_4 = D_1H_4 + a_1D_1H_3 + a_2D_1H_2 + a_3D_1H_1 - a_4D_4H_3 - a_4D_3H_2 - a_4D_2H_1 = 0$$

$$q(\lambda) = \lambda^3 + 4\lambda = \lambda(\lambda^2 + 4)$$

Therefore,

$$\underline{C}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & 0 \end{bmatrix}$$

Using Equation 3.2.8, we find $c_i = 0$ ($i = 0, 1, 2, 3$); i.e., $\text{Re } Z(j\omega) \equiv 0$.

Check the Realizability Conditions:

$\underline{C}_1$ and $\underline{C}_2$ have zeros in alternate rows of the last column, beginning with a zero in the last row. The orders of $\underline{C}_1$ and $\underline{C}_2$ differ by only one. Therefore, the necessary conditions are satisfied.

Every c_i is non-negative ($i = 0, 1, 2, 3$).

$$p(\lambda) + q(\lambda) = \lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4 = \lambda^4 + \lambda^3 + 10\lambda^2 + 4\lambda + 9$$

$$d_1 \; d_2 = 10 > 4 = d_3$$

$$d_1 \; d_2 \; d_3 = 40 > 16 + 9 = 25 = d_3^{\,2} + d_1^{\,2} \; d_4$$

Therefore, the state model is realizable.

Type of Network Which Can be Realized:

The last columns of $\underline{C}_1$ and $\underline{C}_2$ satisfy the LC conditions.

$$(\lambda^2 + 4) \left. \frac{p(\lambda)}{\lambda q(\lambda)} \right|_{\lambda^2 = -4} = \frac{15}{4}$$

$$\lambda \left. \frac{p(\lambda)}{q(\lambda)} \right|_{\lambda = 0} = \frac{9}{4}$$

Therefore, the realization network is LC.

<u>Example 5.4.</u> Fifth-Order System With v Not a Function of i

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -1 & -1 & 2 \\ 1 & 3 & 1 & 1 & -1 \\ -1 & -4 & -2 & -1 & 1 \\ -1 & -4 & -1 & -2 & 1 \\ -2 & -2 & -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} i$$

$$v = \begin{bmatrix} 0 & -1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{5.4.1}$$

Transform $\underline{A}$ to the rational canonical form. (This transformation is done in Example A.1.3 of Appendix A.)

$$\underline{P} = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & -2 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{C}_1 = \begin{bmatrix} 0 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \underline{C}_{11} & \underline{0} & \underline{0} \\ \underline{0} & \underline{C}_{12} & \underline{0} \\ \underline{0} & \underline{0} & \underline{C}_{13} \end{bmatrix}$$

The canonical state equations are partitioned into three sets of equations corresponding to the three blocks of $\underline{C}_1$.

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + [\underline{P}^{-1}\ \underline{e}]_1\ i$$

$$v_1 = \left[\underline{k}\ \underline{P}\right]_1 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad (5.4.2)$$

$$\dot{y}_4 = [-1]\ y_4 + [\underline{P}^{-1}\ \underline{e}]_2\ i$$

$$v_2 = [\underline{k}\ \underline{P}]_2\ y_4 \qquad (5.4.3)$$

$$\dot{y}_5 = [-1]\, y_5 + [\underline{P}^{-1}\, \underline{e}]_3\, i$$

$$v_3 = [\underline{k}\, \underline{P}]_3\, y_5 \tag{5.4.4}$$

The state model is realizable if and only if Equations 5.4.2, 5.4.3, and 5.4.4 are each realizable.

Look at Equation 5.4.2. There is an element in the last column of $\underline{C}_{11}$ which is positive. Therefore, one of the necessary conditions for realizability is violated; and the state model given by Equation 5.4.1 is not realizable.

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Summary. This thesis considers the problem of synthesizing a system of linear differential equations known as the state model. The model which is investigated has a scalar input, or driving function, and a scalar output. The network realization is a passive one-port network with the scalar input and output functions of the state model being the driving-point voltage and current.

The method of solution utilizes linear transformations of the state variables. The first transformation takes the coefficient matrix of the state model to the rational canonical form. The transformed state equations are denoted as the canonical state equations or the canonical state model. The second transformation takes the coefficient matrix of the canonical state model to the Jordan canonical form.

From the rational form of the coefficient matrix, we obtain its reduced characteristic function, $p(\lambda)$. Using $p(\lambda)$ and the canonical state model, another polynomial $q(\lambda)$ is found. If the state model is realizable as a one-port network, then the classical input impedance of the network is the ratio of p and q, expressed as functions of s. This is the important correlation between the state model and the classical network theory. A block diagram showing the relationship of the state equations to the classical s-domain networks is provided.

Necessary and sufficient conditions that the state model be realizable as a one-port network are derived. These conditions require inspections of the transformed coefficient matrices and some short computations using p and q. The realizability conditions are equivalent to the conditions in classical theory that a function be positive real.

Procedures are derived for determining the type of network realization which is possible. These tests also involve inspection of the canonical matrices and, possibly, computations with p and q.

The author's procedure for testing the state model is actually an algorithm. The steps in the algorithm are as follows:

1. The state equations are transformed to canonical forms (both rational and Jordan forms).
2. The coefficient matrices of the canonical forms are inspected to see if the state model is realizable.
3. A check is made for realizability as a one-port LC network.
4. A test is made to see if an RC (or RL) realization is possible.

6.2 Conclusions. There is a strong correlation between the synthesis of the state model and classical network synthesis. This is certainly to be expected since a given network can be described in terms of state variables or in terms of the s-domain variable.

There is a great deal more information available about a network when it is described by the time-domain state model than when the s-domain description is used. The state model gives information about the internal behavior of a system as well as the external characteristics.

It seems that a procedure could be developed which would synthesize a network directly from the state model. However, even in the one-port case, problems arise when we specify not only the terminal characteristics but also the internal topology of the network.

More complications would certainly arise when the input and output functions are not scalars, but rather are n-dimensional vector functions. Even in classical theory, the characteristics and realizability conditions for an n-port network are not well-defined. An investigation, even if it is highly theoretical, of realizability conditions for a general n-port network, using the state-space approach, might also provide a better insight into the s-domain representation of n-ports.

One outstanding advantage of the state model is that it consists of matrix differential equations whose coefficient matrices have constant elements. When the elements of the coefficient matrices are constants (instead of functions of s), a digital computer can be utilized for performing multiplications, inversions, factoring of polynomials, etc. Finding the transformation matrix which gives the rational canonical form is also a mechanical procedure which might be done on the computer.

6.3 Recommendations for Further Study. The first problem which might be investigated is the feasibility of writing a computer program for the algorithm outlined in this thesis. It should be possible to read in the coefficients of the state equations and have the computer do the complete check for realizability.

Another area for further study is the synthesis of a network directly from the state model. This might also involve some transformations of the state variables to get the coefficient matrices into some standard, recognizable form.

A somewhat more lengthy, and probably more difficult, problem is the case where the input and output are not scalar functions. Two-port synthesis should not be overly difficult. However, the general n-port case, with the topological considerations involved, might prove to be a rather formidable obstacle.

A SELECTED BIBLIOGRAPHY

1. Tou, J. T. *Modern Control Theory*. New York: McGraw-Hill, 1964.

2. Zadeh, L. A. and C. A. Desoer. *Linear System Theory, the State Space Approach*. New York: McGraw-Hill, 1963.

3. Van Valkenberg, M. E. *Introduction to Modern Network Synthesis*. New York: John Wiley, 1962.

4. Weinberg, L. E. *Network Analysis and Synthesis*. New York: McGraw-Hill, 1962.

5. Blackwell, W. A. and L. L. Grigsby. "A Simple Formulation Procedure for State-Space Models of Electric Networks." Accepted for publication, *Journal of the Franklin Institute*.

6. Dervisoglu, A. "Realization of the A-Matrix of Half-Degenerate RLC Networks." *Coordinated Science Laboratory Report R-255*, University of Illinois, 1964.

7. Bashkow, T. R. "The A-Matrix, New Network Description." *I.R.E. Transactions on Circuit Theory*. (September, 1957), 117-119.

8. Morgan, B. S. "The Synthesis of Single-Variable Systems by State-Variable Feedback." *Proceedings of the First Allerton Conference on Circuit and System Theory*. (1963), 509-520.

9. Brockett, R. W. "Poles, Zeros, and Feedback: State-Space Interpretation." *I.E.E.E. Transactions on Automatic Control*. (April, 1965), 129-135.

10. Kalman, R. E. "On a New Characterization of Linear Passive Systems." *Proceedings of the First Allerton Conference on Circuit and System Theory*. (1963), 456-470.

11. ______. "Irreducible Realizations and the Degree of a Rational Matrix." *S.I.A.M. Journal*. Volume 13. (June, 1965), 520-544.

12. Browne, E. T. *Introduction to the Theory of Determinants and Matrices*. Chapel Hill: The University of North Carolina Press, 1958.

13. Browne, E. T. "On the Reduction of a Matrix to a Canonical Form." American Mathematical Monthly. Volume 47. (1940), 437-450.

14. Gantmacher, F. R. The Theory of Matrices. Volume I. New York: Chelsea, 1959.

15. Turnbull, H. W. and A. C. Aitken. An Introduction to the Theory of Canonical Matrices. London: Blackie and Son, 1952.

16. Holmes, J. D. "A State-Variable Approach to Control System Design." Ph.D. Dissertation, Oklahoma State University, August, 1965.

17. Guillemin, E. A. The Mathematics of Circuit Analysis. New York: John Wiley, 1949.

18. Ayres, F., Jr. Theory and Problems of Matrices. New York: Schaum, 1962.

APPENDIX A

REDUCTION OF MATRICES TO RATIONAL CANONICAL FORM

Some properties of the minimum polynomial of an n-square matrix $\underline{A}$ will first be discussed. Define:

1. $f(\lambda) = |\lambda\underline{U} - \underline{A}|$ = characteristic function of $\underline{A}$.
2. $D_{n-1}(\lambda)$ = greatest common divisor of all the (n - 1)-rowed minors of $[\lambda\underline{U} - \underline{A}]$.
3. $\phi(\lambda) = f(\lambda)/D_{n-1}(\lambda)$ = reduced characteristic function (or minimum polynomial) of $\underline{A}$.

   Properties of $\phi(\lambda)$:

   a. $\phi(\underline{A}) = 0$

   b. $\phi(\lambda) = 0$ is the scalar equation of lowest degree which is satisfied by $\underline{A}$.

   c. Every root of $f(\lambda) = 0$ is also a root of $\phi(\lambda) = 0$.

A method for finding $D_{n-1}(\lambda)$ and $\phi(\lambda)$ is given by Gantmacher (14) and is summarized below:

Let $\underline{B}(\lambda) = D_{n-1}(\lambda)\,\underline{C}(\lambda)$ = adjoint matrix of $\underline{A}$.

$$f(\lambda) = |\lambda\underline{U} - \underline{A}| = |[\lambda\delta_{ik} - a_{ik}]|$$

Then $\underline{B}(\lambda) = [b_{ik}(\lambda)]$, where b_{ik} is the algebraic complement of the element $(\lambda\delta_{ik} - a_{ik})$.

$\underline{C}(\lambda)$ = reduced adjoint matrix of $\underline{A}$.

If $\delta(\lambda, \mu) = \left(f(\mu) - f(\lambda)\right)/\left(\mu - \lambda\right)$, then $\underline{B}(\lambda) = \delta(\lambda\underline{U}, \underline{A})$. Likewise, $\psi(\lambda, \mu) = \left(\phi(\mu) - \phi(\lambda)\right)/\left(\mu - \lambda\right)$, and $\underline{C}(\lambda) = \psi(\lambda\underline{U}, \underline{A})$.

If $\underline{B}(\lambda)$ is known, then $D_{n-1}(\lambda)$ is easily found; and likewise $\underline{C}(\lambda)$ and $\phi(\lambda)$ are easily found. The procedure will be illustrated with an example.

Example A.1.1.

Let

$$\underline{A} = \begin{bmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{bmatrix}, \quad f(\lambda) = \lambda^3 - 8\lambda^2 + 20\lambda - 16 = (\lambda - 2)^2 (\lambda - 4)$$

$$\delta(\lambda, \mu) = \frac{f(\mu) - f(\lambda)}{\mu - \lambda} = \mu^2 + \mu(\lambda - 8) + \lambda^2 - 8\lambda + 20$$

$$\underline{B}(\lambda) = \underline{A}^2 + (\lambda - 8)\,\underline{A} + (\lambda^2 - 8\lambda + 20)\,\underline{U}$$

$$= \begin{bmatrix} 10 & -18 & 12 \\ -6 & 22 & -12 \\ -6 & 18 & -8 \end{bmatrix} + (\lambda - 8) \begin{bmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{bmatrix}$$

$$+ (\lambda^2 - 8\lambda + 20) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda-3)(\lambda-2) & -3(\lambda-2) & 2(\lambda-2) \\ -(\lambda-2) & (\lambda-1)(\lambda-2) & -2(\lambda-2) \\ -(\lambda-2) & 3(\lambda-2) & (\lambda-6)(\lambda-2) \end{bmatrix}$$

Therefore, $D_2(\lambda) = \lambda - 2$ {the common factor in $\underline{B}(\lambda)$}. Now

$$\underline{C}(\lambda) = \begin{bmatrix} \lambda-3 & -3 & 2 \\ -1 & \lambda-1 & -2 \\ -1 & 3 & \lambda-6 \end{bmatrix}$$

$$\phi(\lambda) = \frac{f(\lambda)}{\lambda - 2} = (\lambda - 2)\ (\lambda - 4)$$

A summary of the theory which underlies the reduction to rational canonical form is given below. Some examples are then used to illustrate the theory.

First, some definitions, due to Ayres (18), are given.

Definition A.1. If $\underline{A}$ is an n-square matrix and $\underline{X}$ is an n-dimensional vector and if $g(\lambda)$ is a polynomial of minimum degree such that $g(\underline{A})\ \underline{X} = \underline{0}$, then with respect to $\underline{A}$ the vector $\underline{X}$ is said to belong to $g(\lambda)$.

Definition A.2. If, with respect to $\underline{A}$, the vector $\underline{X}$ belongs to $g(\lambda)$ of degree p, the linearly independent vectors

$$\underline{X},\ \underline{A}\ \underline{X},\ \underline{A}^2\ \underline{X},\ \ldots,\ \underline{A}^{p-1}\ \underline{X} \qquad (A.1)$$

are called a chain having $\underline{X}$ as leader.

To find the matrix $\underline{P}$ such that $\underline{P}^{-1}\ \underline{A}\ \underline{P} = \underline{C}$, the rational canonical form, a chain of vectors is found as in Definition A.2. These vectors are taken as the first p columns of the transformation matrix $\underline{P}$. The vectors of Equation A.1 constitute a basis of a linear vector space of dimension p.

If $p = n$, the vectors constitute a basis of the entire n-space; and they will be taken as the columns of $\underline{P}$. Then $\phi(\lambda) = f(\lambda)$, and $\underline{C}$ contains only one block of the form of $\underline{C}_1$.

$$\underline{C}_1 = \begin{bmatrix} 0 & 0 & \cdot & \cdot & 0 & -a_p \\ 1 & 0 & & & \cdot & -a_{p-1} \\ 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & 0 & -a_2 \\ 0 & \cdot & \cdot & 0 & 1 & -a_1 \end{bmatrix} \qquad (A.2)$$

where

$$\phi(\lambda) = \lambda^p + a_1 \lambda^{p-1} + \ldots + a_{p-1} \lambda + a_p .$$

If $p < n$, there is a common factor $D_{n-1}(\lambda)$ in all of the $(n - 1)$-rowed minors of $\underline{A}$. Then $\underline{C}$ will have more than one block of the form of $\underline{C}_1$ and

$$f(\lambda) = D_{n-1}(\lambda) \; [\lambda^p + a_1 \lambda^{p-1} + \ldots + a_{p-1}\lambda + a_p]$$

When $p < n$, a vector $\underline{Y}$ independent of the vectors in Equation A.1 can be found. Assume that the set of $p + q$ vectors consisting of Equation A.1 and the vectors

$$\underline{Y}, \; \underline{A}\,\underline{Y}, \; \underline{A}^2\,\underline{Y}, \; \ldots, \; \underline{A}^{q-1}\,\underline{Y}$$

are linearly independent but that $\underline{A}^q\,\underline{Y}$ is a linear combination of them. Then

$$\theta_1(\underline{A})\,\underline{X} + \theta_2(\underline{A})\,\underline{Y} = \underline{0} \qquad (A.3)$$

where θ_1 and θ_2 are scalar polynomials; θ_1 of degree $p - 1$ at most and

θ_2 of degree q. Also, no scalar polynomial θ_2 of degree less than q satisfies Equation A.3.

Now, θ_1 is divisible by θ_2 (see Browne (12), page 211), and we can write Equation A.3 as

$$\theta_2(\underline{A})\ [\underline{Y} - \psi(\underline{A})\ \underline{X}] = \underline{0}$$

Let $\underline{X}_2 = \underline{Y} - \psi(\underline{A})\ \underline{X}$. Then $\underline{X}_2$ has the reduced characteristic function θ_2, and the set of p + q vectors consisting of Equation A.1 and the q vectors

$$\underline{X}_2,\ \underline{A}\ \underline{X}_2,\ \ldots,\ \underline{A}^{q-1}\ \underline{X}_2 \tag{A.4}$$

are linearly independent.

If p + q = n, take the vectors in Equations A.1 and A.4 as the columns of $\underline{P}$. If p + q < n, then the above procedure is repeated with another vector $\underline{Z}$ which is independent of the above vectors. The procedure is continued until a basis for the n-space is found. Then we have

$$\underline{P}^{-1}\ \underline{A}\ \underline{P} = \underline{C} = \begin{bmatrix} \underline{C}_1 & 0 & \cdot & \cdot & 0 \\ 0 & \underline{C}_2 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & \underline{C}_k \end{bmatrix}$$

where each $\underline{C}_i$ (i = 1, 2, ..., k) has the form of Equation A.2.

Look back at Example A.1.1. Choose a vector $\underline{X}$ which belongs to $\phi(\lambda)$. Let

$$\underline{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{A}\ \underline{X} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \ \underline{A}^2\ \underline{X} = \begin{bmatrix} 10 \\ -6 \\ -6 \end{bmatrix} = 6\ \underline{A}\ \underline{X} - 8\ \underline{X}$$

$\underline{X}$ and $\underline{A}\ \underline{X}$ are linearly independent, but $\underline{A}^2\ \underline{X}$ is a linear combination of $\underline{X}$ and $\underline{A}\ \underline{X}$ ($p = 2 < n = 3$). The minimum polynomial of $\underline{A}$ can be found by observing that $\underline{A}^2\ \underline{X} - 6\ \underline{A}\ \underline{X} + 8\ \underline{X} = \underline{0}$.

Therefore, $(\underline{A}^2 - 6\ \underline{A} + 8\ \underline{U})\ \underline{X} = \underline{0}$; and $\phi(\lambda) = \lambda^2 - 6\ \lambda + 8$.

The vectors $\underline{X}$ and $\underline{A}\ \underline{X}$ are taken as the first two columns of $\underline{P}$. To obtain the last column of $\underline{P}$, another vector $\underline{Y}$, independent of $\underline{X}$ and $\underline{A}\ \underline{X}$, must be found.

Choose

$$\underline{Y} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

The matrix

$$[\underline{X}, \underline{A}\ \underline{X}, \underline{Y}] = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

has rank three, so $\underline{Y}$ is independent of $\underline{X}$ and $\underline{A}\ \underline{X}$, as required.

Then

$$\underline{A}\ \underline{Y} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} = 2\ \underline{Y}$$

Now, $[\underline{A} - 2\ \underline{U}]\ \underline{Y} = \underline{0}$, which means that $D_{n-1}(\lambda) = \lambda - 2$.

$$\underline{P} = [\underline{X}, \underline{A}\,\underline{X}, \underline{Y}] = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\underline{P}^{-1} = \begin{bmatrix} 1 & 5 & -2 \\ 0 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\underline{P}^{-1}\,\underline{A}\,\underline{P} = \underline{C} = \begin{bmatrix} 0 & -8 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example A.1.2.

$$\underline{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 5 & 3 & -1 \end{bmatrix}$$

Choose

$$\underline{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \underline{A}\,\underline{X} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \underline{A}^2\,\underline{X} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix}, \underline{A}^3\,\underline{X} = \begin{bmatrix} -1 \\ 6 \\ 4 \\ 9 \end{bmatrix}$$

These four vectors are linearly independent, so $\underline{A}^4\,\underline{X}$ must be a linear combination of them.

$$\underline{A}^4\,\underline{X} = \begin{bmatrix} 1 \\ 15 \\ 17 \\ 33 \end{bmatrix} = 2\,\underline{A}^3\,\underline{X} + 3\,\underline{A}^2\,\underline{X}$$

$$\underline{P} = [\underline{X}, \underline{A}\,\underline{X}, \underline{A}^2\,\underline{X}, \underline{A}^3\,\underline{X}] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 5 & 9 \end{bmatrix}$$

Therefore, $[\underline{A}^4 - 2\,\underline{A}^3 - 3\,\underline{A}^2]\,\underline{X} = \underline{0}$, and $\phi(\lambda) = \lambda^4 - 2\,\lambda^3 - 3\,\lambda^2$.

From $\phi(\lambda)$ we know that

$$\underline{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This can be verified since

$$\underline{A}\,\underline{P} = \underline{P}\,\underline{C} = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 6 & 15 \\ 0 & 3 & 4 & 17 \\ 0 & 5 & 9 & 33 \end{bmatrix}$$

Example A.1.3

$$\underline{A} = \begin{bmatrix} -2 & -1 & -1 & -1 & 2 \\ 1 & 3 & 1 & 1 & -1 \\ -1 & -4 & -2 & -1 & 1 \\ -1 & -4 & -1 & -2 & 1 \\ -2 & -2 & -2 & -2 & 3 \end{bmatrix}$$

Choose

$$\underline{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \underline{A}\,\underline{X} = \begin{bmatrix} -2 \\ 1 \\ -1 \\ -1 \\ -2 \end{bmatrix}, \underline{A}^2\,\underline{X} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \underline{A}^3\,\underline{X} = \begin{bmatrix} -1 \\ 2 \\ -2 \\ -2 \\ 0 \end{bmatrix} = 2\underline{A}^2\,\underline{X} - 3\underline{X}$$

Only $\underline{X}$, $\underline{A}\,\underline{X}$, and $\underline{A}^2\,\underline{X}$ are linearly independent. See if $\lambda^3 - 2\lambda^2 + 3$ can be the minimum polynomial.

$$\underline{A}^3 - 2A^2 + 3\underline{U} = \begin{bmatrix} -1 & 6 & 0 & 0 & 2 \\ 2 & 1 & 2 & 2 & -4 \\ -2 & -2 & -3 & -2 & 4 \\ -2 & -2 & -2 & -3 & 4 \\ 0 & 12 & 0 & 0 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 3 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & -2 \\ -1 & -1 & 0 & -1 & 2 \\ -1 & -1 & -1 & 0 & 2 \\ 0 & 6 & 0 & 0 & 3 \end{bmatrix}$$

$$+ \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \underline{0}$$

Therefore, $\lambda^3 - 2\lambda^2 + 3$ is the minimum polynomial, and

$$\underline{C}_1 = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Choose

$$\underline{Y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{A}\,\underline{Y} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = -\underline{Y}$$

Therefore, $\underline{C}_2 = [-1]$ is a block of order one.

Choose

$$\underline{Z} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \underline{A}\,\underline{Z} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = -\underline{Z}$$

Therefore, $\underline{C}_3 = [-1]$ is a block of order one.

$$\underline{P} = [\underline{X}, \underline{A}\,\underline{X}, \underline{A}^2\,\underline{X}, \underline{Y}, \underline{Z}] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & -2 & 0 & 0 & 0 \end{bmatrix}$$

Now

$$\underline{C} = \begin{bmatrix} \underline{C}_1 & \underline{0} & \underline{0} \\ \underline{0} & \underline{C}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{C}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Check:

$$\underline{A}\,\underline{P} = \underline{P}\,\underline{C} = \begin{bmatrix} -2 & 1 & -1 & -1 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ -1 & -1 & -2 & 0 & -1 \\ -1 & -1 & -2 & 1 & 1 \\ -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

APPENDIX B

REDUCTION OF A MATRIX TO JORDAN

CANONICAL FORM

B.1 Reduction Directly to Jordan Form. Any n-square matrix $\underline{A}$ can be transformed by a nonsingular matrix $\underline{T}$ to the Jordan canonical form.

$$\underline{T}^{-1}\ \underline{A}\ \underline{T} = \underline{J} = \begin{bmatrix} \underline{J}_1 & 0 & . & . & 0 \\ 0 & \underline{J}_2 & & & . \\ . & & . & & . \\ . & & & . & 0 \\ 0 & . & . & 0 & \underline{J}_k \end{bmatrix}$$

where

$$\underline{J}_i = \begin{bmatrix} \alpha_i & 1 & 0 & . & . & 0 \\ 0 & \alpha_i & 1 & & & . \\ . & & . & . & & . \\ . & & & . & . & 0 \\ . & & & & . & 1 \\ 0 & . & . & . & 0 & \alpha_i \end{bmatrix}$$

Each $\underline{J}_i$ is of order ν_i, where

$$\sum_{i=1}^{k} \nu_i = n .$$

The α_i's may not be distinct, as will be explained in the next section.

A non-derogatory matrix is defined to be one for which the characteristic function is equal to the reduced characteristic function. This means that the rational canonical form contains only one block.

When $\underline{A}$ is non-derogatory, finding $\underline{T}$ is relatively simple, once the eigenvalues are known. However, when $\underline{A}$ is derogatory, finding $\underline{T}$ directly from $\underline{A}$ becomes more difficult.

The remainder of Section B.1 is devoted to the reduction of $\underline{A}$ to Jordan form when $\underline{A}$ is non-derogatory. In Section B.2 it is shown that when $\underline{A}$ is derogatory, it can first be reduced to a rational canonical form containing k blocks; and then each of the k blocks can be reduced to Jordan form using the methods of this section.

Two procedures for obtaining $\underline{T}$ are presented. The first is the method used in basic matrix theory to reduce a matrix to diagonal form, with one extension. The second method is due to Gantmacher (14).

Method No. 1. For every eigenvalue λ_i, $[\underline{A} - \lambda_i \underline{U}]\ \underline{X} = \underline{0}$, where $\underline{X}$ is the eigenvector corresponding to λ_i. If there are repeated eigenvalues, $\underline{X}_1$ is found from $[\underline{A} - \lambda_i \underline{U}]\ \underline{X}_1 = \underline{0}$. Then $[\underline{A} - \lambda_i \underline{U}]\ \underline{X}_2 = \underline{X}_1$, $[\underline{A} - \lambda_i \underline{U}]\ \underline{X}_3 = \underline{X}_2$, ..., $[\underline{A} - \lambda_i \underline{U}]\ \underline{X}_k = \underline{X}_{k-1}$, for λ_i with multiplicity k.

From this argument we see that, if $\underline{X}_k$ can be found, the other (k - 1) vectors corresponding to λ_i, along with $\underline{X}_k$, form a chain with $\underline{X}_k$ as leader:

$$\underline{X}_k,\ [\underline{A} - \lambda_i \underline{U}]\ \underline{X}_k,\ [\underline{A} - \lambda_i \underline{U}]^2\ \underline{X}_k,\ \ldots,\ [\underline{A} - \lambda_i \underline{U}]^{k-1}\ \underline{X}_k$$

If such a chain is found for each eigenvalue, the transformation matrix is given as follows:

$$\underline{T} = \Big[[\underline{A} - \lambda_1 \underline{U}]^{\nu_1 - 1} \underline{T}_1, \ldots, [\underline{A} - \lambda_1 \underline{U}] \underline{T}_1, \underline{T}_1, \ldots,$$

$$[\underline{A} - \lambda_k \underline{U}]^{\nu_k - 1} \underline{T}_k, \ldots, [\underline{A} - \lambda_k \underline{U}] \underline{T}_k, \underline{T}_k \Big]$$

where there are k distinct eigenvalues, and each eigenvalue λ_i has multiplicity ν_i.

Method No. 2. Form the function $\underline{C}(\lambda)$ as defined in Appendix A. Since a non-derogatory matrix $\underline{A}$ is assumed, we have $f(\lambda) = \phi(\lambda)$.

$$\delta(\lambda, \mu) = \frac{f(\mu) - f(\lambda)}{\mu - \lambda} = \psi(\lambda, \mu) = \frac{\phi(\mu) - \phi(\lambda)}{\mu - \lambda}$$

$$\underline{C}(\lambda) = \psi(\lambda \underline{U}, \underline{A})$$

The columns of $\underline{T}$ are found from $\underline{C}(\lambda)$ and its derivatives.

$$\underline{T} = \Big[\underline{C}_i(\lambda_1), \underline{C}_i'(\lambda_1), \ldots, \underline{C}_i^{\nu_1 - 1}(\lambda_1), \ldots, \underline{C}_j(\lambda_k),$$

$$\underline{C}_j'(\lambda_k), \ldots, \underline{C}_j^{\nu_k - 1}(\lambda_k) \Big]$$

where $\underline{C}'(\lambda) = \frac{d}{d\lambda} \underline{C}(\lambda)$, etc., and the subscripts i and j indicate the columns of $\underline{C}(\lambda)$ which are used (see Gantmacher (14), p. 164).

Some examples are given below to illustrate the two procedures.

Example B.1.1.

$$\underline{A} = \begin{bmatrix} -4 & -2 \\ 7 & 5 \end{bmatrix} \qquad \begin{array}{l} f(\lambda) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) \\ \phi(\lambda) = f(\lambda) \end{array}$$

Use Method 1:

$$[\underline{A} - 3\underline{U}]\ \underline{T}_1 = 0 \text{ , choose } \underline{T}_1 = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$$

$$[\underline{A} + 2\underline{U}]\ \underline{T}_2 = 0 \text{ , choose } \underline{T}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\underline{T} = [\underline{T}_1, \underline{T}_2] = \begin{bmatrix} -2 & -1 \\ 7 & 1 \end{bmatrix} , \underline{T}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -7 & -2 \end{bmatrix}$$

$$\underline{T}^{-1}\ \underline{A}\ \underline{T} = \underline{J} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

<u>Example B.1.2.</u>

$$\underline{A} = \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} \qquad \begin{aligned} f(\lambda) &= \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 \\ \phi(\lambda) &= f(\lambda) \end{aligned}$$

Use Method 2:

$$\psi(\lambda, \mu) = \mu + (\lambda - 6)$$

$$\underline{C}(\lambda) = \underline{A} + (\lambda - 6)\ \underline{U} = \begin{bmatrix} 5 & 2 \\ -2 & 1 \end{bmatrix} + (\lambda - 6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{C}'(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \underline{C}(+3) = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix} , \quad \underline{C}'(+3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Choose

$$\underline{T} = [\underline{C}_1(+3), \underline{C}_1'(+3)] = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$$

$$\underline{T}^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$$

$$\underline{T}^{-1}\,\underline{A}\,\underline{T} = \underline{J} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Example B.1.3. (Same matrix as in Example A.1.2)

$$\underline{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \qquad \begin{aligned} f(\lambda) = \phi(\lambda) &= \lambda^4 - 2\lambda^3 - 3\lambda^2 \\ &= \lambda^2\,(\lambda - 3)\,(\lambda + 1) \end{aligned}$$

Use Method 1:

$$\underline{A}\,\underline{T}_1 = \underline{0}, \text{ choose } \underline{T}_1 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\underline{A}\,\underline{T}_2 = \underline{T}_1, \text{ choose } \underline{T}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

$$[3\underline{U} - \underline{A}]\,\underline{T}_3 = \underline{0}, \; \underline{T}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$[-\underline{U} - \underline{A}]\ \underline{T}_4 = \underline{0},\ \underline{T}_4 = \begin{bmatrix} 4 \\ -3 \\ 5 \\ 6 \end{bmatrix}$$

$$\underline{T} = [\underline{T}_1, \underline{T}_2, \underline{T}_3, \underline{T}_4] = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 1 & -3 \\ -2 & 1 & 1 & 5 \\ -1 & 4 & 2 & 6 \end{bmatrix}$$

$$\underline{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Check:

$$\underline{A}\ \underline{T} = \underline{T}\ \underline{J} = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 0 & 1 & 3 & 3 \\ 0 & -2 & 3 & -5 \\ 0 & -1 & 6 & -6 \end{bmatrix}$$

Example B.1.4.

$$\underline{A} = \begin{bmatrix} 3 & -1 & -4 & 2 \\ 2 & 3 & -2 & -4 \\ 2 & -1 & -3 & 2 \\ 1 & 2 & -1 & -3 \end{bmatrix} \qquad f(\lambda) = \phi(\lambda) = \lambda^4 - 2\lambda^2 + 1 = (\lambda - 1)^2 (\lambda + 1)^2$$

Use Method 2:

$$\psi(\lambda, \mu) = \mu^3 + \lambda\mu^2 + (\lambda^2 - 2)\,\mu + \lambda^3 - 2\lambda$$

$$\underline{C}(\lambda) = \underline{A}^3 + \lambda\underline{A}^2 + (\lambda^2 - 2)\,\underline{A} + (\lambda^3 - 2\lambda)\,\underline{U}$$

$$\underline{C}(\lambda) = \begin{bmatrix} 3 & -3 & -4 & 6 \\ 6 & 3 & -6 & -4 \\ 2 & -3 & 2 & 6 \\ 3 & 2 & -3 & -3 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 2 & 0 & -4 \\ 4 & 1 & -4 & 0 \\ 0 & 2 & 1 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$$

$$+ (\lambda^2 - 2) \begin{bmatrix} 3 & -1 & -4 & 2 \\ 2 & 3 & -2 & -4 \\ 2 & -1 & -3 & 2 \\ 1 & 2 & -1 & -3 \end{bmatrix} + (\lambda^3 - 2\lambda) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{C}'(\lambda) = \underline{A}^2 + (2\lambda)\,\underline{A} + (3\lambda^2 - 2)\,\underline{U}$$

$$= \begin{bmatrix} 1 & 2 & 0 & -4 \\ 4 & 1 & -4 & 0 \\ 0 & 2 & 1 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} + 2\lambda \begin{bmatrix} 3 & -1 & -4 & 2 \\ 2 & 3 & -2 & -4 \\ 2 & -1 & -3 & 2 \\ 1 & 2 & -1 & -3 \end{bmatrix}$$

$$+ (3\lambda^2 - 2) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{C}(+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 8 & 0 & -8 & 0 \\ 0 & 0 & 5 & 0 \\ 4 & 0 & 4 & 0 \end{bmatrix}, \quad \underline{C}(-1) = \begin{bmatrix} 0 & -4 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C'(+1) = \begin{bmatrix} 8 & 0 & -8 & 0 \\ 8 & 8 & -8 & -8 \\ 4 & 0 & -4 & 0 \\ 4 & 4 & -4 & -4 \end{bmatrix}, \quad \underline{C}'(-1) = \begin{bmatrix} -4 & 4 & 8 & -8 \\ 0 & -4 & 0 & 8 \\ -4 & 4 & 8 & -8 \\ 0 & -4 & 0 & 8 \end{bmatrix}$$

$$[\underline{C}_1(+1), \underline{C}_1'(+1), \underline{C}_2(-1), \underline{C}_2'(-1)] = \begin{bmatrix} 0 & 8 & -4 & 4 \\ 8 & 8 & 0 & -4 \\ 0 & 4 & -4 & 4 \\ 4 & 4 & 0 & -4 \end{bmatrix}$$

This can be simplified by dividing out a (+4) in the first two columns and a (-4) in the last two columns.

$$\underline{T} = \begin{bmatrix} 0 & 2 & 1 & -1 \\ 2 & 2 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad \underline{J} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Check:

$$\underline{A}\,\underline{T} = \underline{T}\,\underline{J} = \begin{bmatrix} 0 & 2 & -1 & 2 \\ 2 & 4 & 0 & -1 \\ 0 & 1 & -1 & 2 \\ 1 & 2 & 0 & -1 \end{bmatrix}$$

B.2 Transformation From Rational Canonical Form to Jordan Canonical Form. When the n-square matrix $\underline{A}$ is non-derogatory, the methods of Section A.2 are used to find the Jordan form. Even when $\underline{A}$ is non-derogatory, it sometimes appears easier to first make a reduction to rational form. There are two good reasons for doing this:

1. In general, the Jordan form cannot be obtained by rational operations since the characteristic roots (or eigenvalues) are, in general, complex. The reduction to Jordan form involves the determination of the eigenvalues. The characteristic function $f(\lambda)$, from which the eigenvalues are obtained, is immediately obvious from the rational form.
2. The rational form, since it contains only $(2n - 1)$ non-zero terms at most, is easier to manipulate than the matrix $\underline{A}$, which has n^2 non-zero terms at most. For matrices of order greater than four, the amount of work saved in either of the two methods of Section B.1 is considerable if the reduction to Jordan form starts with $\underline{C}$ instead of $\underline{A}$. (Obviously, if $\underline{J}$ is the Jordan form of $\underline{A}$ and $\underline{C}$ is the rational form of $\underline{A}$, then $\underline{J}$ is the Jordan form of $\underline{C}$.)

The matrix used in Examples A.1.2 and B.1.3 will be used to illustrate the above remarks.

<u>Example B.2.1.</u>

$$\underline{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 3 & 2 & -1 \\ 0 & 5 & 3 & -1 \end{bmatrix} \quad \text{and } \underline{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

From $\underline{C}$, we have

$$f(\lambda) = \phi(\lambda) = \lambda^2 (\lambda^2 - 2\lambda - 3) = \lambda^2 (\lambda - 3)(\lambda + 1)$$

$$C\,T_1 = 0,\ T_1 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \quad C\,T_2 = T_1,\ T_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ -1 \end{bmatrix}$$

$$[C + U]\,T_3 = 0,\ T_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \quad [C - 3U]\,T_4 = 0,\ T_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$T = [T_1, T_2, T_3, T_4] = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ 2 & 1 & 3 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Check:

$$C\,T = T\,J = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 2 & -3 & -3 \\ 0 & -1 & 1 & -3 \end{bmatrix}$$

When A is derogatory, it is much easier to find J from C than to find J directly from A. The two reasons given above are still valid; and, of even greater importance, Methods 1 and 2 of Section B.1 cannot be applied in this case.

When $\underline{A}$ is derogatory, the eigenvalues corresponding to the various blocks of $\underline{J}$ will not be distinct. A summary of the reason for this statement will be given. (For a more complete explanation, see pages 89-94 and pages 141-144 of Gantmacher (14).)

From Appendix A, we know that $f(\lambda) = D_{n-1}(\lambda)\ \phi(\lambda)$. Every root of $f(\lambda) = 0$ is also a root of $\phi(\lambda) = 0$. Therefore, every root of $D_{n-1}(\lambda) = 0$ is also a root of $\phi(\lambda) = 0$.

Let $A(\lambda)$ be a matrix whose elements are polynomials in λ. $A(\lambda)$ is of order n and has rank $r \leq n$. Let $D_j(\lambda)$ be the greatest common divisor of all minors of order j in $A(\lambda)$, $(j = 1, 2, \ldots, r)$.

Form the series

$$D_r(\lambda),\ D_{r-1}(\lambda),\ \ldots,\ D_1(\lambda),\ D_o(\lambda) \equiv 1 \qquad \text{(B.2.1)}$$

Each polynomial in the series is divisible by the succeeding polynomial.

$$i_1(\lambda) = \frac{D_r(\lambda)}{D_{r-1}(\lambda)},\ i_2(\lambda) = \frac{D_{r-1}(\lambda)}{D_{r-2}(\lambda)},\ \ldots,\ i_r(\lambda) = \frac{D_1(\lambda)}{D_o(\lambda)} = D_1(\lambda) \qquad \text{(B.2.2)}$$

The polynomials $i_1(\lambda)$, $i_2(\lambda)$, ..., $i_r(\lambda)$ defined in Equation B.2.2 are called the <u>invariant polynomials</u> of $A(\lambda)$.

$A(\lambda)$ is always equivalent to a canonical diagonal matrix which is of rank r.

$$\begin{bmatrix} i_1(\lambda) & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & i_2(\lambda) & & & & & \cdot \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & i_r(\lambda) & & & \cdot \\ \cdot & & & & 0 & & \cdot \\ \cdot & & & & & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \end{bmatrix}$$

In the sequence of invariant polynomials, Equation B.2.2, every polynomial from the second onwards divides the preceding one.

Decompose the invariant polynomials into irreducible factors:

$$i_1(\lambda) = [\xi_1(\lambda)]^{c_1} [\xi_2(\lambda)]^{c_2} \ldots [\xi_s(\lambda)]^{c_s}$$

$$i_2(\lambda) = [\xi_1(\lambda)]^{d_1} [\xi_2(\lambda)]^{d_2} \ldots [\xi_s(\lambda)]^{d_s} \qquad \text{(B.2.3)}$$

$$\vdots \qquad \left\{ \begin{array}{c} c_k \geq d_k \geq \ldots \geq \ell_k \geq 0 \\ k = 1, 2, \ldots, s \end{array} \right\}$$

$$i_r(\lambda) = [\xi_1(\lambda)]^{\ell_1} [\xi_2(\lambda)]^{\ell_2} \ldots [\xi_s(\lambda)]^{\ell_s}$$

where $\xi_1(\lambda)$, $\xi_2(\lambda)$, ..., $\xi_s(\lambda)$ are all of the distinct irreducible factors that occur in $i_1(\lambda)$, ..., $i_r(\lambda)$.

All the powers among $[\xi_1(\lambda)]^{c_1}$, ..., $[\xi_s(\lambda)]^{\ell_s}$ in Equation B.2.3, as far as they are distinct from unity, are called the <u>elementary divisors</u> of the matrix $A(\lambda)$.

For every elementary divisor, $[\xi_i(\lambda)]^{k_1}$, of $\underline{A}$ contained in $D_{n-1}(\lambda)$, there is an elementary divisor, $[\xi_i(\lambda)]^{k_2}$, of $\underline{A}$ contained in $\phi(\lambda)$, where $k_2 \geq k_1$. There are blocks in $\underline{J}$ which correspond to both of the elementary divisors $[\xi_i(\lambda)]^{k_1}$ and $[\xi_i(\lambda)]^{k_2}$. Call these blocks $\underline{J}_i$ and $\underline{J}_j$. The eigenvalues of $\underline{A}$ corresponding to $\underline{J}_i$ and $\underline{J}_j$ are identical. For example:

$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -2 & 3 & -3 \\ 0 & 0 & -1 & 2 & -1 \\ 1 & -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 2 \end{bmatrix}$$

$$f(\lambda) = (\lambda - 1)^4 (\lambda + 1)$$

$$\phi(\lambda) = (\lambda - 1)^2 (\lambda + 1)$$

$$D_{n-1}(\lambda) = (\lambda - 1)^2$$

The elementary divisors are $(\lambda - 1)^2$, $(\lambda - 1)^2$, $(\lambda + 1)$. Then

$$\underline{C} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{and } \underline{J} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Some other examples are now given which illustrate the transformation from $\underline{C}$ to $\underline{J}$.

Example B.2.2. (Same $\underline{A}$ as in Example A.1.3)

$$\underline{A} = \begin{bmatrix} -2 & -1 & -1 & -1 & 2 \\ 1 & 3 & 1 & 1 & -1 \\ -1 & -4 & -2 & -1 & 1 \\ -1 & -4 & -1 & -2 & 1 \\ -2 & -2 & -2 & -2 & 3 \end{bmatrix} \quad \text{and } \underline{C} = \begin{bmatrix} 0 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

The last two blocks of $\underline{C}$ are already in Jordan form so work on $\underline{C}_{11}$.

$$\underline{C}_{11} = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \qquad \begin{aligned} \phi(\lambda) &= \lambda^3 - 2\lambda^2 + 3 = (\lambda + 1)(\lambda^2 - 3\lambda + 3) \\ &= (\lambda + 1)\left(\lambda - \frac{3}{2} + j\frac{\sqrt{3}}{2}\right)\left(\lambda - \frac{3}{2} - j\frac{\sqrt{3}}{2}\right) \end{aligned}$$

$$\psi(\lambda, \mu) = \mu^2 + \mu(\lambda - 2) + \lambda^2 - 2\lambda$$

$$\underline{C}(\lambda) = \underline{A}^2 + (\lambda - 2)\,\underline{A} + (\lambda^2 - 2\lambda)\,\underline{U}$$

$$= \begin{bmatrix} 0 & -3 & -6 \\ 0 & 0 & -3 \\ 1 & 2 & 4 \end{bmatrix} + (\lambda - 2) \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} + (\lambda^2 - 2\lambda) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{C}_1(-1) = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \quad \underline{C}_1\left(\frac{3}{2} + j\,\frac{\sqrt{3}}{2}\right) = \begin{bmatrix} -\frac{3}{2} + j\,\frac{\sqrt{3}}{2} \\ -\frac{1}{2} + j\,\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

$$\underline{C}_1\left(\frac{3}{2} - j\,\frac{\sqrt{3}}{2}\right) = \begin{bmatrix} -\frac{3}{2} - j\,\frac{\sqrt{3}}{2} \\ -\frac{1}{2} - j\,\frac{\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

$$\underline{T} = \begin{bmatrix} 3 & -\frac{3}{2} + j\,\frac{\sqrt{3}}{2} & -\frac{3}{2} - j\,\frac{\sqrt{3}}{2} \\ -3 & -\frac{1}{2} + j\,\frac{\sqrt{3}}{2} & -\frac{1}{2} - j\,\frac{\sqrt{3}}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

$$\underline{J}_{11} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{3}{2} + j\,\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{3}{2} - j\,\frac{\sqrt{3}}{2} \end{bmatrix}$$

Check:

$$\underline{C}_{11}\,\underline{T} = \underline{T}\,\underline{J}_{11} = \begin{bmatrix} -3 & -3 & -3 \\ 3 & -\frac{3}{2} + j\,\frac{\sqrt{3}}{2} & -\frac{3}{2} - j\,\frac{\sqrt{3}}{2} \\ 1 & \frac{3}{2} + j\,\frac{\sqrt{3}}{2} & \frac{3}{2} - j\,\frac{\sqrt{3}}{2} \end{bmatrix}$$

Therefore,

$$\underline{J} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} + j\,\frac{\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} - j\,\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

<u>Example B.2.3.</u>

$$\underline{A} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -3 & 3 & -5 & 4 \\ 8 & -4 & 3 & -4 \\ 15 & -10 & 11 & -11 \end{bmatrix}$$

Let

$$\underline{X} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \underline{A}\,\underline{X} = \begin{bmatrix} 1 \\ -3 \\ 8 \\ 15 \end{bmatrix}, \quad \underline{A}^2\,\underline{X} = \begin{bmatrix} -3 \\ 8 \\ -16 \\ -32 \end{bmatrix}$$

$$\underline{A}^3\,\underline{X} = \begin{bmatrix} 5 \\ -15 \\ 24 \\ 51 \end{bmatrix} = -3\underline{A}^2\,\underline{X} - 3\underline{A}\,\underline{X} - \underline{X}$$

$$\phi(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$$

Since the sum of the eigenvalues of $\underline{A}$ must equal the trace of $\underline{A}$, we can find the other elementary divisor.

$$\lambda_4 = tr(\underline{A}) + 3 = 1 + 3 + 3 - 11 + 3 = -1$$

The elementary divisors are $(\lambda + 1)^3$ and $(\lambda + 1)$

$$\underline{C} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The last block is already in Jordan form so work on

$$\underline{C}_1 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix}$$

$$[\underline{C}_1 + \underline{U}]\ \underline{T}_1 = \underline{0},\ \underline{T}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\quad [\underline{C}_1 + \underline{U}]\ \underline{T}_2 = \underline{T}_1,\ \underline{T}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$[\underline{C}_1 + \underline{U}]\ \underline{T}_3 = \underline{T}_2,\ \underline{T}_3 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$\underline{T} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 1 & 1 & 1 \end{bmatrix},\quad \underline{T}^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ -1 & 2 & -3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\underline{T}^{-1}\,\underline{C}_1\,\underline{T} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Therefore,

$$\underline{J} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Example B.2.4.

A final example is a 6 by 6 matrix which has

$$f(\lambda) = \lambda^6 + 5\lambda^5 - 2\lambda^4 - 40\lambda^3 - 32\lambda^2 + 80\lambda + 96$$

By reduction to rational form, it is found that

$$\phi(\lambda) = \lambda^3 + 3\lambda^2 - 4\lambda - 12 = (\lambda^2 - 4)\,(\lambda + 3)$$

$$= (\lambda + 2)\,(\lambda - 2)\,(\lambda + 3)$$

Other elementary divisors are found to be $(\lambda + 2)$, $(\lambda - 2)$, and $(\lambda + 2)$. Therefore,

$$\underline{C} = \begin{bmatrix} \underline{C}_1 & \underline{0} & \underline{0} \\ \underline{0} & \underline{C}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{C}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 12 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$\underline{C}_3$ is already in Jordan form. Corresponding to $\underline{C}_2$ is

$$\underline{J}_2 = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

Corresponding to $\underline{C}_1$ is

$$\underline{J}_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Therefore,

$$\underline{J} = \begin{bmatrix} \underline{J}_1 & \underline{0} & \underline{0} \\ \underline{0} & \underline{J}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{J}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

VITA

Marvin Emmett Daniel

Candidate for the Degree of

Doctor of Philosophy

Thesis: STATE-SPACE SYNTHESIS OF PASSIVE ONE-PORT NETWORKS

Major Field: Engineering

Biographical:

Personal Data: Born near Dexter, Kansas, December 19, 1938, the son of Emmett P. and Eva M. Daniel.

Education: Attended grade school in Dexter, Kansas; graduated from Dexter High School in 1956; attended Arkansas City Junior College, Arkansas City, Kansas; received the Bachelor of Science degree in Electrical Engineering from Kansas State University in January, 1961; received the Master of Science degree in Electrical Engineering from the University of New Mexico in June, 1963; completed requirements for the Doctor of Philosophy degree in May, 1966.

Professional experience: Employed by Boeing Airplane Company, Wichita, Kansas, during the summers of 1959 and 1960. Employed by Sandia Corporation in the design and development of field test equipment from February, 1961, until September, 1963. Employed by the School of Electrical Engineering of Oklahoma State University since September, 1963, as an instructor.

Professional organizations: Member of the Institute of Electrical and Electronic Engineers, Phi Kappa Phi, Eta Kappa Nu, and Sigma Tau.