ELASTIC BOUNDARY LAYERS IN AXISYMMETRIC CYLINDRICAL BODIES

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CYLINDRICAL BODIES

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PREFACE

The nomenclature I have used in this thesis is frequently used in mechanics. Subscripts following commas denote partial differentiation and whenever an eigenfunction is subscripted, the subscripts are the subscripts of the eigenvalues that are arguments of the eigenfunction. These eigenvalues are actually roots of transcendental equations. Calling these roots eigenvalues may lack in mathematical rigor, but, it is a standard practice in mechanics. The subscript, b, is used when the axial coordinate, z, is zero. I have taken all other symbols and conventions from reference (16) or have explained then where I first introduced them.

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The immediate families of most graduate students make significant contributions to their education. I am an exception only because the contributions of my family far exceed any usual average.

I only wish that my graduation could be a larger repayment of all these kindnesses.

Bart Childs

Bart Childs

CHAPTER I

THE PROBLEM

Barrie de Saint Venant, the French elastician, published his well-known principle of linear elasticity in 1853 as a part of his memoirs on torsion (see reference (1)). An interpretation of this principle given by the English elastician Love (2) is:

> "The strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part."

In this thesis, a method for finding the stresses and deformations in the semi-infinite circular bar shown in Figure 1.1 will be proposed. The bar will be assumed to be elastic, isotropic, and

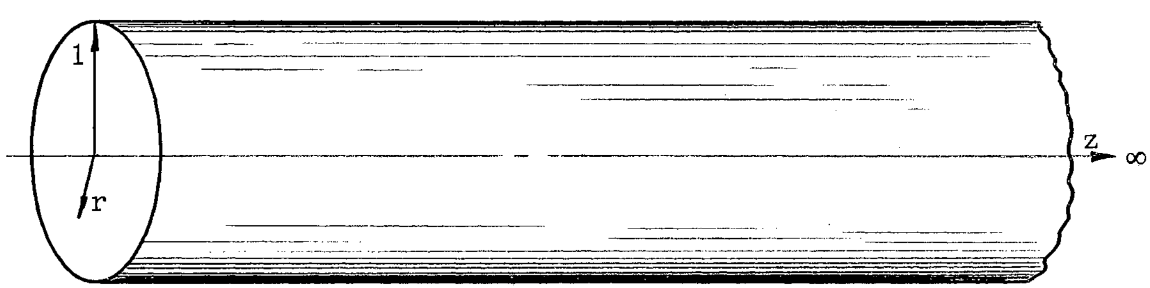

Figure 1.1

homogenous. The curved surface, $r = 1$, will be assumed to be stress free and a set of self-equilibrating axisymmetric tractions and/or displacements will be given on the finite end, $z = 0$. If the tractions acting upon the finite end are mathematically difficult, they can be replaced by a statically equivalent, mathematically simple, set of tractions plus a self-equilibrating set of tractions. It may be observed that Saint Venant's principle assumes that the stresses and strains due to the self-equilibrating tractions are approximately zero for large values of z.

There are four different sets of boundary conditions that may be specified on the finite end. These are:

(a) The normal axial stress and the radial displacement.

(b) The radial shearing stress and the axial displacement.

(c) The normal axial and the radial shearing stresses.

(d) The axial and the radial displacements.

Sets (a) and (b) are called mixed boundary conditions, while (c) and (d) are called stress and displacement boundary conditions, respectively.

The elastic boundary layer is defined as that part of the body appreciably deformed by a set of self-equilibrating tractions. There are many problems in which the elastic boundary layer must be analyzed. Some examples of these problems are punches, thrust bearings, thermal stresses, shrink fits, and numerous others where the stress and displacement distributions near the end of a cylinder are of concern. The solution of the elastic boundary layer may also be used as a correction to approximate solutions employing the Saint

Venant approximation.

There have been many attempts to solve this problem and many related problems. Horvay and Mirabal (3) discarded the rigorous solution and used a variational approximation based on Sadowsky-Sternberg stress functions which were modified to identically satisfy the equilibrium equations. The principal criterion for judgment of the degree of approximation achieved is the comparison of eigenvalues. The authors had only the first two eigenvalues given by Murray (4) for comparison. This writer has compiled an extensive list of the eigenvalues for the rigorous solution and these are shown in Appendix A. The comparison of the approximate eigenvalues with those in Appendix A shows significant differences to be present. As pointed out by the authors, their variational approximation does not satisfy the compatibility equations.

Hodgkins (5) used a finite difference method to obtain numerical results for stress distributions due to certain end conditions. A different approach was used by Mendelson and Roberts (6). Their approach was to reduce the governing equations to a single integro-differential equation in the shear stress. This equation was then solved by a collocation procedure.

Several publications have been concerned with mathematically similar thermal stress problems. Murray (4) formulated his solution in terms of a series of eigenfunctions. He retained only the first two terms of this series and adjusted the coefficients to approximately satisfy the condition that the real part of the shearing stress be zero. Horvay, Giaever, and Mirabal (7) extended the variational approximation of Horvay and Mirabal (3) to a thermal stress problem. Youngdahl and Sternberg (8) solved a thermal stress problem in which their final answer was a Fourier-Bessel series plus a Fourier integral. They evaluated the integral numerically. Had they evaluated the integral by the calculus of residues, their solution should reduce to the same form as Murray's eigenfunction formulation (4).

Horvay and Mirabal (3) give an excellent discussion of some of the difficulties associated with this problem which may be summed up in the following quotes:

> "..., a rigorous solution of the end problem is readily obtained in principle.
> ..., compatibility and equilibrium are satisfied if the well known 'Love function', L, is picked such that
>
> $$\nabla^2 \nabla^2 L = 0 . \tag{1.1}$$
>
> It is readily seen that
>
> $$L(r, z) = e^{-\gamma z} J_o(\gamma r) + \frac{e^{-\gamma z} r J_1(\gamma r)}{2(1-\nu) + J_o(\gamma)/J_1(\gamma)} \tag{1.2}$$
>
> is biharmonic and satisfied the stress free boundary conditions on the outer surface if γ satisfies the eigenvalue equation
>
> $$J_o^2(\gamma) + \left[1 - 2(1-\nu)/\gamma^2\right] J_1^2(\gamma) = 0 . \tag{1.3}$$
>
> ... Why, then, if the exact solutions L of the problem are known, is the method not used?
> There are two reasons: (a) The labor involved in calculating the complex roots γ_k and the corresponding stress and displacement equations is enormous.
> (b) The real and imaginary parts of L_k give rise to two sets of normal and shearing end tractions which are hopelessly intermingled."

For the similar two-dimensional problem, difficulties very similar to those just discussed by Horvay and Mirabal also existed. These difficulties were overcome by Johnson and Little (9). The solution given by Johnson and Little was originally formulated in terms of a Fourier series and a Fourier integral. They evaluated the integral by the calculus of residues and their solution reduced to a single eigenfunction expansion. This formulation automatically incorporated an inner biorthogonality into the eigenfunction eoefficients that allowed an explicit solution for these coefficients for mixed boundary conditions. They solved the stress problem by truncation of an infinite set of infinite equations.

The same two-dimensional problem was also discussed by Buchwald (10). Buchwald mentions that Koiter (11) also used a truncation idea successfully.

With this insight, the solution will be formulated as a Fourier-Bessel series plus a Fourier integral. The integral will be evaluated by the calculus of residues and the solution will reduce to an eigenfunction expansion which is equivalent to equation (1.2). A digital computer has been programmed to calculate the numerical results. The eigenvalues, which Horvay and Mirabal did not have, are listed in Appendix A and the truncation ideas of Johnson, Little, and Koiter will be used to solve the stress boundary conditions.

CHAPTER II

THE LOVE FUNCTION

In a paper published in 1930 (12), the Russian elastician B. Galerkin showed that the solution for the general problem of three-dimensional elasticity could be obtained in terms of three biharmonic strain functions. In later papers (13), Galerkin also showed the applicability of this method. Papkovitch (14) and Westergaard (15) showed that the three biharmonic functions can be interpreted as components of a vector and they named this vector the Galerkin vector. If the problem is axisymmetric and expressed in cylindrical coordinates, two components of the Galerkin vector are zero and the component in the z direction is identical to Love's function (2).

The Love function (or Galerkin vector) has often been called a stress function, see for example F. H. Murray (4). This name does not give an accurate description of the function. Westergaard (15) and Yu (16) refer to the Galerkin vector as sets of strain functions and displacement functions, respectively, because they directly yield displacements.

The governing differential equation for the Love function is the biharmonic equation

$$\nabla^2 \nabla^2 \psi = 0 \tag{2.1}$$

where ∇^2 is the Laplacian operator, in cylindrical coordinates,

$$\nabla^2(\ ) = (\ )_{,rr} + \frac{1}{r}(\ )_{,r} + (\ )_{,zz} \quad . \tag{2.2}$$

The nonzero stresses and deformations are expressed in terms of ψ by the following equations:

$$\sigma_r = \left[\nu \nabla^2 \psi - \psi_{,rr}\right]_{,z} \tag{2.3a}$$

$$\sigma_\theta = \left[\nu \nabla^2 \psi - \frac{1}{r}\psi_{,r}\right]_{,z} \tag{2.3b}$$

$$\sigma_z = \left[(2 - \nu)\nabla^2 \psi - \psi_{,zz}\right]_{,z} \tag{2.3c}$$

$$\tau = \tau_{rz} = \left[(1 - \nu)\nabla^2 \psi - \psi_{,zz}\right]_{,r} \tag{2.3d}$$

$$u = -\frac{1 + \nu}{E}\ \psi_{,rz} \tag{2.3e}$$

$$w = \frac{1 + \nu}{E}\left[2(1 - \nu)\nabla^2 \psi - \psi_{,zz}\right] \tag{2.3f}$$

It is known that the product of two harmonic functions is biharmonic. Thus, the Love function, ψ, may be assumed to be the product

$$\psi = \varphi H \tag{2.4}$$

where φ and H are harmonic functions. the governing equations for φ and H are

$$\nabla^2 \varphi = 0 \ , \quad \nabla^2 H = 0 \tag{2.5}$$

Assuming φ to be the product

$$\varphi = RZ \tag{2.6}$$

where R is a function of r only and Z is a function of z only, the usual separation of variables techniques leads to the equations

$$\frac{\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}}{R} = -\frac{\frac{d^2Z}{d^2z}}{Z} = \lambda \tag{2.7}$$

where λ is a constant and is referred to as the separation constant. To satisfy the boundary conditions at $z = \infty$, λ must be real and negative. Assuming $\lambda = -\beta^2$ and $\beta > 0$,

$$R = c_1 J_o(\beta r) + c_2 Y_o(\beta r)$$

and (2.8)

$$Z \; 4 \; c_3 d^{-\beta z} + c_4 e^{\beta z} \; .$$

To satisfy the conditions that the stresses and strains always be finite,

$$c_2 = c_4 = 0 \; . \tag{2.9}$$

H will be assumed to be of the form

$$H = c_5 + c_6 z \; . \tag{2.10}$$

Thus, a part of ψ is

$$(A + B\beta z) J_o(\beta r) e^{-\beta z} \; . \tag{2.11}$$

Later, β will become subscripted and defined by a transcendental equation necessary for the specification of certain boundary conditions. Motivation for the form of H and the specification of β is from the work of Pickett (17).

Expression (2.11) cannot satisfy all the boundary conditions. Another part of ψ will be taken to be a Fourier integral. This part of ψ will be denoted by χ. Thus,

$$\nabla^2 \nabla^2 \chi = 0 . \tag{2.12}$$

Performing the Fourier cosine transformation on the last equation yields

$$\int_0^\infty \nabla^2 \nabla^2 \chi \cos (\alpha z) dz = \overline{\nabla}^2 \overline{\nabla}^2 X = 0 \tag{2.13}$$

where

$$\overline{\nabla}^2 (\ ) = \int_0^\infty \nabla^2 (\ ) \cos (\alpha z) dz = (\ )_{,rr} + \frac{1}{r} (\ )_{,r} - \alpha^2 \tag{2.14}$$

and

$$X = \int_0^\infty \chi \cos (\alpha z) dz . \tag{2.15}$$

A solution of (2.13) is

$$X = c_7(\alpha) I_0(\alpha r) + c_8(\alpha) \alpha r I_1 (\alpha r) . \tag{2.16}$$

Since X is the Fourier transform of χ, X may be derived by taking the inverse transform of equation (2.16). Thus,

$$\chi = \int_0^\infty \left\{ C(\alpha) I_0(\alpha r) + D(\alpha)\, \alpha r\, I_1(\alpha r) \right\} \cos(\alpha z)\, d\alpha \tag{2.17}$$

where $C(\alpha)$ and $D(\alpha)$ differ from $c_7(\alpha)$ and $c_8(\alpha)$ by a $\sqrt{2/\pi}$ multiple.

Thus, the Love function is taken in the following form:

$$\begin{aligned} \psi = & \sum_m (A_m + B_m \beta_m z) J_0(\beta_m r) e^{-\beta_m z} \\ & + \int_0^\infty \left\{ C(\alpha) I_0(\alpha r) + D(\alpha)\, \alpha r\, I_1(\alpha r) \right\} \cos(\alpha z)\, d\alpha \end{aligned} \tag{2.18}$$

There are four sets of constants of integration in the last equation, namely, A_m, B_m, $C(\alpha)$, and $D(\alpha)$. The sets of constants A_m and B_m will be selected to satisfy the boundary conditions on the finite end while $C(\alpha)$ and $D(\alpha)$ will be picked to satisfy the stress-free boundary conditions on the curved surface. If the Fourier sine transform had been used, the Love function would be taken in the following form:

$$\begin{aligned} \psi = & \sum_m (A_m + B_m \beta_m z) J_0(\beta_m r) e^{-\beta_m z} \\ & + \int_0^\infty \left\{ C(\alpha) I_0(\alpha r) + D(\alpha)\, \alpha r\, I_1(\alpha r) \right\} \sin(\alpha z)\, d\alpha \end{aligned} \tag{2.19}$$

CHAPTER III

MIXED BOUNDARY CONDITIONS

Boundary condition (a) will be solved first. Assuming the stress function given by equation (2.18), differentiation and substitution into equation (2.3d) yields

$$\tau = \sum_m \beta_m^3 \left[A_m + B_m(\beta_m z - 2\nu)\right] J_1(\beta_m r) e^{-\beta_m z}$$

$$+ \int_0^\infty \left\{ C I_1(\alpha r) + D\left[\alpha r I_0(\alpha r) + 2(1-\nu) I_1(\alpha r)\right]\right\} \alpha^3 \cos(\alpha z)\, d\alpha \qquad (3.1)$$

Specifying $\tau(1, z) = 0$ may be satisfied by the following equations

$$J_1(\beta_m) = 0 \qquad (3.2)$$

$$C = -D\left[2(1-\nu) + \frac{\alpha I_0(\alpha)}{I_1(\alpha)}\right] \qquad (3.3)$$

The following expression may be obtained from equations (2.18) and (2.3a):

$$\sigma_r = \sum_m \beta_m^3 \left[(A_m + B_m(\beta_m z - 1))(-J_0(\beta_m r) + \frac{J_1(\beta_m r)}{\beta_m r} + 2\nu B_m J_0(\beta_m r)\right] e^{-\beta_m z}$$

$$+ \int \left\{ C\alpha^3 \left[I_0(\alpha r) - \frac{I_i(\alpha r)}{\alpha r}\right] + D\alpha^3\left[(1-2\nu) I_0(\alpha r) + \alpha r I_1(\alpha r)\right]\right\} \sin(\alpha z)\, d\alpha \qquad (3.4)$$

The boundary condition $\sigma_r(1,z) = 0$ leads to

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ C\alpha^3\left[I_0(\alpha) - \frac{I_1(\alpha)}{\alpha}\right] + D\alpha^3\left[1 - 2\nu)I_0(\alpha) + \alpha I_1(\alpha)\right]\right\} \sin(\alpha z)\, d\alpha$$

$$= \sqrt{\frac{2}{\pi}} \sum_m \left[-A_m + B_m(1 + 2\nu) - B_m\beta_m z\right]\beta_m^3 J_0(\beta_m) e^{-\beta_m z} \qquad (3.5)$$

The LHS of equation (3.5) is a Fourier transformation. Taking the inverse Fourier transform and use of equation (3.3) yields

$$D\frac{\alpha^4}{I(\alpha)}\left[-I_0^2(\alpha) + I_1^2(\alpha)\left\{1 + \frac{2(1-\nu)}{\alpha^2}\right\}\right]$$

$$= \frac{2}{\pi}\sum_m \left\{(-A_m + B_m(1 + 2\nu))\int_0^\infty e^{-\beta_m z} \sin(\alpha z)\, dz\right.$$

$$\left. - B_m\beta_m \int_0^\infty z e^{-\beta_m z} \sin(\alpha z)\, dz\right\}\beta_m^3 J_0(\beta_m) \qquad (3.6)$$

Completing the inverse transformation and rearranging equation (3.6) gives

$$D = \frac{2}{\pi}\sum_m \frac{\beta_m^3 J_0(\beta_m) I_1(\alpha)\left[A_m(\alpha^2 + \beta_m^2) + B_m((1 - 2\nu)(\alpha^2 + \beta_m^2) - 2\alpha^2)\right]}{\alpha^3(\alpha^2 + \beta_m^2)^2\, T(\alpha)} \qquad (3.7)$$

where

$$T(\alpha) = -I_0^2(\alpha) + I_1^2(\alpha)\left[1 + \frac{2(1-\nu)}{\alpha^2}\right]. \qquad (3.8)$$

$$\psi = \sum_m (A_m + B_m \beta_m z) J_o(\beta_m r)^{-\beta_m z}$$

$$+ \frac{2}{\pi} \sum_m \int_o^\infty \frac{\beta_m^3 J_o(\beta_m) L(\alpha, \beta_m) M(\alpha, r) \cos(\alpha z) dx}{\alpha^2 (\alpha^2 + \beta_m^2)^2 T(\alpha)} \quad (3.9)$$

where

$$L(\alpha, \beta_m) = A_m(\alpha^2 + \beta_m^2) + B_m((1 - 2\nu)(\alpha^2 + \beta_m^2) - 2\alpha^2) \quad (3.10)$$

and

$$M(\alpha, r) = I_1(\alpha)\, r I_1(\alpha r) - \left[2(1-\nu)\frac{I_1(\alpha)}{\alpha} + I_o(\alpha)\right] I_o(\alpha r) \quad (3.11)$$

Since the integrand of equation (3.9) is even in α and

$$e^{i\alpha z} = \cos(\alpha z) + i \sin(\alpha z) \quad (3.12)$$

equation (3.9) may be rewritten as

$$\psi = \sum_m (A_m + B_m \beta_m z) J_o(\beta_m r) e^{-\beta_m z}$$

$$+ \frac{1}{\pi} \sum_m \int_{-\infty}^\infty \frac{\beta_m^3 J_o(\beta_m) L(\alpha, \beta_m) M(\alpha, r) e^{i\alpha z} d\alpha}{\alpha^2 (\alpha^2 + \beta_m^2)^2 T(\alpha)} \quad (3.13)$$

The integration indicated in equation (3.13) may be done by calculus of residues. The contour integration must be done around the poles in the upper half plane. The presence of the exponential term will give unbounded values for contour integration in the lower half plane.

There is a pole of order two at the origin which does not contribute to the solution. There is also a pole of order two at $\alpha = i\beta_m$

and there are simple poles at the roots of the transcendental equation, $T(\alpha)$.

If the first quadrant eigenvalues of $T(\alpha)$ are denoted by α_j. there will also be eigenvalues at $\bar{\alpha}_j$, $-\bar{\alpha}_j$, and $-\alpha_j$. Only the poles α_j and $-\bar{\alpha}_j$ are in the upper half plane

Residues will be evaluated by the following differentiation formulae:

Simple poles of $w(\alpha) = \frac{f(\alpha)}{g(\alpha)}$ at $\alpha = a$,

$$\text{Residue} = \left. \frac{f(\alpha)}{g'(\alpha)} \right|_{\alpha = a}, \tag{3.14}$$

Poles of order two of $w(\alpha) = \dfrac{f(\alpha)}{g(\alpha)(\alpha - a)^2}$,

$$\text{Residue} = \left. \frac{d}{d\alpha} \left\{ \frac{f(\alpha)}{g(\alpha)} \right\} \right|_{\alpha = a}. \tag{3.15}$$

The following equation shows the relation between the integral and the residues.

$$\int_{-\infty}^{\infty} \left\{ \right\} d\alpha = 2\pi i \ \Sigma \ (\text{Residues of } \left\{ \right\}) \tag{3.16}$$

First, the residues at the second order poles $i\beta_m$ will be investigated. The relevant part of ψ may be written as

$$\Sigma \, \text{Residues} = 2i \sum_m \left\{ \frac{d}{d\alpha} \left[\frac{\beta_m^3 J_o(\beta_m) L(\alpha, \beta_m) M(\alpha, r) e^{i\alpha z}}{\alpha^2 (\alpha + i\beta_m)^2 T(\alpha)} \right] \right\} \tag{3.17}$$

which when evaluated at

$$\alpha = i\beta_m \tag{3.18}$$

gives

$$\Sigma \text{Residues} = -\sum_m (A_m + B_m \beta_m z) J_o(\beta_m r) e^{-\beta_m z} . \tag{3.19}$$

Since the residue shown in (3.19) due to the poles at $i\beta_m$ cancel the Fourier-Bessel series, the final value of ψ will be

$$\psi = \sum_j \sum_m \frac{i\beta_m^3 J_o(\beta_m) L_{jm} M_j e^{i\alpha_j z}}{(\alpha_j^2 + \beta_m^2)^2 F_j} \tag{3.20}$$

where

$$F_j = \frac{\alpha_j^2}{2} \frac{dT_j}{d\alpha_j} = -2\alpha_j I_o^2(\alpha_j) + \alpha_j I_1^2(\alpha_j) + 2(1-\nu) I_o(\alpha_j) I_1(\alpha_j) \tag{3.21}$$

It should be noted again that the summation of j is done over the eigenvalues of $T(\alpha_j)$ in the first and second quadrants.

For the assumed stress function, the boundary conditions on the finite end will now be satisfied and the biorthogonal functions for the mixed boundary conditions derived.

Substitution of equation (2.18) into equations (2.3c) and (2.3e) and setting z equal to zero yields

$$\sigma_{zb}\Big|_{z=0} = \sum_m \left\{ A_m + B_m(1-2\nu) \right\} \beta_m^3 J_o(\beta_m r) \tag{3.22}$$

$$u_b\Big|_{z=0} = \sum_m \left\{A_m - B_m\right\} \beta_m^{\,2} J_1(\beta_m r) \tag{3.23}$$

Denoting the specified boundary conditions in equations (3.22) and (3.23) by $f(r)$ and $\frac{(1+\nu)\,g(r)}{E}$, respectively, the orthogonality of Bessel functions facilitates unique solutions for each A_m and B_m, viz.,

$$A_m + (1-2\nu)B_m = \int_0^1 f(r)\,J_o(\beta_m r)\,r\,dr \Big/ \tfrac{1}{2}\,\beta_m^{\,3} J_o^2(\beta_m)\,, \tag{3.24}$$

$$A_m - B_m = -\int_0^1 g(r)\,J_1(\beta_m r)\,r\,dr \Big/ \tfrac{1}{2}\,\beta_m^{\,2} J_o^2(\beta_m)\,, \tag{3.25}$$

and

$$A_m = \frac{\int_0^1 f(r)\,J_o(\beta_m r)\,r\,dr - (1-2\nu)\int_0^1 g(r)\,J_1(\beta_m r)\,r\,dr}{(1-\nu)\,\beta_m^{\,3} J_o^2(\beta_m)}\,, \tag{3.26}$$

$$B_m = \frac{\int_0^1 f(r)\,J_o(\beta_m r)\,r\,dr + \beta_m\int_0^1 g(r)\,J_1(\beta_m r)\,r\,dr}{(1-\nu)\,\beta_m^{\,3} J_o^2(\beta_m)}\,. \tag{3.27}$$

To facilitate the derivation of the biorthogonal functions, it is convenient to write equation (3.20) as

$$\psi = \sum_j \frac{M_j\, e^{i\alpha_j z}}{F_j}\; i\sum_m \frac{L_{jm}\,\beta_m^{\,3} J_o(\beta_m)}{(\alpha_j^2 + \beta_m^2)^2} \tag{3.28}$$

Denoting

$$c_j = i \sum_m \frac{L_{jm} \beta_m^3 J_o(\beta_m)}{(\alpha_j^2 + \beta_m^2)^2} \tag{3.29}$$

and substituting equations (3.26) and (3.27) into (3.29) will enable considerable simplification of c_j after interchange of the order of integration and summation. Thus,

$$c_j = i \sum_m \frac{\left\{(\alpha_j^2 + \beta_m^2)A_m + \left[(1-2\nu)(\alpha_j^2+\beta_m^2) - 2\alpha_j^2\right]B_m\right\}\beta_m^3 J_o(\beta_m)}{(\alpha_j^2 + \beta_m^2)^2} . \tag{3.30}$$

Substituting and interchanging the order of integration and summation

$$c_j = i \int_o^1 \left\{ \frac{f(r)}{1-\nu} \sum_m \frac{\left[2(1-\nu)(\alpha_j^2+\beta_m^2) - 2\alpha_j^2\right] J_o(\beta_m r)}{(\alpha_j^2 + \beta_m^2)^2 J_o(\beta_m)} \right.$$

$$\left. + \frac{g(r)}{1-\nu} \sum_m \frac{(-2\alpha_j^2) J_1(\beta_m r)\beta_m}{(\alpha_j^2 + \beta_m^2)^2 J_o(\beta_m)} \right\} rdr \tag{3.31}$$

and

$$c_j = i \int_o^1 \left\{ \frac{f(r)}{1-\nu} \left[2(1-\nu) \sum_m \frac{J_o(\beta_m r)}{(\alpha_j^2+\beta_m^2)} - 2\alpha_j^2 \sum_m \frac{J_o(\beta_m r)}{(\alpha_j^2 + \beta_m^2)^2} \right. \right.$$

$$\left. \left. + \frac{g(r)}{1-\nu} \left[-2\alpha_j^2 \sum_m \frac{\beta_m J_1(\beta_m r)}{(\alpha_j^2 + \beta_m^2)^2} \right] \right\} rdr . \tag{3.32}$$

Noting the following transforms,

$$\sum_m \frac{\beta_m J_1(\beta_m r)}{(\alpha_j^2 + \beta_m^2)^2 J_o(\beta_m)} = \frac{r I_o(\alpha_j r)}{4\alpha_j I_1(\alpha_j)} - \frac{I_o(\alpha_j) I_1(\alpha_j r)}{4\alpha_j I_1^2(\alpha_j)} , \tag{3.33}$$

$$\sum_m \frac{J_o(\beta_m r)}{(\alpha_j^2 + \beta_m^2)^2 J_o(\beta_m)} = \frac{I_o(\alpha_j) I_o(\alpha_j r)}{4\alpha_j^2 I_1^2(\alpha_j)} - \frac{r I_1(\alpha_j r)}{4\alpha_j^2 I_1(\alpha_j)} , \tag{3.34}$$

and

$$\sum_m \frac{J_o(\beta_m r)}{(\alpha_j^2 + \beta_m^2) J_o(\beta_m)} = \frac{I_o(\alpha_j r)}{2\alpha_j I_1(\alpha_j)} , \tag{3.35}$$

the expression for c_j may be written as

$$c_j = i \int_o^1 \frac{f(r)}{(1-\nu)} \left[\left\{ \frac{(1-\nu)}{\alpha_j I_1(\alpha_j)} - \frac{I_o(\alpha_j)}{2I_1^2(\alpha_j)} \right\} I_o(\beta_j r) + \frac{r I_1(\alpha_j r)}{2I_1(\alpha_j)} \right] r dr$$

$$+ i \int_o^1 \frac{g(r)}{(1-\nu)} \left[\frac{\alpha_j r I_o(\alpha_j r)}{2 I_1(\alpha_j)} + \frac{\alpha_j I_o(\alpha_j) I_1(\alpha_j r)}{2I_1^2(\alpha_j)} \right] r dr . \tag{3.36}$$

Now, ψ may be written as follows:

$$\psi = \sum_j \frac{c_j M_j e^{i\alpha_j z}}{F_j} . \tag{3.37}$$

The final stress function should have real values. Since the coefficient of the exponential term is odd in α, summation of evaluations of the jth term at α_j and $-\overline{\alpha}_j$ will give only a real number, viz.,

$$\psi(\alpha_j) + i\,\psi(-\overline{\alpha}_j) = 2 \text{ Real } (i\,\psi(\alpha_j)) . \tag{3.38}$$

In the other mixed boundary condition, a similar form will appear except it will be even and without the imaginary multiplier.

For the second mixed boundary condition, equation (2.19) is used for the Love function instead of equation (2.18). Using a procedure similar to the one used in deriving equation (3.37), equation (2.19) can be reduced to

$$\psi = \sum_j \frac{d_j M_j e^{i\alpha_j z}}{F_j} \tag{3.39}$$

where

$$d_j = \int_0^1 \frac{h(r)}{(1-\nu)} \left\{ \frac{-r I_o(\alpha_j r)}{2I_1(\alpha_j)} + \left[\frac{I_o(\alpha_j)}{2I_1^2(\alpha_j)} + \frac{(1-\nu)}{\alpha_j I_1(\alpha_j)} \right] I_1(\alpha_j r) \right\} r\,dr$$

$$+ \int_0^1 \frac{k(r)}{(1-v)} \left\{ \left[\frac{\alpha_j I_o(\alpha_j)}{2I_1^2(\alpha_j)} - \frac{1}{I_1(\alpha_j)} \right] I_o(\alpha_j r) - \frac{\alpha_j}{2I_1(\alpha_j)} r\, I_1(\alpha_j r) \right\} r\,dr \tag{3.40}$$

The functions $h(r)$ and $\frac{(1+\nu)k(r)}{E}$ are the shearing stress and axial displacement functions specified on the finite end.

CHAPTER IV

GENERAL BOUNDARY CONDITIONS

A general solution will now be formulated as a combination of the two solutions given by equations (3.37) and (3.39).

Before doing this, the complex constants α_j will be replaced by an equivalent constant $i\gamma_j$. This changes the expressions containing modified Bessel functions of an argument α_j to similar expressions containing ordinary Bessel functions of an argument γ_j.

The following notations will be used for convenience in writing the complex mathematical expressions:

$$M_j = -\left[2(1-\nu)\frac{J_1(\gamma_j)}{\gamma_j} + J_o(\gamma_j)\right] J_o(\gamma_j r) - J_1(\gamma_j) r\, J_1(\gamma_j r) \qquad (4.1)$$

$$N_j = -2iF_j = -4\gamma_j J_o^2(\gamma_j) - 2\gamma_j J_1^2(\gamma_j) + 4(1-\nu) J_o(\gamma_j) J_1(\gamma_j) \qquad (4.2)$$

$$W_{1j} = \left[\frac{J_o(\gamma_j)}{2J_1^2(\gamma_j)} - \frac{1-\nu}{\gamma_j J_1(\gamma_j)}\right] J_o(\gamma_j r) + \frac{r\, J_1(\gamma_j r)}{2\, J_1(\gamma_j)} \qquad (4.3a)$$

$$W_{2j} = \frac{r\, J_o(\gamma_j r)}{2\, J_1(\gamma_j)} - \left[\frac{(1-\nu)}{\gamma_j J_1(\gamma_j)} + \frac{J_o(\gamma_j)}{2\, J_1^2(\gamma_j)}\right] J_1(\gamma_j r) \qquad (4.3b)$$

$$W_{3j} = -\frac{\gamma_j r\, J_o(\gamma_j r)}{2\, J_1(\gamma_j)} + \frac{\gamma_j J_o(\gamma_j) J_1(\gamma_j r)}{2\, J_1^2(\gamma_j)} \qquad (4.3c)$$

$$W_{4j} = \left[\frac{1}{J_1(\gamma_j)} - \frac{\gamma_j J_o(\gamma_j)}{2 J_1^2(\gamma_j)}\right] J_o(\gamma_j r) - \frac{\gamma_j r J_1(\gamma_j r)}{2 J_1(\gamma_j)} \qquad (4.3d)$$

The Love function will now be taken to be

$$\psi = \sum_j \frac{a_j M_j e^{-\gamma_j z}}{N_j} \qquad (4.4)$$

where $a_j = -i(c_j + d_j)$

$$= \int_o^1 \frac{1}{1-\nu} \left\{ \sigma_{zb} W_{1j} + \tau_b W_{2j} + \frac{E}{(1+\nu)} \left[u_b W_{3j} + w_b W_{4j} \right] \right\} r\,dr. \qquad (4.5)$$

The following equations result from substituting the derivatives of equation (4.4) into equations (2.3).

$$\sigma_r = \sum_j \frac{a_j e^{-\gamma_j z}}{N_j} \left\{ \left[\gamma_j^2 J_1(\gamma_j) + \gamma_j^3 J_o(\gamma_j) \right] J_o(\gamma_j r) + \gamma_j^3 J\ (\gamma_j) r J_1(\gamma_j r) \right.$$

$$\left. - \left[2(1-\nu)\gamma_j J_1(\gamma_j) + \gamma_j^2 J_o(\gamma_j) \right] J_1(\gamma_j r)/r \right\} \qquad (4.6a)$$

$$\sigma_\theta = \sum_j \frac{a_j e^{-\gamma_j z}}{N_j} \left\{ -(1-2\nu)\gamma_j^2 J_1(\gamma_j) J_o(\gamma_j r) \right.$$

$$\left. + \left[2(1-\nu)\gamma_j J_1(\gamma_j) + \gamma_j^2 J_o(\gamma_j) \right] J_1(\gamma_j r)/r \right\} \qquad (4.6b)$$

$$\sigma_z = \sum_j \frac{a_j e^{-\gamma_j z}}{N_j} \left\{ \left[2\gamma_j^2 J_1(\gamma_j) - \gamma_j^3 J_o(\gamma_j) \right] J_o(\gamma_j r) - \gamma_j^3 J_1(\gamma_j) r J_1(\gamma_j r) \right\} \qquad (4.6c)$$

$$\tau = \sum_j \frac{a_j e^{-\gamma_j z}}{N_j} \left\{ \gamma_j^3 J_1(\gamma_j) r J_o(\gamma_j r) - \gamma_j^3 J_o(\gamma_j) J_1(\gamma_j r) \right\} \tag{4.6d}$$

$$u = \frac{1+\nu}{E} \sum_j \frac{a_j e^{-\gamma_j z}}{N_j} \left\{ -\gamma_j^2 J_1(\gamma_j) r J_o(\gamma_j r) + \left[2(1-\nu)\gamma_j J_1(\gamma_j) + \gamma_j^2 J_o(\gamma_j) \right] J_1(\gamma_j r) \right\} \tag{4.6e}$$

$$w = \frac{(1+\nu)}{E} \sum_j \frac{a_j e^{-\gamma_j z}}{N_j} \left\{ \left[\gamma_j^2 J_o(\gamma_j) - 2(1-\nu)\gamma_j J_1(\gamma_j) \right] J_o(\gamma_j r) + \gamma_j^2 J_1(\gamma_j) J_1(\gamma_j r) \right\} \tag{4.6f}$$

Equations (4.5) express the coefficients a_j in terms of an integral expression of the four stresses and displacements on the finite end. Only two of these stresses and displacements may be arbitrary self-equilibrating functions. The stress and displacement expressions resulting from differentiation of (4.4) will be used for the two unspecified boundary functions. These expressions can be obtained by setting z equal to zero in equations (4.6). For convenience, these functions will be written as

$$\sigma_{zb} = \sum_k a_k R_{1k} / N_k \tag{4.7a}$$

$$\tau_b = \sum_k a_k R_{2k} / N_k \tag{4.7b}$$

$$u_b = \frac{(1+\nu)}{E} \sum_k a_k R_{3k} / N_k \tag{4.7c}$$

$$w_b = \frac{(1+\nu)}{E} \sum_k a_k R_{4k} / N_k \; . \tag{4.7d}$$

Equations (4.5) will now be written in the following form

$$a_j = G_j + \frac{1}{2} \sum_k a_k S_{kj} \tag{4.8}$$

where G_j is the part of the integral in (4.5) that contains the specified boundary functions and S_{kj} is twice the part of that integral that contains the unspecified boundary conditions. For all four permissible boundary conditions, S_{jj} is unity. Thus, equations (4.5) may be written in the following form:

$$a_j = 2\, G_j + \sum_{\substack{k \\ k \neq j}} a_k S_{kj} \; . \tag{4.9}$$

To solve the first mixed boundary conditions, the following equations give the specified and unspecified boundary functions:

$$\sigma_{zb} = f(r) \qquad \tau_b = \sum_k a_k R_{2k} / N_k \tag{4.10}$$

$$u_b = \frac{(1+\nu)}{E} g(r) \qquad w_b = \frac{(1+\nu)}{E} \sum_k a_k R_{2k} / N_k \; .$$

Thus,

$$G_j = \frac{1}{(1-\nu)} \int_0^1 \left\{ f(r) W_{1j} + g(r) W_{3j} \right\} r\,dr$$

and (4.11)

$$S_{kj} = \frac{2}{(1-\nu) N_k} \int_0^1 \left\{ R_{2k} W_{2j} + R_{4k} W_{4j} \right\} r\,dr \; .$$

Because of the inner biorthogonality of the R and W functions, this mixed boundary condition yields

$$S_{kj} = 0 \quad \text{for} \quad k \neq j \tag{4.12}$$

and

$$a_j = 2\,G_j = \frac{2}{(1-\nu)} \int_0^1 \left\{ f(r) W_{1j} + g(r) W_{3j} \right\} r\,dr \,. \tag{4.13}$$

Similarly, the other mixed boundary condition exhibits an inner biorthogonality and yields the following form for a_j:

$$a_j = \frac{2}{(1-\nu)} \int_0^1 \left\{ h(r) W_{2j} + k(r) W_{4j} \right\} r\,dr \,. \tag{4.14}$$

The stress and displacement boundary conditions do not exhibit an inner biorthogonality. Equations (4.9) become an infinite set of equations for these boundary conditions. This infinite set of equations can be solved by truncation and almost any desired numerical accuracy may be obtained. For the stress boundary condition,

$$G_j = \frac{1}{1-\nu} \int_0^1 \left\{ f(r) W_{1j} + h(r) W_{2j} \right\} r\,dr \tag{4.15}$$

$$S_{kj} = \frac{2}{(1-\nu) N_k} \int_0^1 \left\{ R_{3k} W_{3j} + R_{4k} W_{4j} \right\} r\,dr \,. \tag{4.16}$$

CHAPTER V

SUMMARY AND CONCLUSIONS

A Love function of the form of a series of Bessel functions times exponential functions plus an improper integral, a Fourier integral, was considered as a possible solution. It was assumed that the series and the improper integral were convergent and that the derivatives of the Love function were equal to the corresponding term by term derivatives of the series plus the integral of the corresponding derivatives of the integrand. Thus, the solutions are eigenfunctions.

The boundary conditions assumed are that the finite end is loaded by a self-equilibrating set of tractions and/or displacements while the curved surface is taken to be stress free.

The Fourier coefficients for these eigenfunctions are derived. For mixed boundary conditions, an inner biorthogonality enables the Fourier coefficients to be expressed in terms of simple integrals. For the stress and displacement boundary conditions, the coefficients must be determined from an infinite set of equations.

Appendix B shows the decay rates of the normalized displacements due to a typical loading. The decay rates are in good agreement with those presented by Hodgkins (5).

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APPENDIX A

Table A.1

EIGENVALUES OF THE TRANSCENDENTAL EQUATION
J(0,Z)**2+(1.0-2.0*(1.0-NU)/Z**2)*J(1,Z)**2 = 0.0
WHERE Z IS THE EIGENVALUE

NO.	NU = 0.25		NU = 0.30	
1	2.6976517	1.3673571	2.7221754	1.3621971
2	6.0512223	1.6381471	6.0600832	1.6376245
3	9.2612734	1.8285342	9.2668352	1.8282558
4	12.438444	1.9674281	12.442529	1.9672411
5	15.602204	2.0764211	15.605440	2.0762837
6	18.759055	2.1660392	18.761738	2.1659331
7	21.911845	2.2421083	21.914137	2.2420232
8	25.062035	2.3081732	25.064037	2.3081039
9	28.210448	2.3665588	28.212225	2.3665000
10	31.357593	2.4188579	31.359191	2.4188075
11	34.503802	2.4662104	34.505253	2.4661672
12	37.649295	2.5094829	37.650625	2.5094444
13	40.794230	2.5493157	40.795456	2.5492823
14	43.938722	2.5862162	43.939862	2.5861865
15	47.082854	2.6205841	47.083917	2.6205567
16	50.226690	2.6527444	50.227687	2.6527209
17	53.370283	2.6829655	53.371220	2.6829436
18	56.513666	2.7114657	56.514552	2.7114457
19	59.656876	2.7384307	59.657714	2.7384124
20	62.799934	2.7640172	62.800730	2.7640009

NO.	NU = 0.35		NU = 0.40	
1	2.7456664	1.3568473	2.7681892	1.3513369
2	6.0688667	1.6369739	6.0775704	1.6361996
3	9.2723747	1.8279215	9.2778910	1.8275319
4	12.446604	1.9670226	12.450669	1.9667729
5	15.608672	2.0761262	15.611897	2.0759488
6	18.764418	2.1658130	18.767094	2.1656789
7	21.916428	2.2419278	21.918716	2.2418227
8	25.066038	2.3080256	25.068036	2.3079402
9	28.214001	2.3664353	28.215776	2.3663641
10	31.360788	2.4187528	31.362385	2.4186935
11	34.506704	2.4661199	34.508153	2.4660687
12	37.651954	2.5094036	37.653282	2.5093591
13	40.796684	2.5492461	40.797910	2.5492076
14	43.941000	2.5861540	43.942137	2.5861188
15	47.084980	2.6205282	47.086042	2.6204979
16	50.228683	2.6526952	50.229679	2.6526674
17	53.372155	2.6829199	53.373093	2.6828945
18	56.515438	2.7114240	56.516323	2.7114010
19	59.658552	2.7383933	59.659390	2.7383719
20	62.801527	2.7639833	62.802323	2.7639641

APPENDIX B

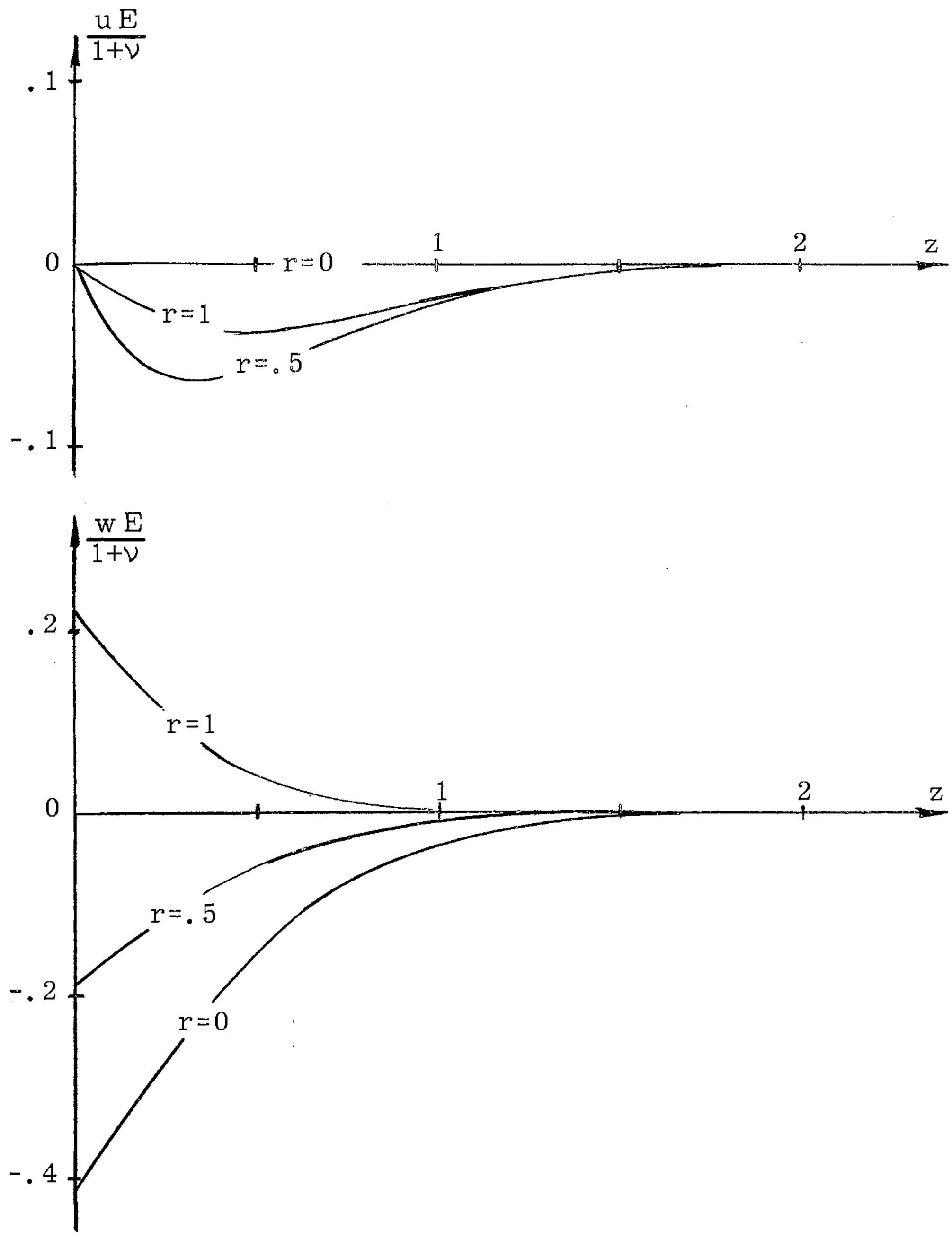

Figure B. 1 Normalized displacements due to the end loading

$\sigma_{zb} = 1 - 2r^2$, $u_b = 0$.

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