ON A METHOD OF CONSTRUCTING PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN

By

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CHAPTER I

INTRODUCTION

Statistical analysis of many types of experimental data may be facilitated by proper planning of the experiment. Partially Balanced Incomplete Block Designs (PBIBD's) are a particular class of arrangements for this purpose. A simple example will be used to illustrate some of the concepts involved.

The average yields of seven new varieties of corn are to be compared in a field experiment. A possible plan is to divide the available land into seven plots and to plant one variety in each plot, as indicated by the following figure.

1	2	3	4	5	6	7

Throughout this example, varieties (treatments) will be indicated by numbers 1 to 7. Under conditions of strict control of soil, fertility, water supply, drainage, and other extraneous factors, this might furnish the desired information on the treatment differences, but in experiments in biological and social sciences such control is not usually possible. It will be impossible with this arrangement to know whether an observed difference between two plots can be attributed to differences in the two varieties or whether it is due to differences between plots of ground. If the effects of extraneous factors cannot be controlled, the next best thing to do is estimate their importance.

This can be done by planting several plots of each variety and observing the variation among them. It is intuitively reasonable and proves to simplify analysis of the data to plant the same number of plots of each variety so that in effect we have a number of repetitions, or replications, of the original experiment. Three replications will be used in this example. Comparison of the varieties grown under similar conditions will be easier if the 21 plots are grouped into blocks of seven plots, each block to contain a complete replication. Soil conditions are likely to be more homogeneous within a block than over the entire experimental area and will have correspondingly small effect on comparisons made within a block. The blocks may or may not be contiguous in the field. This design is indicated by the following diagram. A defect of this plan is that the same arrangement of

Block 1:	1	2	3	4	5	6	7
Block 2:	1	2	3	4	5	6	7
Block 3:	1	2	3	4	5	6	7

varieties is used in each block, so that effects of location within blocks may be impossible to distinguish from differences between varieties. For instance, an observed difference between varieties 1 and 7 could have been caused by a gradient in soil fertility from left to right. Other extraneous sources of variation which are less obvious may introduce a similar bias in favor of certain varieties. To insure that no variety or group of varieties will be systematically favored in all replications of the experiment, a device known as randomization may be used. In the example this would mean assigning the number 1 to 7 to each block in such a way that each of the 7 !

possible arrangements is equally likely to result. The effect is that in each replication, each variety has an equal chance of being tested under favorable conditions. While the results of any particular randomization may favor certain treatments, this happens only to an extent that can be allowed for in the analysis and interpretation of data.

The plan that results is called a randomized complete block design. It might appear as follows:

Block 1:	6	2	3	5	4	7	1
Block 2:	2	5	7	4	1	6	3
Block 3:	3	6	5	7	1	4	2

R. A. Fisher (12) was the first to realize the importance of randomization as a scientific technique and to introduce it into designs for experiments.

It frequently happens that, within a block which includes an entire replication of an experiment, there is too much variability of conditions to allow useful measurements to be made. This may make it necessary to arrange the experimental plots in blocks of smaller size, with direct comparisons made only between varieties in the same block. In the example it is supposed that it is necessary to reduce the block size to three plots. There is some loss of information here, as suggested by the fact that the number of possible direct comparisons is reduced from $3\binom{7}{2} = 63$ to $7\binom{3}{2} = 21$, but the gain in precision of comparisons may more than offset this. If some of the comparisons are less important than others, it may be possible to arrange the blocks so that the unimportant information is lost and the important information is mostly retained. However, in many situations all comparisons may be considered equally important; it will be assumed in this example that information is desired on the comparative yields of each pair of varieties. The term incomplete block design covers any experimental design in which the blocks are of a size smaller than the number of treatments, while the term Balanced Incomplete Block Design (BIBD) is used for the important special case in which equal amount of information is retained on each pair of treatments. A BIBD may be defined as an arrangement of t varieties or treatments into b blocks each containing k distinct varieties, each variety being used the same number of times r, and each pair of distinct varieties occurring in all blocks the same number λ of times. It is easily verified that the following arrangement of the example satisfies these requirements, with $t = b = 7$, $r = k = 3$, $\lambda = 1$.

Block 1:	1, 2, 3	Block 5:	2, 5, 7
2:	1, 4, 5	6:	3, 4, 7
3:	1, 6, 7	7:	3, 5, 6
4:	2, 4, 6		

Randomization would be applied to this design by assigning the numbers 1, 2, . . . , 7 to the varieties at random, assigning the three numbers in each block to the three plots in a random way, and assigning the blocks to the seven positions in the field by a third random process.

BIBD's were introduced by Yates (26) in 1936. The construction of a BIBD for a given set of values of t, k, b, r, λ is a combinatorial problem which may be considered apart from the analysis of experimental data. It is clear that the five parameters are not all independent. Considering the total number of plots gives $rt = bk$, and by counting pairs of varieties two ways $\lambda\binom{t}{2} = b\binom{k}{2}$ is obtained. These two results may be combined to give a more useful result

$$\lambda = r(k - 1)/(t - 1)$$

Other necessary conditions for the existence of these designs have been obtained, along with some methods for constructing large classes of them. In 1938, Fisher and Yates (13) published all BIBD's then known, with a list of the possible parameters of other designs of practical interest. The construction of many of these designs was made possible by methods introduced by R. C. Bose (6) in 1939.

The set of existing BIBD's was soon found to be inadequate for the needs of the experiments. A simple case in which no convenient balanced design is available is obtained from the first example by considering eight varieties of corn instead of seven, again to be planted in blocks of three plots. With $t = 8$ and $k = 3$, the smallest value of r which can be used to give integral values of b and λ is found to be 21, and the blocks of the design are all combinations of eight varieties three at a time. It was to provide useful designs for such values of t and k that arrangements like the following were introduced.

Block	1:	1, 2, 3	Block	5:	2, 5, 8
	2:	1, 4, 6		6:	3, 4, 5
	3:	1, 7, 8		7:	3, 6, 8
	4:	2, 4, 7		8:	5, 6, 7

This is not a balanced design because the pairs of distinct varieties do not all occur equally often. Every pair occurs once with the exceptions (1, 5), (2, 6), (3, 7), (4, 8), which do not appear at all in the same block. The remaining requirements for a balanced design are satisfied. This is an example of a Partially Balanced Incomplete Block Design (PBIBD) which may be defined as an experimental plan

(1) having t treatments arranged in b blocks such that each block contains k experimental units $(k < t)$,

(2) where each treatment is replicated r times and no treatment occurs more than once in any block,

(3) such that with respect to any treatment T, the remaining treatments can be divided into m associate classes such that the i^{th} class contains n_i treatments and T occurs in λ_i blocks with each of the treatments in the i^{th} class,

(4) such that if two treatments are i^{th} associates, the number of treatments common to the j^{th} associates of one and the k^{th} associates of the other is p^i_{jk} $(i, j, k = 1, 2, \ldots, m)$, with $p^i_{jk} = p^i_{kj}$, and is independent of the particular pair of treatments.

All parameters except the p^i_{jk} are referred to as parameters of the first kind; the p^i_{jk} are called parameters of the second kind.

PBIBD's were introduced by Bose and Nair (6) in 1939. They are generalizations of BIBD's and include them as a special case. The methods used to construct PBIBD's are many and varied. The paper by Bose and Nair (6) gives many construction devices. Bose and Connor (4) employ the device of replacing each treatment of a BIBD with a set of n treatments to construct a certain subset of PBIBD's. P. M. Roy (19) was first to state that if this procedure were employed with respect to a PBIBD, then another PBIBD was generated. M. Zelen (27) later proved the same theorem. More will be said concerning methods of construction in Chapter III.

The procedure of replacing a treatment of a BIBD or PBIBD by n treatments holds a special significance to the material of this thesis in that the present work might be regarded as a generalization of this procedure. The central objective of this thesis is to investigate the ramifications of replacing each treatment of a BIBD or PBIBD by a BIBD or a PBIBD. This procedure will be called "composition" and will be defined later.

Chapter II contains definitions, abbreviations, and theorems from the literature which will be used throughout. Chapter III contains a review of literature with some examples of construction methods illustrated. Chapter IV contains the main theorem relative to the composition of two BIBD's, as well as, the relationship of composition to the types of Partially Balanced Incomplete Block Designs having two associate classes, PBIBD (2). Chapter V contains some results relating complementary designs and composition. In Chapter VI are found the main theorems relating the composition of BIBD's and PBIBD's, as well as, PBIBD's and PBIBD's. Chapter VII is a summary with some conjectures about further research.

CHAPTER II

SOME PRELIMINARY DEFINITIONS AND THEOREMS

Certain abbreviations, definitions, and theorems to be used in the sequel are stated in this chapter. The theorems are given without proof, with the appropriate references noted.

Incomplete Block Design is abbreviated IBD; Balanced Incomplete Block Design, BIBD; Partially Balanced Incomplete Block Design, PBIBD. A Partially Balanced Incomplete Block Design having m associate classes is abbreviated PBIBD(m).

Definition 2. 1: An IBD is an experimental design in which the block size is smaller than the number of treatments.

Definition 2. 2: A BIBD is an experimental plan

(1) having t treatments arranged in b blocks such that each block contains k experimental units $(k < t)$,

(2) where each treatment is replicated r times and no treatment occurs more than once in any block,

(3) where every pair of treatments occurs in the same number of blocks; this number is denoted by λ.

Definition 2. 3: A PBIBD(m) is an experimental plan

(1) having t treatments arranged in b blocks such that each block contains k experimental units $(k < t)$,

(2) where each treatment is replicated r times and no

treatment occurs more than once in every block,

(3) such that with respect to any treatment T, the remaining treatments can be divided into m associate classes such that the i^{th} class contains n_i treatments and T occurs in λ_i blocks with each of the treatments in the i^{th} class (i = 1, 2, . . . , m),

(4) such that if two treatments are i^{th} associates, the number of treatments common to the j^{th} associates of one and the k^{th} associates of the other is p^i_{jk} (for i, j, k = 1, 2, . . . , m), with $p^i_{jk} = p^i_{kj}$, and is independent of the particular pair of treatments.

It has been shown that the following relations hold between the parameters of the design:

$$bk = rt$$

$$\sum_{i=1}^{m} n_i = t - 1$$

$$\sum_{i=1}^{m} n_i \lambda_i = r(k - 1)$$

$$\sum_{k=1}^{m} p^i_{jk} = \begin{cases} n_i & \text{for } i \neq j \\ n_i - 1 & \text{for } i = j \end{cases}$$

$$n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij}.$$

Definition 2.4: Two PBIBD's are said to be equivalent if they differ only in the naming of their associate classes; i.e., all parameters are identical except those which depend on the names given the associate classes.

Definition 2.5: A PBIBD (2) is said to be Group Divisible (GD) if t = mn, and the treatments can be divided into m groups of n each, such that any two treatments of the same group are first associates while two treatments from different groups are second associates.

Theorem 2.1: A necessary and sufficient condition for a PBIBD to be GD is the vanishing of p^1_{12} or p^2_{12}. If $p^i_{12} = 0$ then the treatments in the same group are i^{th} associates, $(i = 1, 2)$.

Definition 2.6: A PBIBD (2) is said to be Simple (S1) if $\lambda_1 \neq 0$, $\lambda_2 = 0$.

Definition 2.7: A PBIBD (2) is said to be Triangular if the number of treatments $t = n(n-1)/2$ and the association scheme is an array of n rows and n columns with the following properties.

(1) The positions in the principal diagonal are left blank.

(2) The $n(n-1)/2$ positions above the principal diagonal are filled by the numbers $1, 2, \ldots, n(n-1)/2$ corresponding to the treatments.

(3) The $n(n-1)/2$ positions below the principal diagonal are filled so that the array is symmetrical about the principal diagonal.

(4) For any treatment i the first associates are exactly those treatments which lie in the same row (or in the same column) as i.

The following relations hold:

$$n_1 = 2n - 4, \qquad n_2 = (n-2)(n-3)/2$$

$$(p^1_{jk}) = \begin{bmatrix} n-2 & n-3 \\ n-3 & (n-3)(n-4)/2 \end{bmatrix}.$$

$$(p^2_{jk}) = \begin{bmatrix} 4 & 2n-8 \\ 2n-8 & (n-4)(n-5)/2 \end{bmatrix}$$

Definition 2.8: If in a non-GD PBIBD (2) having n^2 treatments it is possible to form a square array of n rows and n columns filled with numbers $1, 2, \ldots, n^2$ corresponding to the n^2 treatments so that two treatments are first associates if they occur in the same row or same column of the array and are second associates otherwise, then the design is said to belong to the sub-type L_2 of the Latin Square type design.

Theorem 2.2: If the parameters of the second kind for a PBIBD (2) with S^2 treatments are given by $n_1 = 2S - 2$, $p^1_{11} = S - 2$, $p^2_{11} = 2$, then the design has a L_2 association scheme if and only if $S = 2, 3$ or $S > 4$. If $S = 4$ the condition is necessary but not sufficient.

Definition 2.9: Consider a PBIBD (2) having parameters $t, k, b, r, \lambda_i, (p^i_{jk}), i, j, k = 1, 2$. Let the treatments be designated by integers $1, 2, \ldots, t$. The design is said to be Cyclic if the first associates of the treatment i are $i + d_1, i + d_2, \ldots, i + d_{n_1}$ (mod t) where the d's satisfy the conditions:

(1) the d's are all different and $0 < d_j < t$ for $j = 1, 2, \ldots, n_1$,

(2) among the $n_1(n_1 - 1)$ differences $d_j - d_{j'}$, $j, j' = 1, 2, \ldots, n_1$, $j \neq j'$, reduced mod t each of the numbers $d_1, d_2, \ldots, d_{n_1}$ occurs A times, whereas each of the numbers $e_1, e_2, \ldots, e_{n_2}$ occurs B times where $d_1, d_2, \ldots, d_{n_1}, e_1, e_2, \ldots, e_{n_2}$ are all the different $t - 1$ numbers $1, 2, \ldots, t-1$. Necessarily $n_1 A + n_2 B = n_1(n_1 - 1)$.

Theorem 2.1 is due to Bose and Connor (4); Theorem 2.2 is due

S. S. Shrikhande (23); Definition 2.9 is given by Bose and Shimamoto (8).

Theorem 2.3: If in a BIBD having parameters t^*, k^*, b^*, r^*, λ^* each treatment is replaced by a group of n treatments, the resulting design is a Singular GD design with parameters $t = nt^*$, $k = nk^*$, $b = b^*$, $r = r^*$, $\lambda_1 = r^*$, $\lambda_2 = \lambda^*$, $m = v^*$, $n = n$. Conversely, every Singular GD design is obtainable in this way from a corresponding BIBD.

Theorem 2.4: If, in a PBIBD(m) having parameters t^*, k^*, b^*, r^*, λ_i^*, n_i^*, p_{ij}^{*k} (i, j, k = 1, 2, . . . , m), such that $\lambda_i^* \neq r^*$ (i = 1, 2, . . . , m), each treatment is replaced by n different treatments, the derived design will be a PBIBD (m + 1) having parameters

$$t = nt^*, \qquad k = nk^*, \qquad b = b^*, \qquad r = r^*$$

$$\lambda_i = \lambda_i^*, \qquad n_i = nn_i^*, \qquad i = 1, 2, \ldots, m,$$

$$\lambda_{m+1} = r^*, \qquad n_{m+1} = n - 1$$

$$p_{ij}^k = np_{ij}^{*k} \qquad i, j, k = 1, 2, \ldots, m,$$

$$p_{k,m+1}^k = n - 1 \qquad k = 1, 2, \ldots, m,$$

$$p_{i,m+1}^k = 0 \qquad i \neq k \quad i, k = 1, 2, \ldots, m,$$

$$p_{ij}^{m+1} = 0 \quad i \neq j, \quad p_{ii}^{m+1} = nn_i^* \text{ for } i = 1, 2, \ldots, m,$$

$$p_{m+1,m+1}^{m+1} = n - 2$$

Theorem 2.3 is the work of Bose and Connor (4); Theorem 2.4 was first given by P. M. Roy (19), later being proved by M. Zelen (27).

Chapter II has given some of the tools necessary for the work which follows. In the next chapter is found a review of some methods for constructing PBIBD's.

CHAPTER III

CONSTRUCTION METHODS

Of the many methods used to construct PBIBD's those set forth by Bose and Nair (6) in the paper introducing the PBIBD seem to be the most often systematized and generalized. Not only are the methods given in that paper ones which have often been enlarged upon, but they constitute the largest number of methods found in the literature in any one paper. In the following paragraphs methods of construction are explained and illustrated with examples. Those methods which do not have their source given explicity belong to Bose and Nair (6). It is not presumed that this list is complete.

Geometrical Configurations

Simple geometrical configurations often yield designs of interest. Consider the Pappus configuration of nine points and nine lines illustrated by Figure 1. Considering the lines as blocks and points as treatments gives the following nine blocks: (1, 2, 3), (4, 5, 6), (7, 8, 9), (1, 7, 5), (2, 9, 6), (1, 8, 6), (2, 7, 4), (3, 9, 5), (3, 8, 4). The parameters for this design are:

$$t = 9, \quad k = 3, \quad b = 9, \quad r = 3,$$

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad n_1 = 6, \quad n_2 = 2,$$

$$(p_{ij}^1) = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}, \quad (p_{ij}^2) = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$$

The simplest space configurations are provided by the regular polyhedra. PBIBD's may be obtained from these by considering the faces as blocks and vertices as treatments. Thus, the following six blocks are obtained from the configuration of Figure 2: (1, 2, 3, 4), (5, 6, 7, 8), (1, 4, 8, 5), (2, 3, 7, 6), (1, 2, 6, 5), (4, 3, 7, 8). The parameters for this design are as follows:

$$t = 8, \quad k = 4, \quad r = 3, \quad b = 6,$$

$$\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 0, \quad n_1 = 3, \quad n_2 = 3, \quad n_3 = 1,$$

$$(p_{ij}^1) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad (p_{ij}^2) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad (p_{ij}^3) = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

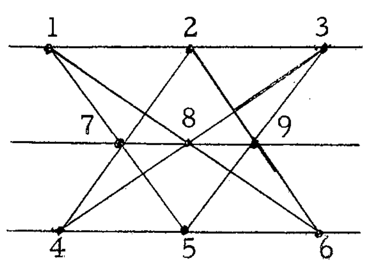

Figure 1: Pappus

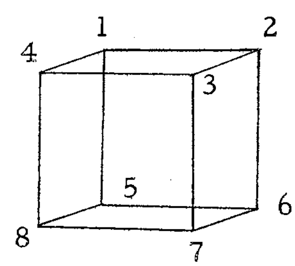

Figure 2: Cubic

Other PBIBD's which can be obtained from geometrical configurations are given in Table I.

TABLE I

DESIGNS FROM GEOMETRICAL CONFIGURATIONS

Geometrical Configuration	t	r	b	k	λ_1	λ_2	n_1	n_2
Desargue, 10 pts., 10 lines	10	3	10	3	1	0	6	3
Complex cube roots unity, 8 pts., 8 lines	8	3	8	3	1	0	6	1
Octahedron, 6 pts., 8 triangles	6	4	8	3	2	0	4	1
Icosahedron, 12 pts., 20 triangles	12	5	20	3	2	0	5	6
Two tetrahedra, 8 pts., 8 planes	8	4	8	4	2	0	6	1

Applications of Finite Geometry

The finite geometries $PG(N, p^n)$ and $EG(N, p^n)$, i.e., the Projective and Euclidean N-dimensional geometries associated with the Galois field $GF(p^n)$ provide many configurations leading to PBIBD's.

A brief review of terminology will be given before proceeding with methods and examples of construction. (1) Any ordered set of N elements $(x_1, \ldots, x_N)$ belonging to $GF(p^n)$ may be called a point of the finite N-dimensional Euclidean geometry $EG(N, p^n)$. The number of points in $EG(N, p^n)$ is s^N where $s = p^n$. All points which satisfy a set of N-m, consistent and independent linear equations may be said to form an m-flat of $EG(N, p^n)$ represented by these equations. (2) Any ordered set of $N + 1$ elements $(x_1, x_2, \ldots, x_{N+1})$ where the x_i's belong to $GF(p^n)$ and are not all simultaneously zero, may be called a point of the finite N-dimensional projective geometry $PG(N, p^n)$, it being understood that the set

$(x_1, x_2, \ldots, x_{N+1})$ represents the same point as the set $(y_1, y_2, \ldots, y_{N+1})$ if and only if there is a $d \neq 0$ of $GF(p^n)$ such that $y_i = dx_i$ for $i = 1, 2, \ldots, N+1$. The number of points in $PG(N, p^n)$ is $s^N + s^{N-1} + \ldots + s + 1 = (s^{N+1} - 1)/(s - 1)$.

All points which satisfy a set of $N - m$ independent linear homogeneous equations may be said to form an m-flat in $PG(N, p^n)$ represented by these equations. (3) Whichever of the two geometries $EG(N, p^n)$ or $PG(N, p^n)$ is considered, as usual, a 1-flat is called a line, and a 2-flat, a plane. Setting

$$\phi(N, m, s) = \frac{(s^{N+1}-1)(s^N-1)(s^{N-1}-1)\ldots(s^{N-m+1}-1)}{(s^{m+1}-1)(s^m-1)(s^{m-1}-1)\ldots(s-1)},$$

it can be shown that the number of m-flats in $PG(N, p^n)$ is $\phi(N, m, s)$ and the number of m-flats in $EG(N, p^n)$ is $\phi(N, m, s) - \phi(N-1, m, s)$.

Suppose from the space $EG(N, p^n)$ one point is deleted, namely the origin $(0, 0, \ldots, 0)$, and all the $(N - m)$-flats passing through this point. Take the retained $(N-m)$ flats as blocks and the retained points as treatments, a treatment occurring in a block when and only when the corresponding point occurs on the corresponding $(N-m)$-flat.

Consider the particular case $N = 2$, $m = 1$. The number of retained points, as well as of retained lines, is $s^2 - 1$, where $s = p^n$. Hence, $b = t = s^2 - 1$. On each of the retained lines there lie s points, and through each retained point there pass s retained lines, as the one joining the point to the origin is to be rejected. Thus, $r = k = s$. Two points (treatments) are first or second associates according as the line joining them does or does not pass through the origin. To every retained point there are thus s^2-s first associates, and $s - 2$ second associates. Thus, $\lambda_1 = 1$, $n_1 = s^2 - s$, $\lambda_2 = 0$, $n_2 = s - 2$. Let 0 be the origin and P and Q be any two first associates. Then all points lying on lines other than PO and QO are common first associates of P and Q. Thus $p_{11}^1 = (s-1)^2$. In the same way the values of other parameters of the second kind are determined. Thus, the following designs are obtained.

$$t = s^2 - 1, \; k = s, \; b = s^2 - 1, \; r = s$$

$$\lambda_1 = 1, \; n_1 = s^2 - s, \; \lambda_2 = 0, \; n_2 = s - 2$$

$$(p_{ij}^1) = \begin{bmatrix} (s-1)^2 & s - 2 \\ s - 2 & 0 \end{bmatrix}, \quad (p_{ij}^2) = \begin{bmatrix} s^2 - s & 0 \\ 0 & s - 3 \end{bmatrix}.$$

Method of Differences

Bose (6) originally applied the method of differences to construct PBIBD's where $b = t$, $k = r$. Sprott (24), (25), generalized some of this work and consequently produced many series of PBIBD's. Chowla and Ryser's (10) study of the combinatorial problem of how to arrange t elements into t sets such that every set contains exactly k distinct elements and such that every pair of sets has exactly $\lambda = k(k-1)/(t-1)$ elements in common $(0 < \lambda < k < t)$ contributed to the work of Sprott (25).

A set of elements is said to form a module M, when there exists a law of composition, viz. the addition, denoted by +, satisfying the following axioms:

(1) To any two elements a and b of M, there exists a unique element s of M defined by $a + b = s$

(2) $a + b = b + a$

(3) $a + (b + c) = (a + b) + c$

(4) To any two elements a and b of M there exists an element x belonging to M, satisfying $a + x = b$.

On the basis of these axioms it can be proved that the element x in (4) is unique. Also there is a unique element 0 with the property that c being any element of M, $c + 0 = c$. If $c + d = 0$, d is denoted by $-c$. $a + (-c)$ may be denoted $a - c$. The element x in (4) is the equal to $b - a$, and may be said to be the difference of b and a. The method of differences has its basis in the following theorem.

Theorem 3.1: Consider a finite module with exactly t elements. Suppose it is possible to find k different elements, $x_1, x_2, \ldots, x_k$, out of the t elements of M satisfying the following:

(1) Among the $k(k-1)$ differences $x_i - x_j$, $(i, j = 1, 2, \ldots, k;\ i \neq j)$, just n_i of the nonzero elements of M are repeated λ_i times $(i = 1, 2, \ldots, m)$. Clearly in this case

$$n_1 + n_2 + \ldots + n_m = t - 1$$

$$n_1\lambda_1 + n_2\lambda_2 + \ldots + n_m\lambda_m = k(k-1)$$

(2) Denote by $a^i_1, a^i_2, \ldots, a^i_{n_i}$, the elements of M, which occur just λ_i times among the differences $x_i - x_j$ $(i, j = 1, 2, \ldots, k;\ i \neq j)$. Then among the $n_i(n_i - 1)$ differences $a^i_u - a^i_w$ $(u, w = 1, 2, \ldots, n_i;\ u \neq w)$ every number of the set $a^q_1, a^q_2, \ldots, a^q_{n_q}$ should be repeated exactly p^q_{ii} times $(q = 1, 2, \ldots, m)$. Also among the $n_i n_j$ differences

$a_u^i - a_w^j$ ($u = 1, 2, \ldots, n_i$; $w = 1, 2, \ldots, n_j$), the numbers of the set $a_1^q, a_2^q, \ldots, a_{n_q}^q$ occur exactly p_{ij}^q times ($q = 1, 2, \ldots, m$; $i, j = 1, 2, \ldots, m$; $i \neq j$). When these conditions are satisfied, the design in which t treatments are t elements of M, and t blocks are $x_1 + \theta, x_2 + \theta, \ldots, x_k + \theta$ where θ is any one of the elements of M, is a PBIBD with $t = b$, $r = k$, n_i, λ_i as the parameters of the first kind and p_{ij}^q as the parameters of the second kind.

The following is an example of the theorem. Let $t = 15$. Consider the classes of residues (mod 15). Denote the 15 treatments as 0, 1, . . . , 14. Let $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 8$. Then the 12 differences $x_i - x_j$ ($i, j = 1, 2, 3, 4$; $i \neq j$) are 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14. Denote these by $a_1^1, a_2^1, \ldots, a_{12}^1$. Denote 5 and 10 by a_1^2, a_2^2 respectively. Thus, the numbers of the set $(a_1^1, a_2^1, \ldots, a_{12}^1)$ occur once and the numbers of the set (a_1^2, a_2^2) occur zero times in the differences $x_i - x_j$. Call these sets, the sets I and II respectively. Hence, $\lambda_1 = 1$, $\lambda_2 = 0$, $n_1 = 12$, $n_2 = 2$.

Now among the 132 differences $a_u^1 - a_w^1$ ($u, w = 1, 2, \ldots, 12$; $u \neq w$), the numbers of set I each occurs 12 times. Among the two differences $a_u^2 - a_w^2$ ($u, w = 1, 2$; $u \neq w$), each number of set I occurs zero times, and each number of set II occurs once. Finally in the 24 differences $a_u^1 - a_w^2$ ($u = 1, 2, \ldots, 12$; $w = 1, 2$) each number of the set I occurs twice, and each number of the set II occurs zero times. By taking the 15 blocks $1 + \theta$, $2 + \theta$, $4 + \theta$, $8 + \theta$, where $\theta = 0, 1, \ldots, 14$, the design with parameters as follows is obtained.

$$t = 15, \quad k = 4, \quad b = 15, \quad r = 4$$

$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad n_1 = 12, \quad n_2 = 2$$

$$(p^1_{ij}) = \begin{bmatrix} 9 & 2 \\ 2 & 0 \end{bmatrix}, \qquad (p^2_{ij}) = \begin{bmatrix} 12 & 0 \\ 0 & 1 \end{bmatrix}.$$

The complete design can be written as follows:

(1, 2, 4, 8), (2, 3, 5, 9), (3, 4, 6, 10), (4, 5, 7, 11), (5, 6, 8, 12), (6, 7, 9, 13), (7, 8, 10, 14), (8, 9, 11, 0), (9, 10, 12, 1), (10, 11, 13, 12), (11, 12, 14, 3), (12, 13, 0, 4), (13, 14, 1, 5), (14, 0, 2, 6), (0, 1, 3, 7).

Miscellaneous Methods

If $t\ (=pq)$ is composite, a PBIBD having pq blocks may be constructed by forming a rectangular lattice with these treatments, having p rows and q columns. Every block has a treatment associated with it and will comprise that treatment and all treatments placed in the same row and column as that treatment. Assuming $p > q > 2$ the parameters of such a design are as follows:

$$t = b = pq \qquad r = k = p + q - 1$$

$$\lambda_1 = p \qquad n_1 = p - 1$$

$$\lambda_2 = q \qquad n_2 = q - 1$$

$$\lambda_3 = 2 \qquad n_3 = (p-1)(q-1)$$

$$(p^1_{jk}) = \begin{bmatrix} p-2 & 0 & 0 \\ 0 & 0 & q-1 \\ 0 & q-1 & (p-2)(q-1) \end{bmatrix}$$

$$(p_{jk}^{2}) = \begin{bmatrix} 0 & 0 & p-1 \\ 0 & q-2 & 0 \\ p-1 & 0 & (p-1)(q-2) \end{bmatrix}$$

$$p_{jk}^{3}) = \begin{bmatrix} 0 & 1 & p-2 \\ 1 & 0 & q-2 \\ p-2 & q-2 & (p-2)(q-2) \end{bmatrix}$$

It may be seen that if $q = 2$, the preceding design degenerates into a PBIBD (2).

If in the above PBIBD (3) blocks had been formed by taking all treatments in the same row and column as that treatment, except itself, the parameters would be:

$$t = b = pq \qquad r = k = p + q - 2$$

$$\lambda_1 = p - 2 \qquad n_1 = p - 1$$

$$\lambda_2 = q - 2 \qquad n_2 = q - 1$$

$$\lambda_3 = 2 \qquad n_3 = (p - 1)(q - 1)$$

Parameters of the second kind are the same as those of the previous design.

If $t\ (=p^2)$ is a perfect square, designs can be constructed by forming blocks such that with respect to every treatment a block is formed with all treatments occurring in the same row, column, and having the same Latin letter as itself in each of s orthogonalized squares ($s = 0, 1, \ldots, p-1$). If each treatment is included in the block associated with it, the parameters of the design are:

$$t = b = p^2 \qquad r = k = (s+2)p - (s+1)$$

$$\lambda_1 = p + s(s+1) \qquad n_1 = (s+2)(p-1)$$

$$\lambda_2 = (s + 1)(s + 2), \qquad n_2 = (p - 1)(p - s - 1),$$

$$(p^1_{jk}) = \begin{bmatrix} p + (s + 2)(s - 1) & (s + 1)(p - s - 1) \\ (s + 1)(p - s - 1) & (p - s - 1)(p - s - 2) \end{bmatrix}$$

$$(p^2_{jk}) = \begin{bmatrix} (s + 1)(s + 2) & (s + 2)(p - s - 2) \\ (s + 2)(p - s - 2) & (p - s - 2)^2 + s \end{bmatrix}$$

If in the above design, the treatment associated with each block is deleted from it, the parameters become:

$$t = b = p^2 \qquad r = k = (s + 2)(p - 1)$$

$$\lambda_1 = p - 2 + s(s + 1) \qquad n_1 = (s + 2)(p - 1)$$

$$\lambda_2 = (s + 1)(s + 2) \qquad n_2 = (p - 1)(p - s - 1)$$

Parameters of the second kind remain unaltered.

Designs can be obtained by interchanging blocks and treatments. In a BIBD or PBIBD number the treatments $1, 2, \ldots, t$ and blocks $1, 2, \ldots, b$. Call treatment 1 block 1 and vice versa; in some cases a design with t blocks and b treatments, r plots per block and k replications of each treatment is formed. This procedure is referred to as inversion. For example, the inverse of the BIBD having parameters $t = 6$, $b = 10$, $r = 5$, $k = 3$, $\lambda = 2$ is a PBIBD having parameters $t = 10$, $k = 5$, $b = 6$, $r = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$, $n_1 = 3$, $n_2 = 6$, $(p^1_{jk}) = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}$, $(p^2_{jk}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

Bose and Nair (6) gave five examples of this type. Subsequently a few examples of PBIBD's obtained by the method of inversion from the corresponding BIBD's have been given by Nair (14), (15), (16), and Bose (3). Roy (19) produced some general results which not only cover the particular cases discussed by Bose and Nair (6), Nair (14), (15), (16), and Bose (3), but provide general solutions for a large number of PBIBD's. Shrikhande (21) obtained at least two of the same results as Roy (19) in a paper submitted just slightly later. Roy and Laha (20) obtained a necessary and sufficient condition for the inverse of a BIBD to be a PBIBD (2). Rao (17) found a general result regarding the inverse of a BIBD, and the results of Shrikhande (21) and Roy (19) are obtained as special cases.

A paper by Bose, Shrikahnde, and Bhattacharya (9) is devoted to constructing Group Divisible PBIBD's. Some of the methods employed coincide with those already mentioned; those which do not are listed.

A method referred to as "omitting varieties" is contained in the following:

Theorem 3.2: By omitting a particular treatment θ from a BIBD with parameters t^*, k^*, b^*, r^*, $\lambda^* = 1$, a GD PBIBD is obtained having parameters $t = t^* - 1$, $k = k^*$, $b = b^* - r^*$, $r = r^* - 1$, $m = r^*$, $n = k^* - 1$, $\lambda_1 = 0$, $\lambda_2 = 1$. Two treatments belong to the same group if they occur in the same block as θ.

A method referred to as "addition of GD designs" consists of getting a new GD design by taking together the blocks of two suitable GD designs with the same t and k.

Clatworthy (11) has given some construction methods for PBIBD's with $k = 2$, $0 < r \leq 10$ having two associate classes. The following example illustrates a technique employed by him to construct GD PBIBD's of a specified type. Before proceeding, however, observe that from the definition of a GD PBIBD, and from the fact that $k = 2$, each treatment of a group is paired with each of the other $n - 1$ treatments of the same group λ_1 times and with each of the treatments of the other $m - 1$ groups λ_2 times to form the blocks of the design. Now the construction will be given for the design with parameters $t = 6$, $k = 2$, $b = 9$, $r = 3$, $m = 2$, $n = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$.

Let the six treatments be represented by the integers 1, 2, 3, 4, 5, 6, and let the $m = 2$ groups be <u>1, 2, 3,</u> and <u>4, 5, 6.</u> By the rule just given for forming blocks, no treatment of any group can occur in a block with any other treatment belonging to the same group (since $\lambda_1 = 0$), and each treatment must occur once in a block with each of the treatments not in its group (since $\lambda_2 = 1$). Thus, the $b = 9$ blocks of the design are the treatment pairs in the following columns:

1	1	1	2	2	2	3	3	3
4	5	6	4	5	6	4	5	6

Recall that in a triangular type PBIBD two treatments lying in the same column are first associates, whereas treatments that do not, are second associates. A construction for the triangular design having parameters $t = 10$, $k = 2$, $b = 30$, $r = 6$, $\lambda_1 = 1$, $\lambda_2 = 0$ is given by use of the association scheme for the design and the rules for the formation of the blocks stated earlier. Its association scheme is given on the next page.

*	1	2	3	4
1	*	5	6	7
2	5	*	8	9
3	6	8	*	10
4	7	9	10	*

Since $\lambda_1 = 1$ and $\lambda_2 = 0$, the 30 blocks are formed by writing down all paris of numbers lying in the same row (or column) of the association scheme.

1 2	1 5	2 5	3 6	4 7
1 3	1 6	2 8	3 8	4 9
1 4	1 7	2 9	3 10	4 10
2 3	5 6	5 8	6 8	7 9
2 4	5 7	5 9	6 10	7 10
3 4	6 7	8 9	8 10	9 10

Similar methods for other types of PBIBD's are also given.

Archbold and Johnson (2) use a variation on the method based on incidence properties of finite geometries. Whereas in finite projective geometry the coordinates are elements of a finite field, they allow the coordinates to belong to a linear associative algebra, of finite order n and with modulus over a finite field F. Even the simplest example of this construction is too lengthy for presentation here.

Bose and Ray-Chaudhuri (7) have shown how to apply the geomtry of quadrics in finite projective hyperspace to construct some series of PBIBD's having 2 and 3 classes. A brief explanation and example follow.

Let (C) and (D) be two classes of linear spaces such that spaces of a given class stand in the same geometrical relation to a quadric Q in PG(n, s), $s = p^m$ where p is a prime. Then the incidence relationship of (C) and (D), provides a PBIBD in many instances. For example, if (C) is taken as the class of points on a non-degenerate quadric Q, and (D) as the class of lines contained in Q, then a PBIBD with the following parameters is produced: $t = N(0, n)$, $k = s + 1$, $b = N(1, n)$, $r = N(0, n-2)$, $\lambda_1 = 1$, $\lambda_2 = 0$, $n_1 = s\,N(0, n-2)$, $n_2 = N(0, n) - sN(0, n-2) - 1$, $p_{11}^1 = (s-1) + s^2 N(0, n-4)$, $p_{11}^2 = N(0, n-2)$, where N(p, n) denotes the number of p-flats in Q and is given by the formulae

$$(1) \qquad N(p, n) = \prod_{r=0}^{p} [(s^{n-2p+2r}-1)/(s^{p+1-r}-1)],$$

if $n = 2k$, $p \leq k - 1$;

$$(2) \qquad N(p, n) = \prod_{r=0}^{p} [(s^{n-2p+2r}-s^{k-p+r}+s^{k-p+r-1}-1)/(s^{p+1-r}-1)],$$

if $n = 2k - 1$, $p \leq k - 2$ and Q is elliptic;

$$(3) \qquad N(p, n) = \prod_{r=0}^{p} [(s^{n-2p+2r}+s^{k-p+r}-s^{k-p+r-1}-1)/(s^{p+1-r}-1)],$$

if $n = 2k - 1$, $p \leq k - 1$ and Q is hyperbolic.

Addleman and Bush (1) have used the array of numbers which represent the treatment combinations of factorial arrangements to construct PBIBD(2)'s. Theirs is a systematic procedure of selecting portions of these arrays. These partial arrays lead to the construction of various incomplete block designs when the columns of the array denote the blocks (treatments), the rows of the array denote the treatments (blocks), and the presence of a specific number in a particular column (row) denotes the presence of a treatment in the block which that column (row) represents. For example, consider the treatment combinations obtained by forming a 7 x 7 array of 0's and 1's, generated by the effects and interactions of the 2^3 factorial system, apart from the control, namely: a, b, ab, c, ac, bc, and abc.

TABLE II

ARRAY GENERATED BY TREATMENT COMBINATIONS OF THE 2^3 FACTORIAL ARRANGEMENT

Trt. Comb.	Factor Representation						
	A	B	AB	C	AC	BC	ABC
a	1	0	1	0	1	0	1
b	0	1	1	0	0	1	1
ab	1	1	0	0	1	1	0
c	0	0	0	1	1	1	1
ac	1	0	1	1	0	1	0
bc	0	1	1	1	1	0	0
abc	1	1	0	1	0	0	1

In every row and in every column of the above array there are exactly three 0's and four 1's. If the columns denote blocks, the rows, treatments, and the 1's in the array, the presence of a treatment in a block, the following BIBD is obtained with parameters $t = 7$, $k = 4$, $b = 7$, $r = 4$, and $\lambda = 2$: (1, 3, 5, 7), (2, 3, 6, 7), (1, 2, 5, 6), (4, 5, 6, 7), (1, 3, 4, 6), (2, 3, 4, 5), (1, 2, 4, 7).

Now consider the first three columns of the array in the table. Note that each row except the row denoted by c contains two 1's and one 0. If this row is eliminated the following PBIBD is obtained, by denoting the columns as blocks and the rows as treatments, where the presence of a 1 denotes the presence of a treatment in a block: (1, 3, 4, 6), (2, 3, 5, 6), (1, 2, 4, 5). The parameters of the design are: $t = 6$, $k = 4$, $b = 3$, $r = 2$, $\lambda_1 = 1$, $\lambda_2 = 2$, $n_1 = 4$, $n_2 = 1$.

The foregoing chapter has presented various techniques which have been used previously to construct PBIBD's. In the next chapter a new method, the subject of this thesis, is defined and investigated.

CHAPTER IV

THE COMPOSITION OF BIBD'S

A method for constructing PBIBD's is defined and investigated in this chapter. The main result, relative to the use of BIBD's in construction, is stated in the first theorem. Then an investigation of each of the types of PBIBD(2) is made to determine which types can be constructed by the method of this thesis.

It was mentioned in the introduction that the construction method under investigation might be regarded as a generalization of Theorems 2.3 and 2.4. This view is taken because the method is to replace each treatment of a BIBD or PBIBD by (instead of n treatments) a BIBD or PBIBD. The method is referred to as "composition," and is denoted symbolically by o placed between two design symbols: $D_1 \circ D_2$.

Definition 4.1: If D_1 and D_2 are either BIBD's or PBIBD's, then $D_1 \circ D_2$ is the design formed by replacing each treatment of D_1 by the design D_2.

As there is a one-to-one correspondence between a design and its incidence matrix, the design D and its incidence matrix are regarded as synonomous and will be used interchangeably. The correspondence is illustrated in the Figures 3 and 4. In view of this design-incidence matrix correspondence, the above definition might be rephrased as ... $D_1 \circ D_2$ is the design formed by replacing each 1 of D_1 by the design D_2 and each 0 of D_1 by $\emptyset$, where $\emptyset$ is the null matrix having dimensions the same as those of D_2.

x	x	
x		x
	x	x

Figure 3: Design

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Figure 4: Matrix

Theorem 4.1: If D_1 and D_2 are BIBD's having parameters

D_1: t_1, k_1, b_1, r_1, λ_1

D_2: t_2, k_2, b_2, r_2, λ_2,

then $D = D_1 \circ D_2$ is a PBIBD having at most three associate classes and parameters

$t = t_1 t_2$, $k = k_1 k_2$, $b = b_1 b_2$, $r = r_1 r_2$, $\lambda_1' = r_1 \lambda_2$, $\lambda_2' = r_2 \lambda_1$, $\lambda_3' = \lambda_1 \lambda_2$, $n_1' = t_2 - 1$, $n_2' = t_1 - 1$, $n_3' = (t_1 - 1)(t_2 - 1)$,

$$(p_{jk}^1) = \begin{bmatrix} t_2 - 2 & 0 & 0 \\ 0 & 0 & t_1 - 1 \\ 0 & t_1 - 1 & (t_1 - 1)(t_2 - 2) \end{bmatrix}$$

$$(p_{jk}^2) = \begin{bmatrix} 0 & 0 & t_2 - 1 \\ 0 & t_2 - 2 & 0 \\ t_2 - 1 & 0 & (t_1 - 2)(t_2 - 1) \end{bmatrix}$$

$$(p^3_{jk}) = \begin{bmatrix} 0 & 1 & t_1 - 2 \\ 1 & 0 & t_1 - 2 \\ t_1 - 2 & t_1 - 2 & (t_1-2)(t_2-2) \end{bmatrix} .$$

<u>Proof:</u> From the definition of the method of construction, it is clear that $t = t_1 t_2$, $k = k_1 k_2$, $r = r_1 r_2$, $b = b_1 b_2$. The problem then is to identify the associate classes and establish the validity of the other parameters. It is suggested that the reader refer to Example 4.1 as he reads through this proof.

Let the first column of D_1 correspond to treatment A_1 of D_1, the first column of D_2 correspond to treatment A_2 of D_2, and the first column of D correspond to treatment A of D. The column corresponding to A_1 in D_1 contains r_1 ones and $b_1 - r_1$ zeros. To form D, each of these r_1 ones is replaced by D_2 and each zero by a $b_2 x t_2$ null matrix $\emptyset$. Thus the first t_2 columns of D contain r_1 repetitions of the design D_2. Considering one of these repetitions of D_2, it is seen that its first column corresponds to A_2 in D_2. In this D_2 matrix there are $t_2 - 1 = n_2$, (say), associates of A_2, and each of them occurs with A_2 in λ_2 blocks. As there are r_1 repetitions of D_2 in the first t_2 columns of D, each of the n_2 associates of A_2 in D_2, considered as treatments of D, occur with A in $r_1\lambda_2$ blocks of D. Define these n_2 treatments of D_2 to be the first associates of A in D. Having identified the first associate class of A in D, the associated parameters are seen to be $n'_1 = n_2 = t_2 - 1$, $\lambda'_1 = r_1 \lambda_2$.

Now consider the associates of A_1 in D_1. There are $t_1 - 1 = n_1$, (say), associates of A_1 in D_1, and each of them occurs with A_1 in

λ_1 blocks of D_1. Consider one of the n_1 associates of A_1. The column in D_1 corresponding to this associate also contains r_1 ones and all other elements are zero. In constructing D each of these r_1 ones is replaced by the matrix D_2, each zero, by the corresponding null matrix $\emptyset$. Hence, the column in D corresponding to the associate of A_1 under consideration repeats D_2 exactly r_1 times. Of these r_1 repetitions of D_2 only λ_1 can be paired with similar repetitions of D_2 in D, because A_1 occurs with any of its associates in λ_1 blocks of D_1. Consider one of these λ_1 pairs of repetitions of D_2. The first column in each D_2 of this pair is the same as that corresponding to A_2. One of these first columns is a part of that corresponding to A in D. Define the treatment in D corresponding to the other first column to be a second associate of A in D. Since A_2 is replicated r_2 times in D_2, it follows that in the pair of D_2 considered above, A and the other treatment, defined to be a second associate of A in D, occur in r_2 blocks of D_2. As there are λ_1 pairs of D_2, it follows that A and its second associates in D occur in $r_2\lambda_1$ blocks of D. Just as the number of associates of A_1 in D_1 is $n_1 = t_1 - 1$, the number of second associates of A in D is n_1. Thus, the second associate class of A in D is identified and the parameters are $n_2' = n_1 = t_1 - 1$, $\lambda_2' = r_2\lambda_1$.

Considering the pair of D_2 again, examine that D_2 in this pair which contains the second associates of A in D. The first column of this D_2 corresponds to A_2. Consider the associates of A_2 in this D_2. These, considered as treatments of D, are defined to be third associates of A in D. In the D_2 under consideration there are

$n_2 = t_2 - 1$ third associates and each of them occurs with A in λ_2 blocks of D_2. Since there are λ_1 such repetitions of D_2 corresponding to each of the $n_1 = t_1 - 1$ associates of A_1 in D_1, it follows that there are $n_1 \cdot n_2$ third associates of A in D and each of them occurs with A in $\lambda_1 \cdot \lambda_2$ blocks of D. This identifies the third associate class of A in D, and the associated parameters are:

$$n_3' = n_1 n_2, \; \lambda_3' = \lambda_1 \lambda_2.$$

Now since $n_1 = t_1 - 1$ and $n_2 = t_2 - 1$, the number of treatments of D accounted for in the above identification of the three associate classes of A in D is $n_1' + n_2' + n_3' = t_1 t_2 - 1$; these and A constitute all of the $t_1 \cdot t_2$ treatments of D.

Let B be a first associate of A in D. The column in D corresponding to B is one of the first t_2 columns of D. Each of the other treatments of D corresponding to the first t_2 columns of D is a first associate of B. This being the case, A and B have $p_{11}^1 = t_2 - 2$ first associates in common.

Consider the pair of D_2 matrices which were used to define the class of second associates of A again. Recall that one of the first columns, corresponding to treatment A_2 of D_2, is a part of that column in D corresponding to treatment A in D; the first column of the other D_2 is a part of the column in D corresponding to that treatment of D which is a second associate of A. This second repetition of D_2, which contains the second associate of A, contains a column B_2, (say), which is a part of that treatment of D that corresponds to the second associate of B. Also recall that each of the other $t_2 - 1$ treatments in the D_2 of the pair which contains the second associate of A is a third associate of A. Likewise, each of the $t_2 - 1$ treatments in this D_2 other than that treatment corresponding to the second associate of B is a third associate of B. As all first associates of A and B are found in the first t_2 columns of D, no treatment which is a first associate of A can be a second or third associate of B; i.e., $p^1_{12} = p^1_{13} = 0$. Since $A_2 \neq B_2$, it is seen that no two treatments which are first associates can have second associates in common; i.e., $p^1_{22} = 0$. From the foregoing statements, it is seen that each second associate of A is a third associate of B and vice versa. Therefore, the second and third associates of two treatments which are first associates intersect once in each of the sets of t_2 treatment of D corresponding to the $t_1 - 1$ associates of A_1 in D_1; i.e., $p^1_{23} = t_1 - 1$. Within those sets of t_2 columns where the second and third associate classes intersect, the remaining t_2-2 treatments are each third associates of A and B. This means that the third associates of A and B intersect at $(t_1-1)(t_2-2)$ places, i.e., $p^1_{33} = (t_1-1)(t_2-2)$. The symmetrical nature of the statements about p^1_{jk} show that $p^1_{jk} = p^1_{kj}$. Observations similar to those just made lead to the stated values for p^2_{jk} and (p^3_{jk}).

<u>Theorem 4.2:</u> If D_1 and D_2 are BIBD's having parameters

$$D_1: \quad t_1, k_1, b_1, r_1, \lambda_1$$

$$D_2: \quad t_2, k_2, b_2, r_2, \lambda_2,$$

then $D = D_1 \circ D_2$ is a PBIBD(2) having parameters

$t^* = t_1 t_2$, $b^* = b_1 b_2$, $k^* = k_1 k_2$, $r^* = r_1 r_2$, $\lambda^*_1 = r_1\lambda_2 = r_2\lambda_1$,

$\lambda_2^* = \lambda_1\lambda_2$, $n_1^* = 2(t_2 - 1) = 2(t_1 - 1)$, $n_2^* = (t_1 - 1)(t_2 - 1)$,

$$(p_{jk}^{1*}) = \begin{bmatrix} t_2 - 2 & t_2 - 1 \\ t_2 - 1 & (t_2 - 1)(t_2 - 2) \end{bmatrix}$$

$$(p_{jk}^{2*}) = \begin{bmatrix} 2 & 2(t_2 - 2) \\ 2(t_2 - 2) & (t_2 - 2)^2 \end{bmatrix}$$

if and only if $r_1\lambda_2 = r_2\lambda_1$ and $t_1 = t_2$.

Proof: If $r_1 \lambda_2 = r_2 \lambda_1$, then in Theorem 4.1 $\lambda_1' = \lambda_2'$. Consequently the classes of first and second associates are combined into one class having $n_1' + n_2' = 2(t_1 - 1) = 2(t_2 - 1) = n_1^*$ elements. The sets of first and second associates being combined, together with the definition of p_{jk}^i, implies that (p_{jk}^i) of Theorem 4.1 become (p_{jk}^{iT}), shown below, i, j, k = 1, 2, 3, due to the following relations

$p_{11}^{1T} = p_{11}^1 + p_{12}^1 + p_{21}^1 + p_{22}^1$, $p_{12}^{1T} = p_{13}^1 + p_{23}^1$, $p_{33}^{1T} = p_{33}^1$,

$p_{11}^{2T} = p_{11}^2 + p_{12}^2 + p_{21}^2 + p_{22}^2$, $p_{12}^{2T} = p_{13}^2 + p_{23}^2$, $p_{33}^{2T} = p_{33}^2$.

$$(p_{jk}^{1T}) = \begin{bmatrix} t_2 - 2 & t_1 - 1 \\ t_1 - 1 & (t_1 - 1)(t_2 - 2) \end{bmatrix}$$

$$(p_{jk}^{2T}) = \begin{bmatrix} t_2 - 2 & t_2 - 1 \\ t_2 - 1 & (t_1 - 2)(t_2 - 1) \end{bmatrix}$$

$$(p_{jk}^{3T}) = \begin{bmatrix} 2 & 2(t_1 - 2) \\ 2(t_1 - 2) & (t_1 - 2)(t_2 - 2) \end{bmatrix}$$

If $t_1 = t_2$ then $(p_{jk}^{1T}) = (p_{jk}^{2T}) = (p_{jk}^{1*})$ and $(p_{jk}^{3T}) = (p_{jk}^{2*})$.

The necessity of $r_1\lambda_2 = r_2\lambda_1$ is obvious. That t_1 must equal t_2 is seen once (p^i_{jk}) are transformed to (p^{iT}_{jk}), which is necessitated by $r_1\lambda_2 = r_2\lambda_1$.

Corollary 4.2.1: For every BIBD, D_1, $D = D_1 \circ D_1$ is a PBIBD(2).

Corollary 4.2.2: $D = D_1 \circ D_2$ is not a PBIBD(2) of the Simple type.

This follows from the requirements that $\lambda_1 \neq 0$ and $\lambda_2 = 0$ for a Simple PBIBD(2), r_1 and r_2 are positive integers, and $\lambda_1^* = r_1\lambda_2 = r_2\lambda_1$, $\lambda_2^* = \lambda_1\lambda_2$.

Corollary 4.2.3: $D = D_1 \circ D_2$ is not a PBIBD(2) of the GD type.

This follows from Theorem 2.1, an examination of (p^{i*}_{jk}), and the fact that no BIBD's exist which have one or two treatments.

Corollary 4.2.4: $D = D_1 \circ D_2$ is not a PBIBD(2) of the Triangular type.

An examination of (p^{2*}_{jk}) shows that $p^{2*}_{11} = 2$ whereas Bose and Shimamoto (8) have shown this number to be 4 for the Triangular type PBIBD(2).

Theorem 4.3: The requirements $r_1\lambda_2 = r_2\lambda_1$ and $t_1 = t_2$ of Theorem 4.2 are equivalent to $r_1\lambda_2 = r_2\lambda_1$ and $k_1 = k_2$.

Proof: The following relationships hold for the BIBD's D_1 and D_2, respectively:

$$\lambda_1(t_1 - 1) = r_1(k_1 - 1)$$

$$\lambda_2(t_2 - 1) = r_2(k_2 - 1).$$

Hence, $r_2\lambda_1(t_1 - 1)(k_2 - 1) = r_1\lambda_2(t_2 - 1)(k_1 - 1)$ from which it follows that $k_1 = k_2$.

Theorem 4.4: If D_1 and D_2 are BIBD's having parameters

$$D_1: t_1, k_1, b_1, r_1, \lambda_1$$

$$D_2: t_2, k_2, b_2, r_2, \lambda_2,$$

and if in $D = D_1 \circ D_2$, $t_1 = t_2 = 2, 3$, or $t_1 = t_2 > 4$ and $r_1\lambda_2 = r_2\lambda_1$, then D is a sub-type L_2 Latin Square PBIBD(2).

Proof: The proof follows from Theorem 2.2 and Theorem 4.2.

Corollary 4.4.1: $D = D_1 \circ D_1$ is a PBIBD(2) of the Latin Square type, sub-type L_2 provided $t_1 = 2, 3$ or $t_1 > 4$.

Corollary 4.4.2: If $D = D_1 \circ D_2$ is a PBIBD(2) of the Latin Square type, L_2 sub-type, then D_1 and D_2 differ only in the parameters b, r, and λ; further $b_1r_2 = b_2r_1$.

Theorem 4.5: If $D = D_1 \circ D_2$ is a PBIBD(2) of the Latin Square type, sub-type L_2 and D_1 and D_2 are symmetrical BIBD's, then D_1 and D_2 are identical.

Proof: D is PBIBD(2) of the Latin Square type, sub-type L_2 implies $r_1\lambda_2 = r_2\lambda_1$, $t_1 = t_2 = 2, 3$ or $t_1 = t_2 > 4$. D_1 is symmetrical implies $b_1 = t_1$, $k_1 = r_1$; likewise, $b_2 = t_2$, $k_2 = r_2$. By Theorem 4.3 $k_1 = k_2$, hence $r_1 = r_2$. By Corollary 4.4.2 $b_1 = b_2$.

Theorem 4.6: Let D_1 and D_2 be any BIBD's such that $D_1 \circ D_2$ is a PBIBD(2). Then $D = D_1 \circ D_2$ is not a Cyclic PBIBD(2).

Proof: An examination of $D_1 \circ D_2$ shows the first associates of treatment 1 to be

$$2, 3, \ldots, t_2, t_2 + 1, 2t_2 + 1, \ldots, (t_1 - 1)t_2 + 1.$$

If D is Cyclic, then by definition the first associates of treatment 1 are:

$$1 + d_1, 1 + d_2, \ldots, 1 + d_{n_1}.$$

Hence $d_1 = 1$, $d_2 = 2$, ..., $d_{n_1} = (t-1)t$ where $t = t_1 = t_2$. In D there are t treatments which are consecutive, hence, $d_1 = 1$ appears exactly $t - 1$ times in the differences $d_j - d_{j'}$, $j \neq j'$, for this design. Also, if D is Cyclic the condition $n_1 A + n_2 B = n(n_1 - 1)$ holds, where A and B are positive integers stipulated in Definition 3.9. In D $n_1 = 2(t-1)$, $n_2 = (t-1)^2$, so that this condition takes the form

$$A = (2t - 3) - 1/2(t - 1)B.$$

For positive integral values of B, A has only one meaningful solution: $A = t - 2$. This means that if D is Cyclic, then $d_1 = 1$ must appear $t - 2$ times in the differences $d_j - d_{j'}$, $j \neq j'$. In fact, $d_1 = 1$ appears $t - 1$ times in D, hence D is not Cyclic.

Theorem 4.7: If D_1 and D_2 are BIBD's having parameters

$$D_1: t_1, k_1, b_1, r_1, \lambda_1$$

$$D_2: t_2, k_2, b_2, r_2, \lambda_2,$$

then $D_1 \circ D_2$ is equivalent to $D_2 \circ D_1$.

Proof: $D_1 \circ D_2$ has parameters of the first kind:

$t = t_1 t_2$, $k = k_1 k_2$, $b = b_1 b_2$, $r = r_1 r_2$, $\lambda_1' = r_1 \lambda_2$, $\lambda_2' = r_2 \lambda_1$, $\lambda_3' = \lambda_1 \lambda_2$, $n_1 = t_2 - 1$, $n_2 = t_1 - 1$, $n_3 = (t_1 - 1)(t_2 - 1)$.

$D_2 \circ D_1$ has corresponding parameters:

$t' = t_2 t_1$, $k' = k_2 k_1$, $b' = b_2 b_1$, $r' = r_2 r_1$, $\lambda_1'' = r_2 \lambda_1$, $\lambda_2'' = r_1 \lambda_2$, $\lambda_3'' = \lambda_2 \lambda_1$, $N_1 = t_1 - 1$, $N_2 = t_2 - 1$, $N_3 = (t_2 - 1)(t_1 - 1)$.

Evidently the only difference in the two designs is that the first and second classes of associates are permuted. Thus, according to Definition 2.4, the designs are equivalent.

Example 4. 1: D_1: $t_1 = 4$, $k_1 = 2$, $b_1 = 6$, $r_1 = 3$, $\lambda_1 = 1$

D_2: $t_2 = 3$, $k_2 = 2$, $b_2 = 3$, $r_2 = 2$, $\lambda_2 = 1$

$D_1 \circ D_2$: $t = 12$, $k = 4$, $b = 18$, $r = 6$, $\lambda_1' = 3$, $\lambda_2' = 2$, $\lambda_3' = 1$, $n_1 = 2$, $n_2 = 3$, $n_3 = 6$,

$$(p_{jk}^1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 3 \end{bmatrix}, \quad (p_{jk}^2) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix}, \quad (p_{jk}^3) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

D_2:

B \ T	1	2	3
1	x	x	
2	x		x
3		x	x

D_1:

B \ T	1	2	3	4
1	x	x		
2			x	x
3	x		x	
4		x		x
5	x			x
6		x	x	

B \ T	1	2	3	4	5	6	7	8	9	10	11	12
1	x	x		x	x							
2	x		x	x		x						
3		x	x		x	x						
4							x	x		x	x	
5							x		x	x		x
6								x	x		x	x
7	x	x					x	x				
8	x		x				x		x			
9		x	x					x	x			
10				x	x					x	x	
11				x		x				x		x
12					x	x					x	x
13	x	x								x	x	
14	x		x							x		x
15		x	x								x	x
16				x	x		x	x				
17				x		x	x		x			
18					x	x		x	x			

$D_1 \circ D_2$

<u>Example 4.2:</u> D_1: $t_1 = 4$, $k_1 = 3$, $b_1 = 4$, $r_1 = 3$, $\lambda_1 = 2$

$D_1 \circ D_1$: $t = 16$, $k = 9$, $b = 16$, $r = 9$, $\lambda_1' = 6$, $\lambda_2' = 4$, $n_1 = 6$, $n_2 = 9$,

$$(p_{jk}^1) = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}, \quad (p_{jk}^2) = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}.$$

B \ T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	x	x	x		x	x	x		x	x	x					
2	x		x	x	x		x	x	x		x	x				
3	x	x		x	x	x		x	x	x		x				
4		x	x	x		x	x	x		x	x	x				
5	x	x	x						x	x	x		x	x	x	
6	x		x	x					x		x	x	x		x	x
7	x	x		x					x	x		x	x	x		x
8		x	x	x						x	x	x		x	x	x
9	x	x	x		x	x	x						x	x	x	
10	x		x	x	x		x	x					x		x	x
11	x	x		x	x	x		x					x	x		x
12		x	x	x		x	x	x						x	x	x
13					x	x	x		x	x	x		x	x	x	
14					x		x	x	x		x	x	x		x	x
15					x	x		x	x	x		x	x	x		x
16						x	x	x		x	x	x		x	x	x

$D_1 \circ D_1$

It is interesting to note that this Latin Square type PBIBD(2), which falls in the range $r \leq 10$, $3 \leq k \leq 10$, is not listed by Bose, Clatworthy, and Shrikhande (5).

In Chapter IV a method for constructing PBIBD's has been defined and investigated. PBIBD(2)'s have been singled out for examination in relation to the construction method. The study revealed that only one type of PBIBD(2) can be produced by the construction method. In Chapter V attention is focused on the composition of BIBD's and their complementary designs.

CHAPTER V

SOME RESULTS ON COMPLEMENTARY DESIGNS

This chapter concerns the construction of PBIBD's using the composition operation in conjunction with the complementary design of a given BIBD. Conditions pertinent to their construction and the parameters, as well as some other relations, are determined. Some attention is given to BIBD's and their complementary designs in relation to the construction of a certain class of disconnected PBIBD(2)'s.

Definition 5.1: By the complement $\underline{D}$ of a given BIBD D is meant that BIBD which has ones where there are zeros and zeros where there are ones in the incidence matrix of D.

It is easily seen that if D has parameters t, k, b, r, λ, then $\underline{D}$ has parameters $t' = t$, $k' = t - k$, $b' = b$, $r' = b - r$, $\lambda' = b - 2r + \lambda$. The expression for λ' follows from the well known relations $\lambda = r(k-1)/(t-1)$, $\lambda' = r'(k'-1)/(t'-1)$ for the BIBD's involved.

Definition 5.2: A BIBD is said to be self-complementary when the parameters of the complementary design are the same as those of the original one.

From this definition and the previous one it is seen that if D is a self-complementary design then $b = 2r$, $t = 2k$.

A brief explanatory note regarding notation for the work to follow will now be given. Suppose D_1 and D_2 are BIBD's having parameters

$$D_1:\ t_1,\ k_1,\ b_1,\ r_1,\ \lambda_1$$

$$D_2:\ t_2,\ k_2,\ b_2,\ r_2,\ \lambda_2,$$

then the parameters of $D_1 \circ \underline{D}_2$ are named so that they reflect the designs being composed. For example, the numbers of treatments of $D_1 \circ \underline{D}_2$ is designated $t_{1\underline{2}}$, the subscript $1\underline{2}$ of $t_{1\underline{2}}$ indicating that designs D_1 and $\underline{D}_2$ are being composed. All the other parameters follow suit: $k_{1\underline{2}}$, $b_{1\underline{2}}$, $r_{1\underline{2}}$, $\lambda_{1\underline{2}i}$, $(p_{jk}^{i1\underline{2}})$.

Theorem 5.1: If D_1 and D_2 are BIBD's having parameters

$$D_1:\ t_1,\ k_1,\ b_1,\ r_1,\ \lambda_1$$

$$D_2:\ t_2,\ k_2,\ b_2,\ r_2,\ \lambda_2,$$

then (1) $D_1 \circ \underline{D}_1$ is a PBIBD with at most three associate classes and having parameters

$t_{1\underline{1}} = t_1^2$, $k_{1\underline{1}} = k_1(t_1 - k_1)$, $b_{1\underline{1}} = b_1^2$, $r_{1\underline{1}} = r_1(b_1 - r_1)$,

$\lambda_{1\underline{1}1} = (b_1 - r_1)\lambda_1$, $\lambda_{1\underline{1}2} = r_1(b_1 - 2r_1 + \lambda_1)$, $\lambda_{1\underline{1}3} = \lambda_1(b_1 - 2r_1 + \lambda_1)$,

$n_{1\underline{1}1} = t_1 - 1$, $n_{1\underline{1}2} = t_1 - 1$, $n_{1\underline{1}3} = (t_1 - 1)^2$,

$$(p_{jk}^{11\underline{1}}) = \begin{bmatrix} t_1-2 & 0 & 0 \\ 0 & 0 & t_1-1 \\ 0 & t_1-1 & (t_1-1)(t_1-2) \end{bmatrix},$$

$$(p_{jk}^{21\underline{1}}) = \begin{bmatrix} 0 & 0 & t_1-1 \\ 0 & t_1-2 & 0 \\ t_1-1 & 0 & (t_1-1)(t_1-2) \end{bmatrix},$$

$$(p_{jk}^{31\underline{1}}) = \begin{bmatrix} 0 & 1 & t_1-2 \\ 1 & 0 & t_1-2 \\ t_1-2 & t_1-2 & (t_1-2)(t_1-2) \end{bmatrix},$$

(2) $D_1 \circ \underline{D}_2$ is a PBIBD with at most three associate classes and having parameters

$t_{1\underline{2}} = t_1 t_2$, $k_{1\underline{2}} = k_1(t_2 - k_2)$, $b_{1\underline{2}} = b_1 b_2$, $r_{1\underline{2}} = r_1(b_2 - r_2)$,

$\lambda_{1\underline{2}1} = (b_2 - r_2)\lambda_1$, $\lambda_{1\underline{2}2} = r_1(b_2 - 2r_2 + \lambda_2)$, $\lambda_{1\underline{2}3} = \lambda_1(b_2 - 2r_2 + \lambda_2)$,

$n_{1\underline{2}1} = t_2 - 1$, $n_{1\underline{2}2} = t_1 - 1$, $n_{1\underline{2}3} = (t_1 - 1)(t_2 - 1)$,

$$(p_{jk}^{11\underline{2}}) = \begin{bmatrix} t_2 - 2 & 0 & 0 \\ 0 & 0 & t_1 - 1 \\ 0 & t_1 - 1 & (t_1 - 1)(t_2 - 2) \end{bmatrix},$$

$$(p_{jk}^{21\underline{2}}) = \begin{bmatrix} 0 & 0 & t_2 - 1 \\ 0 & t_2 - 2 & 0 \\ t_2 - 1 & 0 & (t_2 - 1)(t_1 - 2) \end{bmatrix},$$

$$(p_{jk}^{31\underline{2}}) = \begin{bmatrix} 0 & 1 & t_1 - 2 \\ 1 & 0 & t_1 - 2 \\ t_1 - 2 & t_1 - 2 & (t_1 - 2)(t_2 - 2) \end{bmatrix},$$

(3) $\underline{D}_1 \circ D_2$ is a PBIBD with at most three associate classes and having parameters

$t_{\underline{1}2} = t_1 t_2$, $k_{\underline{1}2} = (t_1 - k_1)k_2$, $b_{\underline{1}2} = b_1 b_2$, $r_{\underline{1}2} = (b_1 - r_1)r_2$,

$\lambda_{\underline{1}21} = r_2(b_1 - 2r_1 + \lambda_1)$, $\lambda_{\underline{1}22} = (b_1 - r_1)\lambda_2$, $\lambda_{\underline{1}23} = (b_1 - 2r_1 + \lambda_1)\lambda_2$,

$n_{\underline{1}21} = n_{1\underline{2}1}$, $n_{\underline{1}22} = n_{1\underline{2}2}$, $n_{\underline{1}23} = n_{1\underline{2}3}$, $(p_{jk}^{i\underline{1}2}) = (p_{jk}^{i1\underline{2}})$, i, j, k = 1, 2, 3.

(4) $\underline{D}_1 \circ \underline{D}_2$ is a PBIBD with at most three associate classes

and having parameters

$t_{\underline{12}} = t_1 t_2$, $k_{\underline{12}} = (t_1 - k_1)(t_2 - k_2)$, $b_{\underline{12}} = b_1 b_2$, $r_{\underline{12}} = (b_1 - r_1)(b_2 - r_2)$,

$\lambda_{\underline{12}1} = (b_2 - r_2)(b_1 - 2r_1 + \lambda_1)$, $\lambda_{\underline{12}2} = (b_1 - r_1)(b_2 - 2r_2 + \lambda_2)$,

$\lambda_{\underline{12}3} = (b_1 - 2r_1 + \lambda_1)(b_2 - 2r_2 + \lambda_2)$, $n_{\underline{12}1} = n_{1\underline{2}1}$, $n_{\underline{12}2} = n_{1\underline{2}2}$,

$n_{\underline{12}3} = n_{1\underline{2}3}$, $(p^{i\underline{12}}_{jk}) = (p^{i1\underline{2}}_{jk})$.

Proof: The proof follows directly from Theorem 4. 1. and Definition 5. 1.

It can be seen from Theorem 5. 1 that $D_1 \circ \underline{D}_2$ is not equivalent to $\underline{D}_1 \circ D_2$ in general. From the standpoint of generating new designs, this holds some interest. Too, it poses the question: Are they ever equivalent? To answer this question recall that according to Theorem 4. 2, $D_1 \circ D_2$ is a PBIBD(2) if and only if $r_1\lambda_2 = r_2\lambda_1$ and $t_1 = t_2$. In the case of $D_1 \circ \underline{D}_2$ it is seen that this first condition takes the form

$$(b_2 - r_2)\,\lambda_1 = r_1(b_2 - 2r_2 + \lambda_2)$$

$$b_2\lambda_1 - r_2\lambda_1 = b_2 r_1 - 2r_2 r_1 + \lambda_2 r_1$$

$$\lambda_1(b_2 - 2r_2) = r_1(b_2 - 2r_2).$$

Hence, the condition requires that $b_2 = 2r_2$. Similar statements can be made for $D_1 \circ \underline{D}_1$, $\underline{D}_1 \circ D_2$, and $\underline{D}_1 \circ \underline{D}_2$. Thus, the following theorem has been proved.

Theorem 5. 2: If D_1 and D_2 are BIBD's having parameters

$$D_1:\ t_1,\ k_1,\ b_1,\ r_1,\ \lambda_1$$

D_2: t_2, k_2, b_2, r_2, λ_2

such that D_1 o D_2 is a PBIBD(2), then

(1) D_1 o $\underline{D}_1$ is a PBIBD(2) if and only if $b_1 = 2r_1$

(2) D_1 o $\underline{D}_2$ is a PBIBD(2) if and only if $b_2 = 2r_2$

(3) $\underline{D}_1$ o D_2 is a PBIBD(2) if and only if $b_1 = 2r_1$

(4) $\underline{D}_1$ o $\underline{D}_2$ is a PBIBD(2) if and only if

$b_1r_2 + b_2\lambda_1 = b_2r_1 + b_1\lambda_2$.

Part (1) of the above theorem might be stated: Under the stated hypothesis D_1 o $\underline{D}_1$ is a PBIBD(2) if and only if D_1 is self-complementary. Similar statements can be made for parts (2) and (3).

Since the conditions $t_1 = t_2$ and $r_2\lambda_1 = r_1\lambda_2$ are equivalent to $k_1 = k_2$, $r_2\lambda_1 = r_1\lambda_2$, and in view of the parameters of D_1 o $\underline{D}_2$ and $\underline{D}_1$ o D_2, the following theorem holds.

Theorem 5.3: If D_1 and D_2 are BIBD's having parameters

D_1: t_1, k_1, b_1, r_1, λ_1

D_2: t_2, k_2, b_2, r_2, λ_2

such that D_1 o D_2 is a PBIBD(2), then D_1 o $\underline{D}_2$ is equivalent to $\underline{D}_1$ o D_2.

Now a special class of disconnected PBIBD(2)'s will be investigated. Before proceeding further, however, a definition and two theorems due to Roy (18) will be given.

Definition 5.3: A BIBD is said to be unreduced if for any t and k such that b, r, and λ have no common factor, $b=\binom{t}{k}$, $r=\binom{t-1}{k-1}$, $\lambda=\binom{t-2}{k-2}$.

Theorem 5.4: The BIBD's for which $k = 2$, $t - 2$, or $t - 1$, and $t = 8$ and $k = 5$ or 3 are the unreduced designs.

Theorem 5.5: A complementary design is reduced or unreduced according as the original one is reduced or unreduced.

An examination of the two theorems above shows that the complementary design of the unreduced design having $k = 2$ is that unreduced design with $k = t - 2$; likewise with $k = 5$ and $k = 3$. The next theorem resolves the question of what design is the complement of the unreduced design having $k = t - 1$.

Theorem 5.6: If D is an unreduced BIBD such that $k = t-1$, then $b + \lambda = 2r$ and D has no complementary design.

Proof: From the hypothesis and the definition of such a design it follows that $b = \binom{t}{t-1} = t$, $r = \binom{t-1}{t-2} = t - 1$, $\lambda = \binom{t-2}{t-3} = t - 2$ and $b + \lambda = t + t - 2 = 2(t-1) = 2r$. This result coupled with Definition 5.1 shows $\lambda' = 0$. Hence, $\underline{D}$ is not a BIBD and consequently does not exist. In fact, what does exist in this case is a degenerate BIBD which has an incidence matrix equivalent to the identity matrix I_t -- equivalent meaning rank equivalence here.

If "$\underline{D}$" is regarded as a true BIBD for the moment, and the composition operation formally applied, the following theorem results.

Theorem 5.7: If D_1 is a BIBD having parameters t_1, k_1, b_1, r_1, λ_1, and D_2 is an unreduced BIBD having $k_2 = t_2 - 1$, then D_1 o "$\underline{D}_2$" is a PBIBD(2) of the GD type, Simple sub-type which is disconnected.

Proof: The parameters of D_1 o "$\underline{D}_2$" are as follows:

$t = t_1 t_2$, $k = k_1 \cdot 1$, $b = b_1 b_2$, $r = r_1 \cdot 1$, $\lambda_1' = 1 \cdot \lambda_1$, $\lambda_2' = 0$,

$n_1 = t_1 - 1$, $n_2 = t_1(t_2 - 1)$,

$$(p^1_{jk}) = \begin{bmatrix} t_1 - 2 & 0 \\ 0 & t_1(t_2 - 1) \end{bmatrix}$$

$$(p^2_{jk}) = \begin{bmatrix} 0 & t_1 - 1 \\ t_1 - 1 & t_1(t_2 - 2) \end{bmatrix}$$

According to Theorem 2.1 $p^1_{12} = 0$ is necessary and sufficient to show the design is GD; $\lambda_1' \neq 0$, $\lambda_2' = 0$ and Definition 2.6 shows that the design is of the Simple sub-type. Clearly "$\underline{D}_2$" is disconnected. As the construction method replaces each 1 in the incidence matrix of D_1 by "$\underline{D}_2$," and each 0 by $\phi_{b_2 x t_2}$, only those $\tau_{ij} - \tau_{i'j}$ (i, i'= 1, 2, . . . b; j, j' = 1, 2, . . . , t), or linear combinations of such, in which $i = i'$ are estimable. That $\tau_{ij} - \tau_{(i+1)j'}$ is never estimable shows the design D_1 o "$\underline{D}_2$" to be disconnected.

The disconnected character of such designs is demonstrated in the following example.

Example 5.1: D_1: $t_1 = 4$, $k_1 = 2$, $b_1 = 6$, $r_1 = 3$, $\lambda_1 = 1$

D_2: $t_2^* = 4$, $k_2^* = 3$, $b_2^* = 4$, $r_2^* = 3$, $\lambda_2^* = 2$

"$\underline{D}_2$": $t_2 = 4$, $k_2 = 1$, $b_2 = 4$, $r_2 = 1$, $\lambda_2 = 0$

$D_1 \circ$ "$\underline{D}_2$": $t = 16$, $k = 2$, $b = 24$, $r = 3$, $\lambda_1' = 1$,

$\lambda_2' = 0$, $n_1 = 3$, $n_2 = 12$,

$$(p_{jk}^1) = \begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix}, \quad (p_{jk}^2) = \begin{bmatrix} 0 & 3 \\ 3 & 8 \end{bmatrix}.$$

"$\underline{D}_2$"

B \ T	1	2	3	4
1	x			
2		x		
3			x	
4				x

D_1

B \ T	1	2	3	4
1	x	x		
2			x	x
3	x		x	
4		x		x
5	x			x
6		x	x	

B \ T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	x				x											
2		x				x										
3			x				x									
4				x				x								
5									x				x			
6										x				x		
7											x				x	
8												x				x
9	x								x							
10		x								x						
11			x								x					
12				x								x				
13					x								x			
14						x								x		
15							x								x	
16								x								x
17	x												x			
18		x												x		
19			x												x	
20				x												x
21					x				x							
22						x				x						
23							x				x					
24								x				x				

D_1 o "$\underline{D}_2$"

B \ T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	x	x														
2			x	x												
3	x		x													
4		x		x												
5	x			x												
6		x	x													
7					x	x										
8							x	x								
9					x		x									
10						x		x								
11					x			x								
12						x	x									
13									x	x						
14											x	x				
15									x		x					
16										x		x				
17									x			x				
18										x	x					
19													x	x		
20															x	x
21													x		x	
22														x		x
23													x			x
24														x	x	

"$\underline{D}_2$" o D_1

Theorem 4.2 shows that $r_1\lambda_2 = r_2\lambda_1$ and $t_1 = t_2$ are necessary and sufficient conditions for $D_1 \circ D_2$ to be a PBIBD(2), where D_1 and D_2 are BIBD's. One might inquire into the possibility of obtaining a BIBD from the composition of two BIBD's. Clearly, a necessary condition would be that $r_1\lambda_2 = r_2\lambda_1 = \lambda_1\lambda_2$. This implies that $r_1 = \lambda_1$ and $r_2 = \lambda_2$. If this latter condition prevailed, then D_1 and D_2 would not be BIBD's. One the other hand, if $r_2 = \lambda_2$ but $r_1 \neq \lambda_1$, then formal application of the composition operation to the BIBD D_1 and the $b \times t$ array of treatments D_2 leads to a GD Singular PBIBD(2). This result is stated in the following theorem.

<u>Theorem 5.8:</u> If in a BIBD with parameters t_1, k_1, b_1, r_1, λ_1 each treatment is replaced by a $p \times q$ array of treatments, a Singular GD PBIBD(2) results having parameters

$$t = qt_1, \quad k = qk_1, \quad b = pb_1, \quad r = pr_1, \quad \lambda_1' = pr_1, \quad \lambda_2' = p\lambda_1,$$

$$m = t_1, \quad n = q$$

Conversely, every Singular GD designs is obtainable in this way from a corresponding BIBD.

As this theorem is altogether "isomorphic" to the one given by Bose and Connor (4), its proof will not be given. Instead, consider an illustration in the following example.

<u>Example 5.2:</u> D_1: $t_1 = 3$, $k_1 = 2$, $b_1 = 3$, $r_1 = 2$, $\lambda_1 = 1$

"D_2" is a 2×3 factorial arrangement of treatments, i.e., $p = 2$, $q = 3$

$D_1 \circ$ "D_2": $t = 9$, $k = 6$, $b = 6$, $r = 4$, $\lambda_1' = 4$, $\lambda_2' = 2$,

$m = 3$, $n = 3$

B \ T	1	2	3
1	x	x	
2	x		x
3		x	x

D_1

p \ q	1	2	3
1	x	x	x
2	x	x	x

"D_2"

B \ T	1	2	3	4	5	6	7	8	9
1	x	x	x	x	x	x			
2	x	x	x	x	x	x			
3	x	x	x				x	x	x
4	x	x	x				x	x	x
5				x	x	x	x	x	x
6				x	x	x	x	x	x

D_1 o "D_2"

Except where noted near the end of Chapter V, both Chapters IV and V have been concerned with the composition of BIBD's. Intuitively, the next evolutionary step would be the composition of BIBD's and PBIBD's. In Chapter VI both this step and the next, the composition of PBIBD's, are investigated.

CHAPTER VI

THE COMPOSITION OF PBIBD'S

Two major theorems relating the composition of BIBD's and PBIBD's are found in this chapter. The first, which composes a PBIBD with a BIBD, is a special case of the second, which composes two PBIBD's. Some results on commutativity and associativity are also included.

<u>Theorem 6.1:</u> If D_1 is a BIBD and D_2 is a PBIBD(g) having parameters

$$D_1: \; t_1, \; k_1, \; b_1, \; r_1, \; \lambda$$

$$D_2: \; t_2, \; k_2, \; b_2, \; r_2, \; \lambda_{2i_2}, \; n_{2i_2}, \; (p^{i_2}_{2j_2k_2}), \; i_2, j_2, \; k_2 = 1, 2, \ldots, g,$$

then $D_1 \circ D_2$ is a PBIBD having at most $2g + 1$ classes and parameters $t = t_1t_2$, $k = k_1k_2$, $b = b_1b_2$, $r = r_1r_2$, $\lambda_{i_2} = r_1\lambda_{2i_2}$, $\lambda_{g+1} = \lambda r_2$, $\lambda_{g+1+i_2} = \lambda\,\lambda_{2i_2}$, $n_{i_2} = n_{2i_2}$, $n_{g+1} = t_1 - 1 = N_1$(say), $n_{g+1+i_2} = (t_1 - 1)n_{2i_2} = N_1 n_{2i_2}$,

$$(p^{i_2}_{jk}) = \left[\begin{array}{c|c|c} (p^{i_2}_{2j_2k_2}) & 0 & \emptyset_{g\times g} \\ \hline 0 & 0 & (d_{i_2}k_2)N_1 \\ \hline \emptyset_{g\times g} & (d_{i_2}k_2)'N_1 & (p^{i_2}_{2j_2k_2})N_1 \end{array}\right]$$

$$(p_{jk}^{g+1}) = \left[\begin{array}{c|c|c} \emptyset_{gxg} & \emptyset_{gx1} & (\text{diag } n_{2i_2}) \\ \hline \emptyset_{1xg} & \lambda & \emptyset_{1xg} \\ \hline (\text{diag } n_{2i_2}) & \emptyset_{gx1} & (\text{diag } n_{2i_2})\lambda \end{array}\right]$$

$$(p_{jk}^{t+1+i_2}) = \left[\begin{array}{c|c|c} \emptyset_{gxg} & (d_{i_2k_2})' & (p_{2j_2k_2}^{i_2}) \\ \hline (d_{i_2k_2}) & 0 & (d_{i_2k_2})\lambda \\ \hline (p_{2j_2k_2}^{i_2}) & (d_{i_2k_2})'\lambda & (p_{2j_2k_2}^{i_2})\lambda \end{array}\right]$$

$i_2, j_2, k_2 = 1, 2, \ldots, g$; $d_{ik} = 1$ if $i = k$, 0 otherwise; $(d_{ik})'$ is the transpose of the matrix (d_{ik}); $(\text{diag } x_i)$ is the diagonal matrix with non-zero elements x_i.

Proof: It is clear from Definition 4.1 that $t = t_1t_2$, $k = k_1k_2$, $b = b_1b_2$, $r = r_1r_2$; hence, the remainder of the proof is concerned with the identification of associate classes and the determination of the remaining parameters.

Let the first column of D_1 correspond to treatment A_1 of D_1, the first column of D_2 correspond to treatment A_2 of D_2, the first column of D correspond to treatment A of D. Due to the method of construction, the first t_2 columns of D contain r_1 repetitions of the design D_2. Considering one of these r_1 matrices D_2, it is seen that its first column corresponds to A_2 in D_2. In this D_2 matrix there are n_{2i_2} i_2th associates of A_2, and each of them occurs with A_2 in λ_{2i_2} blocks. Since there are r_1 repetitions of D_2 in the first t_2 columns of D, each of the n_{2i_2} associates of A_2 in D_2, considered as treatments in D, occur with A in $r_1 \cdot \lambda_{2i_2}$ blocks of D. These n_{2i_2} treatments of D_2 are defined to be i_2th associates $(i_2 = 1, 2, \ldots, g)$ of A in D. Thus, the parameters n_{i_2} and λ_{i_2} of D are seen to be $n_{i_2} = n_{2i_2}$, $\lambda_{i_2} = r_1\lambda_{2i_2}$.

Consider the associates of A_1 in D_1. There are $N_1 = t_1 - 1$ associates of A_1 in D_1, and each of them occurs with A_1 in λ blocks of D_1. The column in D_1, corresponding to one of the n_1 associates of A_1, contains r_1 ones and all the other elements are zero. Hence, due to the method of construction, the column in D corresponding to the associate of A_1 under consideration repeats D_2 exactly r_1 times in D. Of these r_1 repetitions of D_2 only λ of them can be paired with similar repetitions of D_2 in D, because A_1 occurs with any of its associates in λ blocks of D_1. Consider one of these λ pairs of repetitions of D_2. The first column in each repetition of D_2 in this pair is identical with that corresponding to A_2. Define the treatment in D corresponding to the first column of the second repetition of the pair of repetitions of D_2 to be a $(g + 1)$th associate of A in D. Since A_2 is replicated r_2 times in D_2, it follows that in the pair of repetitions of D_2 considered above, A and the other treatment, the $(g+1)$th associate of A in D, occur in r_2 blocks. As there are λ such pairs of D_2, it follows that A and its $(g+1)$th associates in D occur in λr_2 blocks of D. In the same way that there are $N_1 = t_1 - 1$ associates of A_1 in D_1, the number of $(g+1)$th associates of A in D is N_1. Thus, the $(g+1)$th associate class of A in D is identified, and the associated parameters are

$n_{g+1} = N_1$, $\lambda_{g+1} = \lambda r_2$.

Consider the above pair of D_2 matrices again. Look at that D_2 of the pair which contains the (g+1)th associate of A in D. The first column of this D_2 corresponds to A_2 of D_2. Consider the n_{2i_2} i_2th associates of A_2 in this D_2; these, regarded as treatments of D, are defined to be the (i_2+g+1)th associates of A in D, $i_2 = 1, 2, \ldots, g$. In the D_2 under consideration there are n_{2i_2} (i_2+g+1)th associates, each of them occurring with A in λ_{2i_2} blocks of D_2. Since there are λ such repetitions of D_2 corresponding to each of the $N_1 = t_1 - 1$, (g+1)th associates of A in D, it follows that there are $n_{g+1+i_2} = N_1 n_{2i_2}$, $(i_2 + g + 1)$th associates of A in D, and each of them occurs with A in $\lambda_{g+1+i_2} = \lambda\,\lambda_{2i_2}$ blocks of D.

The total number of treatments of D accounted for in the above identification of associate classes is

$$\sum_{i_2=1}^{g}(n_{i_2} + n_{g+1} + n_{g+1+i_2}) = \sum_{i_2=1}^{g} n_{2i_2} + t_1 - 1 + \sum_{i_2=1}^{g}(t_1 - 1)n_{2i_2}$$

$$= t_1 t_2 - 1.$$

These, together with A, are all of the $t_1 t_2$ treatments of D.

The method of identification of the first g associate classes of D shows that the elements of the first g rows and columns of $(p_{jk}^{i_2})$ are exactly the same as $(p_{2j_2k_2}^{i_2})$. To see this, let B be an i_2th associate of A in D, $i_2 = 1, 2, \ldots, g$. Notice that the column in D corresponding to B is one of the first t_2 columns of D, and that these t_2 columns contain only i_2th associates of A in D. The first column in a repetition of D_2, which is a part of the column corresponding to A in D, is treatment A_2 of D_2. This repetition of D_2 contains a column which is a part of the column corresponding to B in D. Let this column of D_2 correspond to treatment $B_2 \neq A_2$ of D_2. A_2 and B_2 are i_2th associates of each other in D_2, and there are $p^{i_2}_{2j_2k_2}$ treatments of D_2 which are common with the j_2th associates of A_2 and the k_2th associates of B_2 in D_2; j_2, k_2 = 1, 2, . . . , g. However, these $p^{i_2}_{2j_2k_2}$ treatments in the repetition of D_2, when considered as treatments of D, are those in common with the j_2th associates of A and the k_2th associates of B in D.

Because the first t_2 columns of D contain only the i_2th associates of A and B, $p^{i_2}_{j_2x} = 0$ for i_2, j_2 = 1, 2, . . . , g; x = g+ 1, . . . , 2g + 1. Thus, the elements of the first g rows from column g + 1 to column 2g + 1 are all zero in $(p^{i_2}_{jk})$.

Consider the (g+1)th associates of A in D. It will be recalled that these are the treatments of D which correspond to those columns in D which, in turn, correspond to A_2 in the repetitions of D_2; these repetitions of D_2 correspond to the associates of A_1 in D_1. The (g+1)th associates of B in D are the treatments of D which correspond to those columns in D which correspond to B_2 in the repetitions of D_2; each repetition of D_2 corresponds to the associates of A_1 in D_1. Since A_2 is different from B_2, $p^{i_2}_{g+1, g+1} = 0$.

The (g + 1 + k_2)th associates of B in D are the k_2th associates of B_2 in repetitions of D_2 which originate from the associates of A_1 in D_1. To determine $p^{i_2}_{g+1,\ g+1+k_2}$ count the number of treatments of D which are common with the $N_1 = t_1 - 1$, $(g + 1)$th associates of A and the n_{2k_2}, $(g + 1 + k_2)$th associates of B in D. This is seen to be

$$p^{i_2}_{g+1,\ g+1+k_2} = \begin{cases} N_1 & \text{if } i_2 = k_2 \\ 0 & \text{if } i_2 \neq k_2 \end{cases}$$

This can be expressed as $d_{i_2k_2} \cdot N_1$ where $d_{xy} = 1$ if $x = y$, $d_{xy} = 0$ if $x \neq y$.

Next consider the $(g + 1+ j_2)$th associates of A and the $(g + 1 + k_2)$th associates of B in D. The number of treatments in common with these two associate classes is denoted $p^{i_2}_{g+1+j_2,\ g+1+k_2}$. From the identification of these associate classes it can be seen that

$$p^{i_2}_{g+1+j_2,\ g+1+k_2} = N_1\, p^{i_2}_{2j_2k_2}.$$

In matrix form this would be

$$(p^{i_2}_{g+1+j_2,\ g+1+k_2}) = N_1\, (p^{i_2}_{2j_2k_2}) .$$

This completes $(p^{i_2}_{jk})$ as given; other parameters of the second kind are derived similarly.

An example illustrating the preceding theorem will now be given.

<u>Example 6. 1:</u> D_1: $t_1 = 4$, $k_1 = 3$, $b_1 = 4$, $r_1 = 3$, $\lambda = 2$,

D_2: $t_2 = 9$, $k_2 = 4$, $b_2 = 9$, $r_2 = 4$, $\lambda_{21}=1$,

$\lambda_{22}=2$, $N_{21} = 4$, $N_{22} = 4$, $(p^1_{2jk}) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$, $(p^2_{2jk}) = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$.

$D_1 o D_2$: $t = 36$, $k = 12$, $b = 36$, $r = 12$, $\lambda_1 = 3$, $\lambda_2 = 6$,

$\lambda_3 = 8$, $\lambda_4 = 2$, $\lambda_5 = 4$, $n_1 = 4$, $n_2 = 4$, $n_3 = 3$,

$n_4 = 12$, $n_5 = 12$,

$$(p^1_{jk}) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 3 & 6 \\ 0 & 0 & 0 & 6 & 6 \end{bmatrix} \qquad (p^2_{jk}) = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 6 & 6 \\ 0 & 0 & 3 & 6 & 3 \end{bmatrix}$$

$$(p^3_{jk}) = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 0 & 0 \\ 4 & 0 & 0 & 8 & 0 \\ 0 & 4 & 0 & 0 & 8 \end{bmatrix} \qquad (p^4_{jk}) = \begin{bmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 2 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 4 & 4 \end{bmatrix}$$

$$(p^5_{jk}) = \begin{bmatrix} 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 4 & 4 \\ 2 & 1 & 2 & 4 & 2 \end{bmatrix}$$

D_1:

B \ T	1	2	3	4
1	x	x	x	
2	x	x		x
3	x		x	x
4		x	x	x

D_2:

B \ T	1	2	3	4	5	6	7	8	9
1	x	x				x			x
2		x		x		x		x	
3	x			x				x	x
4		x			x		x		x
5		x	x	x			x		
6			x	x	x				x
7	x				x	x	x		
8			x			x	x	x	
9	x		x		x			x	

B \ T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
1	x	x				x			x	x	x				x			x	x	x				x			x									
2		x		x		x		x			x		x		x		x			x		x		x		x										
3	x			x				x	x	x			x				x	x	x			x				x	x									
4		x			x		x		x		x			x		x		x		x			x		x		x									
5		x	x	x			x				x	x	x			x				x	x	x			x											
6			x	x	x				x			x	x	x				x			x	x	x				x									
7	x				x	x	x			x				x	x	x			x				x	x	x											
8			x			x	x	x				x			x	x	x				x			x	x	x										
9	x		x		x			x		x		x		x			x		x		x		x			x										
10	x	x				x			x	x	x				x			x										x	x				x			x
11		x		x		x		x			x		x		x		x												x		x		x		x	
12	x			x				x	x	x			x				x	x										x			x				x	x
13		x			x		x		x		x			x		x		x											x			x		x		x
14		x	x	x			x				x	x	x			x													x	x	x			x		
15			x	x	x				x			x	x	x				x												x	x	x				x
16	x				x	x	x			x				x	x	x												x				x	x	x		
17			x			x	x	x				x			x	x	x													x			x	x	x	
18	x		x		x			x		x		x		x			x											x		x		x			x	
19	x	x				x			x										x	x				x			x	x	x				x			x
20		x		x		x		x												x		x		x		x			x		x		x		x	
21	x			x				x	x										x			x				x	x	x			x				x	x
22		x			x		x		x											x			x		x		x		x			x		x		x
23		x	x	x			x													x	x	x			x				x	x	x			x		
24			x	x	x				x												x	x	x				x			x	x	x				x
25	x				x	x	x												x				x	x	x			x				x	x	x		
26			x			x	x	x													x			x	x	x				x			x	x	x	
27	x		x		x			x											x		x		x			x		x		x		x			x	
28										x	x				x			x	x	x				x			x	x	x				x			x
29											x		x		x		x			x		x		x		x			x		x		x		x	
30										x			x				x	x	x			x				x	x	x			x				x	x
31											x			x		x		x		x			x		x		x		x			x		x		x
32											x	x	x			x				x	x	x			x				x	x	x			x		
33												x	x	x				x			x	x	x				x			x	x	x				x
34										x				x	x	x			x				x	x	x			x				x	x	x		
35												x			x	x	x				x			x	x	x				x			x	x	x	
36										x		x		x			x		x		x		x			x		x		x		x			x	

$D_1 \circ D_2$

Theorem 6.2: If D_1, D_2, and D_3 are BIBD's having parameters

D_1: t_1, k_1, b_1, r_1, λ_1,

D_2: t_2, k_2, b_2, r_2, λ_2,

D_3: t_3, k_3, b_3, r_3, λ_3,

then $D_1 \circ (D_2 \circ D_3)$ is equivalent to $(D_2 \circ D_3) \circ D_1$.

Proof: Applying Theorem 4.1 and 6.1 the parameters of the first kind of $D_1 \circ (D_2 \circ D_3)$ are:

$$t = t_1(t_2t_3),\quad k = k_1(k_2k_3),\quad b = b_1(b_2b_3),\quad r = r_1(r_2r_3),$$

$$\lambda_1' = r_1(r_2\lambda_3),\quad \lambda_2' = r_1(r_3\lambda_2),\quad \lambda_3' = r_1(\lambda_2\lambda_3),\quad \lambda_4' = (r_2r_3)\lambda_1,$$

$$\lambda_5' = (r_2\lambda_3)\lambda_1,\quad \lambda_6' = (r_3\lambda_2)\lambda_1,\quad \lambda_7' = (\lambda_2\lambda_3)\lambda_1.$$

Correspondingly, for $(D_2 \circ D_3) \circ D_1$:

$$t' = (t_2t_3)t_1,\quad k' = (k_2k_3)k_1,\quad b' = (b_2b_3)b_1,\quad r' = (r_2r_3)r_1,$$

$$\lambda_1'' = (r_2r_3)\lambda_1,\quad \lambda_2'' = r_1(r_2\lambda_3),\quad \lambda_3'' = r_1(r_3\lambda_2),\quad \lambda_4'' = r_1(\lambda_2\lambda_3),$$

$$\lambda_5'' = \lambda_1(r_2\lambda_3),\quad \lambda_6'' = \lambda_1(r_3\lambda_2),\quad \lambda_7'' = \lambda_1(\lambda_2\lambda_3).$$

From the preceding parameter lists, it is seen that the designs differ only in the naming of associate classes: $\lambda_1' = \lambda_2''$, $\lambda_2' = \lambda_3''$, $\lambda_3' = \lambda_4''$, $\lambda_4' = \lambda_1''$. Hence, the designs are equivalent.

Theorem 6.3: If D_1, D_2, and D_3 are BIBD's having parameters

D_1: t_1, k_1, b_1, r_1, λ_1,

D_2: t_2, k_2, b_2, r_2, λ_2,

D_3: t_3, k_3, b_3, r_3, λ_3, then $D_1 \circ (D_2 \circ D_3)$ is equivalent to $(D_1 \circ D_2) \circ D_3$.

Proof: The parameters of $D_1 \circ (D_2 \circ D_3)$ are given in Theorem 6.2. Applying Theorems 4.1, 6.1 and 6.2, the parameters of $(D_1 \circ D_2) \circ D_3$ are:

$$t' = (t_1t_2)t_3,\ k' = (k_1k_2)k_3,\ b' = (b_1b_2)b_3,\ r' = (r_1r_2)r_3,$$

$$\lambda_1'' = (r_1r_2)\lambda_3,\ \lambda_2'' = r_3(r_1\lambda_2),\ \lambda_3'' = r_3(r_2\lambda_1),\ \lambda_4''=r_3(\lambda_1\lambda_2),$$

$$\lambda_5'' = \lambda_3(r_1\lambda_2),\ \lambda_6'' = \lambda_3(r_2\lambda_1),\ \lambda_7'' = \lambda_3(\lambda_1\lambda_2).$$

From the preceding, it is seen that the designs differ only in the naming of associate classes: $\lambda_3'' = \lambda_4'$, $\lambda_4'' = \lambda_6'$, $\lambda_5'' = \lambda_3'$, $\lambda_6'' = \lambda_5'$. Therefore, $D_1 \circ (D_2 \circ D_3)$ is equivalent to $(D_1 \circ D_2) \circ D_3$.

Theorem 6.4: If D_1 is a PBIBD(f) and D_2 is a PBIBD(g) having parameters

$$D_1\colon\ t_1,\ k_1,\ b_1,\ r_1,\ \lambda_{1i_1},\ n_{1i_1},\ (p^{i_1}_{1j_1k_1}),\ i_1,\ j_1,\ k_1 = 1,\ 2,\ \ldots,\ f,$$

$$D_2\colon\ t_2,\ k_2,\ b_2,\ r_2,\ \lambda_{2i_2},\ n_{2i_2},\ (p^{i_2}_{2j_2k_2}),\ i_2,\ j_2,\ k_2 = 1,\ 2,\ \ldots,\ g,$$

then $D = D_1 \circ D_2$ is a PBIBD having at most $g + f + gf$ associate classes and parameters

$$t = t_1t_2,\ k = k_1k_2,\ b = b_1b_2,\ r = r_1r_2,\ \lambda_{i_2} = r_1\lambda_{2i_2},$$

$$\lambda_{g+i_1} = \lambda_{1i_1}r_2,\ \lambda_{g+i_1+i_2f} = \lambda_{1i_1}\lambda_{2i_2},\ n_{i_2} = n_{2i_2},$$

$$n_{g+i_1} = n_{1i_1},\ n_{g+i_1+i_2f} = n_{1i_1}n_{2i_2},$$

$$
(p_{jk}^{i_2}) = \left[\begin{array}{c|c|c}
(p_{2j_2k_2}^{i_2}) & \emptyset_{gxf} & \emptyset_{gxgf} \\
\hline
 & \emptyset_{fxf} & (d_{i_2k_2}) \times (\mathrm{diag}\ n_{1i_1}) \\
\hline
 & & (p_{2j_2k_2}^{i_2}) \times (\mathrm{diag}\ n_{1i_1})
\end{array}\right]
$$

$$
(p_{jk}^{g+i_1}) = \left[\begin{array}{c|c|c}
\emptyset_{gxg} & \emptyset_{gxf} & (\mathrm{diag}\ n_{2i_2}) \times (d_{i_1k_1}) \\
\hline
 & (p_{1j_1k_1}^{i_1}) & \emptyset_{fxfg} \\
\hline
 & & (\mathrm{diag}\ n_{2i_2}) \times (p_{1j_1k_1}^{i_1})
\end{array}\right]
$$

$$
(p_{jk}^{g+i_1+i_2f}) = \left[\begin{array}{c|c|c}
\emptyset_{gxg} & (d_{i_2k_2})' \times (d_{i_1k_1}) & (p_{2j_2k_2}^{i_2}) \times (d_{i_1k_1}) \\
\hline
 & \emptyset_{fxf} & (d_{i_2k_2}) \times (p_{1j_1k_1}^{i_1}) \\
\hline
 & & (p_{2j_2k_2}^{i_2}) \times (p_{1j_1k_1}^{i_1})
\end{array}\right]
$$

$i_1, j_1, k_1 = 1, 2, \ldots, f$; $i_2, j_2, k_2 = 1, 2, \ldots, g$; $d_{xy} = 1$ if $x = y$, $d_{xy} = 0$ if $x \neq y$; $(A)'$ is the transpose of (A); x in fxf, e.g., is ordinary multiplication; $(A)X(B)$, where (A) is of dimensions sxm and (B) is txn, is the st x mn matrix $a_{ij}(B)$; $(\mathrm{diag}\ x_i)$ is the diagonal matrix having non-zero x_i.

<u>Proof:</u> The definition of the construction method makes it clear that

$t = t_1 t_2$, $k = k_1 k_2$, $b = b_1 b_2$, $r = r_1 r_2$.

To obtain the other parameters procedures similar to those used in Theorem 6.1 are used; in fact, the first i_2th, ($i_2 = 1, 2, \ldots, g$), associates of A in D are defined exactly as in that theorem. Hence, $n_{i_2} = n_{2i_2}$ and $\lambda_{i_2} = r_1 \lambda_{2i_2}$. A, A_1, A_2, B, and B_2 are defined as in Theorem 6.1.

Now consider the i_1th associates of A_1 in D_1. These are n_{1i_1} in number, and each of them occurs with A_1 in λ_{1i_1} blocks of D_1. The column in D_1, corresponding to one of the n_{1i_1} associates of A_1, contains r_1 ones with all other elements zero. Then due to the method of construction, the column in D corresponding to the associate of A_1 under consideration causes r_1 repetitions of D_2 in D. Of these r_1 repetitions of D_2 only λ_{1i_1} of them can be paired with similar repetitions of D_2 in D, because A_1 occurs with any of its associates in λ_{1i_1} blocks of D_1. Consider one of these λ_{1i_1} pairs of repetitions of D_2. The first column in each repetition of D_2 in this pair is identical with that corresponding to A_2. Define the treatment in D corresponding to the first column of the second repetition of the pair of repetitions of D_2 to be a $(g + i_1)$th associate of A in D. Since A_2 is replicated r_2 times in D_2, it follows that in the pair of repetitions of D_2 considered, A and the treatment corresponding to the $(g + i_1)$th associate of A in D occur in r_2 blocks. As there are λ_{1i_1} such pairs of D_2, it follows that A and its $(g + i_1)$th associates occur in $\lambda_{1i_1} \cdot r_2$ blocks of D. As there are n_{1i_1} associates of

A_1 in D_1, the number of $(g + i_1)$th associates of A in D is n_{1i_1}. Thus, the $(g + i_1)$th associate classes of A in D are determined; pertinent parameters are:

$$n_{g+i_1} = n_{1i_1}, \quad \lambda_{g+i_1} = \lambda_{1i_1} r_2.$$

Again considering the above pair of D_2 matrices, look at that D_2 of the pair which contains the $(g + i_1)$th associate of A in D. The first column of this D_2 corresponds to A_2; the other columns are the n_{2i_2} i_2th associates of A_2. These, regarded as treatments of D, are defined to be the $(g + i_1 + i_2 f)$th associates of A in D. In the D_2 under consideration there are n_{2i_2} $(g + i_1 + i_2 f)$th associates, each of them occurring with A in λ_{2i_2} blocks of D_2. There are λ_{1i_1} such repetitions of D_2 corresponding to each of the n_{1i_1} i_1th associates of A_1 in D_1. Hence, there are $n_{g+i_1+i_2 f} = n_{1i_1} \cdot n_{2i_2}$ $(g + i_1 + i_2 f)$th associates of A in D. Each of these occurs with A in $\lambda_{g+i_1+i_2 f} = \lambda_{1i_1} \lambda_{2i_2}$ blocks of D.

The number of treatments of D accounted for in the above identification scheme is

$$\sum_{i_1=1}^{f} n_{i_1} + \sum_{i_1=1}^{f} n_{g+i_1} + \sum_{i_1=1}^{f} \sum_{i_2=1}^{g} n_{g+i_1+i_2 f} = t_1 t_2 - 1.$$

These $t_1 t_2 - 1$ treatments of D, along with A, are all of the $t = t_1 t_2$ treatments of D.

As in the case of Theorem 6.1, the method of identification of the first g associate classes of D shows that the elements of the first g rows and columns of $(p_{jk}^{i_2})$ are exactly $(p_{2j_2k_2}^{i_2})$. Likewise

$p^{i_2}_{j_2 x} = 0$ for $i_2, j_2 = 1, 2, \ldots, g$; $x = g+1, g+2, \ldots, g+f+gf$. Thus, the first g rows and $g+f+gf$ columns of $(p^{i_2}_{jk})$ are determined.

Consider the $(g+j_1)$th associates of A in D. These are the treatments of D which correspond to those columns in D which correspond to A_2 in the repetitions of D_2; these repetitions of D_2 correspond to the associates of A_1 in D_1. The $(g+j_1)$th associates of B in D are the treatments of D which correspond to those columns in D which correspond to B_2 in the repetitions of D_2; each repetition of D_2 corresponds to the associates of A_1 in D_1. Since A_2 and B_2 are different, $p^{i_2}_{g+j_1, g+k_1} = 0$; $j_1, k_1 = 1, 2, \ldots, f$.

The $(g+j_1+k_2 f)$th associates of B in D are the k_2th associates of B_2 in repetitions of D_2 which originate from the associates of A_1 in D_1. To determine $p^{i_2}_{g+j_1, g+k_1+k_2 f}$ it is necessary to count the number of treatments of D which are common with the n_{1i_1} $(g+j_1)$th associates of A and the $n_{2k_2} \cdot n_{1k_1}$ $(g+k_1+k_2 f)$th associates of B in D. From the class identification scheme, it is seen that

$$p^{i_2}_{g+j_1, g+k_1+k_2 f} = 0 \qquad \text{if } j_1 \neq k_1$$

$$p^{i_2}_{g+j_1, g+j_1+k_2 f} = 0 \qquad \text{if } i_2 \neq k_2$$

$$p^{i_2}_{g+j_1, g+j_1+i_2 f} = n_{1j_1}$$

where $j_1, k_1 = 1, 2, \ldots, f$; $k_2 = 1, 2, \ldots, g$. A more compact form of the above is given in matrix notation as

$$(p^{i_2}_{g+j_1, g+k_1+k_2f}) = (d_{i_2k_2}) \times (\text{diag } n_{1j_1}).$$

Next consider the $(g+j_1+j_2f)$th associates of A in D and the $(g+k_1+k_2f)$th associates of B in D. The number of treatments in common with these two associate classes is denoted $p^{i_2}_{g+j_1+j_2f, g+k_1+k_2f}$. From the definition of these associate classes it can be seen that

$$p^{i_2}_{g+j_1+j_2f, g+k_1+k_2f} = \begin{cases} 0 & \text{if } j_1 \neq k_1 \\ n_{1j_1} p^{i_2}_{2j_2k_2} & \text{if } j_1 = k_1 \end{cases}$$

where j_1, k_1 = 1, 2, . . . , f; j_2, k_2 = 1, 2, . . . , g. This can be expressed in matrix form as

$$(p^{i_2}_{g+j_1+j_2f, g+k_1+k_2f}) = (p^{i_2}_{2j_2k_2}) \times (\text{diag } n_{1j_1}) .$$

This completes $(p^{i_2}_{jk})$ as given in the theorem. The other parameters are derived in a similar fashion.

An application of Theorem 6.4 is illustrated by example 6.2. At the end of the example, Table III gives a listing of typical sets of associate classes. Noting that treatments 5, 2, 19, 10, 23, 14, 20, and 11 are 1st, 2nd, . . . , 8th associates of treatment 1, respectively, one might use the table, for example, to verify the parameters of the second kind of $D_1 \circ D_2$.

<u>Example 6.2:</u>

D_1: $t_1 = 6$, $k_1 = 4$, $b_1 = 6$, $r_1 = 4$, $\lambda_{11} = 3$, $\lambda_{12} = 2$, $n_{11} = 2$, $n_{12} = 3$,

$$(p^1_{1jk}) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad (p^2_{1jk}) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix},$$

D_2: $t_2 = 9$, $k_2 = 4$, $b_2 = 9$, $r_2 = 4$, $\lambda_{21} = 2$, $\lambda_{22} = 1$, $n_{21} = 4$, $n_{22} = 4$,

$$(p^1_{2jk}) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \quad (p^2_{2jk}) = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix},$$

$D_1 \circ D_2$: $t = 54$, $k = 16$, $b = 54$, $r = 16$, $\lambda_1 = 8$, $\lambda_2 = 4$, $\lambda_3 = 12$,

$\lambda_4 = 8$, $\lambda_5 = 6$, $\lambda_6 = 4$, $\lambda_7 = 3$, $\lambda_8 = 2$, $n_1 = 4$, $n_2 = 4$,

$n_3 = 2$, $n_4 = 3$, $n_5 = 8$, $n_6 = 12$, $n_7 = 8$, $n_8 = 12$,

$$(p^1_{jk}) = \left[\begin{array}{cc|cc|cccc}
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
\hline
0 & 0 & 2 & 0 & 2 & 0 & 4 & 0 \\
0 & 0 & 0 & 3 & 0 & 3 & 0 & 6 \\
0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 6
\end{array}\right]$$

$$(p^2_{jk}) = \left[\begin{array}{cc|cc|cccc}
2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
\hline
0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 \\
0 & 0 & 2 & 0 & 4 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 & 0 & 6 & 0 & 3
\end{array}\right]$$

$$(p_{jk}^{3}) = \left[\begin{array}{cc|cc|cccc}
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 12
\end{array}\right]$$

$$(p_{jk}^{4}) = \left[\begin{array}{cc|cc|cccc}
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
\hline
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\
4 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\
0 & 4 & 0 & 0 & 0 & 0 & 8 & 0
\end{array}\right]$$

$$(p_{jk}^{5}) = \left[\begin{array}{cc|cc|cccc}
0 & 0 & 1 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\
\hline
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
\hline
1 & 2 & 1 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 & 0 & 3 & 0 & 6 \\
2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 6
\end{array}\right]$$

$$(p^6_{jk}) = \left[\begin{array}{cc|cc|cccc}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
\hline
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 2 & 0 & 2 & 0 & 4 \\
1 & 2 & 2 & 0 & 2 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 \\
2 & 2 & 0 & 0 & 4 & 0 & 4 & 0
\end{array}\right]$$

$$(p^7_{jk}) = \left[\begin{array}{cc|cc|cccc}
0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
\hline
2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 \\
2 & 1 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 3 & 0 & 6 & 0 & 3
\end{array}\right]$$

$$(p^8_{jk}) = \left[\begin{array}{cc|cc|cccc}
0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 4 & 0 & 4 \\
2 & 2 & 0 & 0 & 4 & 0 & 4 & 0 \\
0 & 0 & 0 & 2 & 0 & 4 & 0 & 2 \\
2 & 1 & 2 & 0 & 4 & 0 & 2 & 0
\end{array}\right]$$

TABLE III

A LIST OF TYPICAL SETS OF ASSOCIATE CLASSES

Associate Class	Treatments 1	5	2	19
1	5, 6, 8, 9	1, 3, 7, 9	4, 6, 7, 9	23, 24, 26, 27
2	2, 3, 4, 7	2, 4, 6, 8	1, 3, 5, 8	20, 21, 22, 25
3	19, 37	23, 41,	20, 38	1, 37
4	10, 28, 46	14, 32, 50	11, 29, 47	10, 28, 46
5	23, 24, 26, 27, 41, 42, 44, 45	19, 21, 25, 27, 37, 39, 43, 45	22, 24, 25, 27, 40, 42, 43, 45	5, 6, 8, 9, 41, 42, 44, 45,
6	14, 15, 17, 18, 32, 33, 35, 36, 50, 51, 53, 54	10, 12, 16, 18, 28, 30, 34, 36, 46, 48, 52, 54	13, 15, 16, 18, 31, 33, 34, 36, 49, 51, 52, 54	14, 15, 17, 18, 32, 33, 35, 36, 50, 51, 53, 54
7	20, 21, 22, 25, 38, 39, 40, 43	20, 22, 24, 26, 38, 40, 42, 44	21, 23, 19, 26, 37, 39, 41, 44	2, 3, 4, 7, 38, 39, 40, 43
8	11, 12, 13, 16, 29, 30, 31, 34, 47, 48, 49, 52	11, 13, 15, 17, 29, 31, 33, 35, 47, 49, 51, 53	10, 12, 14, 17, 28, 30, 32, 35, 46, 48, 50, 53	11, 12, 13, 16, 29, 30, 31, 34, 47, 48, 49, 52
	10	**23**	**14**	**20**
1	14, 15, 17, 18	19, 21, 25, 27	10, 12, 16, 18	22, 24, 25, 27
2	11, 12, 13, 16	20, 22, 24, 26	11, 13, 15, 17	19, 21, 23, 26
3	28, 46	5, 41	32, 50	2, 38
4	1, 19, 37	14, 32, 50	5, 23, 41	11, 29, 47
5	32, 33, 35, 36, 50, 51, 53, 54	1, 3, 7, 9, 37, 39, 43, 45	28, 30, 34, 36, 46, 48, 52, 54	4, 6, 7, 9, 40, 42, 43, 45,
6	5, 6, 8, 9, 23, 24, 26, 27, 41, 42, 44, 45	10, 12, 16, 18, 28, 30, 34, 36, 46, 48, 52, 54	1, 3, 7, 9, 19, 21, 25, 27, 37, 39, 43, 45	13, 15, 16, 18, 31, 33, 34, 36, 49, 51, 52, 54
7	29, 30, 31, 34, 47, 48, 49, 52	2, 4, 6, 8, 38, 40, 42, 44	29, 31, 33, 35, 47, 49, 51, 53	1, 3, 5, 8, 37, 39, 41, 44,
8	2, 3, 4, 7, 20, 21, 22, 25, 38, 39, 40, 43	11, 13, 15, 17, 29, 31, 33, 35, 47, 49, 51, 53	2, 4, 6, 8, 20, 22, 24, 26, 38, 40, 42, 44	10, 12, 14, 17, 28, 30, 32, 35, 46, 48, 50, 53

The material of the foregoing chapter has been concerned with the composition of BIBD's and PBIBD's, as well as the composition of PBIBD's and PBIBD's. The main result is found in Theorem 6.4: If a PBIBD(f) is composed with a PBIBD(g), the resulting design is a PBIBD having at most $g + f + gf$ classes. Chapter VI concludes the investigation. In the subsequent chapter is found a brief summary and some suggestions for further research.

CHAPTER VII

SUMMARY AND CONCLUSIONS

A method of constructing PBIBD's called composition is defined and investigated in this thesis. The construction method, which might be regarded as a generalization of theorems given by Bose and Connor (4), Roy (19), and Zelen (27), replaces each treatment of a BIBD or PBIBD with an entire BIBD or PBIBD.

Theorems regarding the composition of BIBD's are given in Chapter IV. Because PBIBD(2)'s have been studied extensively and classified into a small number of groups, each of these types are studied relative to composition. It is found that only the Latin Square subtype L_2 can be generated by the use of the composition of two BIBD's. However, by relaxing the definition of composition, it is possible to construct a type of GD PBIBD, as well as a disconnected Simple design. In Chapter V some results on the composition of BIBD's and complements of BIBD's are given. In Chapter VI are found the theorems which give the results of combining BIBD's and PBIBD's. The principal result is that the composition of a PBIBD(f) with a PBIBD(g) gives a PBIBD having at most $g+f+gf$ associate classes.

In the paragraphs which follow are found some suggestions which might be appropriate for further investigation of the topic under study. For example, it might be interesting to see if one could determine the minimum number of distinct classes of a composite design, given two PBIBD's of specified distinct classes.

Secondly, the method of inversion used in conjunction with composition might elicit some interesting results. It is possible that inversion, complementation, and composition could be amalgamated into some unifying concept.

The relationship of the resolvability of designs used in construction to that of the composite design could be investigated. For example: If D_1 and D_2 are resolvable PBIBD's, does it follow that $D_1 \circ D_2$ is a resolvable PBIBD? If D_1 is a resolvable PBIBD and D_2 is a PBIBD which is not resolvable, is $D_1 \circ D_2$ a resolvable PBIB? As a PBIBD is resolvable, if and only if, any two blocks of the design have the same number of treatments in common, the answers to such questions probably lie in a study of certain block relationships of composite design in relation to the designs used in construction.

It appears that there might be another evolutionary step to be taken in the development of the composite design. As the construction method investigated leaves a great deal of empty space (small block size) in some resultant designs, it might be possible to build a PBIBD by replacing all the 1's of an incidence matrix of a given design by one PBIBD and all of the 0's by another PBIBD. As "partial balance" must be achieved in the resultant design, a similar design might be achieved by replacing certain letters of a Latin Square or similar symmetrical arrangement by certain PBIBD's.

For example, suppose the 3 x 3 Latin Square shown in Figure 5 has the letter C deleted from it as seen in Figure 6.

A	B	C
C	A	B
B	C	A

Figure 5: Latin Square

A	B	
	A	B
B		A

Figure 6: Deleted Latin Square

Now replace each A with a PBIBD(f); replace each B with a PBIBD(g). The resultant design appears to be a PBIBD. If it is, a paucity of questions regarding its properties should be no problem.

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