THEORETICAL AND EXPERIMENTAL COMPARISON OF MATRIX METHODS FOR STRUCTURAL ANALYSIS

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CHAPTER I

INTRODUCTION

The development of digital computers during the last few years provides an improved capability for the analysis and the design of structural configurations required for the current generation of military and commercial airplanes. The prediction of the stress and the deformation characteristics of actual airframe configurations is one phase of structural analysis for which the elementary theories are often incapable of providing accurate results. Consequently, new analysis capabilities are being developed in terms of matrix operations of algebraic equations. These theories are generally referred to as matrix methods or finite element methods. The finite element methods are the topics of numerous current research efforts.

The two most popular of these methods are called the force and the displacement or stiffness methods because of the assumption of the initial unknown quantities. Both methods require the mathematical development of systems of finite elements, which are joined to form the idealized structure and to develop the necessary algebraic equations. These equations are generally solved by a completely automatic sequence of computer operations originating with the definition of the structural configuration and ending with the calculation of the structural response for the applied external load configurations.

The purpose of this research program is to develop a capability for the analysis of integrally reinforced structural skin panels and to demonstrate this capability by the comparison of experimental and analytical results. Chapters II and III illustrate the two finite element methods of structural analysis and demonstrate some of the different assumptions that are made in deriving the stiffness properties of idealized structural elements. Chapter IV and Appendices A and B describe computer programs that are used in the analytical investigation described in Chapter V. The experimental investigation, which is described in Chapter VI, provides a basis for the comparison of the analytical results. The validity of the analytical results, using the new idealized element derived in Chapter III, is demonstrated in Chapter VII.

The structure considered in this dissertation is limited to a rectangular configuration. The structure is a semi-monocoque rectangular panel with thin webs and integral reinforcements. The structure is idealized as rib and stringer elements transmitting axial loads and thin web elements transmitting shear and axial loads. The web elements may be designated as plate or panel elements; however, in structural analysis the term, plate, is commonly applied to planar structural elements which carry loads applied normal to their plane. The rectangular panel is oriented to lie in the xy plane, and the deflections are produced by loads in both x and y directions. A general arbitrary orientation of the panel in three dimensions is not necessary for this investigation; however, it could easily be analyzed with these finite element methods. The size of the planar structure that is analyzed is significantly increased by limiting the configuration to two dimensions.

One of the first approaches suitable for the computer-type analysis of panels was the solution of problems by a finite difference method (1). This technique involves defining a mesh or network system over the panel. The differential equations of equilibrium and compatibility are expressed in finite difference form based on the assumed stress-strain relations. The resulting large number of finite difference equations describes approximately the behavior of the loaded panel. Boundary conditions corresponding to physical boundary restraints and applied loads are specified in the finite difference equations representing the points on the boundary. The finite difference method was subsequently replaced by the finite element methods which are algebraic approaches that are easily formulated in terms of matrix operations. The finite element method of analysis is not new to structural engineering. For example, in many types of dynamic analyses, structural segments with known properties are connected to form a continuous system of finite elements. The techniques used in these dynamic analyses are similar, but by no means equivalent to the finite element methods of stress analysis described in this investigation.

Beginning in 1954, Argyris (2) described in matrix form the schematic analysis of structures composed of discrete structural elements. Argyris compiled a multitude of special analysis methods which were used for structural analysis. Argyris demonstrated the similarity among many of the analysis methods by using matrix notation to abbreviate the mathematics.

Most of Argyris' work is based on the energy principles of structural analysis. Energy methods are convenient in his developments and are a contrast to a method of direct geometrical relationships used by Turner, et al. (3), to develop stiffness and stress matrices or displacement transformation matrices. The methods using direct geometrical relationships provide a clear, simplified development; however, these methods are limited in the degree of generality possible in the derivations. The energy principles provide an advantage in handling more complicated types of structural elements.

Matrix methods of structural analysis were extended to plate-type structures by Turner, et al. (3). They describe the analysis of plane stress problems using finite elements. Their derivations allow the plane stress element to deform in a combination of certain assumed patterns. This concept eliminates the necessity for knowing the behavior of an element before its stiffness can be developed.

These developments in the finite element approach to the approximate analysis of reinforced panels form the basis for this investigation. The structural behavior of a panel is determined by analyzing the group behavior of small elastic elements connected at common joints to form an idealized structure which approximates the actual panel.

The structural behavior is determined by element idealizations using both the force and stiffness methods of analysis and assuming deformation or stress modes of varying complexity. New stiffness and stress matrices are developed in Chapter III for the rectangular skin panels, representing the model used for the experimental phase of this investigation. The new stiffness and stress matrices, combined with the new digital computer program described in Chapter IV, provide an improved analysis capability for reinforced skin structures.

The digital computing programs, which are described in Chapter IV and Appendices A and B, are being used in other current research programs utilizing matrix operations and experimental data analysis references. These digital computing capabilities include a compatible set of matrix operation programs used for the force method of analysis, an integrated system program based on the displacement method of analysis, and data reduction programs based on the least-squares criterion for the experimental stress and deflection data analysis.

The principal digital computing program developed during this research program is entitled the Stress Analysis System. This system is based on the displacement method of finite element structural analysis. This system is developed in a manner that allows for simple and convenient additions of any type of planar structural elements that may be of interest in future research programs. Since systems of this type which are currently in existence are considered "proprietary" by the originators or are developed with a specific objective or intention, no system is available for study or application of finite element methods that allows the researcher the opportunity to experiment with his mathematical derivations. In addition, the Generalized Stress Calculations phase of the program is unique in that previous systems provide only a single state of stress for the entire finite element. This addition to the system provides for computing the state of stress at any number of interest points within the finite element. This feature is most essential in the direct application of the system to structural analyses.

CHAPTER II

FORCE METHOD OF ANALYSIS

The force method and the stiffness method of structural analysis are similar in that a duality exists between the algebraic forms of the equations. Argyris (4) discussed this duality.

Identical results are obtained by both the force and stiffness methods if the same assumptions are made in the behavior of the idealized elements (5). The following discussion illustrates the application of force and stiffness methods to the analysis of structural panels. A comparison between the two methods illustrates that, while both methods are easily adapted to solutions with the digital computer, the stiffness method is easier to use in a general computer program because no requirement is necessary to determine redundant load paths.

A discussion in the standard longhand notation of the main ideas and methods for the analysis of redundant structures, based on the assumption of forces as unknowns, is given by Argyris (4). The author's work deals only with the matrix formulation of the analysis. The matrix approach clarifies some of the more salient features of the analysis. Although the matrix methods are certainly general and applicable to all classes of aerospace structures, the methods studied in this dissertation apply to the integrally reinforced rectangular panels analyzed in the experimental phase of this program.

An essential characteristic of the force analysis is the degree of redundancy which results from the idealization of the structure and the corresponding definition of the idealized elements and node points on the structure. The system of node points along grid lines is arbitrary; but, in general, the system of node points is assumed to be the intersection of the grid lines formed by the ribs and spars connected to the skin cover.

An assumption widely used in aircraft design idealizes the structure as webs which carry only shear forces and as stringer elements which carry the direct stresses. A fraction of the web area is added to the reinforcements to form the equivalent or effective stringer element area (6).

The amount of web area added to the stringer area depends on the stress level, type of material, and type of loading. For example, by neglecting the Poisson's effect and in assuming the same material for stringers and flat plates, one-sixth to one-half of the web cross-sectional area should be added to the stringer area (4). The former value applies when the field is in pure bending within its own plane, and the latter value applies when it is under uniform axial stress.

Degree of Redundancy of Reinforced Skin Structure

The degree of redundancy is the number of unknown forces minus the number of independent equilibrium equations that are obtained for the idealized structure. The idealization of the structure is completely independent of the actual locations of the ribs and stringers. The structure is divided into several equivalent stringers and shear-web elements. The number of redundancies is determined by assuming the flat structural panel to be fixed at the root section and free along the other edges. If no unstiffened cutouts exist, the number of redundancies N is

$$N = \sum_{Bays} (\beta - 2)$$

where β equals the number of longitudinal effective stringer elements which are continuous across a rib junction (4). The number of bays is the number of transverse sections defined in the structural idealization. If any stringer element is not fixed at the root section, the number of redundancies reduces accordingly. If the web is omitted between two adjoining longitudinal stringers in a bay and if the cutout is not reinforced, the number of redundancies is reduced by the number of missing webs.

The degree of redundancy is illustrated for the two-dimensional integrally reinforced skin panel. The unknown forces shown in Figure 1 are

△ Unknown forces in longitudinal stringers	12
○ Unknown forces in transverse ribs	6
□ Unknown shear forces in the webs	9
Total	27

The equations of equilibrium are

Equilibrium of adjacent stringers and webs	12
Equilibrium of adjacent ribs and webs	9
Total	21

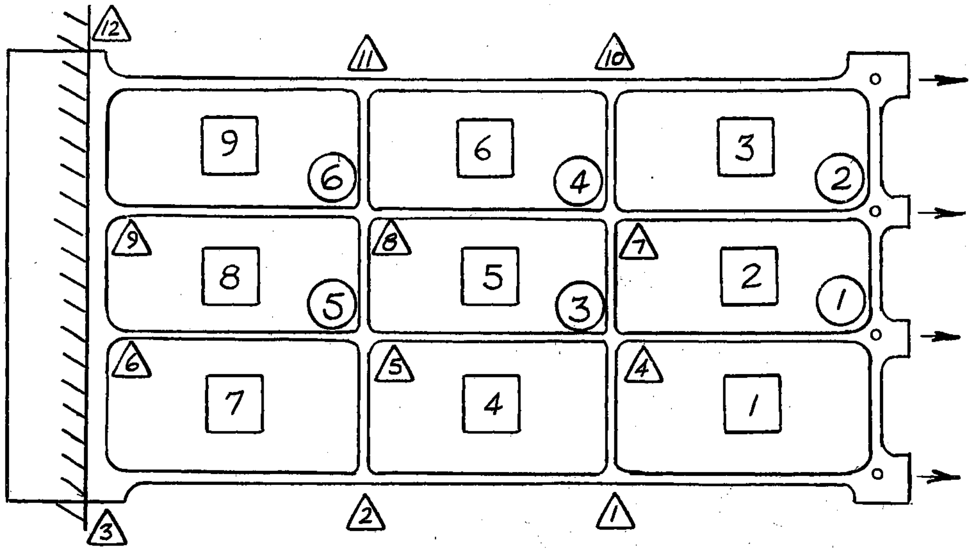


Figure 1. The Unknown Forces in the Integrally Reinforced Skin Panel

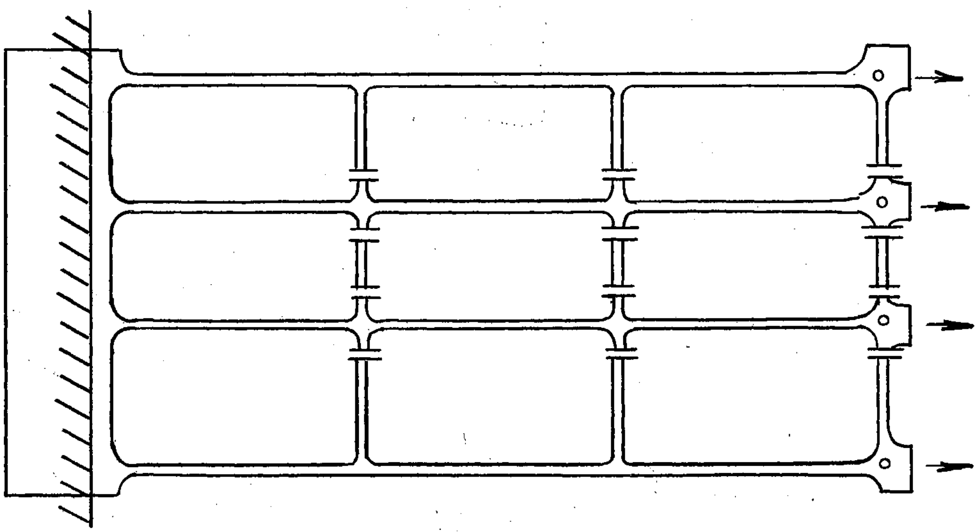


Figure 2. The Statically Determinate Basic System

Thus, for a total of 21 independent equilibrium equations, the degree of redundancy is 27 - 21 = 6.

Also, from the first equation

$$N = \sum_{Bays} (\beta - 2)$$
$$= 3 (4 - 2) = 6.$$

Therefore, six of the unknown internal forces are removed by the use of facticious cuts such that the structure is still stable and statically determinate. For this structural configuration and external load system, the rib forces are relaxed to obtain the statically determinate structure. The statically determinate structure is shown in Figure 2.

Once the idealization is performed, the stresses and deflections are calculated using the force method with matrix algebra operations as shown in Table I. The formulation of the equations used in the digital computer program follows the method of Argyris (7).

Formulation of the Algebraic Equations

The essence of the force method is

1. The redundant forces in the structures are the initially unknown quantities.
2. The internal forces are expressed in terms of both the redundant and external forces.
3. The deformations are determined from assumed stress-strain relationship.
4. The compatibility criterion provides a set of linear algebraic simultaneous equations which can be solved for the redundant forces.

TABLE I

FORTRAN PROGRAM FOR FORCE METHOD OF ANALYSIS

```
C     FORCE METHOD OF ANALYSIS FOR RECTANGULAR PANELS
C     M. U. AYRES
C     MAXIMUM SIZE B1 = 57X6, BO = 51X6, F = 57X57
C     THIS ANALYSIS REQUIRES  5 LOAD  CONDITIONS
      DIMENSION  B1(308), F(3251), BF(308), D(38), DI(38), BO(287),
     1D2(32), D3(32), D4(287), B(287), A(287), FLEX(27), FORCE(7),
     2DELTA(7), FIN(287)
      COMMON KIN, KOUT
      KIN  = 5
      KOUT = 6
    1 CALL RMATNZ (B1)
    2 CALL RMATNZ (F)
    3 CALL MTXM   (B1, F, BF)
    4 CALL MXM    (BF, B1,D)
    5 CALL INVERX (D, DI, DET, IE)
    6 CALL RMATNZ (BO)
    7 CALL MXM    (BF, BO, D2)
    8 CALL MXM    (DI, D2, D3)
    9 CALL MXM    (B1, D3, D4)
   10 CALL MSM    (BO, D4, B)
   11 CALL WRTMAT (B)
   13 CALL MTXM   (B, F, A)
   14 CALL MXM    (A, B, FLEX)
   15 CALL WRTMAT (FLEX)
   16 LOAD = 0
   17 LOAD = LOAD + 1
   18 CALL RMAT   (FORCE)
   19 CALL MXM    (FLEX, FORCE, DELTA)
   20 CALL WRTMAT (DELTA)
   21 CALL MXM    (B, FORCE, FIN)
   22 CALL WRTMAT (FIN)
   23 IF (LOAD .LT. 5 )  GO TO 17
   24 GO TO 1
      END
```

Assume that the structure is subjected to a total of m external forces given by the vector

$$\{F\} = \{F_1 \quad F_2 \quad \cdot \quad \cdot \quad F_m\}.$$

The redundant forces, which are unknown, are denoted by the vector

$$\{X\} = \{X_1 \quad X_2 \quad \cdot \quad \cdot \quad X_n\}.$$

The internal forces S acting within the actual structure are expressed as the total effects of the external forces F and the redundant forces X as

$$\{S\} = [b_0]\{F\} + [b_1]\{X\}$$

where b_0 and b_1 are rectangular matrices with m (number of forces) and n (number of redundants) columns, respectively, and the same number of rows as S. The stress matrix $S_0 = b_0F$ is statically equivalent to the applied loads F, and the stress matrix $S_1 = b_1X$ is self-equilibrating. In the formation of the matrices b_0 and b_1, only equilibrium conditions are considered. When the structure is statically determinate, b_0 is found from the equations of static equilibrium and b_1 does not exist. When the structure is not statically determinate, the matrix b_1 denotes any set of suitable self-equilibrating force systems corresponding to the unit values of the redundant forces.

A suitable self-equilibrating system for a rectangular stiffened panel is shown in Figure 3 (4). The values of stringer loads and shear flows are given in Figure 3 in terms of the forces P and Q. When solving for the b_1 matrix, a unit load is normally applied at the cut; and the induced loads in the surrounding structure are then evaluated relative

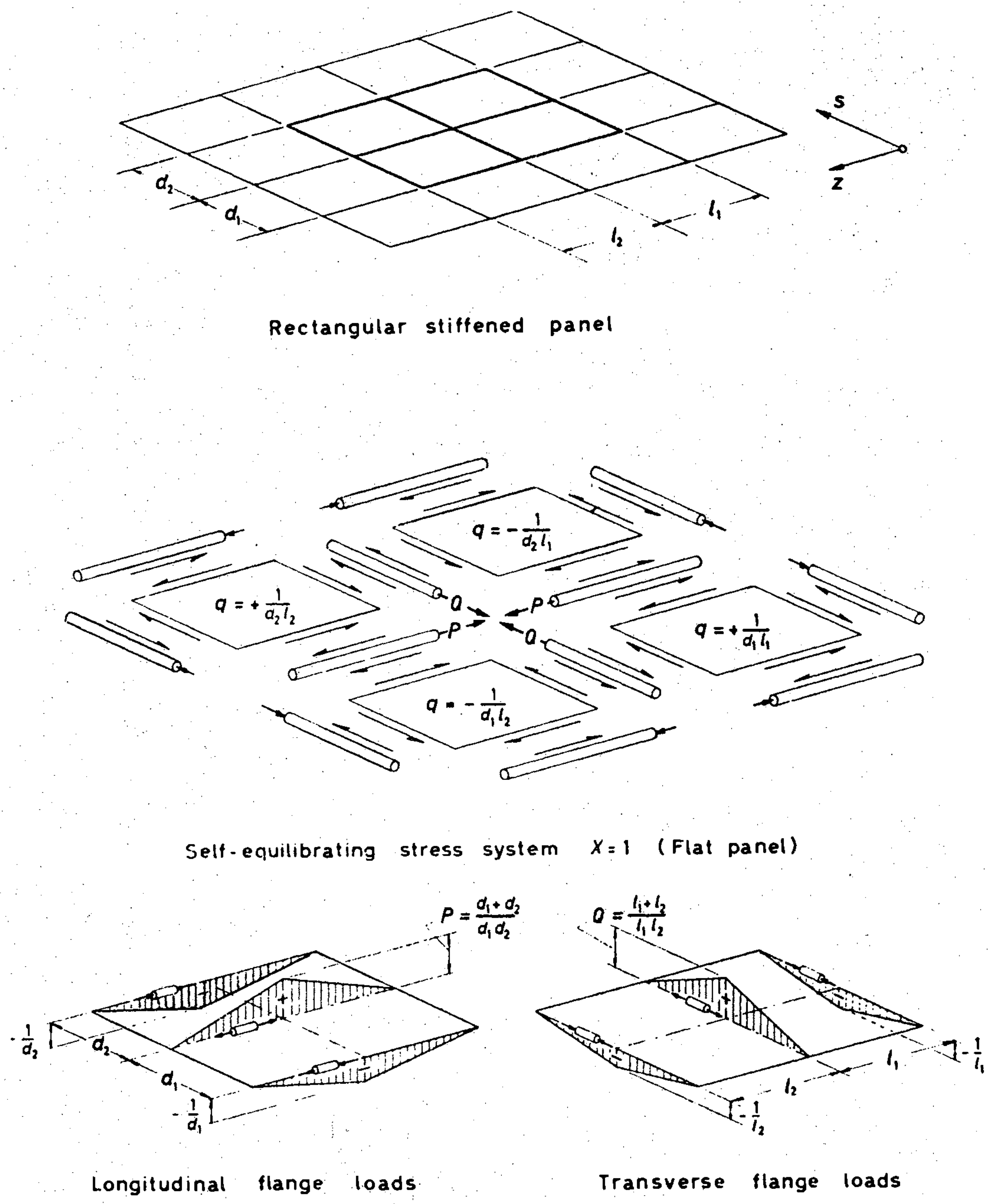


Figure 3. Self-Equilibrating Stress System for the Integrally Reinforced Skin Panel

to the unit load. In actuality, only the relative magnitude of the force at the cut and of the induced loads is required for a complete solution. Hence, the actual magnitude of the force applied at the cut is completely arbitrary. This is shown in page 16.

Compatibility of Deformations

The equation for the compatibility of deformations in the actual structure is

$$\{V_r\} = 0$$

where V_r is a column vector of relative displacements of the redundant forces at the cuts made in the redundant structure.

The deformations V of an element are related to the generalized forces S by the flexibility matrix $\mathcal{F}$ of the element. The coefficients of the flexibility matrix represent the deflections due to unit loads or

$$\{V\} = [\mathcal{F}]\{S\}.$$

To express the compatibility conditions in terms of the applied forces F and the redundant forces X, the relative deformations at the ends or boundaries of the elements are

$$\{V\} = [\mathcal{F}]\{S\} = [\mathcal{F}][b_o]\{F\} + [\mathcal{F}][b_1]\{X\}$$

The compatibility conditions require that the relative displacements of the redundant forces at the cuts made in the redundant structure are zero (4),

$$\{V_r\} = [b_1^T]\{V\} = \{0\}$$

$$[b_1^T][\mathcal{F}][b_1]\{X\} + [b_1^T][\mathcal{F}][b_0]\{F\} = \{0\}.$$

Solving for the redundant forces within the structure,

$$\{X\} = -\left[[b_1^T][\mathcal{F}][b_1]\right]^{-1}\left[[b_1^T][\mathcal{F}][b_0]\{F\}\right].$$

The preceding expression is the general formulation in matrix algebra of the equations for the unknown forces within the structure.

These matrix algebra equations are equivalent to the equations obtained from the application of the unit load method (8). The equations from the unit load methods are of the form

$$\bar{\delta}_a = \bar{\delta}_{ao} + X_a\bar{\delta}_{aa} + X_b\bar{\delta}_{ab} + X_c\bar{\delta}_{ac}$$

$$\bar{\delta}_b = \bar{\delta}_{bo} + X_a\bar{\delta}_{ba} + X_b\bar{\delta}_{bb} + X_c\bar{\delta}_{bc}$$

$$\bar{\delta}_c = \bar{\delta}_{co} + X_a\bar{\delta}_{ca} + X_b\bar{\delta}_{cb} + X_c\bar{\delta}_{cc}$$

where the flexibility coefficients $\bar{\delta}_{ij}$ represent the deflections at point i due to forces at point j.

Comparing this matrix formulation and the unit load method, it is possible to define the matrices D and D_o.

The matrix D is the symmetrical square matrix of the $\bar{\delta}_{ij}$ coefficients or the flexibility matrix for the directions of the unknown forces X in the structure. The matrix D_o is the column matrix of the $\bar{\delta}_{io}$ coefficients for the basic system. The matrix algebra relationships are

$$\left[D\right] = \left[b_1^T\right]\left[\mathcal{F}\right]\left[b_1\right]$$

$$\left[D_o\right] = \left[b_1^T\right]\left[\mathcal{F}\right]\left[b_o\right]\left\{F\right\}.$$

Hence, the expression for the redundant forces is

$$\left\{X\right\} = -\left[D^{-1}\right]\left[D_o\right].$$

Based on the expression for the redundant forces within the structure, the internal loads or stresses are obtained in terms of the applied forces F

$$\left\{S\right\} = \left[b_o\right]\left\{F\right\} + \left[b_1\right]\left[-\left[D\right]^{-1}\left[D_o\right]\right]$$

$$\left\{S\right\} = \left[b\right]\left\{F\right\}$$

where $$\left[b\right] = \left[\left[b_o\right] - \left[b_1\right]\left[\left[b_1^T\right]\left[\mathcal{F}\right]\left[b_1\right]\right]^{-1}\left[\left[b_1^T\right]\left[\mathcal{F}\right]\left[b_o\right]\right]\right].$$

A unit load is generally applied at the cut when determining the distribution of redundant forces within the structure. However, the final solution of the problem requires only the relative magnitude of the induced loads within the structure and the load applied at the cut sections within the structure. This is demonstrated by considering that the matrix b_1 is multiplied by some arbitrary constant C representing something other than a unit load at the cut. Consequently, the internal forces are

$$\{S\} = [b_0]\{F\} + [b_1]\{X\}.$$

Now assume that b_1 is multiplied by some arbitrary constant C, corresponding to a set of redundant forces $\bar{X}$

$$\{S\} = [b_0]\{F\} + c\,[b_1]\{\bar{X}\}$$

$$[\bar{D}] = c\,[b_1^T][\mathcal{F}]\,c\,[b_1] = c^2\,[D]$$

$$[\bar{D}_0] = c\,[b_1^T][\mathcal{F}][b_0] = c\,[D_0]$$

$$\{\bar{X}\} = -[\bar{D}]^{-1}[\bar{D}_0]$$

$$\{\bar{X}\} = \frac{1}{c^2}[D^{-1}]\,c\,[D_0]\{F\} = \frac{1}{c}\{X\}$$

$$\{S\} = [b_0]\{F\} + \not{c}\,[b_1]\frac{1}{\not{c}}\{X\}$$

$$\{S\} = [b_0]\{F\} + [b_1]\{X\}$$

which is identical with the result obtained for a unit load at the cut.

In order to calculate the deflections of points on the structure, it is necessary to determine the flexibility matrix $\bar{\mathcal{F}}$ which relates the applied forces F and their displacements δ according to the equation

$$\{\delta\} = [\bar{\mathcal{F}}]\{F\}$$

which is equivalent to

$$\{F^T\}\{\delta\} = \{F^T\}[\bar{\mathcal{F}}]\{F\}.$$

The work done by the external forces F moving through the displacements δ is $F^T\delta$. The work done by the internal forces S moving through the deformations V is S^T V. If F and S are statically equivalent and δ and V are geometrically compatible, then

$$\{F^T\}\{\delta\} = \{S^T\}\{V\}$$

since

$$\{S\} = [b]\{F\}$$

$$\{S^T\} = \{F^T\}[b^T]$$

and

$$\{F^T\}\{\delta\} = \{F^T\}[b^T]\{V\}$$

but

$$\{V\} = [\mathcal{F}]\{S\} = [\mathcal{F}][b]\{F\}$$

$$\{F^T\}\{\delta\} = \{F^T\}[b^T][\mathcal{F}][b]\{F\}$$

$$[\bar{\mathcal{F}}] = [b^T][\mathcal{F}][b].$$

Analysis of the Test Structure by the Force Method

The application of the force method for the analysis of the rectangular integrally reinforced panel that is described in the experimental investigation, Chapter IV, is shown in Table I. The digital computer program is based on the matrix algebra subroutines in Appendix A. The structure is idealized into the statically determinate basic systems that are described in Figures 4 and 5 (9). The self-equilibrating system, Figure 3, is used for each of the six redundant forces X as in Figure 4.

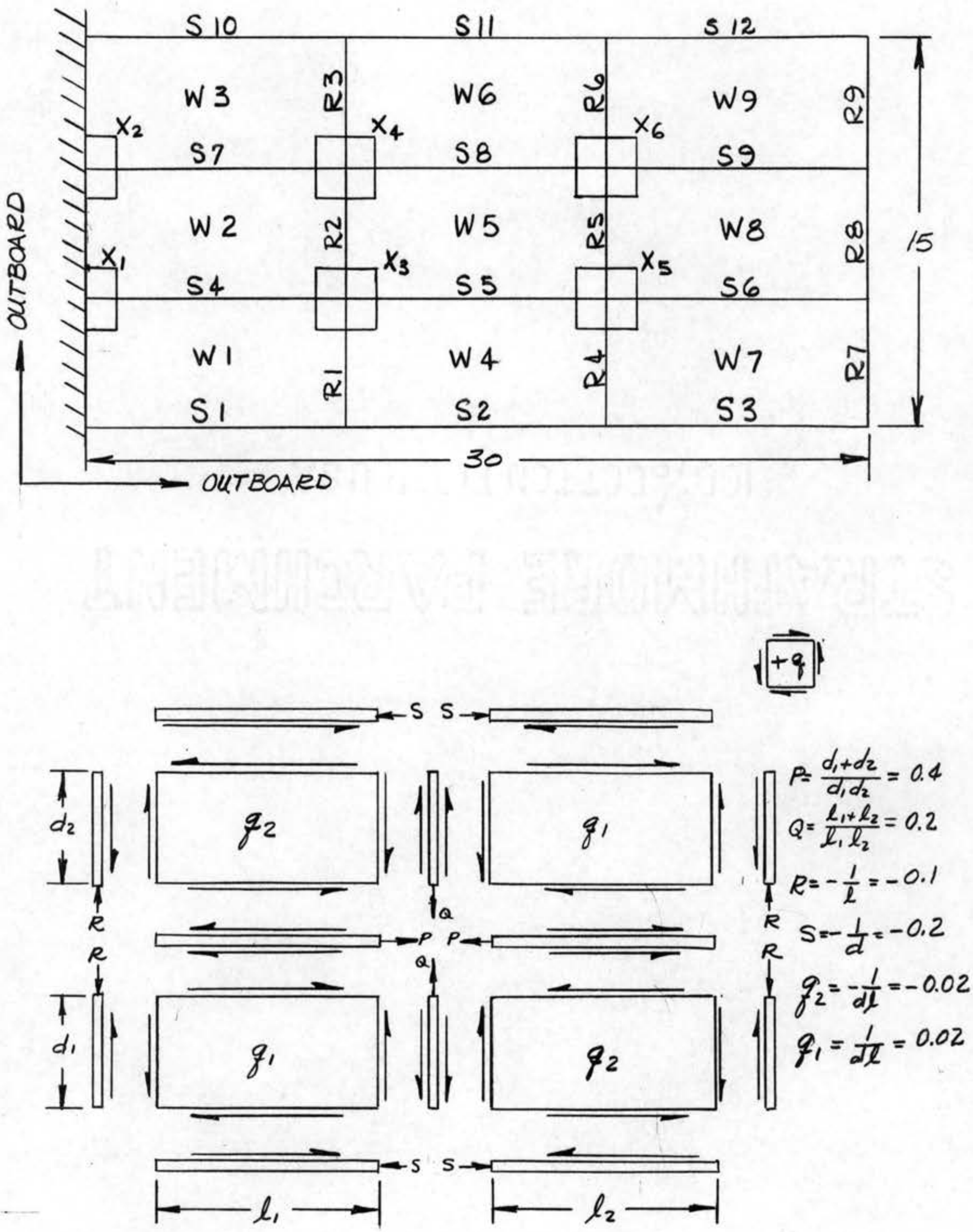


Figure 4. Idealization for Force Method of Analysis

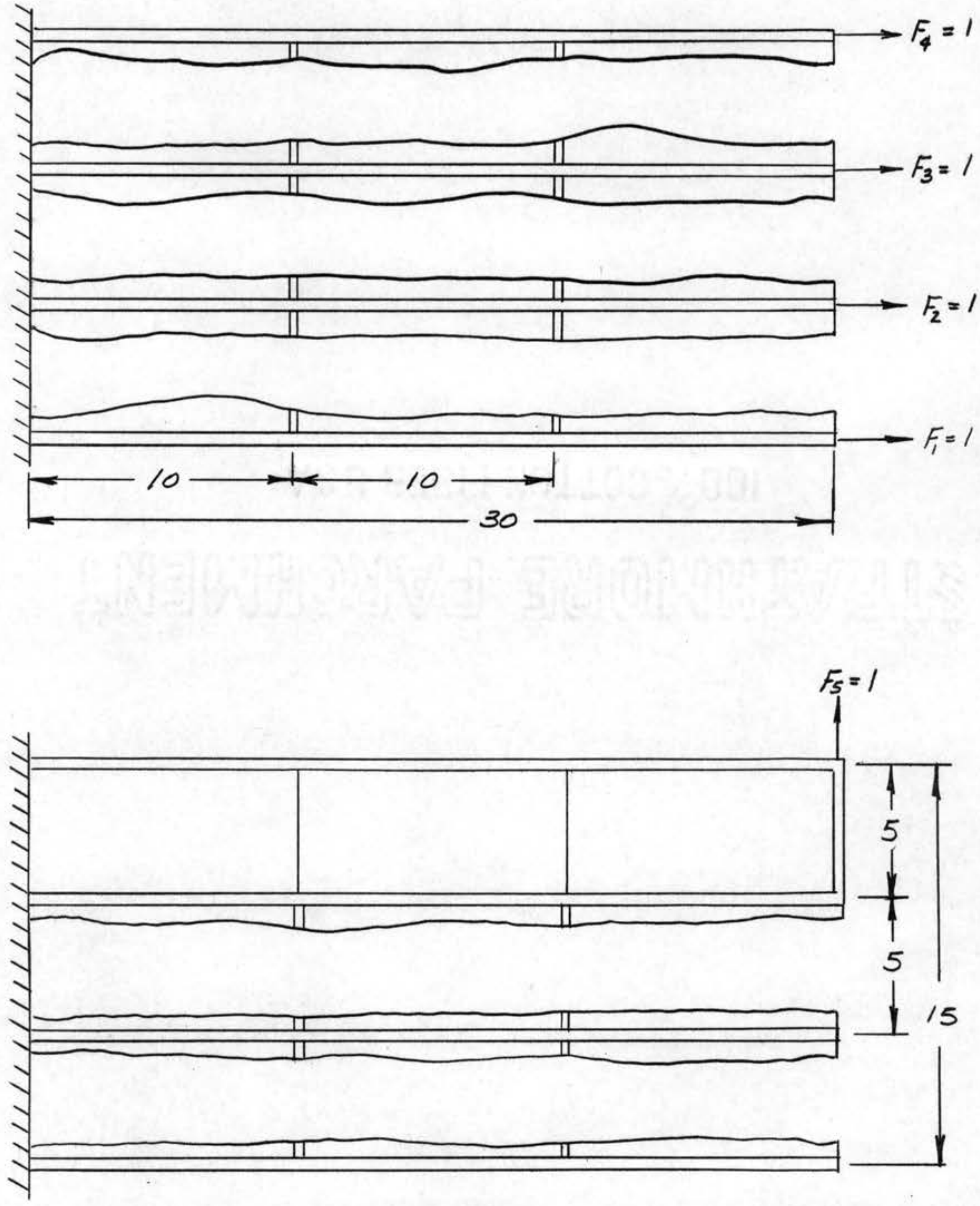


Figure 5. Statically Determinate Systems for Unit External Loads

Unit Load Matrices

The unit external load matrix b_0 and the unit redundant load matrix b_1 are given in tabular form on Table II. In the force method, it is necessary to specify the forces on each side of a junction, although the forces are the same. Therefore, there are 51 rows in the b_0 and the b_1 matrices. The 51 rows correspond to 24 rows for the stringer elements, S1 through S12; 18 rows for the rib elements, R1 through R9; 9 rows for the web elements, W1 through W9.

The element numbering system is shown in Figure 4. Also, the outboard directions are defined in Figure 4. In Table II, the outboard and inboard ends of an element are designated 0 and I, respectively. The unit external load matrix b_0 is formulated by assuming that the external unit loads, F_1 through F_4, are transmitted directly inboard through their respective stringers while the transverse load F_5 is carried by elements S7 through S12, R9, W3, W6, and W9 acting as a cantilever beam. The unit redundant load matrix b_1 is formulated using six of the self-equilibrating systems shown in Figure 3 at the locations shown in Figure 4.

Effective Flange Areas

In accounting for the axial-load-carrying capability of the web elements of the structure, the area of the webs is generally lumped with the stringers and ribs as effective flange areas. The effective flange areas transmit all axial forces acting on the structure; and, consequently, represent the axial stresses in both the actual flanges and the webs.

TABLE II

UNIT LOAD MATRICES

		Unit External Load Matrix b_o					Unit Redundant Load Matrix b_1					
Row	Point	F_1	F_2	F_3	F_4	F_5	X_1	X_2	X_3	X_4	X_5	X_6
1	S1-I	1					-0.20					
2	S1-0	1							-0.20			
3	S2-I	1							-0.20			
4	S2-0	1									-0.20	
5	S3-I	1									-0.20	
6	S3-0	1										
7	S4-I		1				0.40	-0.20				
8	S4-0		1						0.40	-0.20		
9	S5-I		1						0.40	-0.20		
10	S5-0		1								0.40	-0.20
11	S6-I		1								0.40	-0.20
12	S6-0		1									
13	S7-I			1		6	-0.20	0.40				
14	S7-0			1		4			-0.20	0.40		
15	S8-I			1		4			-0.20	0.40		
16	S8-0			1		2					-0.20	0.40
17	S9-I			1		2					-0.20	0.40
18	S9-0			1								
19	S10-I				1	-6		-0.20				
20	S10-0				1	-4				-0.20		
21	S11-I				1	-4				-0.20		
22	S11-0				1	-2						-0.20
23	S12-I				1	-2						-0.20
24	S12-0				1							
25	R1-I											
26	R1-0						-0.10		0.20		-0.10	
27	R2-I						-0.10		0.20		-0.10	
28	R2-0							-0.10		0.20		-0.10
29	R3-I							-0.10		0.20		-0.10
30	R3-0											
31	R4-I											
32	R4-0								-0.10		0.20	
33	R5-I								-0.10		0.20	
34	R5-0									-0.10		0.20
35	R6-I									-0.10		0.20
36	R6-0											
37	R7-I											
38	R7-0										-0.10	
39	R8-I										-0.10	
40	R8-0											-0.10
41	R9-I											-0.10
42	R9-0					1						
43	W1						-0.20		0.20			
44	W2						0.02	-0.02	-0.02	0.02		
45	W3					0.20		0.02		-0.02		
46	W4								-0.02		0.02	
47	W5								0.02	-0.02	-0.02	0.02
48	W6					0.20						
49	W7										-0.02	
50	W8										0.02	-0.02
51	W9					0.20						0.02

The effective areas for the outboard stringer area are 0.375 square inches; for the central stringer area, 0.325 square inches; for the outboard rib area, 0.50 square inches; and for the central rib area, 0.625 square inches.

Element Flexibility Matrix

The flexibility matrix is a partitioned diagonal matrix with 30 submatrices, one for each structural element. The 12-stringer and the 9-rib flexibility matrices are 2 x 2 matrices of the form

$$\mathcal{F} = \frac{1}{E}\begin{bmatrix} \frac{L}{3A} & \frac{L}{6A} \\ \frac{L}{6A} & \frac{L}{3A} \end{bmatrix}.$$

The web flexibility matrices are one-element matrices of the form

$$\mathcal{F} \quad \frac{A}{Gt}.$$

The expanded flexibility matrix is, therefore, a 51 x 51 symmetric matrix with 93 nonzero elements. The flexibility submatrices for the stringer elements are

$$\mathcal{F}_{S1} = \mathcal{F}_{S2} = \mathcal{F}_{S3} = \mathcal{F}_{S10} = \mathcal{F}_{S11} = \mathcal{F}_{S12} = 10^{-7}\begin{bmatrix} 8.386 & 4.193 \\ 4.193 & 8.386 \end{bmatrix}$$

$$\mathcal{F}_{S4} = \mathcal{F}_{S5} = \mathcal{F}_{S6} = \mathcal{F}_{S7} = \mathcal{F}_{S8} = \mathcal{F}_{S9} = 10^{-7}\begin{bmatrix} 9.676 & 4.838 \\ 4.838 & 9.676 \end{bmatrix}.$$

The flexibility submatrices for the rib elements are

$$\mathcal{F}_{R1} = \mathcal{F}_{R2} = \mathcal{F}_{R3} = \mathcal{F}_{R4} = \mathcal{F}_{R5} = \mathcal{F}_{R6} = 10^{-7}\begin{bmatrix} 2.516 & 1.258 \\ 1.258 & 2.516 \end{bmatrix}$$

$$\mathcal{F}_{R7} = \mathcal{F}_{R8} = \mathcal{F}_{R9} = 10^{-7}\begin{bmatrix} 3.145 & 1.572 \\ 1.572 & 3.145 \end{bmatrix}.$$

The flexibility submatrices for the web elements are

$$\mathcal{F}_{W1} \cdots \mathcal{F}_{W9} = \frac{A}{Gt} = 2.516 \times 10^{-7}.$$

These submatrices are combined to form the flexibility matrix for the structure as shown in Table III. The stress and deflection results of the force method of analysis for the five load configurations studied in the experimental investigation are given in Chapter V.

TABLE III

FLEXIBILITY MATRIX FOR STRUCTURAL PANEL ELEMENTS

$$10^7 \left[\mathcal{F} \right]$$

Row	Col.	Coef.	Row	Col.	Coef.	Row	Col.	Coef.
1	1	8.386	17	17	9.676	32	31	1.258
1	2	4.193	17	18	4.838	32	32	2.516
2	1	4.193	18	17	4.838	33	33	2.516
2	2	8.386	18	18	9.676	33	34	1.258
3	3	8.386	19	19	8.386	34	33	1.258
3	4	4.193	19	20	4.193	34	34	2.516
4	3	4.193	20	19	4.193	35	35	2.516
4	4	8.386	20	20	8.386	35	36	1.258
5	5	8.386	21	21	8.386	36	35	1.258
5	6	4.193	21	22	4.193	36	36	2.516
6	5	4.193	22	21	4.193	37	37	3.145
6	6	8.386	22	22	8.386	37	38	1.572
7	7	9.676	23	23	8.386	38	37	1.572
7	8	4.838	23	24	4.193	38	38	3.145
8	7	4.838	24	23	4.193	39	39	3.145
8	8	9.676	24	24	8.386	39	40	1.572
9	9	9.676	25	25	2.516	40	39	1.572
9	10	4.838	25	26	1.258	40	40	3.145
10	9	3.838	26	25	1.258	41	41	3.145
10	10	9.676	26	26	2.516	41	42	1.572
11	11	9.676	27	27	2.516	42	41	1.572
11	12	4.838	27	28	1.258	42	42	3.145
12	11	4.838	28	27	1.258	43	43	2.516
12	12	9.676	28	28	2.516	44	44	2.516
13	13	9.676	29	29	2.516	45	45	2.516
13	14	4.838	20	30	1.258	46	46	2.516
14	13	4.838	30	29	1.258	47	47	2.516
14	14	9.676	30	30	2.516	48	48	2.516
15	15	9.676	31	31	2.516	49	49	2.516
15	16	4.838	31	32	1.258	50	50	2.516
16	15	4.838				51	51	2.516
16	16	9.676						

CHAPTER III

STIFFNESS METHOD OF ANALYSIS

The direct stiffness method is a finite element method of structural analysis which considers a structure to be an assembly of idealized elastic elements which are assumed to be joined only at discrete points called nodes. The stiffness method is a contrast to the force method, which is described in Chapter II, in that displacements, not forces, are the initial unknown quantities. The concept of redundant load paths illustrated in Chapter II is not applicable in the stiffness method of analysis because of the treatments of the node displacements as unknown quantities. The relationship of forces and of displacements is defined for the node points on the structure by the stiffness matrix. The stiffness matrix for the complete structure is obtained by adding the stiffness coefficients for common degrees of freedom of adjacent elements at each node on the structure. The summed stiffness coefficients define the coefficients for the linear algebraic equations relating the nodal forces and the nodal displacements of the complete structure. The general stiffness coefficient K_{jh} is the force in the direction j due to the unit displacement in the direction h, while all other displacements are zero. As a result of equilibrium conditions, the stiffness matrix is a positive definite, symmetric matrix; and the sum of the coefficients along any row or column of the stiffness matrix is equal to zero.

The forces and deflections in each element of the structure are related by an assumed stress-strain relationship for the idealized element. The displacements of the nodes in the structure are considered as the initial unknown quantities. An infinite number of mutually compatible deformations of the elements are possible; the correct pattern of displacements of the elements is the one for which the equations of equilibrium are satisfied.

If the idealized structural elements for which the stiffness coefficients are known are combined for a continuous structure, the composite stiffness matrix for the total structure is assembled as

$$\begin{bmatrix} K_{11} & K_{12} & \cdot & K_{1h} & K_{1M} \\ K_{21} & K_{22} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{j1} & \cdot & \cdot & K_{jh} & K_{jM} \\ K_{M1} & \cdot & \cdot & K_{Mh} & K_{MM} \end{bmatrix}$$

where each K_{jh} is the stiffness coefficient representing the total force component produced at node j due to a corresponding unit displacement component at node h.

The stiffness matrix relates the external forces acting at the nodes on the structure to the displacements of the nodes through the expression

$$\{F\} = [K]\{\delta\}.$$

The expression for nodal displacements δ as a function of the external forces or loads F is obtained by inverting the stiffness matrix and is

$$\{\delta\} = [K^{-1}]\{F\}.$$

A matrix of stress coefficients is derived by using the same strain pattern for the elastic element that is assumed in deriving the stiffness coefficients.

The algebraic equations which express the stresses σ within the elements as a function of its nodal displacement δ are given by the stress coefficient matrix $\hat{S}$

$$\{\sigma\} = [\hat{S}]\{\delta\}.$$

The stresses within the elements are determined subsequent to the calculation of the node displacements. The forces at all nodes on the structure can also be determined from the stiffness matrix once the node displacements are available. Determining the forces at each node is desirable for establishing equilibrium conditions for the structure.

The application of the stiffness method involves determining the stiffness coefficients of the idealized structural elements required to represent accurately a specific structure and using these coefficients to develop the simultaneous equations relating forces and displacements for the structure. Subsequent to the calculation of deflections, the stresses are calculated using stress coefficients based on the same assumptions that are made in deriving the stiffness coefficients. The stiffness and stress coefficients for the integrally reinforced rectangular skin panel used in this research program are derived in the remainder of this chapter. The application of the stiffness method for the analysis of the test structure described in Chapter V is made possible by the Stress Analysis System digital computer program, which is described in Chapter IV. The Stress Analysis System provides a complete analysis and requires only a geometric description of the structure.

The integral reinforcements within the structural skin panel described in Chapter V are represented by idealized axial force elements called stringer or rib elements. The web sections of the test panel are represented by idealized plane stress elements called panels or plates.

The remainder of this chapter describes the derivations of the stiffness and stress matrices for each type of element that is used in the Stress Analysis System digital computer program, which is described in Chapter IV. Additional elements required for different structural configurations are obtained in a similar manner.

The formulation of the stiffness and stress coefficient matrices for idealized structural elements is indicated by the application of the principles of virtual work to the stringer-type element. This method is a contrast to the method of direct geometrical relationships for the same type of idealized element discussed by Turner, et al. (3). However, the results for the first stringer-type element are the same as those obtained by Turner, et al. (3). The method of direct geometrical relationships is very satisfactory for some types of idealized elements; however, the approach becomes less desirable as the assumed behavior of the elements becomes more complex. The subsequent derivations of stiffness and stress coefficient matrices for idealized stringer and plate-type elements are also based on energy methods of structural analysis. However, the basic approach is less difficult conceptually for the stringer-type element.

Stiffness Derivation for Stringer-Type Element

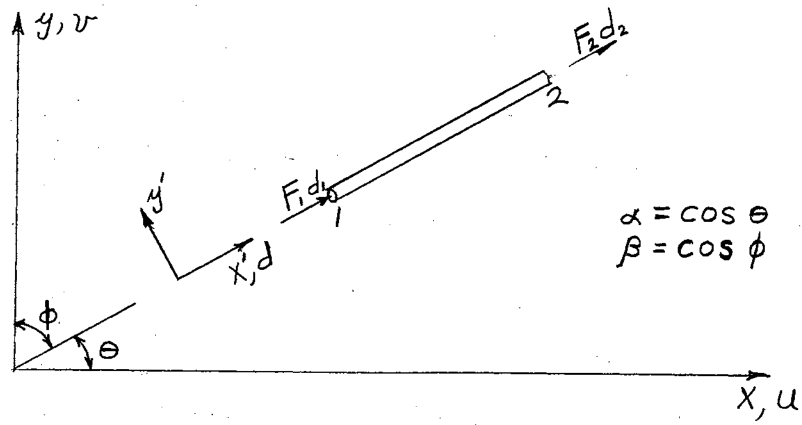


Figure 6. Stringer Element

The assumed stress-strain relationship for the stringer element is

$$\{P\} = [k]\{v\}.$$

The stringer is subjected to a set of external forces

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

and the displacements along their lines of action are represented by

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}.$$

The internal forces in the element are represented by

$$\{P\}.$$

The strain or deformation of the element is

$$\{v\}.$$

The compatibility relation between the strains v and the displacements d is expressed by

$$\{v\} = [a]\{d\}.$$

The coefficients of the ith column of the geometric matrix $[a]$ are the relation between v and $d_i = 1$. These coefficients are interpreted as the values of strain due to a unit displacement d_i when all other displacements remain zero.

The equilibrium condition between the internal forces P and the external forces F are obtained by the principle of virtual work. The statement of the principle is:

> The work done by a set of external forces, F, moving through the associated displacements, d , is equal to the work done by a set of statically equivalent internal forces, P, moving through the associated deformation v (10).

The work done by the external force F moving through the displacement d is

$$\text{work} = \{F\}^T\{d\} = \{d\}^T\{F\}.$$

The work done by the internal forces P moving through the deformation v is

$$\text{work} = \{P\}^T\{v\} = \{v\}^T\{P\}.$$

The forces F and P are statically equivalent; d and v are geometrically compatible.

From the compatibility condition

$$\{v\} = [a]\{d\} \qquad \{v\}^T = \{d\}^T[a]^T$$

$$\{d\}^T\{F\} = \{d\}^T[a]^T\{P\}.$$

For any set of displacements d the equilibrium between internal and external forces is

$$\{F\} = [a]^T\{P\}.$$

Assuming the material obeys Hooke's Law, the stress-strain relationship for the stringer is

$$\{P\} = [k]\{v\}.$$

Since

$$\{F\} = [a]^T\{P\}$$
$$= [a]^T[k]\{v\}$$
$$\{F\} = [a]^T[k][a]\{d\}.$$

Since the stiffness matrix is defined by the equation as

$$\{F\} = [K]\{d\}$$

then the stiffness matrix for the element is

$$[K] = [a]^T[k][a].$$

Assume that the displacement distribution for the stringer is represented by the linear relationship

$$d = C_1 + C_2 x'$$

where x' refers to the local coordinate system along the axis of the stringer element.

The constants C_1 and C_2 are determined from the boundary conditions

$$x' = 0 \;, \quad d = u_1$$
$$x' = \ell \;, \quad d = u_2$$

hence
$$d = u_1 + (u_2 - u_1)\frac{x'}{\ell}$$

and
$$\{v\} = \frac{\partial d}{\partial x'} = \frac{1}{\ell}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}.$$

Thus, the matrix of compatible strains for unit element displacements for a stringer element is given by

$$[a] = \frac{1}{\ell}\begin{pmatrix} -1 & 1 \end{pmatrix}$$

and
$$[K] = \int_{Vol} [a]^T [k] [a] \, dV$$

$$[K] = \int_0^{\ell} [a]^T [k] [a] \cdot Area \cdot dx'$$

$$[K] = \frac{AE}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

To transform into two dimensions, let α, β be the direction cosines for the axis of the stringer and the two coordinate axes as shown in Figure 6.

$$\{\bar{F}\} = \lambda \{F\} \quad \text{and} \quad \{\delta\} = \lambda \{d\}$$

$$\{\bar{F}\} = [\lambda]^{\prime}[K][\lambda]\{\delta\}$$

where
$$[\lambda] = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ 0 & 0 & \alpha & \beta \end{bmatrix}.$$

The stiffness matrix relative to the two-dimensional coordinate system is obtained from the stiffness matrix for the local coordinate system by λ , the transformation matrix of direction cosines. The stiffness matrix for the two-dimensional coordinate system is expressed by the force-deflection relationship

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \frac{AE}{\ell} \begin{bmatrix} \alpha^2 & \alpha\beta & -\alpha^2 & -\alpha\beta \\ \alpha\beta & \beta^2 & -\alpha\beta & -\beta^2 \\ -\alpha^2 & -\alpha\beta & \alpha^2 & \alpha\beta \\ -\alpha\beta & -\beta^2 & \alpha\beta & \beta^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} .$$

The stress within the element is determined from the equation for strain transformed into the two-dimensional coordinate system by the coordinate transformation matrix λ . The coordinate transformation results in the following equation for the stress in the stringer element.

$$\{\sigma\} = \frac{E}{\ell} \lfloor -\alpha \quad -\beta \quad \alpha \quad \beta \rfloor \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Stringer Element With Linear Strain Variation

If the stringer-type and rib-type element experiences a linear change in strain or stress variation due to the effect of shear transfer of load to the web, then the strain function is of the form

$$\epsilon_{x'} = C_1 + C_2 x' .$$

The corresponding displacement function is obtained from integration

$$d = C_1 x' + \frac{C_2}{2} x'^2 + C_3 .$$

The constants are evaluated from the following conditions

1. $d = u_1 \quad @ \quad x' = 0$

2. $\epsilon_x = \frac{\partial d}{\partial x'} = 0 \quad @ \quad x' = 0$

3. $d = u_2 \quad @ \quad x' = \ell$

from 1 $C_3 = u_1$ from 2 $C_1 = 0$ from 3 $C_2 = \frac{(u_2 - u_1)}{\ell^2}$

hence

$$d = u_1 + \left(\frac{u_2 - u_1}{\ell^2}\right) x'^2 .$$

The matrix of compatible strains for unit element displacements for the element is

$$\frac{\partial d}{\partial x} = \frac{2}{\ell^2}\left(-x' \; x'\right) \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [a] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

and

$$[K] = \int_0^{\ell} [a]^T [k] [a] \cdot Area \cdot dx' .$$

Hence, for the local coordinate system along the axis of the element

$$[K] = \frac{AE}{\ell} \begin{bmatrix} \frac{4}{3} & -\frac{4}{3} \\ -\frac{4}{3} & \frac{4}{3} \end{bmatrix} .$$

The stresses within the element are determined from the expression

$$\sigma = \frac{F_1}{A} + \frac{x'}{A\ell}\left(F_2 - F_1\right).$$

The stress at the center of the element is

$$\sigma = \frac{E}{\ell} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}.$$

The stiffness and stress matrices can be obtained relative to the two-dimensional coordinate system using the coordinate transformation matrix λ discussed for the first stringer element.

Stiffness Derivations for Panel Elements

The rectangular web sections of the integrally reinforced rectangular skin panel are idealized as plate- or panel-type elements that resist both shear and axial loads. Different stiffness and stress matrices are obtained depending on the assumed mode of behavior of the element.

The plate-type elements available in the Stress Analysis System program consist of state-of-the-art derivations based on an assumed displacement function, an assumed stress function with five coefficients, and a new rectangular plate stiffness matrix using an assumed stress function with linear variations in two directions. The three different techniques used for deriving these element stiffness matrices may be applied to the development of stiffness and stress coefficient matrices for arbitrary geometric configurations of idealized elements transmitting forces in the plane of the elements.

Rectangular Plate With Assumed Displacements

The origin of the local coordinate system is assumed to be at the lower left-hand corner of the rectangular plate as shown in Figure 7. Nondimensional coordinates

$$\bar{x} = \frac{x}{a} \qquad \bar{y} = \frac{y}{b}$$

are introduced to simplify the analysis. The lengths a and b are the dimensions of the rectangular panel in the x and y directions, respectively,

The deflection of the element is represented by the displacements of the four corners. Consequently, there are eight displacements U_1, V_1, U_2, V_2, U_3, V_3, U_4, V_4; and they are measured positive along the positive x and y axes.

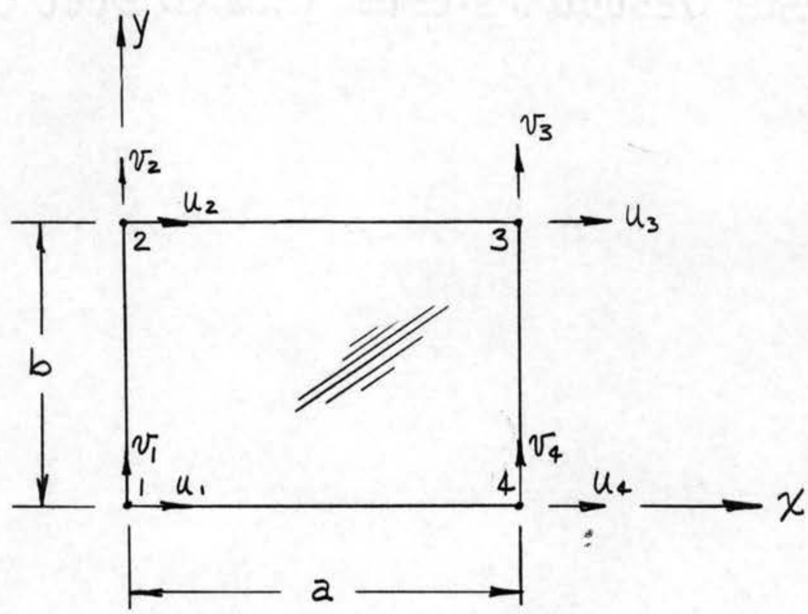


Figure 7. Plate Element With Assumed Displacement Function

A more general derivation of the element stiffness properties for a non-rectangular configuration based on the same element idealization is given by Cook (11).

A simple displacement function based on the assumption of linearly varying boundary displacements and in terms of the dimensionless coordinate is (12)

$$u = C_1 \bar{x} + C_2 \bar{x}\bar{y} + C_3 \bar{y} + C_4$$
$$v = C_5 \bar{x} + C_6 \bar{x}\bar{y} + C_7 \bar{y} + C_8 .$$

The eight arbitrary constants C_1 through C_8 are determined from the displacements in the x and y directions at the four corners of the model.

The unknown constants C_1 through C_8 are evaluated from the boundary conditions

$$u = u_1 \ \& \ v = v_1 \ @ \ (0,0)$$
$$u = u_2 \ \& \ v = v_2 \ @ \ (0,1)$$
$$u = u_3 \ \& \ v = v_3 \ @ \ (1,1)$$
$$u = u_4 \ \& \ v = v_4 \ @ \ (1,0).$$

The displacement functions are

$$u = u_1 (1-\bar{x})(1-\bar{y}) + u_2 (\bar{y})(1-\bar{x}) + u_3 (\bar{x}\bar{y}) + u_4 (\bar{x})(1-\bar{y})$$
$$v = v_1 (1-\bar{x})(1-\bar{y}) + v_2 (\bar{y})(1-\bar{x}) + v_3 (\bar{x}\bar{y}) + v_4 (\bar{x})(1-\bar{y}).$$

The strain of the element is obtained by differentiation. By definition

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \qquad \epsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$= \frac{1}{a}\frac{\partial u}{\partial \bar{x}} \qquad = \frac{1}{b}\frac{\partial v}{\partial \bar{y}} \qquad = \frac{1}{b}\frac{\partial u}{\partial \bar{y}} + \frac{1}{a}\frac{\partial v}{\partial \bar{x}} .$$

The complete strain-displacement relationships are obtained for the strains ϵ_{xx} ϵ_{yy} γ_{xy} in terms of the displacements $\left\{ \begin{matrix} u \\ v \end{matrix} \right\}$

$$\epsilon_{xx} = \frac{1}{a}\left[u_1(1-\bar{y})(-1) + u_2(\bar{y})(-1) + u_3\bar{y} + u_4(1-\bar{y})\right]$$

$$\epsilon_{yy} = \frac{1}{b}\left[v_1(1-\bar{x})(-1) + v_2(1-\bar{x}) + v_3\bar{x} + v_4\bar{x}(-1)\right]$$

$$\begin{aligned}\gamma_{xy} = \frac{1}{b}&\left[u_1(1-\bar{x})(-1) + u_2(1-\bar{x}) + u_3\bar{x} + u_4\bar{x}(-1)\right] + \\ \frac{1}{a}&\left[v_1(1-\bar{y})(-1) + v_2(\bar{y})(-1) + v_3(-\bar{y}) + v_4(1-\bar{y})\right]\end{aligned}$$

or in matrix notation

$$\{\epsilon\} = [A]\left\{ \begin{matrix} u \\ v \end{matrix} \right\}$$

where the coefficients of A contain the dimensionless coordinates on the surface of the element.

$$\left\{ \begin{matrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{matrix} \right\} = \begin{bmatrix} \frac{\bar{y}-1}{a} & 0 & \frac{-\bar{y}}{a} & 0 & \frac{-\bar{y}}{a} & 0 & \frac{1-\bar{y}}{a} & 0 \\ 0 & \frac{\bar{x}-1}{b} & 0 & \frac{1-\bar{x}}{b} & 0 & \frac{\bar{x}}{b} & 0 & -\frac{\bar{x}}{b} \\ \frac{\bar{x}-1}{b} & \frac{\bar{y}-1}{a} & \frac{1-\bar{x}}{b} & \frac{-\bar{y}}{a} & \frac{\bar{x}}{b} & \frac{\bar{y}}{b} & \frac{-\bar{x}}{b} & \frac{1-\bar{y}}{a} \end{bmatrix} \left\{ \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{matrix} \right\}$$

When Hook's Law applies, the stresses are related to the strains by

$$\{\sigma\} = [B]\{\epsilon\}$$

where the coefficients of B are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} .$$

The stresses within the idealized element can now be expressed in terms of the displacements of the corner nodes of the idealized element.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{3E}{8ab} \begin{bmatrix} 3(\bar{y}-b) & \bar{x}-a & -3\bar{y} & a-\bar{x} & 3\bar{y} & \bar{x} & 3(b-\bar{y}) & -\bar{x} \\ \bar{y}-b & 3(x-a) & -\bar{y} & 3(a-\bar{x}) & \bar{y} & 3\bar{x} & b-\bar{y} & -3\bar{x} \\ \bar{x}-a & \bar{y}-b & a-\bar{x} & -\bar{y} & \bar{x} & \bar{y} & -\bar{x} & b-\bar{y} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

The stiffness matrix K is obtained from the unit displacement theorem. Since the matrix A is a function of the position variables, the integration is performed with respect to $\bar{\bar{X}}$ and $\bar{\bar{Y}}$ between the limits $\bar{\bar{X}} = 0$ to 1 and $\bar{\bar{Y}} = 0$ to 1. The unit displacement theorem provides

where A is the relationship between strain and node displacements and B is the relationship between stress and strain.

The stiffness matrix for the panel shown in Figure 7 for $\nu = 1/3$ is

$$\frac{Et}{16ab}\begin{bmatrix} 2a^2+6b^2 & & & & & & & \\ 3ab & 6a^2+2b^2 & & & & & & \\ -2a^2+3b^2 & 0.0 & 2a^2+6b^2 & & & & & \\ 0.0 & -6a^2+b^2 & -3ab & 6a^2+2b^2 & & & & \\ -a^2-3b^2 & -3ab & a^2-6b^2 & 0.0 & 2a^2+6b^2 & & & \\ -3ab & -3a^2-b^2 & 0.0 & 3a^2-2b^2 & 3ab & 6a^2+2b^2 & & \\ a^2-6b^2 & 0.0 & -a^2-3b^2 & 3ab & -2a^2+3b^2 & 0.0 & 6b^2+2a^2 & \\ 0.0 & 3a^2-2b^2 & 3ab & -3a^2-b^2 & 0.0 & -6a^2+b^2 & -3ab & 6a^2+2b^2 \end{bmatrix}$$

Rectangular Plate With Assumed Stresses

A limitation of the results of the previous type of derivation is that the equilibrium conditions are satisfied within the element only for a specific set of relative displacements of the corner nodes.

A second stiffness and stress matrix is derived using an assumed stress variation within the element that can be evaluated using only the boundary conditions expressed in terms of the corner displacements of the element. By using only five undetermined coefficients, the stiffness and stress matrices can be obtained from the node displacements of the element shown in Figure 8.

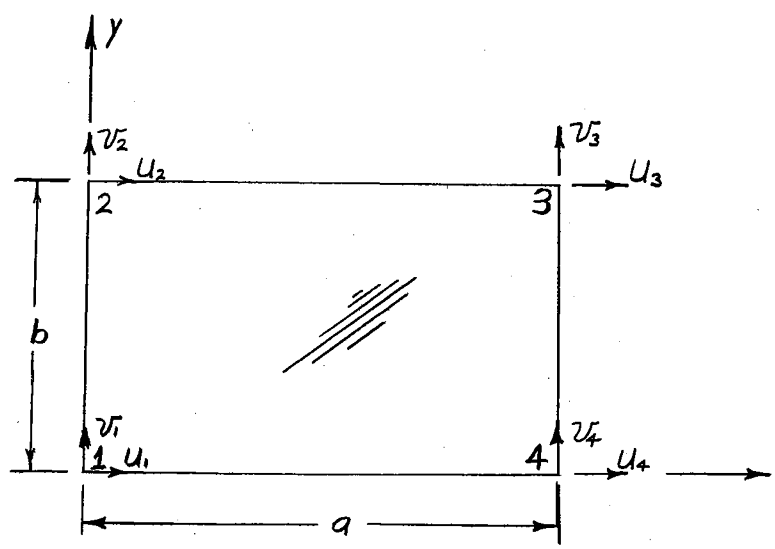


Figure 8. Plate Element With Assumed Stress Function

The stress distribution first used by Turner, et al. (3), is

$$\sigma_x = a_1 + a_2 y$$
$$\sigma_y = a_3 + a_4 x$$
$$\tau_{xy} = a_5$$

This assumption satisfies exactly the stress equilibrium equations within the rectangle; however, the resulting displacement distribution violates the compatibility of boundary displacements on adjacent elements. Using Hooke's Law, the relationship between stresses and strains for the plane stress condition is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}.$$

Defining strain in terms of the displacement functions u and v

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

or in terms of the stresses

$$\frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \nu \sigma_y)$$

$$\frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \nu \sigma_x)$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{2(1+\nu)}{E} \tau_{xy}.$$

Based on the assumed stress distribution, the strains within the element in terms of the five undetermined coefficients are

$$\frac{\partial u}{\partial x} = \frac{1}{E}\left(a_1 + a_2 y - \nu a_3 - \nu a_4 x\right)$$

$$\frac{\partial v}{\partial y} = \frac{1}{E}\left(a_3 + a_4 x - \nu a_1 - \nu a_2 y\right)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{2(1+\nu)}{E}(a_5).$$

The displacement functions u and v are obtained from the integration of the strain functions

$$u_x = \frac{1}{E}\left(a_1 x + a_2 x y - \nu a_3 x - \frac{\nu a_4 x^2}{2} + f(y)\right) = \eta + f(y)$$

where f(y) is some arbitrary function of y and $\eta = \epsilon_x$.

Likewise

$$v_y = \frac{1}{E}\left(a_3 y + a_4 x y - \nu a_1 y - \frac{\nu a_2 y^2}{2} + g(x)\right) = \phi + g(x)$$

where g(x) is some arbitrary function of X and $\phi = \epsilon_y$.

The constants of integration f(y) and g(x) are determined from the shear expression

$$\frac{\partial u}{\partial y} = \frac{1}{E}\left(a_2 x + f'(y)\right) \qquad \frac{\partial v}{\partial x} = \frac{1}{E}\left(a_4 y + g'(x)\right)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{E}\left(a_2 x + f'(y)\right) + \frac{1}{E}\left(a_4 y + g'(x)\right)$$

$$= \frac{2(1+\nu)}{E}(a_5).$$

Solving for g(x) and f(y)

$$f'(y) + a_4(y) = 2(1+\nu)\,a_5 - \left(a_2 x + g'(x)\right) = a_6$$

$$f(y) = a_6 y - \frac{a_4 y^2}{2} + a_7$$

$$g'(x) + a_2(x) = 2(1+\nu)\,a_5 - a_6$$

$$g(x) = \left[2(1+\nu)\,a_5 - a_6\right] x - \frac{a_2 x^2}{2} + a_8$$

where the new undetermined coefficients a_7 and a_8 are rigid body translations and a_6 is the rigid body rotation of the element.

The assumed stress distribution results in displacement functions of the form

$$u_x = \frac{1}{E}\left(a_1 x + a_2 xy - \nu a_3 x - \frac{\nu a_4 x^2}{2} + a_6 y - \frac{a_4}{2} y^2 + a_7\right)$$

$$v_y = \frac{1}{E}\left(a_3 y + a_4 xy - \nu a_1 y - \nu \frac{a_2 y^2}{2} + 2(1+\nu) a_5 x - a_6 x - \frac{a_2 x^2}{2} + a_8\right)$$

which can be arranged in the form

$$u_x = C_1 x + C_2 y - C_3\left(\nu x^2 + y^2\right) + 2 C_4 xy + C_5$$

$$v_y = C_6 x + C_7 y - C_4\left(x^2 + \nu y^2\right) + 2 C_3 xy + C_8$$

where

$$C_1 = \frac{1}{E}(a_1 - \nu a_3) \qquad C_5 = \frac{a_7}{E}$$

$$C_2 = \frac{a_6}{E} \qquad C_6 = \frac{1}{E}\left(2(1+\nu)a_5 - a_6\right)$$

$$C_3 = \frac{a_4}{2E} \qquad C_7 = \frac{1}{E}(a_3 - \nu a_1)$$

$$C_4 = \frac{a_2}{2E} \qquad C_8 = \frac{a_8}{E}\,.$$

Based on the notation and boundary conditions shown for the rectangular plate element shown in Figure 8, the displacement functions for the x and y directions are as follows:

$$u_x = \left\{ \frac{u_4 - u_1}{a} + \frac{\nu}{2b}\left(v_3 - v_4 + v_1 - v_2\right) \right\} x$$

$$+ \left\{ \frac{u_2 - u_1}{b} + \frac{1}{2a}\left(v_3 - v_4 + v_1 - v_2\right) \right\} y$$

$$- \left\{ \frac{1}{2ab}\left(v_3 - v_4 + v_1 - v_2\right) \right\} \left(\nu x^2 + y^2 \right)$$

$$+ \left\{ \frac{1}{2ab}\left(u_3 - u_4 + u_1 - u_2\right) \right\} xy$$

$$+ u_1 .$$

$$v_y = \left\{ \frac{v_4 - v_1}{a} + \frac{1}{2b}\left(u_3 - u_4 + u_1 - u_2\right) \right\} x$$

$$+ \left\{ \frac{v_2 - v_1}{b} + \frac{\nu}{2a}\left(u_3 - u_4 + u_1 - u_2\right) \right\} y$$

$$- \left\{ \frac{1}{2ab}\left(u_3 - u_4 + u_1 - u_2\right) \right\} \left(x^2 + \nu y^2 \right)$$

$$+ \left\{ \frac{1}{ab}\left(v_3 - v_4 + v_1 - v_2\right) \right\} xy$$

$$+ v_1 .$$

The strains in the element are then evaluated from the following relationship.

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = [A] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

where

$$[A] = \frac{1}{6ab} \begin{bmatrix} 6(y-b) & (a-2x) & -6y & -(a-2x) & 6y & (a-2x) & -6(y-b) & -(a-2x) \\ (b-2y) & 6(x-a) & -(b-2y) & -6(x-a) & (b-2y) & 6x & -b-2y & -6x \\ -3a & -3b & 3a & -3b & 3a & 3b & -3a & 3b \end{bmatrix}.$$

The relationship for stresses in terms of node displacements is obtained from

$$\{\sigma\} = [B]\{\epsilon\} = [B][A]\begin{Bmatrix} u \\ v \end{Bmatrix}.$$

For $\nu = 1/3$, the multiplication yields

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{16ab} \begin{bmatrix} y-b & -3 & -y & 3 & y & 3 & b-y & -3 \\ -3b & x-a & -3b & a-x & 3b & x & 3b & -x \\ -3a & -3b & 3a & -3b & 3a & 3b & -3a & 3b \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}.$$

The stiffness matrix K is obtained from the unit displacement theorem. Since the matrix A is a function of the position variables, the integration is performed with respect to $\overline{X}$ and $\overline{Y}$ between the limits $\overline{X} = 0$ to 1 and $\overline{Y} = 0$ to 1. The unit displacement theorem provides

$$[K] = \int_{Vol} [A^T][B][A] \; dVol$$

where A is the relationship between strain and node displacements and B is the relationship between stress and strain.

The stiffness matrix for the panel shown in Figure 8 for $\nu = 1/3$ is

$$\frac{Et}{192ab}\begin{bmatrix}
70b^2+18a^2 & & & & & & & \\
36ab & 70a^2+18b^2 & & & & & & \\
38b^2-18a^2 & 0.0 & 70b^2+18a^2 & & & & & \\
0.0 & -70a^2+18b^2 & -36ab & 70a^2+18b^2 & & & & \\
-38b^2-18a^2 & -36ab & -70b^2+18a^2 & 0.0 & 70b^2+18a^2 & & & \\
-36ab & -38a^2-18b^2 & 0.0 & 38a^2-18b^2 & 36ab & 70a^2+18b^2 & & \\
-70b^2+18a^2 & 0.0 & -38b^2-18a^2 & 36ab & 38b^2-18a^2 & 0.0 & 70b^2+18a^2 & \\
0.0 & 38a^2-18b^2 & 36ab & -38a^2-18b^2 & 0.0 & -70a^2+18b^2 & -36ab & 70a^2+18b^2
\end{bmatrix}$$

Rectangular Plate Element With Higher Stress Variation

The two previous stiffness matrices were developed using assumed stress or displacement patterns which resulted in eight undetermined coefficients in the displacement functions. These eight undetermined coefficients are evaluated by boundary conditions expressed in terms of the eight degrees of freedom of the corner nodes of the elements. These previous assumptions yield stress variations that are constant or linearly varying in only one direction. In addition, for the case of the assumed displacement function, the equilibrium conditions for the element are only satisfied for a particular set of relative displacements of the corner nodes of the elements.

In order to increase the accuracy of the stiffness matrix for a specific size of idealized structural element, the stress or deformation mode of the element is increased by assuming a higher order of variation of stresses within the element or by assuming a less restricted pattern of deformations within the element. Consequently, additional considerations are required to evaluate the additional undetermined coefficients which result from increased variations of stress within the element.

In a recent technical note, Pian (13) has shown that the theorem of complementary energy can be used to obtain stiffness matrices for elements using an unlimited number of undetermined coefficients for the assumed stress variation within the element. In addition, Melosh (14) has recently shown that similar variational methods can be used to develop stiffness matrices for assumed higher deformation modes within the element. Based on these developments, new generations of stiffness matrices can be developed for the numerous types of elements required for structural analyses.

The subsequent development of a stiffness matrix required for the analysis of the integrally reinforced rectangular skin panel assumes a stress pattern that varies linearly in each direction within the idealized element.

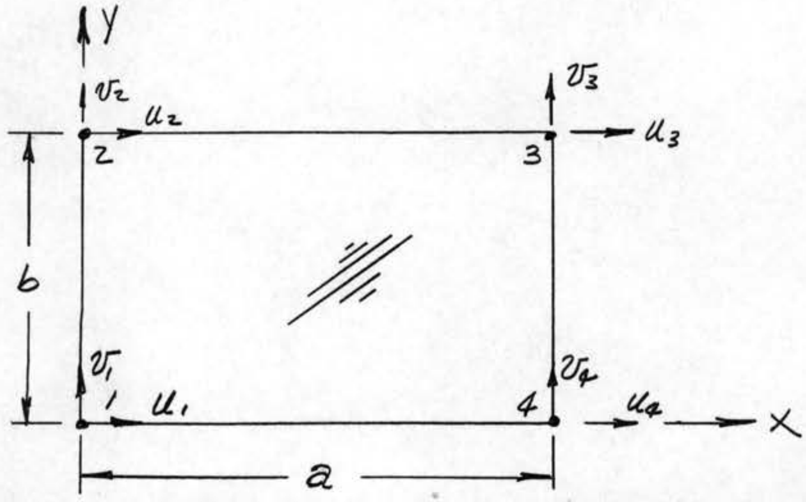


Figure 9. Plate Element With Linear Stress Variation

The assumed stress variation within the element is

$$\sigma_x = a_1 + a_2 y + a_6 x$$
$$\sigma_y = a_3 + a_4 x + a_7 y$$
$$\tau_{xy} = a_5 - a_6 y - a_7 x$$

The stress distribution σ in terms of the undetermined coefficients is

$$\{\sigma\} = [S]\{a_i\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & y & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 1 & x & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 1 & -y & -x \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix}$$

where the coefficients of S are the x and y coordinates of the surface of the element.

The stress-strain relations for the plane stress conditions are

$$\{\sigma\} = [B]\{\epsilon\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

and

$$\{\epsilon\} = [B^{-1}]\{\sigma\}$$

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} .$$

Using the stress distribution in terms of the undetermined coefficients

$$\{\sigma\} = [S]\{a\}$$

and the stress-strain relations

$$\{\epsilon\} = [B^{-1}]\{\sigma\}$$

the internal strain energy for an element can be expressed as (4)

$$U = \frac{1}{2} \iiint_{Vol} \sigma^T B^{-1} \sigma \, dV$$

or

$$U = \frac{1}{2} \iiint_{Vol} \lfloor a^T \rfloor [S^T][B^{-1}][S]\{a\} \, dV.$$

Since the undetermined coefficients $\{a\}$ are not functions of x and y, the internal strain energy for an isotropic plate element of constant thickness is

$$U = \frac{1}{2} \lfloor a^T \rfloor \left[t \int_0^a \int_0^b [S^T][B^{-1}][S]\, dy\, dx \right] \{a\}$$

or

$$U = \frac{1}{2} \lfloor a^T \rfloor [SES] \{a\}$$

where

$$[SES] = t \int_0^a \int_0^b [S^T][B^{-1}][S]\, dy\, dx \, .$$

For Poisson's Ratio of 1/3, the result is shown as follows:

$$\frac{tab}{E} \begin{bmatrix} 1 & & & & & & \\ \frac{b}{2} & \frac{b^2}{3} & & & & & \\ \frac{-1}{3} & \frac{-b}{6} & 1 & & & & \\ \frac{-a}{6} & \frac{-ab}{12} & \frac{a}{2} & \frac{a^2}{3} & & & \\ 0 & 0 & 0 & 0 & \frac{8}{3} & & \\ \frac{a}{2} & \frac{ab}{4} & \frac{-a}{6} & \frac{-a^2}{9} & \frac{-4b}{3} & \frac{3a^2+8b^2}{9} & \\ \frac{-b}{6} & \frac{-b^2}{9} & \frac{b}{2} & \frac{b}{4} & \frac{-4a}{3} & \frac{7ab}{12} & \frac{3b^2+8a^2}{9} \end{bmatrix}$$

The inverse of this matrix is

$$[SES]^{-1} =$$

$$\frac{E}{tab\alpha\beta}\begin{bmatrix}
\frac{3}{8}(1\alpha\beta+9a^2\alpha+b^2\beta) & & & & & & \\
-\frac{24\alpha\beta+3b^2\beta}{4b} & \frac{3b^2\beta+24\alpha\beta}{2b^2} & & & & & \\
\frac{3}{8}(\alpha\beta+3a^2\alpha+3b^2\beta) & -\frac{9b\beta}{4} & \frac{3}{8}(1\alpha\beta+a^2\alpha+9b^2\beta) & & & & \\
-\frac{9a\alpha}{4} & 0 & -\frac{3a^2\alpha+24\alpha\beta}{4a} & \frac{3a^2\alpha+24\alpha\beta}{2a^2} & & & \\
-\frac{9}{8}(3ab\alpha+ab\beta) & \frac{9a\beta}{4} & -\frac{9}{8}(ab\alpha+3ab\beta) & \frac{9b\alpha}{4} & \frac{3}{8}(\alpha\beta+9a^2\beta+9b^2\alpha) & & \\
-\frac{27a\alpha}{4} & 0 & -\frac{9a\alpha}{4} & \frac{9\alpha}{2} & \frac{27b\alpha}{4} & \frac{27\alpha}{2} & \\
-\frac{9b\beta}{4} & \frac{9\beta}{2} & -\frac{27b\beta}{4} & 0 & \frac{27a\beta}{4} & 0 & \frac{27\beta}{2}
\end{bmatrix}_{7\times 7}$$

The forces acting on the boundaries of the elements are expressed in terms of the undetermined coefficients $\{a\}$ by their equilibrium relationship to the stress variations within the element. For the numbering system shown in Figure 9, the surface stresses are

$$\begin{aligned}
(F_x^*)_{12} &= -\sigma_x &&= -a_1 - a_2 y - a_6 x \\
(F_y^*)_{12} &= -\tau_{xy} &&= -a_5 + a_6 y + a_7 x \\
(F_x^*)_{23} &= \tau_{xy} &&= a_5 - a_6 y - a_7 x \\
(F_y^*)_{23} &= \sigma_y &&= a_3 + a_4 x + a_7 y \\
(F_x^*)_{14} &= -\tau_{xy} &&= -a_5 + a_6 y + a_7 x \\
(F_y^*)_{14} &= -\sigma_y &&= -a_3 - a_4 x - a_7 y \\
(F_x^*)_{43} &= \sigma_x &&= a_1 + a_2 y + a_6 x \\
(F_y^*)_{43} &= \tau_{xy} &&= a_5 - a_6 y - a_7 x
\end{aligned}$$

The surface forces are written in terms of the undetermined coefficients in the form

$$\{F^*\} = [C]\{a\}$$

where

$$[C] = \begin{bmatrix} 1 & y & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 1 & x & 0 & 0 & 1 \\ 1 & y & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 1 & x & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -y & -x \\ 0 & 0 & 0 & 0 & 1 & -y & -x \\ 0 & 0 & 0 & 0 & 1 & -y & -x \\ 0 & 0 & 0 & 0 & 1 & -y & -x \end{bmatrix}.$$

The deformation of the element is described in terms of the boundary displacements which must be consistent with the assumed stress distribution in the element.

The deformations of the boundaries are described in terms of the node displacements by the equations

$$\{\delta^*\} = [M]\begin{Bmatrix} u \\ v \end{Bmatrix}$$

where the terms of M represent the linear deformations of the edges in terms of the surface coordinates. The linear edge displacements in terms of the generalized displacements of the nodes are as follows for the edge 1-2:

$$\delta^*_{12}(u) = u_1 + \frac{y}{b}(u_2 - u_1) = \left(1 - \frac{y}{b}\right)u_1 + \left(\frac{y}{b}\right)u_2$$

$$\delta^*_{12}(v) = v_1 + \frac{y}{b}(v_2 - v_1) = \left(1 - \frac{y}{b}\right)v_1 + \left(\frac{y}{b}\right)v_2 .$$

The displacements of the other edges are obtained in a similar manner.

In matrix notation

$$\begin{Bmatrix} \delta^*_{12}(u) \\ \delta^*_{12}(v) \\ \delta^*_{23}(u) \\ \delta^*_{23}(v) \\ \delta^*_{14}(u) \\ \delta^*_{14}(v) \\ \delta^*_{43}(u) \\ \delta^*_{43}(v) \end{Bmatrix} = \begin{bmatrix} 1-y/b & 0 & y/b & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-y/b & 0 & y/b & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-x/a & 0 & x/a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-x/a & 0 & x/a & 0 & 0 \\ 1-x/a & 0 & 0 & 0 & 0 & 0 & x/a & 0 \\ 0 & 1-x/a & 0 & 0 & 0 & 0 & 0 & x/a \\ 0 & 0 & 0 & 0 & y/b & 0 & 1-y/b & 0 \\ 0 & 0 & 0 & 0 & 0 & y/b & 0 & 1-y/b \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}.$$

The strain energy in the element in terms of the generalized displacements is

$$U = \frac{1}{2} \begin{Bmatrix} u \\ v \end{Bmatrix}^T [k] \begin{Bmatrix} u \\ v \end{Bmatrix}.$$

The theorem of minimum complementary energy states (15)

$$\Pi_c = \frac{1}{2} \lfloor a^T \rfloor [SES] \{a\} - \lfloor a^T \rfloor [CM] \begin{Bmatrix} u \\ v \end{Bmatrix}$$

where

$$[CM] = \int_{Area} [C^T][M] \, dA .$$

From the condition for minimum complementary energy

$$\frac{\partial \Pi_c}{\partial a_i} = 0 \quad , (i = 1, \cdots 7)$$

$$[SES]\{a\} = [CM] \begin{Bmatrix} u \\ v \end{Bmatrix}.$$

Consequently, the undetermined coefficients can be expressed in terms of the generalized displacements as

$$\{a\} = [SES]^{-1}[CM]\left\{\begin{matrix}u\\v\end{matrix}\right\}.$$

The internal strain energy within the element in terms of the stiffness matrix for the element is

$$U = \frac{1}{2}\left\{\begin{matrix}u\\v\end{matrix}\right\}^T[K]\left\{\begin{matrix}u\\v\end{matrix}\right\}.$$

The internal strain energy for an isotropic plate element of constant thickness is

$$U = \frac{1}{2}\lfloor a^T\rfloor[SES]\{a\}.$$

Based on the solution for the undetermined coefficients, the strain energy can also be expressed as

$$U = \frac{1}{2}\left\{\begin{matrix}u\\v\end{matrix}\right\}^T[CM][SES]^{-1}[CM]\left\{\begin{matrix}u\\v\end{matrix}\right\}.$$

Consequently, the element stiffness matrix is

$$[K] = [CM]^T[SES]^{-1}[CM].$$

The stiffness matrix for the rectangular plate element is evaluated for $\nu = 1/3$. The coefficients of the stiffness matrix are shown as partitioned matrices for convenience.

$$[K] = \left(\frac{Et}{96\,ab\alpha\beta}\right)\begin{bmatrix}\xi & \eta^T\\ \eta & \xi\end{bmatrix}$$

where $\alpha = 3a^2 + b^2$, $\beta = a^2 + 3b^2$ and

$$
\xi = \begin{bmatrix}
35b^2\alpha\beta+b^4\beta-6a^2b^2\beta+9a^2\alpha\beta+9a^4\beta & & & \\
18ab\alpha\beta & 35a^2\alpha\beta+a^4\alpha-6a^2b^2\alpha+9b^2\alpha\beta+9b^4\alpha & & \\
19b^2\alpha\beta-b^4\beta+6a^2b^2\beta-9a^2\alpha\beta-9a^4\beta & 0.0 & 35b^2\alpha\beta+b^4\beta-6a^2b^2\beta+9a^2\alpha\beta+9a^4\beta & \\
0.0 & -35a^2\alpha\beta-a^4\alpha+6a^2b^2\alpha+9b^2\alpha\beta-9b^4\alpha & -18ab\alpha\beta & 35a^2\alpha\beta+a^4\alpha-6a^2b^2\alpha+9b^2\alpha\beta+9b^4\alpha
\end{bmatrix}
$$

$$
\eta = \begin{bmatrix}
-19b^2\alpha\beta+b^4\beta-6a^2b^2\beta-9a^2\alpha\beta+9a^4\beta & -18ab\alpha\beta & -35b^2\alpha\beta-b^4\beta+6a^2b^2\beta+9a^2\alpha\beta-9a^4\beta & 0.0 \\
-18ab\alpha\beta & -19a^2\alpha\beta+a^4\alpha-6a^2b^2\alpha-9b^2\alpha\beta+9b^4\alpha & 0.0 & 19a^2\alpha\beta-a^4\alpha+6a^2b^2\alpha-9b^2\alpha\beta-9b^4\alpha \\
-35b^2\alpha\beta-b^4\beta+6a^2b^2\beta+9a^2\alpha\beta-9a^4\beta & 0.0 & -19b^2\alpha\beta+b^4\beta-6a^2b^2\beta-9a^2\alpha\beta+9a^4\beta & 18ab\alpha\beta \\
0.0 & 19a^2\alpha\beta-a^4\alpha+6a^2b^2\alpha-9b^2\alpha\beta-9b^4\alpha & 18ab\alpha\beta & -19a^2\alpha\beta+a^4\alpha-6a^2b^2\alpha-9b^2\alpha\beta+9b^4\alpha
\end{bmatrix}.
$$

CHAPTER IV

STRESS ANALYSIS SYSTEM

The Stress Analysis System is a digital computer program using matrix methods based on discrete element idealization for two-dimensional structures. The complete solution for deflections and stresses requires only that the structure be defined in terms of its geometrical characteristics and types of structural elements. The structure is first idealized as an assemblage of discrete structural elements. Each structural element has an assumed form of displacement or stress distribution. The complete solution is obtained by satisfying the force equilibrium and displacement compatibility at the junctions of the elements. Thus, the conditions of equilibrium and compatibility are satisfied at only a finite number of points which do not necessarily imply any appreciable loss of accuracy. When the size of the element is sufficiently small in relation to the overall size of the structure and the variations of stresses within the structure do not exceed those allowed in the mathematical model, the discrete element methods give good approximations to the exact solutions.

The displacement method is the basis for developing this digital computer program for analyzing two-dimensional rectangular panel configurations for arbitrary load and support conditions. The system provides solutions for displacements and internal or external forces at the structural node points and stresses at any stress node points defined for the structural element.

The input data required for the Stress Analysis System consist of node numbers, element numbers, and geometric descriptions of the idealized structure and locations of desired stress results on the elements. The program is divided into the following catagories:

1. Geometric description of the structure
2. Idealized description of the structure
3. Generation of stiffness matrices
4. Generation of stress matrices
5. Deflection solution
6. Reaction force solution
7. Generalized stress calculations
8. Printing of analysis results

The data required under item number 1 are shown in Table IV. The data for item number 2 are described in Table V. The data required for item number 7 are shown in Table VI.

The first step for preparing the input data for the analysis is to simulate the actual structure as an assemblage of idealized elements, which is commonly referred to as the idealized structure shown in Figure 10. The structure is formed from available elements, i.e., stringers and rectangular plates, so that it is capable of representing the deflection behavior of the actual structure. The idealized structure is described in terms of the node data and the structural data. The node data, Table IV, consist of the number of the node point, the coordinates of the node point, the external forces acting on the node point, and the definition of the boundary condition at the node point. The structural data consist of the location of the idealized elements relative to the node points, the type of structural element, and the description of its material properties.

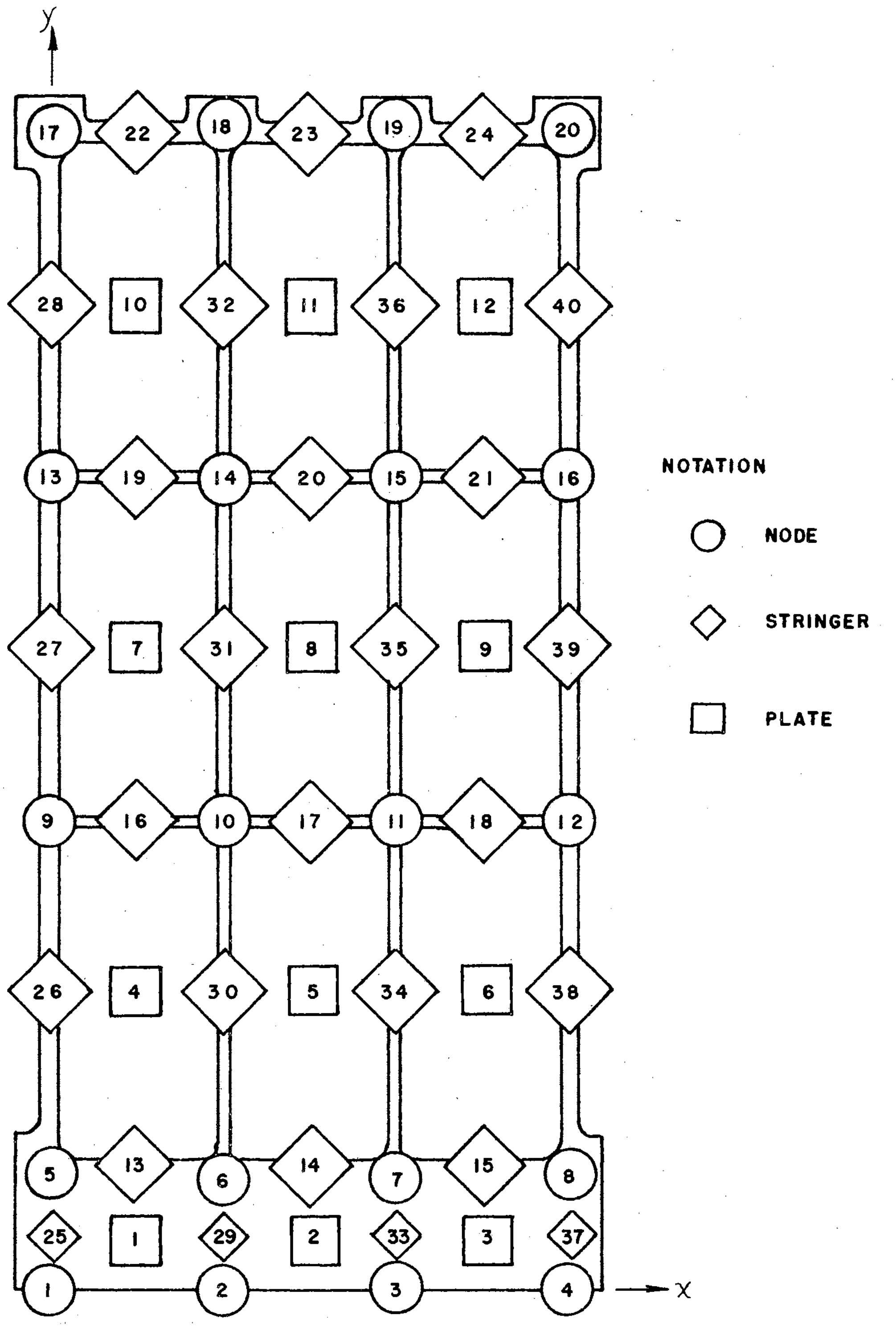


Figure 10. Idealization for Stiffness Method of Analysis

TABLE IV

NODE DATA FOR STRESS ANALYSIS SYSTEM

NODE POINT		COORDINATES	LOADING CONDITIONS					SUP-PORT
			Case 1	Case 2	Case 3	Case 4	Case 5	
1 5	6	7 18	19 30	31 42	43 54	55 66	67 78	80
1	x	0.0						1
1	y	0.0						1
2	x	5.0						1
2	y	0.0						1
3	x	10.0						1
3	y	0.0						1
4	x	15.0						1
4	y	0.0						1
5	x	0.0						0
5	y	2.0						0
6	x	5.0						0
6	y	2.0						0
7	x	10.0						0
7	y	2.0						0
8	x	15.0						0
8	y	2.0						0
9	x	0.0						0
9	y	12.0						0
10	x	5.0						0
10	y	12.0						0
11	x	10.0						0
11	y	12.0						0
12	x	15.0						0
12	y	12.0						0
13	x	0.0						0
13	y	22.0						0
14	x	5.0						0
14	y	22.0						0
15	x	10.0						0
15	y	22.0						0
16	x	15.0						0
16	y	22.0						0
17	x	0.0						0
17	y	32.0	2500.					0
18	x	5.0						0
18	y	32.0	2500.	5000.				0
19	x	10.0						0
19	y	32.0	2500.	5000.				0
20	x	15.0				1000.	1000.	0
20	y	32.0	2500.		1000.		1000.	0

TABLE V

STRUCTURAL DATA FOR STRESS ANALYSIS SYSTEM

ELEMENT NUMBER	ELEMENT LOCATION (NODE POINTS)				TYPE	STIFFNESS DATA			
	P	Q	R	S		YOUNG'S MODULUS		POISSON'S RATIO	AREA OR THICKNESS
1 5	6 9	10 13	14 17	18 21	24	26	35	36 41	42 47
1	1	5	6	2	7	10.6	+6	0.3333	0.5
2	2	6	7	3	7	10.6	+6	0.3333	0.5
3	3	7	8	4	7	10.6	+6	0.3333	0.5
4	5	9	10	6	7	10.6	+6	0.3333	0.05
5	6	10	11	7	7	10.6	+6	0.3333	0.05
6	7	11	12	8	7	10.6	+6	0.3333	0.05
7	9	13	14	10	7	10.6	+6	0.3333	0.05
8	10	14	15	11	7	10.6	+6	0.3333	0.05
9	11	15	16	12	7	10.6	+6	0.3333	0.05
10	13	17	18	14	7	10.6	+6	0.3333	0.05
11	14	18	19	15	7	10.6	+6	0.3333	0.05
12	15	19	20	16	7	10.6	+6	0.3333	0.05
13	5	6			1	10.6	+6		0.25
14	6	7			1	10.6	+6		0.25
15	7	8			1	10.6	+6		0.25
16	9	10			1	10.6	+6		0.125
17	10	11			1	10.6	+6		0.125
18	11	12			1	10.6	+6		0.125
19	13	14			1	10.6	+6		0.125
20	14	15			1	10.6	+6		0.125
21	15	16			1	10.6	+6		0.125
22	17	18			1	10.6	+6		0.25
23	18	19			1	10.6	+6		0.25
24	19	20			1	10.6	+6		0.25
25	1	5			1	10.6	+6		0.25
26	5	9			1	10.6	+6		0.25
27	9	13			1	10.6	+6		0.25
28	13	17			1	10.6	+6		0.25
29	2	6			1	10.6	+6		0.125
30	6	10			1	10.6	+6		0.125
31	10	14			1	10.6	+6		0.125
32	14	18			1	10.6	+6		0.125
33	3	7			1	10.6	+6		0.125
34	7	11			1	10.6	+6		0.125
35	11	15			1	10.6	+6		0.125
36	15	19			1	10.6	+6		0.125
37	4	8			1	10.6	+6		0.25
38	8	12			1	10.6	+6		0.25
39	12	16			1	10.6	+6		0.25
40	16	20			1	10.6	+6		0.25

TABLE VI

STRESS NODE DATA FOR STRESS ANALYSIS SYSTEM

ELEMENT NUMBER		NODE 1	NODE 2	NODE 3	NODE 4	NODE 5	
1 5	6	7 18	19 30	31 42	43 54	55 66	
1	x	2.5	1.25	1.25	3.75	3.75	
1	y	1.0	0.5	1.5	1.5	0.5	
2	x	2.5	1.25	1.25	3.75	3.75	
2	y	1.0	0.5	1.5	1.5	0.5	
3	x	2.5	1.25	1.25	3.75	3.75	
3	y	1.0	0.5	1.5	1.5	0.5	
4	x	2.5	1.25	1.25	3.75	3.75	
4	y	5.0	2.5	7.5	7.5	2.5	
5	x	2.5	1.25	1.25	3.75	3.75	
5	y	5.0	2.5	7.5	7.5	2.5	
6	x	2.5	1.25	1.25	3.75	3.75	
6	y	5.0	2.5	7.5	7.5	2.5	
7	x	2.5	1.25	1.25	3.75	3.75	
7	y	5.0	2.5	7.5	7.5	2.5	
8	x	2.5	1.25	1.25	3.75	3.75	
8	y	5.0	2.5	7.5	7.5	2.5	
9	x	2.5	1.25	1.25	3.75	3.75	
9	y	5.0	2.5	7.5	7.5	2.5	
10	x	2.5	1.25	1.25	3.75	3.75	
10	y	5.0	2.5	7.5	7.5	2.5	
11	x	2.5	1.25	1.25	3.75	3.75	
11	y	5.0	2.5	7.5	7.5	2.5	
12	x	2.5	1.25	1.25	3.75	3.75	
12	y	5.0	2.5	7.5	7.5	2.5	

The location of the node points is given relative to a two-dimensional rectangular coordinate system. The n node points are numbered consecutively from 1 to n in the direction of the minimum width.

The boundary conditions are specified by restricting the displacement of the supported node point in the directions of the intended supports. This is achieved by placing a 1 in column 80 of each node data card for the degrees of freedom which are to be restrained. If insufficient boundary conditions are defined, the stiffness of the general structure is zero in that direction. Consequently, the stiffness matrix is singular; and the analysis cannot be completed.

The loading conditions are given as part of the node data as shown in Table IV. Five loading conditions can be considered in each analysis. The loads are entered in Table IV by listing the x and y components of the applied load in the x and y rows of the node points on which the loads are acting. The actual external loads acting on the real structure are represented by concentrated loads acting at the node points of the idealized structure.

The locations of the idealized elements are given relative to the node points in the structural data. The idealized elements are numbered consecutively. No specific grouping is required between stringer or rectangular plate elements. If an integer is assigned to a stringer, the next integer can be assigned to a rectangular plate. For stringer elements, the connecting node point numbers are given in columns 6 through 9 and 10 through 13 of the structural data cards and are called nodes P and Q. For rectangular plates, the nodes are called P, Q, R, and S and are listed in consecutive order clockwise around the rectangular plate. The implication in listing the corner node point numbers is that it automatically assigns a local xy coordinate system for the rectangle. The local x axis extends from node P to node S; the local y axis extends from node P to node Q.

The stress components are calculated and printed out relative to the local coordinate system. For example, if the structure has grid lines parallel to the x and y axis of the general coordinate system, a PQRS sequence is chosen so that the coordinate axes for each rectangular plate have directions identical to those of the general coordinate axes. In this case, the stresses are then relative to the external coordinate axes and are the same for all rectangular plates. The stress results for the stringer elements are given relative to the axis of the stringers. As additional elements are added to this program, the common element coordinate system should be maintained.

The type of idealized element is specified in the structural data by entering the type number in column 24. The type numbers for each element are given in Appendix B.

The elastic properties of the material are defined in the structural data and consist of modulus of elasticity and Poisson's ratio. They are entered in Table V for each element.

Stresses are calculated for the stress node points defined for each element relative to the local coordinate system of the element. The characteristic dimensions of the idealized elements are defined by the coordinates of their end or corner node points. The coordinates of the stress node points are given in inches relative to the local coordinate system for the element. A maximum of five stress nodes can be used in each analysis. If no stress nodes are specified, stresses are automatically computed for the coordinates of the centroid of the element.

Node numbers, element numbers, element-type numbers, and support conditions are always entered as integers. All other data are entered with a decimal point in the proper place. An example of the input data for the test structure is given in TablesIV, V, and VI.

Once the idealized structure and the loading conditions are defined, the computational sequence follows from the stiffness method. The stiffness and stress matrices are generated for each element using the structural material properties and the dimensions obtained from the node data. The rows and columns of the stiffness matrix and stress and load matrices are in the order of x and y for each node point on the structure. In general, if P is the number of the node point, the x and y degrees of freedom at P are labeled 2P-1 and 2P, respectively. These numbers are then used as indices to denote a displacement or force component acting at node P in either x or y direction.

The matrix $\bar{K}$ (BARK) is the stiffness matrix of the idealized structure in lower symmetric form. It is obtained by simply summing up the contributions of the various element stiffness coefficients in the direction of each displacement. To facilitate this summation, the MPQRS numbering scheme is used to denote the x and y directions of each of the nodes (16).

Once the element stiffness matrices have been computed based on the stiffness properties and the node locations of each element, the coefficients of the stiffness matrix are assigned indices according to the MPQRS scheme. The indices designate the position of the stiffness matrix for the individual composite stiffness matrix for the total structure. The total stiffness matrix $\bar{K}$ is obtained by summing the stiffness matrix elements with common indices obtained by the MPQRS scheme. As the stiffness matrix for each element is generated, it is added to the large $\bar{K}$ matrix.

The coefficients of $\overline{K}$ are the forces generated at the node points in the x and y directions, when one node is displaced a unit distance in the x or y direction and all other displacements are restrained. The sum of the coefficients in every row and column is zero since the forces generated at restrained node points and the force developed due to the unit displacement are in equilibrium. If the structure is restrained from rotation and translation degrees of freedom by removing the rows and columns of the $\overline{K}$ matrix that represent the displacement of boundary conditions, the matrix is subsequently nonsingular. Removing these rows and columns decreases the size of the matrix and consequently changes the indices of the coefficients of $\overline{K}$. Consequently, one has the choice of using the reduced matrix and changing the indices of the rows and column designations or removing the rows and columns except on the diagonal. The diagonal element is replaced by a 1. The result is that the stiffness matrix will contain a unit matrix which will not effect the solution of the simultaneous equations obtained by performing the inverse operation. This technique does save the numbering scheme but, of course, retains the size of the stiffness matrix. This method of modification rather than reduction of the stiffness matrix is utilized in this program because it simplifies the bookkeeping problems throughout the calculations; and, for these types of structures, the decrease in the size of the stiffness matrix obtained by reducing the matrix for the boundary conditions is not a significant advantage.

After the stiffness matrices for each element have been added to the total stiffness matrix $\overline{K}$, the matrix $\overline{K}$ is modified, as mentioned in the previous paragraph, according to the defined boundary conditions. The modified stiffness matrix is then inverted and the node point deflections are calculated from the equation

$$\{\delta\} = [K^{-1}]\{F\}.$$

The deflection matrix δ is a complete listing of the node displacements, including the zero displacements at the boundaries.

The stresses in each idealized element are calculated from the deflections δ for the element, which must be obtained from the total δ matrix. The stresses are computed by generating the stress matrix for the coordinates of the stress node point and postmultiplying the element stress matrix by the element displacements. The stresses within the idealized element are based on the assumptions made for deriving the stiffness and stress matrices. Consequently, the stresses at any number of points in a single plate may be obtained through the stress coefficient matrix and the corner displacements of the plate or stringer element. The components of the stress tensor at the stress node points defined in the stress node data are calculated relative to the local coordinate system of the plate element.

The reaction forces at the boundary node points are computed from the equation

$$\{F\} = [\bar{K}]\{\delta\}$$

by evaluating the right-hand side of the equation where $\bar{K}$ is the original stiffness matrix before boundary conditions are applied. The reaction forces are used for checking the original input data or the accumulation of numerical errors in the computing process and do provide a solution for the reactions in the directions of the specified boundary conditions.

The output data are presented in two forms, an abbreviated form containing only the basic results of the analysis and an extended form including all of the individual plate and stringer stiffness and stress coefficient matrices and bookkeeping arrays in the analysis. The output is controlled by placing a numeral 1 in column 30 of the program control card. If no parameter is used in column 30, the abbreviated form of the analysis will be printed.

CHAPTER V

ANALYTICAL INVESTIGATION

The structural panel used in this investigation was designed so that the idealization used in the stiffness analysis corresponded as accurately as possible to the actual test model. In the case of complex structural configurations, the analysis problem should be divided into two phases: the idealization of the complex structure; the analysis of the idealized structure.

In the first phase, large errors may occur due to computer size limitations because it is necessary to approximate large structural configurations with a relatively few number of structural elements. In addition, thick panels are idealized as thin panels which carry no out-of-plane loads; and tapered bar elements are idealized into constant area sections that carry constant loads. These discrepancies occur in the idealization phase of the analysis.

The second phase, the comparison between the structural behavior of the panel and the mathematical analysis of the idealized panel, is hopefully limited to errors in the mathematical representation of the characteristics of the structural elements. It is first necessary to prove that an idealized structural configuration behaves in a manner similar to an actual structural configuration of approximately the same geometric characteristics. After this comparison is made, the errors resulting from idealization procedures can be more accurately investigated.

The design of the research model shown in Figure 11 is based on the idealization of actual structural configurations that are commonly encountered in aerospace structural analysis. This structural configuration results in a convenient idealization for both the force and the stiffness methods of analysis.

The analysis of the panel by the force method described in Chapter II is based on the assumption that the shear forces are transmitted only by the web elements and the axial forces are transmitted only by the rib and stringer elements. The cross-sectional areas of the rib and stringer elements are increased to account for the axial forces that are also transmitted by the web elements. This procedure is desirable in the force analysis since the consideration of additional axial forces in the web elements increases the degree of redundancy of the structure.

The force method was used for the analysis of the structure based on the nominal dimensions of the structure shown in Figure 11. The structure was analyzed for the five load conditions used in the experimental investigation. A complete description of these load configurations is given in Chapter VI. A numbering system of points and elements on the structural panel is shown in Figure 12. This sequence of numbers is used to identify the analytical results shown in Tables VII and VIII for the force method of analysis described in Chapter II.

A more extensive analysis of the structure was performed using the stiffness method of analysis described in Chapter III. A complete analysis of the structure was performed using each of the idealized elements described in Chapter III for each of the load configurations used in the experimental investigation.

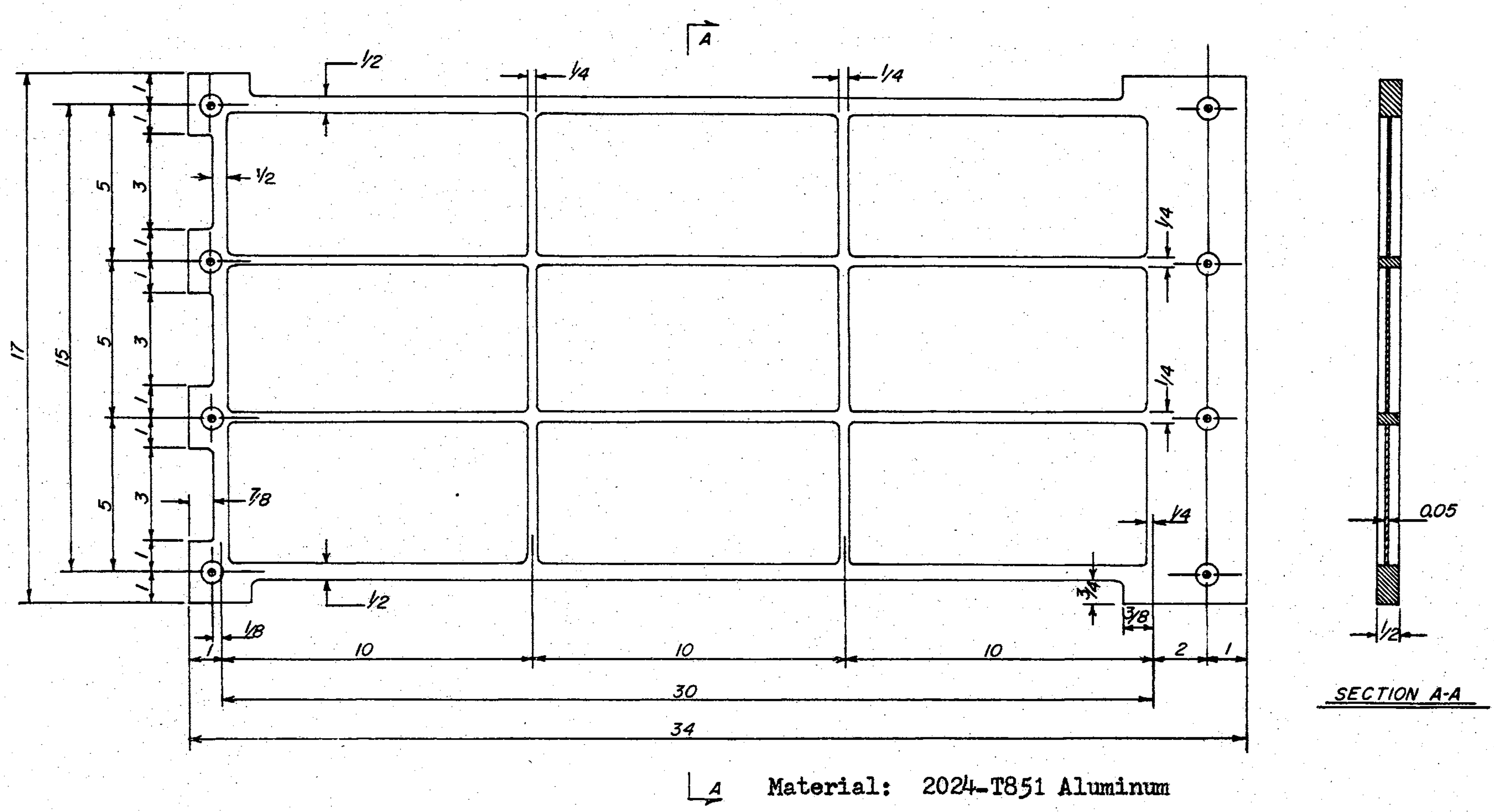


Figure 11. Integrally Reinforced Skin Panel Structure

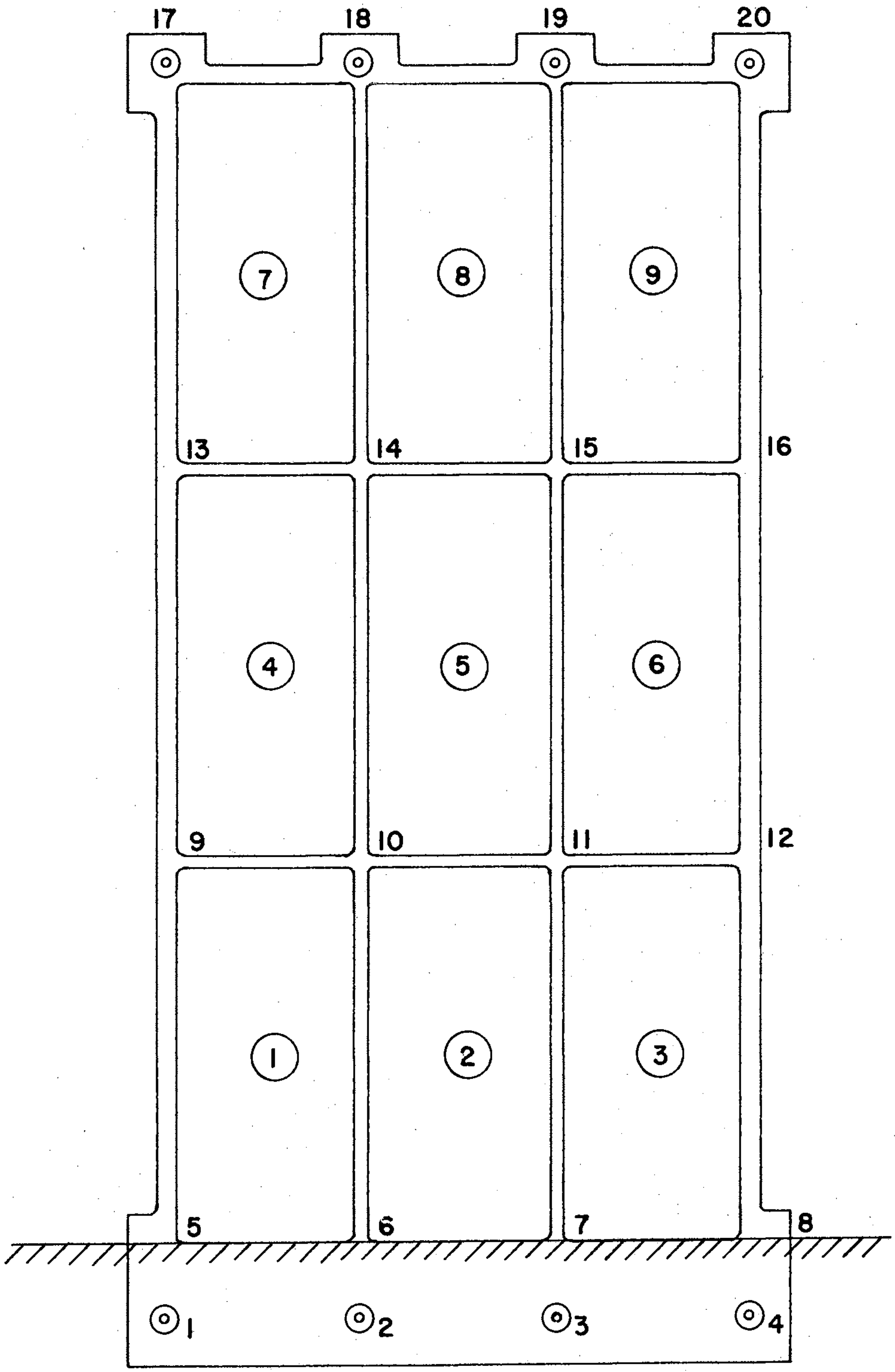


Figure 12. Notation for Analytical Data

TABLE VII

STRESSES FROM FORCE ANALYSIS

Node Numbers	Uniform Case 1 10000 lb	Center Case 2 10000 lb	Node 1 Case 3 1000 lb	Shear Case 4 1000 lb	Combined Case 5 1000 lb
Stringers					
5	7120	7077	-497	4930	4432
9	7120	6860	-479	3231	2755
13	7040	5500	-370	1559	1189
17	6670	0	0	0	0
6	7150	7218	302	1386	1688
10	7170	7471	283	1078	1361
14	7270	9037	181	596	777
18	7690	15384	0	0	0
7	7150	7218	1113	-1374	-261
11	7170	7471	1089	-1038	51
15	7270	9037	919	-436	483
19	7690	15384	0	0	0
8	7120	7077	1936	-4930	-3002
12	7120	6860	1956	-3266	-1311
16	7040	5500	2083	-1697	385
20	6670	0	2666	0	2666
Ribs					
5	0	0	0	0	0
10	-23	-341	27	7	34
11	-23	-341	32	-30	2
12	0	0	0	0	0
13	0	0	0	0	0
14	-83	-1243	79	34	113
15	-83	-1243	137	38	175
16	0	0	0	0	0
17	0	0	0	0	0
18	137	2064	-138	584	466
19	137	2064	-218	1363	1144
20	0	0	-0	2000	2000
Webs					
1	11	164	-14	1272	1259
2	0	0	0	1473	1472
3	-11	-164	15	1254	1269
4	68	1018	-80	1254	1174
5	0	0	-15	1568	1553
6	-68	-1018	95	1177	1272
7	27	4125	-277	1169	981
8	0	0	-160	1557	1397
9	-27	-4125	438	1273	1711

TABLE VIII

DEFLECTIONS FROM FORCE ANALYSIS

Deflection	Load Conditions: Uniform Case 1 10000 lb	Center Case 2 10000 lb	Node 1 Case 3 1000 lb	Shear Case 4 1000 lb	Combined Case 5 1000 lb
V_{17}	0.0199	0.0150	-0.0010	0.0068	0.0058
V_{18}	0.0206	0.0262	0.0006	0.0023	0.0028
V_{19}	0.0206	0.0262	0.0023	-0.0021	0.0004
V_{20}	0.0199	0.0150	0.0059	-0.0070	-0.0010
U_{20}	0.0001	0.0010	-0.0070	0.0295	0.0225

The analyses based on the stiffness method are easily performed using the Stress Analysis System described in Chapter IV. Since the concept of redundant load paths is not a consideration in the stiffness method of analysis, few restrictions are placed on the idealized form of the structure. The web elements are assumed to transmit axial forces as well as shear forces. The rib and stringer elements transmit only axial forces. The amount of the axial forces transmitted by each element depends on the relative stiffness of the elements. The stiffness properties are formulated within the Stress Analysis System using a geometric description of the structure as described in Chapter IV.

The analytical results using the new stiffness matrix derived in Chapter III, based on an assumed linear stress variation in each direction, are shown in Tables IX, X, and XI. This analysis was performed using the nominal dimensions of the structure shown in Figure 11 and the structural idealization illustrated in Chapter IV. The data shown in Tables IX, X, and XI are relative to the numbering system of points and elements on the structural panel shown in Figure 12.

Each analysis yields different results for the same structural idealization because of the initial assumptions that are made for the derivation of stiffness properties. The most obvious differences result from the assumed behavior of the web elements. For example, the web element used in the force method of analysis transmits only shear forces. The three plate elements representing the webs for the stiffness method of analysis transmit both axial and shear forces. However, the three plate stiffness matrices provide different results because of the following limitations. The stress distribution within the first plate element described in Chapter III based on an assumed displacement function

TABLE IX

WEB STRESSES FROM STIFFNESS ANALYSIS

Web Element	Stress	Load Conditions Case 1	Case 2	Case 3	Case 4	Case 5
	σ_x	1278	1238	-18	563	545
1	σ_y	6879	6873	-169	2823	2654
	τ_{xy}	299	332	20	1290	1310
	σ_x	1291	1230	135	-18	117
2	σ_y	6787	6792	678	0	676
	τ_{xy}	0	0	12	1431	1443
	σ_x	1278	1238	282	-550	-268
3	σ_y	6879	6873	1546	-2818	-1272
	τ_{xy}	-300	-332	-33	1279	1246
	σ_x	273	-291	34	37	72
4	σ_y	6708	6655	-161	1659	1497
	τ_{xy}	-51	979	-104	1258	1154
	σ_x	357	-955	167	-61	106
5	σ_y	6878	7452	630	71	701
	τ_{xy}	0	0	-7	1668	1660
	σ_x	274	-291	133	-216	-83
6	σ_y	6708	6655	1513	-1692	-179
	τ_{xy}	51	-979	112	1074	1186
	σ_x	817	995	-23	185	162
7	σ_y	6804	6782	-80	551	471
	τ_{xy}	-86	3857	-300	1113	813
	σ_x	791	1911	-33	547	514
8	σ_y	6811	11485	214	130	343
	τ_{xy}	0	0	-207	1581	1374
	σ_x	817	995	151	883	1034
9	σ_y	6804	6782	1445	-374	1071
	τ_{xy}	87	-3857	506	1306	1813

TABLE X

STRINGER AND RIB STRESSES FROM STIFFNESS ANALYSIS

Element Number	Between Nodes	Load Conditions: Case 1	Case 2	Case 3	Case 4	Case 5
Stringers						
26	5-9	6549	6539	-560	4094	3535
27	9-13	6473	5733	-504	2387	1884
28	12-17	6517	2052	-210	745	535
30	6-10	6356	6382	234	1177	1410
31	10-14	6759	7771	157	906	1064
32	14-18	6547	10849	66	233	299
34	7-11	6356	6382	1032	-1168	-136
35	11-15	6759	7771	992	-723	268
36	15-19	6547	10848	383	-338	45
38	8-12	6550	6540	1872	-4102	-2230
39	12-16	6473	5733	1946	-2517	-571
40	16-20	6517	2052	2406	-999	1406
Ribs						
16	9-10	-2029	-2106	759	-756	-679
17	10-11	-1941	-2068	181	-35	-217
18	11-12	-2029	-2106	-466	778	312
19	13-14	-1894	-2912	100	-275	-175
20	14-15	-1929	-4811	95	-135	-39
21	15-16	-1894	-2913	-276	-83	-358
22	17-18	-1009	382	-94	278	185
23	18-19	-1028	977	-303	1142	839
24	19-20	-1008	382	-386	2090	1713

TABLE XI

DEFLECTIONS FROM STIFFNESS ANALYSIS

	Load Conditions				
Deflection	Uniform Case 1 10000 lb	Center Case 2 10000 lb	Node 1 Case 3 1000 lb	Shear Case 4 1000 lb	Combined Case 5 1000 lb
V_{17}	0.0184	0.014	-0.0012	0.0068	0.0056
V_{18}	0.0185	0.024	0.0004	0.0022	0.0026
V_{19}	0.0185	0.024	0.0023	-0.0021	0.0002
V_{20}	0.0189	0.014	0.0059	-0.0072	-0.0013
U_{20}	-0.0007	0.000	-0.0072	0.0292	0.0221

does not satisfy equilibrium conditions except for a specific set of relative node displacements. The second plate element derived in Chapter III based on an assumed stress distribution does not provide compatible displacements between adjacent elements at their boundaries. The new plate element derived in Chapter III does not violate either of these conditions.

As a result of the manufacturing tolerances on the structure, the actual dimensions of the structure are slightly different than the nominal dimensions of the structure. The actual thickness of the test structure is the only significant variation from the nominal dimensions. Consequently, an additional analysis using the new stiffness matrix is performed based on the same idealization described in Chapter IV and using the actual dimensions of the structure based on the measured thicknesses shown in Figure 13.

The validity of the analysis is demonstrated by comparing the analytical data using the actual structural dimensions with the test data obtained during the experimental investigation. These comparisons are shown in Chapter VII.

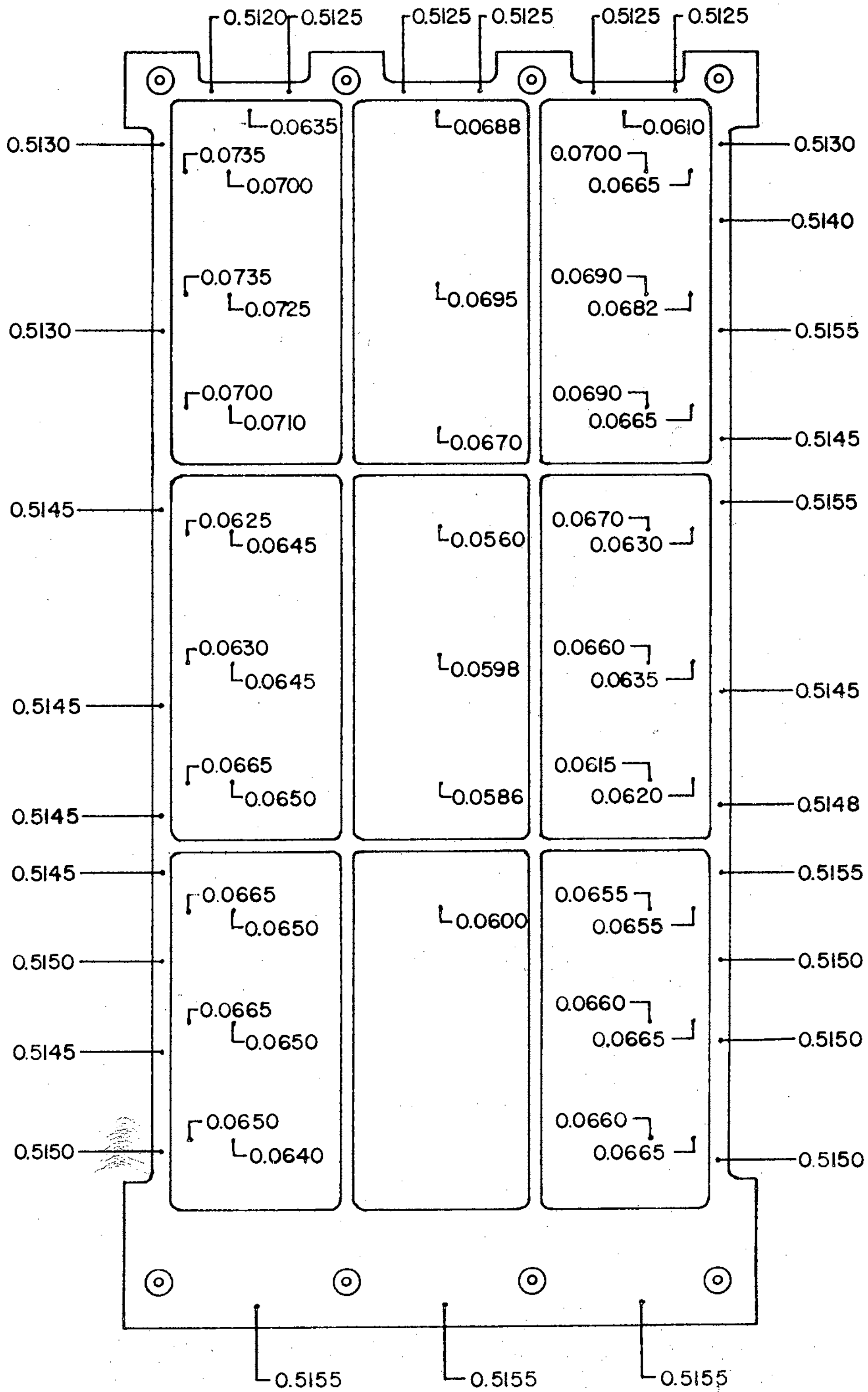


Figure 13. Measured Thicknesses of Test Panel

CHAPTER VI

EXPERIMENTAL INVESTIGATION

Concurrent with the development of analytical methods is a requirement for the development of test techniques to provide experimental verification of the theory.

The purpose of the experimental investigation is to provide data for direct comparison to the analytical methods. Since the structural idealization techniques provide a unique and somewhat unrealistic structural configuration, prior experimental data are unavailable for comparison purposes. The experimental facility and the structural skin panel that were developed for this investigation are shown in Figure 14; a general floor plan of the facility is given in Figure 15.

One objective of the experimental investigation is the determination of the complete state of strain at various points in the model for five conditions of external loading. The strain gages are positioned on the panel at points which correspond with node points easily selected for the analytical solutions. These locations of the strain gages reduce any errors that might occur as a result of extrapolating either the analytical or the experimental data.

The research model was fabricated from a plate of 2024-T851 aluminum alloy by General Dynamics Corporation, Fort Worth, Texas. This material was selected because of its high utilization in current aircraft programs. The panel was machined from one-half-inch-thick plates to eliminate joints.

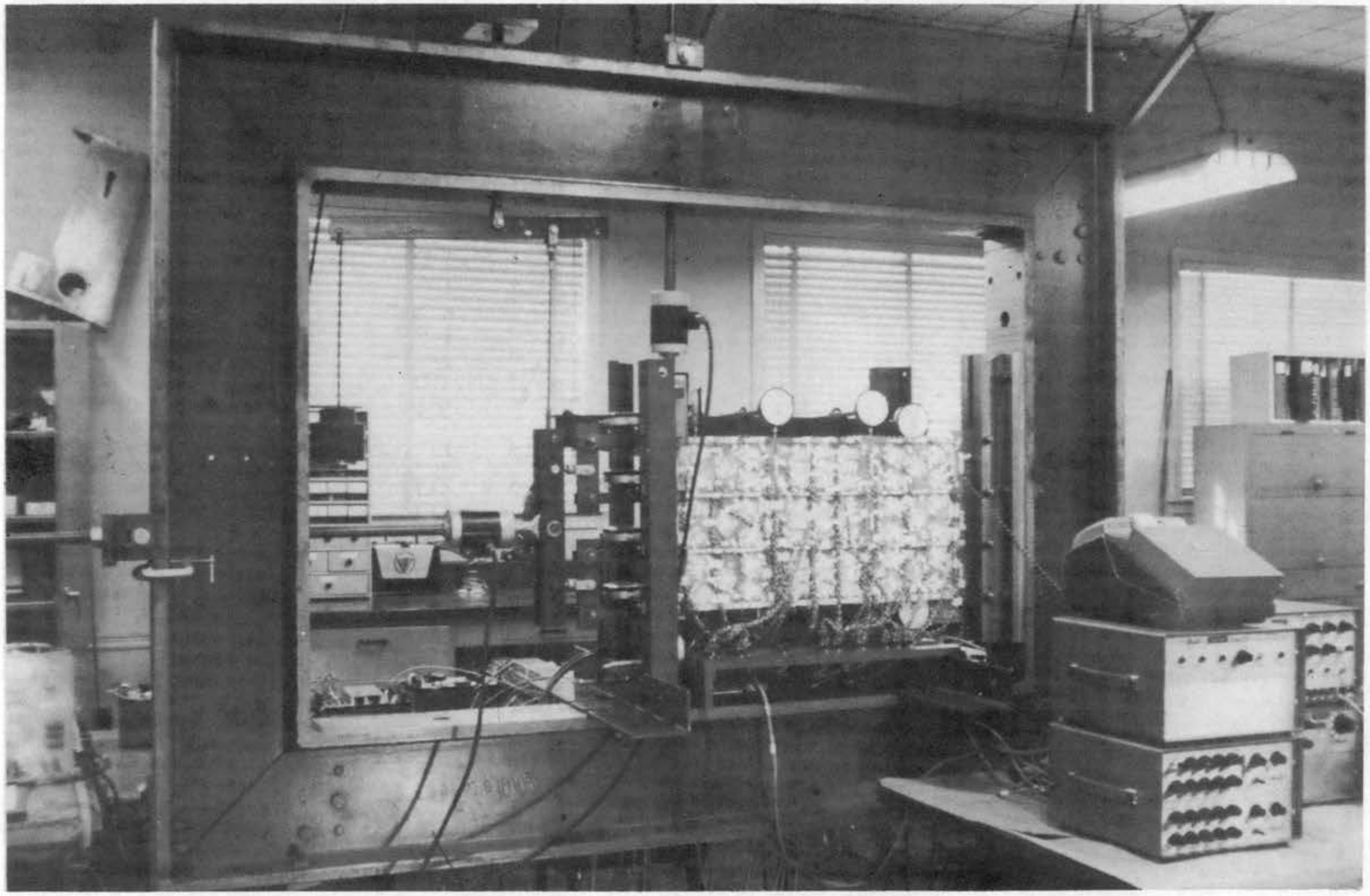


Figure 14. Experimental Facility and Structural Skin Panel

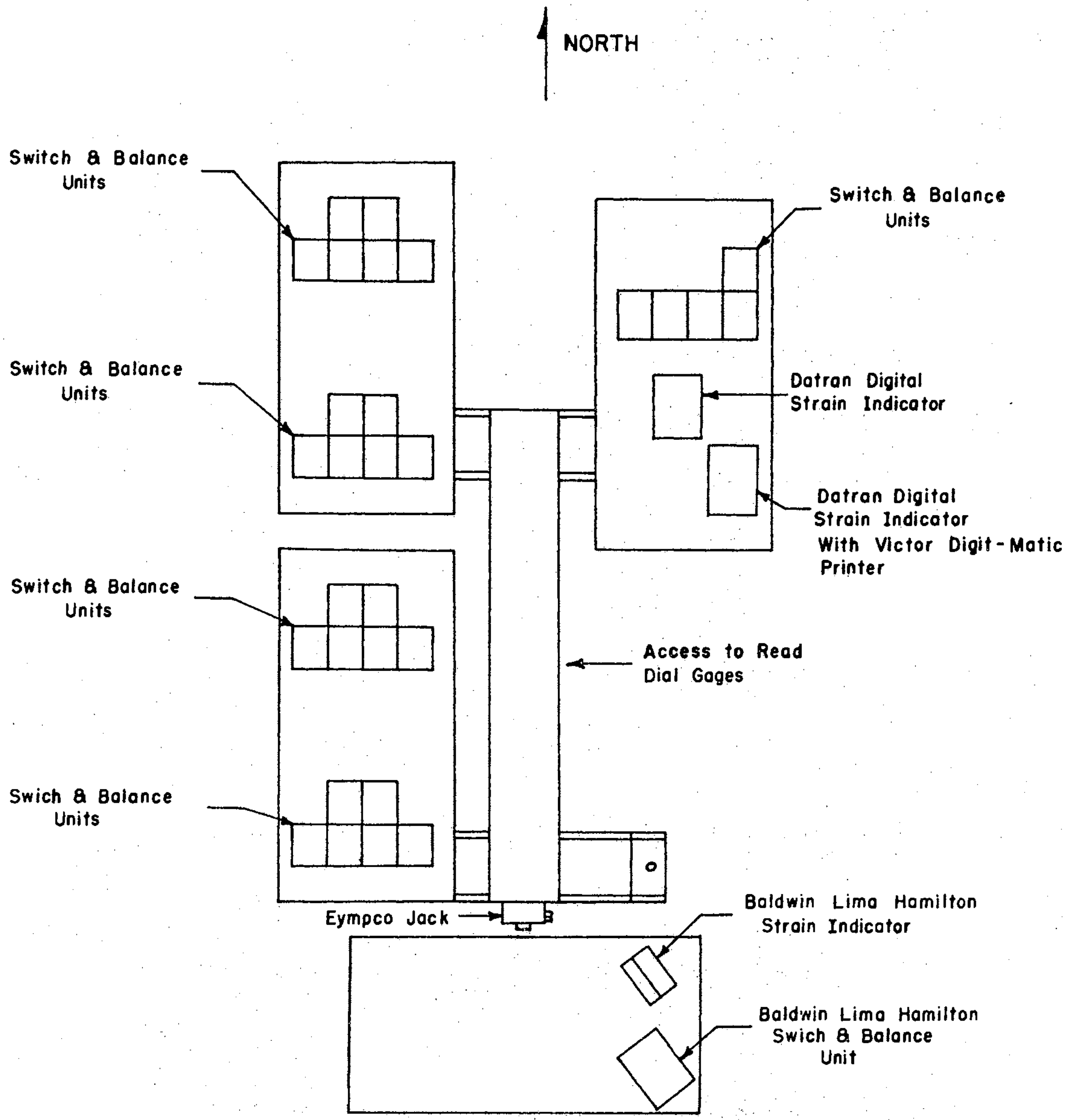


Figure 15. Floor Plan of Experimental Facility

Test Apparatus and Instrumentation

A list of the major equipment used in this test program is given in Appendix D.

The types of strain gages selected for this experimental program were

	Axial	Rosette
Manufacturer	The Budd Co.	The Budd Co.
Type	C12-121-A	C12-121D-R3Y
Gage Factor	2.07 ± 1/2%	2.03 ± 1/2%
Resistance	120 ± 0.2 ohms	120 ± 0.2 ohms

Eastman 910 cement was used to bond the strain gages to the surface of the model after the surface of the model had been prepared using sand-paper, trichlorethylene, and an acid neutralizer. A three-wire system was used to connect the strain gages to the read out instrumentation in order to cancel the effect of changes of wire resistance encountered with changes of room temperature.

The strain gage data recording instrumentation consists of a Datran Digital Strain Indicator with a Victor Digit-Matic Printer shown in Figure 14. In addition, portable strain indicators and switch and balance units, shown in Figure 16, were used to record a total of 300 channels of strain data.

Deflections were measured with Starrett Dial Indicators. The indicators have a range of 0.4 inches and a graduation of 0.0001 inch. The dial indicators were located at the boundary of the panel as shown in Figure 17. Data from these dial indicators were used to determine the deflected shape of the panel.

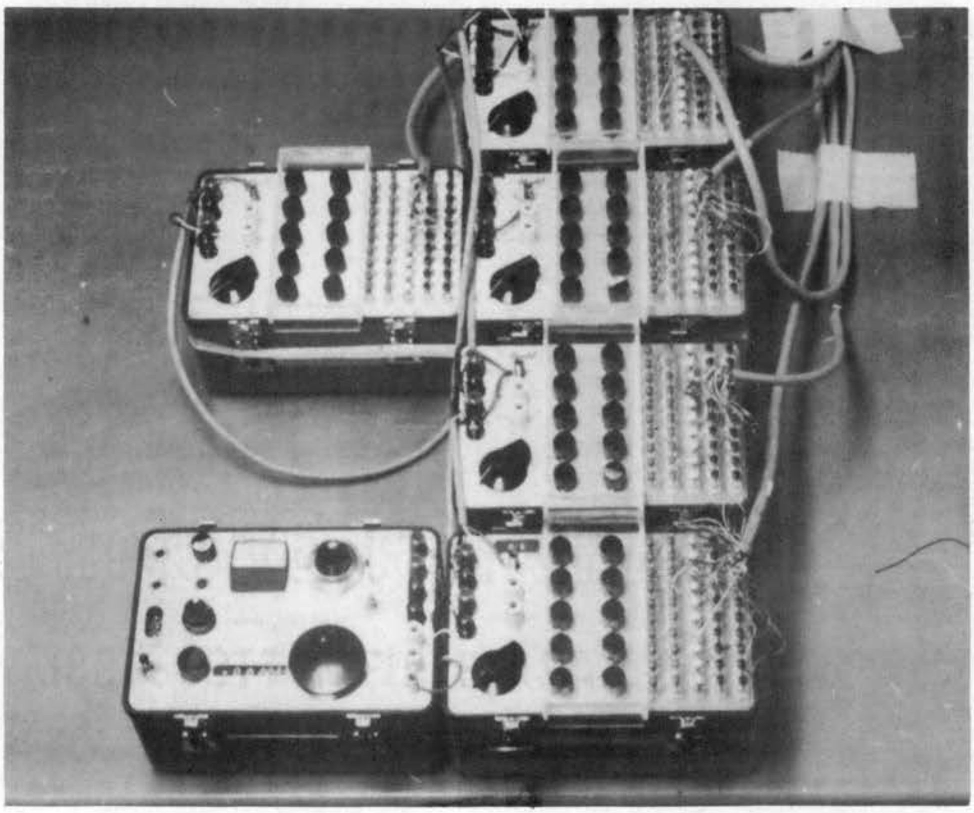


Figure 16. Portable Strain Gage Instrumentation

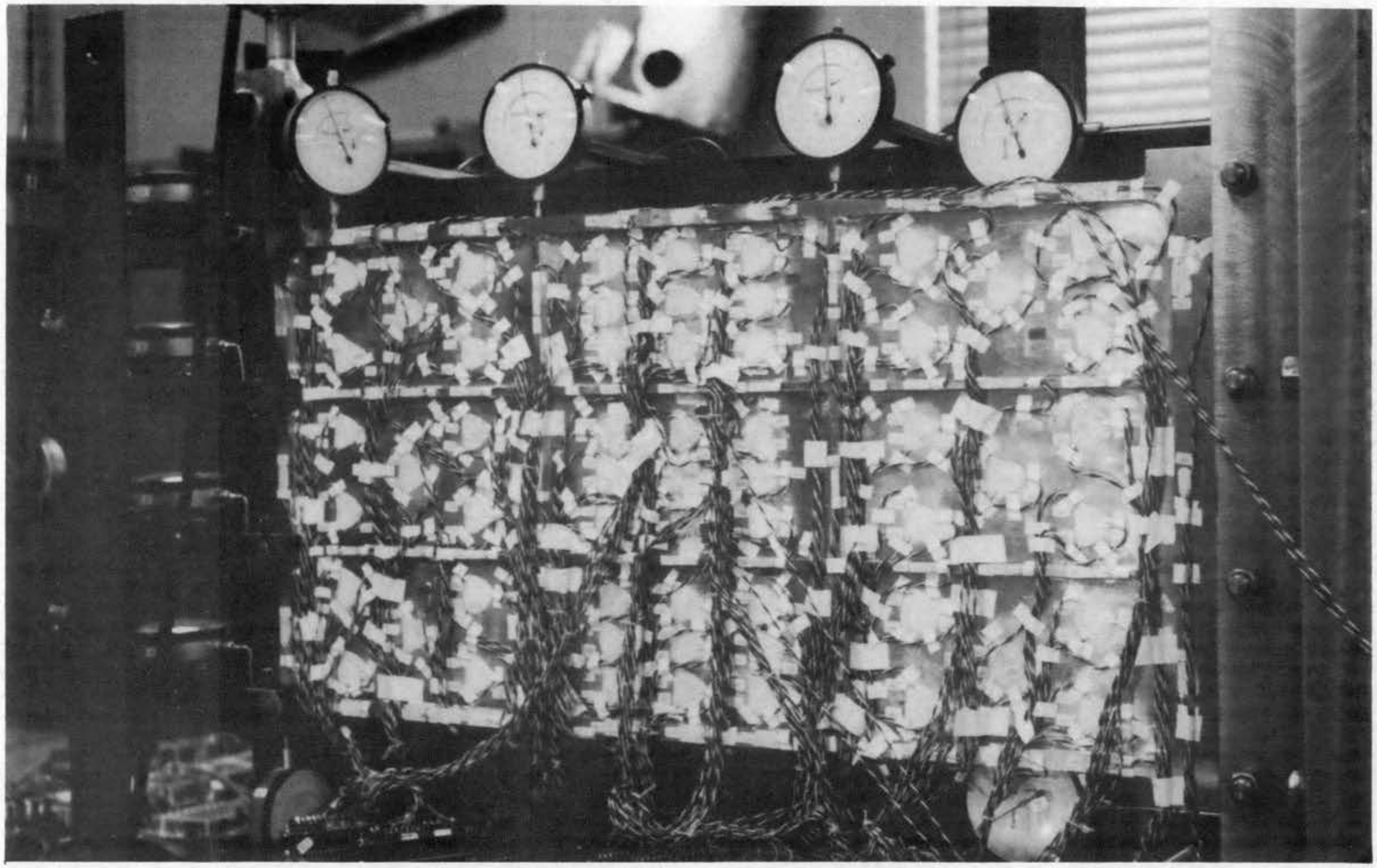


Figure 17. Experimental Reinforced Skin Panel

The loads were applied using an Empco Vertical Motion Jack, Style JH-20, purchased from the Enterprise Machine Parts Corporation. Preliminary tests indicated that these mechanical load devices were satisfactory for this type of static testing. Budd SR-4 Load Cells were used to monitor the external loads on the panel. The loading system is shown in Figure 18. These load cells were calibrated by the manufacturer for an accuracy of ± 0.25 per cent of full scale.

In order to read both load cells on the BLH SR-4 Indicator, the load cells were connected to the indicator through the BLH Switch and Balance Unit, and the system calibrated for a gage factor of 2.0. The SR-4 Load Cells were used to calibrate the BLH, Type N, Indicator against the Budd portable indicators using the calibration factors specified by The Budd Company. The system was also calibrated using test equipment at the Haliburton Oil Company, Duncan, Oklahoma.

The loading system is shown in Figure 18. Load-divider systems shown in Figure 14 were used to divide the load symmetrically to the various load points for load configuration numbers one and two.

The basic loading fixture to be used for the experimental investigation, Figure 14, was designed, fabricated, and used in previous experimental programs at Oklahoma State University (11).

One of the most critical aspects of testing these small structural configurations for deflection and stress characteristics is the manner in which the model is supported in the loading fixture. The support system must not contribute effects at the supports which cannot be represented accurately as boundary conditions. The support system should be rigid enough to minimize the contributions to the panel deflections for maximum loads. Two types of support configurations were considered: A simple

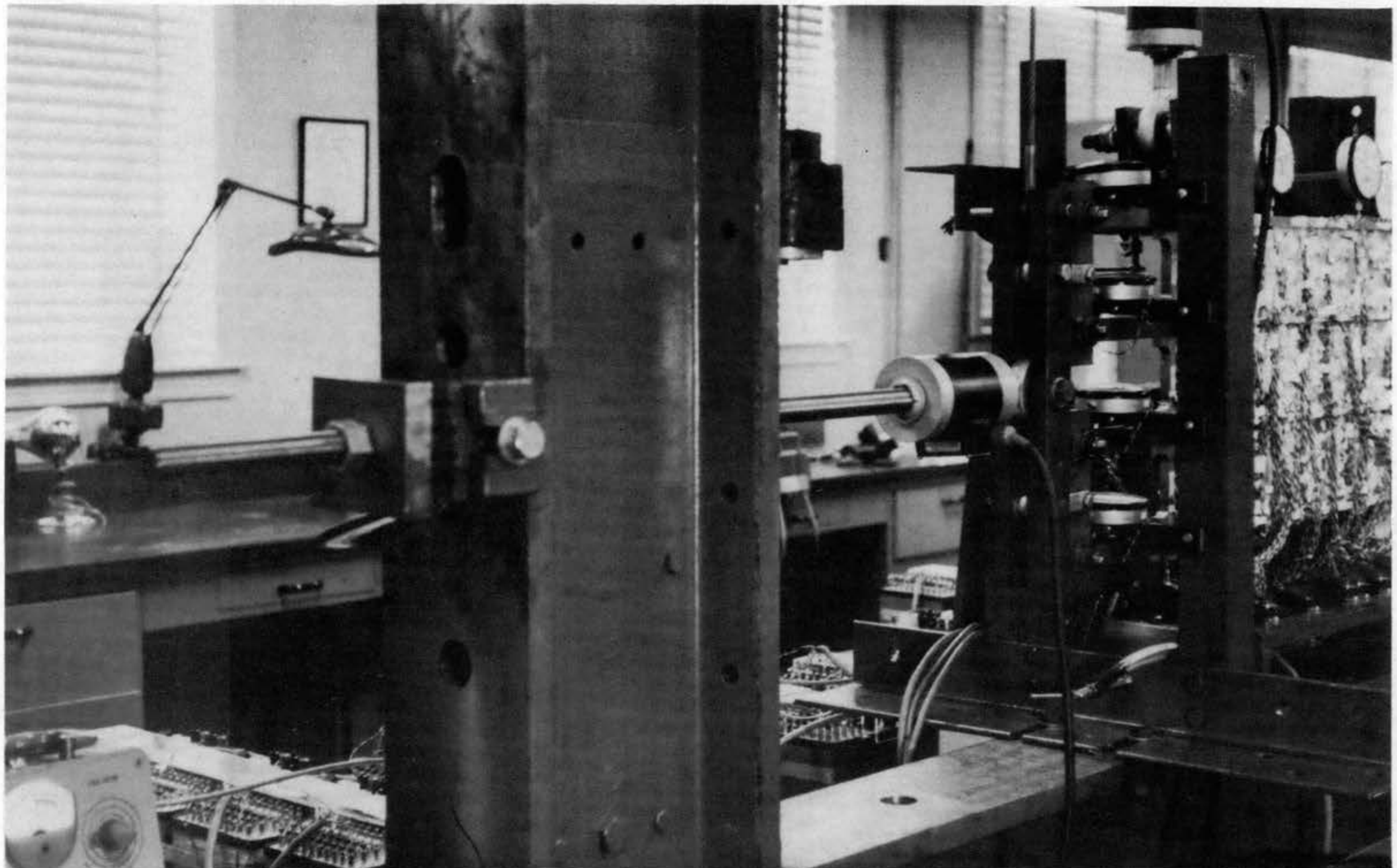


Figure 18. Mechanical Loading System

support configuration, and a fixed-base configuration. Either of these support configurations could be handled accurately in the analysis; however, preliminary experimental test results indicate that the fixed support system, Figure 19, performed more satisfactorily. This was a result of friction in the sliding support which must be assumed friction free.

Preliminary tests were conducted on the panel using twenty strain gages to determine the panel alignment characteristics and to verify the design and application of the related test equipment. The objectives of the preliminary tests were

1. To ascertain the linearity of the load deflection relationships;

2. To determine hysteresis effects;

3. To determine the amount of preload required to remove the initial joint slippage in the model.

The results of these preliminary tests indicated that hysteresis effects were negligible for the load conditions to be investigated. In addition, the model yielded linear results with strains of sufficient magnitude to be recorded easily from the available equipment for the desired load levels. The expected stress concentration effects were observed from both the load divider system and the support system. These unavoidable effects were not excessive and hence did not prejudice the experimental data.

The preliminary tests did indicate that a small amount of out-of-plane deformation was present in the model as a result of the machining operation. This initial deformation had a significant effect on strain measured at the surface of the stringers and ribs. The strain gages on the stringers and ribs were actually one-fourth inch from the center plane of the model. However, excellent results were obtained by using strain

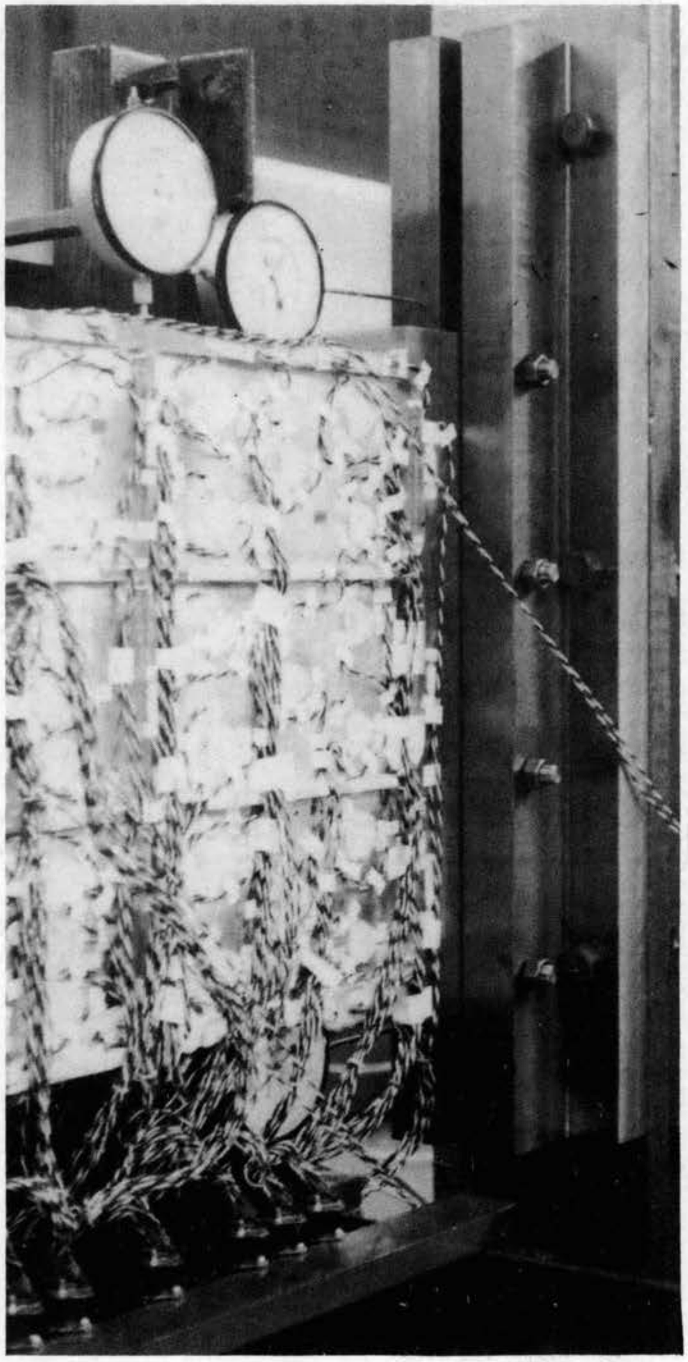


Figure 19. Support System

gages located opposite each other on the ribs and stringers and by using the average of the two readings.

The initial shape of the model also had a significant effect for the shear load configuration. The initial eccentricity resulted in less load capacity than would have been present for a perfect model. This difficulty was overcome by using a 10,000-pound uniform preload to straighten the model for the shear load configuration. Since the combined load was still in the linear load-deformation range, the effect of the 10,000-pound uniform load was easily segregated from the shear load effects.

Subsequent to the completion of the preliminary tests, an additional 280 strain gages were applied to the model at the typical locations shown in Figures 20, 21, and 22. In many cases, redundant gage locations were used to check the symmetry of load distribution. The axial and rosette gages were numbered as shown in Figures 20, 21, and 22. The numbering system was designed to provide maximum flexibility in the adding or in the changing of gages.

Deflections and internal load distributions were determined experimentally for the fundamental types of applied loads that are found on actual aircraft structural skin panel configerations. The most common of these load configurations are the uniform tensile and the combined tensile and shear loads. The test configurations are divided into five load conditions. These five load configurations are shown in Figure 23. Data for each test configuration were obtained after a check out of the test equipment.

The strain gages monitored during each test are indicated in columns two and three of Table XII under the heading, Number of Gages. The rosette gages are divided into three classes. The first class consists of all of

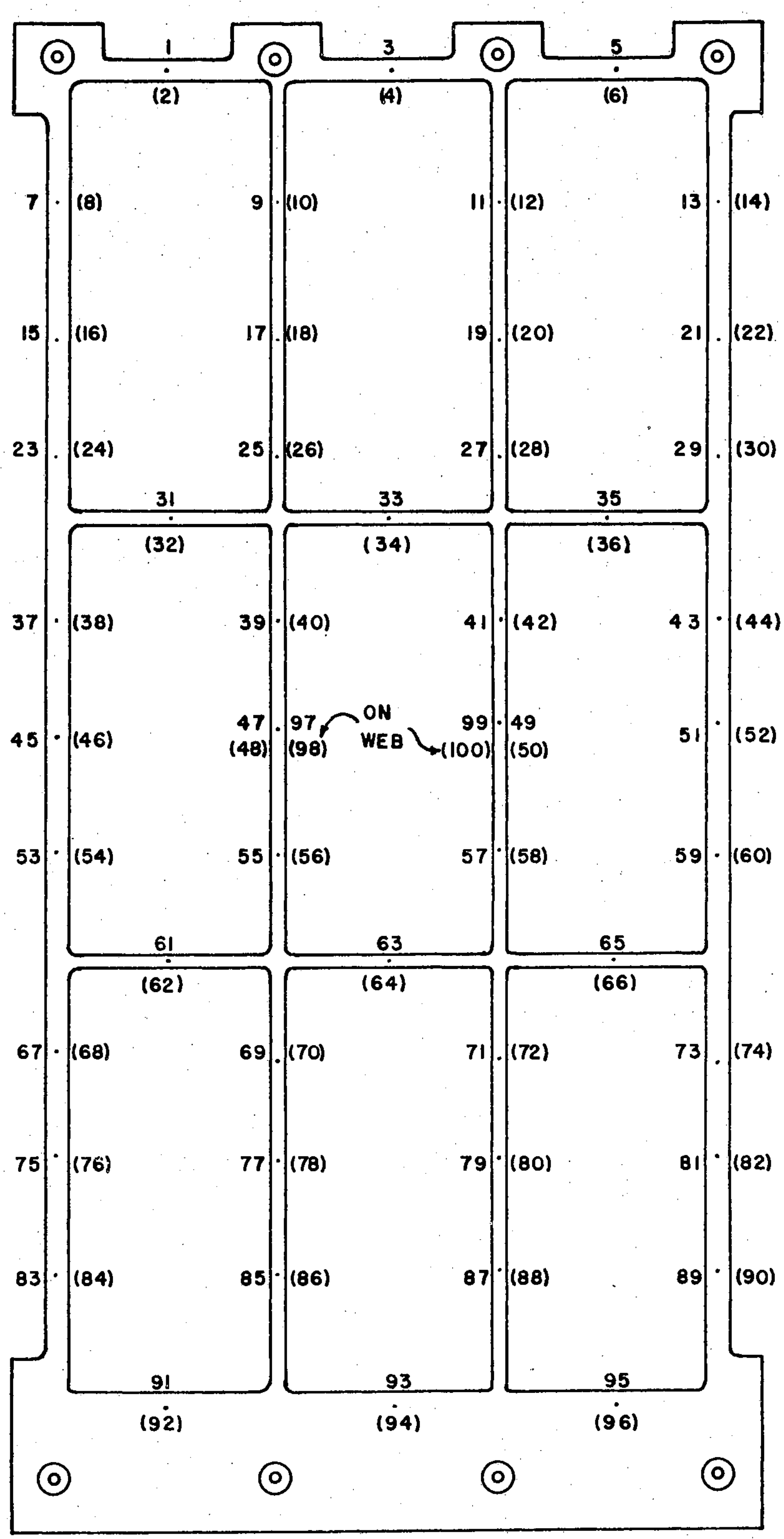


Figure 20. Axial Strain Gage Numbering System

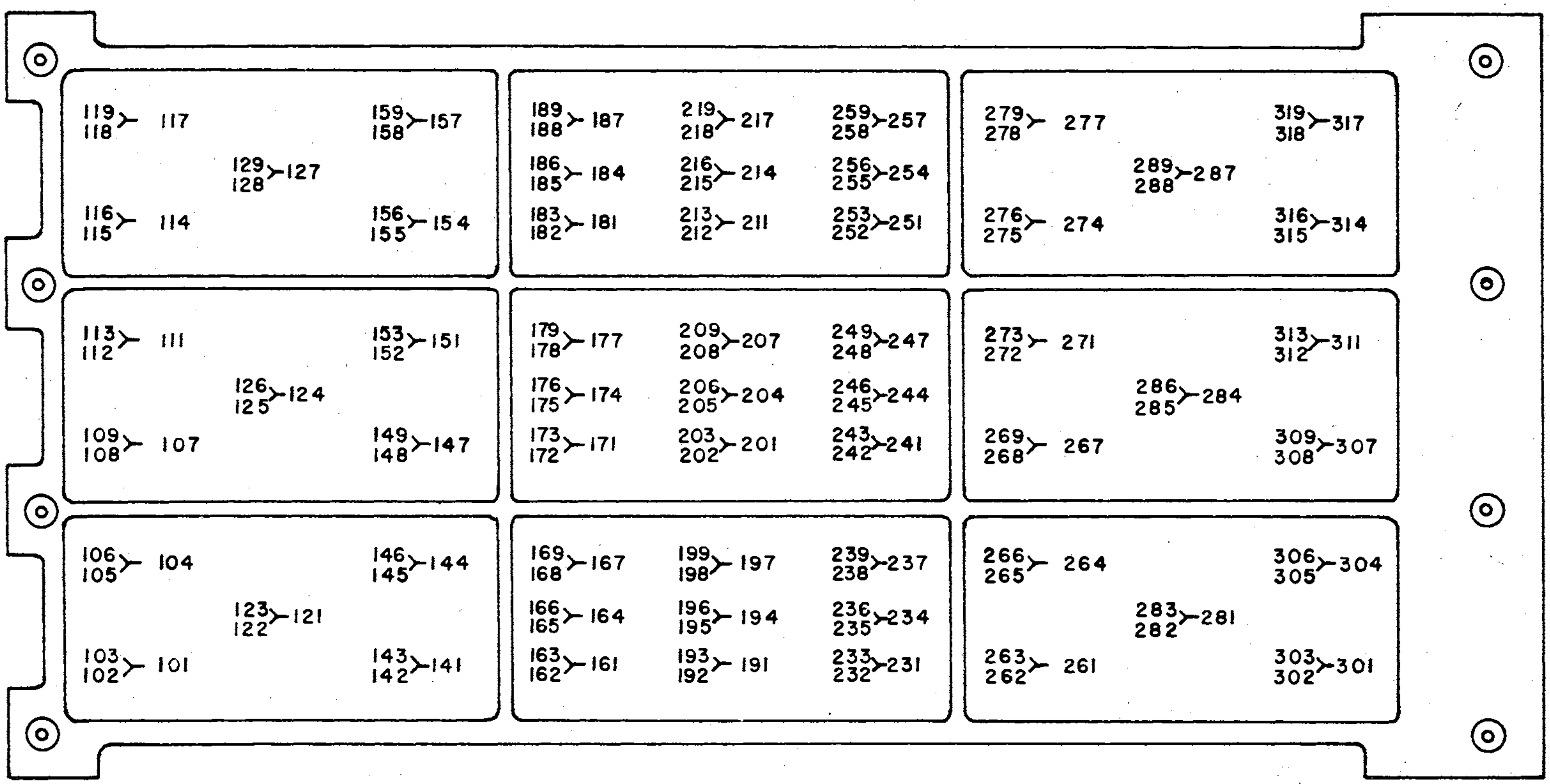


Figure 21. Rosette Strain Gage Numbering System on Front Side

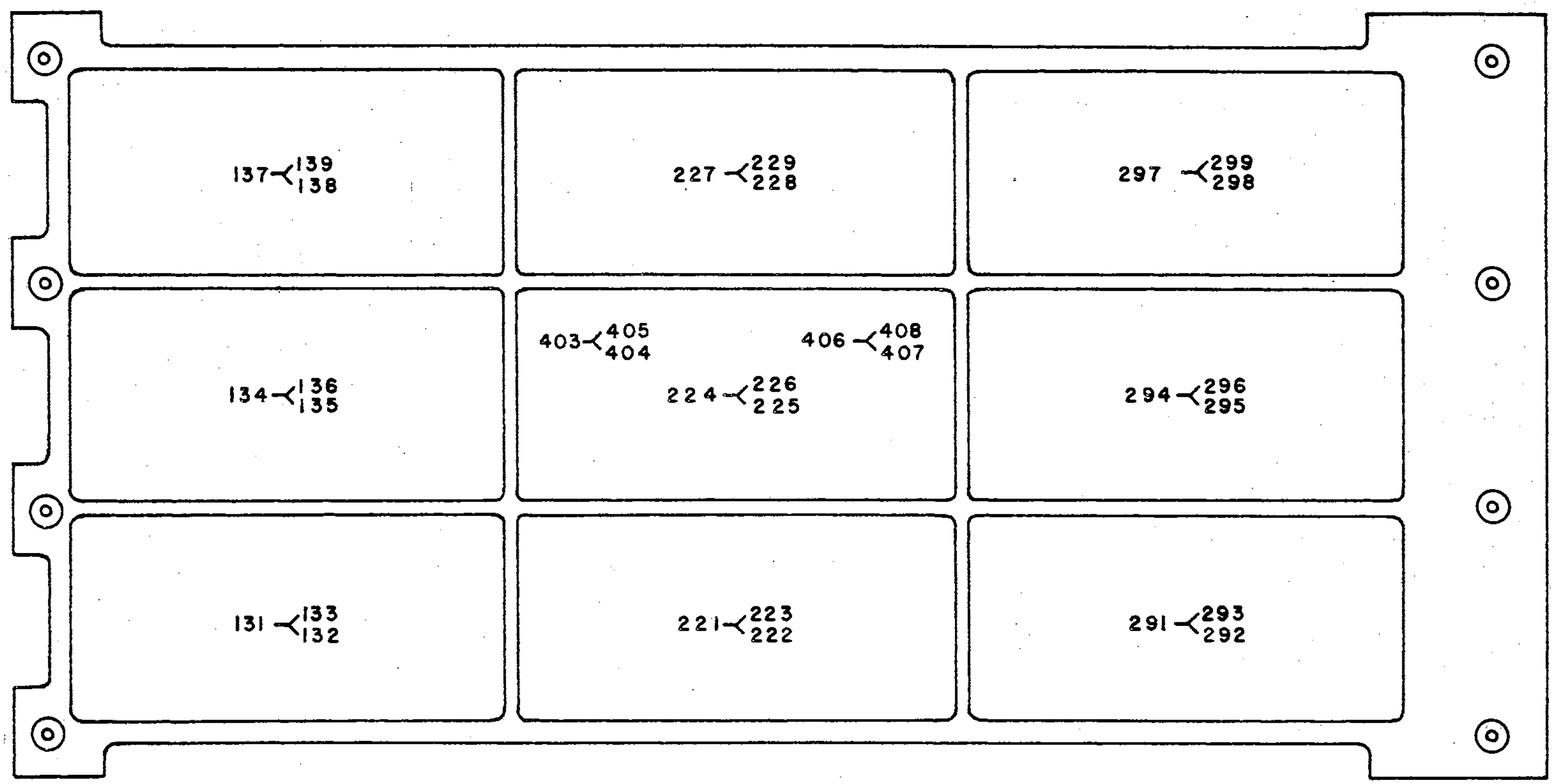


Figure 22. Rosette Strain Gage Numbering System on Back Side

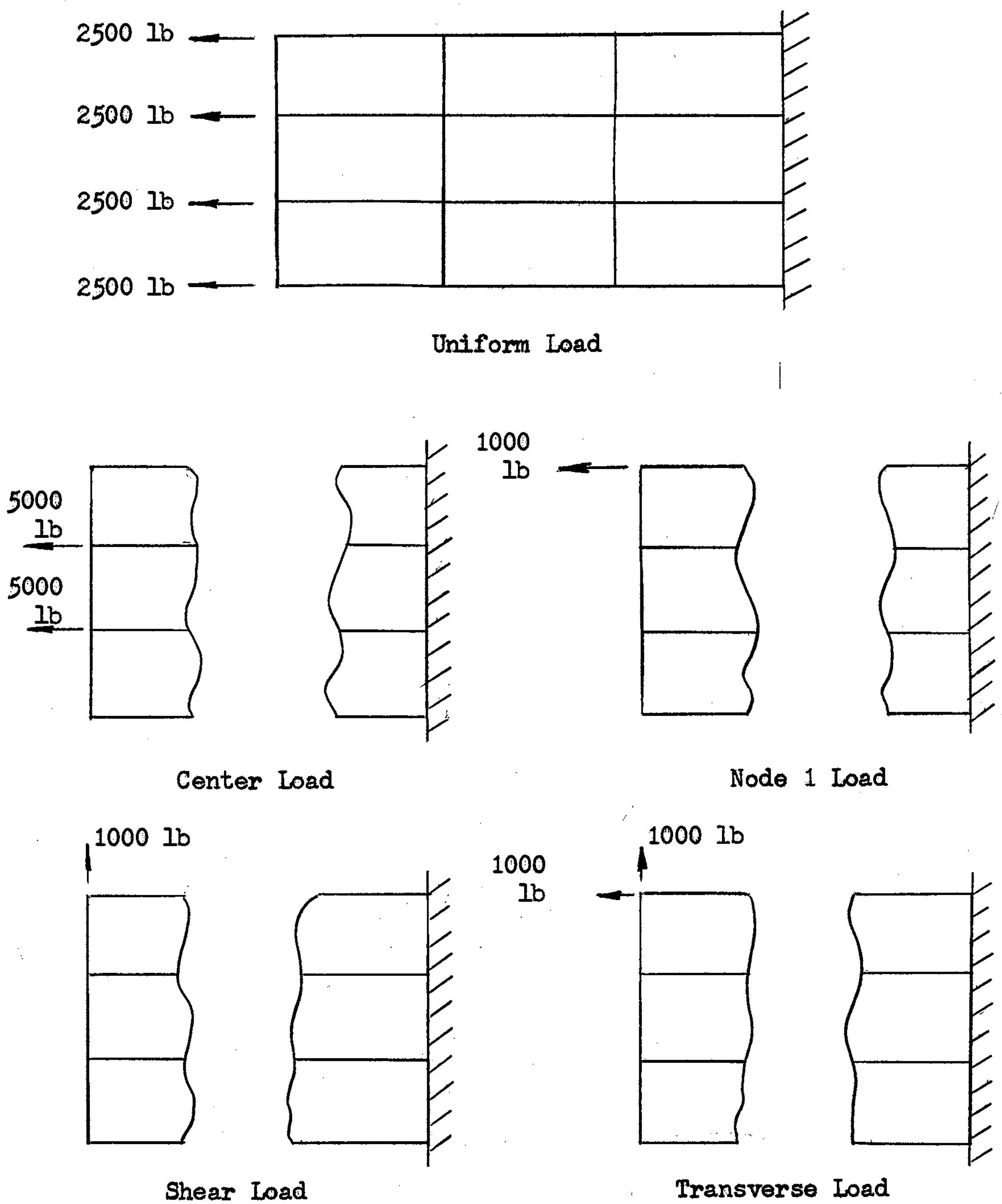


Figure 23. Load Configurations

TABLE XII

TEST CONDITIONS

Test No.	Number of Gages: Axial	Number of Gages: Rosette	Number of Observations	Test Date	Load Interval	Test Description
1	60	All	10	12-13	1000-10000	Uniform Load
2	60	All	4	12-14	500-1500	Shear Load
2A	0	All	9	12-14	2500-500-1500	Shear Load
3	60	All	9	12-14	100-5000	Center Load
4	96	All	9	12-16	1000-5000	Single Load Node 2
5	96	All	10	1-26	1000-10000	Uniform Load
6	96	All	6	1-27	1000-6000	Uniform Load
7	96	0	Use 9	2-2	1000-10000	Uniform Load
8	96	Class 2	4	2-4	500-1500	Combined for Shear
9	96	Class 2	9	2-7	0-250-1750	Combined for Shear
10	96	0	6	2-8	0-1000-5000	Center Load
11	96	0	5	2-8	1000-5000	Single Load Node 1
12	96	0	5	2-8	1000-5000	Single Load Node 1
13	96	0	5	2-9	1000-5000	Center Load
14	96	0	10	2-9	0-3000-0	Transverse
15	96	0	8	2-11	250-2000	Transverse
16	96	Class 2	8	2-14	250-2000	Transverse
17	96	Class 2 & 3	8	2-16	250-2000	Transverse
18	96	Class 2	8	2-28	250-2000	Transverse
19	100	Class 2 & 3	4	2-28	250-2000	Transverse
20	100	All	10	3-2	500-2750	Transverse
21	100	0	10	3-3	0-5000	Single Load Node 1
22	100	All	6	3-3	10,000 Horizontal 500-3000 Shear	Combined for Shear

the rosette gages. The class-two gages are the twelve rosettes located on the center web of the model. The class-three gages are the eighteen gages located at the center of each web of the model.

The strain and deflection data were obtained for the magnitudes of external loads shown in Table XII. Since hysteresis effects were demonstrated to be small in the preliminary tests, data were recorded for increasing loads at equal intervals for the number of observations during each test condition as shown in Table XII. The experimental data were reduced to values per unit of load by the procedures and digital computer programs described in Appendix C.

CHAPTER VII

COMPARISON OF ANALYTICAL AND EXPERIMENTAL RESULTS

The objective of this research effort is to develop the capability for the analytical and experimental investigation of integrally reinforced rectangular skin panels using finite element methods of structural analysis. The analytical capabilities, which are developed, include both the force and displacement methods of structural analysis.

The force method of analysis used in this investigation demonstrates the redundant load paths that are possible in the analysis of complex skin structures. The accuracy of the force analysis is influenced by the choice of the idealized statically determinate system. The idealized systems used in this investigation satisfactorily represent the principal load paths throughout the structure. The idealization resulted in well-conditioned matrices preserving computational accuracy and stress variations that represent the actual structural behavior. Consequently, good results are obtained from the force method of analysis as shown in Figure 26.

The stiffness method of analysis was used for the most extensive investigations of the structural skin panel, because there is no requirement for the choice of statically determinate load paths within the structure. Consequently, the complete analysis can be performed using the digital computer specifying only the geometric and structural configuration of the skin panel. The analysis capability is described in Chapters III, IV, and V. Numerous structural idealizations are used in the investigation;

however, only the results of the most obvious idealization using the best stiffness matrix are reported in this thesis. The analysis capability is available for further study of any class of two-dimensional structural configurations, and the scope of these problems is too broad to be mentioned here.

The experimental capabilities developed during this and previous investigations have provided fundamental procedures and equipment that are applicable for numerous future research programs. Some of these possibilities are suggested in Chapter VIII.

A total of twenty-two tests were performed with the integrally reinforced rectangular panel, using five load conditions applicable for this type of structure. A total of approximately thirty thousand data points were recorded during these twenty-two tests. Only the basic data required for comparison to the analytical results are reported in this thesis. Additional data would only duplicate the basic information shown in this chapter for additional points on the structure. The basic data reported here are sufficient to indicate the excellent agreement between the analytical and experimental results. This agreement demonstrates the applicability of the finite elements methods of structural analysis for integrally reinforced structural skin panels.

A qualitative description of the axial stress variations obtained from the Stress Analysis System are shown for the shear and the transverse load configurations in Figures 24 and 25. The axial stresses are in the direction of the longitudinal axis and were computed at specific points within the structure. A smooth surface is generated through these points. The value of the stress at each point is represented by the distance along the vertical axis. These surfaces demonstrate the large variations in axial stresses that occurred within the panel for the shear and the transverse load conditions. The comparisons of the analytical and experimental stress results at typical points on the panel are shown in Figures 26 through 37. The comparisons of the analytical and experimental deflection results for points on the edge of the panel are shown in Tables XIII, XIV, and XV.

The deflections representing the corner point where the shear load is applied are actually shown for two different points located as close as possible to each other. The analytical data are obtained for the exact point where the shear load is applied. Due to the loading system, it was not possible to place a dial indicator at the same point. Therefore, the experimental data are obtained for a point approximately two inches from the point where the shear load is applied.

The experimental deflection data shown in Tables XIII, XIV, and XV. are corrected based on the measured deflections of the supporting system. However, the data are still different by a constant value as shown in the sketches on Tables XIII, XIV, and XV. This constant value is due to a slight displacement of the complete test panel relative to the support system and occurs possibly in the bolts and self-aligning bearings connecting the panel to the support system.

In general, the accuracy of these comparisons is within the variations resulting from the manufacturing tolerances for the structure. The actual dimensions of the panel are used for the analytical and the experimental comparisons. The actual dimensions are shown in Figure 11 and can be compared to the nominal dimensions shown in Figure 13. The nominal dimensions would normally be used for design calculations.

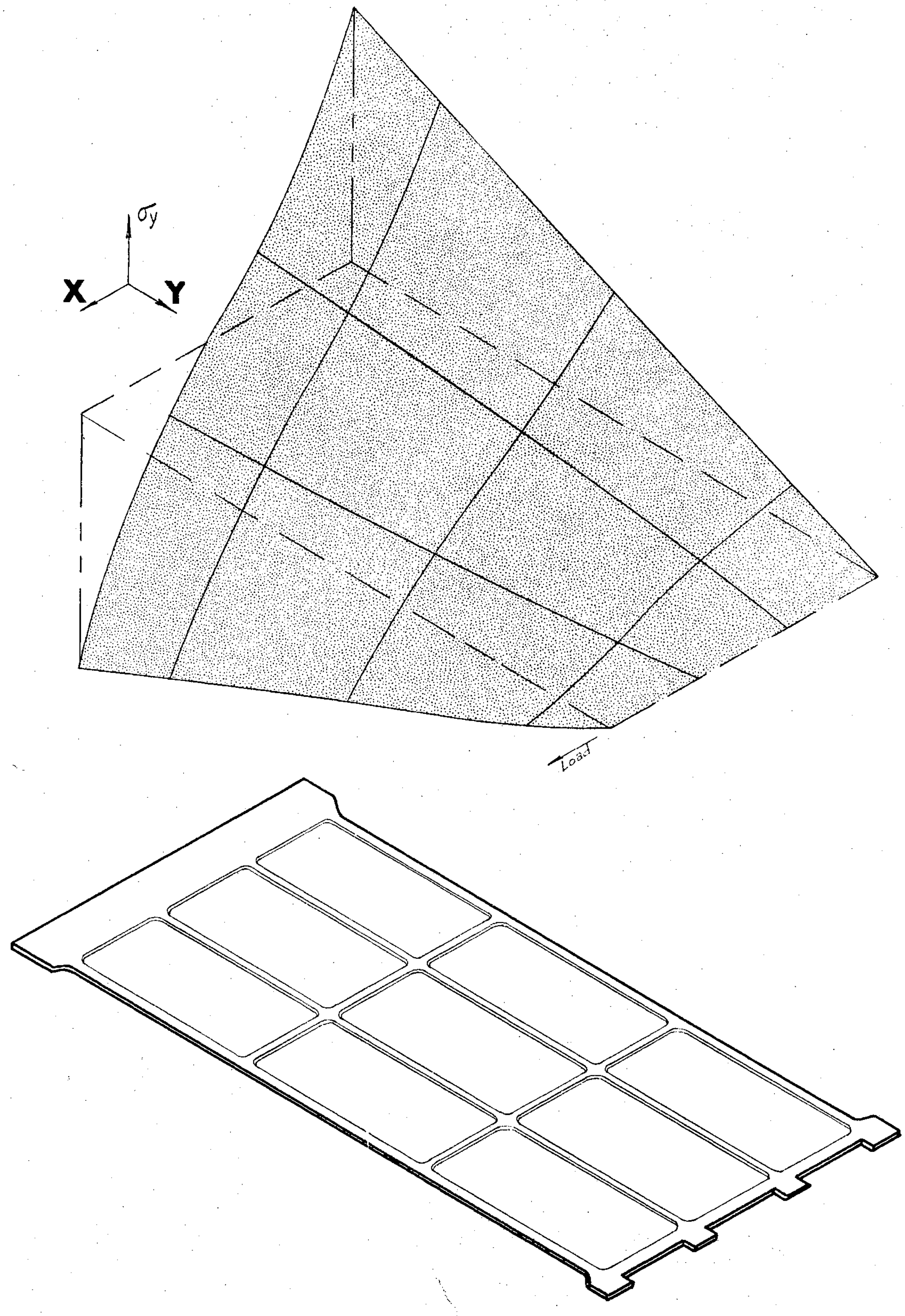


Figure 24. Qualitative Description of the Axial Stress Variation for the Shear Load Condition

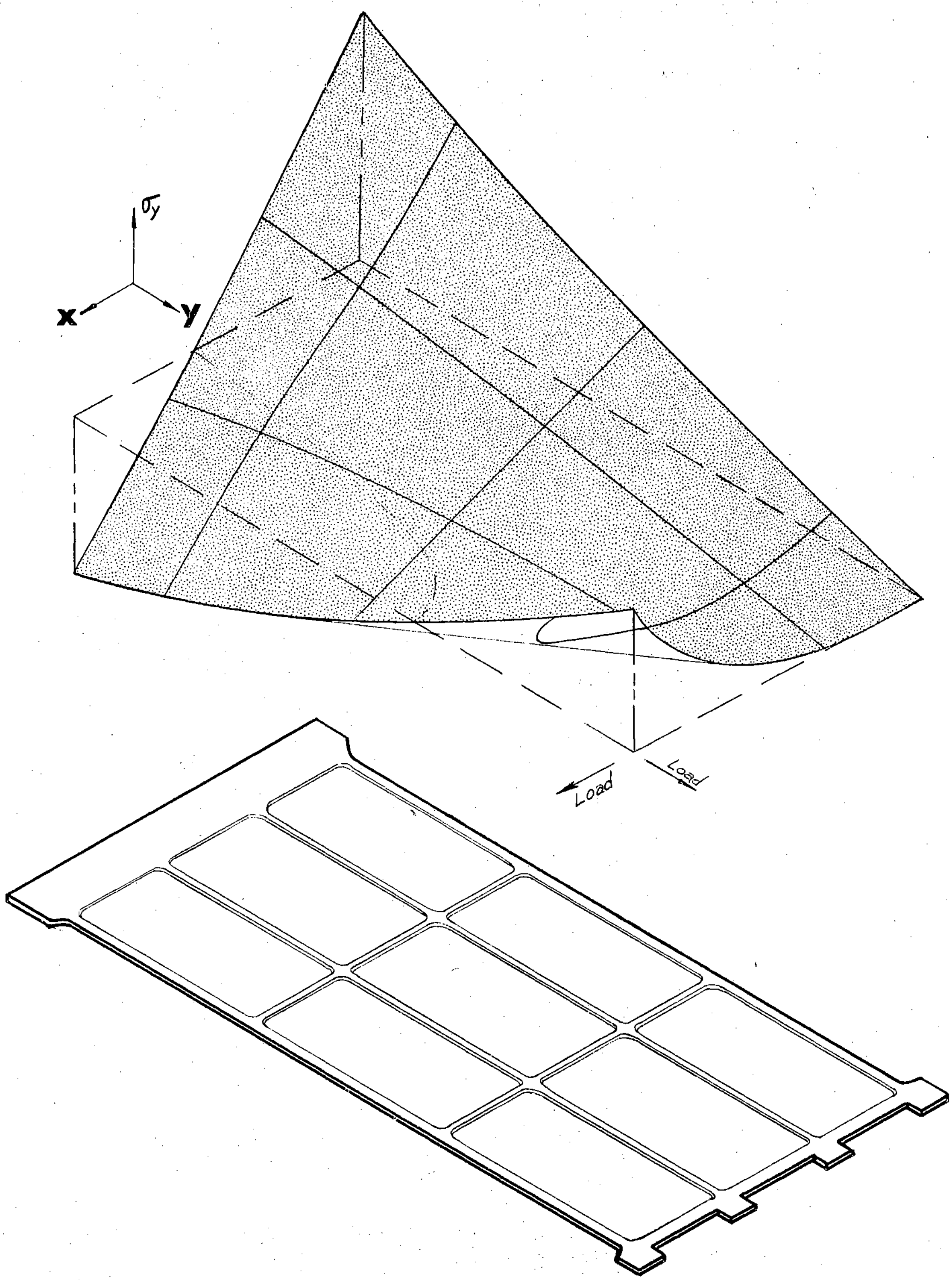


Figure 25. Qualitative Description of the Axial Stress Variation for the Transverse Load Condition

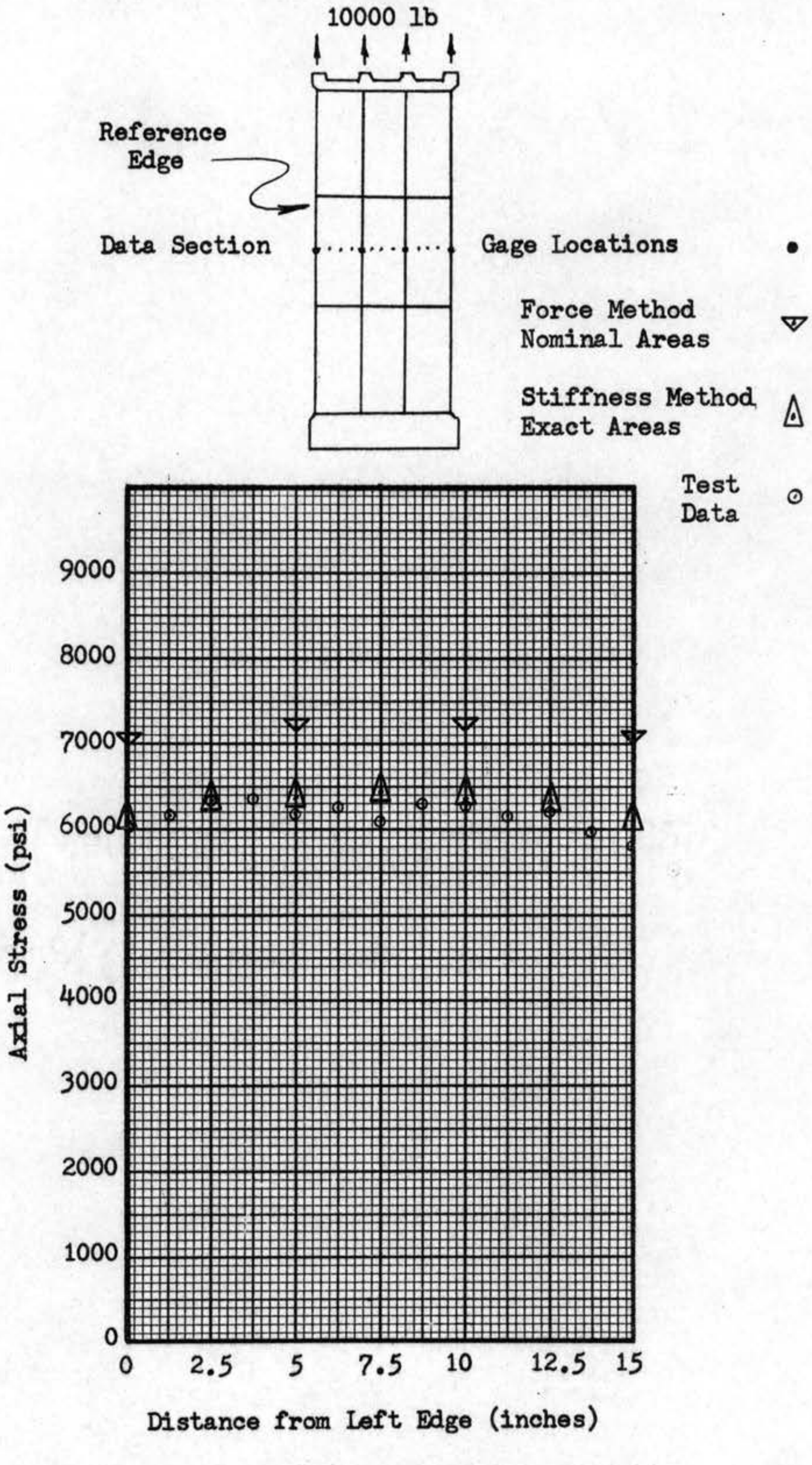


Figure 26. Axial Stresses for Uniform Load Condition

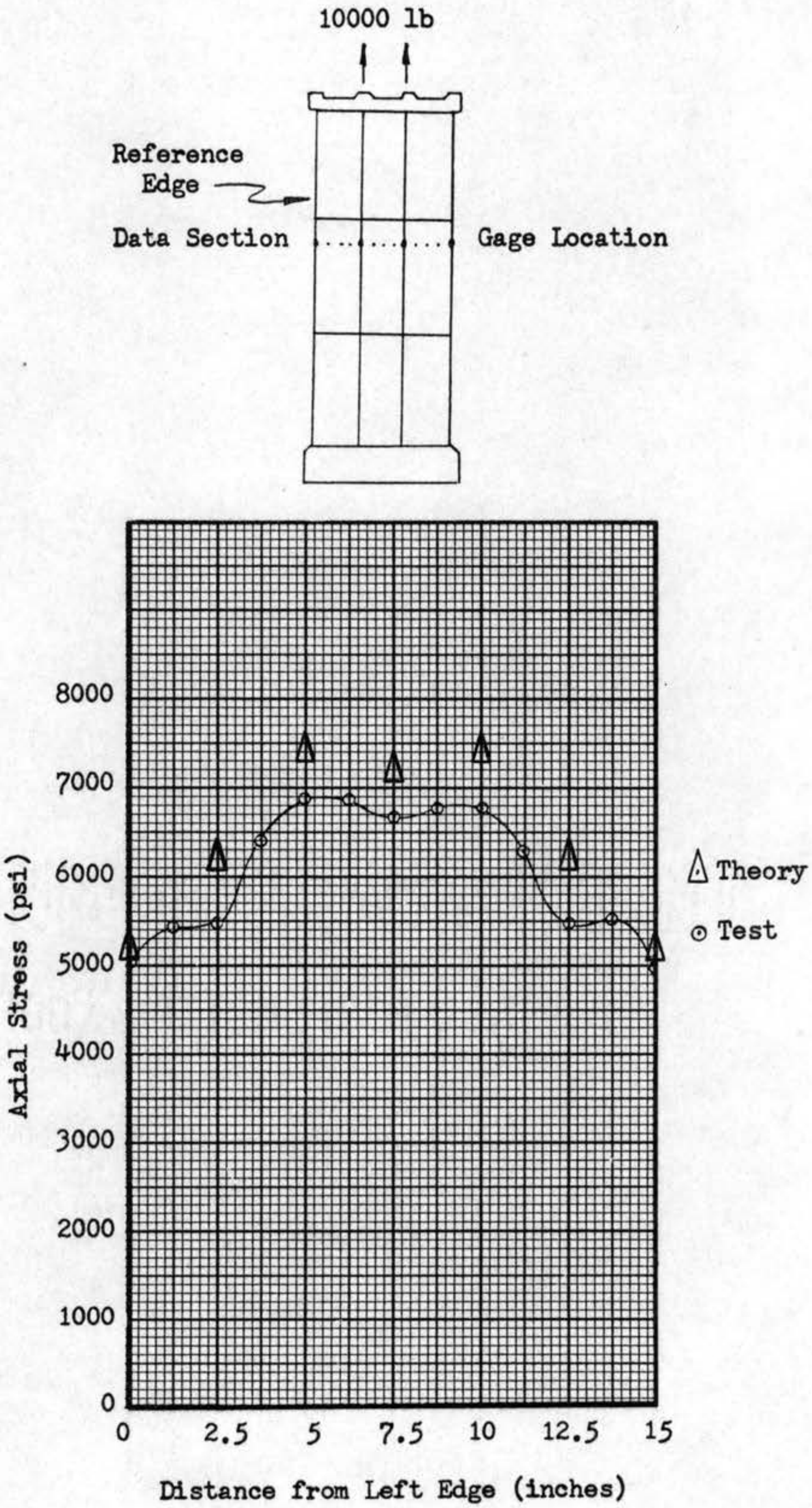


Figure 27. Axial Stresses for Center Load Condition

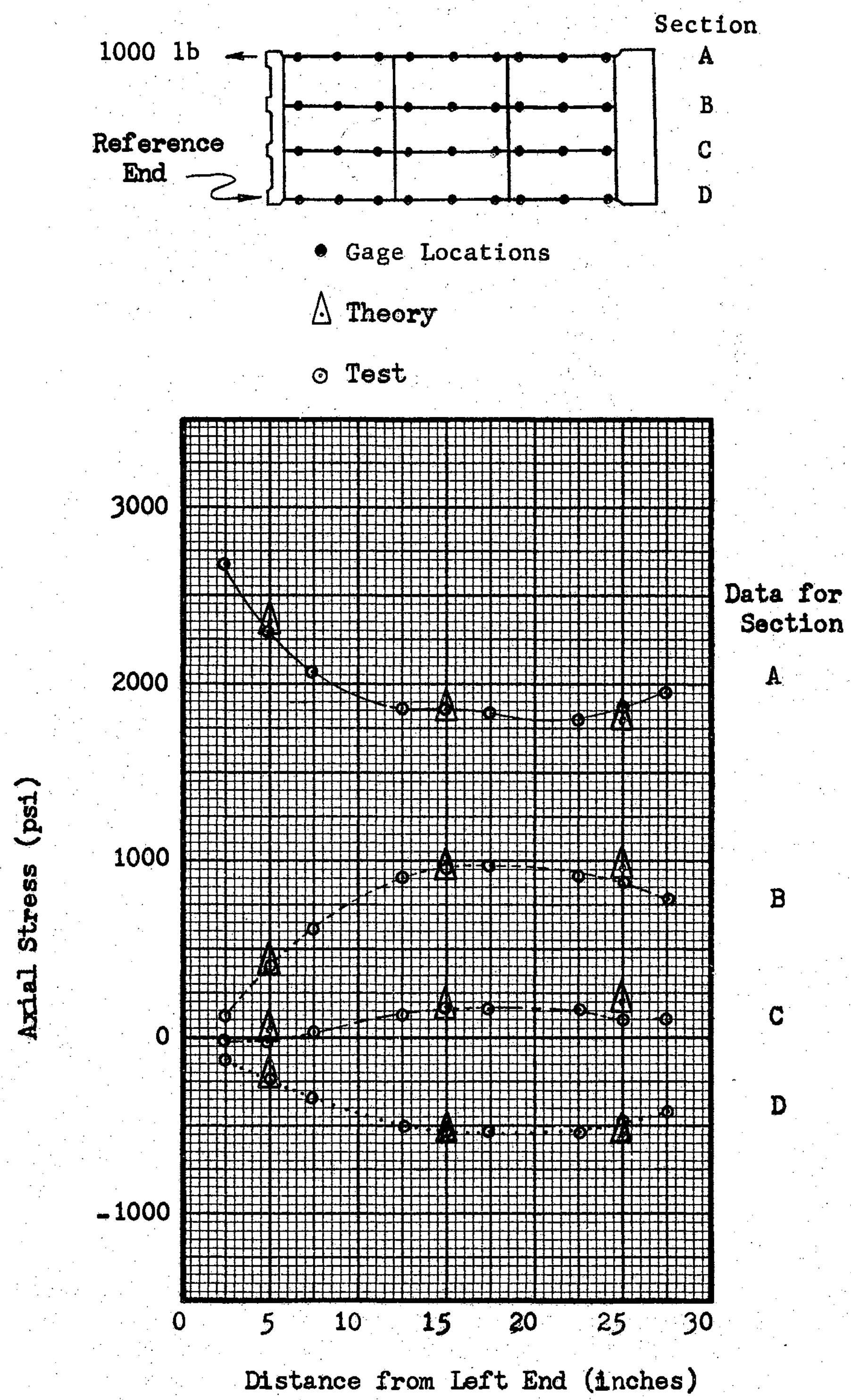
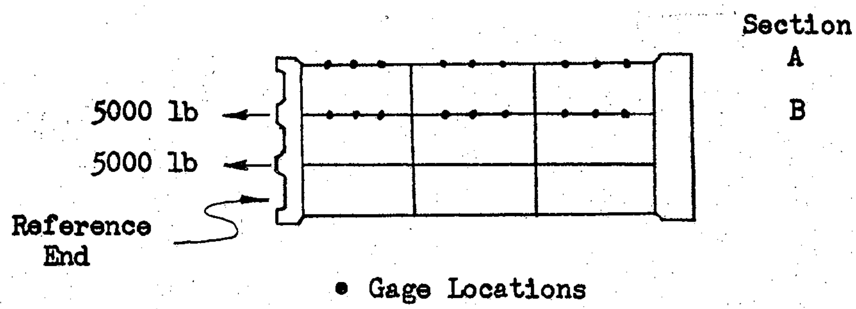


Figure 28. Stringer Stresses for Node 1 Load Condition



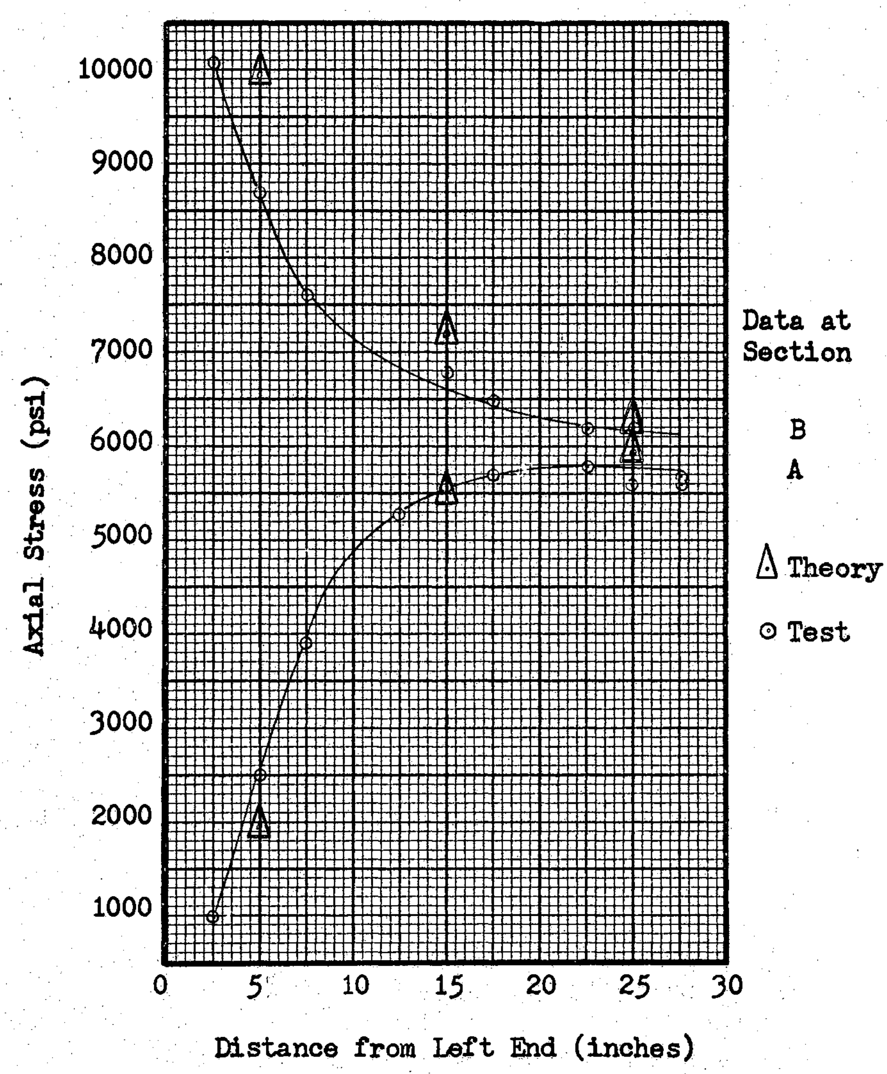


Figure 29. Stringer Stresses for Center Load Condition

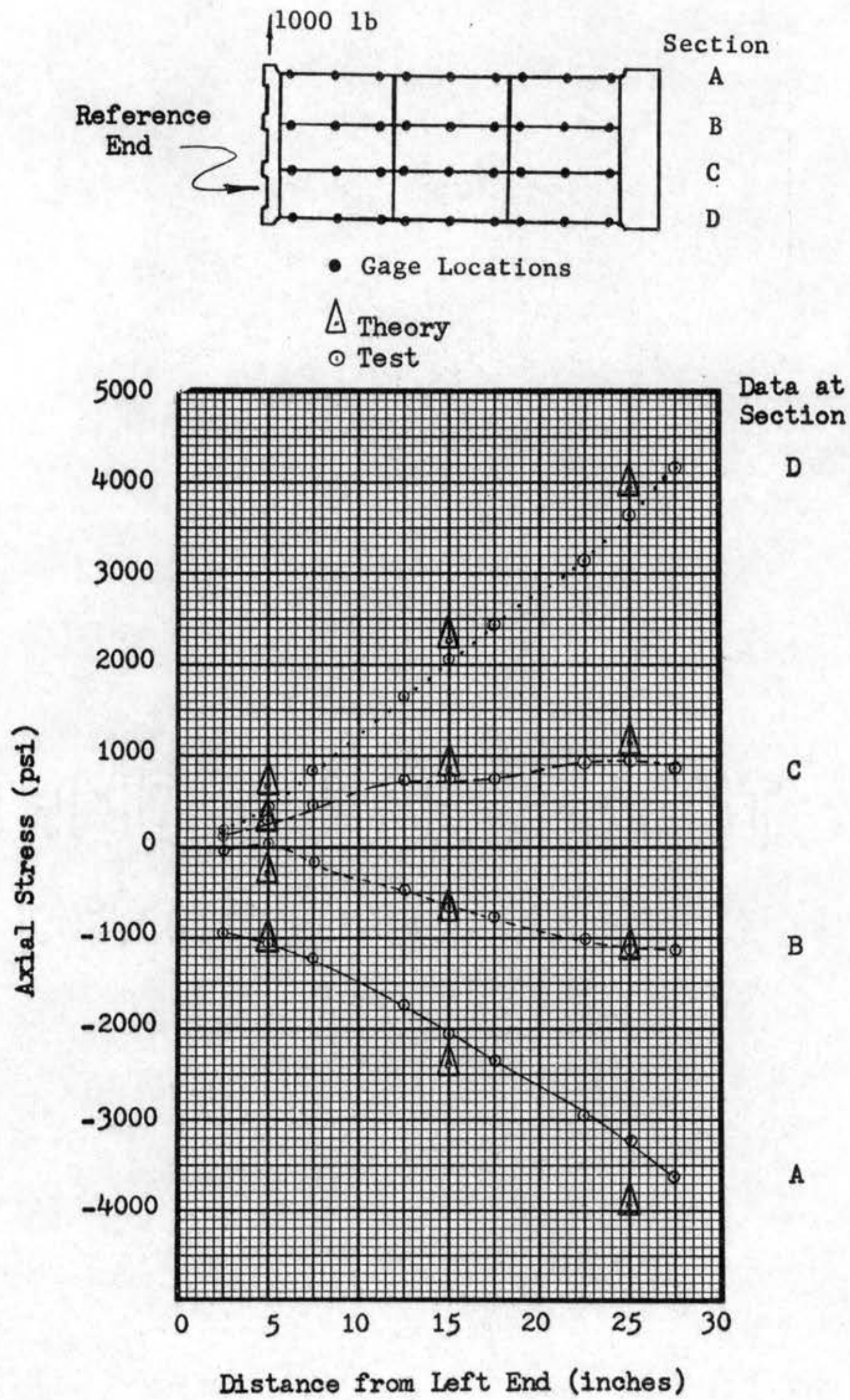


Figure 30. Stringer Stresses for Shear Load Condition

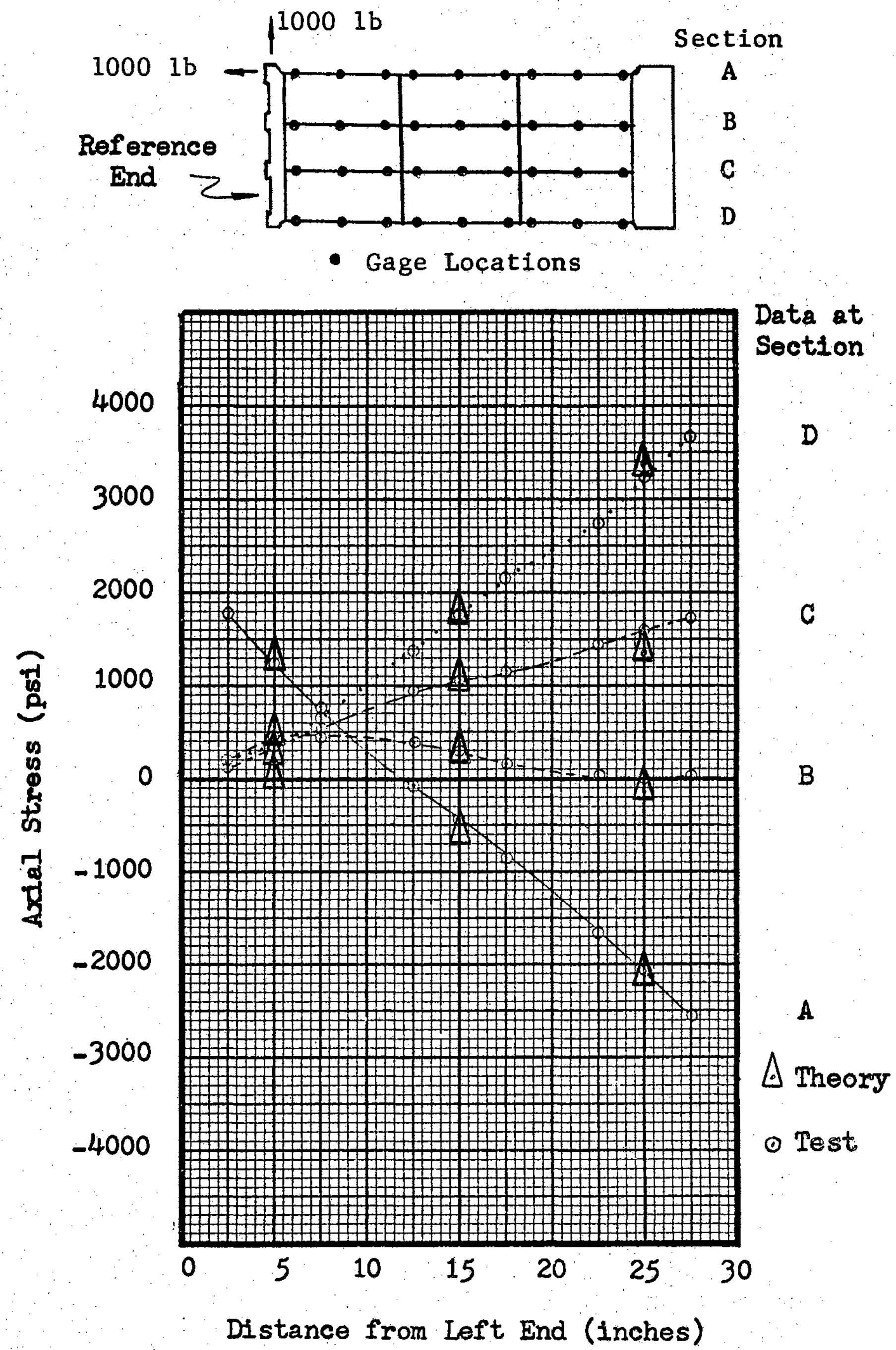


Figure 31. Stringer Stresses for Transverse Load Condition

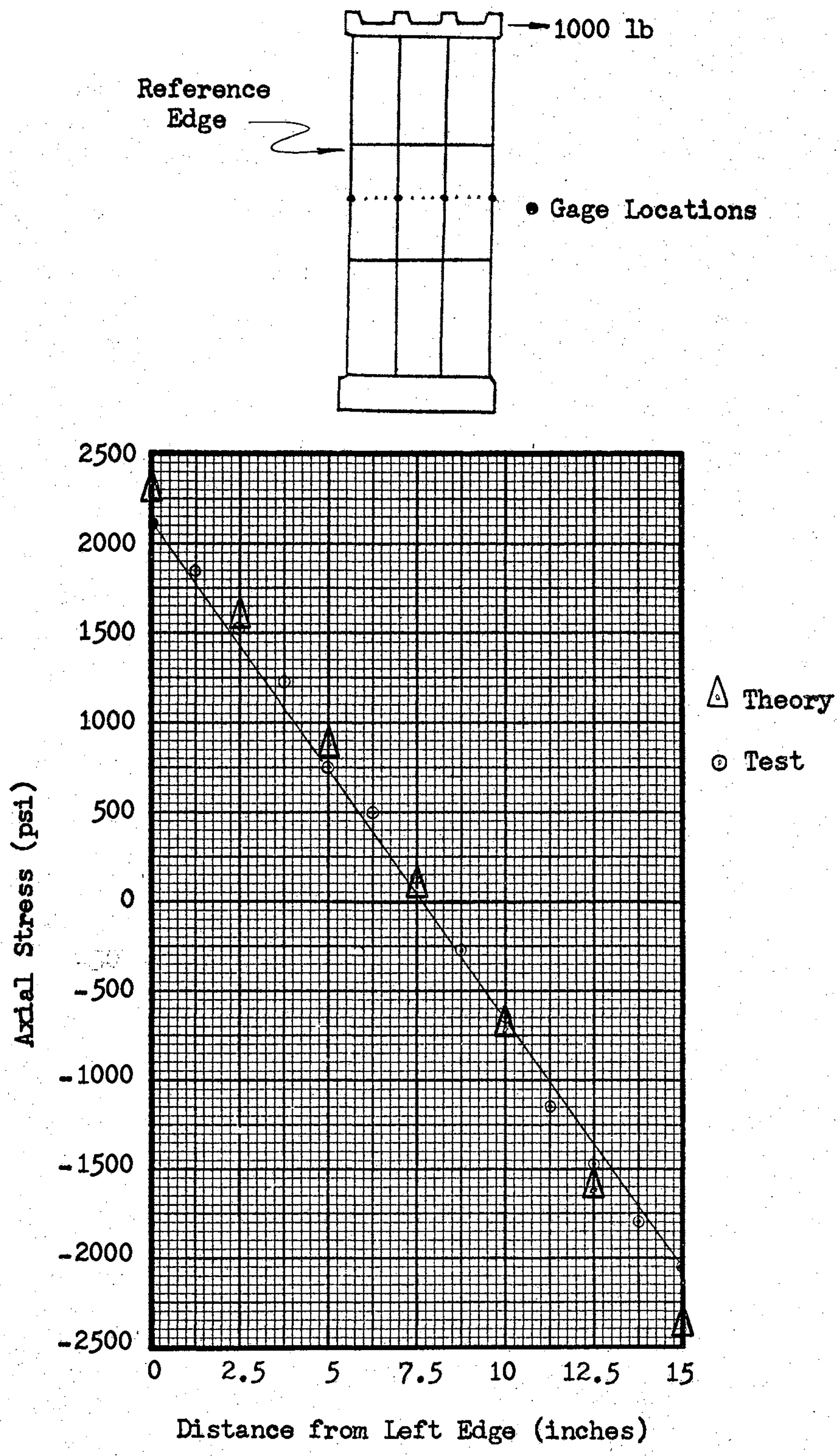


Figure 32. Axial Stresses at Center Section for Shear Load Condition

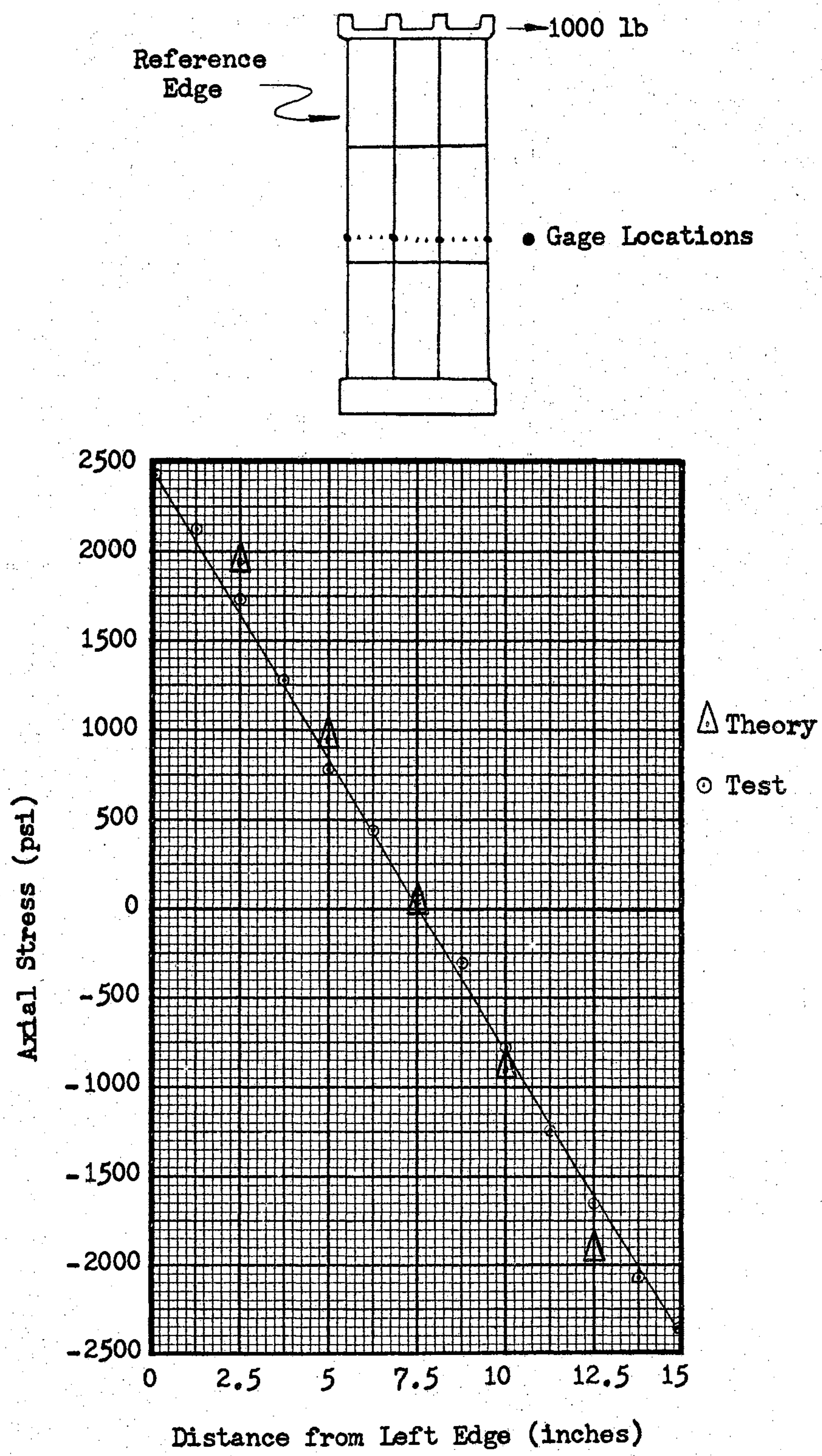


Figure 33. Axial Stresses at Aft Center Section for Shear Load Condition

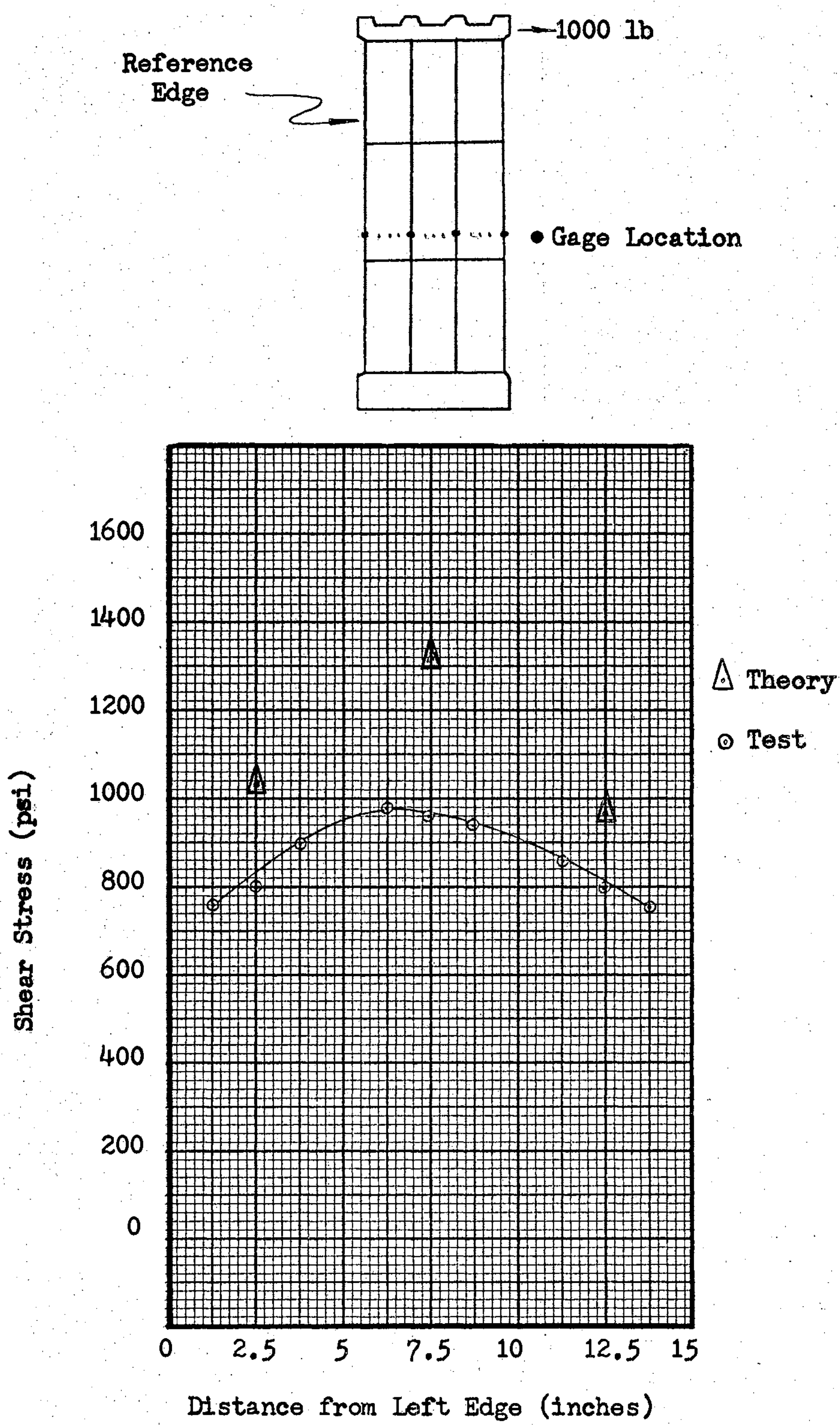


Figure 34. Shear Stresses for Shear Load Condition

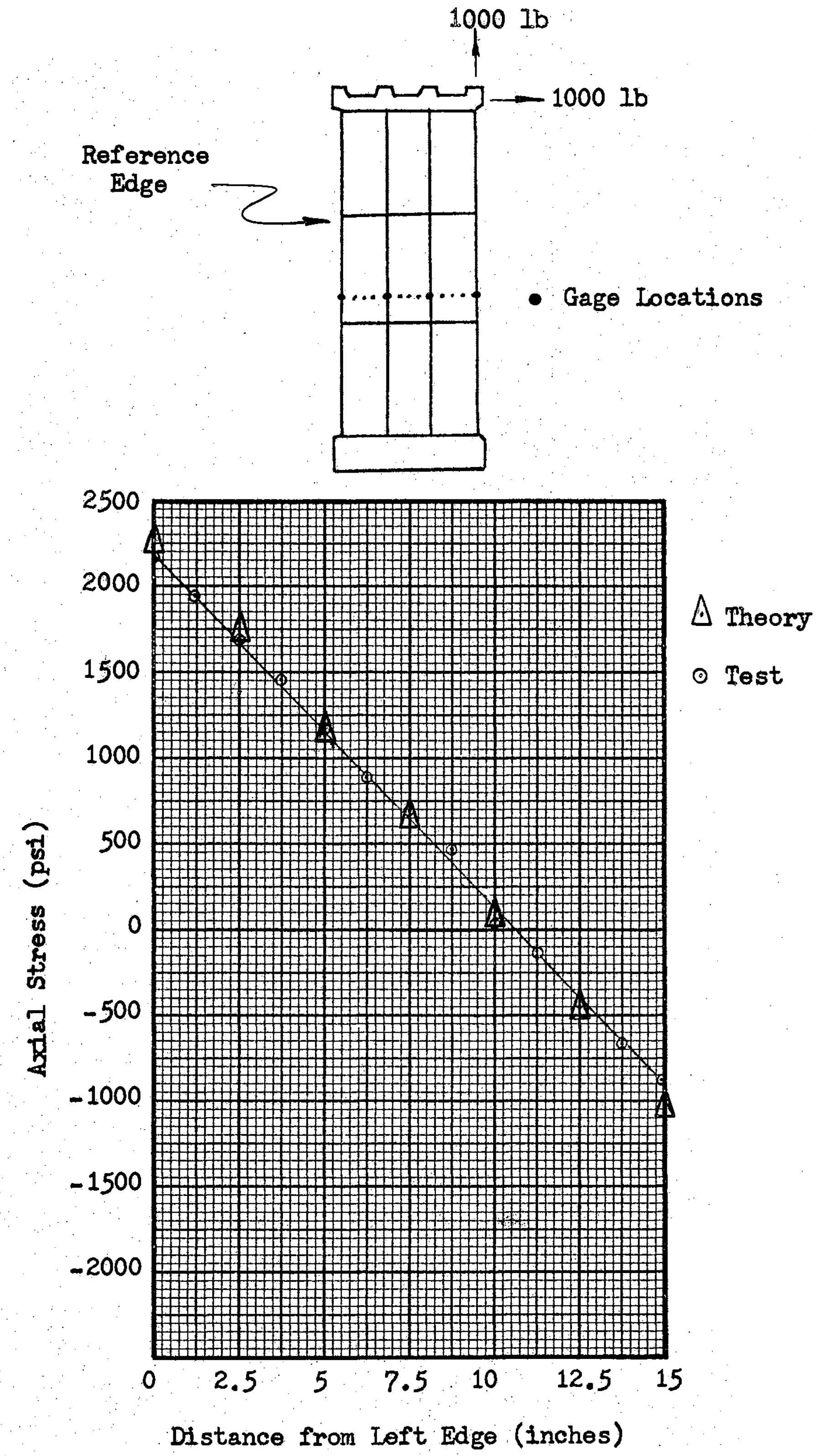


Figure 35. Axial Stresses at Aft Center Section for Transverse Load Condition

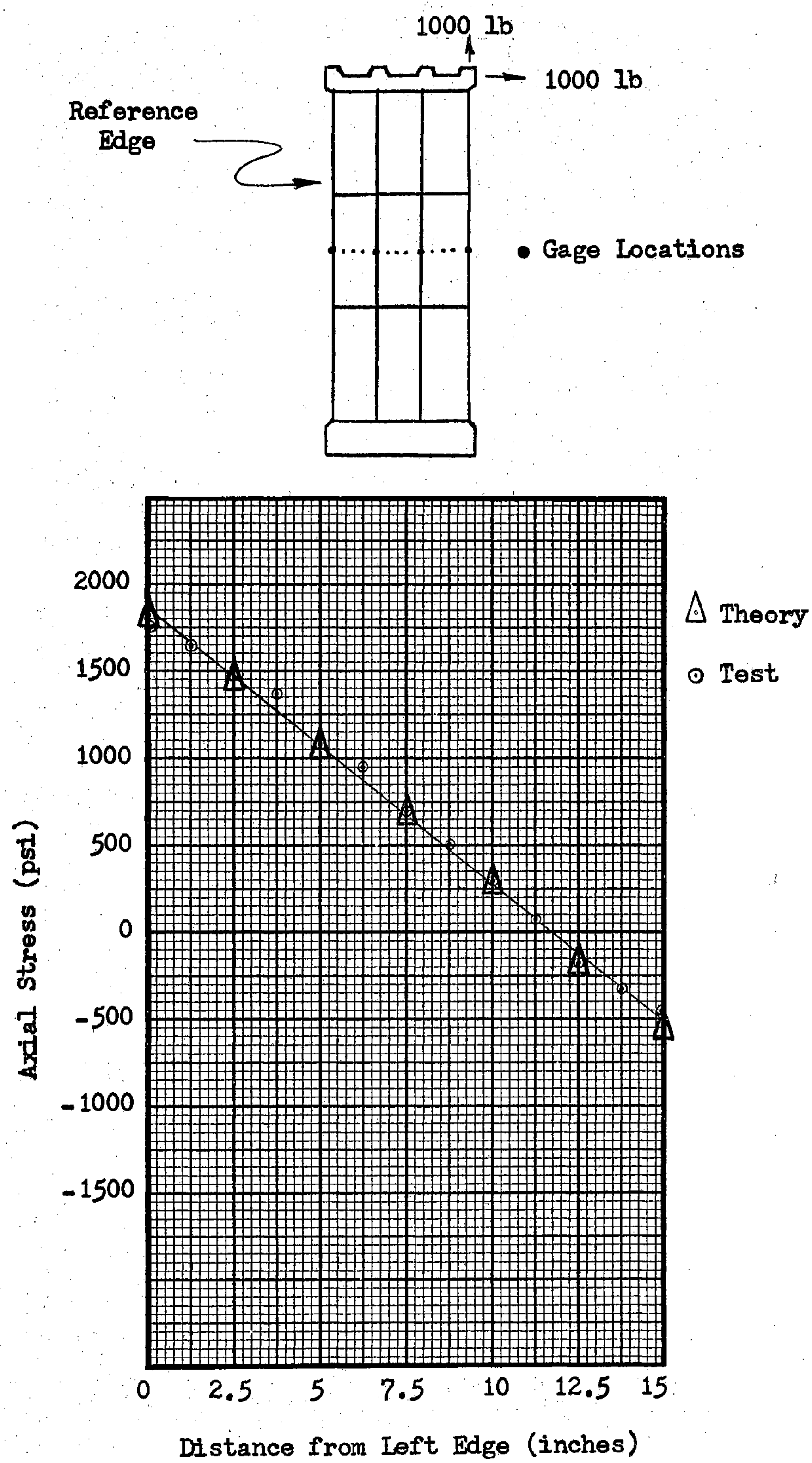


Figure 36. Axial Stresses at Center Section for Transverse Load Condition

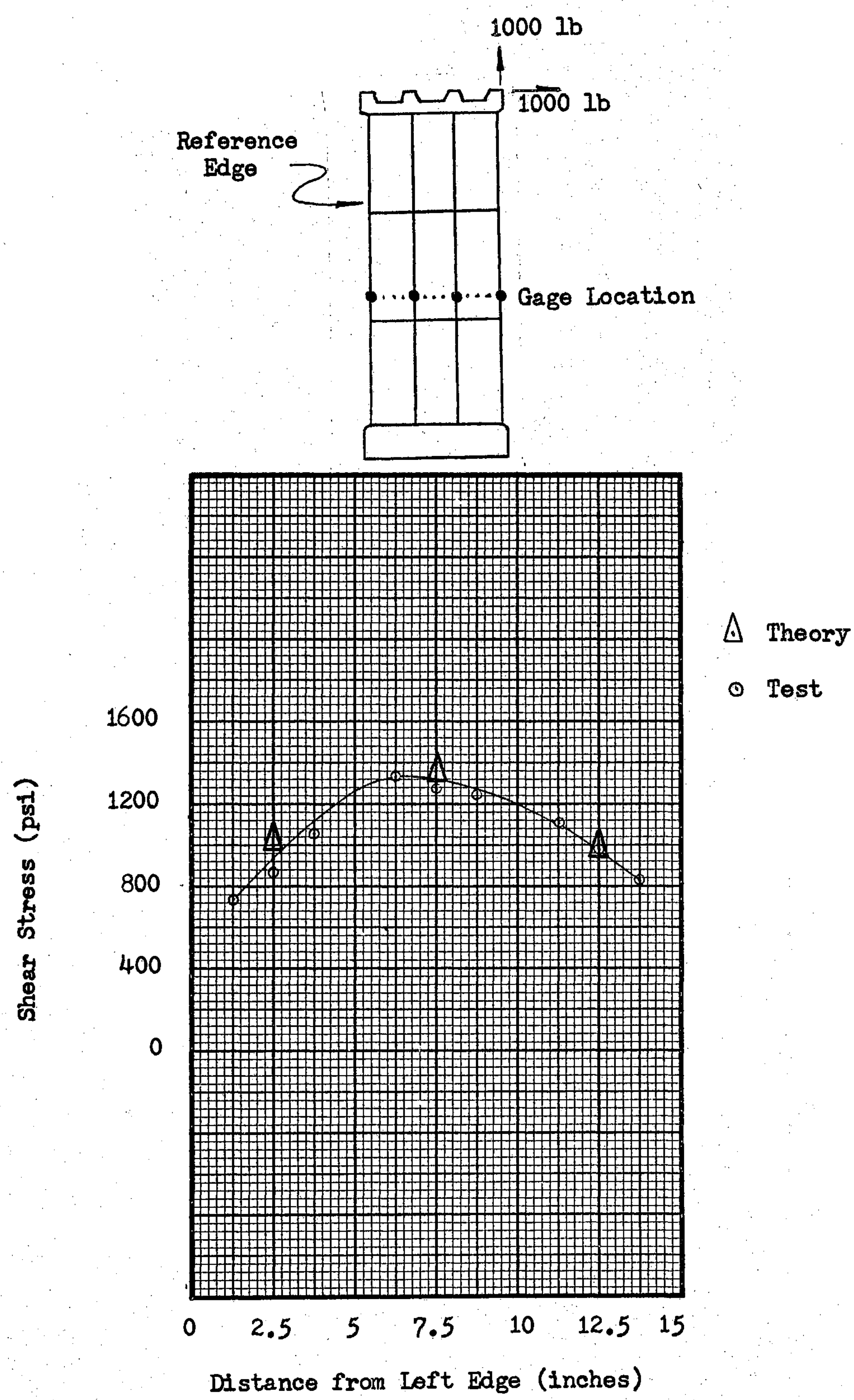
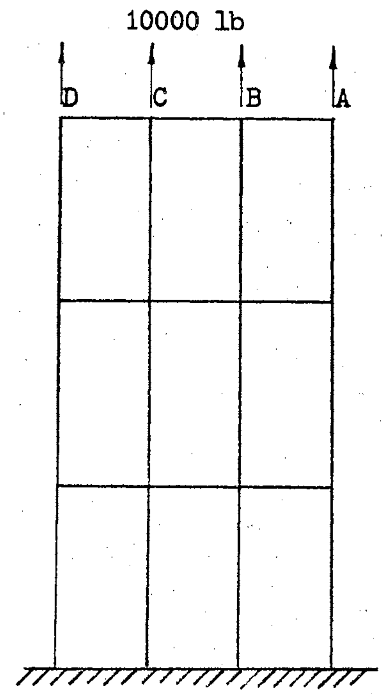


Figure 37. Shear Stresses for Transverse Load Condition

TABLE XIII

COMPARISON OF DEFLECTIONS FOR UNIFORM LOAD CONDITION

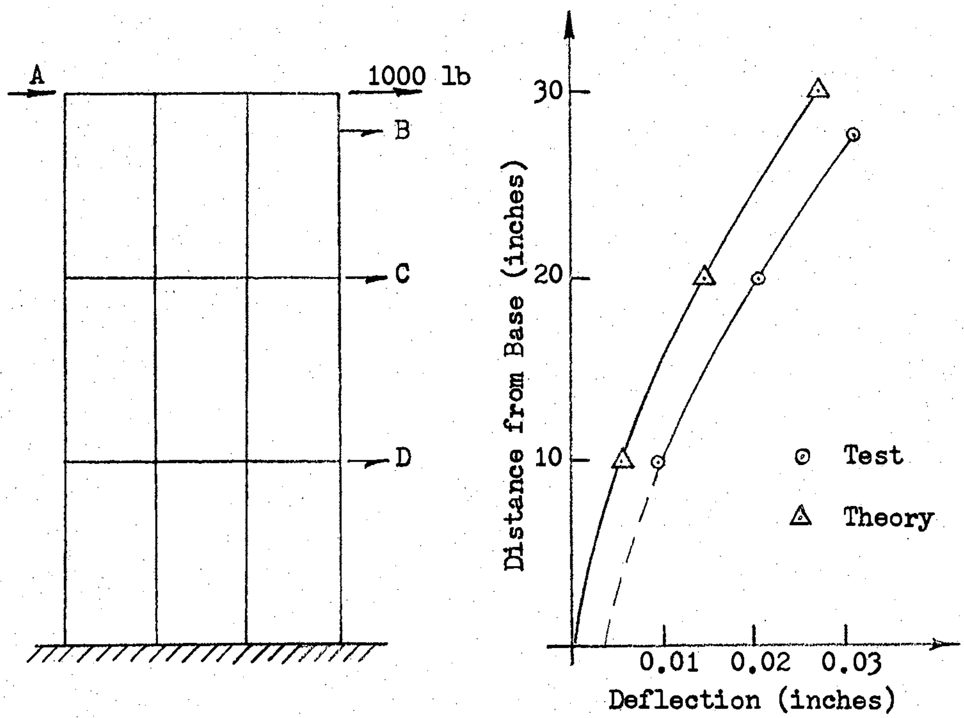


Deflection	Experimental Test 1	Experimental Test 2	Experimental Average*	Theoretical Nominal Areas	Theoretical Exact Areas
A	0.0222	0.0222	0.0188	0.0184	0.0174
B	0.0212	0.0225	0.0184	0.0185	0.0176
C	0.0205	0.0214	0.0173	0.0185	0.0176
D	0.0216	0.0249	0.0194	0.0184	0.0174

*Average deflections are adjusted for measured base deflection.

TABLE XIV

COMPARISON OF DEFLECTIONS FOR SHEAR LOAD CONDITION

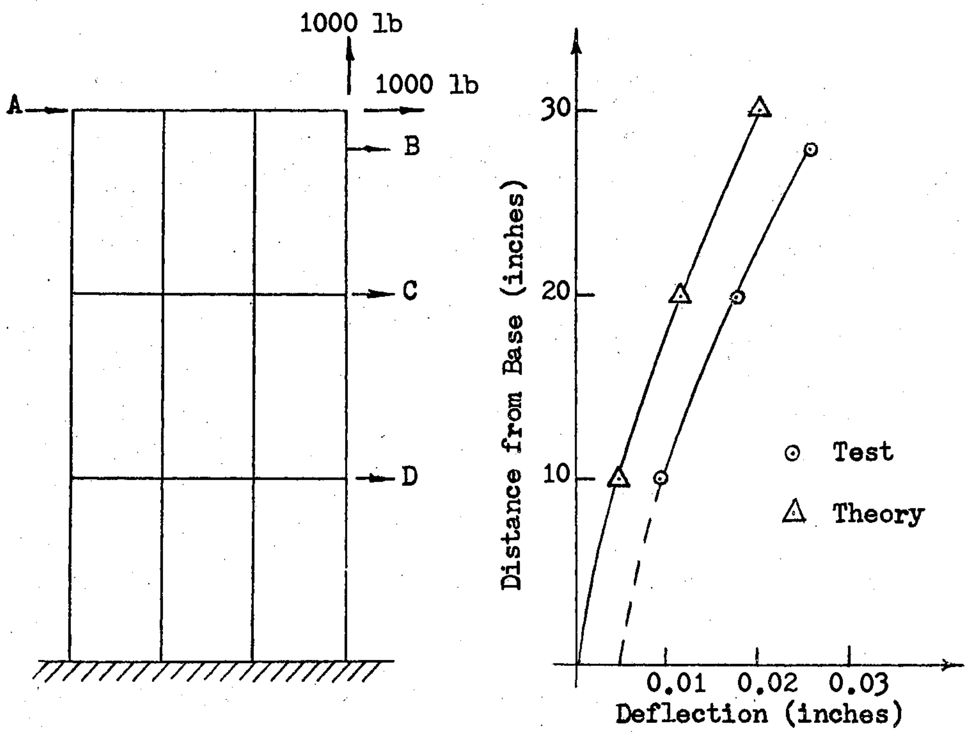


Deflection	Experimental Test 1	Experimental Test 2	Experimental Average*	Theoretical Nominal Areas	Theoretical Exact Areas
A	0.0355	0.0350	0.0305	0.0276	0.0254
B	0.0356	0.0360	0.0312	0.0292	0.0271
C	0.0232	0.0244	0.0205	0.0159	0.0145
D	0.0109	0.0115	0.0095	0.0062	0.0055

*Average deflections are adjusted for measured base deflections.

TABLE XV

COMPARISON OF DEFLECTIONS FOR TRANSVERSE LOAD CONDITION



Deflection	Experimental			Theoretical	
	Test 1	Test 2	Average*	Nominal Areas	Exact Areas
A	0.0319	0.0302	0.0256	0.0207	0.0188
B	0.0307	0.0313	0.0256	0.0221	0.0200
C	0.0212	0.0213	0.0176	0.0128	0.0115
D	0.0103	0.0107	0.0087	0.0052	0.0046

*Average deflections are adjusted for measured base deflections.

CHAPTER VIII

CONCLUSIONS AND RECOMMENDATIONS

The conclusions drawn from the comparisons of the analytical and experimental data are that a satisfactory capability has been developed for the analysis of integrally reinforced skin panels. The least satisfactory of these comparisons is shown in Figure 34 for the shear load condition. The shear stresses predicted from the analysis are in excess of the measured values. In addition, it is observed that the measured values are not in equilibrium with the applied load. Consequently, the panel was repositioned in the load frame for the shear load configuration; and strain rosettes were attached to both sides of the outside stringers at the center section of the panel as shown in Figure 38.

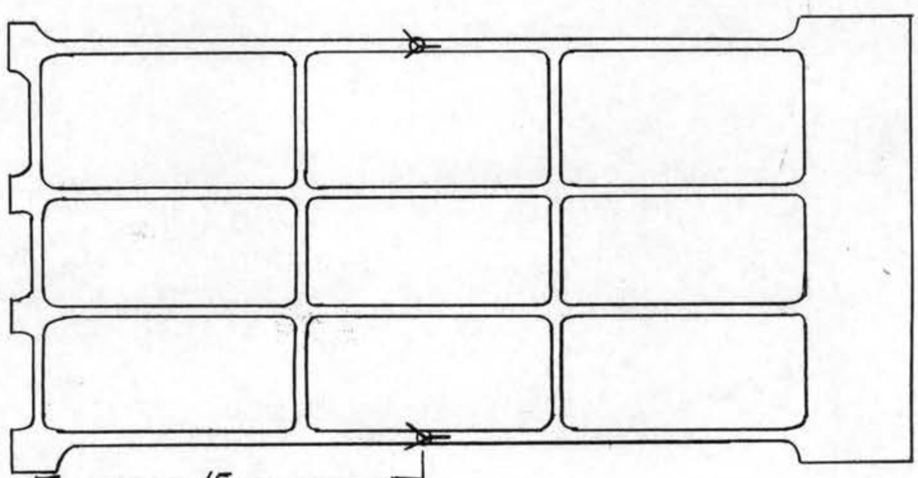


Figure 38. Location of Rosette Gages on Stringer Elements

These gages were used to indicate the portion of the shear force reacted by the stringer elements. The strains observed at these gage locations indicated that the stringers react the remaining portion of the external load not indicated by the shear stresses in the webs. By including the shear forces across the stringers and webs, the total shear forces are in equilibrium. The shear forces in the stringers indicate that the area of the stringers is approximately fifty per cent effective in resisting shear. The amount of shear force reacted by the stringers depends on the shape of the stringer and the method by which it is fastened to the skin structure. A suitable topic for future investigations would be to develop a routine procedure for accounting for the shear forces across the stringers.

Additional topics for future investigations consist of continuing the current investigation with a cutout section in the center panel. The capabilities developed in this program can be used for direct application to the problem of cutout sections. Extending the analysis capability for arbitrary cutout configurations would be valuable for practical aircraft structural design considerations.

A second topic of special significance would be the development of stiffness matrices for arbitrary configurations using the variational approach described in Chapter III. Direct calculation of stiffness matrices could be made using the SES^{-1} matrix and the digital computer matrix subroutines given in Appendix A. It is only necessary to establish the CM matrix of linear-edge displacements for the configurations of interest. The reduction of the stiffness matrices for practical configurations to algebraic expressions would also be a valuable contribution for extending analysis capabilities.

As a result of the broad class of problems encountered in this investigation, it is recommended that future studies make full use of the current computing capabilities and limit the experimental investigation whenever possible. The requirement of additional new stiffness matrices for arbitrary configurations and the development of criterion for evaluation of these matrices is of primary importance.

In addition, a study of idealization techniques and computational procedures would be a valuable contribution, providing significant reductions in computer running time could be accomplished.

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APPENDIX A

MATRIX ALGEBRA SUBROUTINES

The matrix algebra subroutines developed for this investigation are described below. The matrix operations are written in single subscript notation to conserve core space within the computer. The Fortran listings describing the operations are also included for reference.

Fortran listings for the various matrix algebra subroutines are

Subroutine Name	Description
RMAT (A)	Read the matrix from cards with Format 7E10.4.
RMATNZ (A)	Read only the nonzero elements of the matrix from cards with Format 6X, I4, 6X, I4, 6X, E14.8.
WRTMAT (A)	Write matrix A.
PUNCH (A)	Punch the nonzero elements on cards with the format of RMATNZ A.
MAM (A, B, C)	Add matrix A and B. The sum is matrix C.
MSM (A, B, C)	Subtract matrix B from matrix A. The difference is matrix C.
MSCA (Scalar A, C)	Multiply a scalar times the matrix A. The product is matrix C.
MXM (A, B, C)	Postmultiply matrix A by matrix B. The product is matrix C.
TRANSP (A, B)	Transpose matrix A and define $A^t = B$.
MTXM (A, B, C)	Postmultiply the transpose of matrix A by matrix B. The product is matrix C.
INVERX (A, B)	Invert the matrix A and define $A^{-1} = B$.

TABLE XVI

FORTRAN SUBROUTINE RMAT

```
$IBFTC RMAT
      SUBROUTINE RMAT(A)
      DIMENSION A(1)
      COMMON KIN,KOUT
    1 FORMAT(6X,I4,6X,I4)
    2 FORMAT(5E15.8)
      READ (KIN,1) KA1,KA2
      IF(KA1.GT.0) GO TO 6
      WRITE(KOUT,200)
  200 FORMAT(35H WE UNLOADED TAPES FROM MATRIX READ)
      CALL EXIT
    6 CONTINUE
      KA1=A(1)
      KA2=A(2)
      L = A(1)
      L1 = A(2)
      J = L*L1 + 2
      READ(KIN,2)(A(I),I=3,J)
      WRITE(KOUT,100)L,L1
  100 FORMAT(15H1THIS MATRIX IS,I4,3X,1HX,I4)
      L2 = 3
      DO 20 K = 1,L
      L3 = L2 + L1 - 1
      WRITE(KOUT,102)K
  102 FORMAT(10X,5H ROW ,I4)
      WRITE(KOUT,101)(A(I),I=L2,L3)
  101 FORMAT(25X,6E15.6)
      L2 = L3 + 1
   20 CONTINUE
      RETURN
      END
```

TABLE XVII

FORTRAN SUBROUTINE RMATNZ

```
$IBFTC RMATNZ  DECK
      SUBROUTINE RMATNZ (A)
      DIMENSION A(1)
      COMMON KIN, KOUT
  101 FORMAT  (6X,I4,6X,I4   )
  102 FORMAT  (6X,I4,6X,I4,6X, E15.8)
  103 FORMAT  (15H1THIS MATRIX IS, I4,3X,1HX,I4)
  104 FORMAT (10X,5H ROW ,I4)
  105 FORMAT  (25X, 6E15.4)
      READ (KIN, 101) IROW, JCOL
      A(1) = IROW
      A(2) = JCOL
      IJMAX = IROW * JCOL + 2
      DO 1 I = 3, IJMAX
    1 A(I) = 0.0
    2 READ(KIN, 102) M, N, DATA
      IF (N .LE. 0 ) GO TO 1000
      I  = (M-1) * JCOL  +  N  +2
      A(I) = DATA
      GO TO 2
C     PRINT  INPUT MATRIX
 1000 WRITE (KOUT, 103) IROW, JCOL
      L2 = 3
      DO 3  K =1,IROW
      L3 = L2 + JCOL -1
      WRITE (KOUT, 104) K
      WRITE (KOUT, 105)(A(I), I =L2, L3)
      L2 = L3 + 1
    3 CONTINUE
      RETURN
      END
```

TABLE XVIII

FORTRAN SUBROUTINE WRTMAT

```
$IBFTC WRTMAT  DECK
      SUBROUTINE WRTMAT(A)
      DIMENSION A(1)
  100 FORMAT(15H1THIS MATRIX IS,I4,3X,1HX,I4)
  101 FORMAT(20X,1P6E16.7)
  102 FORMAT(10X,5H ROW ,I4)
      COMMON KIN,KOUT
      L = A(1)
      L1 = A(2)
      L2 = 3
      J = L*L1 + 2
      WRITE(KOUT,100)L,L1
      DO 20 K = 1,L
      L3 = L2 + L1 - 1
      WRITE(KOUT,102)K
      WRITE(KOUT,101) (A(I),I=L2,L3)
      L2 = L3 + 1
   20 CONTINUE
      RETURN
      END
```

TABLE XIX

FORTRAN SUBROUTINE PUNCH

```
$IBFTC PUNCH
      SUBROUTINE PUNCH (A)                                          PUNCH001
      DIMENSION A(1)                                                PUNCH002
      COMMON KPUN                                                   PUNCH003
  100 FORMAT(6X,I4,6X,I4)                                           PUNCH004
  101 FORMAT(6X,I4,6X,I4,6X,E14.8)                                  PUNCH005
  102 FORMAT(5H2 END)                                               PUNCH006
      L=A(1)                                                        PUNCH007
      L1=A(2)                                                       PUNCH008
      WRITE(KPUN,100)L,L1                                           PUNCH009
      I=2                                                           PUNCH010
      DO 10 M=1,L                                                   PUNCH011
      DO 10 N=1,L1                                                  PUNCH012
      I=I+1                                                         PUNCH013
      IF(A(I).EQ.0.0) GO TO 10                                      PUNCH014
      WRITE(KPUN,101)M,N,A(I)                                       PUNCH015
   10 CONTINUE                                                      PUNCH016
      WRITE(KPUN,102)                                               PUNCH017
      RETURN                                                        PUNCH018
      END                                                           PUNCH019
```

TABLE XX

FORTRAN SUBROUTINE MAM

```
$IBFTC MAM
      SUBROUTINE MAM (A,B,C)
      DIMENSION A(1),B(1),C(1)
      COMMON KIN,KOUT
    5 FORMAT(1H0,31HTHE MATRIX ADD--INCORRECT SIZE ,I4,2HX ,I4,5HPLUS ,I
     14,2HX ,I4)
      ITEST=0
1     IROWA=A(1)
      ICOLA=A(2)
      IROWB=B(1)
      ICOLB=B(2)
      IF(IROWA.EQ.IROWB) GO TO 3
      IF(IROWA.GT.IROWB) GO TO 8
    7 C(1)=A(1)
      ITEST=1
      GO TO 3
    8 C(1)=B(1)
      ITEST=1
    3 IF(ICOLA.EQ.ICOLB) GO TO 4
      IF(ICOLA.GT.ICOLB) GO TO 10
    9 C(2)=A(2)
      IF(ITEST.NE.0) GO TO 2
   12 C(1)=A(1)
      GO TO 2
   10 C(2)=B(2)
      IF(ITEST.NE.0) GO TO 2
    2 WRITE(KOUT,5) IROW,ICOLA,IROWB,ICOLB
      GO TO 13
    4 IF(ITEST.EQ.0) GO TO 15
  14  C(2)=A(2)
      GO TO 13
15    L=IROWA*ICOLA+2
      DO 6 I=3,L
6     C(I)=A(I)+B(I)
      C(1)=A(1)
      C(2)=A(2)
   13 RETURN
      END
```

TABLE XXI

FORTRAN SUBROUTINE MSM

```
$IBFTC MSM
      SUBROUTINE MSM (A,B,C)
      DIMENSION A(1),B(1),C(1)
      COMMON KIN,KOUT
      ITEST=0
    5 FORMAT(1H0,31HTHE MATRIX MSM--INCORRECT SIZE ,I4,2HX ,I4,5HPLUS ,I
     14,2HX ,I4)
1     IROWA=A(1)
      ICOLA=A(2)
      IROWB=B(1)
      ICOLB=B(2)
      IF(IROWA.EQ.IROWB) GO TO 3
      IF(IROWA.GT.IROWB) GO TO 8
    7 C(1)=A(1)
      ITEST=1
      GO TO 3
    8 C(1)=B(1)
      ITEST=1
    3 IF(ICOLA.EQ.ICOLB) GO TO 4
      IF(ICOLA.GT.ICOLB) GO TO 10
    9 C(2)=A(2)
      IF(ITEST.NE.0) GO TO 2
   12 C(1)=A(1)
      GO TO 2
   10 C(2)=B(2)
      IF(ITEST.NE.0) GO TO 2
    2 WRITE(KOUT,5) IROW,ICOLA,IROWB,ICOLB
      GO TO 13
    4 IF(ITEST.EQ.0) GO TO 15
  14  C(2)=A(2)
      GO TO 13
15    L=IROWA*ICOLA+2
      DO 6 I=3,L
6     C(I)=A(I)-B(I)
      C(1)=A(1)
      C(2)=A(2)
   13 RETURN
      END
```

TABLE XXII

FORTRAN SUBROUTINE MSCA

```
$IBFTC MSCA
      SUBROUTINE MSCA (SCALAR,A,C)                                        MSCA001
      DIMENSION A(1),C(1)                                                 MSCA002
1     IROWA=A(1)                                                          MSCA003
      ICOLA=A(2)                                                          MSCA004
      L=IROWA*ICOLA+2                                                     MSCA005
      DO 2 I=3,L                                                          MSCA006
2     C(I)=SCALAR*A(I)                                                    MSCA007
      C(1)=A(1)                                                           MSCA008
      C(2)=A(2)                                                           MSCA009
      RETURN                                                              MSCA010
      END                                                                 MSCA011
```

TABLE XXIII

FORTRAN SUBROUTINE MXM

```
$IBFTC MXM     DECK
      SUBROUTINE MXM(A,B,C)
      DIMENSION A(1),B(1),C(1)
  100 FORMAT(1H0,49HTHE MATRICES ARE NOT CONFORMAL FOR MULTIPLICATION,2(
     15X,I4,2HX ,I4))
      COMMON KIN,KOUT
      IROWA=A(1)
      ICOLA=A(2)
      IROWB=B(1)
      ICOLB=B(2)
      IF(ICOLA-IROWB.EQ.0) GO TO 4
      WRITE(KOUT,100) IROWA,ICOLA,IROWB,ICOLB
      GO TO 6
4     N=IROWA*ICOLB+2
      DO 5 I=1,N
5     C(I)=0.0
      IX=3
      I=3
      J=3
      K=3
      KX=3
      DO 10 M=1,IROWA
      DO 9 N=1,ICOLB
      DO 8 NX=1,ICOLA
      C(J)=C(J)+A(I)*B(K)
      I=I+1
8     K=K+ICOLB
      I=IX
      J=J+1
      KX=KX+1
9     K=KX
      IX=IX+ICOLA
      I=IX
      K=3
10    KX=3
    6 C(1)=A(1)
      C(2)=B(2)
      RETURN
      END
```

TABLE XXIV

FORTRAN SUBROUTINE TRANSP

```
$IBFTC TRANSP
      SUBROUTINE TRANSP(A,B)                                            TRSP001
      DIMENSION A(1),B(1)                                               TRSP002
      B(1) = A(2)                                                       TRSP003
      B(2) = A(1)                                                       TRSP004
      L1 = B(1)                                                         TRSP005
      L2 = B(2)                                                         TRSP006
      JJ = 3                                                            TRSP007
      J1 = 3                                                            TRSP008
      J2 = L2 + 2                                                       TRSP009
      DO 1 I = 1,L1                                                     TRSP010
      J = JJ                                                            TRSP011
      DO 2 K = J1,J2                                                    TRSP012
      B(K) = A(J)                                                       TRSP013
    2 J = J + L1                                                        TRSP014
      JJ = JJ + 1                                                       TRSP015
      J1 = J2 + 1                                                       TRSP016
    1 J2 = J2 + L2                                                      TRSP017
      RETURN                                                            TRSP018
      END                                                               TRSP019
```

TABLE XXV

FORTRAN SUBROUTINE MTXM

```
$IBFTC MTXM     DECK
      SUBROUTINE MTXM (A,B,C)                                           MTXM001
      DIMENSION A(1),B(1),C(1)                                          MTXM002
      COMMON KIN,KOUT                                                   MTXM003
  100 FORMAT(1H0,49HTHE MATRICES ARE NOT CONFORMAL FOR MULTIPLICATION,2( MTXM004
     15X,I4,2HX ,I4))                                                   MTXM005
      ICOLA=A(1)                                                        MTXM006
      IROWA=A(2)                                                        MTXM007
      IROWB=B(1)                                                        MTXM008
      ICOLB=B(2)                                                        MTXM009
      IF(ICOLA-IROWB.EQ.0) GO TO 4                                      MTXM010
      WRITE(KOUT,100) IROWA,ICOLA,IROWB,ICOLB                           MTXM011
      GO TO 6                                                           MTXM012
    4 N=IROWA*ICOLB+2                                                   MTXM013
      DO 5 I=1,N                                                        MTXM014
    5 C(I)=0.0                                                          MTXM015
      IX=3                                                              MTXM016
      I=3                                                               MTXM017
      J=3                                                               MTXM018
      K=3                                                               MTXM019
      KX=3                                                              MTXM020
      DO 10 M=1,IROWA                                                   MTXM021
      DO 9 N=1,ICOLB                                                    MTXM022
      DO 8 NX=1,ICOLA                                                   MTXM023
      C(J)=C(J)+A(I)*B(K)                                               MTXM024
      I=I+IROWA                                                         MTXM025
    8 K=K+ICOLB                                                         MTXM026
      I=IX                                                              MTXM027
      J=J+1                                                             MTXM028
      KX=KX+1                                                           MTXM029
    9 K=KX                                                              MTXM030
      IX=IX+1                                                           MTXM031
      I=IX                                                              MTXM032
      K=3                                                               MTXM033
   10 KX=3                                                              MTXM034
    6 C(1)=A(2)                                                         MTXM035
      C(2)=B(2)                                                         MTXM036
      RETURN                                                            MTXM037
      END                                                               MTXM038
```

TABLE XXVI

FORTRAN SUBROUTINE INVERX

```
$IBFTC INVERX
      SUBROUTINE INVERX(A,B)                                            INVRT001
      DIMENSION A(1),B(1)                                               INVRT002
      DET = 1.0                                                         INVRT003
      N = A(1)                                                          INVRT004
      L10 = N**2 + 2                                                    INVRT005
      DO 1 I = 1,L10                                                    INVRT006
    1 B(I) = 0.                                                         INVRT007
      B(1) = N                                                          INVRT008
      B(2) = N                                                          INVRT009
      L9 = N + 1                                                        INVRT010
      DO 2 I = 3,L10,L9                                                 INVRT011
    2 B(I) = 1.0                                                        INVRT012
      JK = N - 1                                                        INVRT013
      J = 3                                                             INVRT014
      N1 = 3                                                            INVRT015
      N2 = N + 2                                                        INVRT016
      J0 = N - 1                                                        INVRT017
      J2 = N + 3                                                        INVRT018
      J4 = 3                                                            INVRT019
      DO 300 L1 = 1,JK                                                  INVRT020
      NR = (J + N - 2)/(N + 1)                                          INVRT021
      NR1 = NR                                                          INVRT022
      NRI = N - NR                                                      INVRT023
      JN1 = J + N                                                       INVRT024
      IF(NRI.LT.1) GO TO 900                                            INVRT025
      IF(NRI.GT.1) GO TO 804                                            INVRT026
  800 AMAX=ABS(A(J))                                                    INVRT027
      AMXA=ABS(A(JN1))                                                  INVRT028
      IF(AMAX.GE.AMXA) GO TO 900                                        INVRT029
  801 N5 = J - NR + 1                                                   INVRT030
      N6 = N5 + N - 1                                                   INVRT031
      IAD = N                                                           INVRT032
  802 DO 803 IT = N5,N6                                                 INVRT033
      IT6 = IT + IAD                                                    INVRT034
      ATEM = A(IT)                                                      INVRT035
      A(IT) = A(IT6)                                                    INVRT036
      A(IT6) = ATEM                                                     INVRT037
      ATEM = B(IT)                                                      INVRT038
      B(IT) = B(IT6)                                                    INVRT039
  803 B(IT6) = ATEM                                                     INVRT040
      GO TO 900                                                         INVRT041
  804 J11 = J + N + 1                                                   INVRT042
      J10 = J + N                                                       INVRT043
      AMAX=ABS(A(J))                                                    INVRT044
      DO 807 IT = 1,NRI                                                 INVRT045
      AMXA=ABS(A(J10))                                                  INVRT046
      IF(AMAX.GE.AMXA) GO TO 806                                        INVRT047
  805 AMAX = AMXA                                                       INVRT048
      NR1 = (J11 + N - 2)/(N + 1)                                       INVRT049
  806 J10 = J10 + N                                                     INVRT050
  807 J11 = J11 + N + 1                                                 INVRT051
      N5 = J - NR + 1                                                   INVRT052
      N6 = N5 + N - 1                                                   INVRT053
      ITEM = NR1 - NR                                                   INVRT054
      IAD = ITEM*N                                                      INVRT055
      IF(IAD.GT.0) GO TO 802                                            INVRT056
  900 CONTINUE                                                          INVRT057
      DENOM = A(J)                                                      INVRT058
      IF(DENOM.EQ.0.0) GO TO 51                                         INVRT059
   50 IF(IAD.GT.0) GO TO 701                                            INVRT060
  700 DET = DET*DENOM                                                   INVRT061
```

TABLE XXVI (Continued)

```
      GO TO 702
  701 DET = DET*(-DENOM)
  702 DO 100 J1 = N1,N2
      A(J1) = A(J1)/DENOM
  100 B(J1) = B(J1)/DENOM
      J3 = J4
      N3 = N2 + 1
      N4 = N2 + N
      DO 200 L = 1,JO
      AMULT = A(J2)
      DO 101 J1 = N3,N4
      A(J1) = A(J1) - AMULT*A(J3)
      B(J1) = B(J1) - AMULT*B(J3)
  101 J3 = J3 + 1
      J2 = J2 + N
      J3 = J4
      N3 = N3 + N
  200 N4 = N4 + N
      N1 = N1 + N
      N2 = N2 + N
      JO = JO - 1
      J = J + N + 1
      J2 = J + N
  300 J4 = J4 + N
      DENOM = A(J)
      IF(DENOM.EQ.0.0) GO TO 51
   60 A(J) = A(J)/DENOM
      DET = DET*DENOM
      LT = J - N + 1
      DO 400 J1 = LT,J
  400 B(J1) = B(J1)/DENOM
      JO = JK
      J2 = J - N
      J4 = J - N + 1
      N2 = J2 - N
      DO 600 L1 = 1,JK
      J3 = J4
      N3 = N2 + 1
      N4 = N2 + N
      DO 500 L = 1,JO
      AMULT = A(J2)
      DO 401 J1 = N3,N4
      A(J1) = A(J1) - AMULT*A(J3)
      B(J1) = B(J1) - AMULT*B(J3)
  401 J3 = J3 + 1
      J3 = J4
      J2 = J2 - N
      N3 = N3 - N
  500 N4 = N4 - N
      N2 = N2 - N
      JO = JO - 1
      J = J - N - 1
      J2 = J - N
  600 J4 = J4 - N
      IE = 1
  703 RETURN
   51 IE = 0
      GO TO 703
      END
```

APPENDIX B

STRESS ANALYSIS SYSTEM DIGITAL COMPUTER PROGRAM

The Stress Analysis System described in Chapter IV is based on the stiffness method of structural analysis described in Chapter III. The digital computer requires only a geometric description of the structure to perform the stress and deflection analysis. The program is controlled by the first two data cards, which are called the program control cards.

The first card contains the heading to be placed at the beginning of the program output data section. The second card defines the number of node points, the number of elements, the number of load cases, the number of stress nodes, and the print option. The correct placement of this information on the control cards is shown as follows:

Title Card
Card No. 1

1	2 ***Any available character can go in these spaces.***	72
1	Analysis of Rectangular Panels---------M. U. Ayres------	1/1/66

Control Card
Card No. 2

Number of Node Points	Number of Elements	Number of Load Cases	Number of Stress Nodes	Col 30=1 or 0 Print Option
1 6	7 12	13 18	19 24	30
20	40	5	5	0

If column 30 of card number 2 contains a nonzero number, the element stiffness and stress matrices and the transformation arrays will be printed. A flow diagram for the program is shown in Figure 39.

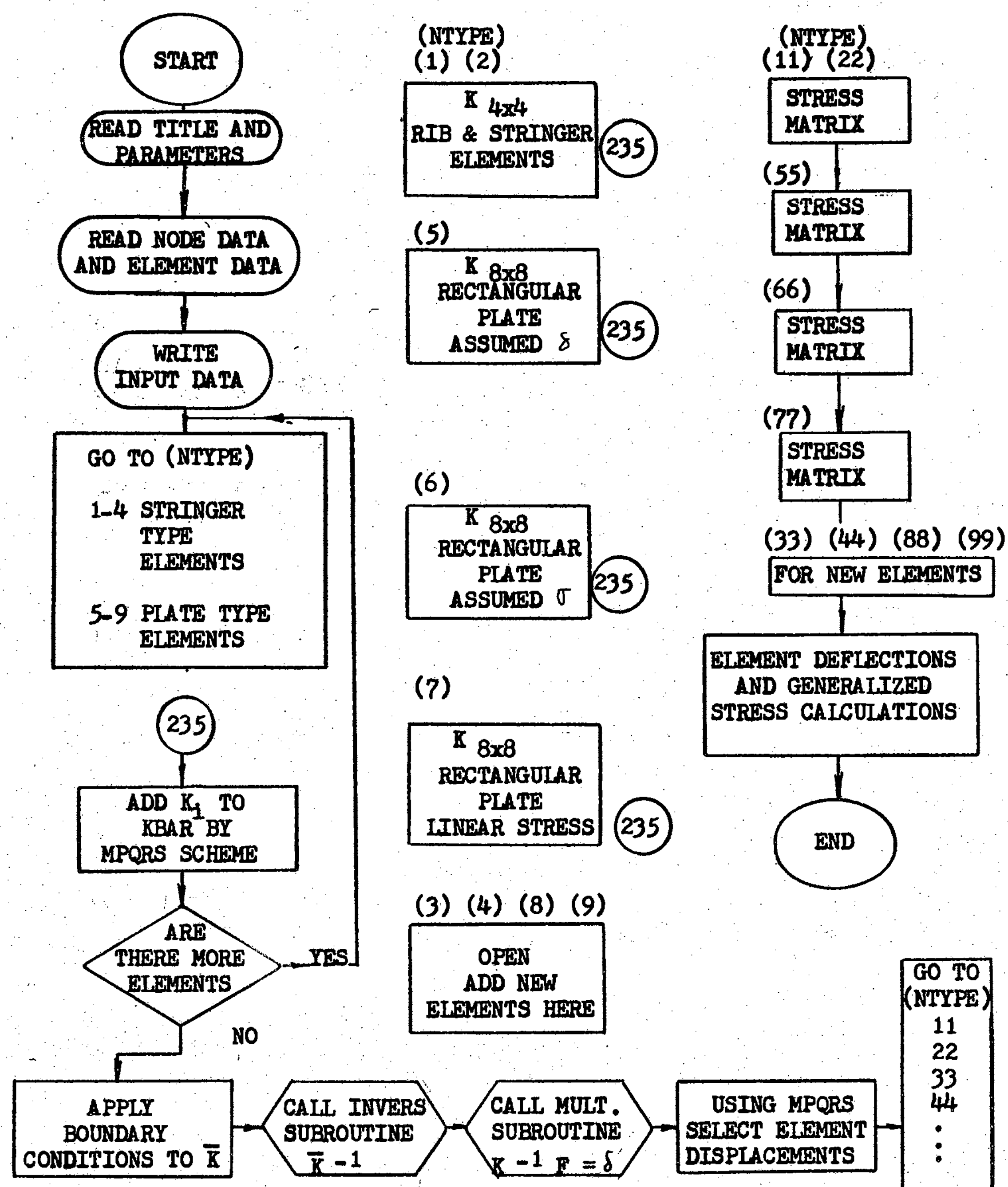

Figure 39. Flow Diagram for Stress Analysis System

The idealized structural elements used for an analysis with the Stress Analysis System are selected depending on the number in column 24 of the structural data card. Numbers 1 through 9 can be used and correspond to the idealized elements shown in Table XXVII.

TABLE XXVII

IDEALIZED ELEMENTS IN STRESS ANALYSIS SYSTEM

Element Number (NTYPE)	Description of Idealized Element
1	Stringer Element With Constant Stress
2	Stringer Element With Linear Strain Variation
3	Available for New Elements
4	Available for New Elements
5	Plate Element With Assumed Displacements
6	Plate Element With Assumed Stresses
7	Plate Element With Linear Stress Variation
8	Available for New Elements
9	Available for New Elements

A Fortran IV listing of the digital computer program for the Stress Analysis System is given in Table XXVIII.

TABLE XXVIII

FORTRAN PROGRAM FOR STRESS ANALYSIS SYSTEM

```
C     SAS PROGRAM BY M U AYRES
      DIMENSION AL(2),AL2(2),AL3(2),IPQRS(4),MPQRS(8),DSK(8,8),STR(3,8),
     1QORU(8,5),STRESS(3,5),R(12),BARK(1830),NBC(60),X(60),Y(60),
     2UBAR(60,5),FORCE(60,5),QBAR(60,5),XN(60,5),YN(60,5)
      EQUIVALENCE(IPQRS(4),IS),(IPQRS(3),IR),(IPQRS(2),IQ),(IPQRS(1),IP)
  101 FORMAT ( 2X, 1P8E16.3)
  102 FOR4AT ( 2X,  1P4E16.3)
  103 FORMAT (1H0, 7HK BAR I   , 1X)
  104 FORMAT (2X,I5)
  105 FORMAT (  6H0 I = , I5, 13H  IPQRS(I) = , I5)
  106 FORMAT (  6H0 K = , I5, 13H  MPQRS(K) = , I5)
  107 FORMAT (  6H0LA = , I5, 19H  KI = MPQRS(LA) = , I5)
  109 FORMAT ( 6H0KJ = , I5)
  110 FORMAT ( 6H0BARK(, I5, 9H ) = DSK(, I5, 2H , , I5, 2H ) )
  111 FORMAT ( 6H0 I = , I5)
  112 FORMAT ( 6H0IJ = , I5, 12H  NBC(IJ) = , I5)
  113 FORMAT ( 7H0 LA = , I5, 7H   I =   , I5, 17H BARK(I)    = 1.0    )
  114 FORMAT ( 41H0 NUMBER OF ROWS AND COLS TO BE ZEROED = , I5)
  115 FORMAT ( 6H0 I =   , I5, 15H BARK(I) = 0.0      )
  116 FCRMAT (2X, I5,5X,3E14.8,5X, I5,5X, 4E14.8, / 2X, 8I10,
     1 / 2X, 4I10)
  200 FORMAT ( 25H0 ELEMENT STRESS MATRIX       )
  201 FORMAT(8H0NODE    ,2(8X,7HTYPE OF),49X,8HSTRESSES)
 202  FOR4AT(1X,6HNUMBER,9X,7HELEMENT,8X,6HSTRESS,10X,6HCASE 1,11X,6HCAS
     1E 2 ,11X,6HCASE 3,11X,6HCASE 4,11X,6HCASE 5)
  203 FORMAT (35H1 GENERALIZED STRESS CALCULATIONS     )
  204 FORMAT  (33H1 DEFLECTIONS FOR ELEMENT NUMBER , I5 )
  205 FORMAT(//43H    STRESSES AT THE CENTROID OF THE ELEMENT//)
  206 FORMAT (30H0 STRESSES FOR ELEMENT NUMBER  ,  I3,  6H TYPE ,I3)
  219 FORMAT(1H0,I4,9X,I5,14X,2HXX,9X,5E17.8)
 221  FORMAT(33X,2HXY, 9X,5(2X,E15.8))
 222  FORMAT(33X,2HYY, 9X,5(2X,E15.8))
  251 FORMAT (I5,1X,5F12.4)
  252 FORMAT(  44H1                     STRESS NODE COORDINATES   , /
     1 52H ELEMENT    NODE 1  NODE 2  NODE 3  NODE 4  NODE 5         )
  253 FORMAT( 1X, I3, 2H X, 5F12.4, )
  254 FORMAT (I5,1X,5F12.4)
  255 FORMAT(1X,I3,2H Y,5F12.4)
  256 FORMAT(1X,30HNO STRESS     MATRIX FOR TYPE   ,I3,2X,7HELEMENT)
  257 FORMAT(1X,30HNO STIFFNESS MATRIX FOR TYPE   ,I3,2X,7HELEMENT)
  258 FORMAT ( 8H ELEMENT, 25X, 16HCOORDINATES  FOR,   /
     1  7H NUMBER,  4X,54HNODE 1      NODE 2      NODE 3      NODE 4
     2  NODE 5 )
  259 FORMAT(1H0,27HNORMALIZED COORDINATES X = ,F12.4,10X,4HY = ,F12.4)
 603  FORMAT(10I6)
 612  FOR4AT(6E13.0)
 687  FORMAT(1X,4HDET=,E14.2,10X,2HL=,I3)
  800 FORMAT(1H1)
 801  FORMAT(1H0,10HNODE POINT,5 X,11HCOORDINATES,47X,
     125HDEFLECTION OF NODE POINTS)
  802 FORMAT(1X,6HNUMBER,40X,6HCASE 1,11X ,
     16HCASE 2,11X,6HCASE 3,11X,6HCASE 4,11X,6HCASE 5  )
 804  FORMAT(1H0,2X,I2,13X,1HX,24X,5E17.8)
 805  FORMAT(18X,1HY,24X,5E17.8)
 809  FORMAT(    11H1NODE POINT,3X,11HCOORDINATES,63X,6HFORCES )
 992  FORMAT(20I4)
 993  FORMAT(6X,6F12.0,I2)
  994 FORMAT(I5,4I4,I3,1X,E10.6,2F6.0)
 995  FCRMAT(1H1,12A6)
 8629 FORMAT(19HAMATRIX IS SINGULAR)
 8798 FORMAT (7H1 K BAR   /1X)
 8799 FORMAT(16H1 K BAR  INVERSE/1X)
```

TABLE XXVIII (Continued)

```
 9603 FORMAT(  7H NODES=,I5,5X,9HELEMENTS=,I5,5X,6HCASES=,I2,5X
     1,13HSTRESS NODES= ,I2/
     2        89H NODE     COORDINATE         LOAD 1        LOAD 2          LOAD 3
     3     LOAD 4       LOAD 5    SUPPORT/1X)
 9993 FORMAT(1X,I3,2H X,F12.3,1X,5F12.3,6X,I1/1X,I3,2H Y,F12.3,1X,5F12.
     13,6X,I1)
 9994 FORMAT(1X,I5,4I4,I3,4X,E11.4,F11.4,F13.4 )
 9995 FORMAT(114H1 ELEM    P    Q    R    S TYPE          E                PR      TH
     1ICKNESS-AREA                                                          )
31009 FORMAT(1X,3HROW,I4,/1X,(1P10E13.4))
99999 FORMAT(1H1,23HEXECUTION COMPLETED FOR)
  839 CONTINUE
      REWIND 3
      REWIND  4
C READ IN TITLE
      READ(5,995) (R(J),J=1,12)
      WRITE(6,995) (R(J),J=1,12)
C READ IN PARAMETERS
      REA)(5,603) NNODES,NELEM,NC,NSN,IWRITE
      WRITE(6,9603) NNODES,NELEM,NC,NSN
      N2=2 *NNODES
      NUM=(N2*(N2+1))/2
C READ IN NODE LOCATIONS, FORCE, AND BOUNDARY CONDITIONS
      DO 7777 I=1,NNODES
      I2=2*I
      READ(5,993)  X(I),  (FORCE(I2-1,J), J=1,5 ),BARK(I2-1),
     1  Y(I), (FORCE (I2,J), J=1,5),   BARK(I2)
 7777 WRITE (6,9993)  I,X(I), (FORCE(I2-1,J),J=1,5), BARK(I2-1),
     1  I, Y(I), (FORCE (I2 , J), J=1,5),BARK(I2)
C  THE NCROSS ROWS AND COLS. TO BE STRUCK FROM K-BAR ,AS DICTATED BY
C BOUNDARY CONDITIONS, ARE STORED IN ARRAY NBC(I).
C      BARK IS USED TO READ THE INDEX OF FIXED BOUNDARY NODES
      IJ=0
      DO 7778 I=1,N2
      IF(BARK(I))7779,7778,7779
 7779 IJ=IJ+1
      NBC(IJ)=I
      IF(IWRITE.EQ.0) GO TO 7778
      WRITE (6,111) I
      WRITE (6,112) IJ, I
 7778 CONTINUE
      NCROSS=IJ
      DO 320 I=1,NUM
      BARK (I)=0.0
  320 CONTINUE
C  READ NODE NUMBER  TYPE ELEMENT   MODULUS  PR    AREA
      WRITE(6,9995)
      DO 236 NN=1,NELEM
      READ(5,994) IE,IP,IQ,IR,IS,NTYPE,E,PR,A
      IF(IWRITE.EQ.0) GO TO 513
      WRITE (6,9995)
  513 CONTINUE
      WRITE(6,9994) IE,IP,IQ,IR,IS,NTYPE,E,PR,A
      GO TO (1,2,3,4,5,6,7,8,9),NTYPE
    1 CONTINUE
C*****************STRINGER AND RIB CALCULATIONS***********************
      JLAM=4
      DO 10004 I=1,4
      DO 10004 J=1,4
10004 DSK(I,J)=0.0
CALCULATE THE PQ DIRECTION COSINES.
      XQP=X(IQ)-X(IP)
```

TABLE XXVIII (Continued)

```
      YQP=Y(IQ)-Y(IP)                                                   SAS125
      D1=SQRT (XQP**2+YQP**2)                                           SAS126
      D2 = D1                                                           SAS127
      AL(1)=XQP/D1                                                      SAS128
      AL(2)=YQP/D1                                                      SAS129
      AE=A*E                                                            SAS130
      DO239I=1,2                                                        SAS131
      DO239J=1,2                                                        SAS132
      DSK (I,J)     = AL(I)*AL(J)*AE/D1                                 SAS133
      DSK(I+2,J)    = -DSK(I,J)                                         SAS134
      DSK (I,J+2)   = -DSK(I,J)                                         SAS135
      DSK(I+2,J+2) = DSK(I,J)                                           SAS136
  239 CONTINUE                                                          SAS137
      IF(IWRITE.EQ.0) GO TO 500                                         SAS138
      WRITE (6,205) NTYPE                                               SAS139
      WRITE (6,103)                                                     SAS140
      WRITE (6,102)  ((DSK(I,J),I=1,4), J=1,4)                          SAS141
  500 CONTINUE                                                          SAS142
      GO TO 235                                                         SAS143
    2 CONTINUE                                                          SAS144
C     ****************STRINGER WITH LINEAR STRESS FUNCTION *************SAS145
      JLAM=4                                                            SAS146
      DO 10005 I=1,4                                                    SAS147
      DO 10005 J=1,4                                                    SAS148
10005 DSK(I,J)=0.0                                                      SAS149
CALCULATE THE PQ DIRECTION COSINES.                                     SAS150
      XQP=X(IQ)-X(IP)                                                   SAS151
      YQP=Y(IQ)-Y(IP)                                                   SAS152
      D1=SQRT (XQP**2+YQP**2)                                           SAS153
      D2 = D1                                                           SAS154
      AL(1)=XQP/D1                                                      SAS155
      AL(2)=YQP/D1                                                      SAS156
      AE=A*E                                                            SAS157
      DO 240 I=1,2                                                      SAS158
      DO 240 J=1,2                                                      SAS159
      DSK(I,J)=AL(I)*AL(J)*(AE/D1)*4.0/ 3.0                             SAS160
      DSK(I+2,J)=-DSK(I,J)                                              SAS161
      DSK(I,J+2)=-DSK(I,J)                                              SAS162
      DSK(I+2,J+2) = DSK(I,J)                                           SAS163
  240 CONTINUE                                                          SAS164
      IF(IWRITE.EQ.0) GO TO 511                                         SAS165
      WRITE (6,205) NTYPE                                               SAS166
      WRITE (6,103)                                                     SAS167
      WRITE (6,102)  ((DSK(I,J),I=1,4), J=1,4)                          SAS168
  511 CONTINUE                                                          SAS169
      GO TO 235                                                         SAS170
    3 CONTINUE                                                          SAS171
    4 CONTINUE                                                          SAS172
      WRITE(6,257) NTYPE                                                SAS173
      GO TO 839                                                         SAS174
    5 CONTINUE                                                          SAS175
C*****************RECTANGULAR*PLATE*CALCULATIONS**********************SAS176
C*******************ASSUMED DISPLACEMENT FUNCTION*********************SAS177
      DO 10003  I = 1,8                                                 SAS178
      DO 10003 J=1,8                                                    SAS179
10003 DSK (I,J) = 0.0                                                   SAS180
      JLAM=8                                                            SAS181
      XQP=X(IQ)-X(IP)                                                   SAS182
      YQP=Y(IQ)-Y(IP)                                                   SAS183
      D1=SQRT (XQP**2+YQP**2)                                           SAS184
      AE=A*E                                                            SAS185
      X2=X(IR)-X(IQ)                                                    SAS186
```

TABLE XXVIII (Continued)

```
      Y2=Y(IR)-Y(IQ)
      D2=SQRT (X2**2+Y2**2)
      AL(1)=XQP/D1
      AL(2)=YQP/D1
      AL2(1)=X2/D2
      AL2(2)=Y2/D2
      BETA=D1/D2
      ET1=AE/(1.-PR**2)
      ET2=AE/(2.+2.*PR)
C     CALCULATE THE KD+KS MATRIX
      PR2=PR**2
      DSK  (1,1)= ET1*BETA/3.+ET2/(3.*BETA)
      DSK  (2,1)=(ET1*PR+ET2)/4.
      DSK  (3,1)=ET1*BETA/6.-ET2/(3.*BETA)
      DSK  (4,1)=(-ET1*PR+ET2)/4.
      DSK  (5,1)=-ET1*BETA/6.-ET2/(6.*BETA)
      DSK  (7,1)=-ET1*BETA/3.+ET2/(6.*BETA)
      DSK  (2,2)=ET1/(3.*BETA)+ET2*BETA/3.
      DSK  (4,2)=-ET1/(3.*BETA)+ET2*BETA/6.
      DSK  (6,2)=-ET1/(6.*BETA)-ET2*BETA/6.
      DSK  (8,2)=ET1/(6.*BETA)-ET2*BETA/3.
      DSK  (3,3)=ET1*BETA/3.+ET2/(3.*BETA)
      DSK  (5,3)=-ET1*BETA/3.+ET2/(6.*BETA)
      DSK  (6,1)=-DSK  (2,1)
      DSK  (8,1)=-DSK  (4,1)
      DSK  (3,2)=-DSK  (4,1)
      DSK  (5,2)=-DSK  (2,1)
      DSK  (7,2)= DSK  (4,1)
      DSK  (4,3)=-DSK  (2,1)
      DSK  (6,3)= DSK  (4,1)
      DSK  (7,3)= DSK  (5,1)
      DSK  (8,3)= DSK  (2,1)
      DSK  (4,4)= DSK  (2,2)
      DSK  (5,4)= DSK  (3,2)
      DSK  (6,4)= DSK  (8,2)
      DSK  (7,4)= DSK  (2,1)
      DSK  (8,4)= DSK  (6,2)
      DSK  (5,5)= DSK  (1,1)
      DO 8620 I=2,4
      DSK  (I+4,5)=DSK  (I,1)
 8620 DSK  (I+4,6)=DSK  (I,2)
      DSK  (7,7)= DSK  (1,1)
      DSK  (8,7)=-DSK  (2,1)
      DSK  (8,8)= DSK  (2,2)
      DO 302 J=1,8
      DO 302 I=1,8
  302 DSK(J,I) = DSK(I,J)
      IF(IWRITE.EQ.0) GO TO 502
      WRITE (6,205) NTYPE
      WRITE (6,103)
      WRITE (6,101)  ((DSK(I,J),I=1,8), J=1,8)
  502 CONTINUE
      GO TO 235
    6 CONTINUE
C******************RECTANGULAR*PLATE*CALCULATIONS***********************
C********ASSUMED STRESS FUNCTION WITH FIVE  COEFFICIENTS****************
      DO 10002  I = 1,8
      DO 10002  J = 1,8
10002 DSK (I,J) = 0.0
      JLAM=8
      XQP=X(IQ)-X(IP)
      YQP=Y(IQ)-Y(IP)
```

TABLE XXVIII (Continued)

```
      D1=SQRT (XQP**2+YQP**2)
      AE=A*E
      X2=X(IR)-X(IQ)
      Y2=Y(IR)-Y(IQ)
      D2=SQRT (X2**2+Y2**2)
      AL(1)=XQP/D1
      AL(2)=YQP/D1
      AL2(1)=X2/D2
      AL2(2)=Y2/D2
      BETA=D1/D2
      ET1=AE/(1.-PR**2)
      ET2=AE/(2.+2.*PR)
      PR2=PR**2
C     CALCULATE THE KD+KS MATRIX
      DSK  (1,1)= (2.*(4.-PR2)*BETA/3.+(1.-PR)/BETA)*ET1/8.
      DSK  (2,1)= (1.+PR)*ET1/8.
      DSK  (3,1)= (2.*(2.+PR2)*BETA/3.-(1.-PR)/BETA)*ET1/8.
      DSK  (4,1)= (1.-3.*PR)*ET1/8.
      DSK  (5,1)= (-2.*(2.+PR2)*BETA/3.-(1.-PR)/BETA)*ET1/8.
      DSK  (7,1)= (-2.*(4.-PR2)*BETA/3.+(1.-PR)/BETA)*ET1/8.
      DSK  (2,2)= (2.*(4.-PR2)/(3.*BETA)+(1.-PR)*BETA)*ET1/8.
      DSK  (4,2)= (-2.*(4.-PR2)/(3.*BETA)+(1.-PR)*BETA)*ET1/8.
      DSK  (6,2)= (-2.*(2.+PR2)/(3.*BETA)-(1.-PR)*BETA)*ET1/8.
      DSK  (8,2)= (2.*(2.+PR2)/(3.*BETA)-(1.-PR)*BETA)*ET1/8.
      DSK  (3,3)= (2.*(4.-PR2)*BETA/3. +(1.-PR)/BETA)*ET1/8.
      DSK  (5,3)= (-2.*(4.-PR2)*BETA/3.+(1.-PR)/BETA)*ET1/8.
      DSK  (6,1)=-DSK  (2,1)
      DSK  (8,1)=-DSK  (4,1)
      DSK  (3,2)=-DSK  (4,1)
      DSK  (5,2)=-DSK  (2,1)
      DSK  (7,2)= DSK  (4,1)
      DSK  (4,3)=-DSK  (2,1)
      DSK  (6,3)= DSK  (4,1)
      DSK  (7,3)= DSK  (5,1)
      DSK  (8,3)= DSK  (2,1)
      DSK  (4,4)= DSK  (2,2)
      DSK  (5,4)= DSK  (3,2)
      DSK  (6,4)= DSK  (8,2)
      DSK  (7,4)= DSK  (2,1)
      DSK  (8,4)= DSK  (6,2)
      DSK  (5,5)= DSK  (1,1)
      DO 8621 I=2,4
      DSK  (I+4,5)=DSK  (I,1)
 8621 DSK  (I+4,6)=DSK  (I,2)
      DSK  (7,7)= DSK  (1,1)
      DSK  (8,7)=-DSK  (2,1)
      DSK  (8,8)= DSK  (2,2)
      DO 301 J=1,8
      DO 301 I=1,8
  301 DSK(J,I) = DSK(I,J)
      IF(IWRITE.EQ.0) GO TO 501
      WRITE (6,205) NTYPE
      WRITE (6,103)
      WRITE (6,101)  ((DSK(I,J),I=1,8), J=1,8)
  501 CONTINUE
      GO TO 235
    7 CONTINUE
C******************RECTANGULAR*PLATE*CALCULATIONS**********************
C********ASSUMED STRESS FUNCTION WITH SEVEN COEFFICIENTS***************
      DO 10006  I = 1,8
      DO 10006  J = 1,8
10006 DSK (I,J) = 0.0
```

TABLE XXVIII (Continued)

```
      JLAM=8
      XQP=X(IQ)-X(IP)
      YQP=Y(IQ)-Y(IP)
      BY=SQRT (XQP**2+YQP**2)
      D1 = BY
      AE=A*E
      AL(1)=XQP/D1
      AL(2)=YQP/D1
      X2=X(IR)-X(IQ)
      Y2=Y(IR)-Y(IQ)
      AX=SQRT (X2**2+Y2**2)
      D2 = AX
      ALP = (3.0*AX*AX) + (BY*BY)
      BET = (AX*AX) + (3.0 * BY*BY)
      DSK(1,1)=+(35.*BY*BY*ALP*BET)+((BY**4)*BET)-(6.*AX*AX*BY*BY*BET)+(
     19.*AX*AX*ALP*BET)+(9.*(AX**4)*BET)
      DSK(2,1)=18.*AX*BY*ALP*BET
      DSK(3,1)=+(19.*BY*BY*ALP*BET)-((BY**4)*BET)+(6.*AX*AX*BY*BY*BET)-(
     19.*AX*AX*ALP*BET)-(9.*(AX**4)*BET)
      DSK(5,1)=-(19.*BY*BY*ALP*BET)+((BY**4)*BET)-(6.*AX*AX*BY*BY*BET)-(
     19.*AX*AX*ALP*BET)+(9.*(AX**4)*BET)
      DSK(7,1)=-(35.*BY*BY*ALP*BET)-((BY**4)*BET)+(6.*AX*AX*BY*BY*BET)+(
     19.*AX*AX*ALP*BET)-(9.*(AX**4)*BET)
      DSK(2,2)=+(35.*AX*AX*ALP*BET)+((AX**4)*ALP)-(6.*AX*AX*BY*BY*ALP)+(
     19.*BY*BY*ALP*BET)+(9.*(BY**4)*ALP)
      DSK(4,2)=-(35.*AX*AX*ALP*BET)-((AX**4)*ALP)+(6.*AX*AX*BY*BY*ALP)+(
     19.*BY*BY*ALP*BET)-(9.*(BY**4)*ALP)
      DSK(6,2)=-(19.*AX*AX*ALP*BET)+((AX**4)*ALP)-(6.*AX*AX*BY*BY*ALP)-(
     19.*BY*BY*ALP*BET)+(9.*(BY**4)*ALP)
      DSK(8,2)=+(19.*AX*AX*ALP*BET)-((AX**4)*ALP)+(6.*AX*AX*BY*BY*ALP)-(
     19.*BY*BY*ALP*BET)-(9.*(BY**4)*ALP)
      DSK(6,1) =-DSK(2,1)
      DSK(5,2) = DSK(6,1)
      DSK(3,3) = DSK(1,1)
      DSK(4,3) = DSK(6,1)
      DSK(5,3) = DSK(7,1)
      DSK(7,3) = DSK(5,1)
      DSK(8,3) = DSK(2,1)
      DSK(4,4) = DSK(2,2)
      DSK(6,4) = DSK(8,2)
      DSK(7,4) = DSK(2,1)
      DSK(8,4) = DSK(6,2)
      DSK(5,5) = DSK(1,1)
      DSK(6,5) = DSK(2,1)
      DSK(7,5) = DSK(3,1)
      DSK(6,6) = DSK(2,2)
      DSK(8,6) = DSK(4,2)
      DSK(7,7) = DSK(1,1)
      DSK(8,7) = DSK(6,1)
      DSK(8,8) = DSK(2,2)
      DO 402 J=1,8
      DO 402 I=1,8
  402 DSK(J,I) = DSK (I,J)
      DO 403 I=1,8
      DO 403 J=1,8
  403 DSK(I,J) = DSK(I,J)* ((E*A)/(96.*ALP*BET*AX*BY))
      IF(IWRITE.EQ.0) GO TO 512
      WRITE (6,205) NTYPE
      WRITE (6,103)
      WRITE (6,101)   ((DSK(I,J),I=1,8), J=1,8)
  512 CONTINUE
      GO TO 235
```

TABLE XXVIII (Continued)

```
    8 CONTINUE
    9 CONTINUE
      WRITE (6,257)
      GO TO 839
C  MPQRS(I) CONTAINS THE SCHEME FOR PLACING THE ELEMENT MATRICES INTO
C THERE LARGER COUNTERPARTS.
  235 CONTINUE
      K=0
      JROW = JLAM / 2
      DO 39 I=1,JROW
      DO 39 J=1,2
      K=K+1
      MPQRS(K)=2*IPQRS(I)-2+J
      IF(IWRITE.EQ.0) GO TO 504
      WRITE (6,106) K, MPQRS(K)
  504 CONTINUE
   39 CONTINUE
C  ADD KBAR I  INTO KBAR
   38 DO 37 LA=1,JLAM
      KI=MPQRS(LA)
      DO 37 I=1,JLAM
      KL=MPQRS(I)
      IF(KI-KL)37 ,374,374
 374  KJ=(KI*(KI-1))/2+KL
      BARK(KJ)=BARK(KJ)+DSK  (LA,I)
      IF(IWRITE.EQ.0) GO TO 505
      WRITE (6,107) LA, KI
      WRITE (6,110) KJ, LA, I
  505 CONTINUE
 37   CONTINUE
C*****WRITE TAPE 4 FOR STRESS CALCULATIONS *****************************
      WRITE (4) NTYPE,E,PR,A,JLAM,D1,D2,AL(1),AL(2),MPQRS, IPQRS
      IF(IWRITE.EQ.0) GO TO 506
      WRITE(6,8798)
      CALL WRT ( BARK, N2)
  506 CONTINUE
  236 CONTINUE
C*******WRITE COMPLETE STIFFNESS MATRIX ON TAPE 3 FOR FORCE CALCULATION*
      WRITE( 3) (BARK(I),I=1,NUM)
      WRITE(6,8798)
      NF=0
      NS=0
      DO 31007 J=1,N2
      NS=NF+1
      NF=NF+J
31007 WRITE (6,31009) J,(BARK(I),  I=NS,NF)
C  REMOVE SINGULARITIES FROM K-BAR BY PLACING 1 ON DIAGONAL AND ZERO
C ELSEWHERE ON DUPLICATED ROWS AND COLUMNS.
      WRITE (6,114) NCROSS
      DO 316 LC=1,NCROSS
      LA=NBC(LC)
      DO 315 I=1,N2
      L=MAX0(LA,I)
      KA=(LA+I)+(L*(L-3))/2
      IF(IWRITE.EQ.0) GO TO 507
      WRITE (6,115) KA
  507 CONTINUE
 315  BARK(KA)=0
      KB=(LA*(LA+1))/2
      IF(IWRITE.EQ. 0) GO TO 508
      WRITE (6,113 ) LA, KB
  508 CONTINUE
```

TABLE XXVIII (Continued)

```
      BARK(KB)=1.
  316 CONTINUE
      IF(IWRITE.EQ. 0) GO TO 509
      WRITE(6,8798)
      CALL WRT ( BARK, N2)
  509 CONTINUE
CALCULATE  K-BAR-INVERSE.  IF ISING IS 0 ON RETURN THE MATRIX IS SINGULA
      CALL SYMINV (N2, BARK, ISING)
      WRITE(6,8799)
      NS=0
      NF=0
      DO 31008 J=1,N2
      NS=NF+1
      NF=NF+J
31008 WRITE(6,31009) J,(BARK(I),I=NS,NF)
30001 IF(ISING)317,8623,317
 8623 WRITE(6,8629)
      GO TO 839
  317 CONTINUE
C     ZERO DIAGONAL ELEMENTS OF BARK INVERSE
      DO 319 LC=1,NCROSS
      LA=(NBC(LC)*(NBC(LC)+1))/2
  319 BARK(LA)=0
      IF(IWRITE.EQ. 0) GO TO 510
      WRITE(6,8799)
      CALL WRT ( BARK, N2)
  510 CONTINUE
      CALL SMMPY(BARK,FORCE,UBAR,N2,NC)
      WRITE(6,800)
      WRITE(6,801)
  900 WRITE(6,802)
      K=0
      DO 638 I=1,N2,2
      K=K+1
      WRITE(6,804) K,(UBAR(I,J),J=1,NC)
  638 WRITE(6,805) (UBAR(I+1,J),J=1,NC)
  637 CONTINUE
C*********WRITE FORCES ACTING ON THE STRUCTURE*************************
      WRITE(6,809)
      WRITE(6,802)
      K=0
      DO 701 I=1,N2,2
      K=K+1
      WRITE(6,804) K,(FORCE(I,J),J=1,NC)
  701 WRITE(6,805)(FORCE(I+1,J),J=1,NC)
C     CALCULATE THE FORCE MATRIX = KBAR * UBAR
      REWIND 3
      READ(3)(BARK(I),I=1,NUM)
      CALL  SMMPY (BARK,UBAR ,QBAR,N2,NC)
      DO 700 I=1,N2
      DO 700 J=1,NC
  700 QBAR(I,J) = QBAR(I,J) + FORCE(I,J)
      WRITE(6,809)
      WRITE(6,802)
      K=0
      DO 640 I=1,N2,2
      K=K+1
      WRITE (6,804) K, (QBAR(I,J), J=1,NC)
  640 WRITE(6,805)(QBAR(I+1,J),J=1,NC)
C**************ELEMENT GENERALIZED STRESS CALCULATIONS*****************
      IF(NSN.EQ.0) GO TO 642
      WRITE (6,203)
```

TABLE XXVIII (Continued)

```
  642 CONTINUE
      REWIND 4
      DO 370 NN=1,NELEM
      READ  (4) NTYPE,E,PR,A,JLAM,D1,D2,AL(1),AL(2),MPQRS ,IPQRS
      IF(IWRITE.EQ.0) GO TO 641
      WRITE (6,116) NTYPE,E,PR,A,JLAM,D1,D2,AL(1),AL(2),MPQRS ,IPQRS
  641 CONTINUE
C*********************************************************************
C   SELECT U-BAR-I FROM U-BAR AND STORE IT IN QORU(I,J)
      DO 220 I=1,JLAM
      KI=MPQRS(I)
      DO 220 J=1,NC
  220 QORU(I,J)=UBAR(KI,J)
      WRITE (6,204) NN
      WRITE (6,801)
      WRITE (6,802)
      K=0
      DO 223 I = 1,JLAM, 2
      K=K+1
      WRITE (6,804) IPQRS(K), (QORU(I,J),J=1,NC)
      WRITE(6,805)  (QORU(I+1, J),J=1,NC)
  223 CONTINUE
C*********************************************************************
      IF(NSN.EQ.0) GO TO 379
      WRITE (6,258)
      IF(NTYPE.GE. 5) GO TO 375
      READ(5,251) I,(XN(NN,J),J=1,NSN)
      WRITE(6,253) I,(XN(NN,J),J=1,NSN)
      GO TO 376
  375 CONTINUE
      READ (5,251)I,  (XN(NN,J),J=1,NSN)
      READ(5,254) I,(YN(NN,J),J=1,NSN)
      WRITE(6,253)I,  (XN(NN,J),J=1,NSN)
      WRITE(6,255) I,(YN(NN,J),J=1,NSN)
      GO TO 376
  379 CONTINUE
      IF(NSN.EQ.0) NSN1=1
      IF(NSN.NE.0) NSN1=NSN
      XN(NN,1)=D2/2.
      YN(NN,1)=D1/2.
      WRITE(6,205)
  376 CONTINUE
      DO 237 NNSN=1,NSN1
      DO 377 I=1,3
      DO 377 J=1,8
  377 STR  (I,J) = 0.0
      DO 378 I=1,3
      DO 378 J=1,5
  378 STRESS (I,J) = 0.0
      GO TO (11,22,33,44,55,66,77,88,99),NTYPE
   11 CONTINUE
C**************STRESS MATRIX STRINGER ELEMENT************************
      WRITE (6,200)
      STR     (1,1) = -(AL(1)*E)  / D1
      STR     (1,2) = -(AL(2)*E)  / D1
      STR     (1,3) = AL(1)*E  / D1
      STR     (1,4) = AL(2)*E  / D1
      WRITE (6,101) (STR    (1,J),J=1,4)
      CALL MXM (STR,QORU,STRESS,NC)
      GO TO 30
C***********STRINGER STRESS MATRIX  ASSUMED STRESS FUNCTION************
   22 CONTINUE
```

TABLE XXVIII (Continued)

```
      XX = XN(NN,NNSN) / D2
      WRITE(6,101) XX
      STR   (1,1)=-(AL(1)*E)*(1.0-XX)    / D1
      STR   (1,2)=-(AL(2)*E)*(1.0-XX)   / D1
      STR   (1,3)=AL(1)*E*XX     / D1
      STR   (1,4)=AL(2)*E*XX    / D1
      WRITE(6,200)
      WRITE(6,101)(STR    (1,J),J=1,4)
      CALL MXM (STR,QORU,STRESS,NC)
      GO TO 30
   33 CCNTINUE
   44 CONTINUE
      WRITE (6,256)
      GO TO 839
   55 CONTINUE
C******************STRESS MATRIX ASSUMED DISPLACEMENTS*****************
      XX = XN(NN,NNSN) / D2
      YY = YN(NN,NNSN) / D1
      WRITE(6,259) XX,YY
      XA = D2
      YB = D1
      EPR0=1.0-PR**2
      EPR1=E/EPR0
      STR(1,1)=-EPR1*(1.0-YY)/XA
      STR(1,2)=-EPR1*PR*(1.0-XX)/YB
      STR(1,3)=-EPR1*XX/XA
      STR(1,4)=  -(STR(1,2))
      STR(1,5)=  -(STR(1,3))
      STR(1,6)=EPR1*PR*XX/YB
      STR(1,7)=  -(STR(1,1))
      STR(1,8)=  -(STR(1,6))
      STR(2,1)=-EPR1*PR*(1.0-YY)/XA
      STR(2,2)=-EPR1*(1.0-XX)/YB
      STR(2,3)=-EPR1*PR*YY/XA
      STR(2,4)=  -(STR(2,2))
      STR(2,5)=  -(STR(2,3))
      STR(2,6)=EPR1*XX/YB
      STR(2,7)=  -(STR(2,1))
      STR(2,8)=  -(STR(2,6))
      STR(3,1)=-EPR1*(1.0-PR)*(1.0-XX)/(2.0*YB)
      STR(3,2)=-EPR1*(1.0-PR)*(1.0-YY)/(2.0*XA)
      STR(3,3)=  -(STR(3,1))
      STR(3,4)=-EPR1*YY*(1.0-PR)/(2.0*XA)
      STR(3,5)=EPR1*XX*(1.0-PR)/(2.0*YB)
      STR(3,6)=  -(STR(3,4))
      STR(3,7)=  -(STR(3,5))
      STR(3,8)=  -(STR(3,2))
      WRITE (6,200)
      WRITE (6,101)((STR(I,J), J=1,8), I=1,3)
      CALL MXM (STR,QORU,STRESS,NC)
      GO TO 30
   66 CONTINUE
C*********STRESS MATRIX ASSUMED STRESS FUNCTION WITH 5 COEFFICIENTS*****
      XX = XN(NN,NNSN) / D2
      YY = YN(NN,NNSN) / D1
      WRITE(6,259) XX,YY
      XA = D2
      YB = D1
      EPR0=1.0-PR**2
      EPR1=E/EPR0
      EPR2=2.0*YY-1.0
      EPR3=1.0-2.0*YY
```

TABLE XXVIII (Continued)

```
      EPR4=2.0*XX-1.0
      EPR5=1.0-2.0*XX
      STR(1,1)=EPR1*((EPR0*EPR2)-1.0)/(2.0*XA)
      STR(1,2)=-EPR1*PR/(2.0*YB)
      STR(1,3)=EPR1*((EPR0*EPR3)-1.0)/(2.0*XA)
      STR(1,4)=EPR1*PR/(2.0*YB)
      STR(1,5)=EPR1*((EPR0*EPR2)+1.0)/(2.0*XA)
      STR(1,6)=STR(1,4)
      STR(1,7)=EPR1*((EPR0*EPR3)+1.0)/(2.0*XA)
      STR(1,8)=-STR(1,4)
      STR(2,1)=-EPR1*PR/(2.0*XA)
      STR(2,2)=EPR1*((EPR0*EPR4)-1.0)/(2.0*YB)
      STR(2,3)=STR(2,1)
      STR(2,4)=EPR1*((EPR0*EPR5)+1.0)/(2.0*YB)
      STR(2,5)=-STR(2,1)
      STR(2,6)=EPR1*((EPR0*EPR4)+1.0)/(2.0*YB)
      STR(2,7)=STR(2,5)
      STR(2,8)=EPR1*((EPR0*EPR5)-1.0)/(2.0*YB)
      STR(3,1) = -(EPR1*(1.0-PR)/(4.0 * YB))
      STR(3,2) = -(EPR1*(1.0-PR)/(4.0 * XA))
      STR(3,3)=-STR(3,1)
      STR(3,4)=STR(3,2)
      STR(3,5)=STR(3,3)
      STR(3,6)=-STR(3,2)
      STR(3,7)=STR(3,1)
      STR(3,8)=STR(3,6)
      WRITE(6,200)
      WRITE(6,101)((STR(I,J),J=1,8),I=1,3)
      CALL MXM (STR,QORU,STRESS,NC)
      GO TO 30
   77 CONTINUE
C*****STRESS MATRIX - WITH SEVEN COEFFICIENTS***************************
      BY = D1
      AX = D2
      XX= XN(NN,NNSN)
      YY= YN(NN,NNSN)
      WRITE(6,259) XX,YY
      ALP =  (3.*D2*D2 + D1*D1)
      BET=(3.*D1*D1)+(D2*D2)
      DO 371 I=1,3
      DO 371 J=1,8
  371 STR(I,J) = 0.0
      STR(1,1)= -(102.*BY*ALP*BET)-( 6.*(BY**3)*BET)+(18.*AX*AX*BY*BET)
     1+YY*((96.*ALP*BET)+(12.*BY*BY*BET)-(36.*AX*AX*BET))
      STR(2,1)= -( 18.*BY*ALP*BET)-(18.*(BY**3)*BET)+(54.*AX*AX*BY*BET)
     1+YY*(( 36.*BY*BY*BET) - (108.*AX*AX*BET))
      STR(3,1)= -( 18.*AX*ALP*BET)-(54.*(AX**3)*BET)+(18.*AX*BY*BY*BET)
     1-XX*(( 36.*BY*BY*BET) - (108.*AX*AX*BET))
      STR(1,2)= -( 18.*AX*ALP*BET)-(18.*(AX**3)*ALP)+(54.*AX*BY*BY*ALP)
     1+XX*(( 36.*AX*AX*ALP) - (108.*BY*BY*ALP))
      STR(2,2)= -(102.*AX*ALP*BET)-( 6.*(AX**3)*ALP)+(18.*AX*BY*BY*ALP)
     1+XX*((96.*ALP*BET) - (36.*BY*BY*ALP) + (12.*AX*AX*ALP))
      STR(3,2)= -( 18.*BY*ALP*BET)-(54.*(BY**3)*ALP)+(18.*AX*AX*BY*ALP)
     1-YY*(( 36.*AX*AX*ALP) - (108.*BY*BY*ALP))
      STR(1,3)= -(  6.*BY*ALP*BET)+( 6.*(BY**3)*BET)-(18.*AX*AX*BY*BET)
     1+YY*((-96.*ALP*BET)-(12.*BY*BY*BET)+(36.*AX*AX*BET))
      STR(2,3)= -( 18.*BY*ALP*BET)+(18.*(BY**3)*BET)-(54.*AX*AX*BY*BET)
     1+YY*((-36.*BY*BY*BET) + (108.*AX*AX*BET))
      STR(3,3)= +( 18.*AX*ALP*BET)+(54.*(AX**3)*BET)-(18.*AX*BY*BY*BET)
     1-XX*((-36.*BY*BY*BET) + (108.*AX*AX*BET))
      STR(1,4)= +( 18.*AX*ALP*BET)+(18.*(AX**3)*ALP)-(54.*AX*BY*BY*ALP)
     1+XX*((-36.*AX*AX*ALP) + (108.*BY*BY*ALP))
```

TABLE XXVIII (Continued)

```
      STR(2,4)= +(102.*AX*ALP*BET)+( 6.*(AX**3)*ALP)-(18.*AX*BY*BY*ALP)      SAS683
     1+XX*((-96.*ALP*BET)+(36.*BY*BY*ALP)-(12.*AX*AX*ALP))                  SAS684
      STR(3,4)= -( 18.*BY*ALP*BET)+(54.*(BY**3)*ALP)-(18.*AX*AX*BY*ALP)      SAS685
     1-YY*((-36.*AX*AX*ALP) + (108.*BY*BY*ALP))                             SAS686
      STR(1,5)= +(  6.*BY*ALP*BET)-( 6.*(BY**3)*BET)+(18.*AX*AX*BY*BET)      SAS687
     1+YY*((96.*ALP*BET)+(12.*BY*BY*BET)-(36.*AX*AX*BET))                   SAS688
      STR(2,5)=  ( 18.*BY*ALP*BET)-(18.*(BY**3)*BET)+(54.*AX*AX*BY*BET)      SAS689
     1+YY*(( 36.*BY*BY*BET) - (108.*AX*AX*BET))                             SAS690
      STR(3,5)= +( 18.*AX*ALP*BET)-(54.*(AX**3)*BET)+(18.*AX*BY*BY*BET)      SAS691
     1-XX*(( 36.*BY*BY*BET) - (108.*AX*AX*BET))                             SAS692
      STR(1,6)= +( 18.*AX*ALP*BET)-(18.*(AX**3)*ALP)+(54.*AX*BY*BY*ALP)      SAS693
     1+XX*(( 36.*AX*AX*ALP) - (108.*BY*BY*ALP))                             SAS694
      STR(2,6)= +(  6.*AX*ALP*BET)-( 6.*(AX**3)*ALP)+(18.*AX*BY*BY*ALP)      SAS695
     1+XX*((96.*ALP*BET) - (36.*BY*BY*ALP) + (12.*AX*AX*ALP))               SAS696
      STR(3,6)=  ( 18.*BY*ALP*BET)-(54.*(BY**3)*ALP)+(18.*AX*AX*BY*ALP)      SAS697
     1-YY*((+36.*AX*AX*ALP) - (108.*BY*BY*ALP))                             SAS698
      STR(1,7)=  (102.*BY*ALP*BET)+( 6.*(BY**3)*BET)-(18.*AX*AX*BY*BET)      SAS699
     1+YY*((-96.*ALP*BET)-(12.*BY*BY*BET)+(36.*AX*AX*BET))                  SAS700
      STR(2,7)=  ( 18.*BY*ALP*BET)+(18.*(BY**3)*BET)-(54.*AX*AX*BY*BET)      SAS701
     1+YY*((-36.*BY*BY*BET) + (108.*AX*AX*BET))                             SAS702
      STR(3,7)= -( 18.*AX*ALP*BET)+(54.*(AX**3)*BET)-(18.*AX*BY*BY*BET)      SAS703
     1-XX*((-36.*BY*BY*BET) + (108.*AX*AX*BET))                             SAS704
      STR(1,8)= -( 18.*AX*ALP*BET)+(18.*(AX**3)*ALP)-(54.*AX*BY*BY*ALP)      SAS705
     1+XX*((-36.*AX*AX*ALP) + (108.*BY*BY*ALP))                             SAS706
      STR(2,8)= -(  6.*AX*ALP*BET)+( 6.*(AX**3)*ALP)-(18.*AX*BY*BY*ALP)      SAS707
     1+XX*((-96.*ALP*BET)+(36.*BY*BY*ALP)-(12.*AX*AX*ALP))                  SAS708
      STR(3,8)=  ( 18.*BY*ALP*BET)+(54.*(BY**3)*ALP)-(18.*AX*AX*BY*ALP)      SAS709
     1-YY*((-36.*AX*AX*ALP) + (108.*BY*BY*ALP))                             SAS710
      DO 404 I=1,3                                                           SAS711
      DO 404 J=1,8                                                           SAS712
  404 STR(I,J)= STR(I,J)*(E/(96.*ALP*BET *AX*BY))                            SAS713
      WRITE(6,200)                                                           SAS714
      WRITE(6,101)((STR(I,J),J=1,8),I=1,3)                                   SAS715
      CALL MXM (STR,QORU,STRESS,NC)                                          SAS716
      GC TO 30                                                               SAS717
   88 CONTINUE                                                               SAS718
   99 CONTINUE                                                               SAS719
      WRITE (6,256)                                                          SAS720
      GO TO 839                                                              SAS721
   30 CONTINUE                                                               SAS722
      WRITE(6,206) NN,NTYPE                                                  SAS723
      WRITE (6,201)                                                          SAS724
      WRITE (6,202)                                                          SAS725
      WRITE (6,219) NNSN, NTYPE, (STRESS(1,I), I=1,NC)                       SAS726
      IF(NTYPE.LE.4) GO TO 237                                               SAS727
      WRITE (6,222)          (STRESS(2,I), I=1,NC)                           SAS728
      WRITE (6,221)          (STRESS(3,I), I=1,NC)                           SAS729
  237 CONTINUE                                                               SAS730
  370 CONTINUE                                                               SAS731
      REWIND 3                                                               SAS732
      REWIND 4                                                               SAS733
         WRITE(6,99999)                                                      SAS734
      WRITE(6,995)(R(J),J=1,12)                                              SAS735
19999  GO TO 839                                                             SAS736
11999 CALL EXIT                                                              SAS737
      END                                                                    SAS738
$IBFTC SYMINV
      SUBROUTINE SYMINV ( IO, A, ISING)                                      SMINV001
      DIMENSION A(1830),COL(60)                                              SMINV002
      IF(IO-1)800,810,97                                                     SMINV003
C     ----INVERSE OF 2X2----                                                 SMINV004
 97   C=A(1)*A(3)-A(2)*A(2)                                                  SMINV005
```

TABLE XXVIII (Continued)

```
      IF(C)98,900,98                                                    SMINV006
 98   A(2)=-A(2)/C                                                      SMINV007
      COL(1)=A(1)/C                                                     SMINV008
      A(1)=A(3)/C                                                       SMINV009
      A(3)=COL(1)                                                       SMINV010
      IF(IO-2)800,720,99                                                SMINV011
 99   K=1                                                               SMINV012
      M=IO-1                                                            SMINV013
      DO700IO11=2,M                                                     SMINV014
      K=K+IO11                                                          SMINV015
C     ----L.L.H.OFSYMMETRICMATRIX*COLUMN----                            SMINV016
      N=C                                                               SMINV017
      DO100I=1,IO11                                                     SMINV018
 100  COL(I)=0                                                          SMINV019
      DO300I=1,IO11                                                     SMINV020
      IA=K+I                                                            SMINV021
      DO300J=1,I                                                        SMINV022
      N=N+1                                                             SMINV023
      COL(J)=COL(J)+A(N)*A(IA)                                          SMINV024
      IF(J-I)200,300,800                                                SMINV025
 200  IB=K+J                                                            SMINV026
      COL(I)=COL(I)+A(N)*A(IB)                                          SMINV027
 300  CONTINUE                                                          SMINV028
C     ----COMPUTEB22----                                                SMINV029
      C=0                                                               SMINV030
      DO400I=1,IO11                                                     SMINV031
      IA=K+I                                                            SMINV032
 400  C=C+A(IA)*COL(I)                                                  SMINV033
      IA=IA+1                                                           SMINV034
      C=A(IA)-C                                                         SMINV035
      IF(C)410,900,410                                                  SMINV036
 410  C=1.0/C                                                           SMINV037
      A(IA)=C                                                           SMINV038
C     ----COMPUTEB21----                                                SMINV039
      DO500I=1,IO11                                                     SMINV040
      IA=K+I                                                            SMINV041
 500  A(IA)=-C*COL(I)                                                   SMINV042
C     ----COMPUTEB11----                                                SMINV043
      N=0                                                               SMINV044
      DO600I=1,IO11                                                     SMINV045
      DO600J=1,I                                                        SMINV046
      N=N+1                                                             SMINV047
      IA=K+J                                                            SMINV048
 600  A(N)=A(N)-A(IA)*COL(I)                                            SMINV049
 700  CONTINUE                                                          SMINV050
 720  ISING=1                                                           SMINV051
 710  RETURN                                                            SMINV052
 900  ISING=0                                                           SMINV053
      GOTO710                                                           SMINV054
 810  A(1)=1.0/A(1)                                                     SMINV055
      GO TO 720                                                         SMINV056
 800  ISING = 2                                                         SMINV057
      RETURN                                                            SMINV058
      END                                                               SMINV059
$IBFTC SMMPY
      SUBROUTINE SMMPY(A,B,C,N3,NC)                                     SMMPY001
C     (KINVERSE)*(FORCE)***DEFLECTIONS****NO OF ROWS****NO OF FORCES    SMMPY002
      DIMENSION A(1830),B(60,5),C(60,5)                                 SMMPY003
      DO 100 I=1,N3                                                     SMMPY004
      DO 100 J=1,NC                                                     SMMPY005
      C(I,J)=0                                                          SMMPY006
      DO 100 K1=1,N3                                                    SMMPY007
```

TABLE XXVIII (Continued)

```
      L=MAXO(I,K1)                                                      SMMPY008
      K=(L*(L-3))/2+(I+K1)                                              SMMPY009
 100  C(I,J)=A(K)*B(K1,J)+C(I,J)                                        SMMPY010
      RETURN                                                            SMMPY011
      END                                                               SMMPY012
$IBFTC WRT
      SUBROUTINE WRT(A, N3)                                               WRT001
      DIMENSION A(1)                                                      WRT002
31009 FORMAT(1X,3HROW,I4,/1X,(1P10E13.4))                                 WRT003
      NF=0                                                                WRT004
      NS=0                                                                WRT005
      DO 31010 J=1,N3                                                     WRT006
      NS=NF+1                                                             WRT007
      NF=NF+J                                                             WRT008
31010 WRITE (6,31009) J,(A(I), I=NS,NF)                                   WRT009
      RETURN                                                              WRT010
      END                                                                 WRT011
$IBFTC MXM
      SUBROUTINE MXM ( A, B, C, NC)                                       MXM001
      DIMENSION A(3,8),B(8,5),C(3,5)                                      MXM002
      DO 20 I=1,3                                                         MXM003
      DO 20 J=1,NC                                                        MXM004
   20 C(I,J) = 0.0                                                        MXM005
      DO 10 I=1,3                                                         MXM006
      DO 10  J=1,NC                                                       MXM007
      DO 10 N=1,8                                                         MXM008
   10 C(I,J) = C(I,J) + A(I,N) * B(N,J)                                   MXM009
      RETURN                                                              MXM010
      END                                                                 MXM011
```

APPENDIX C

TREATMENT OF EXPERIMENTAL DATA

The experimental stress and deflection data were processed by the IBM 7040 Digital Computer. The basic data obtained from the strain gages and dial indicators are reduced to values per unit loads for each of the load configurations, and these values are used for comparisons with the analytical predictions.

The unit stress and unit deflection values are obtained by finding the most reliable linear relationship using the least-squares criterion. The method of least squares provides that the most probable function for a quantity obtained from a set of measurements is the function which minimizes the sum of the squares of the deviations of these measurements. The deviation d_i is defined as the difference between any measurement y_i and the predicted value $\hat{y}_i$ (17).

$$d_i = y_i - \hat{y}_i$$

The least-squares criterion produces a system of equations for finding a functional relationship for the experimental data. Since this experimental investigation is restricted to the linear load-deflection range, the data can be expressed by the relation

$$\hat{y}_i = C_1 + C_2 X_i$$

It is necessary to find C_1 and C_2 in order to minimize

$$S = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{N} (y_i - C_1 - C_2 x_i)^2.$$

The minimum of S, considered as a function of C_1, is obtained from the partial derivative of S with respect to C_1 equal to zero. The result is

$$\frac{\partial S}{\partial C_1} = -2 \sum_{i=1}^{N} (y_i - C_1 - C_2 x_i) = 0$$

since

$$\frac{d}{dx} \sum_{i=1}^{N} f(x_i) = \sum_{i=1}^{N} \frac{d}{dx} f(x_i)$$

rearranging

$$\sum_{i=1}^{N} y_i = N C_1 + \left(\sum_{i=1}^{N} x_i\right) C_2 .$$

Similarly, for the minimum of S, considered as a function of C_2

$$\frac{\partial S}{\partial C_2} = -2 \sum_{i=1}^{N} x_i (y_i - C_1 - C_2 x_i) = 0$$

rearranging

$$\sum_{i=1}^{N} (x_i y_i) = \left(\sum_{i=1}^{N} x_i\right) C_1 + \left(\sum_{i=1}^{N} x_i^2\right) C_2 .$$

The two simultaneous equations in two unknowns are called normal equations (18).

To find the best linear function for the given data, it is necessary to perform the summations and solve the system of two equations for C_1 and C_2. The constant C_1 is the intercept of the straight line; the constant C_2 is the slope of the straight line. The slope is the unit stress of the influence coefficient value. The intercept is merely a function of the value at which the indicators are initially balanced or zeroed.

The solution for the constants C_1 and C_2 assuming the linear variation of strain or deflection versus load is

$$C_1 = \frac{(\Sigma y_i)(\Sigma x_i^2) - (\Sigma y_i x_i)(\Sigma x_i)}{N(\Sigma x_i^2) - (\Sigma x_i)^2}$$

$$C_2 = \frac{N(\Sigma y_i x_i) - (\Sigma y_i)(\Sigma x_i)}{N(\Sigma x_i^2) - (\Sigma x_i)^2}$$

where Σ is $\sum_{i=1}^{N}$.

Correlation of Experimental Data

The least-squares criterion is used to obtain a linear equation relating the two variables, load and stress, or deflection by using pairs of observations (x_i, y_i) of these variables. It is assumed in advance that such a linear relationship exists. In the event of a spread in the experimental data, there would be a question if a linear correlation exists between the load and the stress or deflection data. If a linear correlation does exist, the values for C_1 and C_2 are obtained as described previously.

A graphical interpretation of the procedure is described by using Figure 40. The data points in Figure 40 are determined experimentally, and it is necessary to represent the best straight line through the points. The slope of the lines is C_2, and its intercept on the y axis is C_1.

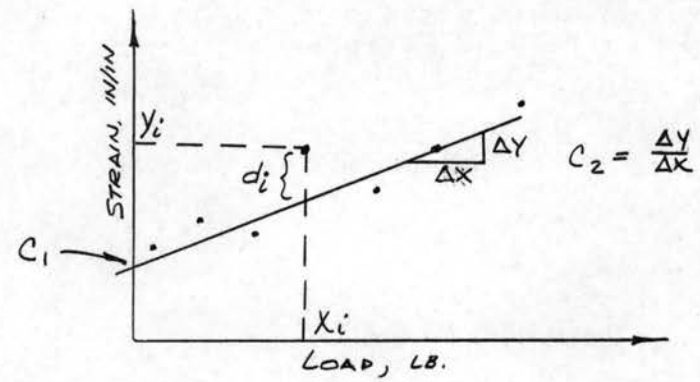

Figure 40. Typical Experimental Data

The deviations used in the method of least squares are

$$d_i = y_i - C_1 - C_2 x_i$$

where d_i represents the vertical distance between the point (x_i, y_i) and the straight line described by the constants C_1 and C_2. The method of least squares minimizes the sum of the squares of the vertical distances between the point and the straight line. The line determined by this procedure is sometimes called the line of regression of y on x (17).

An estimate of how well the linear function represents the experimental data is given by the correlation coefficient R (18).

$$R = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{\left[(N \sum x_i^2) - (\sum x_i)^2\right]^{1/2} \left[(N \sum y_i^2) - (\sum y_i)^2\right]^{1/2}} .$$

Thus, R = 1 means perfect correlation, and R = 0 means no correlation. Consequently, for imperfect correlation, $0 \geq |R| < 1$.

The interpretation of the correlation coefficient R is based on experience. The question is how large a value of R indicates a significant correlation between the variables x and y. Because of random fluctuations in the experimental data, R would not be exactly equal to zero, even if the data were completely erroneous. And, in addition, due to experimental fluctuations, R would not be exactly equal to one. However, since the nature of the problem dictates that a linear relationship exists and the experimental errors are hopefully minimized, then one should expect to get values in the neighborhood of $R = 1$. The criterion used to determine if the linear correlation is substantial is to consider the probability of obtaining a value of R as large as possible purely by chance from the observations of two variables which are not related. Table XXIII has been calculated to give the probability of obtaining a given value of R for various numbers of pairs of observations (18).

From Table XXIII for ten observations, N equals ten. The probability P is 0.10 of finding a correlation coefficient of 0.549 or larger and a probability of 0.01 of finding R greater than or equal to 0.765 if the variables are not related. If, for ten observations, the correlation coefficient $R = 0.9$, there is reasonable assurance that this indicates a true correlation and not an accident. Conversely, if $R = 0.5$, this would mean that the data were questionable since there is more than a ten per cent chance that this value would occur for random data. A commonly used rule of thumb for interpreting values of the correlation coefficient is to regard the correlation as significant if there is less than one chance in twenty, $P = 0.05$, that the value will occur by chance (18). For any value of the correlation coefficient greater than the value given in the Table XXIII for $P = 0.05$, the experimental data should be regarded as showing a significant correlation.

TABLE XXIX

CORRELATION COEFFICIENTS*

	Probability				
N	0.10	0.05	0.02	0.01	0.001
3	0.988	0.997	0.999	1.000	1.000
4	0.900	0.950	0.980	0.990	0.999
5	0.805	0.878	0.934	0.959	0.992
6	0.729	0.811	0.882	0.917	0.974
7	0.669	0.754	0.833	0.874	0.951
8	0.621	0.707	0.789	0.834	0.925
10	0.549	0.632	0.716	0.765	0.872
12	0.497	0.576	0.658	0.708	0.823
15	0.441	0.514	0.592	0.641	0.760
20	0.378	0.444	0.516	0.561	0.679

*This table is adapted from Table V of H. Young, Statistical Treatment of Experimental Data published by McGraw-Hill Book Company, Inc., New York.

TABLE XXX

SAMPLE DATA SHEET

LOAD INTERVAL

Load										
Load										
Load										
Load										

Recorded by		Date		AEROSPACE LABORATORY	
Punched		Date		SCHOOL OF MECHANICAL ENGINEERING	
RECTANGULAR PANEL DATA				OKLAHOMA STATE UNIVERSITY	
TEST CONFIGURATION NO. ___					PAGE
				M. U. Ayres Ex. 7223	OF

The linear correlation coefficient is only a measure of the best fit of a linear relationship to the experimental data and is in no way an indication that the experimental data accurately represent the physical phenomena. It is merely an indication that a linear correlation exists between the variables x and y.

Data Reduction Digital Computer Programs

Separate digital computer programs are used for the deflection indicator data, the axial strain gage data, and the rosette strain gage data. The programs are used to calculate the best linear relationship based on the least-squares criterion; however, each program is different in the manner in which the data are finally presented. The data analysis is controlled by the parameters specified on the control cards.

The experimental data for the axial gage are keypunched directly from the Victor printer tape or from the data forms shown in Table XXX . The experimental data for the rosette gages are punched from the data forms in Table XXX . The punched data are arranged in ascending gage numbers for the gage numbering system shown in Figures 20, 21, and 22 by use of the IBM card sorter. Data must be given for each gage number since in the current configuration the program expects the data to be in sets of two for axial gages and set of three for rosette gages. If no data are available for one axial gage or one leg of a rosette, a card containing only the gage number should be used. Each two sets of axial gage data is averaged to give the back-to-back readings for the stringers and ribs. Each three sets of rosette gage data is used for the calculation of axial and principal stresses from the following equations.

Stress-Strain Relations for Equiangular Rosette Gages

For the general case of plane stress, strains must be measured in at least three directions to find the principal strains and their directions.

The strain along an axis at an angle ϕ with the x axis is (19)

$$\epsilon_\phi = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\phi + \frac{\gamma_{xy}}{2} \sin 2\phi .$$

For the equiangular, or delta, rosette, the angles are

$$\phi_1 = 0^\circ \qquad \phi_2 = 60^\circ \qquad \phi_3 = 120^\circ .$$

Solving for the strains ϵ_x, ϵ_y, γ_{xy} from the equations above

$$\epsilon_x = \epsilon_1$$

$$\epsilon_y = \frac{-\epsilon_1 + 2\epsilon_2 + 2\epsilon_3}{3}$$

$$\gamma_{xy} = \frac{2(\epsilon_2 - \epsilon_3)}{\sqrt{3}} .$$

Consequently, the stresses are

$$\sigma_x = \frac{E}{1-\nu^2}\left(\epsilon_x + \nu \epsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\epsilon_y + \nu \epsilon_x\right)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\left(\gamma_{xy}\right) .$$

The principal stresses are given by

$$\sigma_{\substack{max \\ MIN}} = E \left\{ \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3(1+\nu)} \pm \frac{1}{1+\nu} \sqrt{\left(\epsilon_1 - \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}\right)^2 \left(\frac{\epsilon_2 - \epsilon_3}{\sqrt{3}}\right)^2} \right\}$$

$$\tau_{max} = \frac{E}{1+\nu} \left\{ \sqrt{\left(\epsilon_1 - \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}\right)^2 + \left(\frac{\epsilon_2 - \epsilon_3}{\sqrt{3}}\right)^2} \right\}$$

$$2\theta = \tan^{-1} \frac{\sqrt{3}(\epsilon_2 - \epsilon_3)}{2\epsilon_1 + \epsilon_2 + \epsilon_3} .$$

The axial and rosette strain gage data reduction programs require control cards containing the following information:

CARD 1

Column 3	The number of different sets of data to be analyzed.
Column 11	The parameter Iwrite = 1 if only a summary of the data consisting of gage number, correlation coefficient, and stress is to be printed. If Iwrite = 0, the complete data reductions are printed.

CARD 2

Column 1	The numeral 1.
Columns 2-30	Contain alphabetic or numeric description for the test identification.

CARD 3

Column 3	Contains the number of observations for each gage.
Column 13	Contains the number of active gages.
Columns 21-30	Contain the cross-sectional area of the stringer or rib element if forces are desired.
Column 32	Contains a numeral 1 if the data are keypunched from the Victor printer tape , and is blank if the data are punched from the data forms in Table XXX.

CARD 4	Contains the load data in FORMAT (7x, 10F7.0).
CARDS 5 to N	Contain the gage number and strain data in FORMAT (I7, 10F7.0).

The program prints the test data in tabular form for each indicator. The correlation coefficient and stress data are summarized at the end of the analysis to provide a more rapid analysis of the experimental results. The validity of the data is indicated by the correlation coefficient.

The flow diagram for the axial strain gage data program is shown in Figure 41. A Fortran listing of the program is given in Table XXXI. The flow diagram for the rosette strain gage data program is shown in Figure 42 A Fortran listing of the program is given in Table XXXII.

The deflection data reduction program requires the same control cards as the stress data programs, except for card 3 which requires only the information in columns 1 through 13. The flow diagram for the deflection data reduction program is shown in Figure 43. A Fortran listing of the program is given in Table XXXIII.

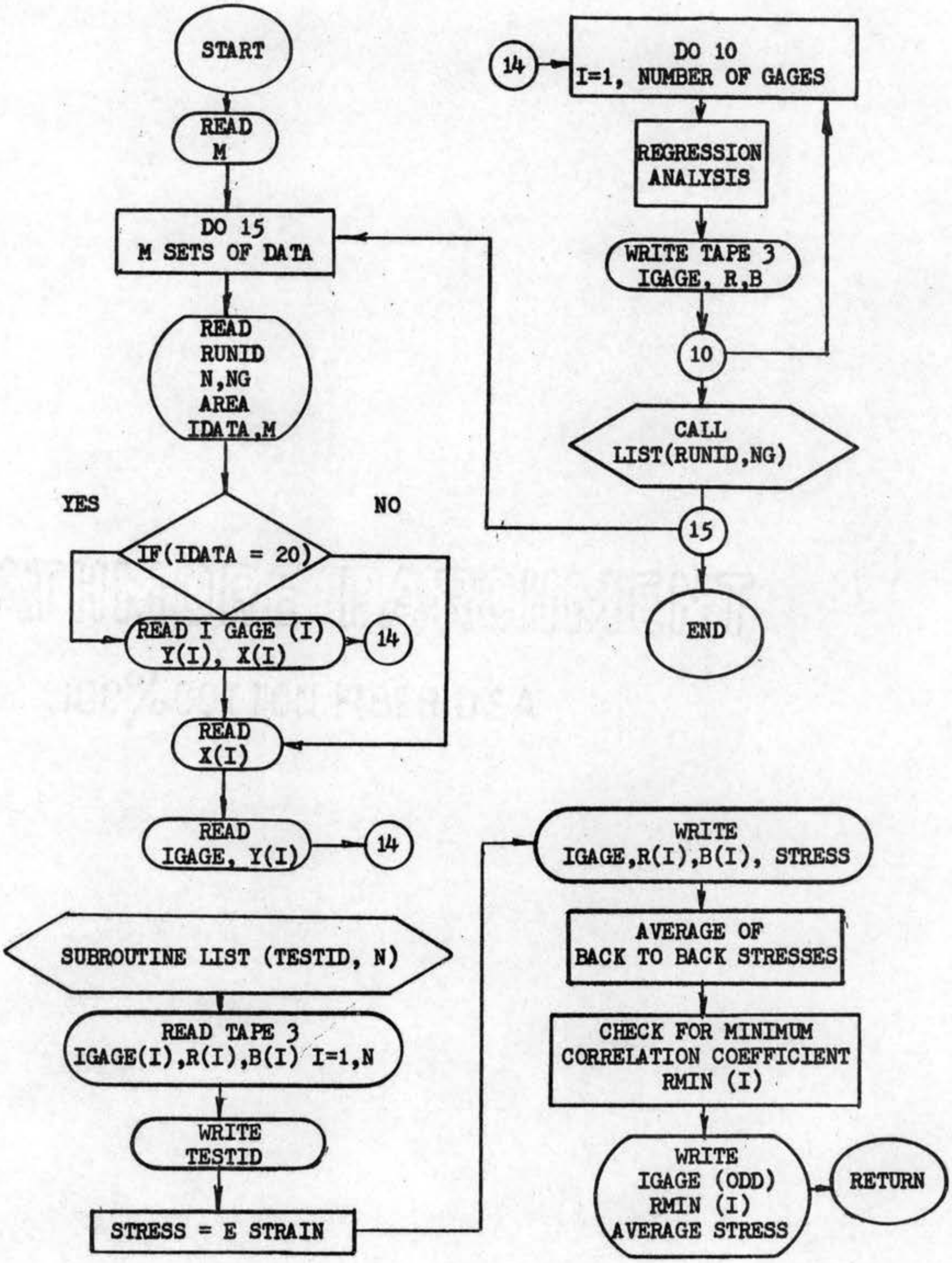

Figure 41. Flow Diagram for Axial Gage Program

TABLE XXXI

AXIAL STRAIN GAGE DATA REDUCTION PROGRAM

```
C     AXIAL TEST DATA REDUCTION PROGRAM    M.U.AYRES********************AXIAL001
      DIMENSION X(100),Y(100),RUNID(5),PROID(3),SUM(11),STRS(100)       AXIAL002
     1 ,  SAVE(100)                                                     AXIAL003
      COMMON TITLE(12),  MOP(18), NCH(40), TAB1, TAB2, ND, NP, NM, NB   AXIAL004
     1 , TAB3                                                           AXIAL005
      EQUIVALENCE (A,SUM(1)),(B,SUM(2)),(R,SUM(5)),(STD,SUM(6)),        AXIAL006
     1(US,SUM(3)),(UF,SUM(4)),(SX,SUM(7)),(SY,SUM(8)),(SXY,SUM(9)),     AXIAL007
     2(SXS,SUM(10)),(SYS,SUM(11))                                       AXIAL008
1     FORMAT(12A6)                                                      AXIAL009
    2 FORMAT(58A1,3A6,4A1)                                              AXIAL010
  100 FORMAT(5A6/I3,7X,I3,7X,F10.3,I2)                                  AXIAL011
  101 FORMAT(I2,4X,F4.0,F10.0)                                          AXIAL012
  102 FORMAT(I3,7X,I1)                                                  AXIAL013
  200 FORMAT(1H1)                                                       AXIAL014
  201 FORMAT(26X,29H****STRESS DATA REDUCTION****,19X,5HPAGE ,I3//      AXIAL015
     220X,10HTEST ID...,5A6/20X,10HGAGE ID...,I2//,20X,                 AXIAL016
     324HNUMBER OF OBSERVATIONS =,I3//,10X,4HLOAD,9X,10H STRAIN   ,10X, AXIAL017
     410H STRESS    ,/(5X,F10.0,5X,F10.0,10X,F10.0))                    AXIAL018
  202 FORMAT(//20X,12HINTERCEPT = ,F13.4,/18X,                          AXIAL019
     114HUNIT STRAIN = ,F17.8/,18X,14HUNIT STRESS = ,F13.4/,            AXIAL020
     219X,13HUNIT FORCE = ,F13.4/,6X,26HCORRELATION COEFFICIENT = ,     AXIAL021
     3F13.4/,11X,21HSTANDARD DEVIATION = ,F13.4)                        AXIAL022
  203 FORMAT(I3,7X,F13.4,10X,F17.8)                                     AXIAL023
  204 FORMAT(1H1,5A6///I3,7X,I3,7X,F10.3,7X,I2,7X,I3)                   AXIAL024
 1001 FORMAT(I7,10F7.0)                                                 AXIAL025
1002  FORMAT(I7,10F7.0/(7X,10F7.0))                                     AXIAL026
 1101 FORMAT(7X,10F7.0)                                                 AXIAL027
C     ***************** READ CONTROL DATA ******************************AXIAL028
      READ(5,102) M,IWRITE                                              AXIAL029
      DO 15 IT=1,M                                                      AXIAL030
C*****READ PLOTTER TITLES***********************************************AXIAL031
       READ(5,1)(TITLE(I),I=1,12)                                       AXIAL032
                  READ(5,2)(MOP(I),I=1,18),(NCH(I),I=1,40),TAB1,TAB2,TABAXIAL033
     13,ND,NP,NM,NB                                                     AXIAL034
      IPG=0                                                             AXIAL035
      REWIND 3                                                          AXIAL036
      READ(5,100)RUNID,N,NG,AREA,IDATA                                  AXIAL037
      WRITE(6,204)RUNID,N,NG,AREA,IDATA,M                               AXIAL038
C     ***************** READ EXPERIMENTAL DATA *************************AXIAL039
      IF(IDATA.EQ.20) GO TO 12                                          AXIAL040
      READ (5, 1101)(X(I), I=1,N)                                       AXIAL041
      DO 10 II = 1,NG                                                   AXIAL042
      IF (N .LE. 10) GO TO 1003                                         AXIAL043
      READ (5, 1002) IGAGE,  (Y(I), I = 1,N)                            AXIAL044
      GO TO 1004                                                        AXIAL045
 1003 READ (5, 1001) IGAGE, (Y(I) , I = 1,N)                            AXIAL046
 1004 IF (IGAGE .EQ. 0) GO TO 15                                        AXIAL047
      GO TO 14                                                          AXIAL048
   12 DO 10 IK= 1,NG                                                    AXIAL049
      READ (5,101) (IGAGE, Y(I), X(I),  I=1,N)                          AXIAL050
      IF ( Y(I) .LT. 0.0)  Y(I) = 1000.  + Y(I)                         AXIAL051
      IGAGE = IGAGE + 1                                                 AXIAL052
C     ***************** REGRESSION ANALYSIS ****************************AXIAL053
   14 DO 9 I=1,11                                                       AXIAL054
```

TABLE XXXI (Continued)

```
    9 SUM(I) = 0.0                                                      AXIAL055
      DO 3 I = 1,N                                                      AXIAL056
      STRS (I) = Y(I)*10.6                                              AXIAL057
      SX = SX + X(I)                                                    AXIAL058
      SY = SY + Y(I)                                                    AXIAL059
      SXY=SXY+X(I)*Y(I)                                                 AXIAL060
      SXS=SXS+X(I)*X(I)                                                 AXIAL061
    3 SYS=SYS+Y(I)*Y(I)                                                 AXIAL062
      AN=N                                                              AXIAL063
      B=(AN*SXY-SX*SY)/(AN*SXS-SX*SX)                                   AXIAL064
      CALLDVCHK(K)                                                      AXIAL065
      GO TO (6,4),K                                                     AXIAL066
    6 B=000.000                                                         AXIAL067
    4 A=(SY-B*SX)/AN                                                    AXIAL068
      R=(AN*SXY-SX*SY)/SQRT((AN*SXS-SX*SX)*(AN*SYS-SY*SY))              AXIAL069
      CALLDVCHK(K)                                                      AXIAL070
      GO TO (7,5),K                                                     AXIAL071
    7 R=0.0                                                             AXIAL072
    5 STD = SQRT((SYS-A*SY-B*SXY)/AN)                                   AXIAL073
      IPG = IPG + 1                                                     AXIAL074
      US = B*10.6                                                       AXIAL075
      UF = US*AREA                                                      AXIAL076
      R = ABS  (R)                                                      AXIAL077
      WRITE (3,203) IGAGE, R, B                                         AXIAL078
C     IF COLUMN 11 = 1 SKIP TO THE SUMMARY OF THE RESULTS***************AXIAL079
      IF(IWRITE.EQ.1) GO TO 10                                          AXIAL080
C     PRINT RESULTS OF THE REGRESSION ANALYSIS *************************AXIAL081
      WRITE(6,200)                                                      AXIAL082
      WRITE(6,201)IPG,RUNID,IGAGE,N,(X(I),Y(I),STRS(I),I=1,N)           AXIAL083
      WRITE (6,202)(SUM(I),I=1,6)                                       AXIAL084
C     PLOT THE EXPERIMENTAL DATA ***************************************AXIAL085
      DO 302 I = 1,N                                                    AXIAL086
  302 SAVE(I) = X(I)                                                    AXIAL087
      DO 300 I = 2,N                                                    AXIAL088
  300 Y(I)=ABS(Y(I)-Y(1))                                               AXIAL089
      Y(1) = 0.0                                                        AXIAL090
      X(N+1) = X(N) + 500.0                                             AXIAL091
      CALL PLOT (X,0.0,X(N+1),0,Y,0.0,Y(N),0,0.0,0.0,0.0,0,N,1,1,0,2)   AXIAL092
      DO 301 I = 1,N                                                    AXIAL093
  301 X(I) = SAVE (I)                                                   AXIAL094
   10 CONTINUE                                                          AXIAL095
      END FILE 3                                                        AXIAL096
      REWIND 3                                                          AXIAL097
      CALL LIST (RUNID , NG)                                            AXIAL098
   15 CONTINUE                                                          AXIAL099
      CALL EXIT                                                         AXIAL100
      END                                                               AXIAL101
$IBFTC LIST                                                             AXIAL102
      SUBROUTINE LIST (TESTID, N)                                       AXIAL103
      DIMENSION IGAGE(100), R(100), B(100), C(100), BAVG(100), CAVG(100)AXIAL104
     2,  RMIN(100), TESTID (5)                                          AXIAL105
   99 FORMAT (5A6)                                                      AXIAL106
  100 FORMAT(I3,7X,F13.4,10X,F17.8)                                     AXIAL107
  200 FORMAT(1H1,25X,5A6////,21X,11HCORRELATION/,5X,11HGAGE NUMBER,5X,  AXIAL108
```

TABLE XXXI (Continued)

```
   211HCOEFFICIENT,5X,6HSTRAIN,12X,6HSTRESS)                            AXIAL109
  201 FORMAT(8X,I3,10X,F8.4,7X,F11.8,6X,F11.8)                           AXIAL110
  202 FORMAT  ( 65HOSTRAIN DATA IS LISTED AS MICROINCHES  PER POUND OF EAXIAL111
     2XTERNAL LOAD    /,5X,  57H STRESS DATA IS LISTED AS  PSI PER POUND AXIAL112
     3OF EXTERNAL LOAD )                                                 AXIAL113
C     N = NUMBER OF GAGES TO BE USED ***********************************AXIAL114
      E=10.6                                                             AXIAL115
      READ (3,100) (IGAGE(I), R(I), B(I), I=1,N)                         AXIAL116
      WRITE (6,200)  TESTID                                              AXIAL117
      WRITE (6,202)                                                      AXIAL118
      LINES = 0                                                          AXIAL119
      DO 10 I = 1,N                                                      AXIAL120
      LINES = LINES + 1                                                  AXIAL121
      IF (LINES .LT. 40 )  GO TO 30                                      AXIAL122
      WRITE (6,200)  TESTID                                              AXIAL123
      WRITE (6,202)                                                      AXIAL124
      LINES = 0                                                          AXIAL125
   30 C(I) = B(I) * E                                                    AXIAL126
   10 WRITE(6,201) IGAGE(I),R(I),B(I),C(I)                               AXIAL127
C     WRITE OUT THE AVERAGE OF THE BACK TO BACK GAGE READINGS **********AXIAL128
      WRITE (6,200) TESTID                                               AXIAL129
      WRITE (6,202)                                                      AXIAL130
      LINES = 0                                                          AXIAL131
      DO 20 I=1,N,2                                                      AXIAL132
      BAVG(I) = (B(I)+B(I+1))/2.0                                        AXIAL133
      CAVG(I) = BAVG(I) * E                                              AXIAL134
      RMIN(I) = AMIN1(R(I),R(I+1))                                       AXIAL135
      LINES= LINES + 1                                                   AXIAL136
      IF (LINES .LT. 40 )  GO TO 50                                      AXIAL137
      WRITE (6,200) TESTID                                               AXIAL138
      WRITE (6,202)                                                      AXIAL139
      LINES = 0                                                          AXIAL140
   50 WRITE (6,201) IGAGE(I), RMIN(I), BAVG(I), CAVG(I)                  AXIAL141
   20 CONTINUE                                                           AXIAL142
      RETURN                                                             AXIAL143
      END                                                                AXIAL144
```

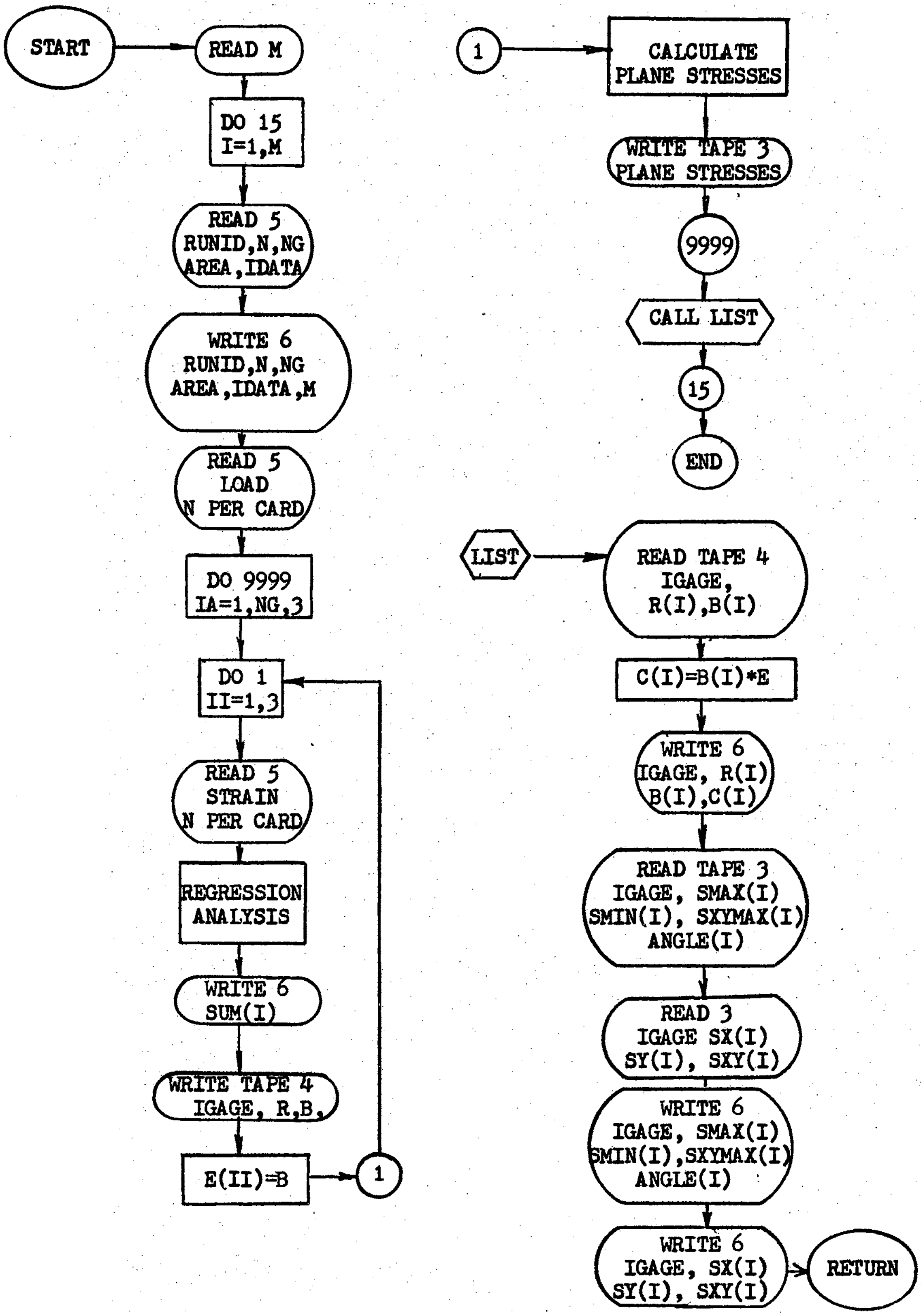

Figure 42. Flow Diagram for Rosette Gage Program

TABLE XXXII

ROSETTE STRAIN GAGE DATA REDUCTION PROGRAM

```
C      ROSETTE TEST DATA REDUCTION PROGRAM BY M.U.AYRES                   ROSET001
       DIMENSION X(100),Y(100),RUNID(5),PROID(3),SUM(11),STRS(500),E(3)   ROSET002
      1  ,  SAVE(100)                                                     ROSET003
       EQUIVALENCE (A,SUM(1)),(B,SUM(2)),(R,SUM(5)),(STD,SUM(6)),         ROSET004
      1(US,SUM(3)),(UF,SUM(4)),(SX,SUM(7)),(SY,SUM(8)),(SXY,SUM(9)),      ROSET005
      2(SXS,SUM(10)),(SYS,SUM(11))                                        ROSET006
       COMMON TITLE(12),  MOP(18), NCH(40), TAB1, TAB2, ND, NP, NM, NB    ROSET007
      1 , TAB3                                                            ROSET008
1      FORMAT(12A6)                                                       ROSET009
     2 FORMAT(58A1,3A6,4A1)                                               ROSET010
   100 FORMAT(5A6/I3,7X,I3,7X,F10.3,I2)                                   ROSET011
   101 FORMAT(I2,4X,F4.0,F10.0)                                           ROSET012
   102 FORMAT(I3,7X,I1)                                                   ROSET013
   103 FORMAT (1H1, 38X, 18HPRINCIPAL STRESSES/// 20X,  8HGAGE NO.        ROSET014
      13X,11HMAX. STRESS,   4X, 11HMIN. STRESS,   7X, 10HMAX. SHEAR,      ROSET015
      26X, 5HANGLE)                                                       ROSET016
   111 FORMAT (1H1, 40X, 14HAXIAL STRESSES/// 20X,  8HGAGE NO.            ROSET017
      13X, 11HX-DIRECTION,   4X, 11HY-DIRECTION,   8X,  5HSHEAR)          ROSET018
   200 FORMAT(1H1)                                                        ROSET019
   201 FORMAT(26X,29H****STRESS DATA REDUCTION****,19X,5HPAGE ,I3//       ROSET020
      220X,10HTEST ID...,5A6/20X,10HGAGE ID...,I3//,20X,                  ROSET021
      324HNUMBER OF OBSERVATIONS =,I3//,10X,4HLOAD,9X,10H STRAIN    ,10X, ROSET022
      410H STRESS    ,/(5X,F10.0,5X,F10.0,10X,F10.0))                     ROSET023
   202 FORMAT(//20X,12HINTERCEPT = ,F13.4,/18X,                           ROSET024
      114HUNIT STRAIN = ,F17.8/,18X,14HUNIT STRESS = ,F13.4/,             ROSET025
      219X,13HUNIT FORCE = ,F13.4/,6X,26HCORRELATION COEFFICIENT = ,      ROSET026
      3F13.4/,11X,21HSTANDARD DEVIATION = ,F13.4)                         ROSET027
   203 FORMAT(I3,7X,F13.4,10X,F17.8)                                      ROSET028
   204 FORMAT(1H1,5A6///I3,7X,I3,7X,F10.3,7X,I2,7X,I3)                    ROSET029
  1001 FORMAT(I7,10F7.0)                                                  ROSET030
  1002 FORMAT(I7,10F7.0/(7X,10F7.0))                                      ROSET031
  1101 FORMAT(7X,10F7.0)                                                  ROSET032
   105 FORMAT ( 15X, I10, 3F15.5)                                         ROSET033
   106 FORMAT( 15X, I10, 4F15.5)                                          ROSET034
C      READ CONTROL DATA *********************************************ROSET035
       READ(5,102) M,IWRITE                                               ROSET036
       DO 15 IT=1,M                                                       ROSET037
C      READ PLOTTER TITLES*********************************************ROSET038
        READ(5,1)(TITLE(I),I=1,12)                                        ROSET039
                 READ(5,2)(MOP(I),I=1,18),(NCH(I),I=1,40),TAB1,TAB2,TABROSET040
      13,ND,NP,NM,NB                                                      ROSET041
       REWIND 2                                                           ROSET042
       REWIND 3                                                           ROSET043
       REWIND 4                                                           ROSET044
       IPG=0                                                              ROSET045
C      READ EXPERIMENTAL DATA ******************************************ROSET046
       READ(5,100)RUNID,N,NG,AREA,IDATA                                   ROSET047
       WRITE(6,204)RUNID,N,NG,AREA,IDATA,M                                ROSET048
       READ (5, 1101)(X(I), I=1,N)                                        ROSET049
       DO 9999  IA = 1,NG ,3                                              ROSET050
       DO 10 II = 1,3                                                     ROSET051
       IF (N .LE. 10) GO TO 1003                                          ROSET052
       READ (5, 1002) IGAGE,  (Y(I), I = 1,N)                             ROSET053
       GO TO 1004                                                         ROSET054
```

TABLE XXXII (Continued)

```
 1003 READ (5, 1001) IGAGE, (Y(I) , I = 1,N)
 1004 CONTINUE
C     REGRESSION ANALYSIS ********************************************
   14 DO 9 I=1,11
    9 SUM(I) = 0.0
      DO 3 I = 1,N
C     Y(I) = SCAFAC * Y(I) IF GAGE FACTORS NOT EQUAL FOR ALL GAGES *****
      SX = SX + X(I)
      SY = SY + Y(I)
      SXY=SXY+X(I)*Y(I)
      SXS=SXS+X(I)*X(I)
    3 SYS=SYS+Y(I)*Y(I)
      AN=N
      B=(AN*SXY-SX*SY)/(AN*SXS-SX*SX)
      CALLDVCHK(K)
      GO TO (6,4),K
    6 B=1.000000000
    4 A=(SY-B*SX)/AN
      R=(AN*SXY-SX*SY)/SQRT((AN*SXS-SX*SX)*(AN*SYS-SY*SY))
      CALLDVCHK(K)
      GO TO (7,5),K
    7 R=0.00000000000
    5 STD = SQRT((SYS-A*SY-B*SXY)/AN)
      IPG = IPG + 1
      US = B*10.6
      UF = US*AREA
      R = ABS  (R)
C     IF COLUMN 11 = 1    PRINT ONLY THE SUMMARY OF THE RESULTS  *******
      IF(IWRITE.EQ.1) GO TO 8
C     PRINT EXPERIMENTAL DATA ******************************************
      WRITE(6,200)
      WRITE(6,201)IPG,RUNID,IGAGE,N,(X(I),Y(I),STRS(I),I=1,N)
      WRITE (6,202)(SUM(I),I=1,6)
C     PLOT  EXPERIMENTAL DATA ******************************************
      DO 302 I = 1,N
  302 SAVE(I) = X(I)
      DO 300 I = 2,N
  300 Y(I)=ABS(Y(I)-Y(1))
      Y(1) = 0.0
      X(N+1) = X(N) + 500.0
      CALL PLOT (X,0.0,X(N+1),0,Y,0.0,Y(N),0,0.0,0.0,0.0,0,N,1,1,0,2)
      DO 301 I = 1,N
  301 X(I) = SAVE (I)
    8 CONTINUE
      WRITE (4,203) IGAGE, R, B
   10 E(II) = B
C     USE E1, E2, AND E3, FROM THE REGRESSION ANALYSIS FOR PLANE STRESS
      EE = 10.6
      PR =  0.333333
      E1 = E(1)
      E2 = E(2)
      E3 = E(3)
      EX  = E1
      EY =(-(E1)+(2.0*E2) +(2.0*E3)) / 3.0
```

TABLE XXXII (Continued)

```
      EXY = (2.0*(E2 - E3))/ 1.73214                                    ROSET109
      SX=((EE/(1.-(PR**2)))*(EX+(PR*EY)))                               ROSET110
      SY=((EE/(1.-(PR**2)))*(EY+(PR*EX)))                               ROSET111
      SXY=(EE/(2.*(1.+PR)))*EXY                                         ROSET112
      A = SQRT  (((E1-((E1+E2+E3)/3.0))*(E1-((E1+E2+E3)/3.0))) +        ROSET113
     1(((E2-E3)/1.73214)*((E2-E3)/1.73214)))                            ROSET114
      SXYMAX  = (EE/(1.0 + PR))* A                                      ROSET115
      B = (EE*(E1+E2+E3))/(3.0 *(1.0 - PR))                             ROSET116
      SMAX = B+SXYMAX                                                   ROSET117
      SMIN = B-SXYMAX                                                   ROSET118
      TAN2O =  ((E2 - E3) * 1.73214) / ((2.0*E1)+E2+E3)                 ROSET119
      ANGLE = 0.5 *  ATAN (TAN2O)                                       ROSET120
      WRITE (2,105)  IGAGE,   SX, SY, SXY                               ROSET121
      WRITE (3,106)IGAGE,SMAX,SMIN,SXYMAX,  ANGLE                       ROSET122
 9999 CONTINUE                                                          ROSET123
      END FILE 2                                                        ROSET124
      REWIND 2                                                          ROSET125
      END FILE 3                                                        ROSET126
      REWIND 3                                                          ROSET127
      END FILE 4                                                        ROSET128
      REWIND 4                                                          ROSET129
      CALL LIST (NG,RUNID)                                              ROSET130
   15 CONTINUE                                                          ROSET131
      CALL EXIT                                                         ROSET132
      END                                                               ROSET133
$IBFTC LIST                                                             ROSET134
      SUBROUTINE LIST (NG,RUNID)                                        ROSET135
      DIMENSION  IGAGE(500), SMAX(500), SMIN(500), SXYMAX(500),         ROSET136
     1ANGLE (500),SX(500),SY(500),SXY(500),R(500),B(500),C(500),        ROSET137
     2BAVG(500),CAVG(500),RUNID(5)                                      ROSET138
   99 FORMAT (5A6)                                                      ROSET139
  100 FORMAT(I3,7X,F13.4,10X,F17.8)                                     ROSET140
  200 FORMAT(1H1,25X,5A6////,21X,11HCORRELATION/,5X,11HGAGE NUMBER,5X,  ROSET141
     211HCOEFFICIENT,5X,6HSTRAIN,12X,6HSTRESS)                          ROSET142
  201 FORMAT(8X,I3,10X,F8.4,7X,F11.8,6X,F11.8)                          ROSET143
  202 FORMAT  ( 65HOSTRAIN DATA IS LISTED AS MICROINCHES  PER POUND OF EROSET144
     2XTERNAL LOAD   /,5X,  57H STRESS DATA IS LISTED AS  PSI PER POUND ROSET145
     3OF EXTERNAL LOAD )                                                ROSET146
  111 FORMAT (1H1, 40X, 14HAXIAL STRESSES/// 20X,  8HGAGE NO.           ROSET147
     13X, 11HX-DIRECTION,   4X, 11HY-DIRECTION,   8X,  5HSHEAR)         ROSET148
  102 FORMAT ( 15X, I10, 3F15.5)                                        ROSET149
  103 FORMAT (1H1, 38X, 18HPRINCIPAL STRESSES/// 20X,  8HGAGE NO.       ROSET150
     13X,11HMAX. STRESS,   4X, 11HMIN. STRESS,   7X, 10HMAX. SHEAR,     ROSET151
     26X, 5HANGLE)                                                      ROSET152
  104 FORMAT( 15X, I10, 4F15.5)                                         ROSET153
C     NG= NUMBER OF GAGES *********************************************ROSET154
      E=10.6                                                            ROSET155
      READ (4,100) (IGAGE(I), R(I), B(I), I=1,NG)                       ROSET156
      WRITE (6,200)   RUNID                                             ROSET157
      WRITE (6,202)                                                     ROSET158
      LINES = 0                                                         ROSET159
      DO 40 I = 1,NG                                                    ROSET160
      LINES = LINES + 1                                                 ROSET161
      IF (LINES .LT. 40 )  GO TO 30                                     ROSET162
```

TABLE XXXII (Continued)

```
      WRITE (6,200) RUNID                                                ROSET163
      WRITE (6,202)                                                      ROSET164
      LINES = 0                                                          ROSET165
   30 C(I) = B(I) * E                                                    ROSET166
   40 WRITE (6,201) IGAGE(I), R(I), B(I), C(I)                           ROSET167
      NG = NG/3                                                          ROSET168
      READ(2,102)(IGAGE(I), SX(I),SY(I),SXY(I),I=1,NG)                   ROSET169
      READ(3,104)(IGAGE(I),SMAX(I),SMIN(I),SXYMAX(I),ANGLE(I),I=1,NG)    ROSET170
      WRITE (6,111)                                                      ROSET171
      WRITE (6,202)                                                      ROSET172
      LINES=0                                                            ROSET173
      DO 10  I = 1,NG                                                    ROSET174
      LINES = LINES + 1                                                  ROSET175
      IF (LINES .LT. 40) GO TO 10                                        ROSET176
      WRITE (6,111)                                                      ROSET177
      LINES = 0                                                          ROSET178
   10 WRITE (6, 102) IGAGE(I), SX(I), SY(I), SXY(I)                      ROSET179
      WRITE (6,103)                                                      ROSET180
      WRITE (6,202)                                                      ROSET181
      LINES=0                                                            ROSET182
      DO 20  I = 1,NG                                                    ROSET183
      LINES = LINES + 1                                                  ROSET184
      IF (LINES .LT. 40) GO TO 20                                        ROSET185
      WRITE (6,103)                                                      ROSET186
      LINES = 0                                                          ROSET187
   20 WRITE (6,104) IGAGE(I), SMAX(I), SMIN(I), SXYMAX(I),ANGLE(I)       ROSET188
      RETURN                                                             ROSET189
      END                                                                ROSET190
```

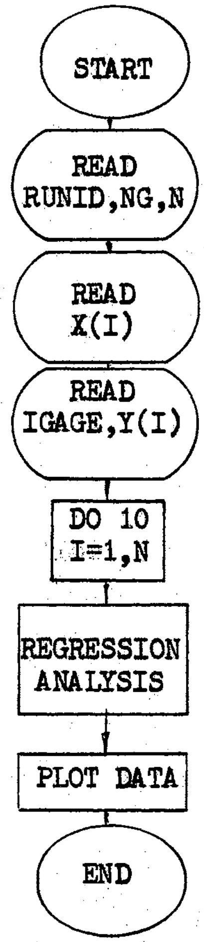

Figure 43. Flow Diagram for Deflection Data Program

TABLE XXXIII

DEFLECTION DATA REDUCTION PROGRAM

```
C       DEFLECTION DATA REDUCTION PROGRAM    M U AYRES                    DELTA001
        DIMENSION X(100), Y(100), RUNID(5), PROID(3), SUM(9)              DELTA002
       1 , SAVE (100)                                                     DELTA003
        COMMON TITLE(12),  MOP(18), NCH(40), TAB1, TAB2, ND, NP, NM, NB   DELTA004
       1 , TAB3                                                           DELTA005
        EQUIVALENCE (A,SUM(1)),(B,SUM(2)),(R,SUM(3)),(STD,SUM(4))         DELTA006
       2 ,(SX,SUM(5)),(SY,SUM(6)),(SXY,SUM(7)),(SXS,SUM(8)),(SYS,SUM(9))  DELTA007
1       FORMAT(12A6)                                                      DELTA008
      2 FORMAT(58A1,3A6,4A1)                                              DELTA009
    100 FORMAT (5A6)                                                      DELTA010
    101 FORMAT(I3)                                                        DELTA011
    200 FORMAT ( 1H1)                                                     DELTA012
    201 FORMAT(  25X,29H**DEFLECTION DATA REDUCTION**,19X,5HPAGE ,I3/     DELTA013
       220X,10HTEST ID...,5A6/20X,10HGAGE ID...,I3 //                     DELTA014
       3 11X,10HINPUT DATA ,30X,4HLOAD,9X,10HDEFLECTION /                 DELTA015
       4 15X,24HNUMBER OF OBSERVATIONS = ,I3,4X,F10.0,5X,                 DELTA016
       5F10.4/ (46X,F10.0,5X,F10.4))                                      DELTA017
    202 FORMAT (//20X, 12HINTERCEPT = ,F10.4/                             DELTA018
       28X, 24HINFLUENCE COEFFICIENT = ,F14.8//                           DELTA019
       3 6X, 26HCORRELATION COEFFICIENT =  ,F10.4/                        DELTA020
       4 11X,21HSTANDARD DEVIATION =   , F10.4 )                          DELTA021
   1001 FORMAT ( I7, 10F7.0)                                              DELTA022
   1002 FORMAT (I7, 10F7.0 / (7X, 10F7.0))                                DELTA023
   1101 FORMAT (7X, 10F7.0)                                               DELTA024
      9 CONTINUE                                                          DELTA025
C       READ PLOTTER TITLES *********************************************DELTA026
         READ(5,1)(TITLE(I),I=1,12)                                       DELTA027
                     READ(5,2)(MOP(I),I=1,18),(NCH(I),I=1,40),TAB1,TAB2,TABDELTA028
       13,ND,NP,NM,NB                                                     DELTA029
        IPG = 0                                                           DELTA030
        READ (5,100)RUNID                                                 DELTA031
        READ (5,101) NGAGES                                               DELTA032
        READ (5, 101) N                                                   DELTA033
        READ (5,1101) (X(I), I=1,N)                                       DELTA034
     10 CONTINUE                                                          DELTA035
        IF ( NGAGES .EQ. 0 ) GO TO 9                                      DELTA036
        NGAGES = NGAGES - 1                                               DELTA037
        WRITE (6,200)                                                     DELTA038
        IF (N .LE. 10) GO TO 1003                                         DELTA039
        READ (5, 1002) IGAGE,  (Y(I), I = 1,N)                            DELTA040
        GO TO 1004                                                        DELTA041
   1003 READ (5, 1001) IGAGE, (Y(I) , I = 1,N)                            DELTA042
C       REGRESSION ANALYSIS *********************************************DELTA043
   1004 DO 11 I=1,9                                                       DELTA044
     11 SUM(I)=0.0                                                        DELTA045
        DO 3 I = 1,N                                                      DELTA046
        SX = SX + X(I)                                                    DELTA047
        SY = SY + Y(I)                                                    DELTA048
        SXY=SXY+X(I)*Y(I)                                                 DELTA049
        SXS=SXS+X(I)*X(I)                                                 DELTA050
      3 SYS=SYS+Y(I)*Y(I)                                                 DELTA051
        AN=N                                                              DELTA052
        B=(AN*SXY-SX*SY)/(AN*SXS-SX*SX)                                   DELTA053
        CALLDVCHK(K)                                                      DELTA054
```

TABLE XXXIII (Continued)

```
      GO TO (6,4),K
    4 A=(SY-B*SX)/AN
      R=(AN*SXY-SX*SY)/SQRT((AN*SXS-SX*SX)*(AN*SYS-SY*SY))
      CALLDVCHK(K)
      GO TO (7,5),K
    5 STD = SQRT((SYS-A*SY-B*SXY)/AN)
      IPG = IPG + 1
C     PRINT EXPERIMENTAL DATA ******************************************
      WRITE(6,201)IPG,RUNID,IGAGE,N,(X(I),Y(I),I=1,N)
      WRITE (6,202) (SUM(I),I=1,4)
C     PLOT EXPERIMENTAL DATA *******************************************
      DO 302 I = 1,N
  302 SAVE(I) = X(I)
      DO 300 I = 2,N
  300 Y(I)=ABS(Y(I)-Y(1))
      Y(1) = 0.0
      X(N+1) = X(N) + 500.0
      CALL PLOT (X,0.0,X(N+1),0,Y,0.0,Y(N),0,0.0,0.0,0.0,0,N,1,1,0,2)
      DO 301 I = 1,N
  301 X(I) = SAVE (I)
      GO TO 10
    6 B=1.000000000
      GO TO 4
    7 R=0.00000000
      GO TO 5
      END
```

APPENDIX D

LIST OF MAJOR INSTRUMENTATION

Victor DigitMatic Printing Unit	
Datran Switch & Balance Unit	Budd Model C10LCT
Datran Printer Control Unit	Budd Model E140
Digital Strain Indicator	Budd Model A110
Datran Switch & Balance Unit	Budd Model C10T
Strain Indicator (4)	Budd Model P350
Switch & Balance Unit (25)	Budd Model SB-1
Switch & Balance Unit	BLH Type PSBA20 Model 3
Switch & Balance Unit	BLH Type 225
SR-4 Strain Indicator	BLH Type N
10,000-lb. Load Cell	BLH Type U3G1
5,000-lb. Load Cell	BLH Type U3G1
Dial Indicators (10)	Starrett No. 656-617
Calibration Unit	BLH Model 625

APPENDIX E

CALIBRATION OF STRAIN GAGE SYSTEMS

Once the strain gages are attached to the panel, it is not possible to attain a calibration by the use of a known strain situation. The strain gages are manufactured under carefully controlled conditions, and the gage factor for each lot of gages is within about ± 0.27 per cent. The gage factor and the gage resistance make possible a simple method for calibrating the resistance strain gage system. This method consists of determining the system's response to the introduction of a specific small resistance change at the gage and of calculating the resulting equivalent strain. The resistance change is introduced by shunting a relatively high value precision resistor across the gage as shown in the following figure.

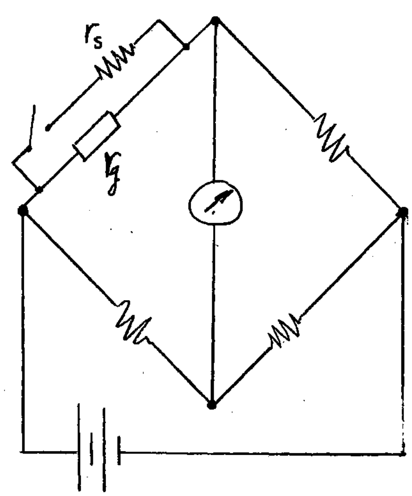

Figure 44. Strain Gage Bridge With Calibration Resistor

The equivalent strain for the shunt resistor in parallel with the active gage is

$$\epsilon = \frac{1}{GF}\left(\frac{r_g}{r_g + r_s}\right)$$

where GF = Gage factor
r_g = Gage resistance, ohms
r_s = Shunt resistance, ohms.

The Budd Model A-110 Digital Strain Indicator has a push button labeled Calibration Check for the purpose of shunting a 60K ohm $\pm$ 0.1 per cent resistor across one arm of the input bridge. For a gage factor of 2.00, multiplier at 1, coarse balance switch to Ext., the 60K calibration resistor should provide exactly 1001 counts for a 120 ohm gage. If the indicator calibration is found to be in error, readjustment of the internal calibration potentiometer is required.

The Budd portable strain indicator systems were calibrated using the same 60K-ohm resistor that was used in calibrating the strain gages for the Model A-110 Digital Strain Indicator. The resistor was shunted across each active gage.

Direct calibration of an external bridge input by using a known resistance assures maximum accuracy if the gage resistances are known accurately and load resistances are insignificant. The shunt calibration circuit is also helpful to ascertain the error caused by load resistance when long input leads are used.

The maximum variation for any single gage was less than three per cent, and the majority of gages were within one per cent of the calibration value. Typical results from the calibration tests are shown in the following table.

TABLE XXXIV

TYPICAL INDICATOR READINGS DURING

CALIBRATION TESTS

Gage Number	Indicator Reading Zero Level	Indicator Reading with Shunt Resistor	Net Change
121	1337	330	1007
122	1366	360	1006
123	1271	262	1009
124	1205	198	1007
125	1210	204	1006
126	1208	202	1006
127	1222	214	1008
128	1215	207	1008
129	1215	207	1008
303	1229	222	1007

Calibration of Load Recording Equipment

A calibration of the load recording equipment was performed to determine the accuracy of the load application system. The BLH U-3G1 type load cells have strain gages with a gage factor of 2.0 and a resistance of 350 ohms. Using a 60K calibration resistor, the computed strain should be 2900.

The calibration was performed from the zero reading from the 5000-pound load cell of 11050. The 60K resistor was shunted across each leg of the strain gage bridge, and the following records were obtained:

Shunt	Dial Reading	Net Change
P_1 to S_1	13915	2865
P_1 to S_2	8240	2810
P_2 to S_1	8180	2870
P_2 to S_2	13860	2810

The same procedure was used in calibrating the system for the 10,000-pound load cell. Again, the gage factor of 2.0 and a gage resistance of 350 ohms provide a strain input of 2900. The 60K resistor was shunted across the four arms of the bridge, one arm at a time. The following records were obtained:

Shunt	Dial Reading	Net Change
P_1 to S_1	13770	2870
P_2 to S_2	8100	2800
P_2 to S_1	8030	2870
P_2 to S_2	13715	2815

In general, a value of approximately 2800 to 2870 was obtained for each leg of the strain gage bridge. This is a variation of approximately three per cent or corresponds to a gage factor change of from 2.00 to 2.07, which might actually be the gage factor for the strain gages used in the load cell.

The load indicator system was subsequently calibrated with a BLH Model 625 voltage divider unit. A linear change in indicator reading was obtained for a linear change in MV/V input. The load cells have a 3 MV/V full scale output which corresponds to 6000 units on the BLH SR-4 indicator.

The various calibration techniques are redundant and are only a substitute for a dead weight test of the complete system. However, based on the calibration information, the load cells are sufficiently accurate.

APPENDIX F

ADDITIONAL EXPERIMENTAL DATA

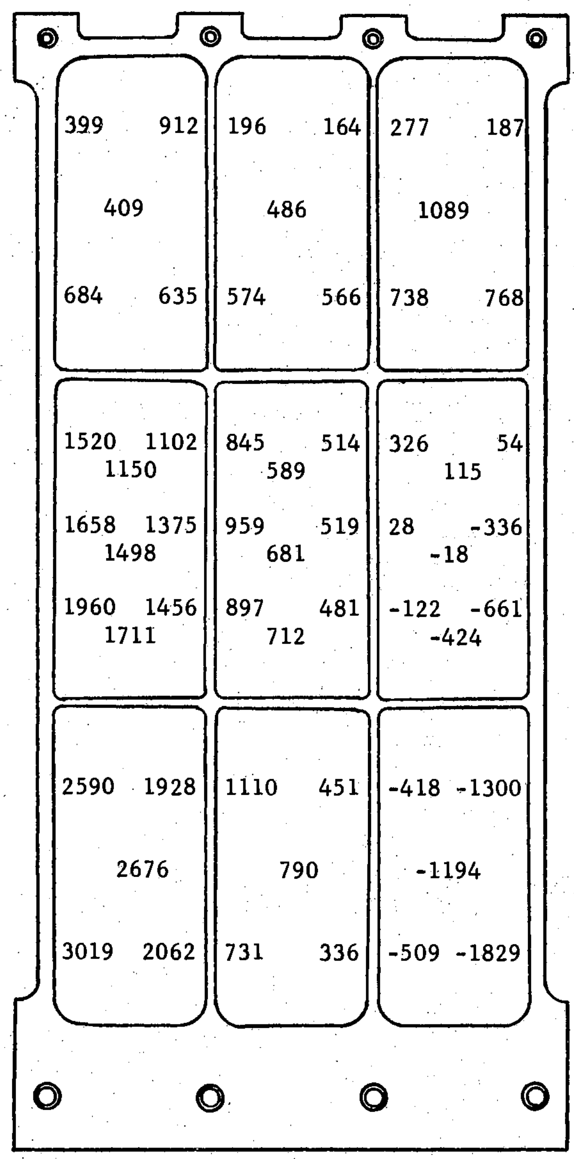

Figure 45. σ_y **Stress for Transverse Load Condition, Test 20**

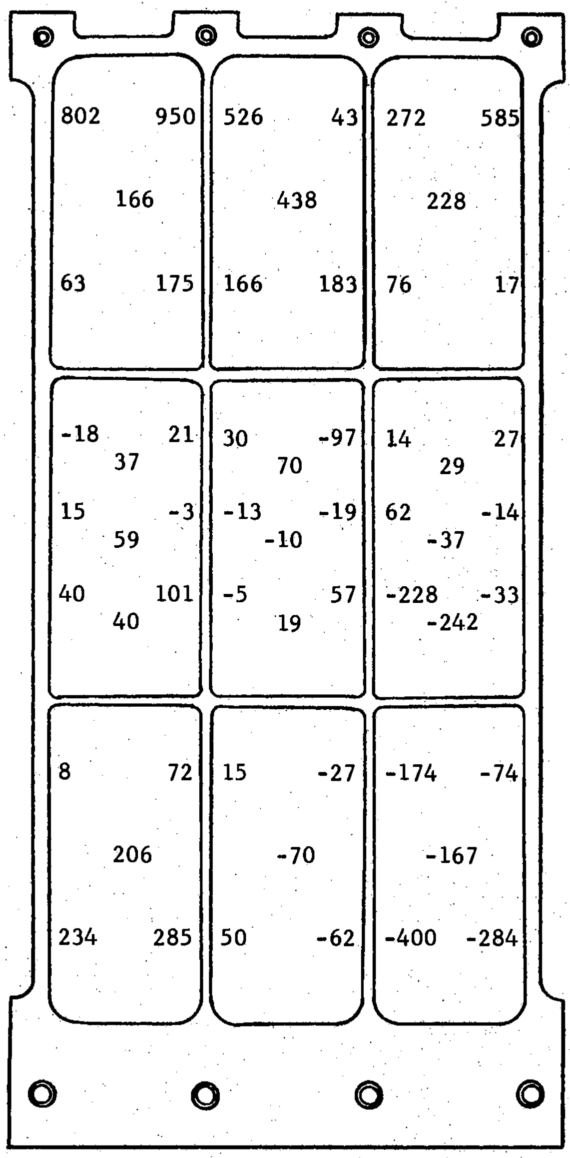

Figure 46. σ_x Stress for Transverse Load Condition, Test 20

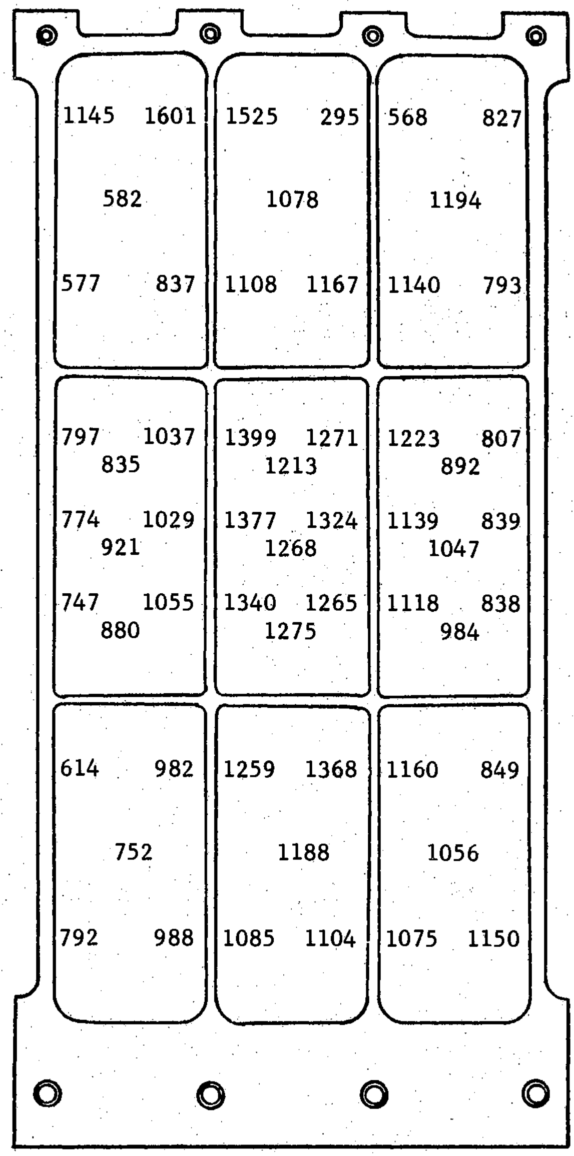

Figure 47. Shear Stress for Transverse Load Condition, Test 20

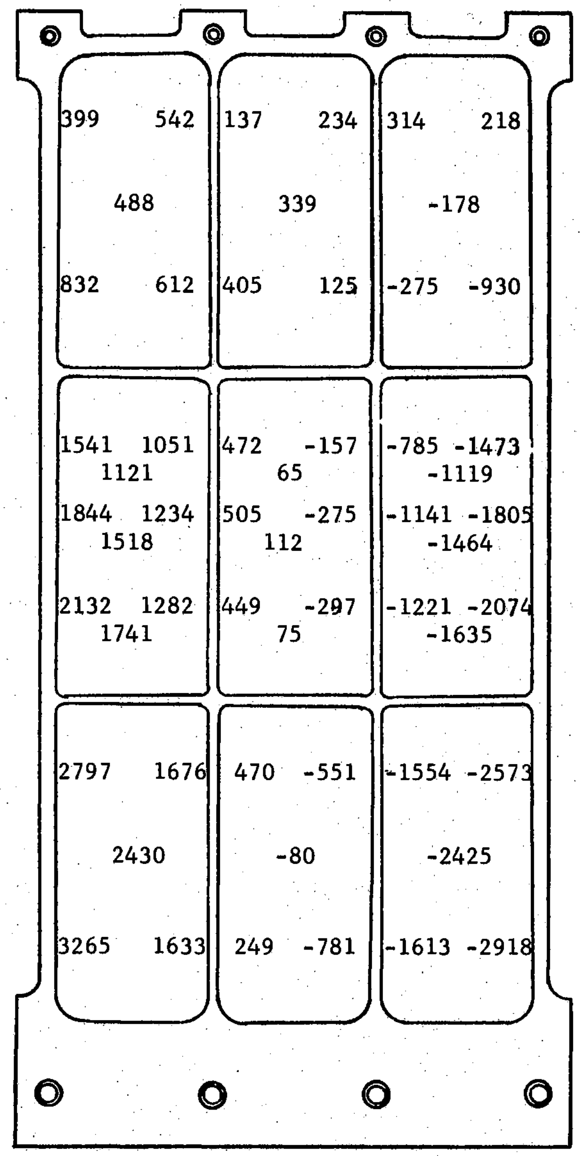

Figure 48. σ_y **Stress for Shear Load Condition, Test 22**

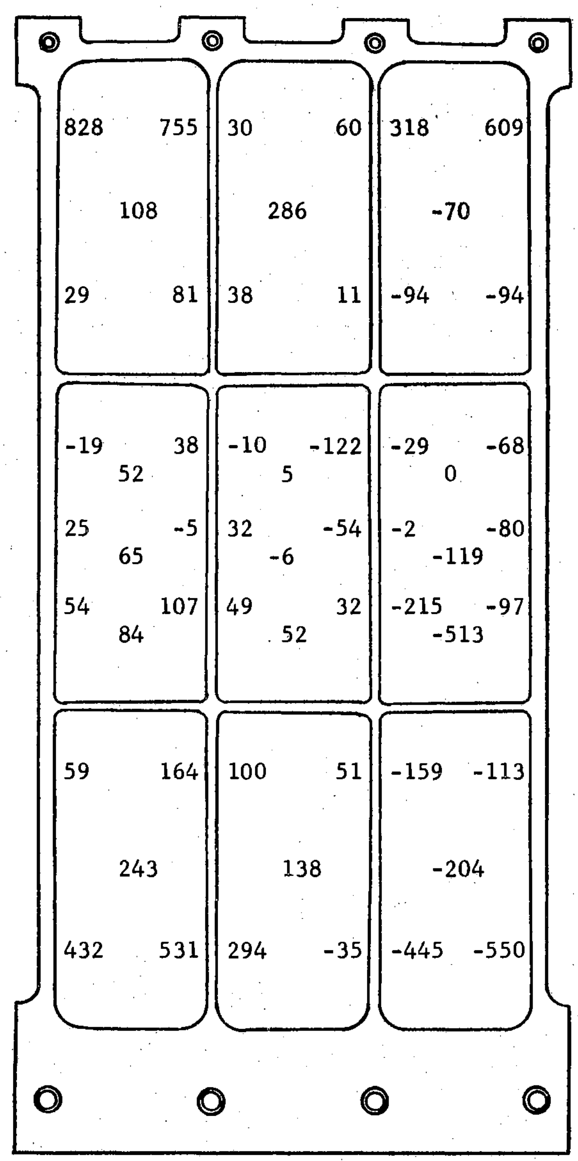

Figure 49. σ_x Stresses for Shear Load Condition, Test 22

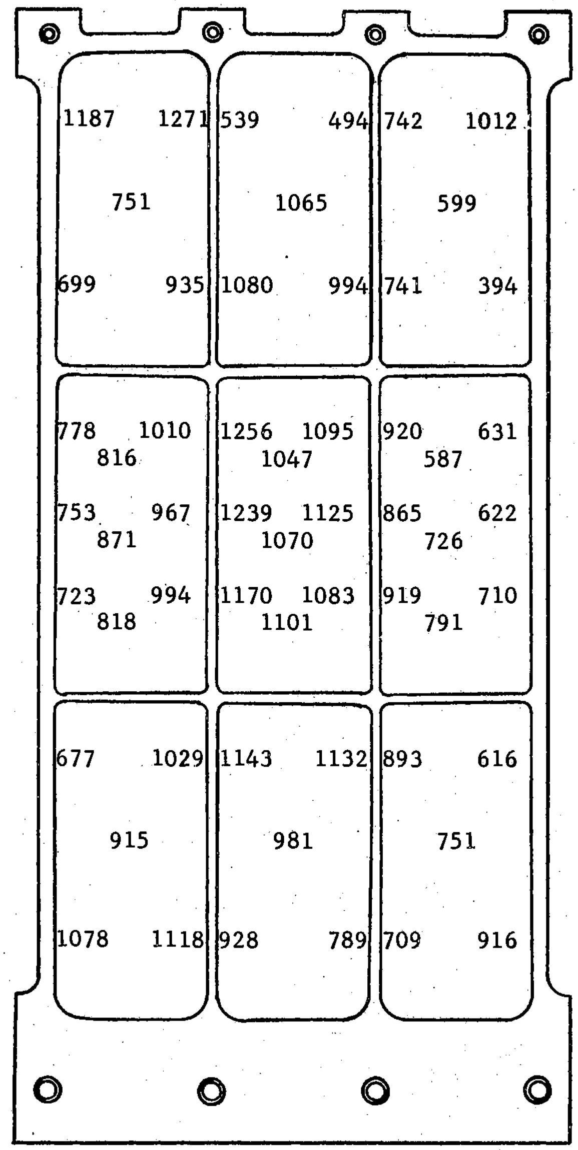

Figure 50. Shear Stress for Shear Load Condition, Test 22

VITA

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