A PROBABILISTIC APPROACH TO INVESTMENT ANALYSIS

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CHAPTER I

INTRODUCTION

Among the most challenging and crucial decisions that the business executive must make are those concerning alternative capital investment opportunities. In making such decisions, several standard methods of analysis may be utilized, including measures of the payback period, the simple rate of return, and the time adjusted rate of return. However, most of these methods fail to take into consideration the range of possible outcomes, that is, the risk of the occurrence of an outcome other than the one expected or most likely. Other methods which do take into account the possibility and probability of many outcomes, such as the one proposed by Friedrich Lutz,[1] are impractical because they require prior knowledge of probabilities of given happenings. Because the outcomes of events which affect the inflow from an investment are not always known with certainty, this information concerning the probabilities is not usually available. Without it, these methods are of questionable value.

There remains, then, a need for an approach to investment analysis that not only incorporates the consideration of more than one outcome

[1] Friedrich Lutz and Vera Lutz, The Theory of Investment of the Firm (New Jersey, 1951), p. 180.

from an investment, but is also applicable to most analyses. After a brief review of current practices and methods, this study will show how some of the procedures found in the projects management tool known as Program Evaluation and Review Technique may be applied to the discounted cash flow method of investment analysis to yield a usable probabilistic approach.

CHAPTER II

PRESENT DAY INVESTMENT DECISION-MAKING PRACTICES AND PROBLEMS

Before any analysis process can be successfully accomplished, two major steps must be taken: information must be gathered and differences between alternative investments measured. The decision-maker must make certain that the information collected is correct for investment analysis, and he must then use a good and practical method of evaluation.

Faulty Practices in Decision-Making

Several highly questionable practices often appear in financial decision making that either prevent or impair the collection of correct information. In the first place, the problem to be solved may not be fully understood. Answers to unrelated questions contribute nothing of value to the analysis. Also, some companies ignore the fact that much of the needed information is in the form of estimates, omitting from their solutions the affect that the use of estimates may have on the final results. Although solutions appear to be precise, they are actually based in part on inexact approximations. In addition, not enough attention may be given to the question of what constitutes an adequate profit return. If standards have not been set, management cannot be sure even of what information it needs. And perhaps the worst of current practices and yet the most frequent to occur is the haphazard and inaccurate handling of the data which makes up the information to be used. All too often not enough attention is given to the methods of collection. Proper data, carelessly assembled, may obscure needed information or present a false picture.[1]

Such faulty practices must be avoided. The decision-maker should be aware of the possibility of their occurrence when he chooses a method of analysis, for a good analysis method is one with the capability of pointing out or eliminating them.

Measures of Investment Acceptability

The process of analyzing and choosing between alternative investments has been presented in many forms. Joel Dean points out that there are four common measures of acceptability in use in comparing investment opportunities. They are (1) the question of necessity or postponability, (2) the length of the payback period, (3) the simple rate of return expected from an investment, and (4) the time adjusted rate of return which may be expected.[2] The question of necessity or postponability can be a poor basis for a decision. An asset that must be replaced may be no longer needed, and an investment that can be postponed may show the greatest return. The use of the payback period as a criterion for decision lacks any measure of long-term profitability since it takes into account neither the life

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[1] Ross B. Walker, "The Judgement Factor in Investment Decision," Harvard Business Review, Vol. 39, No. 2 (March-April, 1961), p. 95.

[2] Joel Dean, "Profitability Indexes for Capital Investment," The Controller, Vol. XXVI, No. 2 (February, 1958), pp. 66-67.

of the investment nor the total income over that life. The profitability of an investment is also not considered in the simple rate of return method since it too does not consider the investment life or income. Only the fourth measure, that of the time adjusted rate of return, takes into account the consideration of the time dimension of annual cash flows.[3]

The possibility of more than one rate of return as an outcome from an investment has been recognized by Friedrich and Vera Lutz. They argue that a rate of return is made up of many cost and revenue factors, each having a probability distribution of its own. The method which they advocate requires the use of discounting to present value, as in the calculation of the time adjusted rate of return. It also requires the process of simulation to obtain the true expected rate of return. This method presupposes that the distribution for each factor is known and that there is a given standard rate upon which to base calculations.[4] Several other methods have been presented as refinements of this one.[5] All of these methods require the knowledge of the probability distribution of each cost and revenue affecting the investment. In the real world, this type of information is not available. If it were, there would be little need for elaborate analysis methods since accurate predictions could be easily made without them.

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[3]Donald F. Istvan, "The Economic Evaluation of Capital Expenditures," Journal of Business, Vol. XXXIV, No. 1 (January, 1961), pp. 47-50.

[4]Lutz, p. 180.

[5]Neil R. Payne, "Uncertainty and Capital Budgeting," Accounting Review, Vol. XXXIX, No. 2 (April, 1964), p. 330. Also, see David B. Hertz, "Risk Analysis in Capital Investment," Harvard Business Review, Vol. XLII, No. 1 (January-February, 1964), pp. 95-106.

Several other methods have been proposed which are of limited scope. For example, the use of decision trees has been suggested as a means of determining the probability of receiving various rates of return when probabilities are assumed for discrete cash flows that together make up the total return from an investment.[6] Another possible approach involves the use of game theory.[7] A two-player zero-sum game is set up. The various rates of return represent one player, and the various possible growth rates represent the other. As a way of better representing the real world, the game is played only once. If this requirement were not in force, there would be the question of how to handle a game calling for mixed strategies. Both the decision tree and the game theory method require the knowledge of the probability of any given value of a contributing cash flow, and they are both limited in the number of investment analyses to which they can be applied since only discrete functions may be considered by either method, and most investments are not of this type.

The Discounted Cash Flow Method

The method which appears to be presently most advocated for measuring the acceptability of an investment is the discounted cash flow method. As a time adjusted rate of return process, this method

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[6] John F. Magee, "How to Use Decision Trees in Capital Investment," Harvard Business Review, Vol. 42, No. 5 (September-October, 1964), pp. 77-96.

[7] William G. Nelson, "Could Game Theory Aid Capital Budgeting?" N.A.A. Bulletin, Vol. XLIII, No. 10, (June, 1962), p. 53.

recognizes the time value of money by making allowances for the differences in the time at which investments generate their income. It also determines the rate of return generated by an investment rather than basing comparisons on a given rate.[8] It gives results that are in a compatible form with those used in banking to quote rates and yields. This identity of form allows direct comparison of the expected return of a project with the cost of capital.[9]

The mechanics of the discounted cash flow method consist essentially of finding that rate of interest which equates the discounted future earnings of an investment to the discounted cost of the investment. This rate is taken as the rate of return for the investment. The method is based on the idea that when management makes an investment outlay, it is actually buying a series of future annual incomes. There is an investment in each of these incomes that compounds in value until the cash flow is realized.[10]

The discounted cash flow method is not in itself the complete answer. It enables management to calculate a rate of return to a fraction of a percent. Yet this exact return is based on estimates, and estimates are rarely, if ever, exact. Other information must be considered concerning the amount of uncertainty present in each estimate in order to give management some indication of how nearly

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[8] Wendell M. Childs, "Capital Budgeting for Improved Profits," N.A.A. Bulletin, Vol. XLV, No. 9 (May, 1964), p. 56.

[9] John C. McLean, "How to Evaluate New Capital Investment," Harvard Business Review, Vol. 36, No. 6 (November-December, 1958), p. 63.

[10] Joel Dean, "Measuring the Productivity of Capital," Harvard Business Review, Vol. 32, No. 1 (January-February, 1954), p. 128.

precise the rate of return really is. The less certainty there is in the set of estimates that make up the basis of the analysis, the greater chance there is that a return other than the one predicted will occur. If management must choose between two otherwise identical investments, it will rationally choose the one which seems to offer the best chance of actually generating the return that has been predicted.[11]

If the actual rate of return generated by an investment can take on a wide range of values, there is a danger that the one-valued solution of the discounted cash flow method might give management a false impression of the amount of risk present in making the investment. As an example, suppose that a company is considering an investment of one million dollars. The single-valued estimate of the return predicts an income of two hundred thousand dollars. However, the income prediction is based on three possible returns--a one-in-three chance of obtaining two hundred thousand dollars per year, a one-in-three chance of obtaining four hundred thousand dollars per year, and a one-in-three chance of obtaining no return at all. Furthermore, if the company earns no return on the investment, it would be put out of business. If the estimate presented to management shows a return which is greater than the cut-off requirement, then the project may be undertaken without any consideration of the one-in-three chance of failure of the company.[12]

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[11] David B. Hertz, "Risk Analysis in Capital Investment," Harvard Business Review, Vol. 42, No. 1 (January-February, 1964), p. 101.

[12] Ibid., p. 98.

Limited Improvements Over the Discounted Cash Flow Method Forecast

There are several ways in which allowances for risk can be incorporated into the rate of return estimates as they are being made. Friedrich and Vera Lutz suggest the use of a correction factor based in part on management's attitude toward risk.[13] The greater the aversion of management to risk, the smaller the correction factor fraction will be. This factor is applied to the annual cash flow estimates by multiplying each of them by it. Joel Dean expands the basis of this factor, which he chooses to call the "probability multiplier."[14] The multiplier is not only based on management's attitude toward risk, but also on the distance in the future of a cash flow for which an estimate is made. For more distant estimates, the multiplier is smaller to reflect greater uncertainty. As in the case of the correction factor, the multiplier is applied to the annual cash flow estimates. This reduces the total cash flow estimate and, therefore, the expected rate of return. The same effect could be realized by raising the cut-off requirements for investment return except that such a raise would be uniform with respect to all investments, while the application of the multiplier would have the greatest effect on the longer term investments. Dean also suggests three other ways of allowing for risk, the first of them being to inflate the rate of discount. This has the same effect as does the application of the correction factor suggested

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[13] Lutz, p. 180.

[14] Joel Dean, Capital Budgeting (New York, 1951), pp. 30-31.

by the Lutzes. Both provide a uniform safety factor or buffer against a rate of return falling below what is expected. The second suggestion is to shorten the estimate of the economic life of the proposed asset, again lowering the rate that will be determined in the analysis. His final suggestion is to simply use good judgment. If subjectivity must be present in a decision, then good judgment is a necessity on the part of the decision-maker.

Other approaches made to protect against the faults of one-valued forecasts have been employed. Attempts have been made to increase the accuracy of forecasts by reducing the error in estimates. Cut-off rates have been revised upwards to provide a buffer against receiving lower returns than expected. As a means of indicating the possible range of outcomes, some companies require the use of three level estimates for each cash flow. From this set of high, medium, and low estimates are calculated rates of return based on various combinations of possibilities. But the question still remains regarding the probability of an outcome actually occurring. None of these improvements aid the presentation of all possible outcomes.[15]

Therefore, each of these methods of protection fails to correct the problem of not showing the possibility of more than one outcome. They offer only a safety factor which provides some protection against not obtaining the expected return. It becomes apparent that a method of evaluation is needed which will incorporate all of the advantages of the discounted cash flow method and also predict the possibility of

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[15] Hertz, p. 98.

various outcomes. There is a definite need for the use of a measure of acceptability such as that proposed by the Lutzes, which incorporates the concept of probability distributions.

The presently suggested methods of measuring investment acceptability which incorporate the concept of probability distributions for the various cost and revenue factors are, however, not sufficient in that they all require the knowledge of the shape of the distribution or the cash flow of each factor, or at least the probability of various discrete values resulting from any factor. As has already been pointed out, the information necessary to apply these methods is nearly always not available, nor should management envision having such information. Another way of incorporating probability considerations must be found.

The problem of working with unknown probability distributions has been faced in other areas than investment analysis with satisfactory results. One such area is in projects management, where a tool known as Program Evaluation and Review Technique (PERT) is used as a means of forecasting the time which will be needed to complete a project. The total project time depends on many events, each having completion time distributions, just as the total cash flow from an investment depends on many events, each having a distribution of cash flows. It is possible to apply a PERT type approach to investment analysis. The remainder of this paper will deal with the application to the investment analysis process of some of the procedures which make up this management tool.

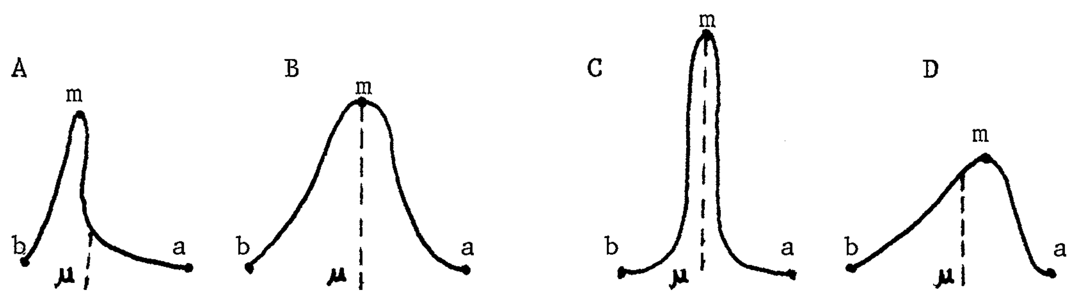
CHAPTER III

THE PROPOSAL OF A PROBABILISTIC METHOD OF INVESTMENT ANALYSIS

The development of a different approach to investment analysis which will take into account the probability distribution of the possible outcomes of an investment has centered around the discounted cash flow method in order to maintain the advantages that it possesses. The approach also includes some parts of the method of Program Evaluation and Review Technique, thereby allowing the single-valued solution normally associated with the discounted cash flow method to be replaced with a description of all possible outcomes.

A General Discussion of the Proposed Method

The "expected value method," as the author has chosen to call it, requires three estimates for each factor that will affect the cash flow resulting from the investment. These estimates of the high, most likely, and low possible values are combined in a set of equations which give as their solutions an approximation of the mean and variance of the distribution for each factor. This relationship is shown in Figure 1. It is necessary to typify the results in terms of their mean and variance in order to make statistical inferences about the many possible actual outcomes. The mean (μ) or expected value is a statistical term that corresponds to the average of a set of values



Problem: Given three estimates of cash flow, find μ (mean) and σ^2 (variance) of the distribution when the distribution form varies as shown above.

a--Optimistic Estimates of the Interval
m--Most Likely Flow of the Interval
b--Pessimistic Estimate of the Interval

} Obtained for Each Interval

Solution: The mean and variance for the range of distributions to be encountered can be approximated by the use of the following equations applied to each interval.

$$\mu = \frac{a + 4m + b}{6}$$

$$\sigma^2 = \frac{(b - a)^2}{36}$$

Fig. 1. Determining the Mean and Variance of Cash Flow Distributions

such that, for a random variable,

$$\mu = \sum_{i=1}^{n} X_i P(X_i) = \bar{X}$$

where X_i is the ith value of the random variable and $P(X_i)$ is the probability of the ith value occurring. The variance (σ^2) or square of the standard deviation (σ) is commonly used as a measure of dispersion. It is defined as the sum of the squared deviations of the values of the random variable from its mean, weighted by the probability of the deviation such that

$$\sigma^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 P(X_i)$$

where X_i is the ith value of the random variable, $\bar{X}$ is the mean, and $P(X_i)$ is the probability of the ith value occurring.[1] In this way the basis for the consideration of probability is added to the process without the requirement of a prior knowledge of complete distributions for each cash flow.

The means and variances for all cash flows of a given year can be summed individually to give a mean and variance for the flow for each year. The mathematical reasoning behind this is presented in Appendix A. The yearly values in turn are discounted just as in the discounted cash flow method. Then a mean and variance for the life of the project are derived simply by adding the discounted yearly values. This mean and variance for the life of the investment will describe the distribution of the possible outcome. As will be

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[1]Harold Bierman, Lawrence E. Fouraker, and Robert K. Jaedicke, Quantitative Analysis for Business Decisions (Illinois, 1961), p. 20.

discussed later, this distribution is approximated by a normal distribution having the same mean and variance. Therefore, it is quite simple to find the cumulative probability of reaching any given rate of return by consulting a normal distribution table. Consequently, management may be presented with these two parameters, the mean and the variance, which together provide a description of the expected return from an investment and the probability of realizing any particular rate of return. Also of major importance is the fact that neither the knowledge of probability distributions nor the use of simulation is required, thus simplifying the process in comparison to other methods of applying probability considerations.

The Proposed Method in Detail

The procedure used in applying the expected value method may be broken down into four steps.

Step 1. Obtaining Estimates

The first thing to be done is to determine the variables that will affect the cash flow for each year. These variables may include such things as the price of labor, the cost of raw materials, the demand for goods produced, or the salvage values. The more variables considered, the more accurate the overall picture will be. Estimates of the three different cash flows must be made for each of these variables. The first of the estimates is that of the most likely flow. The value of this estimate will be interpreted technically as the mode of the flow distribution. The second estimate gives the most optimistic possibility. The value of this estimate should be the most favorable cash flow which the company could realistically expect. The remaining estimate is that of the pessimistic flow. This is the worst flow that could be experienced barring an unforeseen disaster.

Since it will often be difficult to secure more than a single authoritative estimate, the problem of personal bias is serious. Therefore, the estimates should be made following careful explanations of exactly what is needed. These explanations should be made by a highly qualified interviewer. Then when the company becomes certain that the estimator understands just what is wanted, the making of these estimates may be incorporated into the normal business routine.

Step 2. Obtaining the Yearly Mean and Variance

After the individual estimates have been taken, they must be combined into a yearly mean and variance. There are two ways of doing this. The obvious way is to calculate the mean and variance for each variable for a given year by using the equations to be presented. The means and the variances may then be summed to give the yearly values for each. A mathematical proof of this statement is presented in Appendix A. The same results may be obtained by independently summing all most likely, pessimistic, and optimistic figures for the variables affecting that year and then applying the three totals to the equations for the mean and variance only once. This shortens the number of calculations considerably and is the procedure followed in this paper. In either case one handles the costs (outflows) as negative values and the revenues (inflows) as positive.

Equations for the mean and variance are based on studies done by the United States Bureau of Naval Weapons in connection with the development of Program Evaluation and Review Technique. In PERT the same three types of estimates are taken for event times. The total project completion time depends on the actual combination of event time values just as the total return from an investment depends on the actual combination of cash flow values.[2] In both PERT and the expected value method, it is assumed that there is relatively little chance of realizing either the pessimistic or optimistic value. It is also assumed that the most likely value may lie anywhere between the two end points of the range.

The variance (σ^2) is given by

$$\sigma^2 = \frac{(a - b)^2}{36}$$

where a is the optimistic estimate and b is the pessimistic estimate. The equation is a simplification of that used to find the variance of any unimodal distribution. Since the standard deviation of such a distribution is roughly one-sixth of the range, the standard deviation for the cash flow should be one-sixth of the difference between the optimistic and pessimistic estimates.[3]

The mean (μ) is given by

$$\mu = \frac{(a + 4m + b)}{6}$$

where a is the optimistic estimate, m, the most likely estimate, and b, the pessimistic estimate. The equation is derived from the beta

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[2] Program Evaluation Research Task, Summary Report, Phase 1, Bureau of Naval Weapons, Department of the Navy, Washington, D. C. (July, 1958), pp. 5-9.

[3] Ibid., Appendix B(2).

distribution, and the selection of constants for this distribution is based in part on an attempt to correct for the inherent bias in the making of estimates. The derivation of this equation and a discussion of the bias correction are presented in Appendix B.

The three estimates for each variable, or for the year, are substituted into the two equations. By solving for the expected value and variance, the data are reduced to two quantities which describe the distribution for that year.

Step 3. Discounting the Expected Values

The expected value for each year is treated in the same manner as the single yearly estimates of the discounted cash flow method. Various discount rates are applied to these estimates until one is found which equates the discounted future earnings of the investment to its discounted costs. This rate is considered the rate of return from the investment.

Step 4. Determining the Variance of the Cash Return on Investment

In order to add the total variance to the solution, the variance for each year is multiplied by the square of the present value factor associated with the year and the rate that was found in Step 3. The resulting values are then added together to give a total variance for the cash return on investment. The reasoning behind the method of discounting the yearly variance is presented in Appendix C. A proof that the sum of the yearly variances gives the variance of the cash return on investment is presented in Appendix A. This total variance, the total cash flow, and the rate of return associated with them make up the solution to a problem handled by the expected value method.

Using the Final Solution of the Proposed Method

The final solution will be in the form of a cash value for the expected cash flow total amount and a cash value for the variance, both in present value terms and both associated with a specific rate of return. The distribution of the many actual outcomes possible is approximated by a normal distribution having the mean and variance of the solution. The use of the normal distribution as an approximation of the distribution of the outcome is based on the central limit theorem, and the use of that theorem here is discussed in Appendix D. The accuracy of this approximation depends on the number of variables which have been evaluated in reaching the final solution. Again because of the central limit theorem, as the number of the variables for which estimates are made increases, the accuracy of the final approximation generally also increases. If the number of variables considered is small, the use of the normal distribution as an approximation is no longer valid. (See Appendix D.) However, since a well handled analysis of an investment opportunity requires the consideration of as many variables as are practical, the possibility of a poor approximation is considerably reduced.

A normal distribution table will provide the probability of any dollar variation from the expected dollar value (μ_T). If it is desirable to give the expected value and variance in terms of percent return, it is quite easy to convert such information from the final solution to the percent form. The variance in percent is given by

$$\left(\frac{\mu_{\%}}{\mu_T}\right)^2 \times \sigma_T^2 = \sigma_{\%}^2$$

where μ_T is the expected total cash flow, $\mu_{\%}$ is the expected rate of return, σ_T^2 is the cash flow variance, and $\sigma_{\%}^2$ is the variance in percent.[4]

The solution of the expected value method contains all of the information that is presented by the use of the discounted cash flow method. It fulfills the requirement of showing the flow of costs and revenues over the life of the investment. It also presents, with a minimum of effort, the possibility and probability of outcomes other than that of the expected return.

An Illustration of the Proposed Method

In order to demonstrate the use of the expected value method, a hypothetical investment proposal is analyzed in Table I. The exhibit data contain the yearly estimates of cash flows which are totals of all the estimates for each year. The investment requires an immediate outlay of one thousand dollars to purchase assets which have a probable life of either nine or ten years. Table I also presents the expected values and variances of the three estimates for each year. These parameters were found by substituting the estimates into the equations for μ and σ^2.[5]

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[4]When a distribution with mean (μ) and variance (σ^2) is multiplied by a constant (c), the result is a distribution with mean ($c\mu$) and variance ($c^2\sigma^2$). If $(\mu_T)(c) = \mu_{\%}$, then $(c^2 \sigma_T^2) = \sigma_{\%}^2$.

[5]It can easily be seen that the expected values and most likely values are not always the same. However, this should be anticipated since the expected value represents the average value of the distribution while the most likely estimate represents the high point or mode of that distribution. The mode and mean of a distribution do not have to be equal.

TABLE I

A PROPOSED INVESTMENT

Immediate Investment Required = $1000.00

	Yearly Totals of Estimates of Cash Flow				
Year	Pessimistic Estimate	Most Likely Estimate	Optimistic Estimate	Expected Value	Variance
1	$ 10	$ 50	$ 70	$ 46.67	$ 100
2	100	150	170	145.00	136
3	250	300	350	300.00	278
4	300	400	450	391.67	625
5	350	400	450	400.00	278
6	350	400	450	400.00	278
7	350	400	450	400.00	278
8	150	300	400	291.67	1,740
9	75	100	150	104.12	156
10	0	50	70	45.00	136

TABLE II

EVALUATION BY THE DISCOUNTED CASH FLOW METHOD

Year	Cash Flow	Factor @20%	Present Worth @20%	Factor @22%	Present Worth @22%
1	$ 50	.833	$ 41.65	.820	$ 41.00
2	150	.694	104.10	.672	100.80
3	300	.579	173.70	.551	165.30
4	400	.482	192.80	.451	180.40
5	400	.402	160.80	.370	148.00
6	400	.335	134.00	.303	112.12
7	400	.279	111.60	.249	99.60
8	300	.233	69.90	.204	61.20
9	100	.194	19.40	.167	16.70
10	50	.162	8.10	.137	6.85
			$1016.05		$ 931.97
			-1000.00		-1000.00
		Approx. =	$ 16.00		$- 68.03

$$\frac{(16.00)(2)}{16.00 + 68.03} = .38$$, giving an estimated return of 20% + .38%

Discounted Cash Flow Rate of Return = 20.38%.

TABLE III

EVALUATION BY THE PROPOSED METHOD

Year	Expected Value	Factor @20%	Present Value	Variance	Factor Squared	Present Value
1	$ 46.67	.833	$ 38.88	$ 100	.694	$ 69.40
2	145.00	.694	100.63	136	.482	65.55
3	300.00	.579	173.70	278	.335	93.13
4	391.67	.482	188.78	625	.232	145.00
5	400.00	.402	160.80	278	.162	45.03
6	400.00	.336	134.00	278	.112	31.14
7	400.00	.279	111.60	278	.078	21.68
8	291.67	.233	67.96	1740	.054	93.96
9	104.12	.194	20.20	156	.038	5.93
10	45.00	.162	7.29	136	.026	3.54
			$1003.84			$574.36

Approx. = 1000.

$$\sigma^2 = \left(\frac{20\%}{1003.84}\right)^2 \times 574.36 = .23\%$$

The proposed investment has a mean expected rate of return of twenty percent with a variance of .23 percent.

In Table II the investment is evaluated using the regular discounted cash flow method. Here the most likely estimate has been used as the cash flow amount. The rate of return is found to be slightly greater than twenty percent.

Table III presents the result of the application of the expected value method. Here the expected values and variances are discounted, and their present values are totaled. The final results give an expected rate of return of twenty percent with a variance of .23 percent. This means that there is a probability of .5 that the investment will show a rate of return of twenty percent or greater. Because of the small variance, there is very little chance of a deviation of more than one percent in either direction from the expected rate. By consulting a normal distribution table, it is found that the probability of such a deviation is only .04. Therefore, if a twenty percent return is satisfactory, the investment is seen as sound with little chance of its not succeeding.

If the results of the application of the expected value method had yielded a different variance, the soundness of making the investment might be questioned. Suppose that the variance was found to be nine percent rather than .23 percent. If the cut-off point had been set at eighteen percent, then there would be a probability of .25 of the return being substandard. Management would have to decide if the investment were worth the risk.

The amount of the expected rate of return should also be compared with that found by using the discounted cash flow method. Although the two are approximately equal in this particular example, this is seldom the case. How close these two values are clearly depends upon how close the most likely and expected values are for each year.

The Importance of Each Estimate

In many instances there will be quite a bit of uncertainty regarding the validity of the estimates even if extra care has been taken in obtaining them. It is a simple matter to find out what effect the variables having estimates with a high degree of uncertainty have on the final solution. If there is a question about any variable, its estimated values may be changed, causing a change in the final solution. The amount of change appearing in the solution is an indicator of the degree to which any given variable will affect the total return. It may easily turn out that a factor about which very little is known is causing management a lot of worry when it actually has little effect upon the long-range outlook.

CHAPTER IV

CONCLUSION

There are several methods of evaluating alternative investment opportunities in use today, ranging from the simple use of common sense to complex processes which utilize probability considerations and simulation techniques. While present methods based on the time adjusted rate of return principle coupled with considerations of probability appear to offer the greatest amount of information in such analyses, their usefulness is limited. Each requires a knowledge of the distribution of cash flows for each cost and revenue factor, and management often does not have such information available. There is a need for a method that retains the advantages of presently used methods while providing a more practical means of considering probability distributions.

The probability consideration has been used successfully in some areas. One of these areas, projects management, provides a method which utilizes a similar structure of data as that required in investment analyses of the discounted cash flow type. By applying part of this method to the discounted cash flow method, the author has developed an investment analysis tool which recognizes the time value of money and takes into consideration the variation of possible outcomes from a given investment. This tool differs from those in use since it does not require any knowledge of factor distributions. It can also function in almost any investment analysis, and it is simple to apply.

This expected value method which the author is proposing consists of four steps, as follows:

1. Obtain estimates of the most likely, pessimistic, and optimistic cash flows for each variable which affects the total cash flow from an investment.

2. Sum each type of estimate to give the yearly values of the pessimistic, most likely, and optimistic flows, treating costs as negative and revenues as positive. Calculate the expected value or mean (μ) and the variance (σ^2) for each year by using the following relations:

$$\mu = \frac{a + 4m + b}{6}$$

$$\sigma^2 = \frac{(a - b)^2}{36}$$

where a is the optimistic estimate, b, the pessimistic estimate, and m, the most likely estimate.

3. Discount the expected values to a present value equal to the cost of the investment.

4. Calculate the variance of the present value found in Step 3 by multiplying the variance for each year by the square of the discount factor for that year and then totaling the resulting products.

This method, while requiring only a little more information than is necessary for the frequently used discounted cash flow method, offers the decision-maker a chance to discriminate between investments with expected values based on all possible outcomes and to evaluate risk not only between investments, but also within a single investment. It enables management to have this added information while requiring the addition of only one more value, the variance, to the final set needed in making an analysis of alternatives.

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APPENDIX A

THE CALCULATION OF THE VARIANCE[1]

Let $x_1, x_2, \ldots, x_n$ be a set of independent variables with means $u_1, u_2, \ldots, u_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ and let

$$z = x_1 + x_2 + \ldots + x_n$$

Now

$$u_z = E(z)$$

and

$$z - u_z = (x_1 - u_1) + (x_2 - u_2) + \ldots + (x_n - u_n)$$

It follows that for $i \neq j$

$$E(z-u_z)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} E(x_i - u_i)(x_j - u_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} E(x_i - u_i)\ E(x_j - u_j)$$

But, $E(x_i - u_i) = 0$; therefore,

$$E(z-u_z)^2 = \sum_{i=1}^{n} E(x_i - u_i)^2$$

Since $E(x_i - u_i)^2 = \sigma_i^2$, this result can be expressed as

$$z^2 = \sum_{i=1}^{n} \sigma_i^2$$

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[1]Paul G. Hoel. Introduction to Mathematical Statistics, John Wiley & Sons, Inc., New York, 1962, p. 136.

APPENDIX B

THE DERIVATION OF THE EQUATION FOR THE EXPECTED VALUE[1]

Assumptions:

1. The standard deviation is approximated by one-sixth of the range.
2. The beta distribution is an adequate model of the distribution of a cash flow.

The beta distribution used is one having its mode as the most likely value, its range as the interval between the optimistic and pessimistic values, and its standard deviation as one-sixth of the range. The probability density is

$$f(t) = (\text{Constant})(t-a)^{\alpha}\,(b-t)^{\gamma}$$

where _t_ is the cash flow, _a_, the pessimistic estimate, and _b_, the optimistic estimate; and _k_ (the constant), α, and γ are functions of _a_, _b_, and _M_ (the most likely flow).

To reduce the probability density to the standard form of the beta distribution, let _x_ be a random variable such that

$$x = \frac{t - a}{b - a}$$

If _r_ is the mode of _x_, then

$$r = \frac{M - a}{b - a}$$

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[1] _Program Evaluation Research Task, Summary Report, Phase 1_, Bureau of Naval Weapons, Department of the Navy, Washington, D. C., (July, 1958), Appendix B.

and also

$$r = \frac{\alpha}{\alpha + \gamma}$$

If E(x) and V(x) are respectively the expected value and variance of $\underline{x}$,

$$E(x) = \frac{\alpha + 1}{\alpha + \gamma + 2}$$

and

$$V(x) = \frac{(\alpha + 1)(\gamma + 1)}{(\alpha + \gamma + 2)^2(\alpha + \gamma + 3)}$$

If the variance of $\underline{t}$ is $(b-a)^2/36$, the variance of $\underline{x}$ is 1/36 or

$$V(x) = 1/36$$

By eliminating γ from $\underline{r}$ and V(x), the following is obtained:

$$\alpha^3 + (36r^3 - 36r^2 + 7r)\alpha^2 - 20r^2\alpha - 24r^2 = 0$$

Given $\underline{M}$, $\underline{a}$, and $\underline{b}$, and that E(t) = a + (b-a)E(x), the values of $\underline{r}$, α, and γ can be calculated from these formulas. When E(x) is plotted as a function of $\underline{r}$, it is found that the relation between these variables is approximately linear. A satisfactory approximation of this is

$$E(x) = (4r + 1)/6$$

From the relations between E(x) and E(t) and between $\underline{r}$ and $\underline{M}$, the approximation reduces to

$$E(t) = (a + 4M + b)/6$$

This means that E(t) is the weighted mean of $\underline{M}$ and the mid-range (a + b)/2, with weights of 2 and 1 respectively. In other words, E(t) is located one-third of the way from the likely flow $\underline{M}$ to the mid-range.

The placement of E(t) is a result of an attempt to correct for bias on the part of the estimator. The Bureau of Naval Weapons found that the value of the constants used in the beta distribution as described here best adjusted for this bias. Individual experience will show if the weighted mean is appropriate in any given situation and should also suggest what changes, if any, should be made in its placement. However, this equation for E(t) has proved to be a sound starting point.

APPENDIX C

DISCOUNTING OF THE MEAN AND VARIANCE

One way of presenting correctly the data of the expected value method in present value terms is to discount each estimate before any calculations are made. If the three estimates are a, m, and b, and the discount factor is c, then the values to be used in the equations for the expected value and variance are ca, cm, and cb. The equations become

$$\mu = (ca + 4cm + cb)/6 = c(a + 4m + b)/6$$

and

$$\sigma^2 = (ca - cb)^2/36 = c^2(a - b)^2/36$$

Therefore, it is just as correct to calculate the mean and variance of the cash flow for each year and then multiply the values respectively by the discount factor and the discount factor squared as it would be to discount each estimate before making any calculations.

APPENDIX D

THE NORMAL DISTRIBUTION APPROXIMATION

The assumption that the mean and variance of the solution of the expected value method describe a normal distribution is based on the central limit theorem. One form of this theorem states that

> If $\bar{z}$ is the sum of n independent random variables all having the same probability distribution, then as n increases the distribution of $\bar{z}$ is more and more closely approximated by a normal distribution with mean equal to the expected value of $\bar{z}$ and variance equal to the variance of $\bar{z}$; and this is true regardless of the nature of the distribution of the individual variables.[1]

This statement also applies under certain conditions even to sums of independent variables which do not have identical individual distributions.[2] The mathematical identification of such conditions can be found in most advanced statistics texts. While these conditions will not be presented here, it is true that they are met by the structure of the data presented by the expected value method.[3] Therefore, the normal distribution is used as an approximation of the possible outcomes, and the accuracy of the approximation depends directly on the number _n_ of the independent random variables that have been included.

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[1] Robert Schlaifer. _Probability and Statistics for Business Decisions_ (New York, 1959), p. 285.

[2] Ibid.

[3] _Program Evaluation Research Task, Summary Report, Phase 1_, Bureau of Naval Weapons, Department of the Navy, Washington, D. C. (July, 1958) p. 12.

VITA

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Report: A PROBABILISTIC APPROACH TO INVESTMENT ANALYSIS

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Scope and Method of Study: This report has been undertaken to introduce probability considerations in evaluating prospective investments. The scope of the report is limited to the consideration of the type of investment made by a profit motivated firm. The method of study includes a survey of journals and books which present currently used methods of investment analysis and suggestions about future analytical approaches. The main emphasis of this report is placed on a presentation of a method of analysis which yields a complete probability distribution of the outcome from any given investment.

Findings and Conclusions: A firm is able to choose more wisely an investment that will increase its net worth if it knows the probability of realizing any given return from each investment it considers. By using the method of investment analysis presented in this report, management can derive estimates of the most likely rate of return and of the probability of realizing any other rate. The procedure requires three estimates for each revenue and cost expected during the life of the investment. It also requires the use of the beta distribution to approximate the actual distribution of each of these costs and revenues. The final result is a distribution of the expected rate of return approximated by the normal distribution. Such a solution, containing only two parameters, the mean and the variance, describes all possible outcomes and the probability of their occurrence.

ADVISER'S APPROVAL ______________________