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Title of Study: BENDING AND VIBRATION OF PLATES BY FINITE ELEMENT METHOD.

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Scope of Study: The purpose of this study is to use finite element approach to bending and vibration problems and compare results obtained with those of the classical approach.

The finite element method assumes to start with either stress distribution or displacement distribution. In this report both the appraoches are utilized and results compared for the case of a square plate fixed on four sides. Only rectangular elements are used.

Conclusions: The results obtained from both approaches compare well with the results obtained by classical approach for the specific problems discussed.

The finite element method is suitable for computer use.

ADVISER'S APPROVAL ________________________________

BENDING AND VIBRATION OF PLATES

BY FINITE ELEMENT METHOD

By

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Deepak Laxman Naik

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NOMENCLATURE

The following symbols have been adopted for use in this report:

A ....Coefficients, assumed displacement field;

a, b ....Plate element dimensions;

D ....Modulus of rigidity of plate;

E ....Modulus of elasticity;

F ....Nodal actions;

M_x, M_y ....Moment stress resultants;

Q_x, Q_y ....Transverse shear stress resultants;

q ....Displacement when stress field is assumed;

t ....Plate thickness;

U ....Strain energy;

$u, v, w,$ ....Displacements;

v ....Displacement when displacement field is assumed;

W ....External work;

$x, y, z,$ ....Coordinates

β ....Coefficients of assumed stress field;

γ ....Shear strain;

ϵ ....Extensional strain;

ϕ ....Rotation about x axis;

θ ....Rotation about y axis;

λ ....Eigen value for vibration

ν ....Poisson's ratio;

σ_x, σ_y, ...Normal stresses;

τ_{xy} ....In-plane shear stresses;

τ_{xz}, τ_{yz}, ...Transverse shear stresses;

ρ ....Mass density.

CHAPTER I

INTRODUCTION

1.1 Statement of the Problem

The finite element method is applied for bending and vibration of rectangular plates of uniform thickness. The method of approach is developed for assumed displacement function as well as assumed stress function. Numerical examples are given for both assumptions and results obtained are compared with those obtained by classical approach.

1.2 Review of Literature

The finite element method has been recently developed by Turner, Clough (1) and others. Two approaches to calculate the elemental matrices reflecting the element characteristics are available. This can be done by either assuming a displacement field over the element or a stress field. However, most of the work done in this area utilizes an assumed displacement field. Among those who used an assumed displacement field are Zienkiewicz and Cheung (2) and Melosh (3) who analyzed bending of plates, Kapur and Hartz (4) who analyzed stability of plates. Also, Dawe (5) investigated the vibration of plates. L. Archer (6) derived the mass matrix for vibration of bars. Pian (7) was the first to suggest an assumed stress field. In a later paper Severn and Taylor (8) analyzed problems of bending of plates by assuming stress fields over the element.

In their work they presented a comparison of results for rectangular and triangular elements. Recently Lundgren (9) extended the same principle to analyze stability of multilayer sandwich plates. Also R. A. Apanian (10) used it for analyzing bending of hyperbolic paraboloid shell structures.

The purpose of this report is to present a comprehensive account of the finite element method as applied to bending and free vibration of rectangular plates and to compare results obtained when stress field is assumed versus results obtained when a displacement field is assumed.

CHAPTER II

PHILOSOPHY OF FINITE ELEMENT METHOD

2.1 General

The finite element method is a generalized procedure which permits the analysis of two or three dimensional structures by the same techniques which are applied in ordinary framed structures.

Originally developped in aircraft industry, this method is now finding increasing use in non-aeronautical applications. The success of this method is due to its ability to obtain solutions to problems which are difficult to handle by other methods.

2.2 Basic Concepts

The basic principles of the method are as follows:

The structure is physically subdivided into small element of finite dimensions of such shapes and in such a manner so that the assemblage of the elements would resemble the original structure as closely as possible. It is this finite character of structural connectivity which makes possible the analysis by means of a matrix methods. Also it distinguishes a structural system from problems of continuous mechanics.

With the properties of the material being known, the response characteristics of the element to a given type of loading, static or dynamic can be established in terms of the displacements and slopes of of the element at the nodes. This can be done by assuming either displacement or stress fields over the elements. By applying the well-established energy principles, the desired characteristics, known as the elemental matrices, can be obtained. The elemental matrices for all neighboring elements are then related through the generated coordinates (displacements and slopes) and applied forces at the nodes. Finally, the elements are assembled to yield the structural matrices reflecting the response characteristics of the structure as a whole to the given forces. The desired information then can be found from these last matrices. If the assembled structure behaves exactly like the original structure then an exact solution is achieved. This can be done for one dimentional structure, but is not quite possible for other complex structures. Therefore, the result, as expected, is an approximate solution.

The technique: The techniques used to attack this problem are general and are the same for one, two or three dimensional structures.

The following three requirements are to be satisfied simultaneously, as in any structural problem.

(a) Equilibrium -- the internal element forces acting at each nodal point must equal the externally applied nodal forces.

(b) Compatibility -- the element deformation must be such that elements continue to meet at the nodal points.

(c) Force-deflection relationship -- the internal forces and internal displacements within each element must be as required by the individual geometric and material property characteristics of each element.

The superiority of the method over other methods can be summarized in the following.

1. Discontinuities in the geometry of the structure such as openings, reentrant corners, thickness, and shape can be handled very easily. The same can be said about considerations for thermal stresses and material properties.
2. The insertion of prescribed boundary conditions is a trivial matter.
3. The stiffness matrices obtained are well conditioned compared with those obtained by a method such as finite difference.
4. The source of errors are generally controlable, and the results are usually quite satisfactory.
5. The governing differential equation of the problem need not be known to solve the problem.

CHAPTER III

OUTLINE OF METHOD ASSUMING DISPLACEMENTS

3.1 Bending

Consider a rectangular element as shown in Figure 1 and assume the displacement over the element to take the form

$$W = A_1 + A_2\frac{x}{a} + A_3\frac{y}{b} + A_4\frac{x^2}{a^2} + A_5\frac{xy}{ab} + A_6\frac{y^2}{b^2} + A_7\frac{x^3}{a^3}$$

$$+ A_8\frac{x^2y}{a^2b} + A_9\frac{xy^2}{ab^2} + A_{10}\frac{y^3}{b^3} + A_{11}\frac{x^3y}{a^3b} + A_{12}\frac{xy^3}{ab^3} \qquad (1)$$

where

a and b are plate element dimensions in x and y direction respectively.

x , y are co-ordinate distances from origin and A_1, A_2, ----- A_{12} are constants to be determined.

In matrix notation, this can be written as

$$\{W\} = [m]\{A\}$$

The constants $\{A\}$ can be evaluated in terms of nodal displacements and slopes by substituting the co-ordinates of nodes in equation (1) and its partial derivatives as follows;

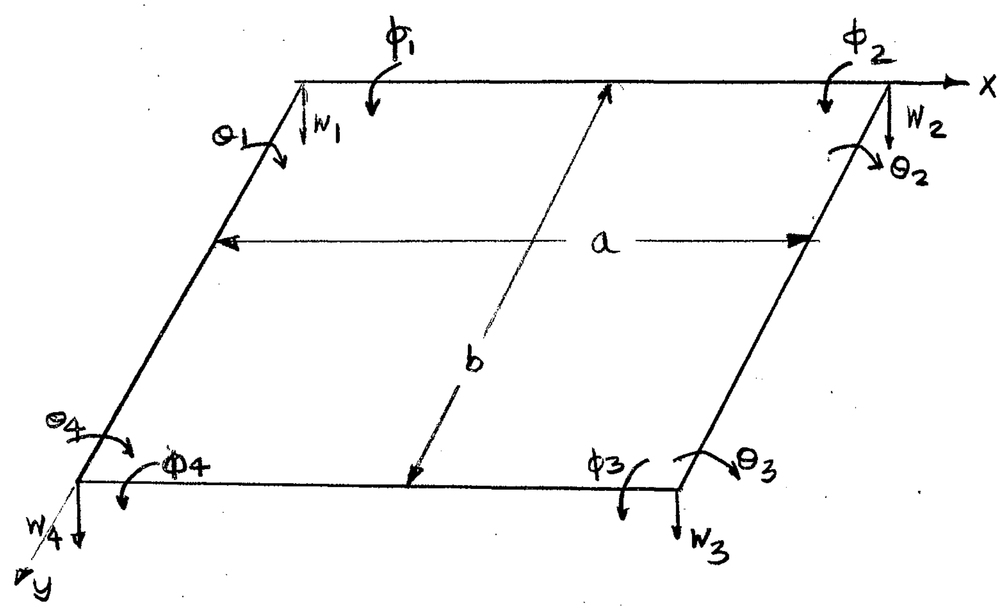

Figure 1. Rectangular Element Assuming Displacements

$$W = W_i \qquad \frac{\partial w}{\partial y} = \phi_i \qquad \frac{\partial w}{\partial x} = \Theta_i \text{ at the node points}$$

Figure (1) shows the displacements and slopes at the node points Inserting these conditions, and denoting the column matrix of nodal displacements and shapes by v ;

$$\{v\} = [B]\{A\}$$

or

$$\{A\} = [B^{-1}]\{v\}$$

where

$$[B^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & -2a & 3 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -b & -a & 1 & b & 0 & -1 & 0 & 0 & 1 & 0 & a \\ -3 & -2b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -b & 0 \\ 2 & 0 & a & -2 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 2a & -3 & 0 & a & 3 & 0 & -a & -3 & 0 & -2a \\ 3 & 2b & 0 & -3 & -2b & 0 & 3 & -b & 0 & -3 & b & 0 \\ 2 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & b & 0 \\ -2 & 0 & -a & 2 & 0 & -a & -2 & 0 & a & 2 & 0 & a \\ -2 & -b & 0 & 2 & b & 0 & -2 & b & 0 & 2 & -b & 0 \end{bmatrix}$$

The bending strain energy of an isotropic plate of rigidity D is given by

$$U = \frac{D}{2}\int_0^b\int_0^a (W_{xx})^2 + (W_{yy})^2 + 2\nu W_{xx}\cdot W_{yy} + 2(1-\nu)(W_{xy})^2 \; dxdy$$

In matrix form,

$$U = \frac{D}{2} \int_0^b \int_0^a [C]^T [D] \{C\} \; dxdy \qquad (3)$$

where

$$\{C\} = \{W_{xx}, W_{yy}, W_{xy}\}$$

$$W_{xx} = \frac{\partial^2 W}{\partial x^2}, \quad W_{yy} = \frac{\partial^2 W}{\partial y^2}, \quad W_{xy} = \frac{\partial^2 W}{\partial x \partial y}$$

and

$$[D] = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 2(1-\nu) \end{bmatrix}$$

The curvatures (W_{xx} etc.) can easily be obtained by differentiating equation (1)

Thus in matrix form,

$$\{C\} = [E]\{A\} = [E][B^{-1}]\{v\} \qquad (4)$$

where

$$[E] = \begin{bmatrix} 0 & 0 & 0 & \frac{2}{a^2} & 0 & 0 & \frac{6x}{a^3} & \frac{2y}{a^2 b} & 0 & 0 & \frac{6xy}{a^3 b} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{b^2} & 0 & 0 & \frac{2x}{ab^2} & \frac{6y}{b^3} & 0 & \frac{6xy}{ab^3} \\ 0 & 0 & 0 & 0 & \frac{1}{ab} & 0 & 0 & \frac{2x}{a^2 b} & \frac{2y}{ab^2} & 0 & \frac{3x^2}{a^3 b} & \frac{3y^2}{ab^3} \end{bmatrix}$$

Substitution of equation (4) into equation (3) gives

$$U = \frac{D}{2}[v]^T \left\{ [B^{-1}]^T \left(\int_0^b \int_0^a [E]^T [D] [E]\, dxdy \right) [B^{-1}] \right\} \{v\}$$

or in matrix form

$$U = [v]^T [K] \{v\} \tag{5}$$

Applying Castigliano's theorem,

$$\frac{\partial U}{\partial V_i} = F_i$$

Differenting equations (5) where F_i are actions corresponding to displacements and slopes

$$\{F\} = [K]\{v\} \tag{6}$$

where

$$\{F\} = \{Q_1,\ M_{y_1},\ M_{x_1},\ Q_2, \text{- - - - - - - - -}, M_{x4}\}$$

and K = Elemental stiffness matrix is given by

$$K = \frac{D}{2} \left\{ [B^{-1}]^T \left(\int_0^b \int_0^a [E]^T [D][E]\, dxdy \right) [B^{-1}] \right\}$$

The $[K]$ matrix is listed in appendix 1 a.

The actual loading can be transferred as nodal loading N^A. The criterion is that the work done by the actual loading should be equal to work done by equivalent nodal loading.

Consider a virtual displacement $\delta v^A = I$

The work done by nodal forces is $W_N = (\delta v^A)N^A = IN^A = N^A$

And the work done by distributed loading q is

$$W_D = \int_0^b \int_0^a (\delta W)^T \quad qdxdy$$

since

$$W = [m]\{A\}$$

therefore

$$\delta W = [m][B]^{-1}\delta v^A = [m][B]^{-1}$$

and

$$W_D = N^A = [B]^{-1^T} \int\int [m]^T \, qdxdy \qquad (7)$$

For uniform load q , on a rectangular plate

$$N^A = qab\left\{\frac{1}{4}, \frac{b}{24}, \frac{a}{24}, \frac{1}{4}, \frac{b}{24}, \frac{-a}{24}, \frac{1}{4}, \frac{-b}{24}, \frac{-a}{24}, \frac{1}{4}, \frac{-b}{24}, \frac{a}{24},\right\}$$

The structural stiffness matrix is obtained by assembling the elemental matrices. The force vector F becomes 'External Load Vector' which is a known quantity.

Therefore $\{v\} = [K]^{-1}\{F\}$ gives the displacements. Knowing the displacements, the actions at nodes are found with the help of the elemental stiffness matrix.

This completes the solution to the problem of bending.

3.2 Vibrations

The equation of motion for a structure which is freely vibrating sinusoidally with a circular frequency P is

$$[K]\{v\} - \rho P^2 [\bar{M}]\{v\} = 0 \qquad (8)$$

where

$[K]$ is the previously defined stiffness matrix

and

$[\bar{M}]$ is the mass matrix whose elements represent the inertia forces at the i^{th} node due to a unit acceleration at the j^{th} node.

The mass matrices can be formed by replacing the inertia loading distributed over the element by equivalent nodal inertia forces. The criterion adopted here is that external work done by the equivalent nodal inertia forces $\{F_{in}\}$ in moving through $\{V\}$ is equal to the external work done over whole element area by actual distributed inertia loading in moving through a virtual deflection.

In matrix form,

$$\{W_v\} = [m] [B]^{-1} \{V_v\} \tag{9}$$

Where the subscript v stands for virtual displacement.

The inertia force per unit area of the plate element is

$$\{f\} = \rho p^2 \{W_v\} = \rho p^2 [m] [B]^{-1} \{V_v\} \tag{10}$$

Thus equating work done in both cases

$$[V_v]^T \{F_{in}\} = \int_0^b \int_0^a [W_v]^T \{f\} dxdy$$

$$= \rho p^2 [V_v]^T [B]^{-1T} \left(\int_0^b \int_0^b [m]^T [m] dxdy \right) \cdot [B]^{-1} \{V_v\} \tag{11}$$

Each virtual displacement can be given the value of unity while other displacements are held equal to zero. That is,

$$\{V_v\} = [I] \qquad \text{unit matrix}$$

Therefore, equation (10) becomes

$$\{F_{in}\} = \rho P^2 [B^{-1}]^T \left(\int_{y=0}^{b} \int_{x=0}^{a} [m]^T [m] \, dxdy \right) [B^{-1}] \{V\}$$

$$= \lambda [M] \{V\}$$

where λ is proportioned to P^2,

$[M]$ is the elemental mass matrix and $\{F_{in}\}$ column matrix of nodal forces acting due to inertia loading only. It is in the same order as $\{F\}$ in the case of bending.

Matrix $[M]$ is listed in appendix 1b.

The structural mass matrix is obtained by assembling the elemental matrices as in the case of bending.

With the mass matrix obtained, equation (8) can be solved for the eigen values from which the natural frequencies can be calculated.

CHAPTER IV

OUTLINE OF METHOD ASSUMING STRESSES

4.1 Bending

Consider a rectangular element as shown in Figure (2a). At each of the nodes 1, 2, 3 and 4, the transverse displacement, W, and two components of rotations are specified. In Figure (2b) the actions on the sides of the element are shown. Having specified the direction of positive displacement and rotation in Figure (2a) it is important that the directions of the actions should be consistent with these directions.

At any point in the element there are five components of stress and they will now be assumend as follows :

$$\sigma_x = \left\{ B_1 + B_2 x + B_3 y + B_4 x^2 + B_5 xy + B_6 y^2 \right\} 8z/t \tag{12a}$$

$$\sigma_y = \left\{ B_7 + B_8 x + B_9 y + B_{10} x^2 + B_{11} xy + B_{12} y^2 \right\} 8z/t \tag{12b}$$

$$\tau_{xy} = \left\{ B_{13} + B_{14} x + B_{15} y + B_{16} x^2 + B_{17} xy + B_{18} y^2 \right\} 8z/t \tag{12c}$$

$$\tau_{zx} = \left\{ B_{19} + B_{20} x + B_{21} y \right\} (1-4z^2/t^2) \tag{12d}$$

and

$$\tau_{zy} = \left\{ B_{22} + B_{23} x + B_{24} y \right\} (1-4z^2/t^2) \tag{12e}$$

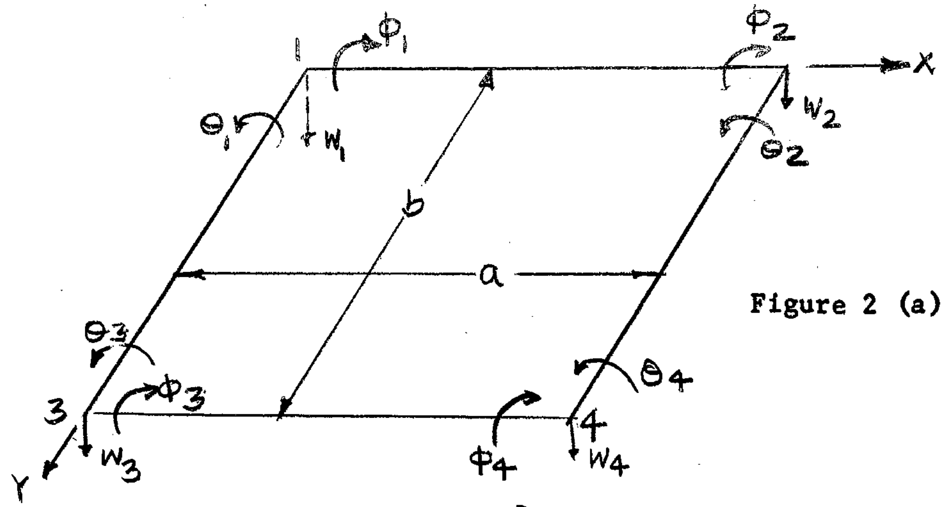

Figure 2 (a)

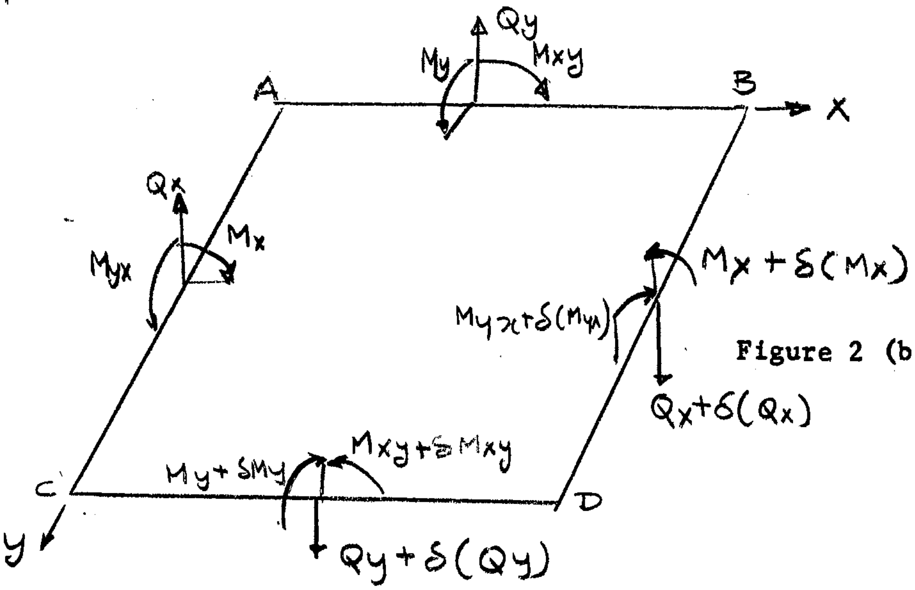

Figure 2 (b)

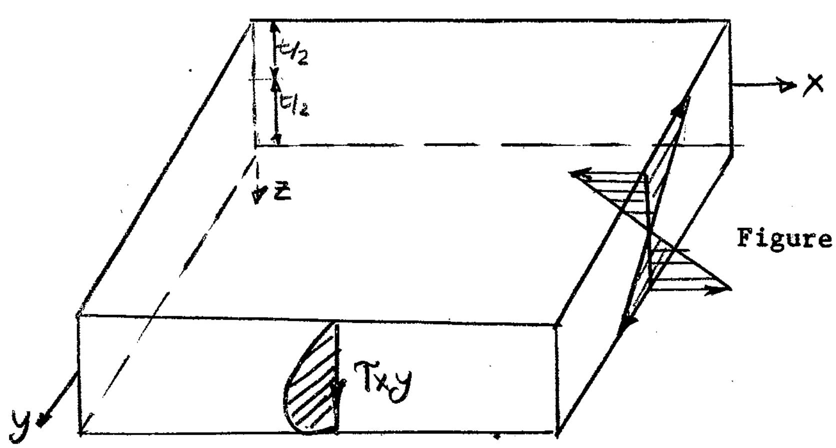

Figure 2 (c)

Figure 2. Rectangular Element Assuming Stresses

Where z is the co-ordinate direction transverse to the plane of the element, that is, in the direction of W. In equation (12) B_1, B_2, -- B_{24} are yet to be found. Also $\bar{x}$ and $\bar{y}$ are written to represent x/a and y/b respectively. In principle, no restriction is placed on the assumed form of stresses. Quadratic form is assumed for the three more important stress components. For the relatively minor shear stresses, τ_{zx} and τ_{zy}, a linear variation is assumed with the in-plane co-ordinates and a quadratic variation is assumed with z, where as the three major stress components are assumed to vary linearly with z. The assumptions for the minor stresses are shown diagramatically in Figure 2c.

From equation (12) taking notice of directions set out in Figure 2b.

$$M_x = \int \sigma_x \, zdz \qquad (13a) \qquad Q_x = \int \tau_{zx} \, dz \qquad (13b)$$

$$M_y = \int \sigma_y \, zdz \qquad (13c) \qquad Q_y = \int \tau_{zy} \, dz \qquad (13d)$$

and $$M_{xy} = \int \tau_{xy} \, zdz \qquad (13e)$$

where the integration is performed in each case between the limits -t/2 and t/2, where t is the plate thickness.

Having assumed (12) quite arbitrarily, it is necessary to satisfy the three conditions of equilibrium of the element. These are

$$\frac{\partial Qx}{\partial x} + \frac{\partial Qy}{\partial y} = 0 \qquad (14a)$$

$$\frac{\partial Mxy}{\partial x} + \frac{\partial My}{\partial y} - Qy = 0 \qquad (14b)$$

$$\frac{\partial Mxy}{\partial y} + \frac{\partial Mx}{\partial x} - Qx = 0 \qquad (14c)$$

Making use of equations (13) and (12), these three equilibrium conditions to be satisfied for all values of x and y, allow seven of the B coefficients namely B_{17}, B_{19}, B_{20}, B_{21}, B_{22}, B_{23}, and B_{24}, to be written in terms of the remaining seventeen. In fact equation (12) is replaced by equation (15) in matrix form concisely as

$$\{\sigma\} = [P]\{B\} \tag{15}$$

In the extended form given on the next page, where $\bar{x}$ and $\bar{y}$ are written for x/a and y/b, respectively, $\bar{z}$ is written for 8z/t and z' for $t(1-4z^2/t^2)$. Similarly the strain components can be written in terms of the stresses in the form:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \\ \gamma_{zx} \\ \gamma_{zy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 & 0 & 0 \\ -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \bar{\nu} & 0 & 0 \\ 0 & 0 & 0 & \bar{\nu} & 0 \\ 0 & 0 & 0 & 0 & \bar{\nu} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{zx} \\ \tau_{zy} \end{Bmatrix}$$

where $\bar{\nu} = 2(1+\nu)$

More concisely

$$\{\epsilon\} = [N]\{\sigma\} \tag{16}$$

The internal strain energy U of the element, is given by

$$U = \frac{1}{2}\int \{\sigma\}^T [N]\{\sigma\}\, dV \tag{17}$$

where the integration is performed over the volume of the element.

In equation (17) it is convenient to let

$$[H] = \int [P^T]\,[N]\,[P]\, dV \tag{18}$$

$$
\begin{Bmatrix} \sigma_y/\bar{Z} \\ \sigma_y/\bar{Z} \\ \tau_{xy}/\bar{Z} \\ \tau_{zx}/Z' \\ \tau_{zy}/Z' \end{Bmatrix} =
\begin{bmatrix}
1 & \bar{x} & \bar{y} & \bar{x}^2 & \bar{x}\bar{y} & \bar{y}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \bar{x} & \bar{y} & \bar{x}^2 & \bar{x}\bar{y} & \bar{y}^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\bar{x}\bar{y}b/a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{x}\bar{y}a/b & 1 & \bar{x} & \bar{y} & \bar{x}^2 & \bar{y}^2 \\
0 & 1/a & 0 & \bar{x}/a & \bar{y}/a & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{x}a/b^2 & 0 & 0 & 1/b & 0 & 2\bar{y}/b \\
0 & 0 & 0 & -\bar{y}b/a^2 & 0 & 0 & 0 & 0 & 1/b & 0 & \bar{x}/b & \bar{y}/b & 0 & 1/a & 0 & 2\bar{x}/a & 0
\end{bmatrix}
\begin{Bmatrix} B_1 \\ B_2 \\ \vdots \\ \vdots \\ B_{16} \\ B_{18} \end{Bmatrix}
$$

$$
\{\sigma\} = [P]\{B\}
$$

then

$$U = \frac{1}{2} \left[B\right]^T \left[H\right] \left\{B\right\} \tag{19}$$

both $\left[P\right]$ and $\left[N\right]$ are known so that $\left[H\right]$ can be calculated. $\left[H\right]$ is shown on next page.

At each node of the element (Fig. 2) three generalized displacements, θ, Φ, and W, have been specified and along each edge the displacements can be written in terms of these specified nodal displacements and the special coordinates. Thus returning to Fig. 2, along AB

$$W = W_1 \ (1 - 3\bar{x}^2 + 2\bar{x}^3) + W_2 (3\bar{x}^2 - 2\bar{x}^3) + a\Phi_1 \ (-\bar{x} + 2\bar{x}^2 - \bar{x}^3) + a \ \Phi_2 (\bar{x}^2 - \bar{x}^3) \tag{20a}$$

$$\Phi = \frac{6}{a}(W_1 - W_2)(\bar{x} - \bar{x}^2) + \Phi_1 (1 - 4\bar{x} + 3\bar{x}^2) + \Phi_2 (-2\bar{x} + 3\bar{x}^2) \tag{20b}$$

$$\theta = \theta_1 (1 - \bar{x}) + \theta_2 (\bar{x}) \tag{20c}$$

Corresponding expressions can be written for the remaining three edges of the element. Particular note should be taken of equation (20c) because it is this equation that insures continuity of slope across the junction between two elements. Equations (20) for all edges can be written in matrix form concisely as

$$\left\{u\right\} = \left[L\right] \left\{q\right\} \tag{21}$$

The extended form of equation (21) is given on page 21. Corresponding to the generalized displacements as given by equation 20, are the generalized forces. In matrix form, they can be written as

$$\left\{S\right\} = \left[R\right] \left\{B\right\} \tag{21}$$

It is important to note that $\left\{S\right\}$ and $\left\{u\right\}$ must correspond, item by item.

$$
[H] = \frac{16abt}{3E}
\left[\begin{array}{ccccccccccccccccc}
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & -\nu & \frac{-\nu}{2} & \frac{-\nu}{2} & \frac{-\nu}{3} & \frac{-\nu}{4} & \frac{-\nu}{3} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{1}{3}+q & \frac{1}{4} & \frac{1}{4}+\frac{q}{2} & \frac{1}{6}+\frac{q}{2} & \frac{1}{6} & \frac{-\nu}{2} & \frac{-\nu}{3} & \frac{-\nu}{4} & \frac{-\nu}{4} & \frac{-\nu}{6} & \frac{\nu}{6}-\frac{r}{2} & 0 & 0 & s & 0 & s \\
 & & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{4} & \frac{-\nu}{2} & \frac{-\nu}{4} & \frac{-\nu}{3} & \frac{-\nu}{6} & \frac{-\nu}{6} & \frac{-\nu}{4} & 0 & 0 & 0 & 0 & 0 \\
 & & & g & \frac{1}{8}+\frac{q}{4} & \frac{1}{9} & \frac{-\nu}{3} & \frac{-\nu}{4} & \frac{-\nu}{6}-\frac{q}{2} & \frac{\nu}{5} & \frac{-\nu}{8}-\frac{q}{4} & e & \frac{-\bar{\nu}}{4h} & \frac{-\bar{\nu}}{6h}-\frac{q}{2h} & \frac{-\bar{\nu}}{6h} \quad \frac{s}{2} & \frac{-\bar{\nu}}{8h}-\frac{q}{2h} & \frac{-\bar{\nu}}{8h} \quad \frac{s}{2} \\
 & & & & \frac{1}{9}+\frac{q}{3} & \frac{1}{8} & \frac{-\nu}{4} & \frac{-\nu}{6} & \frac{-\nu}{6} & \frac{-\nu}{8} & \frac{-\nu}{9} & \frac{-\nu}{8}-\frac{r}{4} & 0 & 0 & \frac{s}{2} & 0 & \frac{2s}{3} \\
 & & & & & \frac{1}{5} & \frac{-\nu}{3} & \frac{-\nu}{6} & \frac{-\nu}{4} & \frac{-\nu}{9} & \frac{-\nu}{8} & \frac{-\nu}{5} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & \frac{1}{3} \quad r & \frac{1}{6} & \frac{1}{6} \quad \frac{r}{2} & \frac{1}{4}+\frac{r}{2} & 0 & s & 0 & s & 0 \\
 & & & & & & & & & \frac{1}{5} & \frac{1}{8} & \frac{1}{9} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & \frac{1}{9}+\frac{r}{3} & \frac{1}{8}+\frac{r}{4} & 0 & \frac{s}{2} & 0 & \frac{2s}{3} & 0 \\
 & & & & & & & & & & & f & \frac{-\bar{\nu}h}{4} & \frac{-\bar{\nu}h}{6}+\frac{s}{2} & \frac{-\bar{\nu}h}{6}-\frac{rh}{2} & \frac{-\bar{\nu}h}{8}+\frac{s}{2} & \frac{-\bar{\nu}h}{8}-\frac{rh}{2} \\
 & & & & & & & & & & & & \bar{\nu} & \frac{\bar{\nu}}{2} & \frac{\bar{\nu}}{2} & \frac{\bar{\nu}}{3} & \frac{\bar{\nu}}{3} \\
 & & & & & & & & & & & & & \frac{\bar{\nu}}{3} \quad q & \bar{\nu} & \frac{\bar{\nu}}{4}+q & \frac{\bar{\nu}}{6} \\
 & & & & & & & & & & & & & & \frac{\bar{\nu}}{3}+r & \frac{\bar{\nu}}{6} & \frac{\bar{\nu}}{4}+r \\
 & & & & & & & & & & & & & & & \frac{\bar{\nu}}{5}+\frac{4q}{3} & \frac{\bar{\nu}}{9} \\
 & & & & & & & & & & & & & & & & \frac{\bar{\nu}}{5}+\frac{4r}{3}
\end{array}\right]
$$

SYMMETRICAL

$r = \bar{\nu}t^2/10b^2$

$s = \nu t^2/10ab$

$h = a/b$

$e = \frac{1}{9}(\bar{\nu}-\nu) - \frac{1}{3}(r+q)$

$f = \frac{1}{5} + \frac{\bar{\nu}h^2}{9} + \frac{r}{3}(h^2 + 1)$

$g = \frac{1}{5} + \frac{\bar{\nu}}{9h^2} + \frac{q}{3}(\frac{1}{h^2} + 1)$

$q = \nu t^2/10a^2$

$\bar{\nu} = 2(1+\nu)$

$$
\begin{Bmatrix} \Phi AB \\ \theta AB \\ WAB \\ \Phi BD \\ \theta BD \\ WBD \\ \Phi CD \\ \theta CD \\ WCD \\ \Phi AC \\ \theta AC \\ WAC \end{Bmatrix}
=
\left[\begin{array}{cccccc|cccccc}
1-\bar{x} & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-4\bar{x}+3\bar{x}^2 & 6(\bar{x}-\bar{x}^2)/a & 0 & -2\bar{x}+3\bar{x}^2 & 6(-\bar{x}+\bar{x}^2)/a & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a(-\bar{x}+2\bar{x}^2-\bar{x}^3) & 1-3\bar{x}^2+2\bar{x}^3 & 0 & a(\bar{x}^2-\bar{x}^3) & 3\bar{x}^2-2\bar{x}^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-4\bar{y}+3\bar{y}^2 & 0 & 6(\bar{y}-\bar{y}^2)/b & 0 & 0 & 0 & -2\bar{y}+3\bar{y}^2 & 0 & 6(-\bar{y}+\bar{y}^2)/b \\
0 & 0 & 0 & 0 & 1-\bar{y} & 0 & 0 & 0 & 0 & 0 & \bar{y} & 0 \\
0 & 0 & 0 & b(-\bar{y}+2\bar{y}^2-\bar{y}^3) & 0 & 1-3\bar{y}^2-2\bar{y}^3 & 0 & 0 & 0 & b(\bar{y}^2-\bar{y}^3) & 0 & 3\bar{y}^2-2\bar{y}^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-\bar{x} & 0 & 0 & x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-4\bar{x}+3\bar{x}^2 & 6(\bar{x}-\bar{x}^2)/a & 0 & -2\bar{x}+3\bar{x}^2 & 6(-\bar{x}+\bar{x}^2)/a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a(-\bar{x}+2\bar{x}^2-\bar{x}^3) & 1-3\bar{x}^2+2\bar{x}^3 & 0 & a(\bar{x}^2-x^3) & 3\bar{x}^2-2\bar{x}^3 \\
1-4\bar{y}+3\bar{y}^2 & 0 & 6(\bar{y}-\bar{y}^2)/b & 0 & 0 & 0 & 2\bar{y}+3\bar{y}^2 & 0 & 6(-\bar{y}+\bar{y}^2)/b & 0 & 0 & 0 \\
0 & 1-\bar{y} & 0 & 0 & 0 & 0 & 0 & \bar{y} & 0 & 0 & 0 & 0 \\
b(-\bar{y}+2\bar{y}^2-\bar{y}^3) & 0 & 1-3\bar{y}^2+2\bar{y}^3 & 0 & 0 & 0 & b(\bar{y}^2-\bar{y}^3) & 0 & 3\bar{y}^2-2\bar{y}^3 & 0 & 0 & 0
\end{array}\right]
\begin{Bmatrix} \Phi_1 \\ \theta_1 \\ W_1 \\ \Phi_2 \\ \theta_2 \\ W_2 \\ \Phi_3 \\ \theta_3 \\ W_3 \\ \Phi_4 \\ \theta_4 \\ W_4 \end{Bmatrix}
$$

$$\{u\} = [L]\{q\}$$

The work W, done by the edge forces, is now given by

$$W = \oint \lfloor U \rfloor \{S\} \, ds$$

It is convenient to let

$$[T] = \oint [R^T] [L] \, ds \tag{22}$$

Matrix $[T]$ is listed on next page

$$W = -\{B\} [T] \{q\}$$

and the total complimentary energy, $U + W$ can be written as

$$U + W = \frac{1}{2} \{B\} [H] \{B\} - \{B\}^T [T] \{q\} \tag{23}$$

The principle of minimum complimentary energy now requires that

$$\partial (U + W) / \partial B_i = 0 \qquad \text{for all values of } i$$

From equation (23)

$$[H] \{B\} = [T] \{q\} \tag{24}$$

From equation (24), $\{B\}$ can be substituted into equation (19) to obtain

$$U = \frac{1}{2} \{q\}^T [T]^T [H^{-1}] [T] \{q\} \tag{25}$$

Now the stiffness matrix of the element, $[K]$ can be defined so that

$$U = \frac{1}{2} \{q\}^T [K] \{q\} \tag{26}$$

comparison of equations (25) and (26) gives

$$[K] = [T^T] [H^{-1}] [T] \tag{27}$$

$$
[T] = \frac{t^2}{90}\left[\begin{array}{cccccccccccc}
0 & -30 & 0 & 0 & 30 & 0 & 0 & -30 & 0 & 0 & 30 & 0 \\
5h^{-2} & 0 & -30h^{-1} & -5h^{-2} & 30 & 30h^{-1} & -5h^{-2} & 0 & -30h^{-1} & 5h^{-2} & 30 & 30h^{-1} \\
0 & -10 & 0 & 0 & 10 & 0 & 0 & -20 & 0 & 0 & 20 & 0 \\
0 & 0 & 0 & 0 & 30 & 0 & 0 & 10 & -60h^{-1} & 0 & 20 & 60h^{-1} \\
2h^{-2} & 0 & -9h^{-1} & -2h^{-2} & 10 & 9h^{-1} & -3h^{-2} & 0 & -21h^{-1} & 3h^{-2} & 20 & 21h^{-1} \\
0 & -5 & 0 & 0 & 5 & 0 & 0 & -15 & 0 & 0 & 15 & 0 \\
-30 & 0 & 0 & -30 & 0 & 0 & 30 & 0 & 0 & 30 & 0 & 0 \\
-10 & 0 & 0 & -20 & 0 & 0 & 10 & 0 & 0 & 20 & 0 & 0 \\
0 & 5h^2 & -30h & 0 & -5h^2 & -30h & 30 & -5h^2 & 30h & 30 & 5h^2 & 30h \\
-5 & 0 & 0 & -15 & 0 & 0 & 5 & 0 & 0 & 15 & 0 & 0 \\
0 & 2h^2 & -9h & 0 & -3h^2 & -21h & 10 & -2h^2 & 9h & 20 & 3h^2 & 21h \\
0 & 0 & 0 & 10 & 0 & -60h & 30 & 0 & 0 & 20 & 0 & 60h \\
0 & 0 & -120 & 0 & 0 & 120 & 0 & 0 & 120 & 0 & 0 & -120 \\
0 & 10h & -60 & 0 & -10h & 60 & 0 & -10h & 60 & 0 & 10h & -60 \\
-10h^{-1} & 0 & -60 & -10h^{-1} & 0 & 60 & -10h^{-1} & 0 & 60 & 10h^{-1} & 0 & -60 \\
0 & 8h & -36 & 0 & -12h & 36 & 0 & -8h & 36 & 0 & 12h & -36 \\
8h^{-1} & 0 & -36 & -8h^{-1} & 0 & 36 & -12h^{-1} & 0 & 36 & 12h^{-1} & 0 & -36
\end{array}\right]
$$

Relating generalized forces, $\{Q\}$, to their corresponding generalized displacements, $\{q\}$, gives

$$\{Q\} = [K] \{q\} \tag{28}$$

The structural stiffness matrix is obtained as outlined in Chapter III and the problem is approached in the same manner.

4.2 Vibration

The inertia matrix is obtained in exactly same manner as in Chapter III The elements are suitably arranged to correspond to elements of $\{q\}$. The natural frequency is obtained by

$$[K] \{v\} = \lambda [M] \{v\}$$

where $[K]$ and $[M]$ are as derived in Chapter IV.

CHAPTER V

NUMERICAL EXAMPLE

For the square plate shown in Figure 3 clamped on all sides and loaded with uniformly distributed load q, the following will be determined.

1) Displacements and actions at nodes.

2) Natural frequency of vibration.

This will be done by both methods outlined in Chapters III and IV. The results of both will then be compared. The plate is subdivided into sixteen elements as shown in Figure 3. Because of symmetry it is necessary to consider only one quadrant of the plate. Young's modulus of elasticity of the plate material is taken equal to $E = 30 \times 10^6$ psi while its Poisson's ratio is taken as $\nu = 0.3$.

5.1 Assumed Displacement Field for Bending

Substituting numerical values in the elemental stiffness matrix developed in Chapter III,

$$[F] = [K]\{v\}$$

$[K]$ is listed in (Appendix 1c).

The structural stiffness matrix is obtained with the help of key plan on page 27 and is listed in Appendix 1d.

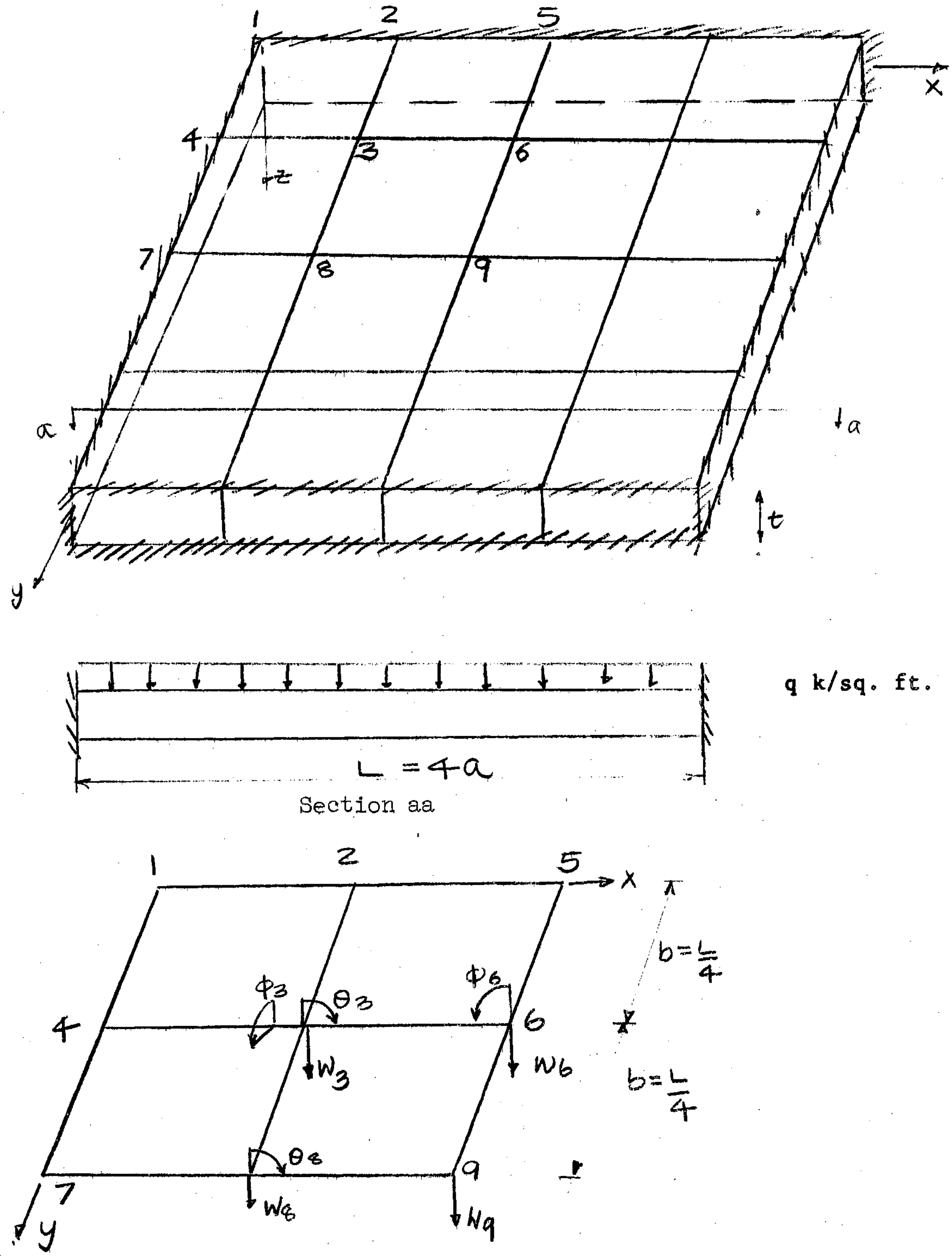

Figure 4. Square Plate Assuming Displacements

K_{11}	K_{12}	K_{13}	K_{14}	0	0	0	0	0
	K_{22} (1) (2)	K_{23} (1) (2)	K_{34}	K_{25}	K_{26}	0	0	0
		K_{33} (1) (2) (3) (4)	K_{34} (1) (4)	K_{35}	K_{36} (2) (3)	K_{37}	K_{38} (3) (4)	K_{39}
			K_{44} (1) (4)	0	0	K_{47}	K_{48}	K_{49}
				K_{55}	K_{56}	0	0	0
					K_{66} (2) (3)	0	K_{68}	K_{69}
	SYMMETRICAL					K_{77}	K_{78}	0
							K_{88} (3) (4)	K_{89}
								K_{99}

Taking advantage of symmetry only the top left quadrant of the plate is considered. From the overall stiffness matrix, only those columns corresponding to unknown displacement are used. The rows are rearranged so that top eight rows correspond to the columns and the rest are arranged in order.

This can be written as

$$\left\{ \frac{F_o}{F_u} \right\} = \left[\frac{K}{K_d} \right] \left\{ v \right\}$$

where $\{F_o\}$ is known load vector

$$= \left\{ 400,0,0,200,0,200,0,100 \right\} \frac{qab}{400}$$

$\{F_u\}$ are unknown node actions

$$\{v\} = [K]^{-1} \{F_o\}$$

and $$\{F_u\} = [K_d] \{v\}$$

in this way, both $\{v\}$ and $\{F_u\}$ are obtained.

$$\{v\} = \{51.0, 63.5, 63.5, 85, 104, 85, 104, 140\} \frac{qL^4}{D} \times 10^{-5}$$

The maximum central deflection is $140 \times 10^{-5} \frac{qL^4}{D}$

Solving for $\{F_u\}$ various actions are obtained. They are listed in Table I.

5.2 Assumed Displacement Field for Vibration

The mass matrix is obtained with the same key plan as for bending, solving

$$[K]\{v\} = \lambda[M]\{v\}$$

where

$$\lambda = \frac{\rho a^2 b^2 p^2}{25200D}$$

The eigen values λ obtained are 0.02428, 0.01790, 0.00818, 0.000187, the lowest natural frequency is

$$p = 33.9 \sqrt{\frac{D}{\rho L^4}}$$

5.3 Assumed Stress Field for Bending

The plate is subdivided as shown in Figure 4.

$$[K] = [T]^T [H]^{-1} [T]$$

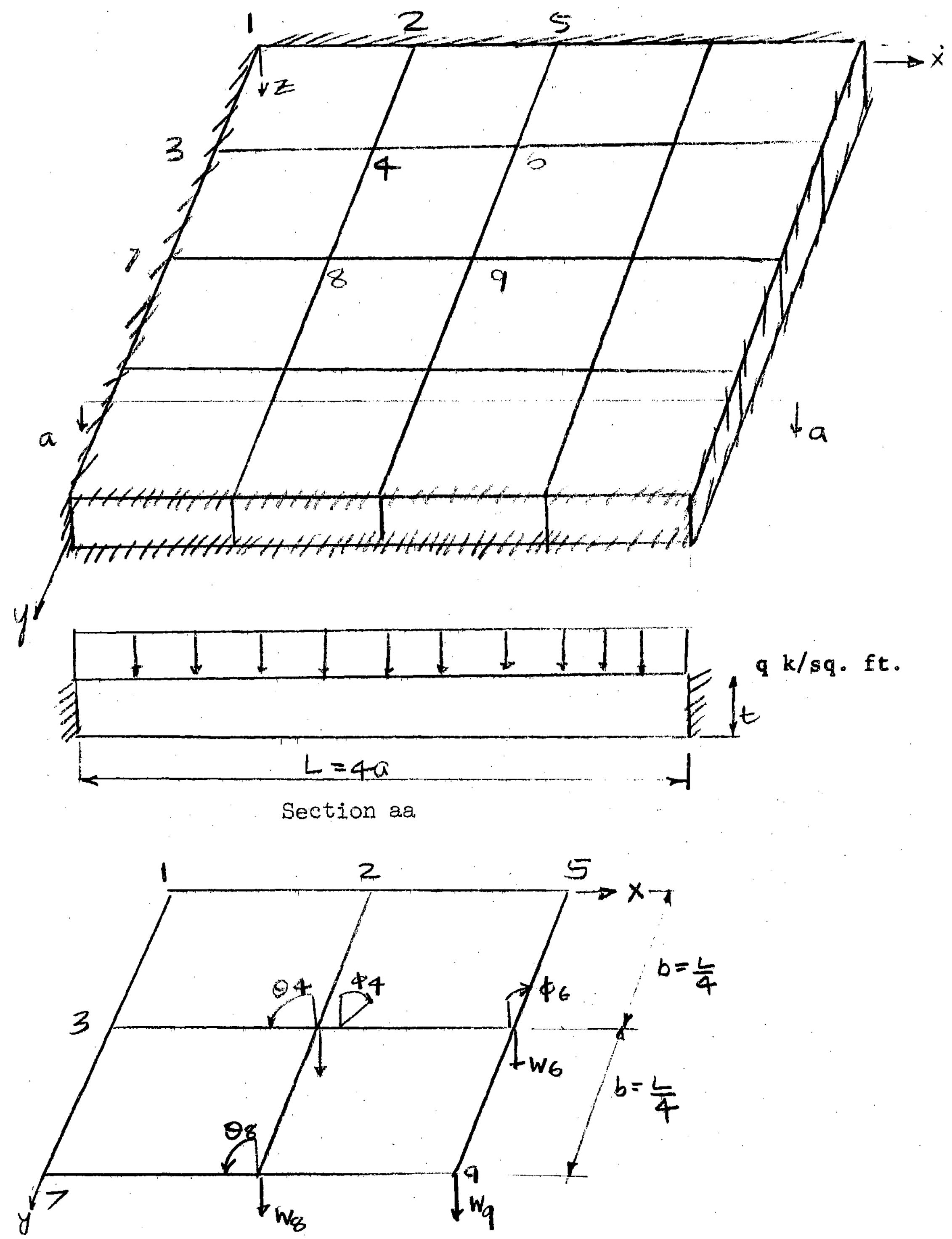

Figure 3. **Square Plate Assuming Stresses**

Substituting numerical values in the right hand side, elemental stiffness matrix is obtained and is listed in Appendix 1e.

Structural matrix nodal displacements and actions are obtained as in previous case. The structural matrix is shown in Appendix 1f. Solving for displacements

$$\{q\} = \{-70,\ -70,\ 40,\ -100,\ 78,\ -100,\ 78,\ 126\} \ \frac{qL^4}{D} \times 10^{-5}$$

Maximum deflection is $126 \times 10^{-5} \ \frac{qL^4}{D}$

The actions obtained are as listed in Table I.

5.4 Assumed Stress Field for Vibration

The approach to solution is the same as in section 5.2. Eigen values obtained are 0.00711, 0.00404, 0.00008. Lowest natural frequency is

$$p = 36.0\sqrt{\frac{D}{\rho L^4}}$$

TABLE I

ACTIONS AT NODES*

		Node								
Assumption	Action	1	2	3	4	5	6	7	8	9
Displacements	M_x/b	-532			-3380	-588	-1050	-4820		-2820
	M_y/a	-532	-3380			-4820		-588	-1050	-2820
Stresses	M_x/b	-535		3400		560	-1020	4790		2590
	M_y/a	-535	3400			4790		560	1020	2590

* Multiplier $= \frac{qL^2}{8} \times 10^{-5}$

TABLE II

SUMMARY OF RESULTS FOR THE NUMERICAL EXAMPLE

Assumption	Central Deflection	Maximum Negative Moment	Maximum Positive Moment	Natural Frequency
Displacement	140	482.0	282.0	33.9
Stresses	126	479.0	259.0	36.0
Multiplier	$\frac{qL^4}{D} \times 10^{-5}$	$-qL^2 \times 10^{-4}$	$qL^2 \times 10^{-4}$	$\sqrt{\frac{D}{\rho L^4}}$

TABLE III

COMPARISON OF RESULTS OBTAINED BY ZIENKIEWICZ SEVERN AND TAYLOR AND DAWE

		Central Deflection			Maximum Negative Moment			Maximum Positive Moment			Natural Frequency
		(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)	(d)
Finite element	2x2	148	133	106	355	440	289	460	461	139	
	4x4	140	124	116	476	473	322	278	256	168	334
	6x6	133	125	121	496	491	376	249	244	199	
	8x8	130	126	124	503	500	406	240	237	208	
Exact Soln			127			513			231		36
Multiplier			$10^{-5} \frac{qL^4}{D}$			$-10^{-4} qL^2$			$10^{-4} qL^2$		$\sqrt{\frac{D}{\rho L^4}}$

(a) Results obtained by Zienkiewicz (2) assuming displacements.

(b) Results obtained by Severn and Taylor (8) assuming stresses.

(c) Assuming triangular elements.

(d) Natural frequency obtained by Dawe (5).

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Summary

The finite element method for determining displacements, actions and natural frequency of rectangular plates is discussed in this report. The method is developed assuming displacements once and again assuming stresses. Only isotropic plates are considered. The results for the numerical example of a square plate fixed on four sides from the displacement method are compared with the stress method and both are compared with the results of the classical approach.

6.2 Discussion of Results

The results obtained are approximate but compare well with those obtained by Severn and Taylor (8) and Zienkiewicz (2). The natural frequency obtained by both methods compare well with those obtained assuming displacements.

6.3 Conclusion

A systematic approach to the problem of bending and vibration of the plates is given by the finite element method. This method is especially suitable for computer use. A program can be written for developing the stiffness matrix and mass matrix which can solve then the problem of bending and vibration.

The finite element method has wide applications and is not limited to problems discussed in this report. It can be used to study stability, static as well as dynamic problems, multiple-layer plates, concentrated and variable loadings, etc. It is a valuable tool in the hands of structural engineers whenever computer facilities are available.

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APPENDIX

$$\frac{D}{ab}$$

$4P+4P^{-1}$ $\frac{14}{5}-\frac{4}{5}\nu$											
$2P+\frac{1}{5}+\frac{4}{5}\nu$	$\frac{4}{3}P+\frac{2}{15}\beta$										
$2P^{-1}+\frac{1}{5}+\frac{4}{5}\nu$	ν	$\frac{4}{3}P^{-1}+\frac{2}{15}\beta$									
$2P-4P^{-1}$ $-\frac{14}{5}+\frac{4}{5}\nu$	$P-\frac{1}{5}-\frac{4}{5}$	$-2P^{-1}-\frac{1}{10}\beta$	$4P+4P^{-1}$ $\frac{14}{5}-\frac{4}{5}\nu$								
$P-\frac{1}{5}-\frac{4}{5}\nu$	$\frac{2}{3}P-\frac{2}{15}\beta$	0	$2P+\frac{1}{5}+\frac{4}{5}\nu$	$\frac{4}{3}P+\frac{2}{15}\beta$							
$2P^{-1}+\frac{1}{10}\beta$	0	$\frac{2}{3}P^{-1}-\frac{1}{30}\beta$	$-2P^{-1}-\frac{1}{5}-\frac{4}{5}\nu$	$-\nu$	$\frac{4}{3}P^{-1}+\frac{2}{15}\beta$						
$-2P-2P^{-1}$ $\frac{14}{5}-\frac{4}{5}\nu$	$-P+\frac{1}{10}\beta$	$-P^{-1}+\frac{1}{10}\beta$	$-4P+2P^{-1}$ $-\frac{14}{5}+\frac{4}{5}\nu$	$-2P-\frac{1}{10}\beta$	$-P^{-1}+\frac{1}{5}+\frac{4}{5}\nu$	$4P+4P^{-1}$ $\frac{14}{5}-\frac{4}{5}\nu$					
$P-\frac{1}{10}\beta$	$\frac{1}{3}P+\frac{1}{30}\beta$	0	$2P+\frac{1}{10}\beta$	$\frac{2}{3}P-\frac{1}{30}\beta$	0	$-2P-\frac{1}{5}-\frac{4}{5}\nu$	$\frac{4}{3}P+\frac{2}{15}\beta$				
$P^{-1}-\frac{1}{10}\beta$	0	$\frac{1}{3}P^{-1}-\frac{1}{5}-\frac{4}{5}\nu$	$-P^{-1}+\frac{1}{5}+\frac{4}{5}\nu$	0	$\frac{2}{3}P^{-1}-\frac{2}{15}\beta$	$-2P^{-1}-\frac{1}{5}$ $-4/5\,\nu$	ν	$\frac{4}{3}P^{-1}+\frac{2}{15}\beta$			
$-4P+2P^{-1}$ $-\frac{14}{5}+\frac{4}{5}\nu$	$-2P-\frac{1}{10}\beta$	$P^{-1}-\frac{1}{5}-\frac{4}{5}\nu$	$-2P-2P^{-1}$ $+\frac{14}{5}-\frac{4}{5}\nu$	$-P+\frac{1}{10}\beta$	$P^{-1}-\frac{1}{10}\beta$	$2P-4P-1$ $-\frac{14}{5}+\frac{4}{5}\nu$	$-P+\frac{1}{5}+\frac{4}{5}$	$2P^{-1}+\frac{1}{10}\beta$	$4P+4P^{-1}$ $+\frac{14}{5}-\frac{4}{5}\nu$		
$2P+\frac{\beta}{10}$	$\frac{2}{3}P-\frac{1}{30}\beta$	0	$P-\frac{1}{10}\beta$	$\frac{1}{3}P+\frac{1}{30}\beta$	0	$-P+\frac{1}{5}+\frac{4}{5}\nu$	$\frac{2}{3}P-\frac{2}{15}\beta$	0	$-2P-\frac{1}{5}-\frac{4}{5}\nu$	$\frac{4}{3}P+\frac{2}{15}\beta$	
$P^{-1}-\frac{1}{5}-\frac{4}{5}\nu$	0	$\frac{2}{3}P^{-1}-\frac{2}{15}\beta$	$-P^{-1}+\frac{1}{10}\beta$	0	$\frac{1}{3}P^{-1}+\frac{1}{30}\beta$	$-2P^{-1}+\frac{1}{10}\beta$	0	$\frac{2}{3}P^{-1}-\frac{1}{30}\beta$	$2P^{-1}+\frac{1}{5}+\frac{4}{5}\nu$	$-\nu$	$\frac{4}{3}P^{-1}+\frac{2}{15}\beta$

SYMMETRICAL

$P=\frac{a^2}{b^2}$

$D=\frac{Et^3}{12(1-\nu^2)}$

$\beta=2(1+\nu)$

ELEMENTAL STIFFNESS MATRIX

Appendix 1a

$$
\left\{F_{in.}\right\} = \frac{\rho\, ab\, p^2}{25200}
\begin{bmatrix}
3454 & & & & & & & & & & & \\
461 & 80 & & & & & & & & & & \\
461 & 63 & 80 & & & & & & & \text{SYMMETRICAL} & & \\
1226 & 199 & 274 & 3454 & & & & & & & & \\
199 & 40 & 42 & 461 & 80 & & & & & & & \\
-274 & -42 & -60 & -461 & -63 & 80 & & & & & & \\
394 & 116 & 116 & 1226 & 274 & -199 & 3454 & & & & & \\
-116 & -30 & -28 & -274 & -60 & 42 & -461 & 80 & & & & \\
-116 & -28 & -30 & -199 & -42 & 40 & -461 & 63 & 80 & & & \\
1226 & 274 & 199 & 394 & 116 & -116 & 1226 & -199 & -274 & 3454 & & \\
-274 & -60 & -42 & -116 & -30 & 28 & -199 & 40 & 42 & -461 & 80 & \\
199 & 42 & 40 & 116 & 28 & -30 & 274 & -42 & -60 & 461 & -63 & 80
\end{bmatrix}
\begin{Bmatrix}
W_1 \\ \Phi_1 \\ \Theta_1 \\ W_2 \\ \Phi_2 \\ \Theta_2 \\ W_3 \\ \Phi_3 \\ \Theta_3 \\ W_4 \\ \Phi_4 \\ \Theta_4
\end{Bmatrix}
$$

ELEMENTAL MASS MATRIX

Appendix 1b

$[K] = \frac{D}{ab}$

10.56											
2.44	1.52										
2.44	0.3	1.52					SYMMETRICAL				
-4.56	0.56	-2.14	10.56								
0.56	0.48	0	2.44	1.52							
2.14	0	0.63	-2.44	-0.3	1.52						
-1.44	-0.86	-0.86	-4.56	-2.14	-0.56	10.56					
0.86	0.38	0	2.14	0.63	0	-2.44	1.52				
0.86	0	0.38	-0.56	0	0.48	-2.44	0.3	1.52			
-4.56	-2.14	0.56	-1.44	-0.86	0.86	-4.56	-0.56	2.14	10.56		
2.14	0.63	0	0.86	0.38	0	-0.56	0.48	0	-2.44	1.52	
0.56	0	0.48	-0.86	0	0.38	-2.14	0	0.48	2.44	-0.3	1.52

ELEMENTAL STIFFNESS MATRIX $[K]$ NUMERICAL

Appendix 1c

$$\{Q, M\} = \frac{400D}{qab^2} [K] \{W\}$$

Q_3	42.24	0	0	-9.12	0	-9.12	0	-1.44
M_{y3}	0	6.08	0	0	0.96	-4.28	0	-0.86
M_{x3}	0	0	6.08	-4.28	0	0	0.96	-0.86
Q_6	-9.12	0	-4.28	21.12	0	-1.44	-0.86	-4.56
M_{y6}	0	0.96	0	0	3.04	-0.86	0	-2.14
Q_8	-9.12	-4.28	0	-1.44	-0.86	21.12	0	-4.56
M_{x8}	0	0	0.96	-0.86	0	0	3.04	-2.14
Q_9	-1.44	-0.86	-0.86	-4.56	-2.14	-4.56	-2.14	10.56
Q_1	-1.44	0.86	0.86	0	0	0	0	0
M_{y1}	-0.86	0.38	0	0	0	0	0	0
M_{x1}	-0.86	0	0.38	0	0	0	0	0
Q_2	-9.12	4.28	0	-1.44	0.86	0	0	0
M_{y2}	-4.28	1.26	0	-0.86	0.38	0	0	0
M_{x2}	0	0	0.96	-0.86	0	0	0	0
Q_4	-9.12	0	4.28	0	0	-1.44	0.86	0
M_{y4}	0	0.96	0	0	0	-0.86	0	0
M_{x4}	-4.28	0	1.26	0	0	-0.86	0.38	0
Q_5	-1.44	0.86	-0.86	-4.56	2.14	0	0	0
M_{y5}	-0.86	0.38	0	-2.14	0.63	0	0	0
M_{x5}	0.86	0	0.38	-0.56	0	0	0	0
M_{x6}	4.28	0	1.26	4.88	0	0.86	0.38	-0.56
Q_7	-1.44	-0.86	0.86	0	0	-4.56	2.14	0
M_{y7}	0.86	0.38	0	0	0	-0.56	0	0
M_{x7}	-0.86	0	-0.38	0	0	-2.14	0.63	0
M_{y8}	4.28	1.26	0	0.86	0.38	-4.88	0	-0.56
M_{y9}	0.86	0.38	0	2.14	0.63	-0.56	0	-2.44
M_{x9}	+0.86	0	0.38	-0.56	0	2.14	0.63	-2.44

$\{W\} = \{W_3, \Phi_3, \theta_3, W_6, \Phi_6, W_8, \theta_8, W_9\}$

[K] STRUCTURAL Appendix 1d

$$[K] = \frac{D}{3920ab}$$

239.5											
1760	827.3										
5109	4617	-8010									
2860	1260	-4502	5831								
2209	1511	-8677	-2955	2803							
-8182	-3074	8970	-8831	9368	24237						
-5791	-5217	12522	3336	2135	-12655	-3639			SYMMETRICAL		
8415	7825	-22188	-3095	-1845	13977	7808	-5836				
-20102	-20000	50397	-9675	6852	-33314	-22689	26316	-75897			
5774	4154	14074	1306	-2101	-3571	6243	-4908	16046	4314		
6130	5222	-16138	-382.6	-644.3	4072	5440	-4271	12334	884.3	3498	
23175	18459	-51356	3660	-7544	1068	22822	-18106	58814	1599	-267.6	-7565

ELEMENTAL STIFFNESS MATRIX [K] NUMERICAL

Appendix 1e

$$
\begin{Bmatrix} M_{y4} \\ M_{x4} \\ Q_4 \\ M_{y6} \\ M_{x6} \\ Q_6 \\ Q_8 \\ Q_9 \end{Bmatrix}
= \frac{D}{3920ab}
\begin{bmatrix}
6746 & & & & & & & \\
3977 & 1292 & & & \text{SYMMETRICAL} & & & \\
-24811 & 40033 & -67235 & & & & & \\
9103 & -3648 & 11544 & 10145 & & & & \\
14640 & -21180 & 67784 & -7232 & 16672 & & & \\
8032 & 7181 & -18166 & -3095 & 13977 & -2338 & & \\
-16442 & -27544 & 50504 & 9675 & -33314 & 26045 & -83462 & \\
23175 & 18459 & -51356 & 3660 & 107 & -18106 & 58814 & -7565
\end{bmatrix}
\begin{Bmatrix} \Phi_4 \\ \theta_4 \\ W_4 \\ \Phi_6 \\ W_6 \\ \theta_8 \\ W_8 \\ W_9 \end{Bmatrix}
$$

STRUCTURAL STIFFNESS MATRIX $[K]$ STRUCTURAL

Appendix 1f-a

$\{M_{y1}, M_{x1}, Q_1, M_{y2}, M_{x2}, Q_2, M_{y3}, M_{x3}, Q_3, M_{y5}, M_{x5}, Q_5, M_y, M_{y7}, M_{x7}, Q_7, M_{y8}, M_{y9}, M_{x9}\} = \frac{D}{3920ab}\,[K]\,\{\Phi_4, \theta_4, W_4, \Phi_6, W_6, \theta_8, W_8, W_9\}$

M_{y1}	5774	6130	23175	0	0	0	0	0
M_{x1}	4154	5222	18454	0	0	0	0	0
Q_1	-14074	-16138	-51356	0	0	0	0	0
M_{y2}	-4485	8032	-16442	9110	32850	0	0	0
M_{x2}	-7318	7181	-27544	6289	25811	0	0	0
Q_2	8951	-18166	50504	-26729	-84670	0	0	0
M_{y3}	9104	7649	14640	0	0	6130	23175	0
M_{x3}	-3648	-2760	-21180	0	0	5222	18656	0
Q_3	11544	3657	69784	0	0	-16138	-51356	0
M_{y5}	3336	-3095	9675	1306	3660	0	0	0
M_{x5}	2135	-1845	6852	-2101	-7544	0	0	0
Q_5	-12655	13971	-33310	-3571	107	0	0	0
M_y	7649	-2760	3657	-2071	9100	-1895	6852	-7544
M_{y7}	3336	2135	-12655	0	0	5440	22522	0
M_{x7}	-3095	-1545	13977	0	0	-4271	-18106	0
Q_7	9675	6852	-33314	0	0	12304	58814	0
M_{y8}	-4485	-7318	8951	3336	-12655	8693	-21090	22822
M_{y9}	5774	4154	-14074	1306	-3571	-4908	16096	1599
M_{x9}	6130	5222	-16138	-383	4072	-4271	12334	-2676

STRUCTURAL STIFFNESS MATRIX [K] STRUCTURAL

Appendix 1f-b

$$
\begin{bmatrix}
42.24 & 0 & -18.24 & 0 & -1.44 \\
 & 12.16 & 8.56 & 1.92 & 1.72 \\
 & & 39.36 & 1.72 & -9.12 \\
 & \text{SYMMETRICAL} & & 6.08 & 9.28 \\
 & & & & 10.56
\end{bmatrix}
\begin{Bmatrix} W_3 \\ \Phi_3 \\ W_6 \\ \Phi_6 \\ W_9 \end{Bmatrix}
= \lambda
\begin{bmatrix}
13816 & 0 & 5304 & 0 & 394 \\
 & 640 & 1098 & 160 & 232 \\
 & & 14504 & 232 & 2452 \\
 & \text{SYMMETRICAL} & & 320 & 548 \\
 & & & & 3452
\end{bmatrix}
\begin{Bmatrix} W_3 \\ \Phi_3 \\ W_6 \\ \Phi_6 \\ W_9 \end{Bmatrix}
$$

$$
\lambda = \frac{\rho a^2 b^2 p^2}{25200D}
$$

VIBRATION ASSUMING DISPLACEMENTS

$$
\begin{bmatrix}
1.5922 & 0.5222 & 2.0668 & -5.0726 & 5.4334 \\
 & -6.7235 & -0.6622 & 13.5288 & -5.2356 \\
 & & 0.1517 & 4.1468 & -1.4446 \\
 & \text{SYMMETRICAL} & & -13.3418 & 5.892 \\
 & & & & -0.0756
\end{bmatrix}
\begin{Bmatrix} \Phi_4 \\ W_4 \\ \Phi_6 \\ W_6 \\ W_9 \end{Bmatrix}
= \lambda
\begin{bmatrix}
640 & 0 & 160 & 1096 & 232 \\
 & 13816 & 0 & 5304 & 394 \\
 & & 320 & 232 & 548 \\
 & \text{SYMMETRICAL} & & 14504 & 2452 \\
 & & & & 3452
\end{bmatrix}
\begin{Bmatrix} \Phi_4 \\ W_4 \\ \Phi_6 \\ W_6 \\ W_9 \end{Bmatrix}
$$

$$
\lambda = \frac{\rho a^2 b^2 p^2}{64000D}
$$

VIBRATION ASSUMING STRESSES

Appendix 1g

VITA

Deepak Laxman Naik

Candidate for the Degree of

Master of Science

Report: BENDING AND VIBRATION OF PLATES BY FINITE ELEMENT METHOD

Major Field: Civil Engineering

Biographical:

Personal Data: Born in Supa, Mysore State, India, May 3, 1944, the son of Laxman M. and Tara Naik.

Education: Graduated from Gibb High School Kumta in 1960. Received the Bachelor of Technology degree from the Indian Institute of Technology, Bombay, with major in Civil Engineering in June, 1966; completed requirements for the Master of Science degree in July, 1967.

Professional experience: None.