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JOHN KWANG-MING CHENG

Norman, Oklahoma

1969

APPLICATION OF MATRIX METHOD TO

GENERAL ENGINEERING STRUCTURES

INCLUDING SPACE MEMBERS

APPROVED BY

H. Alden Gillespie

L. A. Comp

Arthur Bernhart

Geo. W. R.

Fulton K. Fears

DISSERTATION COMMITTEE

To My Family

In order to remember each single member of my family, from my grandparents to my children, I dedicate this dissertation to all of them, especially to my grandmother and mother from whom I have been deeply influenced by their spirits of industry and conduct of life, to my third brother James Kwang-T'seng (成匡錚), and to my only sister Mary Mei-P'ing (成梅屏).

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CHAPTER I

INTRODUCTION

Most of the materials used in building structures are elastic, such as steel, wrought iron and wood, and follow Hooke's law when the stress does not exceed the proportional limit.

Reinforced concrete may even be considered elastic if the stresses are not too great. Structures composed of elastic members which are subjected to forces or to imposed deformations will undergo deformations (or small changes in shape). This will cause points within the structure to be displaced to new positions. In general, all points of the structure except immovable points of support will undergo such displacements. The calculation of these displacements is an essential part of structural analysis. Therefore, "the objective in analyzing a structure is the determination of stresses and displacements of all points of the material" (6).

"In the matrix method of analysis, an essential step in developing the matrix method of analysis is the development of the relations between the loads and the deflections of the individual elements"(10). When the relations between the end actions and the end displacements are established, the end actions can be determined through the relations when the end displacements are given or vice versa. And when both the end actions and the end displacements are known, the stress and deflection conditions can be obtained to meet the purpose of the analysis of a structure.

"Under the actions applied to its ends a member undergoes deformation. For any member, relationships will exist between the end actions, p, and end displacements, u, and these will be functions of the shape, size and elastic properties of the member" (6). The resulting relations between the end actions and the end displacements are represented by stiffness and flexibility matrices. Thus stiffness and flexibility matrices are the objectives of the matrix method of analysis of a structure.

Four Kinds of Stiffness under Consideration

The most common stresses under consideration for a structure are normal stress, transverse shearing stress, bending stress, and torsional stress. These stresses are due to axial load, transverse load, bending load, and torsional load respectively.

A structure is built to perform a certain function. In order to perform this function satisfactorily it must have sufficient strength and rigidity; the kinds of stiffnesses which have to be developed for the structural analysis must correspond to the above mentioned stresses, i.e., axial stiffness, shearing stiffness, bending stiffness and torsional stiffness. In other words, the relations between loads and deflections "are dependent on the type of loading to which the element is subjected. The types of member loading to be considered are axial, bending and twist" (10).

All framed structures consist of members and joints, which are points of intersection of the members, points of support, and free ends of members. Therefore, when one analyzes a framed structure by the matrix method, he can consider a framed structure as an assembly of structural members connected at a finite number of points, which will be referred to as nodel points.

Owing to the existing reality, for the purpose of analysis and using the concept of free bodies one may break a framed structure into smaller sections or directly into individual members, i.e., for the simplicity of analysis one may consider each structural member as a unit to be analyzed. It is evident in the matrix theory that any matrix can be partitioned into submatrices of different orders by passing through it some horizontal and vertical lines.

Using the idea of breaking down the members of a framed structure and the idea of partitioning a matrix into any number of submatrices of different orders, one may isolate each member of the frame as an individual unit and thus find the stiffness matrix for each of them by either the displacement method or the force method based on their own local axes. By the transformation of axes, which may be easily done with matrix rotation operations, one may transfer the established stiffness matrices based on the local axes of each member to the stiffness matrices based on the reference axes system for the frame as a whole.

Then, when one combines all the stiffness matrices for the individual elements of the structure, he will get the stiffness matrix for the whole structure. This will relate the displacement vector to the load vector, i.e., by means of the stiffness matrix of a structure one may find the nodal actions from the given nodal displacements, or using the flexibility matrix one may find the nodal displacements from the given nodal actions. If it is necessary, one may invert the stiffness matrix into the flexibility matrix because of the reciprocal relation between stiffness and flexibility. Therefore, with either of these two matrices one may find the other by matrix inversion.

As has been mentioned above, for the purpose of analysis, any framed structure may be considered as an assembly of structural members connected at a finite number of points, and each structural member may be taken as a unit to be analyzed. It has also been pointed out above that any matrix may be partitioned into any number of submatrices of any size. With these two things in mind, one may take just one member for general investigation. Either the stiffness matrix or the flexibility matrix of this member will be used universally on any other members of the same type with some modifications, such as the modulus of elasticity in tension, compression and shearing of material, member size, and the loading type. In other words, the author is replacing the actual continuous structure by a mathematical model made up of structural elements of finite size which are in systematic combination.

By definition a matrix is a rectangular array of numbers arranged in rows and columns. In the array each number which is called an element or quantity in the matrix algebra may be addressed by double subscripts. Therefore, if one considers a system of simultaneous linear equations, it is apparent that one may put all the coefficients in one array, addressed with double subscripts, and the constant terms in another array, addressed with one subscript. Then all investigations of the system may be carried out by working only with the array of coefficients and the array of constant terms by matrix operations.

In the process of analyzing a structure, no matter whether one is dealing with a problem of analysis of a highly statically indeterminate structure to find the redundant components, or with the problem of determining the stress elements and deflection components at the different points of a simply supported beam, one always deals with a system of simultaneous linear equations in which the number of equations can be rather large. With the benefit of the matrix approach and the operations that are performed on a matrix, this can be done rapidly. Besides, in the problems of finding out redundants, stress components, and deflection components, one always needs to indicate each of those quantities with double subscripts when they are analyzed by structural theories, and matrix algebra will meet this need.

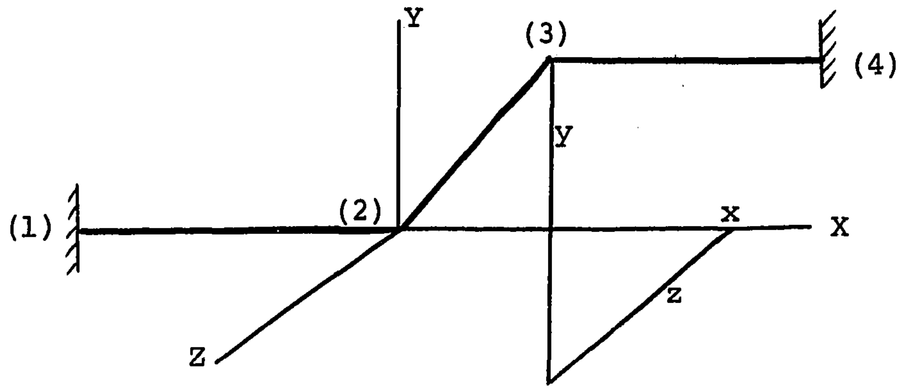
The solution of these equations which are used to express the theory that is used to analyze a highly indeterminate structure can be best achieved as a sequence of numerical matrix calculations. Since the elements in a matrix are in systematic arrangement and are addressed with double subscripts, matrices are useful not only in expressing the theory to analyze a structure, but also in providing convenient means for carrying out the numerical calculations. The high speed digital computer is ideally adopted to carrying out the matrix operation needed in the solution of the equations which express the structural theory used to analyze highly indeterminate structures.

Therefore, once the total stiffness or flexibility matrices for a whole structure are available, the complete solution to the structural analysis problem follows from a routine set of numerical matrix calculations with the use of the high speed digital computer which provides high accuracy and speed to meet the requirements of the design of complex structures.

Representative Problem in Research

Space frames are the most general type of framed structures, inasmuch as there are no restrictions on the locations of joints, directions of members or directions of loads. The individual members of a space frame may carry internal axial forces, bending couples, torsional couples and shearing forces in both principal directions of the cross-section. The members are assumed to have two axes of symmetry in their cross-sections.

The reason the author has chosen this problem for research is that it represents the most general situations in frame structural analysis.



(Fig. 1)

How to Set the Matrix Equation for Matrix Methods of Structural Analysis

"Simultaneous linear equations occur very frequently in structural analysis, . . ." (18) "In the analysis of linear structures. . .we shall also require the solution of linear equations" (18). For instance, "When the forces at the coordinates are given, we shall solve for the corresponding displacements, and when the displacements are given we shall solve for the corresponding forces" (18). No matter whether the energy method or the elastic weight method is going to be used, a set of linear equations will be set up.

If n known displacements are given, then a set of n linear equations can be established, as:

$$\Sigma k_{1i} u_i = F_i$$

$$\Sigma k_{2i} u_i = F_2$$

$$\vdots$$

$$\Sigma k_{ni} u_i = F_n$$

$i = 1,2...n.$ (n = total number of displacements)

With the convenience of the matrix theory, a matrix equation can be set up as

$$[k][u] = [F]$$

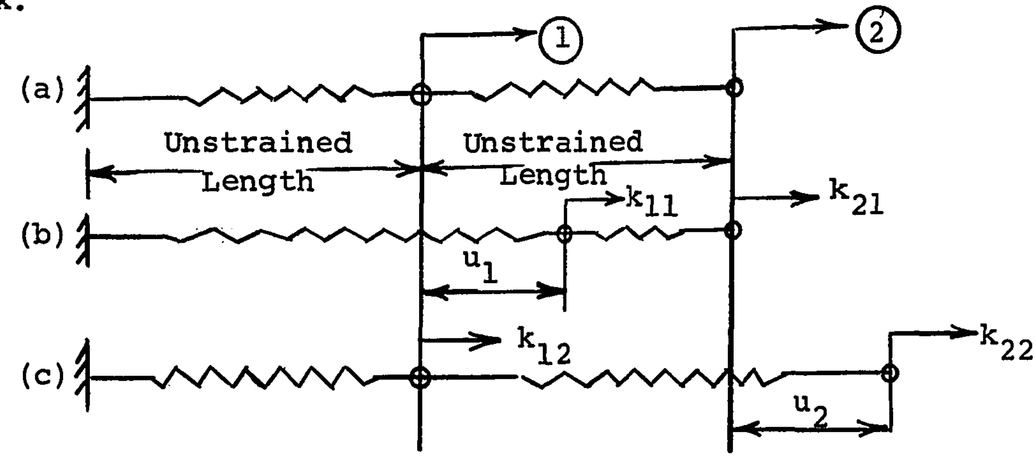
in which $[k]$ is a coefficient matrix which is called a stiffness matrix in the matrix method of structural analysis,

$[u]$ is a column matrix containing all given displacements,

$[F]$ is a column matrix containing the required actions.

"Example" (18)

This example is used to show how to establish a stiffness matrix.



(Fig. 2)

First a unit displacement is applied at 1 only, Fig.(b), and the required forces at 1 and 2 are designated as $k_{1,1}$ and $k_{2,1}$ respectively. Similarly in Fig.(c), forces $k_{1,2}$ and $k_{2,2}$ are required to cause a unit displacement at 2 only. In other words,

$u_1=1$, all other $u_i=0$ generates $k_{i1}(i=1,2)$. The forces required to cause displacement u_1 are $k_{i1}u_1(i=1,2)$.

$u_2=1$, all other $u_i=0$ generates $k_{i2}(i=1,2)$. The forces required to cause displacement u_2 are $k_{i2}u_2(i=1,2)$.

The forces required to produce displacements u_1 and u_2 simultaneously are obtained by a superposition of Figures (b) and (c) which yields the following 2 simultaneous linear equations:

$$F_1 = k_{11}\, u_1 + k_{12}\, u_2$$

$$F_2 = k_{21}\, u_1 + k_{22}\, u_2$$

The above 2 simultaneous linear equations can be put in matrix equation as

$$\begin{bmatrix} F_1 \\ \\ F_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix}$$

In order to generate a stiffness matrix in n coordinates, the steps proceed as follows:

"Apply a unit displacement at coordinate 1 only and compute the forces $k_{i1}(i = 1,2,...n)$ required at all the coordinates; this yields the first column of the stiffness matrix. . . . Finally, to generate the n^{th} column apply a unit displacement at coordinate n only and compute $k_{in}(i = 1,2,...n)$. In general, to generate the j^{th} column of the stiffness matrix, apply a unit displacement at coordinate j only and compute the required forces $k_{ij}(i=1,2,...n)$" (18).

On the other hand, if the action informations are given, the displacements at all the nodal points are required; then a set of simultaneous linear equations can be established, as:

$$\Sigma f_{1i}F_i = u_1$$

$$\Sigma f_{2i}F_i = u_2$$

$$\vdots$$

$$\Sigma f_{ni}F_i = u_n$$

$i = 1,2,...n$ (n=total number of actions)

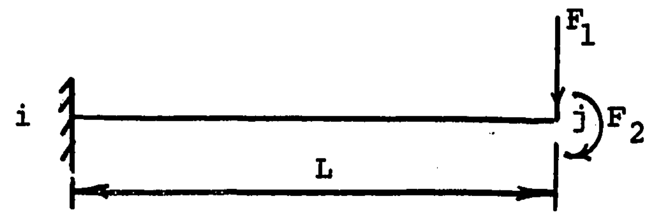
A matrix equation can be formed for the n linear equations as

$$[f][F] = [u]$$

in which [f] is the coefficient matrix which is called the flexibility matrix in the matrix method of structural analysis, [F] is a column matrix containing the given actions, [u] is the required displacement matrix.

Example:

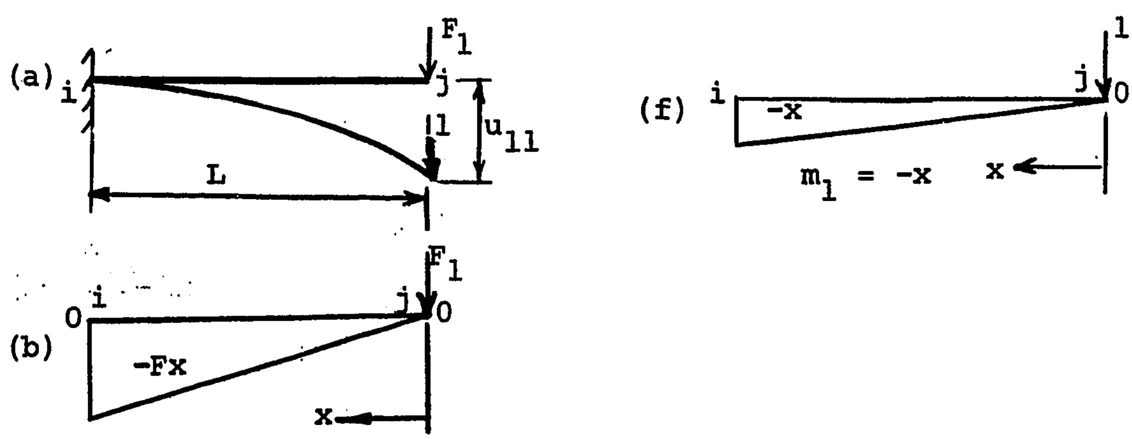
This example is used to show how to establish a flexibility matrix.



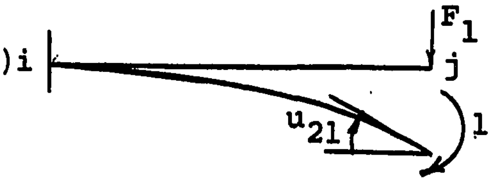
EI = constant

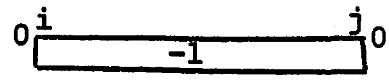
(Fig. 3)

The virtual work method is used to find the flexibility matrix for the end j as follows:



$$u_{11} = \int_0^L \frac{(-x)(-F_1 x)}{EI}\,dx = \frac{F_1}{EI}\int_0^L (x^2)\,dx = \frac{F_1}{EI}\left(\frac{x^3}{3}\right)_0^L = \frac{F_1 L^3}{3EI}$$

(c) 

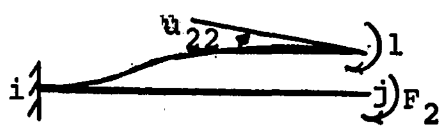
(g) 

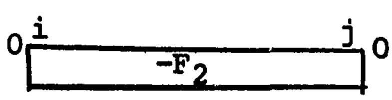
$m_2 = -1$

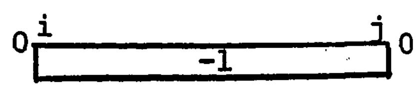
$$u_{21} = \int_0^L \frac{(-1)(-F_1 x)}{EI}\,dx = \frac{F_1}{EI}\int_0^L x\,dx = \frac{F_1}{EI}\left(\frac{x^2}{2}\right)_0^L = \frac{F_1 L^2}{2EI}$$

By reciprocal theorem,

$$u_{12} = u_{21} = \frac{F_1 L^2}{2EI}$$

(d) 

(e) 

(h) 

$$u_{22} = \int_0^L \frac{(-1)(-F_2)}{EI}\,dx = \frac{F_2}{EI}\int_0^L dx = \frac{F_2 L}{EI}$$

(Fig. 4)

The final total displacement at coordinate 1 is u_1

$$u_1 = u_{11} + u_{12} = \frac{F_1 L^3}{3EI} + \frac{F_2 L^2}{2EI}$$

The final total displacement at coordinate 2 is u_2

$$u_2 = u_{21} + u_{22} = \frac{F_1 L^2}{2EI} + \frac{F_2 L}{EI}$$

The above two equations can be put in a matrix equation as

$$\begin{bmatrix} u_1 \\ \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ \\ \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix} \begin{bmatrix} F_1 \\ \\ F_2 \end{bmatrix}$$

Generally, in order to generate a flexibility matrix in n coordinates, one should proceed as follows: "Apply a unit force at coordinate 1 only and compute the displacements a_{i1} $(i = 1,2,\ldots n)$ at all the coordinates; this yields the first column of the flexibility matrix. . . . Finally, to generate the n^{th} column apply the unit force at coordinate n only and compute a_{in} $(i = 1,2,\ldots n)$. In general then, to generate the j^{th} column of the flexibility matrix apply a unit force at coordinate j only and compute a_{ij} $(i = 1,2,\ldots n)$" (18).

Method of Analysis

In this dissertation, the method which is adopted by the author to find the relations between the actions and the displacements is the displacement method.

In the process of using the displacement method for the matrix method of structural analysis, "We focus our attention on displacements at selected coordinates and produce them separately one at a time, then apply superposition to yield a final configuration. The stresses and displacements at any point on the structure in the final configuration will be equal to the sum of the corresponding values in the superposed configurations" (18). In general, the displacements at the coordinates of a member are introduced separately one at a time, and then the actions are computed at the coordinates of the member at each stage of displacement in terms of the displacement introduced. By the superposition of all the displacements at the different stages, the final total stiffness at each coordinate of the final displaced configuration is obtained by summing up the corresponding values in the superposed configuration. The method used to find the stiffness coefficients at each stage of displacement can be either the elastic weight method or the strain energy method. In this dissertation, the elastic weight method, such as the conjugate beam method, was adopted by the author. The procedure of the analysis of a framed structure is performed member by member. This means that the member stiffness matrix of each individual member which is considered as a submatrix of the overall stiffness matrix of the whole structure is obtained first; then the forming of the total structure stiffness matrix is most conveniently accomplished by superimposing the stiffness matrices of the individual structural members.

Since ". . . the rigid body in space, . . . has six degrees of freedom, . . ." (18) the member stiffness matrix of a member in space will be of the size 12 x 12, which is usually shown as the notation on page 17, Table 1.

The meaning of the two subscripts of each stiffness coefficient is that the first subscript is the coordinate where the force is measured, and the second subscript is the coordinate where the unit displacement is applied.

"Since the loading sequence has no effect on the final value of the strain energy, U, this same energy will be obtained in the following two loading sequences in which all forces are applied gradually"(18). "We can use the fact that the strain energy in a structure is independent of the loading history to prove that the stiffness and flexibility matrices are symmetrical" (18).

Owing to the property of symmetry of a stiffness matrix, Prof. L A. Comp made an abbreviation for the stiffness matrix of a member in space, in order to save some labor in doing the computations, as shown on page 18, Table 2.

TABLE 1

NOTATION OF MEMBER STIFFNESS MATRIX OF A MEMBER IN SPACE

	Δ_x^i	Δ_y^i	Δ_z^i	θ_x^i	θ_y^i	θ_z^i	Δ_x^j	Δ_y^j	Δ_z^j	θ_x^j	θ_y^j	θ_z^j
V_x^i	$k_{1,1}$	$k_{1,2}$	$k_{1,3}$	$k_{1,4}$	$k_{1,5}$	$k_{1,6}$	$k_{1,7}$	$k_{1,8}$	$k_{1,9}$	$k_{1,10}$	$k_{1,11}$	$k_{1,12}$
V_y^i	$k_{2,1}$	$k_{2,2}$	$k_{2,3}$	$k_{2,4}$	$k_{2,5}$	$k_{2,6}$	$k_{2,7}$	$k_{2,8}$	$k_{2,9}$	$k_{2,10}$	$k_{2,11}$	$k_{2,12}$
V_z^i	$k_{3,1}$	$k_{3,2}$	$k_{3,3}$	$k_{3,4}$	$k_{3,5}$	$k_{3,6}$	$k_{3,7}$	$k_{3,8}$	$k_{3,9}$	$k_{3,10}$	$k_{3,11}$	$k_{3,12}$
M_x^i	$k_{4,1}$	$k_{4,2}$	$k_{4,3}$	$k_{4,4}$	$k_{4,5}$	$k_{4,6}$	$k_{4,7}$	$k_{4,8}$	$k_{4,9}$	$k_{4,10}$	$k_{4,11}$	$k_{4,12}$
M_y^i	$k_{5,1}$	$k_{5,2}$	$k_{5,3}$	$k_{5,4}$	$k_{5,5}$	$k_{5,6}$	$k_{5,7}$	$k_{5,8}$	$k_{5,9}$	$k_{5,10}$	$k_{5,11}$	$k_{5,12}$
M_z^i	$k_{6,1}$	$k_{6,2}$	$k_{6,3}$	$k_{6,4}$	$k_{6,5}$	$k_{6,6}$	$k_{6,7}$	$k_{6,8}$	$k_{6,9}$	$k_{6,10}$	$k_{6,11}$	$k_{6,12}$
V_x^j	$k_{7,1}$	$k_{7,2}$	$k_{7,3}$	$k_{7,4}$	$k_{7,5}$	$k_{7,6}$	$k_{7,7}$	$k_{7,8}$	$k_{7,9}$	$k_{7,10}$	$k_{7,11}$	$k_{7,12}$
V_y^j	$k_{8,1}$	$k_{8,2}$	$k_{8,3}$	$k_{8,4}$	$k_{8,5}$	$k_{8,6}$	$k_{8,7}$	$k_{8,8}$	$k_{8,9}$	$k_{8,10}$	$k_{8,11}$	$k_{8,12}$
V_z^j	$k_{9,1}$	$k_{9,2}$	$k_{9,3}$	$k_{9,4}$	$k_{9,5}$	$k_{9,6}$	$k_{9,7}$	$k_{9,8}$	$k_{9,9}$	$k_{9,10}$	$k_{9,11}$	$k_{9,12}$
M_x^j	$k_{10,1}$	$k_{10,2}$	$k_{10,3}$	$k_{10,4}$	$k_{10,5}$	$k_{10,6}$	$k_{10,7}$	$k_{10,8}$	$k_{10,9}$	$k_{10,10}$	$k_{10,11}$	$k_{10,12}$
M_y^j	$k_{11,1}$	$k_{11,2}$	$k_{11,3}$	$k_{11,4}$	$k_{11,5}$	$k_{11,6}$	$k_{11,7}$	$k_{11,8}$	$k_{11,9}$	$k_{11,10}$	$k_{11,11}$	$k_{11,12}$
M_z^j	$k_{12,1}$	$k_{12,2}$	$k_{12,3}$	$k_{12,4}$	$k_{12,5}$	$k_{12,6}$	$k_{12,7}$	$k_{12,8}$	$k_{12,9}$	$k_{12,10}$	$k_{12,11}$	$k_{12,12}$

Note: i---member---j

TABLE 2

ABBREVIATED NOTATION OF MEMBER STIFFNESS MATRIX OF A MEMBER IN SPACE

(Abbreviated by Prof. L. A. Comp)

	Δ^i_x	Δ^i_y	Δ^i_z	θ^i_x	θ^i_y	θ^i_z	Δ^j_x	Δ^j_y	Δ^j_z	θ^j_x	θ^j_y	θ^j_z
V^i_x	$k_{1,1}$	$k_{1,2}$	$k_{1,3}$	$k_{1,4}$	$k_{1,5}$	$k_{1,6}$	$-k_{1,1}$	$-k_{1,2}$	$-k_{1,3}$	$k_{1,4}$	$k_{1,5}$	$k_{1,6}$
V^i_y		$k_{2,2}$	$k_{2,3}$	$k_{2,4}$	$k_{2,5}$	$k_{2,6}$	$-k_{1,2}$	$-k_{2,2}$	$-k_{2,3}$	$k_{2,4}$	0	$k_{2,6}$
V^i_z			$k_{3,3}$	$k_{3,4}$	$k_{3,5}$	$k_{3,6}$	$-k_{1,3}$	$-k_{2,3}$	$-k_{3,3}$	$k_{3,4}$	$k_{3,5}$	$k_{3,6}$
M^i_x				$k_{4,4}$	$k_{4,5}$	$k_{4,6}$	$-k_{1,4}$	$-k_{2,4}$	$-k_{3,4}$	$k_{4,10}$	$k_{4,11}$	$k_{4,12}$
M^i_y					$k_{5,5}$	$k_{5,6}$	$-k_{1,5}$	0	$-k_{3,5}$	$k_{4,11}$	$k_{5,11}$	$k_{5,12}$
M^i_z						$k_{6,6}$	$-k_{1,6}$	$-k_{2,6}$	$-k_{3,6}$	$k_{4,12}$	$k_{5,12}$	$k_{6,12}$
V^j_x							$k_{1,1}$	$k_{1,2}$	$k_{1,3}$	$-k_{1,4}$	$-k_{1,5}$	$-k_{1,6}$
V^j_y								$k_{2,2}$	$k_{2,3}$	$-k_{2,4}$	0	$-k_{2,6}$
V^j_z									$k_{3,3}$	$-k_{3,4}$	$-k_{3,5}$	$-k_{3,6}$
M^j_x										$k_{4,4}$	$k_{4,5}$	$k_{4,6}$
M^j_y											$k_{5,5}$	$k_{5,6}$
M^j_z												$k_{6,6}$

Note: $k_{ij} = k_{ji}$, $i \neq j$

i $\underline{\text{member}}$ j

CHAPTER II

NOTATION CONVENTION

Since in the matrix method of structural analysis there are a large number of symbols to be used, it is necessary to have a notation convention set up in order to have the symbols organized so that readers have a system to follow.

The notations that are to be used in this dissertation, arranged in alphabetical order are as follows:

A: cross-sectional area of a member.

A^i: the area of the elastic weight diagram with the bending moment applied at the end i of the member i-j.

A^j: the area of the elastic weight diagram with the bending moment applied at the end j of the member i-j.

A_E: equivalent load vector in structure-axes system.

A_m: action matrix based on local axes system.

A_s: action matrix based on structure-axes system.

D_m: displacement matrix based on the local axes system.

D_s: displacement matrix based on the structure-axes system.

e^i: axial deformation at the end i of the member i-j.

e^j: axial deformation at the end j of the member i-j.

E: modulus of elasticity.

G: shearing modulus of elasticity.

I: identity matrix

$I_x=J$: polar moment of inertia of the cross-section of the member i-j. (circular only)

I_y: moment of inertia w.r.t. the Y-axis of the cross-section of the member i-j.

I_z: moment of inertia, w.r.t. the Z-axis of the cross-section of the member i-j.

k_{nm}: stiffness coefficient at the coordinate n due to a unit displacement at the coordinate m.

L: the length of the member i-j.

M_x^i: moment about the X-axis at the end i of the member i-j.

M_x^j: moment about the X-axis at the end j of the member i-j.

M_y^i: moment about the Y-axis at the end i of the member i-j.

M_y^j: moment about the Y-axis at the end j of the member i-j.

M_z^i: moment about the Z-axis at the end i of the member i-j.

M_z^j: moment about the Z-axis at the end j of the member i-j.

$\bar{M}^i$: submatrix of the action matrix based on the local axes system at the end i of the member i-j (for moment actions).

$\bar{M}^j$: submatrix of the action matrix based on the local axes system at the end j of the member i-j (for moment actions).

M^Δ: couple applied at one end of the conjugate beam to balance the couple formed by the two elastic weights acting on the conjugate beam.

M^i: submatrix of the action matrix based on the structure-axes system at the end i of the member i-j (for the moment actions).

M^j: submatrix of the action matrix based on the structure-axes system at the end j of the member i-j (for moment actions).

r_i: the radius of the cross-section at the end i of a nonprismatic member with circular cross-section.

r_j: the radius of the cross-section at the end j of a nonprismatic member with circular cross-section.

r_x: the radius of a cross-section which is x distance from the origin which is taken at the left end of the member i-j.

R_1: rotational transformation matrix about Y-axis.

R_2: rotational transformation matrix about Z-axis.

R: rotational transformation matrix about both Y- and Z-axes.

R^T: transpose of R.

R^{-1}: inverse of R.

R^i: rotational transformation matrix for the three orthogonal axes at the end i of the member i-j.

R^i_T: total rotational transformation matrix for the end i of the member i-j.

R^j: rotational transformation matrix for the three orthogonal axes at the end j of the member i-j.

R^j_T: total rotational transformation matrix for the end j of the member i-j.

R_T: total rotational transformation matrix for the member i-j.

s^{i-i}: submatrix of a stiffness matrix of a member i-j which relates the stiffnesses at the coordinates at the end i to the unit displacements at the coordinates at the end i.

s^{i-j}: submatrix of a stiffness matrix of a member i-j which relates the stiffnesses at the coordinates at the end i to the unit displacements at the coordinates at the end j.

s^{j-i}: submatrix of a stiffness matrix of a member i-j which relates the stiffnesses at the coordinates at the end j to the unit displacements at the coordinates at the end i.

s^{j-j}: submatrix of a stiffness matrix of a member i-j which relates the stiffnesses at the coordinates at the end j to the unit displacements at the coordinates at the end j.

s_m^{i-j}: stiffness matrix of a member i-j based on the local axes system.

s_s^{i-j}: stiffness matrix of a member i-j based on the structure-axes system.

V_x^i: force in the X-direction at the end i of the member i-j.

V_x^j: force in the X-direction at the end j of the member i-j.

V_y^i: force in the Y-direction at the end i of the member i-j.

V_y^j: force in the Y-direction at the end j of the member i-j.

V_z^i: force in the Z-direction at the end i of the member i-j.

V_z^j: force in the Z-direction at the end j of the member i-j.

$\bar{V}^i$: submatrix of the action matrix based on the local axes system at the end i of the member i-j (for linear actions).

$\bar{V}^j$: submatrix of the action matrix based on the local axes system at the end j of the member i-j (for linear actions).

V^i: submatrix of the action matrix based on the structure-axes system at the end i of the member i-j (for linear actions).

V^j: submatrix of the action matrix based on the structure-axes system at the end j of the member i-j (for linear actions).

Δ_x^i: linear displacement in the X-direction at the end i of the member i-j.

Δ_x^j: linear displacement in the X-direction at the end j of the member i-j.

Δ_y^i: linear displacement in the Y-direction at the end i of the member i-j.

Δ_y^j: linear displacement in the Y-direction at the end j of the member i-j.

Δ_z^i: linear displacement in the Z-direction at the end i of the member i-j.

Δ_z^j: linear displacement in the Z-direction at the end j of the member i-j.

θ_x^i: angular displacement about X-axis at the end i of the member i-j (twisting angle).

θ_x^j: angular displacement about X-axis at the end j of the member i-j (twisting angle).

θ_y^i: angular displacement about Y-axis at the end i of the member i-j (bending angle).

θ_y^j: angular displacement about Y-axis at the end j of the member i-j (bending angle).

θ_z^i: angular displacement about Z-axis at the end i of the member i-j (bending angle).

θ_z^j: angular displacement about Z-axis at the end j of the member i-j (bending angle).

λ_{pq}: direction cosine of the direction angle between q-axis of the structure-axes system and p-axis of the local axes system.

(p: X_m, Y_m and Z_m axes of the local axes system).

(q: X_s, Y_s and Z_s axes of the structure-axes system).

X^i: distance from the centroid of the elastic weight diagram with bending moment applied at the end-i of the member i-j to the origin, which is at the initial end i of the member.

X^j: distance from the centroid of the elastic weight diagram with bending moment applied at the end-j of the member i-j to the origin, which is at the initial end i of the member.

w.r.t.: with respect to.

CHAPTER III

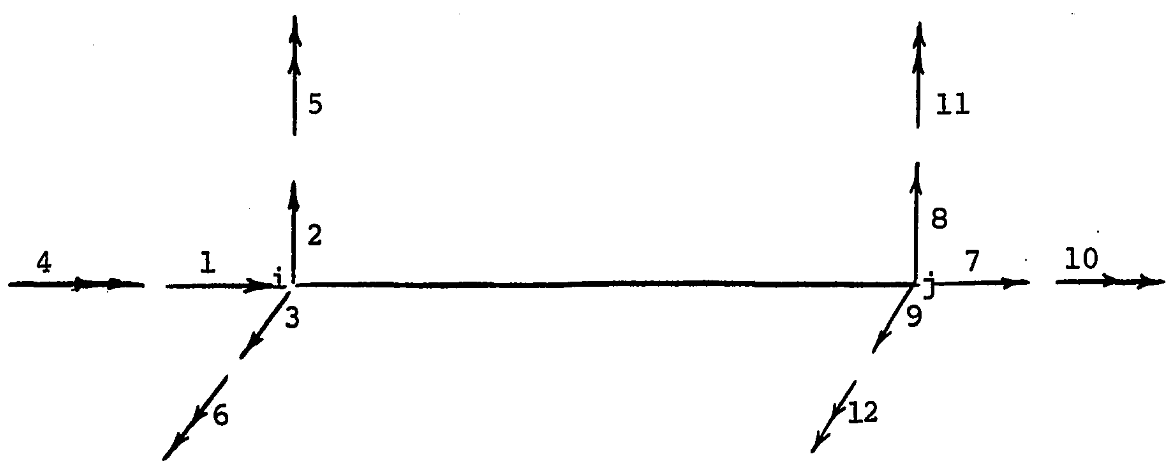
STIFFNESS MATRIX FOR A PRISMATIC MEMBER IN THREE-DIMENSIONAL SPACE

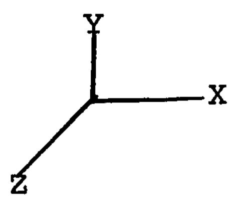
Coordinate System

The coordinate system is employed both to systematize the action and displacement systems of a structure and to make it easy to build up the correspondence-relation between the action system and the displacement system. Thus, with the help of the coordinate system, the measurements (actions or displacements) in a structure, for the purpose of structural analysis, can be easily identified, at which point and in which direction.

In addition to identifying measurements in a structure, the coordinate system can also be used to indicate the address of an element in either a stiffness matrix or a flexibility matrix. Since the matrix method of structural analysis deals with actions and displacements that correspond to one another, one coordinate system in each problem is good for both actions and displacements. Therefore, in the matrix method of structure analysis, a coordinate system must be set up for each problem.

Since a rigid body in a three-dimensional space has six degrees of freedom--that is each end of a member can have six possible components of displacement--a beamlike structural element which is a part of a space frame is defined for 12 coordinates as shown in the following figures.

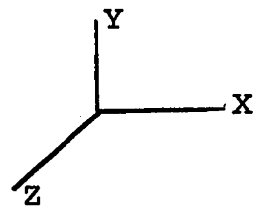




Coordinate System

End i is the initial end, and end j is the terminal end. By the framework sign convention, all the vectors are pointing in the positive directions.

The single arrow-headed vector indicates a linear action or displacement; the double arrow-headed vectors indicate moments or angular displacements. The referring of the coordinate system to the axes system is shown in the following figure



Axes System

To make it easier to calculate the stiffness coefficients of a member in a three-dimensional space, two kinds of axis systems can always be set up: one is a member-axes system; the other is a structure-axes system. Although it is convenient to describe the internal forces and displacements in terms of a member-axes system, the structure-axes system will be required for the consideration of the structure as a whole.

In the member-axes system, the X-axis will be taken along the axis of the member, while the other two axes will constitute the Y- and Z-axes. The structure-axes system is the reference axes system which is conveniently oriented w.r.t. the overall structure.

If the member-axes system for a member is not coincident with the structure-axes system for the structure of which the member is an element, then the rotational transformation of the member-axes system is needed.

Development of Stiffness Coefficients of a Prismatic Member in Space

It is obviously known that the common objective in the analysis of structures is to find the internal forces which result from the application of external loads. Knowing all the internal forces, we can compute the displacements at any point on the structure. In a word, the goal of analyzing a structure is to determine the stress and strain conditions of the structure under a given loading condition. To reach this goal, one should start with the study of the conditions of the actions and displacements at the ends of the members of the structure. When actions are exerted upon a joint by the adjacent members, equal and opposite reactions will be exerted on the members by the joint. Under reactions of this nature at both ends of any given member, the member will suffer deformation, i.e., under the action exerted by the joints on both ends of a member, it will undergo deformation. For any member the relationships will exist between the actions on the ends of the member and the end displacements of the member. Therefore, the main objective of studying the matrix method to analyze a structure is to establish the relationships between the actions and their corresponding displacements at the nodal points, usually the ends of the member. In other words, the relationships which are expressed either in stiffness coefficients or flexibility coefficients between the actions and their corresponding displacements of the nodal points of a member are the results of the matrix method of structural analysis.

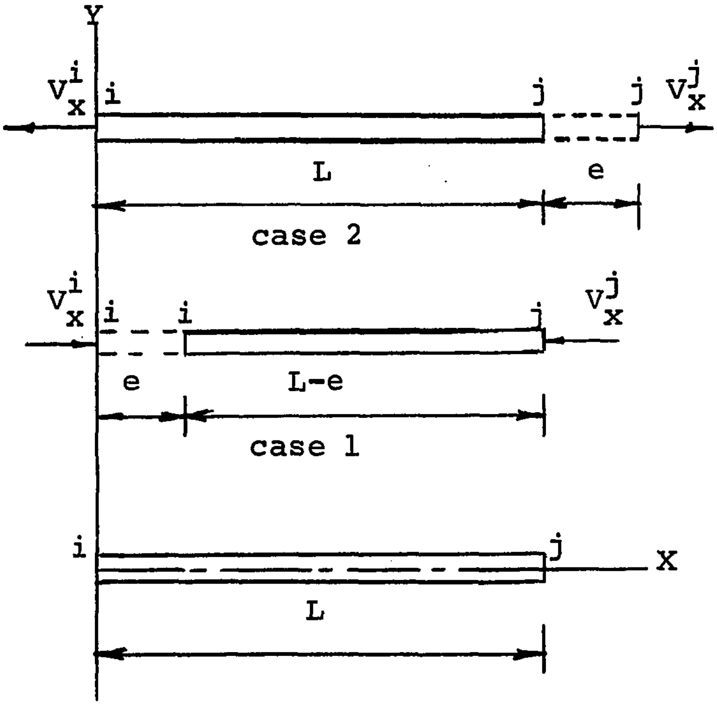
Because in the matrix method of structural analysis the relationships between the actions and the corresponding displacements at the ends of a member must be investigated, each member must be investigated individually. As a matter of fact, actual structures consist of structural components, such as tension rods, compression members, and beams properly fastened together. Therefore, by using the matrix method for analysis, the structure can be considered as an assembly of structural components connected at a finite number of points referred to as nodal points. And in the process of analysis, each member must be taken off as a free body; that is, each member is considered as a unit for the purpose of analysis.

When each member is taken off as a free body for analysis, each individual member will have its own axis. For the sake of convenience and ease, it is not only necessary but also more convenient to use the member-axes system to develop the stiffness coefficients. In the member-axes system, one of the three axes which constitutes a space is representing the member, usually the X-axis.

The stiffness coefficients the author is working on are therefore developed on the basis of the member-axes system, and the unit displacements are introduced in the positive directions of the coordinates. The principle of superposition of the displacements is the basic idea for finding the final total stiffness at each coordinate.

The principal types of deformations to be considered are axial, flexural, torsional, and shearing deformations. The stiffness coefficients to be developed are therefore axial, bending, torsional and shearing.

Axial Stiffness



$(V_x^i = P_i$, and $V_x^j = P_j)$

It is assumed that the i^{th} end of the member is the initial end and the j^{th} end the terminal end.

The axial stiffness at both ends can be developed in two cases:

Case 1

It is assumed that the i^{th} end is free to have an axial displacement in the positive direction of the X-axis by the amount, e_i, and j^{th} end is restrained from displacing.

Assume that the cross-sectional area of the bar is A. Due to the contraction of the total amount, e_i, there must be a force, P_i, applied at the i^{th} end in the same direction as the contraction e.

By Hookes' Law $$E = \frac{\text{stress}}{\text{strain}}$$

$$\text{stress} = \frac{+P_i}{A}$$

$$\text{strain} = \frac{+e_i}{L}$$

Therefore,

$$E = \frac{\frac{+P_i}{A}}{\frac{+e_i}{L}} = \frac{P_i L}{Ae_i}, \quad P_i = \frac{AE}{L}(e_i), \quad e_i = \Delta_x^i$$

Let e_i be a unit displacement,

then

$$P_i = \frac{AE}{L} = k_{1,1}$$

where $k_{1,1}$ is the axial stiffness coefficient in the direction of coordinate 1 due to a unit displacement in the direction of coordinate 1 at the i^{th} end. To satisfy the requirement of equilibrium of the member, there must be a force P_j which is equal in magnitude, opposite in direction, and acting collinear with the force P_i, i.e.,

$$P_j = -P_i = \frac{-AE}{L} = k_{7,1}$$

where $k_{7,1}$ is the axial stiffness coefficient in the direction of coordinate 7 at the j^{th} end due to a unit displacement in the direction of coordinate 1 at the i^{th} end.

Case 2

Next let the j^{th} end of the member be free to displace axially in the positive direction of the X-axis by the amount of e_j and the i^{th} end is kept from displacing. The positive direction of displacement along the X-axis at the j^{th} end will be an elongation.

Using the same idea applied previously there must be a force, P_j, acting axially to respond to the elongation e_j at the j^{th} end.

$$E = \frac{\frac{+P_j}{A}}{\frac{+e_j}{L}}$$

$$E = \frac{P_j L}{e_j A}, \quad P_j = \frac{EA}{L}(e_j), \quad e_j = \Delta_x^j$$

if e_j is unit,

then $$P_j = \frac{EA}{L} = k_{7,7}$$

where $k_{7,7}$ is the axial stiffness coefficient in the direction of coordinate 7, due to a unit displacement in the direction of coordinate 7 at the j^{th} end.

To satisfy the equilibrium condition of the member, there must be a force, P_i, which is equal, opposite, and collinear with P_j, i.e.,

$$P_i = -P_j = \frac{-EA}{L} = k_{1,7}$$

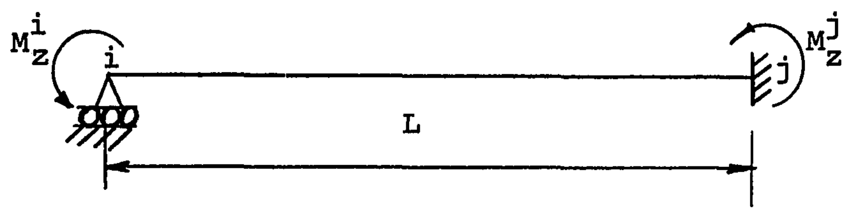
where $k_{1,7}$ is the axial stiffness coefficient in the direction of coordinate 1 at the i^{th} end due to a unit displacement in the direction of coordinate 7 at the j^{th} end.

Bending Stiffness Coefficients

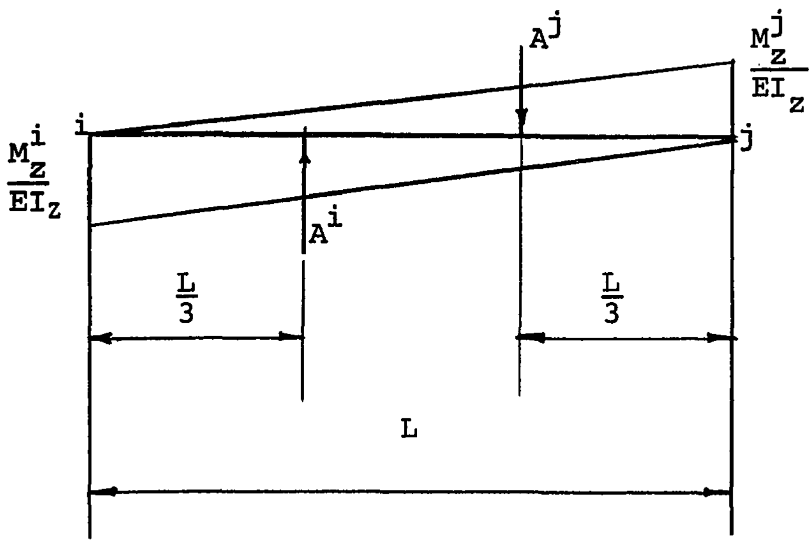
Bending Stiffness Coefficients due to End Rotation

Case 1

Suppose the end-j of the member i-j is fixed, and the rotational restraint at end-i ~~i~~s released, i.e., the end-i is free to rotate when an externally applied rotational action is applied. Now there is a moment M_z^i acting at the end-i to cause an end rotation θ_z^i and there is a moment M_z^j induced at the end-j. The relations between the θ_z^i and M_z^i as well as M_z^j can be developed as follows:



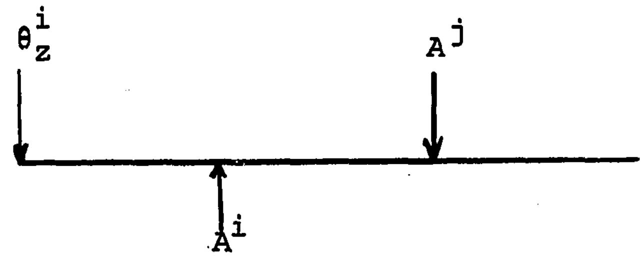
By conjugate beam method, the conjugate member is loaded with $\frac{M_z}{EI_z}$ diagram as shown below:



$$M_z^j = \frac{1}{2} M_z^i$$

$$A^i = \frac{1}{2} (L) (\frac{M_z^i}{EI_z})$$

$$A^j = \frac{1}{2}\,(L)\,(\frac{M_z^j}{EI_z})$$



$$\theta_z^i = A^i - A^j$$

$$\theta_z^i = \frac{1}{2}\,(L)\,(\frac{M_z^i}{EI_z}) - \frac{1}{2}\,(L)\,(\frac{M_z^j}{EI_z})$$

$$= \frac{1}{2}\,(L)\,(\frac{M_z^i}{EI_z}) - \frac{1}{2}\,(L)\,(\frac{M_z^i}{2EI_z}) = \frac{M_z^i L}{4EI_z}$$

from which $$M_z^i\, L = 4EI_z\;(\theta_z^i)$$

or $$M_z^i = \frac{4EI_z}{L}\;(\theta_z^i)$$

If let θ_z^i be unit (one radian), then

$$M_z^i = \frac{4EI_z}{L} = k_{6,6}$$

where $k_{6,6}$ is the bending stiffness coefficient at the coordinate 6 due to a unit displacement at coordinate 6.

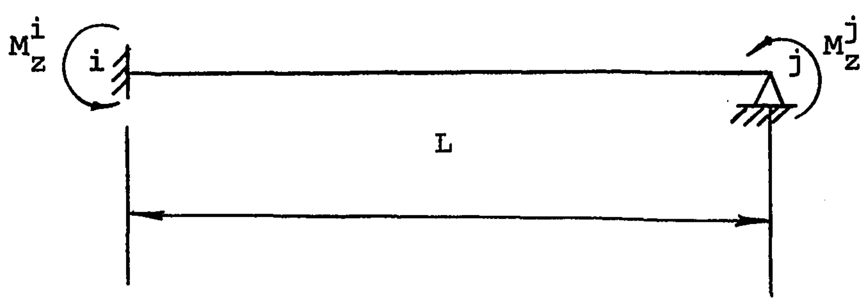
Since

$$M_z^j = \frac{1}{2}(M_z^i),\ M_z^j = \frac{2EI_z}{L} = k_{12,6}$$

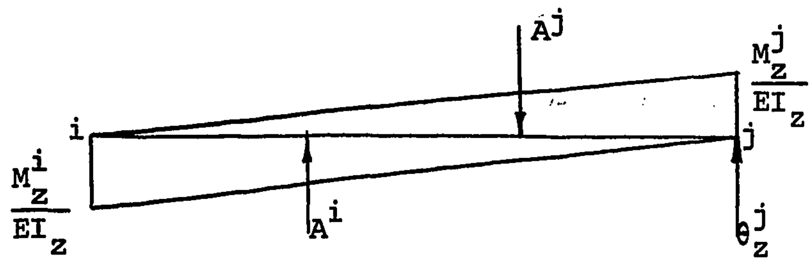
where $k_{12,6}$ is the bending stiffness coefficient at coordinate 12 due to a unit displacement at coordinate 6.

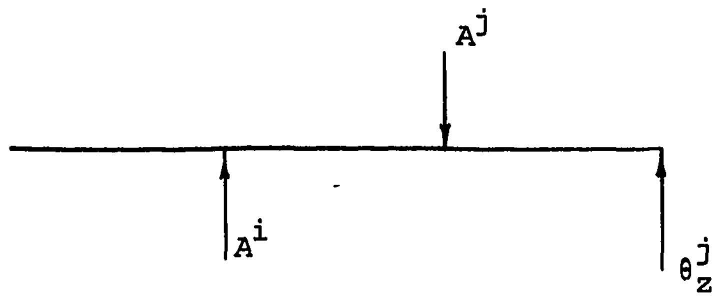
Case 2

Suppose the member i-j is fixed at end-i and simply supported at end j; then, when one releases the rotational restraint at end-j, there is a moment M_z^j acting at end-j to cause a rotation θ_z^j, as well as a moment M_z^i induced at end-i. The relations between θ_z^j and M_z^j and M_z^i can be developed as follows:



By the conjugate beam method, the conjugate member is loaded with $\frac{M_z^j}{EI_z}$ diagram as shown below:





$$A^j = \frac{1}{2}\left(\frac{M_z^j}{EI_z}\right)(L) \downarrow$$

$$A^i = \frac{1}{2}\left(\frac{M_z^i}{EI_z}\right)(L) \uparrow$$

$$M_z^i = \frac{M_z^j}{2}$$

$$A^i = \frac{1}{2}\left(\frac{M_z^j}{2EI_z}\right)(L) = \frac{M_z^j\ L}{4EI_z}$$

$$\theta_z^j = A^j - A^i = \frac{M_z^j\ L}{2EI_z} - \frac{M_z^j\ L}{4EI_z} = \frac{M_z^j\ L}{4EI_z}$$

$$M_z^j\ L = 4EI_z\ \theta_z^j$$

$$M_z^j = \frac{4EI_z}{L}\ (\theta_z^j)$$

If $\theta_z^j = 1^{rad.}$, then $M_z^j = \frac{4EI_z}{L} = k_{12,12}$

where $k_{12,12}$ is the bending stiffness coefficient at coordinate 12 due to a unit displacement at coordinate 12.

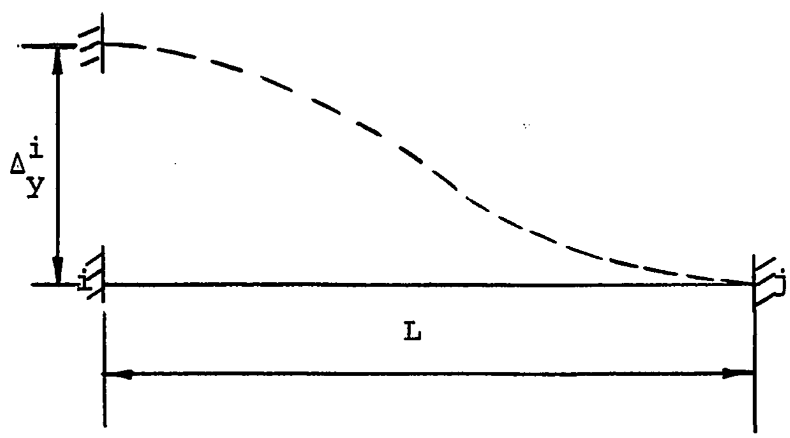
Since

$$M_z^i = \frac{1}{2} M_z^j \ , \quad M_z^i = \frac{2EI_z}{L} = k_{6,12}$$

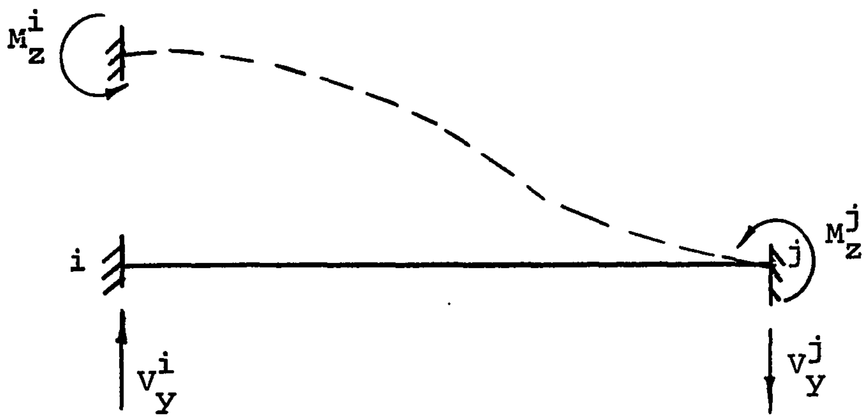
where $k_{6,12}$ is the bending stiffness coefficient at the coordinate 6 due to a unit displacement at coordinate 12.

Bending Stiffness Coefficients due to End Transverse Deflection

Case 1

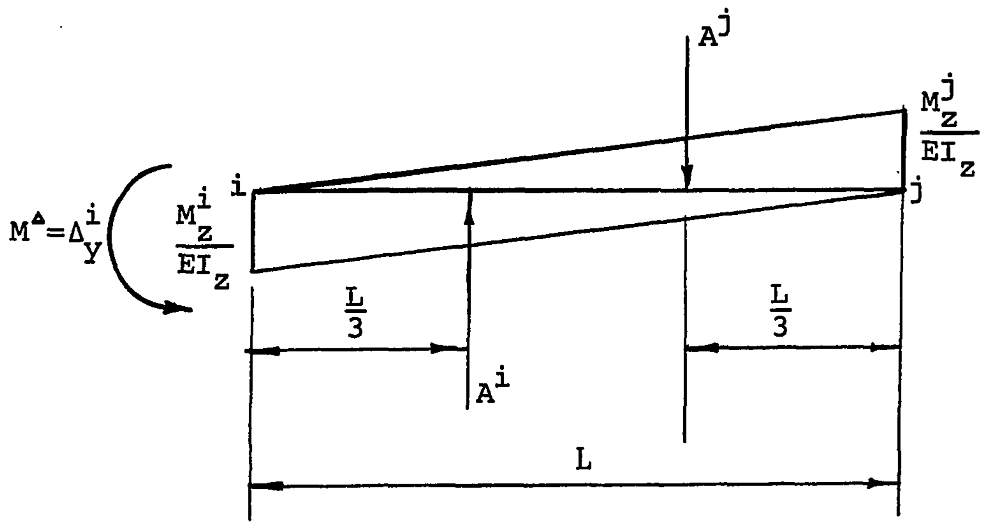


Let the end-i of the member i-j be free to displace vertically in the X-Y plane by the amount Δ_y^i w. r. t. the end-j. It can be assumed that there is an induced moment M_z^i acting at the end-i which is corresponding to the vertical displacement Δ_y^i as shown in the following sketch:



By the conjugate beam method, the relations between Δ_y^i and M_z^i as well as M_z^j can be derived. Let the conjugate beam be loaded with the diagrams of $\frac{M_z^i}{EI_z}$ and $\frac{M_z^j}{EI_z}$.

$$M_z^i = M_z^j$$



$M^\Delta = \Delta_y^i$ = moment applied at the end i of the conjugate beam

$$A^i = \frac{1}{2}\,(L)\,(\frac{M_z^i}{EI_z}) = \frac{M_z^i\,L}{2EI_z}$$

$$A^j = \frac{1}{2}\,(L)\,(\frac{M_z^j}{EI_z}) = \frac{M_z^j\,L}{2EI_z}$$

$$A^i = A^j$$

A^i and A^j form a couple of the value $\frac{M_z^i\,L}{2EI_z}\,(\frac{L}{3}) = \frac{M_z^i\,L^2}{6EI_z}$

To keep this conjugate beam in equilibrium condition, there must be a moment of the value Δ_y^i acting at the end i in the direction of the deflection. Therefore,

$$\Delta_y^i = \frac{M_z^i\,L^2}{6EI_z}\,,\ \text{and}\ M_z^i = \frac{6EI_z}{L^2}\,(\Delta_y^i)$$

Let Δ_y^i be of unit value, then $M_z^i = \frac{6EI_z}{L^2} = k_{6,2}$

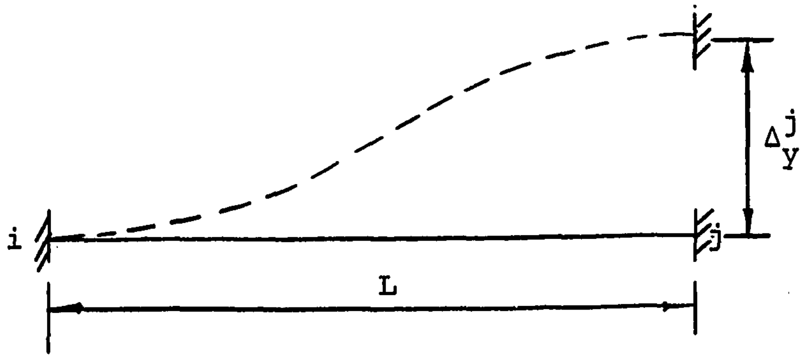
where $k_{6,2}$ is the bending stiffness coefficient at the coordinate 6 due to a unit displacement at coordinate 2.

Since

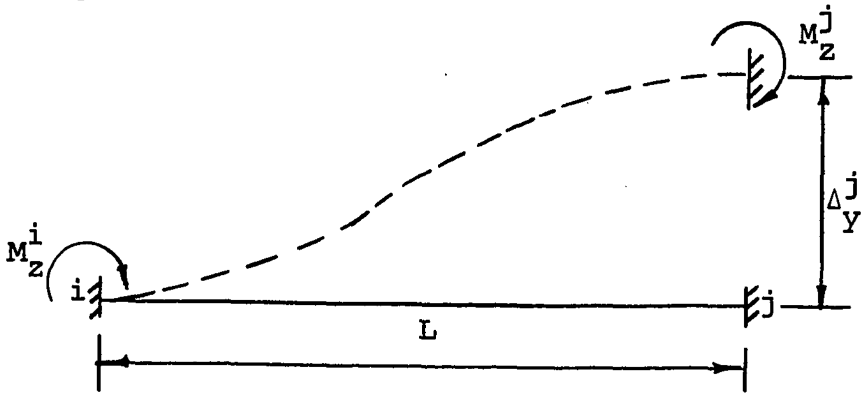
$$M_z^i = M_z^j\ ,\ M_z^j = \frac{6EI_z}{L^2} = k_{12,2}$$

where $k_{12,2}$ is the bending stiffness coefficient at the coordinate 12 due to a unit displacement at coordinate 2.

Case 2



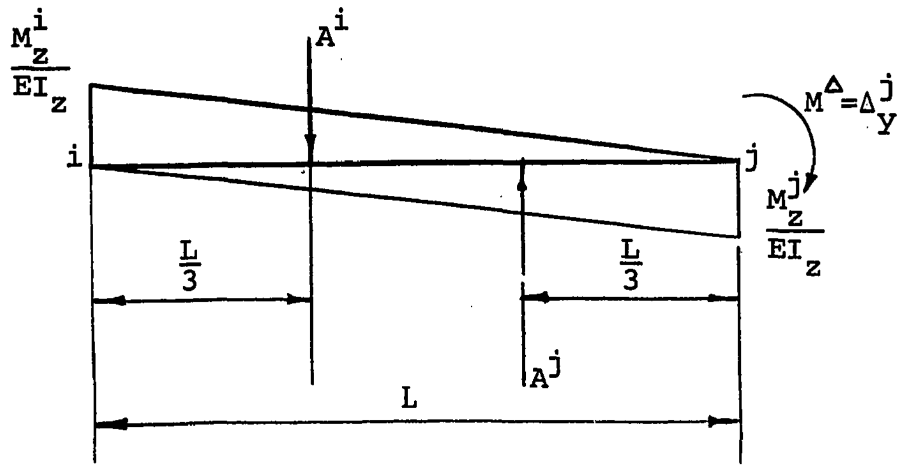
Let the end-j of the member i-j be free to displace vertically in the X-Y plane by the amount Δ_y^j. It is apparent that an induced moment M_z^j is acting at the end-j which is corresponding to the vertical displacement Δ_y^j as shown in the following sketch.



By the conjugate beam method the relations between the Δ_y^j and M_z^i and M_z^j can be developed.

Let the conjugate beam be loaded with the diagrams of $\frac{M_z^j}{EI_z}$ and $\frac{M_z^i}{EI_z}$.

$$M_z^i = M_z^j$$



$$A^i = \frac{1}{2} (L) (\frac{M_z^i}{EI_z})$$

$$A^j = \frac{1}{2} (L) (\frac{M_z^j}{EI_z})$$

$$A^i = A^j$$

A^i and A^j form the couple of the value $\frac{M_z^j L^2}{6EI_z}$.

To keep the conjugate beam in the equilibrium position, there must be a moment of the value Δ_y^j acting at the end-j in the direction of the end deflection, i.e.,

$$\Delta_y^j = \frac{M_z^j L^2}{6EI_z}$$

$$M_z^j = \frac{6EI_z}{L^2} (\Delta_y^j)$$

Let Δ_Y^j be of unit value, then

$$M_z^j = \frac{6EI_z}{L^2} = k_{12,8} \text{ (in clockwise direction)}$$

where $k_{12,8}$ is the bending stiffness coefficient at the coordinate 12 due to a unit displacement at the coordinate 8.

By the framework sign convention, M_z^j is negative; therefore,

$$k_{12,8} = - \frac{6EI_z}{L^2}$$

Since $M_z^i = M_z^j$, $\qquad M_z^i = \frac{-6EI_z}{L^2} = k_{6,8}$

where $k_{6,8}$ is the bending stiffness coefficient at the coordinate 6, due to a unit displacement at the coordinate 8.

Using the same idea the bending stiffness coefficients at both ends around the Y-axis due to the end rotations at both ends around the Y-axis and due to the end transverse displacements at both ends in the direction of Z-axis can be developed.

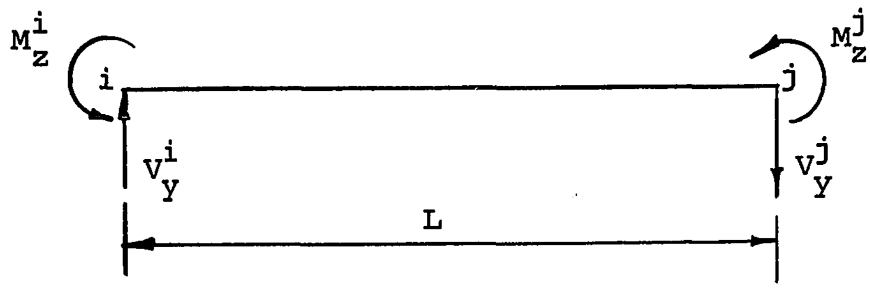
End Shearing Stiffness

Using the idea that the end shears are forming a couple to balance the effects of the moments acting at both ends of a member to keep the member in rotational equilibrium, one can find the end shearing stiffness coefficients with the solved bending stiffness coefficients.

<u>End Shearing Stiffness Coefficients in the Y-Direction at Both Ends of the Member due to the End Rotations θ_z^i and θ_z^j</u>

Case 1: End Shearing Stiffness Coefficients due to the End Rotation θ_z^i

When the unit angular displacement θ_z^i is occurring at the end-i of the member i-j, there will be two moments induced at both ends as the bending stiffness coefficients.



As has been developed, M_z^i due to the unit value of θ_z^i was $\frac{4EI_z}{L}$, M_z^j due to the unit value of θ_z^i was $\frac{2EI_z}{L}$, and both were in positive direction.

Therefore,

$$V_y^i = \frac{M_z^i + M_z^j}{L} = \frac{1}{L}\left(\frac{4EI_z}{L} + \frac{2EI_z}{L}\right) = \frac{6EI_z}{L^2} = k_{2,6}$$

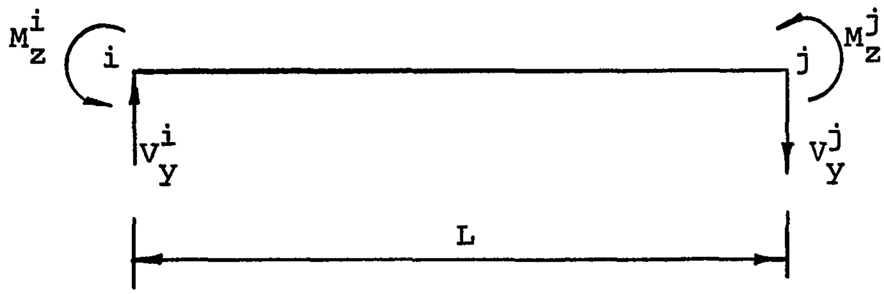
To keep the member i-j in rotational equilibrium, V_y^i and V_y^j must be equal and opposite so that they can form a couple to balance the rotational effect of M_z^i and M_z^j. Therefore,

$$V_y^j = -V_y^i = -\frac{6EI_z}{L^2} = k_{8,6}$$

where V_y^i , $k_{2,6}$, due to θ_z^i is called the end shearing stiffness coefficient at the end-i in the direction of Y due to a unit rotation at end-i around Z-axis and V_y^j due to θ_z^i is called the end shearing stiffness coefficient at the end-j in the direction of Y due to a unit rotation at the end-i around Z-axis.

Case 2: End Shearing Stiffness Coefficients
due to End Rotation θ_z^j

In a like manner, when the end angular displacement θ_z^j is occurring at the end-j of the member i-j, there will be two moments induced at both ends of the member as the bending stiffness coefficients.



Both M_z^i and M_z^j have been developed for the unit value of θ_z^j as

$$M_z^i = \frac{2EI_z}{L}, \quad \text{and} \quad M_z^j = \frac{4EI_z}{L},$$

and both are in the positive direction. Similarly, to satisfy the requirement of the equilibrium condition, V_y^i and V_y^j, due to θ_z^j, must be equal in magnitude and opposite in direction to form a couple to balance the moments M_z^i and M_z^j, i.e.,

$$V_y^i = V_y^j = \frac{M_z^i + M_z^j}{L}$$

Therefore,

$$V_y^i = \frac{1}{L}\left(\frac{2EI_z}{L} + \frac{4EI_z}{L}\right) = \frac{6EI_z}{L^2} = k_{2,12}$$

and

$$V_y^j = -\frac{1}{L}\left(\frac{2EI_z}{L} + \frac{4EI_z}{L}\right) = -\frac{6EI_z}{L^2} = k_{8,12}$$

where V_y^i due to θ_z^j is called $k_{2,12}$, end shearing stiffness coefficient at end-i in the direction of Y due to a unit rotation at end-j around Z-axis, and V_y^j due to θ_z^j is called $k_{8,12}$, end shearing stiffness coefficient at the end-j in the direction of Y due to a unit rotation at end-j around the Z-axis.

<u>End Shearing Stiffness Coefficients in the Y-Direction at Both Ends of the Member due to the End Transverse Displacements, Δ_y^i and Δ_y^j</u>

Case 1: The End Shearing Stiffness Coefficients in the Y-Direction at Both Ends due to the End Transverse Displacement at the End-i in the Direction of Y-axis

The development of the end shearing stiffness coefficients due to the end transverse displacement can be accomplished in the same manner as they were done due to the end angular displacement. As previously developed, both M_z^i and M_z^j due to Δ_y^i were $\frac{6EI_z}{L^2}$; therefore V_y^i due to Δ_y^i will be

$$\frac{1}{L}\left(\frac{6EI_z}{L^2} + \frac{6EI_z}{L^2}\right)$$

i.e., $V_y^i = \frac{12EI_z}{L^3} = k_{2,2}$ (in positive direction)

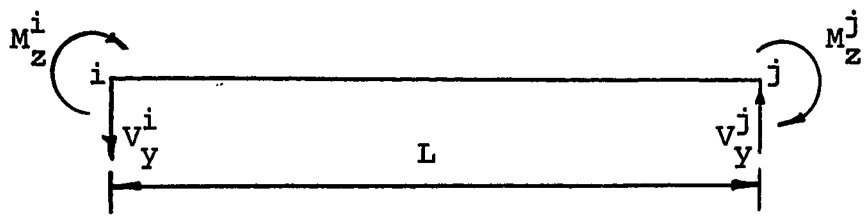
where $k_{2,2}$ is the end shearing stiffness coefficient at the end-i in the direction of the Y-axis due to a unit transverse displacement at the end-i in the direction of the Y-axis,

and $$V_y^j = -V_y^i = \frac{-12EI_z}{L^3} = k_{8,2}$$

where $k_{8,2}$ is the end shearing stiffness coefficient at the end-j in the direction of the Y-axis due to a unit transverse displacement at the end-i in the direction of the Y-axis.

Case 2: The End Shearing Stiffness Coefficients in the Y-Direction at Both Ends due to the End Transverse Displacement at the End-j in the Direction of Y-axis

When the end-j displaces in the positive direction of the Y-axis, there will be moments M_z^i and M_z^j induced as shown in the following sketch:



where $M_z^i = M_z^j = \frac{6EI_z}{L^2}$ (in negative direction)

$$V_y^i = \frac{-12EI_z}{L^3} = k_{2,8} \text{(in negative direction)}$$

where $k_{2,8}$ is the end shearing stiffness coefficient at the end-i in the direction of the Y-axis due to a unit transverse displacement at the end-j in the direction of the Y-axis,

and
$$V_y^j = -V_y^i = \frac{12EI_z}{L^3} = k_{8,8}$$

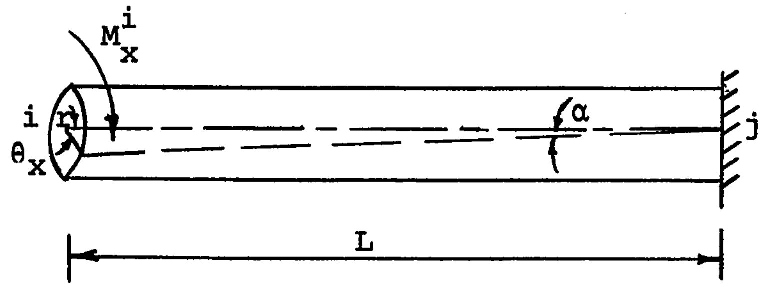
where $k_{8,8}$ is the end-shearing stiffness coefficient at the end-j in the direction of the Y-axis due to a unit transverse displacement at the end-j in the direction of the Y-axis.

In the same way, the end shearing stiffness coefficients at both ends in the direction of the Z-axis due to the end rotations at both ends around the Y-axis and the end transverse displacements at both ends in the direction of the Z-axis can be developed.

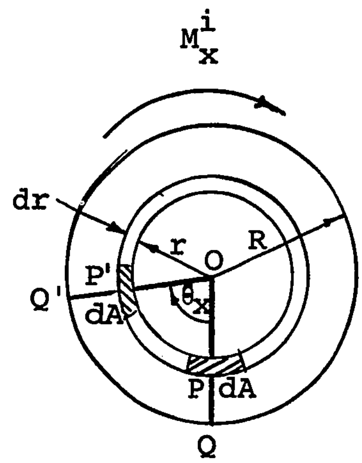
Torsional Stiffness

To find the torsional stiffness of a member, it is assumed that the member is completely fixed at one end, and the other end is free to rotate only about its axis. In this consideration, the member axis is coincident with the X-axis and the member is of a circular cross section.

The i^{th} End of Member i-j Free to Twist



Take the i^{th} end of the member as a free body as shown in the following sketch:



It is known that when a member is subjected to a torsion, there will be shearing stress developed in each cross section, and each cross-section will turn through an angle with reference to the fixed end.

Take a hollow cylinder of radius r and thickness dr for investigation. This hollow cylinder can be developed into a rectangular solid of width $2\pi r$ and thickness dr. The area of the cross-section of this hollow cylinder is $(2\pi r)dr$. Suppose that there is an element of area dA at the position P on the hollow cylinder. When a torque M_x^i is applied at the end-i and the end-i is turned through an angle θ_x radians, this area dA is moved to P'. Its displacement is $r\theta_x$, and the unit shearing displacement is given by

$$\Delta_s = \frac{r\theta_x}{L} .$$

The unit shearing stress on dA is given by $\tau_s = \frac{r\theta_x}{L}(G)$ in which G is the shearing modulus of elasticity. Therefore, the shearing force required to deform this hollow cylinder is given by

$$S = 2\pi r dr\left(\frac{r\theta_x \cdot G}{L}\right) = \frac{2\pi G\theta_x}{L}(r^2 dr)$$

The moment of this shearing force w.r.t. the axis of the cylinder is given by

$$T = \frac{2\pi G\theta_x}{L}(r^3 dr).$$

The entire member may be regarded as being composed of a series of concentric hollow cylinders of thickness dr. The total moment of the shearing stress over the entire cross-section will be equivalent to the externally applied torque which is given by

$$M_x^i = \frac{2\pi G\theta_x}{L} \int_0^R r^3 dr$$

$$= \frac{2\pi G\theta_x}{L} \left(\frac{R^4}{4}\right) = \frac{GI_x}{L} (\theta_x) \qquad \left(I_x = \frac{\pi R^4}{2}\right)$$

If θ_x be unit (one radian), then

$$M_x^i = \frac{GI_x}{L} = k_{4,4}$$

in which I_x represents $\frac{\pi R^4}{2}$ which is the polar moment of inertia of a member of circular cross-section of radius R, where $k_{4,4}$ is the torsional stiffness coefficient at the coordinate 4 due to a unit displacement at the coordinate 4. Since $M_x^j = - M_x^i$,

$$M_x^j = \frac{-GI_x}{L} = k_{10,4}$$

where $k_{10,4}$ is the torsional stiffness coefficient at the coordinate 10 due to a unit displacement at the coordinate 4. Following this idea exactly, one has

$$k_{10,10} = \frac{GI_x}{L}$$

where $k_{10,10}$ is the torsional stiffness coefficient at the coordinate 10 due to a unit displacement at the coordinate 10, and $k_{4,10} = \frac{-GI_x}{L}$ where $k_{4,10}$ is the torsional stiffness coefficient at the coordinate 4 due to a unit displacement at the coordinate 10.

When the member is of any cross-section other than circular, then I_x will not be the polar moment of inertia. Consider, for example, a rectangular shaft; I_x is equal to "βbc^3" (22). In this expression b is the longer and c is the shorter side of the rectangular cross-section, and "β" (22) is a numerical factor depending upon the ratio $\frac{b}{c}$.

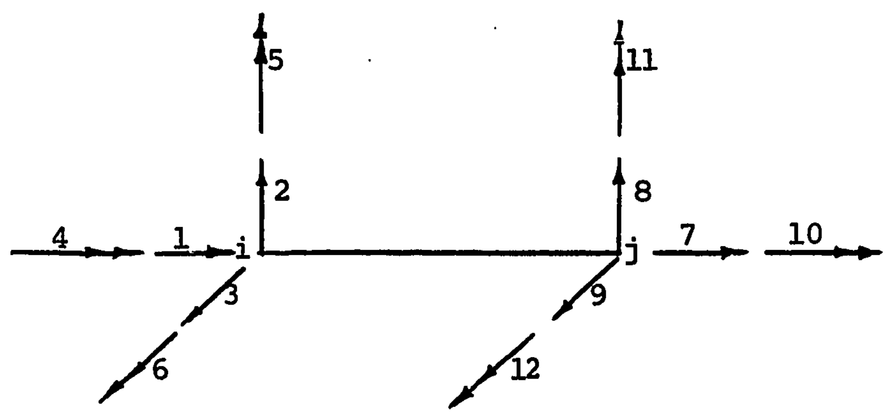
Using the idea of superposition of the displacements, the collection of the stiffness coefficients at each coordinate which are the relations between the stiffness of the coordinate and the unit displacements at all the coordinates of the member can be made to give the final total stiffness at the coordinate which is a row in the stiffness matrix.

Expression of a Stiffness Matrix of a Prismatic Member in Three-Dimensional Space

Using the coordinate system, one can express the stiffness matrix of a member in a three-dimensional space in a partitioned matrix and conveniently write the stiffness matrix.

Each submatrix indicates which two ends are related in the action-displacement relationship.

The convention of expressing a member, coordinate system and the submatrices is given as follows:



end i = initial end of a member,

end j = terminal end of a member,

each arrowed direction represents the positive direction of the coordinate for both action and displacement.

The total stiffness matrix of a member is arranged by the sequence of coordinate number and is partitioned by the ends of the member which matrix is shown as follows:

$$
\begin{array}{c|c:c}
 & \Delta_x^i \;\; \Delta_y^i \;\; \Delta_z^i \;\; \theta_x^i \;\; \theta_y^i \;\; \theta_z^i & \Delta_x^j \;\; \Delta_y^j \;\; \Delta_z^j \;\; \theta_x^j \;\; \theta_y^j \;\; \theta_z^j \\
\hline
V_x^i & & \\
V_y^i & & \\
V_z^i & s^{i-i} & s^{i-j} \\
M_x^i & & \\
M_y^i & & \\
M_z^i & & \\
\hdashline
V_x^j & & \\
V_y^j & & \\
V_z^j & s^{j-i} & s^{j-j} \\
M_x^j & & \\
M_y^j & & \\
M_z^j & & \\
\end{array}
$$

From the superscripts of each submatrix, one can easily tell which two ends are involved in it.

The total stiffness matrix of a prismatic member in a three-dimensional space is expressed in the following four separate submatrices.

TABLE 3

SUBMATRIX s^{i-i} OF A MEMBER STIFFNESS MATRIX OF A PRISMATIC MEMBER IN SPACE

$$
s^{i-i} = \begin{bmatrix}
\frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}
$$

TABLE 4

SUBMATRIX s^{i-j} OF A MEMBER STIFFNESS MATRIX OF A PRISMATIC MEMBER IN SPACE

$$
s^{i-j} = \begin{bmatrix}
\frac{-EA_x}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\
0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & 0 \\
0 & 0 & 0 & \frac{-GI_x}{L} & 0 & 0 \\
0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\
0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L}
\end{bmatrix}
$$

TABLE 5

SUBMATRIX s^{j-i} OF A MEMBER STIFFNESS MATRIX OF A PRISMATIC MEMBER IN SPACE

$$s^{j-i} = \begin{bmatrix} \frac{-EA_x}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{-GI_x}{L} & 0 & 0 \\ 0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \end{bmatrix}$$

TABLE 6

SUBMATRIX s^{j-j} OF A MEMBER STIFFNESS MATRIX OF A PRISMATIC MEMBER IN SPACE

$$s^{j-j} = \begin{bmatrix} \frac{EA_x}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix}$$

Member Stiffness Matrix of a Prismatic Member in Space with the Consideration of Shearing Deformations

By definition, "A beam is the name given to any member of a structure which is exposed to transverse stresses" (24). From the definition, it is obvious that "there are usually shearing forces as well as bending moments acting on the cross-section of a beam" (5). "Usually the effects of shear are small compared to the effects of bending and can be neglected; . . ." (5). If the shearing deformation of a member is considered, then the total transverse displacement at any point along a member will be influenced by both flexural and shearing deformations. In other words, "the unit displacement consists of two parts . . . . The first part is due to the flexural deformations in the member, and the second part is due to shear deformations. . ." (5). Therefore, the stiffness coefficients that relate the linear end displacements in the direction perpendicular to the member axis and the angular end displacement rotating about the axis perpendicular to the member axis to the actions induced at the member ends by these displacements, should be modified by considering the shearing deformations. Modification can be accomplished by adding the shearing effects to the stiffness coefficients due to the bending effects.

For the sake of simplicity, how to find the stiffness coefficients at the coordinates of a member in space for the introduction of unit displacements in the X-Y plane is illustrated. The numbering system of the coordinates and the subscript system for the stiffness coefficients in a member stiffness matrix of a member in space remain the same as before.

Stiffness Coefficients Induced by the Transverse End Displacement due to Bending Deformations and Shearing Deformations

Introduction of Δ_y^i

When the settlement or the relative displacement between the two ends of a member exists, there will be two fixed end moments which are equal and of the same sense induced at both ends of the member; they are denoted as M_z^i in this dissertation. As derived,

$$M_z^i = M_z^j = \frac{6EI_z}{L^2} (\Delta_y^i), \quad V_y^i = \frac{M_z^i + M_z^j}{L} .$$

Therefore, $$V_y^i = \frac{1}{L} (\frac{12EI_z}{L^2}) , \text{ i.e.,}$$

$$V_y^i = \frac{12EI_z}{L^3} (\Delta_{y-b}^i) \text{. . . end action induced by end displacement due to bending deformation.}$$

$$\Delta_{y-b}^i = \frac{L^3}{12EI_z} (V_y^i) \text{. . . end displacement due to bending deformation.}$$

$$\Delta^i_{y-v} = \int_0^L \frac{V^i_y}{GA}\, dx$$

$$= \frac{V^i_y\, Lf}{GA} \;.\;.\;.$$ end displacement due to shearing deformation.

$$\Delta^i_{y-b} + \Delta^i_{y-v} = \frac{L^3\, V^i_y}{12EI_z} + \frac{Lf\, V^i_y}{GA} = \left(\frac{L^3}{12EI_z} + \frac{LF}{GA}\right) V^i_y$$

Let $\Delta^i_{y-b} + \Delta^i_{y-v} = 1$, then

$$V^i_y = \frac{12EI_z\, GA}{12EI_z\, Lf + GAL^3} = k_{2,2}$$

By equilibrium condition, $V^j_y = -V^i_y$

Therefore, $$V^j_y = \frac{-12EI_z\, GA}{12EI_z\, Lf + GAL^3} = k_{8,2}$$

as mentioned above, $M^i_z = \frac{6EI_z}{L^2} (\Delta^i_{y-b})$

$$\Delta^i_{y-b} = \frac{L^2}{6EI_z} (M^i_z) \;.\;.\;.$$ end displacement due to bending deformation.

$$\Delta^i_{y-v} = \frac{Lf}{GA} (V^i_y) \;.\;.\;.$$ end displacement due to shearing deformation.

$$V^i_y = \frac{2M^i_z}{L}$$

$$\Delta^i_{y-v} = \frac{Lf}{GA}\left(\frac{2M^i_z}{L}\right) = \frac{2f}{GA}(M^i_z)$$

$$\Delta^i_{y-b} + \Delta^i_{y-v} = \left(\frac{L^2}{6EI_z} + \frac{2f}{GA}\right)M^i_z$$

Let $\Delta^i_{y-b} + \Delta^i_{y-v} = 1$, then

$$M^i_z = \frac{6EI_z\ GA}{GAL^2 + 12fEI_z} = k_{6,2}$$

Since $\quad M^j_z = M^i_z,\ M^j_z = \dfrac{6EI_z\ GA}{GAL^2 + 12fEI_z} = k_{12,2}$

<u>Introduction of Δ^j_y</u>

Using the similar idea, when the Δ^j_y is introduced,

$$V^j_y = \frac{12EI_z\ GA}{12EI_z\ Lf + GAL^3} = k_{8,8}$$

$$V^i_y = \frac{-12EI_z\ GA}{12EI_z\ Lf + GAL^3} = k_{2,8}$$

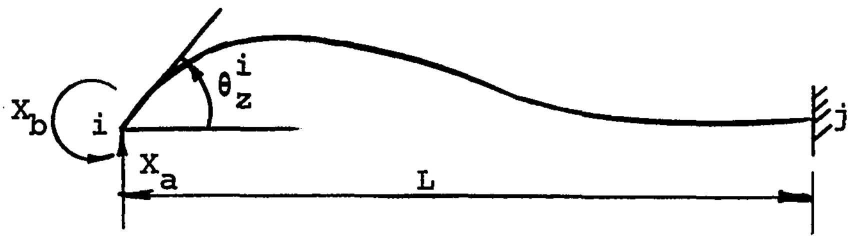
$$M^j_z = \frac{-6EI_z\ GA}{GAL^2 + 12fEI_z} = k_{12,8}$$

and

$$M^i_z = \frac{-6EI_z\ GA}{GAL^2 + 12fEI_z} = k_{6,8}$$

Stiffness Coefficients Induced by the End Rotational Displacement due to Bending Deformations and Shearing Deformations

Introduction of θ_z^i



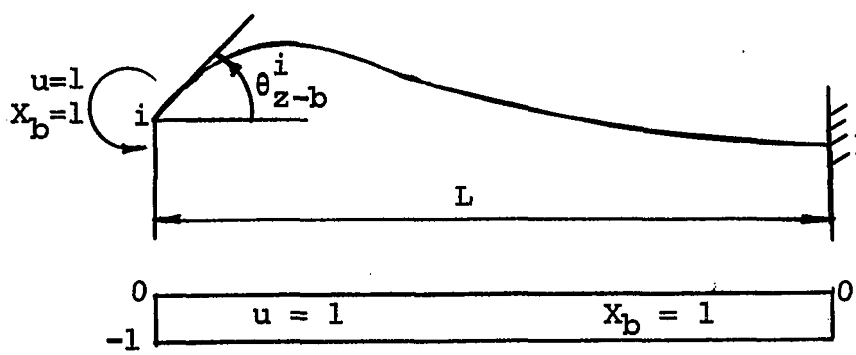
EI = constant

$$X_b = M_z^i$$

$$X_a = V_y^i$$

Find the stiffness factors by the virtual work method.

Rotation due to X_b



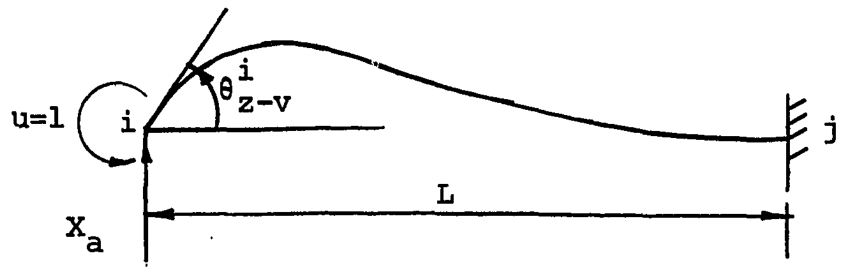
$$m = -1$$

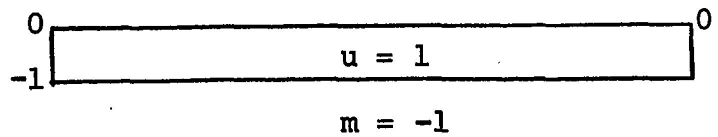
$$M = -1$$

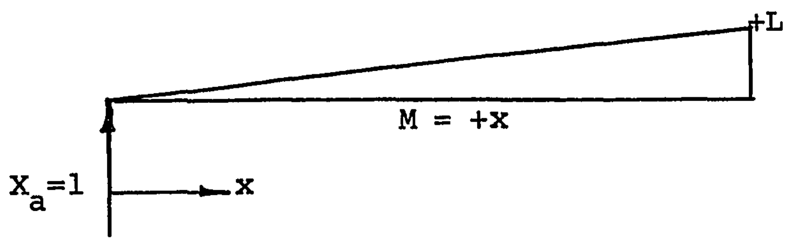
Since the member is subjected to pure bending, there is neither shearing stress nor shearing strain energy.

$$\theta_{bb} = \int_0^L \frac{dx}{EI_z} = \frac{L}{EI_z} \; ; \; \theta^i_{z-b} = X_b(\theta_{bb}) = \frac{X_b L}{EI_z}$$

Rotation due to X_a







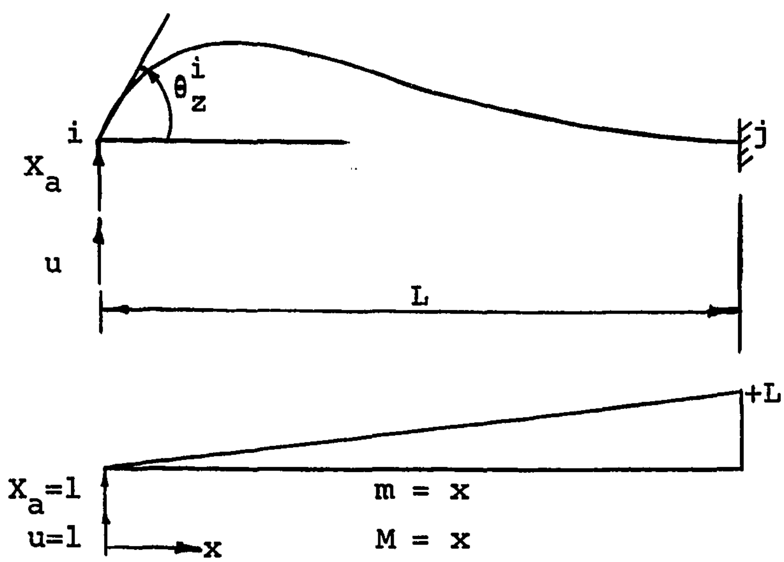
$$\theta_{ba} = \int_0^L \frac{-x dx}{EI_z} = \frac{-L^2}{2EI_z} \; ; \; \theta^i_{z-v} = X_a \theta_{ba} = \frac{X_a(-L^2)}{2EI_z} \, .$$

$$\theta_z^i = \theta_{z-b}^i + \theta_{z-v}^i = \frac{LX_b}{EI_z} + \frac{-L^2X_a}{2EI_z} = 1 \qquad (1)$$

Deflection at joint i due to X_b is equal to the rotation at the joint i due to X_a, i.e.,

$$\delta_{ab} = \theta_{ba} = \frac{-L^2}{2EI_z}$$

Deflection at the Joint-i due to X_a



$$\delta_{aa} = \int_0^L \frac{x^2\,dx}{EI_z} + \int_0^L \frac{dx}{A_wG} = \frac{L^3}{3EI_z} + \frac{L}{A_wG} \;;$$

$$\Delta_{y-v}^i = X_a\delta_{aa} \;, \quad \Delta_{y-b}^i = X_b\delta_{ab} \;.$$

Therefore, $\Delta_y^i = \Delta_{y-b}^i + \Delta_{y-v}^i = \dfrac{L^3 X_a}{3EI_z} + \dfrac{LX_a}{A_wG} - \dfrac{L^2 X_b}{2EI_z} = 0$ (2)

From Equation (1) $X_b = \dfrac{EI_z}{L} + \dfrac{LX_a}{2}$

Substitute $X_b = \dfrac{EI_z}{L} + \dfrac{LX_a}{2}$ into the equation (2)

$$X_a \left(\frac{L^3}{3EI_z} + \frac{L}{A_wG}\right) = \frac{L^2}{2EI_z}\left(\frac{EI_z}{L} + \frac{LX_a}{2}\right)$$

$$X_a \left(\frac{L^3}{12EI_z} + \frac{L}{A_wG}\right) = \frac{L}{2}$$

$$X_a \left(\frac{A\ G\ L^3 + 12EI_zLf}{12EI_z\ A\ G}\right) = \frac{L}{2}$$

$$X_a = \frac{6EI_z\ A\ G}{AGL^2 + 12fEI_z} = V_y^i = k_{2,6}$$

Since $V_y^j = -V_y^i$, $V_y^j = -\dfrac{6EI_z\ A\ G}{AGL^2 + 12fEI_z} = k_{8,6}$

$$X_b = \frac{EI_z}{L} + \frac{L}{2} X_a$$

$$X_b = \frac{EI_z}{L} + \frac{L}{2}\left(\frac{6EI_z\ AG}{AGL^2 + 12fEI_z}\right)$$

$$X_b = \frac{EI_z}{L} + \frac{3EI_z\ AGL}{AG\ L^2 + 12fEI_z} = \frac{4EI_zAGL^2 + 12fE^2I_z^2}{AGL^3 + 12fEI_zL} = M_z^i = k_{6,6}$$

$$M_z^j = V_y^i\ L - M_z^i$$

$$V_y^i(L) = \frac{6EI_zAGL}{AGL^2 + 12fEI_z}$$

$$M_z^j = \frac{2EI_zAGL^2 - 12fE^2I_z^2}{AGL^3 + 12fEI_zL} = k_{12,6}$$

<u>Introduction to θ_z^j</u>

Using the similar idea, when θ_z^j is introduced, the following actions are obtained:

$$V_y^i = \frac{6EI_zAG}{AGL^2 + 12fEI_z} = k_{2,12}$$

$$V_y^j = -\ V_y^i = -\ \frac{6EI_z\ AG}{AGL^2 + 12fEI_z} = k_{8,12}$$

$$M_z^j = \frac{4EI_zL^2AG + 12fE^2I_z^2}{AGL^3 + 12EI_zL} = k_{12,12}$$

$$M_z^i = \frac{2EI_zAGL^2 - 12fE^2I_z^2}{AGL^3 + 12fEI_zL} = k_{6,12}$$

By the same procedure, the stiffness factors due to the unit displacements in the X-Z plane can be obtained.

Note:

"f is a form factor that is dependent upon the shape of the cross-section" (5). The value of f is the ratio of the area of the gross cross-section of a member to the area of the portion of the cross-section assumingly subjected to shearing stress, i.e., $f = \frac{A}{A_w}$.

Stiffness Matrix for a Curved-Beam Element

"Many important engineering structures either represent a curved beam or else contain curved-beam elements used in conjunction with other structural units" (11). Because of the actual need in the engineering structures, the stiffness coefficients for curved-beam elements are necessarily required. The basic idea of deriving the stiffness coefficients for a curved-beam element follows that used for a straight member.

The requirements of satisfying the equilibrium and compatibility conditions are used to solve the different values of stiffness coefficients. The method given in Ref.(11) in paragraph 5.6, page 144, used to derive the relations between the introduced displacements and the corresponding actions at the nodal points is an energy method, such as Castigliano's second theorem.

From the point of view of mathematics in the process of doing calculations, the function of the curved-beam element should be defined; that is, the geometric property of the curved-beam element should be explained.

CHAPTER IV

ROTATIONAL TRANSFORMATION OF A STIFFNESS MATRIX OF A MEMBER IN THREE-DIMENSIONAL SPACE

Define the Direction Cosines of a Member in Three-Dimensional Space

Direction cosines are the cosine values of the direction angles of a vector, in a three-dimensional space, w.r.t. the three reference axes which form the space.

By taking advantage of double subscripts, the Greek letter λ with two subscripts can be used to represent direction cosines of a vector, in any orientation, in a three-dimensional space. From the two subscripts of each direction cosine, it is very obvious that a direction cosine is used as a transforming coefficient; between the two vectors (or axes) specified by the two subscripts. In this dissertation the author defines the direction angles of the axes in member-axes system w.r.t. the structure-axes system as follows:

α refers to an angle w.r.t. X_s

β refers to an angle w.r.t. Y_s

and γ refers to an angle w.r.t. Z_s

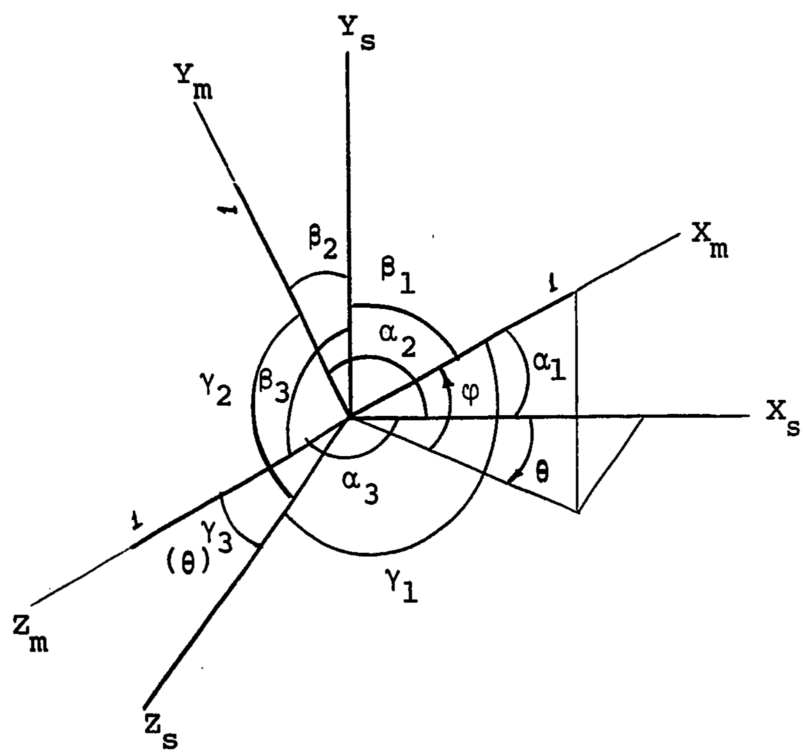
The subscripts of the direction angles are used as the code numbers for the three axes of member-axes system such as

1 represents X_m

2 represents Y_m

3 represents Z_m.

With the above mentioned information, one can tell which two axes of the two axes systems are transformed to each other by the direction angle and its subscripts shown as follows:



$\alpha_1 = X_m\hat{}X_s$ $\qquad \alpha_2 = Y_m\hat{}X_s$ $\qquad \alpha_3 = Z_m\hat{}X_s$

$\beta_1 = X_m\hat{}Y_s$ $\qquad \beta_2 = Y_m\hat{}Y_s$ $\qquad \beta_3 = Z_m\hat{}Y_s$

$\gamma_1 = X_m\hat{}Z_s$ $\qquad \gamma_2 = Y_m\hat{}Z_s$ $\qquad \gamma_3 = Z_m\hat{}Z_s$

Then the direction cosines of the member-axes system w.r.t. the structure-axes system can be defined as follows:

$$\lambda_{11} = \cos\alpha_1 = \cos X_m\hat{}X_s = \cos\varphi\cos\theta$$

$$\lambda_{12} = \cos\beta_1 = \cos X_m\hat{}Y_s = \cos(90^\circ - \varphi)$$

$$\lambda_{13} = \cos\gamma_1 = \cos X_m\hat{}Z_s = \cos\varphi\cos(90^\circ - \theta)$$

$$\lambda_{21} = \cos\alpha_2 = \cos Y_m\hat{}X_s = \cos(90^\circ + \varphi)\cos\theta$$

$$\lambda_{22} = \cos\beta_2 = \cos Y_m\hat{}Y_s = \cos\varphi$$

$$\lambda_{23} = \cos\gamma_2 = \cos Y_m\hat{}Z_s = \cos(90^\circ + \varphi)\cos(90^\circ - \theta)$$

$$\lambda_{31} = \cos\alpha_3 = \cos Z_m\hat{}X_s = \cos(90^\circ + \theta)$$

$$\lambda_{32} = \cos\beta_3 = \cos Z_m\hat{}Y_s = \cos(90^\circ)$$

$$\lambda_{33} = \cos\gamma_3 = \cos Z_m\hat{}Z_s = \cos\theta$$

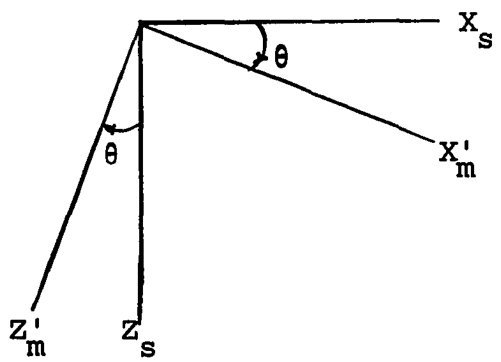
Thus, whenever the orientation of a member in a three-dimensional space is known, the direction cosines of the member-axes system w.r.t. the structure-axes system can be easily obtained by applying the above equations.

To Prove the Rotational Transformation Matrix to Be an Orthogonal Matrix

The rotational transformation matrix for two orthogonal sets of axes is formed with the direction cosines, which are used as the elements in the matrix, of one set of orthogonal axes w.r.t. the other. In this consideration, the former set of orthogonal axes are the axes of the member-axes system, while the latter are the axes of the structure-axes system. The X-axis of the member-axes system, which is usually taken as the member-axis, can be assumed to be oriented from the X-axis of the structure-axes system with two rotations as follows:

Rotation about Y_s-Axis

In this stage of rotation, the Y_s-axis is kept fixed, and both X_s and Z_s axes rotate about Y_s-axis by the same angle θ, as shown in the following sketch:



The rotation relation can be expressed in a matrix equation as follows:

$$\begin{bmatrix} X'_m \\ Y'_m \\ Z'_m \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix}$$

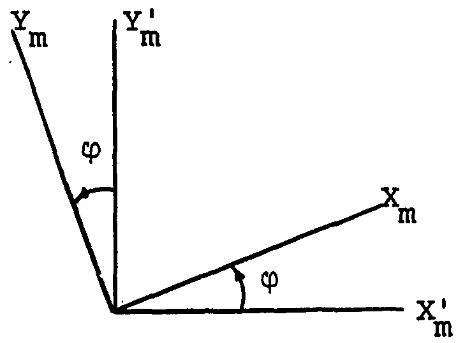
$(Y'_m = Y_s)$

$$\text{Let } R_1 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

By the definition of orthogonal matrix, it is obvious that R_1 is an orthogonal matrix.

Rotation about Z'_m-Axis

In this stage of rotation, the Z'_m-axis is kept fixed, and both X'_m and Y'_m axes rotate about the Z'_m-axis by the same angle φ, as shown in the following sketch:



The rotation relation can also be expressed in a matrix equation as follows:

$$\begin{bmatrix} X_m \\ Y_m \\ Z_m \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X'_m \\ Y'_m \\ Z'_m \end{bmatrix}$$

$$\text{Let } R_2 = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also by the definition of orthogonal matrix, R_2 is an orthogonal matrix. If the orthogonal set of axes of the structure-axes system is rotationally transformed into the orthogonal set of axes of the member-axes system, it must undergo two rotational transformations R_1 and R_2 i.e.,

$$\begin{bmatrix} X_m \\ Y_m \\ Z_m \end{bmatrix} = \begin{bmatrix} R_2 \end{bmatrix} \left\{ \begin{bmatrix} R_1 \end{bmatrix} \begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} \right\}$$

By the associative property of matrix multiplication.

$$\begin{bmatrix} X_m \\ Y_m \\ Z_m \end{bmatrix} = \left\{ \begin{bmatrix} R_2 \end{bmatrix} \begin{bmatrix} R_1 \end{bmatrix} \right\} \begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix}$$

Let $[R] = [R_2]\,[R_1]$.

Therefore,

$$[R] = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta\cos\varphi & \sin\varphi & \sin\theta\cos\varphi \\ -\cos\theta\sin\varphi & \cos\varphi & -\sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

[R] is the rotational transformation matrix which is orthogonal.

To Prove the Inverse of a Rotational Transformation Matrix Equal to Its Transpose

Using the definition of the nonsingularity of a matrix, if a matrix A is said to be nonsingular, then the determinant of A is not equal to 0, and A must be square. With this idea, rotation matrix, R, can be proved nonsingular.

As it is shown, the rotational transformation matrix R is a square matrix; therefore, if the determinant of the rotational transformation matrix R can be proved not equal to 0, then the matrix R can be proved nonsingular.

If the determinant of the matrix R is denoted by $|R|$ then

$$|R| = \begin{vmatrix} \cos\theta\cos\varphi & \sin\varphi & \sin\theta\cos\varphi \\ -\cos\theta\sin\varphi & \cos\varphi & -\sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{vmatrix}$$

Expand $|R|$ by cofactor method,

$$|R| = (-\sin\theta) \begin{vmatrix} \sin\varphi & \sin\theta\cos\varphi \\ \cos\varphi & -\sin\theta\sin\varphi \end{vmatrix} + \cos\theta \begin{vmatrix} \cos\theta\cos\varphi & \sin\varphi \\ -\cos\theta\sin\varphi & \cos\varphi \end{vmatrix}$$

$$= (-\sin\theta)(-\sin^2\varphi \sin\theta - \sin\theta \cos^2\varphi) + (\cos\theta)(\cos\theta \cos^2\varphi + \cos\theta\sin^2\varphi)$$

$$= \sin^2\theta(\sin^2\varphi + \cos^2\varphi) + \cos^2\theta(\sin^2\varphi + \cos^2\varphi) = 1 \neq 0$$

Therefore, the matrix R is nonsingular.

By the theorem of the nonsingularity of a matrix, if a matrix A is said to be nonsingular, its inverse A^{-1} is unique. With this idea, rotation matrix R can be proved to have an inverse. As proved above, the matrix R is nonsingular; therefore [R] must have an inverse which is unique.

By definition, a matrix A which is square and nonsingular there must be a matrix B such that

$$A\,B = I = B\,A$$

any such matrix B is called an inverse of [A].

Now let the rotation matrix [R] be A and the transpose, R^T, of matrix R be B. If $RR^T = I$; then by definition, R^T must be equal to R^{-1} and also by the theorem R^T is the only inverse of R.

It can be proved by direct matrix multiplication as follows:

$$R\,R^T = \begin{bmatrix} \cos\theta\cos\varphi & \sin\varphi & \sin\theta\cos\varphi \\ -\cos\theta\sin\varphi & \cos\varphi & -\sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta\cos\varphi & -\cos\theta\sin\varphi & -\sin\theta \\ \sin\varphi & \cos\varphi & 0 \\ \sin\theta\cos\varphi & -\sin\theta\sin\varphi & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\varphi(\cos^2\theta+\sin^2\theta) + \sin^2\varphi & -\cos\varphi\sin\varphi(\cos^2\theta+\sin^2\theta) + \cos\varphi\sin\varphi & 0 \\ -\cos\varphi\sin\varphi(\cos^2\theta+\sin^2\theta) + \cos\varphi\sin\varphi & \sin^2\varphi(\cos^2\theta+\sin^2\theta)+\cos^2\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, $R^T = R^{-1}$.

To Develop the Matrix Equation for the Rotational Transformation of a Stiffness Matrix of a Member Skew in Any Direction in a Three-Dimensional Space

Using the definition of the rotational transformation matrix R, one can set up the following two matrix equations:

$$[A_m] = [R][A_s] \tag{1}$$

$$[D_m] = [R][D_s] \tag{2}$$

Using the definition of a stiffness matrix of a member, and the action-displacement equation, one can set up the following two matrix equations:

$$[A_m] = [S_m][D_m] \tag{3}$$

$$[A_s] = [S_s][D_s] \tag{4}$$

Substitution:

$$[S_m][D_m] = [R][S_s][D_s]$$

premultiply both sides by the inverse of the rotation matrix R

$$[R^{-1}][S_m][D_m] = [S_s][D_s]$$

substitute $[R][D_s]$ for $[D_m]$

$$[R^{-1}][S_m][R][D_s] = [S_s][D_s]$$

post-multiply both sides by the inverse of $[D_s]$

$$[R^{-1}][S_m][R] = [S_s]$$

as previously proved $[R^{-1}] = [R^T]$

Therefore, $$[S_s] = [R^T][S_m][R]$$

Development of a Rotational Transformation Matrix for a Member Skew in a Three-Dimensional Space

With the advantage of partitioning a matrix, one can develop the rotational transformation matrix for a member skew in a three-dimensional space joint by joint.

For the sake of simplicity, one can perform the rotational transformation upon the end actions of each end of a member skew in a three-dimensional space instead of upon the stiffness matrix of the member.

To Transform the End Actions at the Initial Joint-i from the Structure-Axes System into the Member-Axes System

By applying the direction cosine to get the projection of one vector upon the other, one can get the projections of all the end actions at the joint-i, which are in the directions of the structure axes, upon the member axes by the direction cosines as shown in the following simultaneous equations:

$$\bar{V}_x^i = V_x^i \lambda_{11} + V_y^i \lambda_{12} + V_z^i \lambda_{13}$$

$$\bar{V}_y^i = V_x^i \lambda_{21} + V_y^i \lambda_{22} + V_z^i \lambda_{23}$$

$$\bar{V}_z^i = V_x^i \lambda_{31} + V_y^i \lambda_{32} + V_z^i \lambda_{33}$$

$$\bar{M}_x^i = M_x^i \lambda_{11} + M_y^i \lambda_{12} + M_z^i \lambda_{13}$$

$$\bar{M}_y^i = M_x^i \lambda_{21} + M_y^i \lambda_{22} + M_z^i \lambda_{23}$$

$$\bar{M}_z^i = M_x^i \lambda_{31} + M_y^i \lambda_{32} + M_z^i \lambda_{33}$$

The matrix equation for the above simultaneous equations is

$$\begin{bmatrix} \bar{V}_x^i \\ \bar{V}_y^i \\ \bar{V}_z^i \\ \bar{M}_x^i \\ \bar{M}_y^i \\ \bar{M}_z^i \end{bmatrix}_m = \left[\begin{array}{ccc|ccc} \lambda_{11} & \lambda_{12} & \lambda_{13} & & & \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & & 0 & \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & & & \\ \hline & & & \lambda_{11} & \lambda_{12} & \lambda_{13} \\ & 0 & & \lambda_{21} & \lambda_{22} & \lambda_{23} \\ & & & \lambda_{31} & \lambda_{32} & \lambda_{33} \end{array}\right] \begin{bmatrix} V_x^i \\ V_y^i \\ V_z^i \\ M_x^i \\ M_y^i \\ M_z^i \end{bmatrix}_s$$

If let R^i be the rotation matrix for 1 set of orthogonal components at joint i, then

$$R^i = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

And if let R_T^i be the rotation matrix for 2 sets of orthogonal components at joint i, then

$$R_T^i = \left[\begin{array}{c|c} R^i & 0 \\ \hline 0 & R^i \end{array}\right]$$

The above matrix equation can be abbreviated as

$$\begin{bmatrix} \bar{V}^i \\ \hline \bar{M}^i \end{bmatrix}_m = \left[\begin{array}{c|c} R^i & 0 \\ \hline 0 & R^i \end{array}\right] \begin{bmatrix} V^i \\ \hline M^i \end{bmatrix}_s$$

Premultiply both sides of the above abbreviated matrix equation with the inverse of R_T^i one can get

$$\begin{bmatrix} V^i \\ \hline M^i \end{bmatrix}_s = \left[\begin{array}{c|c} R^i & 0 \\ \hline 0 & R^i \end{array}\right]^{-1} \begin{bmatrix} \bar{V}^i \\ \hline \bar{M}^i \end{bmatrix}_m$$

Since $(R^i)^{-1} = (R^i)^T$, $(R_T^i)^{-1} = (R_T^i)^T$

Therefore,

$$\begin{bmatrix} V^i \\ \hline M^i \end{bmatrix}_s = \left[\begin{array}{c|c} R^i & 0 \\ \hline 0 & R^i \end{array}\right]^T \begin{bmatrix} \bar{V}^i \\ \hline \bar{M}^i \end{bmatrix}_m$$

i.e.,

$$\begin{bmatrix} V^i \\ \hline M^i \end{bmatrix}_s = \left[\begin{array}{c|c} (R^i)^T & 0 \\ \hline 0 & (R^i)^T \end{array}\right] \begin{bmatrix} \bar{V}^i \\ \hline \bar{M}^i \end{bmatrix}_m$$

To Transform the End Actions at the Terminal Joint-j from the Structure-Axes System into the Member-Axes System

Based on the same idea used on joint-i, one can have the following simultaneous equations:

$$\bar{V}_x^j = V_x^j \lambda_{11} + V_y^j \lambda_{12} + V_z^j \lambda_{13}$$

$$\bar{V}_y^j = V_x^j \lambda_{21} + V_y^j \lambda_{22} + V_z^j \lambda_{23}$$

$$\bar{V}_z^j = V_x^j \lambda_{31} + V_y^j \lambda_{32} + V_z^j \lambda_{33}$$

$$\bar{M}_x^j = M_x^j \lambda_{11} + M_y^j \lambda_{12} + M_z^j \lambda_{13}$$

$$\bar{M}_y^j = M_x^j \lambda_{21} + M_y^j \lambda_{22} + M_z^j \lambda_{23}$$

$$\bar{M}_z^j = M_x^j \lambda_{31} + M_y^j \lambda_{32} + M_z^j \lambda_{33}$$

Matrix equation for the above simultaneous equations is

$$\begin{bmatrix} \bar{V}_x^j \\ \bar{V}_y^j \\ \bar{V}_z^j \\ \bar{M}_x^j \\ \bar{M}_y^j \\ \bar{M}_z^j \end{bmatrix}_m = \left[\begin{array}{ccc|ccc} \lambda_{11} & \lambda_{12} & \lambda_{13} & & & \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & & 0 & \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & & & \\ \hline & & & \lambda_{11} & \lambda_{12} & \lambda_{13} \\ & 0 & & \lambda_{21} & \lambda_{22} & \lambda_{23} \\ & & & \lambda_{31} & \lambda_{32} & \lambda_{33} \end{array}\right] \begin{bmatrix} V_x^j \\ V_y^j \\ V_z^j \\ M_x^j \\ M_y^j \\ M_z^j \end{bmatrix}_s$$

If let R^j be the rotation matrix for 1 set of orthogonal components at joint j, then

$$R^j = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}$$

And if let R_T^j be the rotation matrix for 2 sets of orthogonal components at joint j, then

$$R_T^j = \left[\begin{array}{c|c} R^j & 0 \\ \hline 0 & R^j \end{array}\right]$$

The above matrix equation can be abbreviated as

$$\left[\begin{array}{c} \bar{V}^j \\ \hline \bar{M}^j \end{array}\right]_m = \left[\begin{array}{c|c} R^j & 0 \\ \hline 0 & R^j \end{array}\right] \left[\begin{array}{c} V^j \\ \hline M^j \end{array}\right]_s$$

Premultiply both sides of the above abbreviated matrix equation with the inverse of R_T^j, one can get

$$\left[\begin{array}{c} V^j \\ \hline M^j \end{array}\right]_s = \left[\begin{array}{c|c} R^j & 0 \\ \hline 0 & R^j \end{array}\right]^{-1} \left[\begin{array}{c} \bar{V}^j \\ \hline \bar{M}^j \end{array}\right]_m$$

Since $\quad (R^j)^{-1} = (R^j)^T \ , \ (R_T^j)^{-1} = (R_T^j)^T$

Therefore,

$$\begin{bmatrix} V^j \\ \hline M^j \end{bmatrix}_s = \left[\begin{array}{c|c} R^j & 0 \\ \hline 0 & R^j \end{array}\right]^T \begin{bmatrix} \bar{V}^j \\ \hline \bar{M}^j \end{bmatrix}_m$$

i.e.,

$$\begin{bmatrix} V^j \\ \hline M^j \end{bmatrix}_s = \left[\begin{array}{c|c} (R^j)^T & 0 \\ \hline 0 & (R^j)^T \end{array}\right] \begin{bmatrix} \bar{V}^j \\ \hline \bar{M}^j \end{bmatrix}_m$$

The rotation matrix required to transform the orthogonal components of the structure-axes system into the orthogonal components of the member-axes system is R_T:

$$R_T = \left[\begin{array}{c|c} R_T^i & 0 \\ \hline 0 & R_T^j \end{array}\right] = \left[\begin{array}{c|c|c|c} R^i & 0 & 0 & 0 \\ \hline 0 & R^i & 0 & 0 \\ \hline 0 & 0 & R^j & 0 \\ \hline 0 & 0 & 0 & R^j \end{array}\right]$$

i.e.,

$$R_T = \left[\begin{array}{ccc|ccc|ccc|ccc} \lambda_{11} & \lambda_{12} & \lambda_{13} & & & & & & & & & \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & & 0 & & & 0 & & & 0 & \\ \lambda_{31} & \lambda_{32} & \lambda_{33} & & & & & & & & & \\ \hline & & & \lambda_{11} & \lambda_{12} & \lambda_{13} & & & & & & \\ & 0 & & \lambda_{21} & \lambda_{22} & \lambda_{23} & & 0 & & & 0 & \\ & & & \lambda_{31} & \lambda_{32} & \lambda_{33} & & & & & & \\ \hline & & & & & & \lambda_{11} & \lambda_{12} & \lambda_{13} & & & \\ & 0 & & & 0 & & \lambda_{21} & \lambda_{22} & \lambda_{23} & & 0 & \\ & & & & & & \lambda_{31} & \lambda_{32} & \lambda_{33} & & & \\ \hline & & & & & & & & & \lambda_{11} & \lambda_{12} & \lambda_{13} \\ & 0 & & & 0 & & & 0 & & \lambda_{21} & \lambda_{22} & \lambda_{23} \\ & & & & & & & & & \lambda_{31} & \lambda_{32} & \lambda_{33} \end{array}\right]$$

The rotational transformation matrix required to transform the orthogonal components of the member-axes system into the orthogonal components of the structure-axes system is $(R_T)^T$:

$$(R_T)^T = \begin{bmatrix} (R_T^i)^T & 0 \\ 0 & (R_T^j)^T \end{bmatrix} \begin{bmatrix} (R^i)^T & 0 & 0 & 0 \\ 0 & (R^i)^T & 0 & 0 \\ 0 & 0 & (R^j)^T & 0 \\ 0 & 0 & 0 & (R^j)^T \end{bmatrix}$$

$$(R_T)^T = \left[\begin{array}{ccc:ccc:ccc:ccc} \lambda_{11} & \lambda_{21} & \lambda_{31} & & & & & & & & & \\ \lambda_{12} & \lambda_{22} & \lambda_{32} & & 0 & & & 0 & & & 0 & \\ \lambda_{13} & \lambda_{23} & \lambda_{33} & & & & & & & & & \\ \hdashline & & & \lambda_{11} & \lambda_{21} & \lambda_{31} & & & & & & \\ & 0 & & \lambda_{12} & \lambda_{22} & \lambda_{32} & & 0 & & & 0 & \\ & & & \lambda_{13} & \lambda_{23} & \lambda_{33} & & & & & & \\ \hdashline & & & & & & \lambda_{11} & \lambda_{21} & \lambda_{31} & & & \\ & 0 & & & 0 & & \lambda_{12} & \lambda_{22} & \lambda_{32} & & 0 & \\ & & & & & & \lambda_{13} & \lambda_{23} & \lambda_{33} & & & \\ \hdashline & & & & & & & & & \lambda_{11} & \lambda_{21} & \lambda_{31} \\ & 0 & & & 0 & & & 0 & & \lambda_{12} & \lambda_{22} & \lambda_{32} \\ & & & & & & & & & \lambda_{13} & \lambda_{23} & \lambda_{33} \end{array}\right]$$

<u>Rotationally Transformed Stiffness Matrix of a Prismatic Member Skew in a Three-Dimensional Space</u>

The stiffness matrix of a prismatic member skew in a three-dimensional space established on the basis of local axes will be rotationally transformed into a matrix on the basis of reference axes which will be expressed in the same way as it was in Chapter III; that is, the rotationally transformed stiffness matrix of a prismatic member i-j is expressed as

$$S^{i-j} = \left[\begin{array}{c|c} s^{i-i} & s^{i-j} \\ \hline s^{j-i} & s^{j-j} \end{array}\right]$$

and

$$S = [k_{n\,m}] \qquad n,m = 1,2,\ldots.12$$

in which the s's are the submatrices of the stiffness matrix, i, j represent the two ends of the member, and n and m are the numbers of the coordinates at the two ends of the member. The s's are expressed in the following tables:

TABLE 7

SUBMATRIX s^{i-i} OF A ROTATIONALLY TRANSFORMED STIFFNESS MATRIX

$s^{i-i} =$

$\frac{EA_x}{L}(\lambda_{11})^2$ $\frac{12EI_z}{L^3}(\lambda_{21})^2$ $\frac{12EI_y}{L^3}(\lambda_{31})^2$	$\frac{EA_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{22})$	$\frac{EA_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{23})$ $\frac{12EI_y}{L^3}(\lambda_{31})(\lambda_{33})$	$\frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{31})$ $\frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{31})$	$\frac{-6EI_y}{L^2}(\lambda_{22})(\lambda_{31})$	$\frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{31})$
$\frac{EA_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{22})$	$\frac{EA_x}{L}(\lambda_{12})^2$ $\frac{12EI_z}{L^3}(\lambda_{22})^2$	$\frac{EA_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{12EI_z}{L^3}(\lambda_{22})(\lambda_{23})$	$\frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{31})$	0	$\frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{33})$
$\frac{EA_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{23})$ $\frac{12EI_y}{L^3}(\lambda_{31})(\lambda_{33})$	$\frac{EA_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{12EI_z}{L^3}(\lambda_{22})(\lambda_{23})$	$\frac{EA_x}{L}(\lambda_{13})^2$ $\frac{12EI_z}{L^3}(\lambda_{23})^2$ $\frac{12EI_y}{L^3}(\lambda_{33})^2$	$\frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{31})$ $\frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{33})$	$\frac{-6EI_y}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{33})$ $\frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{33})$
$\frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{31})$ $\frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{31})$	$\frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{31})$	$\frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{31})$ $\frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{33})$	$\frac{GI_x}{L}(\lambda_{11})^2$ $\frac{4EI_z}{L}(\lambda_{31})^2$ $\frac{4EI_y}{L}(\lambda_{21})^2$	$\frac{GI_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{4EI_y}{L}(\lambda_{21})(\lambda_{22})$	$\frac{GI_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{4EI_y}{L}(\lambda_{21})(\lambda_{23})$ $\frac{4EI_z}{L}(\lambda_{31})(\lambda_{33})$
$\frac{-6EI_y}{L^2}(\lambda_{22})(\lambda_{31})$	0	$\frac{-6EI_y}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{GI_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{4EI_y}{L}(\lambda_{21})(\lambda_{22})$	$\frac{GI_x}{L}(\lambda_{12})^2$ $\frac{4EI_y}{L}(\lambda_{22})^2$	$\frac{GI_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{4EI_y}{L}(\lambda_{22})(\lambda_{23})$
$\frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{33})$ $\frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{33})$	$\frac{GI_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{4EI_y}{L}(\lambda_{21})(\lambda_{23})$ $\frac{4EI_z}{L}(\lambda_{31})(\lambda_{33})$	$\frac{GI_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{4EI_y}{L}(\lambda_{22})(\lambda_{23})$	$\frac{GI_x}{L}(\lambda_{13})^2$ $\frac{4EI_y}{L}(\lambda_{23})^2$ $\frac{4EI_z}{L}(\lambda_{33})^2$

TABLE 8

SUBMATRIX s^{i-j} OF A ROTATIONALLY TRANSFORMED STIFFNESS MATRIX

$s^{i-j} =$

$\frac{-EA_x}{L}(\lambda_{11})^2$ $\frac{-12EI_z}{L^3}(\lambda_{21})^2$ $\frac{-12EI_y}{L^3}(\lambda_{31})^2$	$\frac{-EA_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{-12EI_z}{L^3}(\lambda_{21})(\lambda_{22})$	$\frac{-EA_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{-12EI_z}{L^3}(\lambda_{21})(\lambda_{23})$ $\frac{-12EI_y}{L^3}(\lambda_{31})(\lambda_{33})$	$\frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{31})$ $\frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{31})$	$\frac{-6EI_y}{L^2}(\lambda_{22})(\lambda_{31})$	$\frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{31})$
$\frac{-EA_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{-12EI_z}{L^3}(\lambda_{21})(\lambda_{22})$	$\frac{-EA_x}{L}(\lambda_{12})^2$ $\frac{-12EI_z}{L^3}(\lambda_{22})^2$	$\frac{-EA_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{-12EI_z}{L^3}(\lambda_{22})(\lambda_{23})$	$\frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{31})$	0	$\frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{33})$
$\frac{-EA_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{-12EI_z}{L^3}(\lambda_{21})(\lambda_{23})$ $\frac{-12EI_y}{L^3}(\lambda_{31})(\lambda_{33})$	$\frac{-EA_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{-12EI_z}{L^3}(\lambda_{22})(\lambda_{23})$	$\frac{-EA_x}{L}(\lambda_{13})^2$ $\frac{-12EI_z}{L^3}(\lambda_{23})^2$ $\frac{-12EI_y}{L^3}(\lambda_{33})^2$	$\frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{-6EI_y}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{33})$ $\frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{33})$
$\frac{-6EI_z}{L^2}(\lambda_{21})(\lambda_{31})$ $\frac{6EI_y}{L^2}(\lambda_{21})(\lambda_{31})$	$\frac{-6EI_z}{L^2}(\lambda_{22})(\lambda_{31})$	$\frac{6EI_y}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{-6EI_z}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{-GI_x}{L}(\lambda_{11})^2$ $\frac{2EI_y}{L}(\lambda_{21})^2$ $\frac{2EI_z}{L}(\lambda_{31})^2$	$\frac{-GI_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{2EI_y}{L}(\lambda_{21})(\lambda_{22})$	$\frac{-GI_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{2EI_y}{L}(\lambda_{21})(\lambda_{23})$ $\frac{2EI_z}{L}(\lambda_{31})(\lambda_{33})$
$\frac{6EI_y}{L^2}(\lambda_{22})(\lambda_{31})$	0	$\frac{6EI_y}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{-GI_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{2EI_y}{L}(\lambda_{21})(\lambda_{22})$	$\frac{-GI_x}{L}(\lambda_{12})^2$ $\frac{2EI_y}{L}(\lambda_{22})^2$	$\frac{-GI_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{2EI_y}{L}(\lambda_{22})(\lambda_{23})$
$\frac{-6EI_z}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{6EI_y}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{-6EI_z}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{6EI_y}{L^2}(\lambda_{23})(\lambda_{33})$ $\frac{-6EI_z}{L^2}(\lambda_{23})(\lambda_{33})$	$\frac{-GI_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{2EI_y}{L}(\lambda_{21})(\lambda_{23})$ $\frac{2EI_z}{L}(\lambda_{31})(\lambda_{33})$	$\frac{-GI_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{2EI_y}{L}(\lambda_{22})(\lambda_{23})$	$\frac{-GI_x}{L}(\lambda_{13})^2$ $\frac{2EI_y}{L}(\lambda_{23})^2$ $\frac{2EI_z}{L}(\lambda_{33})^2$

TABLE 9

SUBMATRIX s^{j-i} OF A ROTATIONALLY TRANSFORMED STIFFNESS MATRIX

$s^{j-i} =$

$\frac{-EA_x}{L}(\lambda_{11})^2$ $\frac{-12EI_z}{L^3}(\lambda_{21})^2$ $\frac{-12EI_y}{L^3}(\lambda_{31})^2$	$\frac{-EA_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{-12EI_z}{L^3}(\lambda_{21})(\lambda_{22})$	$\frac{-EA_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{-12EI_z}{L^3}(\lambda_{21})(\lambda_{23})$ $\frac{-12EI_y}{L^3}(\lambda_{31})(\lambda_{33})$	$\frac{-6EI_z}{L^2}(\lambda_{21})(\lambda_{31})$ $\frac{6EI_y}{L^2}(\lambda_{21})(\lambda_{31})$	$\frac{6EI_y}{L^2}(\lambda_{22})(\lambda_{31})$	$\frac{-6EI_z}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{6EI_y}{L^2}(\lambda_{23})(\lambda_{31})$
$\frac{-EA_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{-12EI_z}{L^3}(\lambda_{21})(\lambda_{22})$	$\frac{-EA_x}{L}(\lambda_{12})^2$ $\frac{-12EI_z}{L^3}(\lambda_{22})^2$	$\frac{-EA_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{-12EI_z}{L^3}(\lambda_{22})(\lambda_{23})$	$\frac{-6EI_z}{L^2}(\lambda_{22})(\lambda_{31})$	0	$\frac{-6EI_z}{L^2}(\lambda_{22})(\lambda_{33})$
$\frac{-EA_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{-12EI_z}{L^3}(\lambda_{21})(\lambda_{23})$ $\frac{-12EI_y}{L^3}(\lambda_{31})(\lambda_{33})$	$\frac{-EA_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{-12EI_z}{L^3}(\lambda_{22})(\lambda_{23})$	$\frac{-EA_x}{L}(\lambda_{13})^2$ $\frac{-12EI_z}{L^3}(\lambda_{23})^2$ $\frac{-12EI_y}{L^3}(\lambda_{33})^2$	$\frac{6EI_y}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{-6EI_z}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{6EI_y}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{-6EI_z}{L^2}(\lambda_{23})(\lambda_{33})$ $\frac{6EI_y}{L^2}(\lambda_{23})(\lambda_{33})$
$\frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{31})$ $\frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{31})$	$\frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{31})$	$\frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{-GI_x}{L}(\lambda_{11})^2$ $\frac{2EI_y}{L}(\lambda_{21})^2$ $\frac{2EI_z}{L}(\lambda_{31})^2$	$\frac{-GI_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{2EI_y}{L}(\lambda_{21})(\lambda_{22})$	$\frac{-GI_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{2EI_y}{L}(\lambda_{21})(\lambda_{23})$ $\frac{2EI_z}{L}(\lambda_{31})(\lambda_{33})$
$\frac{-6EI_y}{L^2}(\lambda_{22})(\lambda_{31})$	0	$\frac{-6EI_y}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{-GI_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{2EI_y}{L}(\lambda_{21})(\lambda_{22})$	$\frac{-GI_x}{L}(\lambda_{12})^2$ $\frac{2EI_y}{L}(\lambda_{22})^2$	$\frac{-GI_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{2EI_y}{L}(\lambda_{22})(\lambda_{23})$
$\frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{33})$ $\frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{33})$	$\frac{-GI_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{2EI_y}{L}(\lambda_{21})(\lambda_{23})$ $\frac{2EI_z}{L}(\lambda_{31})(\lambda_{33})$	$\frac{-GI_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{2EI_y}{L}(\lambda_{22})(\lambda_{23})$	$\frac{-GI_x}{L}(\lambda_{13})^2$ $\frac{2EI_y}{L}(\lambda_{23})^2$ $\frac{2EI_z}{L}(\lambda_{33})^2$

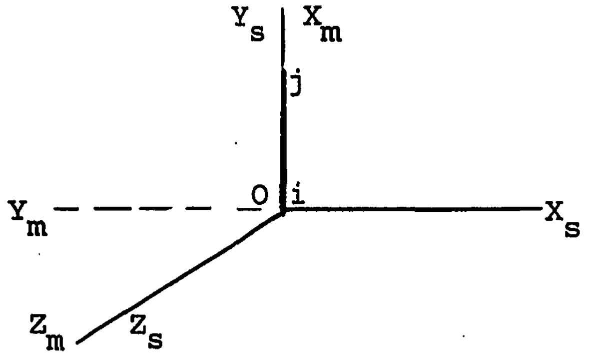
TABLE 10

SUBMATRIX s^{j-j} OF A ROTATIONALLY TRANSFORMED STIFFNESS MATRIX

$s^{j-j} =$

$\frac{EA_x}{L}(\lambda_{11})^2$ $\frac{12EI_z}{L^3}(\lambda_{21})^2$ $\frac{12EI_y}{L^3}(\lambda_{31})^2$	$\frac{EA_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{22})$	$\frac{EA_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{23})$ $\frac{12EI_y}{L^3}(\lambda_{31})(\lambda_{33})$	$\frac{-6EI_z}{L^2}(\lambda_{21})(\lambda_{31})$ $\frac{6EI_y}{L^2}(\lambda_{21})(\lambda_{31})$	$\frac{6EI_y}{L^2}(\lambda_{22})(\lambda_{31})$	$\frac{-6EI_z}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{6EI_y}{L^2}(\lambda_{23})(\lambda_{31})$
$\frac{EA_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{22})$	$\frac{EA_x}{L}(\lambda_{12})^2$ $\frac{12EI_z}{L^3}(\lambda_{22})^2$	$\frac{EA_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{12EI_z}{L^3}(\lambda_{22})(\lambda_{23})$	$\frac{-6EI_z}{L^2}(\lambda_{22})(\lambda_{31})$	0	$\frac{-6EI_z}{L^2}(\lambda_{22})(\lambda_{33})$
$\frac{EA_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{23})$ $\frac{12EI_y}{L^3}(\lambda_{31})(\lambda_{33})$	$\frac{EA_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{12EI_z}{L^3}(\lambda_{22})(\lambda_{23})$	$\frac{EA_x}{L}(\lambda_{13})^2$ $\frac{12EI_z}{L^3}(\lambda_{23})^2$ $\frac{12EI_y}{L^3}(\lambda_{33})^2$	$\frac{6EI_y}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{-6EI_z}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{6EI_y}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{-6EI_z}{L^2}(\lambda_{23})(\lambda_{33})$ $\frac{6EI_y}{L^2}(\lambda_{23})(\lambda_{33})$
$\frac{-6EI_z}{L^2}(\lambda_{21})(\lambda_{31})$ $\frac{6EI_y}{L^2}(\lambda_{21})(\lambda_{31})$	$\frac{-6EI_z}{L^2}(\lambda_{22})(\lambda_{31})$	$\frac{6EI_y}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{-6EI_z}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{GI_x}{L}(\lambda_{11})^2$ $\frac{4EI_y}{L}(\lambda_{21})^2$ $\frac{4EI_z}{L}(\lambda_{31})^2$	$\frac{GI_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{4EI_y}{L}(\lambda_{21})(\lambda_{22})$	$\frac{GI_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{4EI_y}{L}(\lambda_{21})(\lambda_{23})$ $\frac{4EI_z}{L}(\lambda_{31})(\lambda_{33})$
$\frac{6EI_y}{L^2}(\lambda_{22})(\lambda_{31})$	0	$\frac{6EI_y}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{GI_x}{L}(\lambda_{11})(\lambda_{12})$ $\frac{4EI_y}{L}(\lambda_{21})(\lambda_{22})$	$\frac{GI_x}{L}(\lambda_{12})^2$ $\frac{4EI_y}{L}(\lambda_{22})^2$	$\frac{GI_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{4EI_y}{L}(\lambda_{22})(\lambda_{23})$
$\frac{-6EI_z}{L^2}(\lambda_{21})(\lambda_{33})$ $\frac{6EI_y}{L^2}(\lambda_{23})(\lambda_{31})$	$\frac{-6EI_z}{L^2}(\lambda_{22})(\lambda_{33})$	$\frac{6EI_y}{L^2}(\lambda_{23})(\lambda_{33})$ $\frac{-6EI_z}{L^2}(\lambda_{23})(\lambda_{33})$	$\frac{GI_x}{L}(\lambda_{11})(\lambda_{13})$ $\frac{4EI_y}{L}(\lambda_{21})(\lambda_{23})$ $\frac{4EI_z}{L}(\lambda_{31})(\lambda_{33})$	$\frac{GI_x}{L}(\lambda_{12})(\lambda_{13})$ $\frac{4EI_y}{L}(\lambda_{22})(\lambda_{23})$	$\frac{GI_x}{L}(\lambda_{13})^2$ $\frac{4EI_y}{L}(\lambda_{23})^2$ $\frac{4EI_z}{L}(\lambda_{33})^2$

Rotation Matrix for a Vertical Member



Let member i-j be a vertical member which is perpendicular to the X_s-Z_s plane. Then it can be assumed that, θ, the angle of rotation about the Y_s-axis is 0° and ,φ, the angle of rotation about the Z_m axis which is also the Z_s-axis is 90° counter clock-wise. The direction cosines are as follows:

$$\lambda_{11} = \cos\alpha_1 = \cos 0° \cos 90° = 0$$

$$\lambda_{12} = \cos\beta_1 = \cos(90°-90°) = 1$$

$$\lambda_{13} = \cos\gamma_1 = \cos 90° \cos 90° = 0$$

$$\lambda_{21} = \cos\alpha_2 = \cos(90°+90°)\cos 0° = -1$$

$$\lambda_{22} = \cos\beta_2 = \cos\varphi = \cos 90° = 0$$

$$\lambda_{23} = \cos\gamma_2 = \cos(90°+90°)\cos 90° = 0$$

$$\lambda_{31} = \cos\alpha_3 = \cos(90°+0°) = 0$$

$$\lambda_{32} = \cos\beta_3 = \cos 90° = 0$$

$$\lambda_{33} = \cos\gamma_3 = \cos\theta = \cos 0° = 1$$

The rotation matrix, R, which is formed with the λ's, would appear as

$$R = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R^i = R^j = R$

$$R_T^i = \begin{bmatrix} R^i & 0 \\ 0 & R^i \end{bmatrix}$$

$$R_T^j = \begin{bmatrix} R^j & 0 \\ 0 & R^j \end{bmatrix}$$

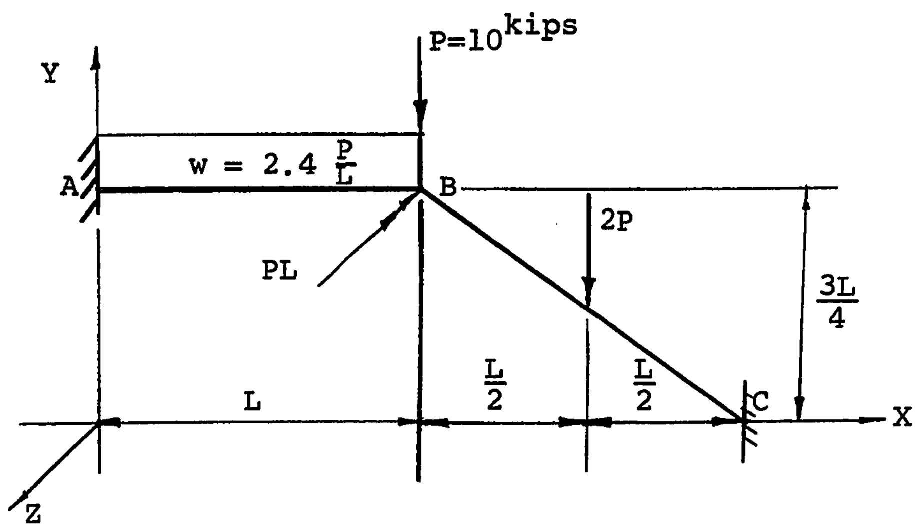
$$R_T = \begin{bmatrix} R_T^i & 0 \\ 0 & R_T^j \end{bmatrix}$$

In conclusion, the developed rotational transformation relation in λ system is good for a member skew in any direction, including a vertical member, in a three-dimensional space.

CHAPTER V

"NUMERICAL EXAMPLE" (5)--PLANE FRAME WITH DISTRIBUTED LOAD AND INTERMEDIATE CONCENTRATED LOAD

A plane frame having two members, three joints, six restraints, and three degrees of freedom is to be analyzed by the displacement method.

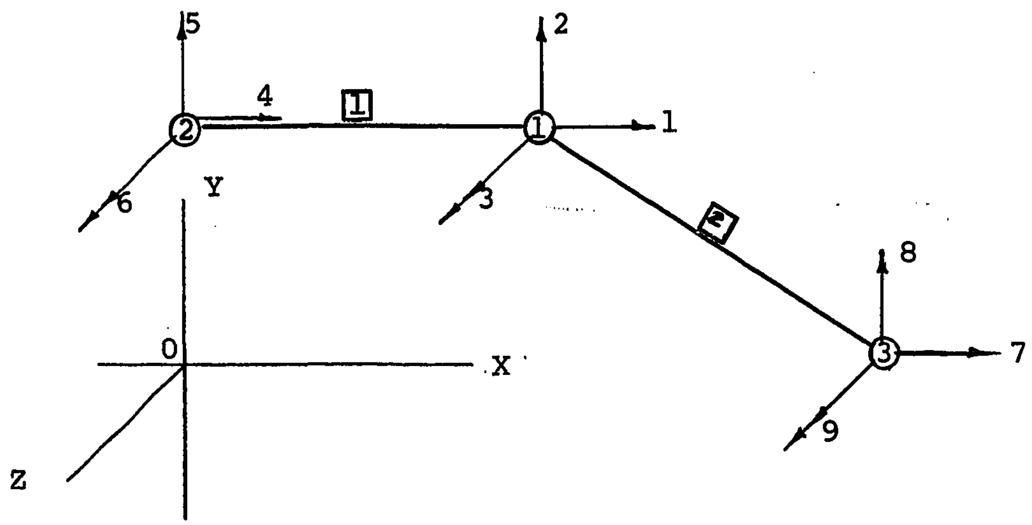


For the purpose of analysis, it is assumed that the cross-sectional area A and the moment of inertia I_z are constant throughout the structure, and units of kips, inches and radians are used throughout the analysis.

The procedure to be followed in the analysis is as follows:

Numbering System

The numbering system for members, joints, and displacement is given in the following figure.



Numbers in circles are the numbers for the joints. Numbers in squares are the numbers for the members. Numbers with the arrow heads are the numbers for the coordinates at the ends of the members, and they are pointing in the positive directions.

Information

Joint Information for the Frame

Joints	Coordinates (in)		Restraint List		
	X	Y	X	Y	Z
1	100	75	0	0	0
2	0	75	1	1	1
3	200	0	1	1	1

Member Information for the Frame

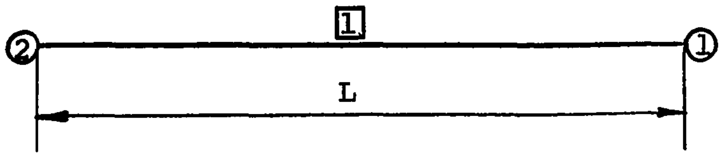
Member Numbers	Initial End	Terminal End	Length (in)	Area (in^2)	I_z (in^4)	E (ksi)
1	2	1	100	10	1,000	10,000
2	1	3	125	10	1,000	10,000

Member Analysis

For the convenience in the processes of analysis, the origin of the axes for the whole structure is taken at the point which is on the same vertical line with the joint 2 and on the same horizontal line with the joint 3. For the member-axes system, the origin is taken at the initial end of each member.

The analysis of the whole structure is done by processing each of the members of the frame separately and is based on the local axes system of each of the members. The displacement method is employed for the analysis of each of the members of the frame as follows:

Analysis of Member 1



Calculations--Section Properties

$$EA = 10{,}000(10) = 100{,}000$$

$$EI_z = 10{,}000(1{,}000) = 10{,}000{,}000$$

$$L^2 = (100)^2 = 10{,}000$$

$$L^3 = (100)^3 = 1{,}000{,}000$$

$$\frac{EA}{L} = \frac{100{,}000}{100} = 1{,}000$$

$$\frac{2EI_z}{L} = \frac{2(10{,}000{,}000)}{100} = 200{,}000$$

$$\frac{4EI_z}{L} = \frac{4(10{,}000{,}000)}{100} = 400{,}000$$

$$\frac{6EI_z}{L^2} = \frac{6(10{,}000{,}000)}{10{,}000} = 6{,}000$$

$$\frac{12EI_z}{L^3} = \frac{12(10{,}000{,}000)}{1{,}000{,}000} = 120$$

Since the member-axis of member 1 is coincident with the X-axis of the structure-axes system, no rotational transformation is needed, i.e., $[S_m^1] = [S_s^1]$.

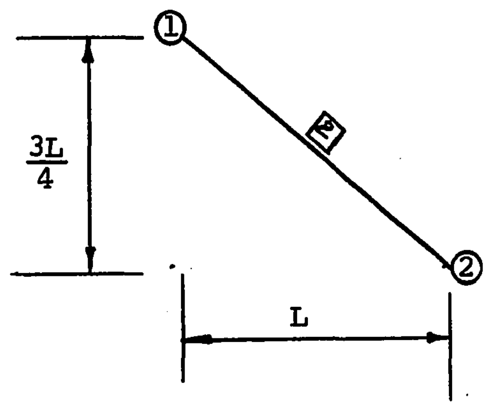
Formulation of Stiffness Matrix on the Basis of Member-Axes System (Also on Structure-Axes System).

$$[S_m] = \begin{bmatrix} \frac{EA_x}{L} & 0 & 0 & \frac{-EA_x}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & \frac{-12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & \frac{2EI_z}{L} \\ \frac{-EA_x}{L} & 0 & 0 & \frac{EA_x}{L} & 0 & 0 \\ 0 & \frac{-12EI_z}{L^3} & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix}$$

Substitution:

$$[S_m^1]=[S_s^1]=\begin{bmatrix} 1{,}000 & 0 & 0 & -1{,}000 & 0 & 0 \\ 0 & 120 & 6{,}000 & 0 & -120 & 6{,}000 \\ 0 & 6{,}000 & 400{,}000 & 0 & -6{,}000 & 200{,}000 \\ -1{,}000 & 0 & 0 & 1{,}000 & 0 & 0 \\ 0 & -120 & -6{,}000 & 0 & 120 & -6{,}000 \\ 0 & 6{,}000 & 200{,}000 & 0 & -6{,}000 & 400{,}000 \end{bmatrix}$$

Analysis of Member 2



Calculations

Section Properties

$$EA = 10{,}000(10) = 100{,}000$$

$$EI_z = 10{,}000(1{,}000) = 10{,}000{,}000$$

$$L^2 = (125)^2 = 15{,}625$$

$$L^3 = (125)^3 = 1{,}953{,}125$$

$$\frac{EA}{L} = \frac{100{,}000}{125} = 800$$

$$\frac{2EI_z}{L} = \frac{20{,}000{,}000}{125} = 160{,}000$$

$$\frac{4EI_z}{L} = \frac{40{,}000{,}000}{125} = 320{,}000$$

$$\frac{6EI_z}{L^2} = \frac{60{,}000{,}000}{15{,}625} = 3{,}840$$

$$\frac{12EI_z}{L^3} = \frac{120,000,000}{1,953,125} = 61.44$$

Direction Cosines

For the calculation of the direction cosines, the origin for the two kinds of axis systems is taken at the joint 1. The coordinates of the joint 3 w.r.t. the structure axes orientated at the joint 1 are $C_x = L$, $C_y = \frac{3L}{4}$.

It can be assumed that the angle θ is 0 and the angle φ is negative. Therefore, $\cos\theta = 1$, $\sin\theta = 0$; $\cos(-\varphi) = 0.8$, $\sin(-\varphi) = -0.6$. The λ's are as follows:

$$\lambda_{11} = \cos\alpha_1 = \cos\theta\cos(-\varphi) = 0.8$$

$$\lambda_{12} = \cos\beta_1 = \cos[90^\circ-(-\varphi)] = -0.6$$

$$\lambda_{13} = \cos\gamma_1 = \cos(90^\circ-\theta)\cos(-\varphi) = 0$$

$$\lambda_{21} = \cos\alpha_2 = \cos\theta\cos[90^\circ+(-\varphi)] = 0.6$$

$$\lambda_{22} = \cos\beta_2 = \cos(-\varphi) = 0.8$$

$$\lambda_{23} = \cos\gamma_2 = \cos(90^\circ-\theta)\cos[90^\circ+(-\varphi)] = 0$$

$$\lambda_{31} = \cos\alpha_3 = \cos(90^\circ+\theta) = 0$$

$$\lambda_{32} = \cos\beta_3 = \cos 90^\circ = 0$$

$$\lambda_{33} = \cos\gamma_3 = \cos\theta = 1$$

Formulation of Rotational Matrix

The rotational transformation matrix of member 2, which is composed of the direction cosines of the member w.r.t. the structure axes, is the combination of the rotation matrices $[R^1]$ and $[R^3]$ of the ends 1 and 3 respectively.

$$[R^1] = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [R^3] = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the rotational transformation matrix for the member 2 is R_T^2:

$$[R_T^2] = \left[\begin{array}{c|c} R^1 & 0 \\ \hline 0 & R^3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 0.8 & -0.6 & 0 & & & \\ 0.6 & 0.8 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 0.8 & -0.6 & 0 \\ & 0 & & 0.6 & 0.8 & 0 \\ & & & 0 & 0 & 1 \end{array}\right]$$

Stiffness Matrix of Member 2

The stiffness matrix for the member 2 based on the local axes system is $[S_m^2]$:

$$[S_m^2] = \begin{bmatrix} 800 & 0 & 0 & -800 & 0 & 0 \\ 0 & 61.44 & 3840 & 0 & -61.44 & 3840 \\ 0 & 3840 & 320000 & 0 & -3840 & 160000 \\ -800 & 0 & 0 & 800 & 0 & 0 \\ 0 & -61.44 & -3840 & 0 & 61.44 & -3840 \\ 0 & 3840 & 160000 & 0 & -3840 & 320000 \end{bmatrix}$$

Rotational Transformation

The stiffness matrix for the member based on the structure-axes system can be obtained by the following rotational transformation equation:

$$[S_s]_2 = [R_T]_2^T [S_m]_2 [R_T]_2$$

The result of this rotational transformation is given as follows:

$$[R_T]_2^T [S_m]_2 = \begin{bmatrix} 640.00 & 36.86 & 2304.00 & -640.00 & -36.86 & 2304.00 \\ -480.00 & 49.15 & 3072.00 & 480.00 & -49.15 & 3072.00 \\ 0.0 & 3840.00 & 320000.00 & 0.0 & -3840.00 & 160000.00 \\ -640.00 & -36.86 & -2304.00 & 640.00 & 36.86 & -2304.00 \\ 480.00 & -49.15 & -3072.00 & -480.00 & 49.15 & -3072.00 \\ 0.0 & 3840.00 & 160000.00 & 0.0 & -3840.00 & 320000.00 \end{bmatrix}$$

$$[S_s]_2 = \begin{bmatrix} 534.12 & -354.51 & 2304.00 & -534.12 & 354.51 & 2304.00 \\ -354.51 & 327.32 & 3072.00 & 354.51 & -327.32 & 3072.00 \\ 2304.00 & 3072.00 & 320000.00 & -2304.00 & -3072.00 & 160000.00 \\ -534.12 & 354.51 & -2304.00 & 534.12 & -354.51 & -2304.00 \\ 354.51 & -327.32 & -3072.00 & -354.51 & 327.32 & -3072.00 \\ 2304.00 & 3072.00 & 160000.00 & -2304.00 & -3072.00 & 320000.00 \end{bmatrix}$$

TABLE 11

Joint Stiffness Matrix for the Frame
(in structure axes system)

1534.12	-354.51	2304.0	-1000.0	0.0	0.0	-534.12	354.51	2304.0
-354.51	447.32	-2928.0	0.0	-120.0	-6000.0	354.51	-327.32	3072.0
2304.0	-2928.0	720000.0	0.0	6000.0	200000.0	-2304.0	-3072.0	160000.0
-1000.0	0.0	0.0	1000.0	0.0	0.0	0.0	0.0	0.0
0.0	-120.0	6000.0	0.0	120.0	6000.0	0.0	0.0	0.0
0.0	-6000.0	200000.0	0.0	6000.0	400000.0	0.0	0.0	0.0
-534.12	354.51	-2304.0	0.0	0.0	0.0	534.12	-354.51	-2304.0
354.51	-327.32	-3072.0	0.0	0.0	0.0	-354.51	327.32	-3072.0
2304.0	3072.0	160000.0	0.0	0.0	0.0	-2304.0	-3072.0	320000.0

Joint Stiffness Matrix of Joint 1

Since only joint 1 will have displacements, the stiffness matrix for joint 1 is needed and can be obtained by combining the stiffness at all the coordinates at joint 1 on member 1 with that on member 2, as shown in the following submatrix $[S_J^1]$ of $[S_J]$:

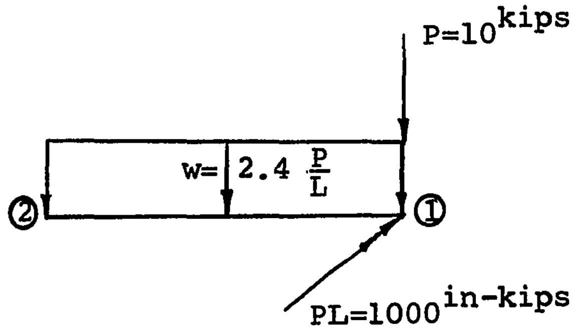
$$[S_J^1] = \begin{bmatrix} 1534.1 & -354.5 & 2304.0 \\ -354.5 & 447.3 & -2928.0 \\ 2304.0 & -2928.0 & 720000.0 \end{bmatrix}$$

Let $[S_J^1]^{-1}$ be the inverse of $[S_J^1]$, then

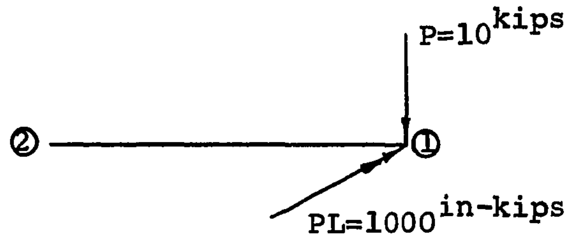
$$[S_J^1]^{-1} = \begin{bmatrix} 0.79799129E-03 & 0.63255767E-03 & 0.18829034E-07 \\ 0.63255767E-03 & 0.27981977E-02 & 0.93551516E-05 \\ 0.18828966E-07 & 0.93551516E-05 & 0.14268728E-05 \end{bmatrix}$$

Loading Analysis

Loading System on Member 1



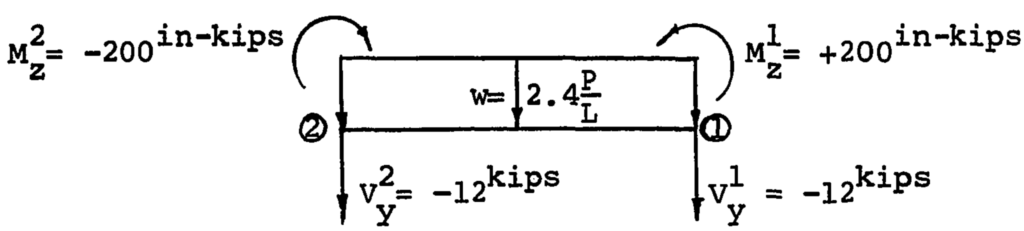
Nodal Actions



Nodal action vector on member 1 is $[A^1]$:

$[A^1] = [0, \;\; -10, \;\; -1000, \;\; 0, \;\; 0, \;\; 0]$.....Column Matrix

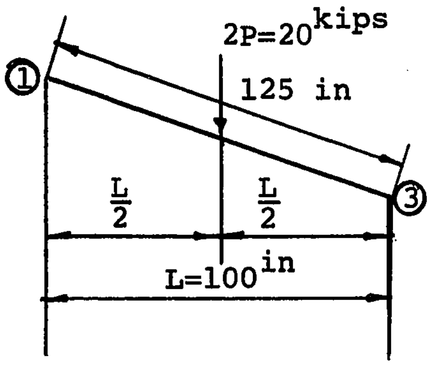
Distributed Loading System



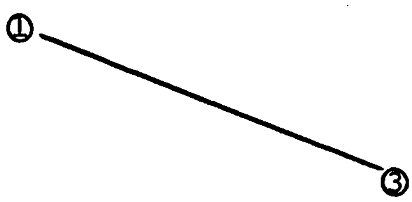
Since member 1 is parallel to the X-axis of the structure-axes system, the equivalent nodal action vector on member 1 can be directly obtained as the following matrix:

$$[A_E^1] = [0, \;\; -12, \;\; 200, \;\; 0, \;\; -12, \;\; -200]$$

Loading System on Member 2



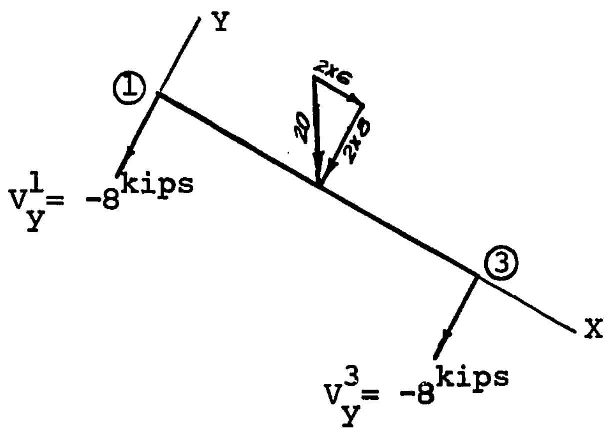
Nodal Actions



Nodal action vector on member 2 is $[A^2]$:

$$[A^2] = [0, \ 0, \ 0, \ 0, \ 0, \ 0]$$

Distributed Loading System
(Intermediate Loading System)



The converted nodal action vector on the basis of member-axes system is $[\overset{o}{A}{}^2_E]$:

$$[\overset{o}{A}{}^2_E] = [6,\ -8,\ -250,\ 6,\ -8,\ 250]$$

The equivalent nodal action vector on member 2, based on the structure-axes system, is $[A^2_E]$:

$$[A^2_E] = [R^2_T]^T[\overset{o}{A}{}^2_E]$$

The result of the above matrix multiplication is given as the following matrix:

$$[A^2_E] = \begin{bmatrix} 0.0 \\ -10.00 \\ -250.00 \\ 0.0 \\ -10.00 \\ 250.00 \end{bmatrix} \begin{array}{l} \text{...in the direction of coordinate X at joint 1} \\ \text{...in the direction of coordinate Y at joint 1} \\ \text{...in the sense of coordinate Z at joint 1} \\ \text{...in the direction of coordinate X at joint 3} \\ \text{...in the direction of coordinate Y at joint 3} \\ \text{...in the sense of coordinate Z at joint 3} \end{array}$$

The final equivalent actions at all the nodal points of the frame is the combination of A^1_E and A^2_E, which is expressed as $[A_E]$:

$$[A_E] = [0,\ -22,\ -50,\ \ 0,\ -12,\ -200,\ \ 0,\ -10,\ 250]$$

The final nodal actions at all the nodal points of the frame is the combination of A and A_E which is expressed as $[A_c]$:

$$A_c = [0,\ -32,\ -1050,\ \ 0,\ -12,\ -200,\ \ 0,\ -10,\ 250]$$

The joint displacements at the rigid joint 1 is calculated by the following matrix equation:

$$[D_j^1] = [S]^{-1}[A_c^1]$$

where $[D_j^1]$ is the matrix of the joint displacements at joint 1, $[S]^{-1}$ is the inverse of the submatrix of the joint stiffness matrix for the whole frame based on the structure-axes system, which relates the nodal actions at the coördinates 1, 2, 3 to the joint displacements at the coordinates 1, 2, 3, $[A_c^1]$ is the matrix of the nodal actions at joint 1 shown below.

$$A_c^1 = \begin{bmatrix} 0 \\ -32 \\ -1050 \end{bmatrix} \begin{array}{l} \text{....nodal action at coordinate 1 of joint 1} \\ \text{....nodal action at coordinate 2 of joint 1} \\ \text{....nodal action at coordinate 3 of joint 1} \end{array}$$

Therefore,

$$[D_j^1] = \begin{bmatrix} -0.0202597 \\ -0.0993653 \\ -0.0017976 \end{bmatrix}$$

The matrix of the joint displacements for the whole frame is $[D_j]$:

$$D_j = [-0.02026, \quad -0.09936, \quad -0.001797, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]$$

The nodal actions at the other two ends due to the given externally distributed loads are shown in the following column matrix $\overset{o}{A}_{RL}$:

$$[\overset{o}{A}_{RL}] = \begin{bmatrix} 0 \\ -12 \\ -200 \\ 0 \\ -10 \\ 250 \end{bmatrix} \begin{array}{l} \text{...nodal action at coordinate 4 of joint 2} \\ \text{...nodal action at coordinate 5 of joint 2} \\ \text{...nodal action at coordinate 6 of joint 2} \\ \text{...nodal action at coordinate 7 of joint 3} \\ \text{...nodal action at coordinate 8 of joint 3} \\ \text{...nodal action at coordinate 9 of joint 3} \end{array}$$

The reactions at the two fixed ends of the frame from the supports are contained in the column matrix $[A_R]$. $[A_R]$ is obtained from the following matrix equation:

$$[A_R] = [A_{RL}] + [S_{RD}][D_j^i]$$

where

$$[A_{RL}]=(-1)[\overset{o}{A}_{RL}]= \begin{bmatrix} 0.0 \\ 12.0 \\ 200.0 \\ 0.0 \\ 10.0 \\ -250.0 \end{bmatrix} \begin{array}{l} \text{...reaction force due to load on member 1 at the coordinate 4 of joint 2} \\ \text{...reaction force, due to load on member 1, at the coordinate 5 of joint 2} \\ \text{...reaction couple, due to load on member 1, at the coordinate 6 of joint 2} \\ \text{...reaction force, due to load on member 2, at the coordinate 7 of joint 3} \\ \text{...reaction force, due to load on member 2, at the coordinate 8 of joint 3} \\ \text{...reaction couple, due to load on member 2, at the coordinate 9 of joint 3} \end{array}$$

$[S_{RD}]$ is the submatrix of the joint stiffness matrix of the whole frame based on the structure-axes system which relates the joint displacements at the coordinates 1, 2, 3 to the nodal actions at the coordinates 4, 5, 6, 7, 8, 9, $[D_j^1]$ is the matrix of the joint displacements at joint 1.

$$[S_{RD}][D_j^1] = \begin{bmatrix} 20.25 \\ 1.13 \\ 236.67 \\ -20.26 \\ 30.86 \\ -639.54 \end{bmatrix}$$

Therefore,

$$[A_R] = \begin{bmatrix} 20.25 \\ 13.13 \\ 436.67 \\ -20.26 \\ 40.86 \\ -889.54 \end{bmatrix}
\begin{array}{l} \text{..Reaction force at the coordinate 4 of joint 2} \\ \text{..Reaction force at the coordinate 5 of joint 2} \\ \text{..Reaction moment at the coordinate 6 of joint 2} \\ \text{..Reaction force at the coordinate 7 of joint 3} \\ \text{..Reaction force at the coordinate 8 of joint 3} \\ \text{..Reaction moment at the coordinate 9 of joint 3} \end{array}$$

The last step in the process of analysis is to determine the end-actions of the members of the frame. The member end-actions on each member is obtained by adding the actions acting on the member ends due to the displacements at the joint 1 to the actions due to the externally applied loading system on each member. The actions acting at both ends of each member due to the joint displacements is obtained by the matrix expression $[S_m]_i[R_T]_i[D_j]_i$, where i is the member number in the numbering system. The actions acting at both ends of member 1 are $[S_m]_1[R_T]_1[D_j]_1$:

$$[S_m]_1[R_T]_1[D_j]_1$$

$$= \begin{bmatrix} -1000 & 0 & 0 \\ 0 & -120 & 6000 \\ 0 & -6000 & 200000 \\ 1000 & 0 & 0 \\ 0 & 120 & -6000 \\ 0 & -6000 & 400000 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.0202597 \\ -0.0993653 \\ -0.0017976 \end{bmatrix}$$

$$= \begin{bmatrix} -1000 & 0 & 0 \\ 0 & -120 & 6000 \\ 0 & -6000 & 200000 \\ 1000 & 0 & 0 \\ 0 & 120 & -6000 \\ 0 & -6000 & 400000 \end{bmatrix} \begin{bmatrix} -0.0202597 \\ -0.0993653 \\ -0.0017976 \end{bmatrix} = \begin{bmatrix} 20.2597008 \\ 1.1382343 \\ 236.6716617 \\ -20.2597008 \\ -1.1382343 \\ -122.8483430 \end{bmatrix}$$

The actions acting at both ends of member 1, due to loads, are $[A_{ML}]_1$:

$$[A_{ML}]_1 = \begin{bmatrix} 0 \\ 12 \\ 200 \\ 0 \\ 12 \\ -200 \end{bmatrix} \begin{array}{l} \text{...in the direction of the coordinate 4} \\ \text{...in the direction of the coordinate 5} \\ \text{...in the sense of the coordinate 6} \\ \text{...in the direction of the coordinate 1} \\ \text{...in the direction of the coordinate 2} \\ \text{...in the sense of the coordinate 3} \end{array}$$

Therefore, the final actions acting at both ends of member 1 are $[A_M]_1$:

$$[A_M]_1 = [A_{ML}]_1 + [S_m]_1[R_T]_1[D_j]_1$$

$$= \begin{bmatrix} 0 \\ 12 \\ 200 \\ 0 \\ 12 \\ -200 \end{bmatrix} + \begin{bmatrix} 20.2597 \\ 1.1382 \\ 236.6717 \\ -20.2597 \\ -1.1382 \\ -122.8483 \end{bmatrix}$$

$$= \begin{bmatrix} 20.2597 \\ 13.1382 \\ 436.6717 \\ -20.2597 \\ 10.8618 \\ -322.8483 \end{bmatrix} \begin{array}{l} \text{...in the direction of the coordinate 4} \\ \text{...in the direction of the coordinate 5} \\ \text{...in the sense of the coordinate 6} \\ \text{...in the direction of the coordinate 1} \\ \text{...in the direction of the coordinate 2} \\ \text{...in the sense of the coordinate 3} \end{array}$$

Similarly, the actions acting at both ends of member 2 are $[S_m]_2[R_T]_2[D_j]_1$:

$$[S_m]_2[R_T]_2[D_j]_1$$

$$= \begin{bmatrix} 800 & 0 & 0 \\ 0 & 61.44 & 3840 \\ 0 & 3840 & 320000 \\ -800 & 0 & 0 \\ 0 & -61.44 & -3840 \\ 0 & 3840 & 160000 \end{bmatrix} \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.0202597 \\ -0.0993653 \\ -0.0017976 \end{bmatrix}$$

$$= \begin{bmatrix} 800 & 0 & 0 \\ 0 & 61.44 & 3840 \\ 0 & 3840 & 320000 \\ -800 & 0 & 0 \\ 0 & -61.44 & -3840 \\ 0 & 3840 & 160000 \end{bmatrix} \begin{bmatrix} 0.0434114 \\ -0.0916480 \\ -0.0017976 \end{bmatrix} = \begin{bmatrix} 34.7291 \\ -12.5336 \\ -927.1606 \\ -34.7291 \\ 12.5336 \\ -639.5446 \end{bmatrix}$$

The actions acting at both ends of member 2 due to the externally applied load on the member are $[A_{ML}]_2$:

$$[A_{ML}]_2 = \begin{bmatrix} -6 \\ 8 \\ 250 \\ -6 \\ 8 \\ -250 \end{bmatrix} \begin{array}{l} \text{... in the direction of the coordinate 1} \\ \text{... in the direction of the coordinate 2} \\ \text{... in the sense of the coordinate 3} \\ \text{... in the direction of the coordinate 7} \\ \text{... in the direction of the coordinate 8} \\ \text{... in the sense of the coordinate 9} \end{array}$$

The final actions acting at both ends of member 2 are $[A_M]_2$:

$$[A_M]_2 = [A_{ML}]_2 + [S_m]_2[R_T]_2[D_j]_1$$

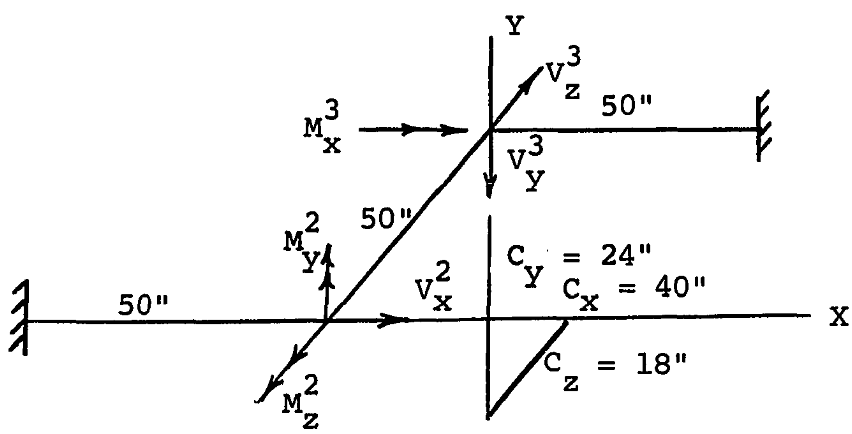
$$= \begin{bmatrix} -6 \\ 8 \\ 250 \\ -6 \\ 8 \\ -250 \end{bmatrix} + \begin{bmatrix} 34.7291 \\ -12.5336 \\ -927.1606 \\ -34.7291 \\ 12.5336 \\ -639.5446 \end{bmatrix}$$

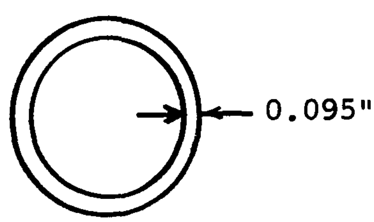
$$= \begin{bmatrix} 28.7291 \\ -4.5336 \\ -677.1606 \\ -40.7291 \\ 20.5336 \\ -889.5446 \end{bmatrix} \begin{array}{l} \text{... in the direction of the coordinate 1} \\ \text{... in the direction of the coordinate 2} \\ \text{... in the sense of the coordinate 3} \\ \text{... in the direction of the coordinate 7} \\ \text{... in the direction of the coordinate 8} \\ \text{... in the sense of the coordinate 9} \end{array}$$

CHAPTER VI

NUMERICAL EXAMPLE: SPACE FRAME

A frame, including a space member, having three members, four joints and twelve degrees of freedom is analyzed by the displacement method.





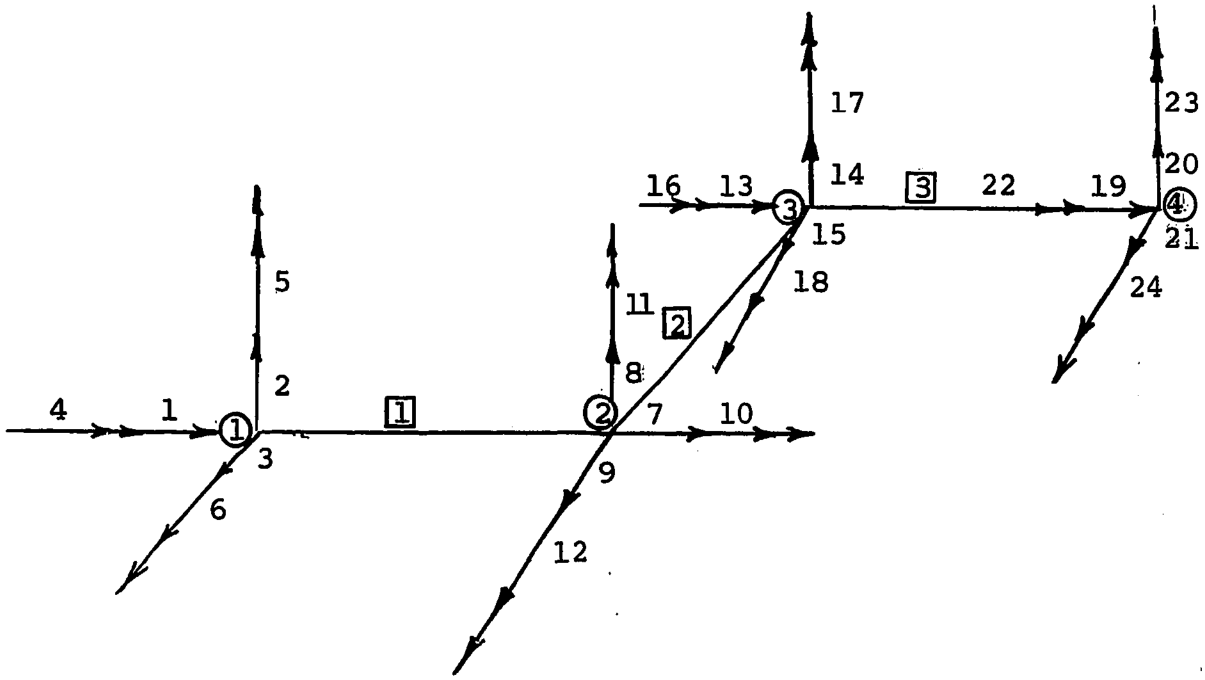
Cross—Section of 2"-0.095" Tube

$A = 0.5685^{in^2}$	$EA = 16486500.0$
$L = 50''$	$\frac{EA}{L} = 329730$
$I = 0.2586^{in^4}$	$EI = 7499400$
$J = 2I = 0.5172^{in^4} (I_x)$	$GJ = 5689200$
$E = 29(10^6)^{lb/in^2}$	$L^2 = (50)^2 = 2500$
$G = 11(10^6)^{lb/in^2}$	$L^3 = (50)^3 = 125000$
$\frac{D}{t} = 21.05$	$\frac{2EI}{L} = 299976$
$V_x^2 = 6{,}000^{lb}$	$\frac{4EI}{L} = 599952$
$M_y^2 = 4{,}800^{in\text{-}lb}$	$\frac{6EI}{L^2} = 17998.56$
$M_z^2 = 3{,}600^{in\text{-}lb}$	
$V_y^3 = -1{,}200^{lb}$	$\frac{12EI}{L^3} = 719.9424$
$V_z^3 = -900^{lb}$	
$M_x^3 = 10{,}000^{in\text{-}lb} (T)$	$\frac{GJ}{L} = 113784$

The procedure of analysis to be followed is as follows:

Numbering System

The numbering system for the members, joints, and the displacements is given in the following figure. Numbers in circles are the numbers for the joints, and numbers in squares are the numbers for the members. The coordinates at every joint are indicated by the arrow headed lines.



<u>Information</u>

Member Information for the Frame

N	i	j	Length (in)	A (in^2)	$I_x=(J)$ (in^4)	$I_y=I_z$ (in^4)	E (psi)	G (psi)
1	1	2	50	0.5685	0.5172	0.2586	$29(10^6)$	$11(10^6)$
2	2	3	50	0.5685	0.5172	0.2586	$29(10^6)$	$11(10^6)$
3	3	4	50	0.5685	0.5172	0.2586	$29(10^6)$	$11(10^6)$

Notes:

N = member number

i = initial end

j = terminal end

A = cross-sectional area

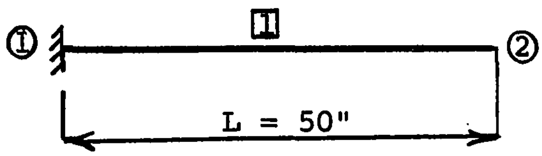
Loading Information for the Frame--
Given Nodal Actions

Joint	Linear Actions (lb)	Moment (in-lb)
2	$V_x^2 = 6000$	$M_y^2 = 4{,}800$ $M_z^2 = 3{,}600$
3	$V_y^3 = -1200$ $V_z^3 = -900$	$M_x^3 = 10{,}000$

Member Analysis

For the purpose of analysis, the origin of the structure-axes system for the whole structure is taken at the joint 1. The origin for the local axes system for each member is taken at the initial end of each of them. The analysis of the whole frame is done by processing each of the members of the frame separately, and is based on the local axes system of each of the members.

Analysis of Member 1



Calculations

Section Properties

$$EA = 29(10^6)(0.5685) = 16,486,500.00$$

$$\frac{EA}{L} = \frac{16,486,500}{50} = 329730$$

$$EI = 29(10^6)(0.2586) = 7,499,400.00$$

$$GI_x = 11(10^6)(0.5172\ in^4) = 5,689,200.00$$

$$L^2 = (50^2) = 2500.00^{in^2}$$

$$L^3 = (50^3) = 125,000.00^{in^3}$$

$$\frac{2EI}{L} = \frac{2(7,499,400)}{50} = 299,976.00$$

$$\frac{4EI}{L} = \frac{4(7,499,400)}{50} = 599,952.00$$

$$\frac{6EI^2}{L} = \frac{6(7,499,400)}{2500} = 17,998.56$$

$$\frac{12EI}{L^3} = \frac{12(7,499,400)}{125000} = 719.9424$$

$$\frac{GI_x}{L} = \frac{5,689,200}{50} = 113,784.00$$

Direction Cosines

For the calculation of the direction cosines, the origins of both the structure-axes system and the local axes system are taken at joint 1. The direction cosines are indicated as the λ's as presented by the author in Chapter IV. The calculation of the λ's is as follows:

$$\theta = 0^\circ,\ \varphi = 0^\circ;\ \cos\theta = 1,\ \cos\varphi = 1$$

$$\lambda_{11} = \cos\alpha_1 = \cos\theta\cos\varphi = 1$$

$$\lambda_{12} = \cos\beta_1 = \cos(90^\circ - \varphi) = 0$$

$$\lambda_{13} = \cos\gamma_1 = \cos(90^\circ - \theta)\cos\varphi = 0$$

$$\lambda_{21} = \cos\alpha_2 = \cos\theta\cos(90^\circ + \varphi) = 0$$

$$\lambda_{22} = \cos\beta_2 = \cos\varphi = 1$$

$$\lambda_{23} = \cos\gamma_2 = \cos(90^\circ - \theta)\cos(90^\circ + \varphi) = 0$$

$$\lambda_{31} = \cos\alpha_3 = \cos(90^\circ + \theta) = 0$$

$$\lambda_{32} = \cos\beta_3 = \cos 90^\circ = 0$$

$$\lambda_{33} = \cos\gamma_3 = \cos\theta = 1$$

<u>Formulation of Rotation Matrix R</u>

From the above values of the direction cosines, it shows that the rotation matrix R is an identical matrix as follows:

$$R = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

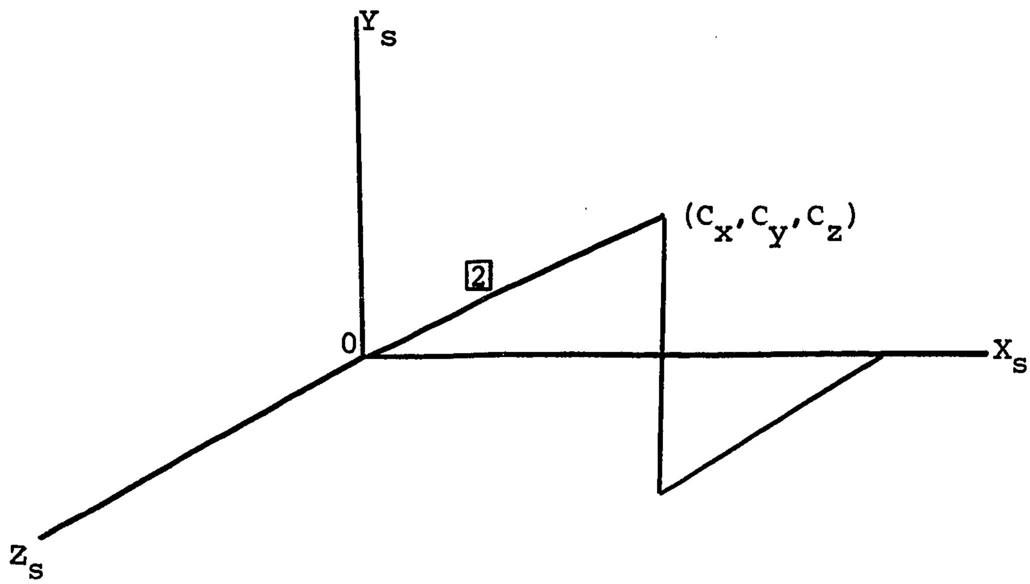
Therefore, the stiffness matrix of member 1 in the member-axes system is identical to that in the structure-axes system, as shown on page 119, Table 12.

TABLE 12

MEMBER STIFFNESS MATRIX OF A HORIZONTAL MEMBER OF A SPACE FRAME

	Δ_x^1	Δ_y^1	Δ_z^1	θ_x^1	θ_y^1	θ_z^1	Δ_x^2	Δ_y^2	Δ_z^2	θ_x^2	θ_y^2	θ_z^2
V_x^1	329730	0	0	0	0	0	-329730	0	0	0	0	0
V_y^1	0	719.9424	0	0	0	17998.56	0	-719.9424	0	0	0	17998.56
V_z^1	0	0	719.9424	0	-17998.56	0	0	0	-719.9424	0	-17998.56	0
M_x^1	0	0	0	113784	0	0	0	0	0	-113784	0	0
M_y^1	0	0	-17998.56	0	599952	0	0	0	17998.56	0	299976	0
M_z^1	0	17998.56	0	0	0	599952	0	-17998.56	0	0	0	299976
V_x^2	-329730	0	0	0	0	0	329730	0	0	0	0	0
V_y^2	0	-719.9424	0	0	0	-17998.56	0	719.9424	0	0	0	-17998.56
V_z^2	0	0	-719.9424	0	17998.56	0	0	0	719.9424	0	17998.56	0
M_x^2	0	0	0	-113784	0	0	0	0	0	113784	0	0
M_y^2	0	0	-17998.56	0	299976	0	0	0	17998.56	0	599952	0
M_z^2	0	17998.56	0	0	0	299976	0	-17998.56	0	0	0	599952

Analysis of Member 2



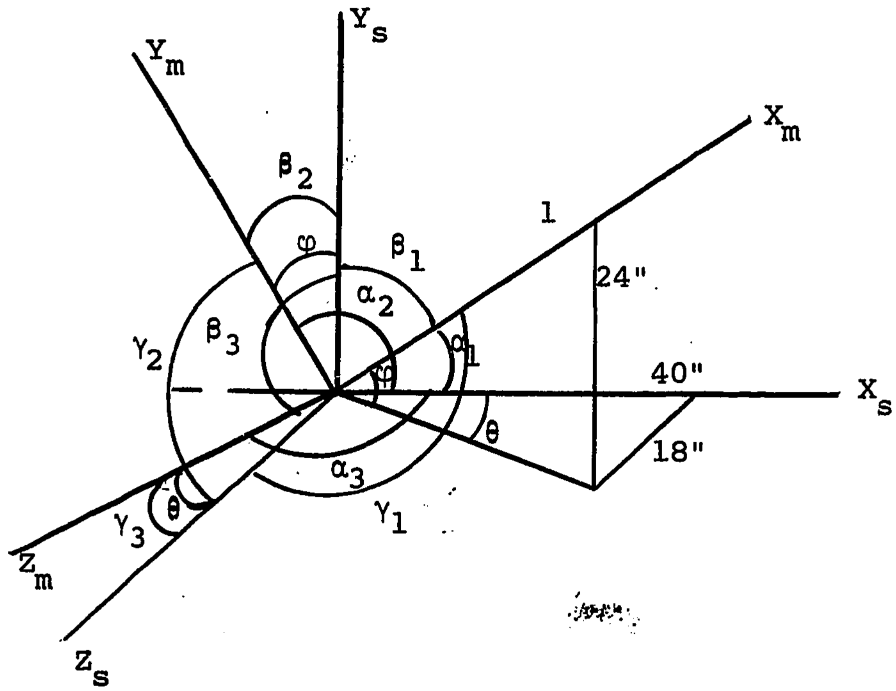
Calculations

Section Properties

Since the member size and the material of member 2 are the same as that of member 1, the section properties of these two members are the same.

Direction Cosines

To calculate the direction cosines the author chose the initial end which is the left end of the member as the origin for both the member-axes system and the structure-axes system. The direction cosines are calculated on the basis of the orientations of the axes of the member-axes system w.r.t. the structure-axes system. The orientation of the member 2 w.r.t. the structure-axes system is shown in the following sketch. (The coordinates, C_x, C_y, and C_z, of the member can be taken as the ratio numbers, 0.8, 0.48, and 0.36 respectively.)



$$\sin\theta = \frac{C_z}{\sqrt{C_x^2+C_z^2}} \qquad \cos\theta = \frac{C_x}{\sqrt{C_x^2+C_z^2}} \qquad C_xC_y = (0.8)(0.48) = 0.384$$

$$\sin\varphi = C_y \qquad \cos\varphi = \sqrt{C_x^2+C_z^2} \qquad C_yC_z=(0.48)(0.36) = 0.1728$$

$$\lambda_{11} = \cos\alpha_1 = \cos\theta\cos\varphi = C_x = 0.8$$

$$\lambda_{12} = \cos\beta_1 = \cos(90^\circ - \varphi) = C_y = 0.48$$

$$\lambda_{13} = \cos\gamma_1 = (\sin\theta\cos\varphi) = C_z = 0.36$$

$$\lambda_{21} = \cos\alpha_2 = (\cos\theta\cos(90^\circ+\varphi)) = \frac{-C_x\ C_y}{\sqrt{C_x^2+C_z^2}} = -0.4377$$

$$\lambda_{22} = \cos\beta_2 = \cos\varphi = \sqrt{C_x^2+C_z^2} = 0.8773$$

$$\lambda_{23} = \cos\gamma_2 = (\cos(90^\circ-\theta)\cos(90^\circ+\varphi)) = \frac{-C_y\ C_z}{\sqrt{C_x^2+C_z^2}} = -0.1970$$

$$\lambda_{31} = \cos\alpha_3 = \cos(90^\circ+\theta) = \frac{-C_z}{\sqrt{C_x^2+C_z^2}} = -0.4103$$

$$\lambda_{32} = \cos\beta_3 = \cos\ 90^\circ = 0$$

$$\lambda_{33} = \cos\gamma_3 = \cos\theta = \frac{C_x}{\sqrt{C_x^2+C_z^2}} = 0.9119$$

<u>Calculation of the Combinations of the λ's</u>

$$(\lambda_{11})^2 = (0.8)^2 = 0.64$$

$$(\lambda_{11})(\lambda_{12}) = (0.8)(0.48) = 0.384$$

$$(\lambda_{11})(\lambda_{13}) = (0.8)(0.36) = 0.288$$

$$(\lambda_{12})^2 = (0.48)^2 = 0.2304$$

$$(\lambda_{12})(\lambda_{13}) = (0.48)(0.36) = 0.1728$$

$$(\lambda_{13})^2 = (0.36)^2 = 0.1296$$

$$(\lambda_{21})^2 = (-0.4377)^2 = 0.1916$$

$(\lambda_{21})(\lambda_{22}) = (-0.4377)(0.8773) = -0.3840$

$(\lambda_{21})(\lambda_{23}) = (-0.4377)(-0.1970) = 0.0862$

$(\lambda_{21})(\lambda_{31}) = (-0.4377)(-0.4103) = 0.1796$

$(\lambda_{21})(\lambda_{33}) = (-0.4377)(0.9119) = -0.3991$

$(\lambda_{22})^2 = (0.8773)^2 = 0.7697$

$(\lambda_{22})(\lambda_{23}) = (0.8773)(-0.1970) = -0.1728$

$(\lambda_{22})(\lambda_{31}) = (0.8773)(-0.4103) = -0.360$

$(\lambda_{22})(\lambda_{33}) = (0.8773)(0.9119) = 0.8$

$(\lambda_{23})^2 = (-0.1970)^2 = 0.0388$

$(\lambda_{23})(\lambda_{31}) = (-0.1970)(-0.4103) = 0.0808$

$(\lambda_{23})(\lambda_{33}) = (-0.1970)(0.9119) = -0.1796$

$(\lambda_{31})^2 = (-0.4103)^2 = 0.1683$

$(\lambda_{31})(\lambda_{32}) = (-0.4103)(0) =0$

$(\lambda_{31})(\lambda_{33}) = (-0.4103)(0.9119) = -0.3742$

$(\lambda_{32})^2 = 0$

$(\lambda_{32})(\lambda_{33}) = 0$

$(\lambda_{33})^2 = (0.9119)^2 = 0.8316$

Calculation of the Stiffness Coefficients in the Stiffness Matrix, $[S_s^2]$, in the Stucture-Axes System

With the advantage of the abbreviation of the stiffness matrix for a member in space proposed by Professor L. A. Comp, only twenty-seven elements need to be calculated instead of 144. These twenty-seven elements are as follows:

$$k_{11} = \frac{EA}{L}(\lambda_{11})^2 + \frac{12EI_z}{L^3}(\lambda_{21})^2 + \frac{12EI_y}{L^3}(\lambda_{31})^2$$

$$= 329730(0.64)+(719.9424)(0.1916)+(719.9424)(0.1683)$$

$$= 211286.3073$$

$$k_{12} = \frac{EA}{L}(\lambda_{11})(\lambda_{12}) + \frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{22})$$

$$= 329730(0.384)-(719.9424)(-0.3840) = 126339.8621$$

$$k_{13} = \frac{EA}{L}(\lambda_{11})(\lambda_{13}) + \frac{12EI_z}{L^3}(\lambda_{21})(\lambda_{23}) + \frac{12EI_y}{L^3}(\lambda_{31})(\lambda_{33})$$

$$= 329730(0.288)+719.9424(0.0862)+719.9424(-0.3742)$$

$$= 94854.8975$$

$$k_{14} = \frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{31}) + \frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{31})$$

$$= -17998.56(0.1796)+17998.56(0.1796) = 0$$

$$k_{15} = \frac{-6EI_y}{L^2}(\lambda_{22})(\lambda_{31}) = -17998.56(-0.36) = 6479.4816$$

$$k_{16} = \frac{6EI_z}{L^2}(\lambda_{21})(\lambda_{33}) + \frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{31})$$

$$= 17998.56(-0.3991)-17998.56(0.0808) = -8637.5089$$

$$k_{22} = \frac{EA_x}{L}(\lambda_{12})^2 + \frac{12EI_z}{L^3}(\lambda_{22})^2 = 329730(0.2304)+719.9424(0.7697)$$

$$= 76523.9317$$

$$k_{23} = \frac{EA_x}{L}(\lambda_{12})(\lambda_{13}) + \frac{12EI_z}{L^3}(\lambda_{22})(\lambda_{23})$$

$$= 329730(0.1728)+719.9424(-0.1728) = 56852.937$$

$$k_{24} = \frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{31}) = 17998.56(-0.356) = -6479.4816$$

$$k^{25} = 0$$

$$k_{26} = \frac{6EI_z}{L^2}(\lambda_{22})(\lambda_{33}) = 17998.56(0.8) = 14398.848$$

$$k_{33} = \frac{EA_x}{L}(\lambda_{13})^2 + \frac{12EI_z}{L^3}(\lambda_{23})^2 + \frac{12EI_y}{L^3}(\lambda_{33})^2$$

$$= 329730(0.1296)+719.9424(0.0388)+719.9424(0.8316)$$

$$= 43359.6459$$

$$k_{34} = \frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{31}) + \frac{-6EI_y}{L^2}(\lambda_{21})(\lambda_{33})$$

$$= 17998.56(0.0808) -1798.56(-0.3991) = 8637.5089$$

$$k_{35} = \frac{-EI_y}{L^2}(\lambda_{22})(\lambda_{33}) = -17998.56(0.8) = -14398.848$$

$$k_{36} = \frac{6EI_z}{L^2}(\lambda_{23})(\lambda_{33}) + \frac{-6EI_y}{L^2}(\lambda_{23})(\lambda_{33})$$

$$= 17998.56(-0.1796)-17998.56(-0.1796) = 0$$

$$k_{44} = \frac{GI_x}{L}(\lambda_{11})^2 + \frac{4EI_z}{L}(\lambda_{31})^2 + \frac{4EI_y}{L}(\lambda_{21})^2$$

$$= 113784(0.64)+599952(0.1683)+599952(0.1916) = 288744.4848$$

$$k_{45} = \frac{GI_x}{L}(\lambda_{11})(\lambda_{12}) + \frac{4EI_y}{L}(\lambda_{21})(\lambda_{22})$$

$$= 113784(0.384)+599952(-0.384) = -186688.512$$

$$k_{46} = \frac{GI_x}{L}(\lambda_{11})(\lambda_{13}) + \frac{4EI_y}{L}(\lambda_{21})(\lambda_{23}) + \frac{4EI_z}{L}(\lambda_{31})(\lambda_{33})$$

$$= 113784(0.288)+599952(0.0862)+599952(-0.3742)$$

$$= -140016.384$$

$$k_{55} = \frac{GI_x}{L}(\lambda_{12})^2 + \frac{4EI_y}{L}(\lambda_{22})^2$$

$$= 113784(0.2304)+599952(0.7697) = 487998.888$$

$$k_{56} = \frac{GI_x}{L}(\lambda_{12})(\lambda_{13}) + \frac{4EI_y}{L}(\lambda_{22})(\lambda_{23})$$

$$= 113784(0.1728)+599952(-0.1728) = -84009.8304$$

$$k_{66} = \frac{GI_x}{L}(\lambda_{13})^2 + \frac{4EI_y}{L}(\lambda_{23})^2 + \frac{4EI_z}{L}(\lambda_{33})^2$$

$$= 113784(0.1296)+599952(0.0391)+599952(0.8316)$$

$$= 537124.6128$$

$$k_{4,10} = \frac{-GI_x}{L}(\lambda_{11})^2 + \frac{2EI_y}{L}(\lambda_{21})^2 + \frac{2EI_z}{L}(\lambda_{31})^2$$

$$= -113784(0.64)+299976(0.1916)+299976(0.1683)$$

$$= 35139.6024$$

$$k_{4,11} = \frac{-GI_x}{L}(\lambda_{11})(\lambda_{12}) + \frac{2EI_y}{L}(\lambda_{21})(\lambda_{22})$$

$$= -113784(0.384)+299976(-0.384) = -158883.84$$

$$k_{4,12} = \frac{-GI_x}{L}(\lambda_{11})(\lambda_{13}) + \frac{2EI_y}{L}(\lambda_{21})(\lambda_{23}) + \frac{2EI_z}{L}(\lambda_{31})(\lambda_{33})$$

$$= -113784(0.288)+299976(0.0862)+299976(-0.3742)$$

$$= -119162.88$$

$$k_{5,11} = \frac{-GI_x}{L}(\lambda_{12})^2 + \frac{2EI_y}{L}(\lambda_{22})^2$$

$$= -113784(0.2304)+299976(0.7697) = 204675.6936$$

$$k_{5,12} = \frac{-GI_x}{L}(\lambda_{12})(\lambda_{13}) + \frac{2EI_y}{L}(\lambda_{22})(\lambda_{23})$$

$$= -113784(0.1728)-299976(-0.1728) = -71497.728$$

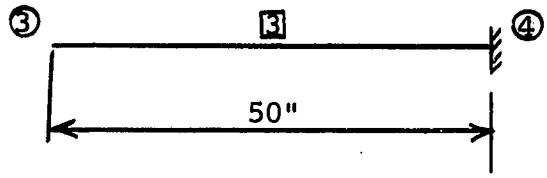
$$k_{6,12} = \frac{-GI_x}{L}(\lambda_{13})^2 + \frac{2EI_y}{L}(\lambda_{23})^2 + \frac{2EI_z}{L}(\lambda_{33})^2$$

$$= -113784(0.1296)+299976(0.0388)+299976(0.8316)$$

$$= 246352.704$$

With these elements the stiffness matrix for member 2 in the structure-axes system can be established and is shown on page 142, Table 13.

Analysis of Member 3



Since member 3 is of the same size and material as that of member 1, the stiffness coefficients in the member stiffness matrix for member 3 are the same as that for member 1. In addition, because the direction of the member axis of this member is coincident with the X-axis of structure-axes system, no rotation transformation needs be performed, i.e., $[S_s^3] = [S_m^3]$. By the inversion of matrix, the inverse, $[S_j^{2-3}]^{-1}$, of the submatrix, $[S_j^{2-3}]$, of the joint stiffness matrix, $[S_j]$, is obtained as shown on page 144, Table 15.

Loading Analysis

Since there is no distributed load or intermediate concentrated loads on any of the three members of the structure, the equivalent load vector is a 0 vector, i.e., $A_E = [a_{i,1}]$ and $a_{i,1} = 0$, $i = 1....24$

$A_c = A = [0,0,0,0,0,0,$ $6000,$ $0,0,0,$ $4800,$ $3600,$ $0,$ $-1200,$
$-900,$ $10000,$ $0,0,0,0,0,0,0,0]$

The nodal actions at the joints 2 and 3 are contained in A_D^{2-3}:

A_D^{2-3} = [6000, 0,0,0, 4800, 3600, 0, -1200, -900, 10000, 0,0,]

A_{RL} = [0], i.e., the column matrix which contains the negative values of the elements contained in A_E is a null matrix.

The joint displacements at joints 2 and 3 are given by the matrix equation $[D_J^{2-3}] = [S_J^{2-3}]^{-1}[A_D^{2-3}]$ in which $[D_J^{2-3}]$ is the displacement vector containing the displacement components at the coordinates at the joints 2 and 3, $[S_J^{2-3}]^{-1}$ is the inverse of the submatrix of the joint stiffness matrix of the structure corresponding to the joints 2 and 3, $[A_D^{2-3}]$ is the action vector containing the nodal actions at the coordinates at the joints 2 and 3.

$$[D_J^{2-3}] = \begin{bmatrix} 0.1529456E-01 \\ -0.1353106E\ 01 \\ -0.1676577E\ 01 \\ 0.3278675E-01 \\ 0.4245323E-01 \\ -0.2008140E-01 \\ 0.2902009E-02 \\ -0.1800001E\ 01 \\ -0.1063856E\ 01 \\ 0.5886848E-01 \\ -0.1715996E-01 \\ 0.3936479E-01 \end{bmatrix}$$

Note:

The numbers in the above displacement vector are the components of the displacements in the directions of the coordinates at the joints 2 and 3 of the framed structure in the order from 7 to 18 numbered from the top to the bottom.

Reactions at the Member Ends of Members 1 and 3

On member 1, since the joint 2 is a rigid connection, there is no reaction that can be developed. The only reactions which can be developed are therefore at the end 1, and are given as the following matrix equation:

$$A_R^1 = [S_m^{1-2}][D_j^2]$$

in which A_R^1 is an action vector containing the reaction components in the directions of the coordinates at joint 1, $[S_m^{1-2}]$ is the sub-matrix of the member stiffness matrix of member 1 containing the stiffnesses relating the actions in the directions of the coordinates at joint 1 to the displacements at joint 2, $[D_j^2]$ is the displacement vector containing the displacement components at the joint 2.

$$A_R^1 = \begin{bmatrix} -0.5043075\text{E } 04 \\ 0.6127219\text{E } 03 \\ 0.4429416\text{E } 03 \\ -0.3730607\text{E } 04 \\ -0.1744101\text{E } 05 \\ 0.1833001\text{E } 05 \end{bmatrix} \begin{matrix} \text{... reaction at coordinate 1} \\ \text{... reaction at coordinate 2} \\ \text{... reaction at coordinate 3} \\ \text{... reaction at coordinate 4} \\ \text{... reaction at coordinate 5} \\ \text{... reaction at coordinate 6} \end{matrix}$$

Similarly, the reactions acting at the end 4 of member 3 are given in the following matrix equation:

$$A_R^4 = [S_m^{4-3}][D_j^3]$$

in which A_R^4 is an action vector containing the reaction components at joint 4, $[S_m^{4-3}]$ is the submatrix of the member stiffness matrix of member 3 containing the stiffnesses relating the actions at joint 4 to the displacements at joint 3, $[D_j^3]$ is the displacement vector containing the displacement components at joint 3.

$$A_R^4 = \begin{bmatrix} -0.9568793E\ 03 \\ 0.5873874E\ 03 \\ 0.4570604E\ 03 \\ -0.6698291E\ 04 \\ 0.1400029E\ 05 \\ -0.2058893E\ 05 \end{bmatrix} \begin{array}{l} \text{... reaction at coordinate 19} \\ \text{... reaction at coordinate 20} \\ \text{... reaction at coordinate 21} \\ \text{... reaction at coordinate 22} \\ \text{... reaction at coordinate 23} \\ \text{... reaction at coordinate 24} \end{array}$$

<u>Member End Actions</u>

Since there is no distributed load or intermediate concentrated load on any of the three members of the framed structure, the member end-actions on each member are just the reaction forces due to the joint displacements of the ends of each member.

Therefore, the member end-actions on the member 1 are obtained by the matrix equation:

$$A_{END}^1 = \begin{bmatrix} S_m^{1-2} \\ \hline S_m^{2-2} \end{bmatrix} [D_j^2]$$

in which A_{END}^{1} is an action vector containing the actions at both ends of member 1,

$\begin{bmatrix} S_m^{1-2} \\ \hline S_m^{2-2} \end{bmatrix}$ is the submatrix of the member stiffness matrix of member 1 which relates the actions at both ends of member 1 to the displacement components at joint 2,

$[D_j^2]$ is the displacement vector containing the displacement components at joint 2.

$$A_{END}^{1} = \begin{bmatrix} -0.5043075E\ 04 \\ 0.6127219E\ 03 \\ 0.4429416E\ 03 \\ -0.3730607E\ 04 \\ -0.1744101E\ 05 \\ 0.1833001E\ 05 \\ 0.5043075E\ 04 \\ -0.6127220E\ 03 \\ -0.4429416E\ 03 \\ 0.3730607E\ 04 \\ -0.4706065E\ 04 \\ 0.1230608E\ 05 \end{bmatrix}$$

Note:

The member end-actions in the above member end-action vector are in the directions of the coordinates ordered from 1 to 12 from the top to the bottom.

The member end-actions on member 2 are obtained from the matrix equation

$$A^2_{END} = [S^2_m][R_T{}^2][D^{2-3}_j]$$

$$[R^2_T][D^{2-3}_j] = \begin{bmatrix} -0.1240822E\ 01 \\ -0.8634884E\ 00 \\ -0.1535145E\ 01 \\ 0.3937764E{-01} \\ 0.2684949E{-01} \\ -0.3176462E{-01} \\ -0.1244666E\ 01 \\ -0.1370831E\ 01 \\ -0.9713208E\ 00 \\ 0.5302932E{-01} \\ -0.4857601E{-01} \\ 0.1174301E{-01} \end{bmatrix}, \quad A^2_{END} = \begin{bmatrix} 0.1267549E\ 04 \\ 0.4897520E\ 01 \\ -0.1487442E\ 02 \\ -0.1553342E\ 04 \\ 0.1168477E\ 05 \\ -0.6403190E\ 04 \\ -0.1267551E\ 04 \\ -0.4897521E\ 01 \\ 0.1487441E\ 02 \\ 0.1553342E\ 04 \\ -0.1094106E\ 05 \\ 0.6648052E\ 04 \end{bmatrix}$$

Notes:

(1) The member end-actions, in the above member end-action vector, are in the directions of the coordinates in the order from 7 to 18 numbered from the top of the column matrix to the bottom.

(2) $[S^2_m]$ is the member stiffness matrix of member 2.

(3) $[R^2_T]$ is the rotation matrix for member 2.

(4) $[D^{2-3}_j]$ is the displacement vector continuing the displacements at joint 2 and 3 which are both ends of this member.

The member end-actions on member 3 are obtained from the matrix equation:

$$A^{3}_{End} = \begin{bmatrix} S_m^{3-3} \\ \hline S_m^{4-3} \end{bmatrix} [D_j^3]$$

in which A^{3}_{END} is an action vector containing the member end-actions at both ends of member 3,

$\begin{bmatrix} S_m^{3-3} \\ \hline S_m^{4-3} \end{bmatrix}$ is the submatrix of the member stiffness matrix of member 3 which relates the actions at both ends of member 3 to the displacement components at joint 3,

$[D_j^3]$ is the displacement vector containing the displacement components at joint 3.

$$A^{3}_{END} = \begin{bmatrix} 0.9568793E\ 03 \\ -0.5873874E\ 03 \\ -0.4570604E\ 03 \\ 0.6698291E\ 04 \\ 0.8852720E\ 04 \\ -0.8780441E\ 04 \\ -0.9568793E\ 03 \\ 0.5873874E\ 03 \\ 0.4570604E\ 03 \\ -0.6698291E\ 04 \\ 0.1400029E\ 05 \\ -0.2058893E\ 05 \end{bmatrix}$$

Note:

The member-end actions in the above member end-action vector are in the directions of the coordinates in the order from 13 to 24 numbered from the top of the matrix to the bottom.

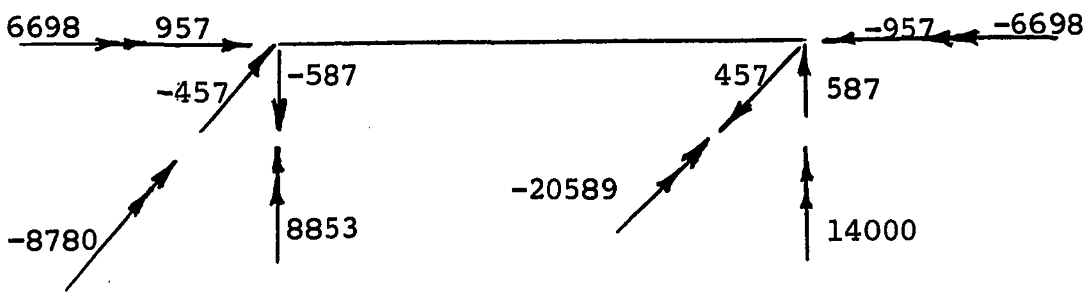
Check the Margin of Safety

To check the margin of safety the comparison of the member end-actions of all the members that compose this frame is necessary. It is necessary because through the comparison an investigation will be made of the maximum combination of the member end-actions that occur at a member of the structure.

Member		1	2	3
Initial End	V_x^i	-5043	1268	957
	V_y^i	613	5	-587
	V_z^i	443	-15	-457
	M_x^i	-3731	-1553	6698
	M_y^i	-17441	11685	8853
	M_z^i	18330	-6403	-8780
Terminal End	V_x^j	5043	-1268	-957
	V_y^j	-613	-5	587
	V_z^j	-443	15	457
	M_x^j	3731	1553	-6698
	M_y^j	-4706	-10941	14000
	M_z^j	12306	6648	-20589

The numbers in the above table are rounded off to the nearest whole number from the computed numbers given by the digital computer.

From the above table, it is seen that the maximum combination of the member end-actions is occurring at the joint 4 of the member 3. Therefore, the check of the margin of safety will be undergone by the member 3 as follows.



The resultant bending moment of M_y^4 and M_z^4 is expressed as M_r^4,

$$M_r^4 = \sqrt{(M_y^4)^2 + (M_z^4)^2}$$

$$M_r^4 = \sqrt{(14000)^2 + (20589)^2} = \sqrt{196,000,000+423,906,921}$$

$$= 24898$$

$$V_x^4 = -957 \qquad M_x^4 = -6698$$

From Mill Handbook 5A, Heat Treated AISI 4130

$$F_{tu} = 125{,}000^{psi}$$

$$F_b = 149{,}000^{psi} \quad \text{(pg. 211)} \quad (D/t = 21.05)$$

$$F_{st} = 63{,}000^{psi} \quad \text{(pg. 218)} \quad (L/D = 50/2 = 25)$$

"By Shanley Stress Ratio Method" (16)

$$f_b = \frac{M_r^4}{\frac{I}{c}} = \frac{24898}{0.2586} = 96280^{psi} \quad (96279.969)$$

$$f_{st} = \frac{T}{\frac{J}{c}} = \frac{-6698}{0.5172} = -12951^{psi} \quad (12950.5027)$$

$$f_c = \frac{P}{A} = \frac{-957}{0.5685} = -1683^{psi} \quad (1683.3773)$$

$$R_c = \frac{f_c}{F_{tu}} = \frac{-1683}{125{,}000} = -0.0135$$

$$R_b = \frac{f_b}{F_b} = \frac{96280}{149{,}000} = 0.6462$$

$$R_{st} = \frac{f_{st}}{F_{st}} = \frac{-12951}{63{,}000} = -0.2506$$

$$R_b^2 = (0.6461)^2 = 0.4176$$

$$R_{st}^2 = (-0.2056)^2 = 0.0423$$

$$R_b^2 + R_{st}^2 = 0.4174 + 0.0422 = 0.4599$$

$$\sqrt{R_b^2 + R_{st}^2} = \sqrt{0.4599} = 0.6782$$

$$\text{"M.S.} = \frac{1}{R_c + \sqrt{R_b^2 + R_{st}^2}} - 1\text{"} \qquad (16)$$

$$\text{M.S.} = \frac{1}{-0.0135 + 0.6782} - 1 = 0.445$$

Check the Axial-Flexural Interaction

From the member end-actions of each member of the frame, it is seen that all of them have the axial end-actions at both ends which will change both the bending moments and bending deflections at the cross sections of each member. Since the axial end-actions on member 1 are tensile forces which will decrease the bending deflection at each cross-section, no attention should be paid to the axial-flexural interaction of the tensile axial forces. In contrast, the axial end-actions of the members 2 and 3 are compressive which will increase the bending deflections of them. The investigation of the change of the bending stiffness factor of each of the two members due to the effect of the axial-flexural interaction of the compressive axial end-actions should be made as follows:

Check Member 2

$$P = 1268$$

$$k = \sqrt{\frac{P}{EI}} = \frac{\sqrt{1268}}{\sqrt{(29x10^6)(0.2586)}} = \sqrt{\frac{1268}{7499400}} = 0.013$$

$$kL = (0.013)(50) = 0.65$$

$$(kL)^2 = (0.65)^2 = 0.4225$$

$$\sin kL = \sin(0.65) = 0.60518641$$

$$\cos kL = \cos(0.65) = 0.79608380$$

$$\csc kL = \frac{1}{\sin kL} = \frac{1}{0.60518641} = 1.65238343$$

$$\cot kL = \frac{\cos kL}{\sin kL} = \frac{0.7960838}{0.60518641} = 1.31543568$$

$$"\alpha = \frac{6(kL \csc kL - 1)}{(kL)^2}" \qquad (16)$$

$$\alpha = \frac{6(0.65x1.65238343-1)}{0.4225} = 1.0516$$

$$"\beta = \frac{3(1-kL \cot kL)}{(kL)^2}" \qquad (16)$$

$$\beta = \frac{3(1-0.65x1.31543568)}{0.4225} = 1.0294$$

$$"C = \frac{3\beta}{(4\beta^2 - \alpha^2)}" \qquad (16)$$

$$C = \frac{3x1.0294}{4(1.0294)^2-(1.0516)^2} = 0.9857$$

$1-c = 0.0143$ . . . the amount of the percentage by which the stiffness factor is decreased.

Modified $k_{12,12} = (599952)(0.9857) = 591372.6864$

Check Member 3

$$P = 957$$

$$k = \sqrt{\frac{P}{EI_z}}$$

$$k = \sqrt{\frac{957}{(29x10^6)(0.2586)}} = 0.0113$$

$$kL = (0.0113)(50) = 0.565$$

$$\alpha = \frac{6(0.565x1.8677073-1)}{0.319225} = 1.0385389$$

$$\beta = \frac{3(1-0.565x1.5774442)}{0.319225} = 1.0219509$$

$$C = \frac{3(1.0219509)}{4(1.0219509)^2-(1.0385389)^2} = 0.989313$$

$1-c = 0.010687$ . . . the amount of the percentage by which the stiffness factor is decreased.

Modified $k_{12,12} = \frac{4EI_z}{L}(c) = (599952)(0.989313) = 593540.313$

It is shown in the above calculations that none of the axial-flexural interactions of the compressive axial end-actions on the two members has resulted in a change of the bending stiffness factor which needs to be considered. The change of the bending stiffness factors of the two members due to the axial-flexural interactions on them are small enough to be neglected. In general, when the value of kL is less than or equal to 1, the effect of the axial-flexural interaction of the compressive axial end-action upon the member can be neglected.

If the value of kL is bigger than 1, then the effect of the axial flexural interaction of the compressive axial end-actions should be considered. "If axial-flexural interactions are to be taken into account in the analysis of plane or space frames, it is necessary to make other modifications of the stiffness method in addition to those already described. The analysis is complicated by the fact that the axial forces in the members are related to the joint displacements. Therefore, the analysis must be conducted in a cyclic fashion. In the first cycle of analysis the stiffness method is applied as explained in Chapter 4. In the second cycle, the axial forces in the members, as obtained from the first cycle, are used in determining the modified member stiffnesses given in Table 6-7, and also in determining the modified fixed-end actions. The second cycle is then completed, using the modified stiffnesses and fixed-end actions, and new values for the axial forces are obtained. This process is repeated until two successive analyses yield approximately the same results" (5).

TABLE 13

MEMBER STIFFNESS MATRIX OF AN INCLINED MEMBER OF A SPACE FRAME

	Δ_x^2	Δ_y^2	Δ_z^2	θ_x^2	θ_y^2	θ_z^2	Δ_x^3	Δ_y^3	Δ_z^3	θ_x^3	θ_y^3	θ_z^3
V_x^2	211286.3073	126339.8621	94854.8975	0	6479.4816	-8637.5089	-211286.3073	-126339.8621	-94854.8975	0	6479.4816	-8637.5089
V_y^2	126339.8621	76523.9317	56852.937	-6479.4816	0	14398.848	-126339.8621	-76523.9317	-56852.937	-6479.4816	0	14398.848
V_z^2	94854.8975	56852.937	43359.6459	8637.5089	-14398.848	0	-94854.8975	-56852.937	-43359.6459	8637.5089	-14398.848	0
M_x^2	0	-6479.4816	8637.5089	288744.4848	-186688.512	-140016.384	0	6479.4816	-8637.5089	35139.6024	-158883.84	-119162.88
M_y^2	6479.4816	0	-14398.848	-186688.512	487998.888	-84009.8304	-6479.4816	0	14398.848	-158883.84	204675.6936	-71497.728
M_z^2	-8637.5089	14398.848	0	-140016.384	-84009.8304	537124.6128	8637.5089	-14398.848	0	-119162.88	-71497.728	246352.704
V_x^3	-211286.3073	-126339.8621	-94854.8975	0	-6479.4816	8637.5089	211286.3073	126339.8621	94854.8975	0	-6479.4816	8637.5089
V_y^3	-126339.8621	-76523.9317	-56852.937	6479.4816	0	-14398.848	126339.8621	76523.9317	56852.937	6479.4816	0	-14398.848
V_z^3	-94854.8975	-56852.937	-43359.6459	-8637.5089	14398.848	0	94854.8975	56852.937	43359.6459	-8637.5089	14398.848	0
M_x^3	0	-6479.4816	8637.5089	35139.6024	-158883.84	-119162.88	0	6479.4816	-8637.5089	288744.4848	-186688.512	-140016.384
M_y^3	6479.4816	0	-14398.848	-158883.84	204675.6936	-71497.728	-6479.4816	0	14398.848	-186688.512	487998.888	-84009.8304
M_z^3	-8637.5089	14398.848	0	-119162.88	-71497.728	246352.704	8637.5089	-14398.848	0	-140016.384	-84009.8304	537124.6128

TABLE 14

JOINT STIFFNESS MATRIX OF A SPACE FRAME

541016.3037	126339.8621	94854.8975	0.0
6479.4816	-8637.5089	-211268.3073	-126339.8621
94854.8975	0.0	6479.4816	-8637.5089
126339.8621	77243.8741	56852.937	-6479.4816
0.0	-3599.712	-126339.8621	-76523.9317
56852.937	-6479.4816	0.0	14398.848
94854.8975	56852.937	44079.5883	8637.5089
3599.712	0.0	-94854.8975	-56852.937
43359.6459	8637.5089	-14398.848	0.0
0.0	-6479.4816	8637.5089	402528.4848
186688.512	-140016.384	0.0	6479.4816
8637.5089	35139.6024	-158883.84	-119162.88
6479.4816	0.0	3599.712	-186688.512
1087950.888	-84009.8304	-6479.4816	0.0
14398.848	-158883.84	204675.6936	-71497.728
8637.5089	-3599.712	0.0	-140016.384
84009.8304	1137076.6128	8637.5089	-14398.848
0.0	-119162.8856	-71497.728	246352.704
211286.3073	-126339.8621	-94854.8975	0.0
6479.4816	8637.5089	541016.3073	126339C8621
94854.8975	0.0	-6479.4816	8637.5089
126339.8621	-76523.9317	-56852.937	6479.4816
0.0	-14398.848	126339.8621	77243.8741
56852.937	6479.4816	0.0	3599.712
94854.8975	-56852.937	-43359.6459	-8637.5089
14398.848	0.0	94854.8975	56852.937
44079.5883	-8637.5089	-3599.712	0.0
0.0	-6479.4816	8637.5089	35139.6024
158883.84	-119162.88	0.0	6479.4816
8637.5089	402528.4848	-186688.512	-140016.384
6479.4816	0.0	-14398.848	-158883.84
204675.6936	-71497.728	-6479C4816	0.0
3599.712	-186688.512	1087950.888	-84009.8304
8637.5089	14398.848	0.0	-119162.88
71497.728	246352.704	8367.5089	3599.712
0.0	-140016.384	-84009.8304	1137076.6128

THE JOINT STIFFNESS MATRIX IS READ IN ROW-WISE, AND IT STARTS WITH THE SEVENTH COORDINATE AND ENDS WITH THE EIGHTEENTH COORDINATE, ACCORDING TO THE NUMBERING SYSTEM DEFINED FOR THIS PROBLEM, EVERY THREE ROWS REPRESENT THE STIFFNESS OF ONE COORDINATE. THE STIFFNESS OF EACH COORDINATE IS STARTED WITH THE DIRECTION OF THE X-AXIS.

TABLE 15

INVERSE OF THE JOINT STIFFNESS MATRIX OF A SPACE FRAME

0.30144169E-05	-0.14337172E-05	-0.13913111E-05
0.15957926E-07	-0.18674526E-07	0.30788250E-07
0.18228195E-07	0.14336492E-05	0.13914761E-05
0.15958278E-07	-0.18671137E-07	0.30789699E-07
-0.14337777E-05	0.15990519E-02	-0.27435942E-03
0.21690040E-04	0.45095585E-05	0.31095434E-04
0.14125957E-05	0.12134518E-02	0.22791905E-03
0.18317547E-04	0.26505445E-05	-0.25840909E-04
-0.13912974E-05	-0.27436693E-03	0.17599633E-02
-0.28955950E-04	-0.33736010E-04	-0.45204560E-05
0.13892267E-05	0.22789972E-03	0.10799595E-02
-0.24458036E-04	0.24301214E-04	-0.26609477E-05
0.15961187E-07	0.21690360E-04	-0.28956179E-04
0.52720815E-05	0.99076192E-06	0.74347281E-06
-0.15465875E-07	-0.18318929E-04	0.24458735E-04
0.25288982E-05	0.81085386E-06	0.60859736E-06
-0.18672114E-07	0.45100787E-05	-0.33735887E-04
0.99074918E-06	0.17743627E-05	0.24675216E-06
0.18813768E-07	-0.26504112E-05	-0.24301360E-04
0.81083067E-06	-0.54713280E-06	0.17235191E-06
0.30783610E-07	0.31094961E-04	-0.45203642E-05
0.74347656E-06	0.24674318E-06	0.16301985E-05
-0.31317164E-07	0.25841458E-04	0.26613752E-05
0.60857905E-06	0.17235561E-06	-0.64718028E-06
0.18366506E-07	0.14335978E-05	0.13912422E-05
-0.15957599E-07	0.18673418E-07	-0.30787404E-07
0.30143924E-05	-0.14336046E-05	-0.13914066E-05
-0.15957848E-07	0.18670053E-07	-0.30787333E-07
0.14334125E-05	0.12134262E-02	0.22791215E-03
-0.18318769E-04	-0.26510038E-05	0.25841403E-04
-0.14590152E-05	0.15990990E-02	-0.27436111E-03
-0.21691306E-04	-0.45102033E-05	-0.31095376E-04
0.13914200E-05	0.22789894E-03	0.10799486E-02
0.24459401E-04	-0.24301080E-04	0.26608272E-05
-0.13878147E-05	-0.27439527E-03	0.17599856E-02
0.28957372E-04	0.33736389E-04	0.45209253E-05
0.15962132E-07	0.18317867E-04	-0.24458280E-04
0.25288986E-05	0.81084533E-06	0.60857326E-06
-0.15356143E-07	-0.21691590E-04	0.28956619E-04
0.52720852E-05	0.99077306E-06	0.74349566E-06
-0.18669389E-07	0.26505126E-05	0.24301105E-04
0.81085954E-06	-0.54712904E-06	0.17235123E-06
0.18872256E-07	-0.45105835E-05	0.33736309E-04
0.99077852E-06	0.17743707E-05	0.24675966E-06
0.30796613E-07	-0.25840447E-04	-0.26608381E-05
0.60859952E-06	0.17237113E-06	-0.64719836E-06
-0.29462068E-07	-0.31096293E-04	0.45199576E-05
0.74349748E-06	0.24676154E-06	0.16301869E-05

THE INVERSE OF THE JOINT MATRIX IS READ IN ROW WISE, AND IT STARTS WITH THE SEVENTH COORDINATE AND ENDS WITH THE EIGHTEENTH COORDINATE, ACCORDING TO THE NUMBERING SYSTEM DEFINED FOR THIS PROBLEM. EVERY FOUR ROWS REPRESENT THE FLEXIBILITY OF ONE COORDINATE. THE FLEXIBILITY OF EACH COORDINATE IS STARTED WITH THE DIRECTION OF THE X-AXIS.

CHAPTER VII

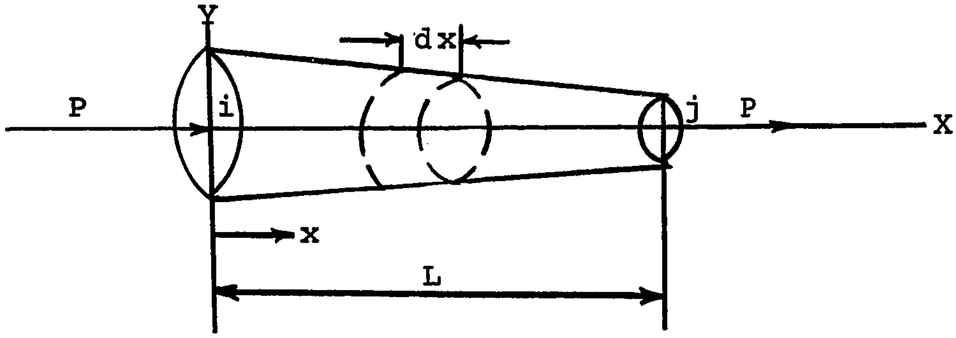
STIFFNESS MATRIX FOR A TAPERED MEMBER OF CIRCULAR CROSS-SECTION IN THREE-DIMENSIONAL SPACE

Development of Stiffness Coefficients

The development of the stiffness coefficients of a tapered member of circular cross-section in a three-dimensional space is based on the member-axes system, and the member axis will be coincident with the X-axis. The displacements at the coordinates are introduced in the positive directions of the coordinates. The framework sign convention is used for the end-actions and end displacements.

As it was in the case for a prismatic member, there are four kinds of stiffness coefficients under consideration. The development of these four stiffness coefficients is given as follows:

Axial Stiffness



From direct experiment, it has been established by Robert Hooke that within the elastic limit of a given material of a structural member there is a linear relationship between the axial force and axial strain. And if structural members of a given material and of different lengths and cross-sectional areas are experimented with, it will be found that the axial strain of the member will be proportional to the axial force and to the length of the member and inversely proportional to the cross-sectional area as well as the elastic property in tension and compression, i.e., $\Delta L = \frac{PL}{EA}$

The above relation can be derived algebraically as shown:

By Hooke's Law, $E = \frac{\sigma}{\epsilon}$ in which

$$\sigma = \frac{P}{A}, \quad \epsilon = \frac{\Delta L}{L}$$

therefore,

$$E = \frac{\frac{P}{A}}{\frac{\Delta L}{L}} = \frac{PL}{A\Delta L}, \text{ and } \Delta L = \frac{PL}{AE}$$

where P = axial force (tension or compression),

σ = axial stress due to axial force,

A = the cross-sectional area of the member,

E = modulus of elasticity.

Since the cross-section of the member under consideration is circular, $A = \pi r^2$, where r is the radius of a cross section, and is a function of the coordinate of the cross-section along the member-axis, i.e., $r_x = r(x)$, as indicated in the following equation:

$$r(x) = (r_j)(1-c)\ \frac{k-x}{cL}$$

where

$$k = \frac{L}{1-c}\ , \quad \text{and } c = \frac{r_j}{r_i}$$

Therefore, since the cross-sectional area is given in term of radius, it also can be expressed as a function of the coordinate of the cross-section along the member axis, i.e., $A_x = A(x)$, and is given by

$$A(x) = \frac{\pi r_j^2}{(cL)^2}\ (1-c)^2(k-x)^2$$

Let the axial strain for a differential length dx of the member x distance away from the joint i be dL. By the stress strain relation as mentioned, $dL = \dfrac{P\ dx}{A(x)E}$,

therefore,

$$dL = \frac{1}{E}\left[\frac{Pdx}{\frac{\pi r_j^2}{(cL)^2}(k-x)^2(1-c)^2}\right] = \left[\frac{P(cL)^2}{E\pi r_j^2(k-x)^2(1-c)^2}\right]dx$$

If let ΔL be the total axial displacement of the member i-j, then $\Delta L = \int dL$

$$\Delta L = \int_0^L \frac{P(cL)^2}{E\pi r_j^2(k-x)^2(1-c)^2}dx = \frac{P(cL)^2}{E\pi r_j^2(1-c)^2}\int_0^L \frac{dx}{(k-x)^2}$$

$$= \frac{P(cL)^2}{E\pi r_j^2(1-c)^2}\left[\frac{1}{(k-x)}\right]_0^L = \frac{P(cL)^2}{E\pi r_j^2(1-c)^2}\left[\frac{L}{k(k-L)}\right]$$

Let $\Delta L = 1$, then

$$P = \frac{E\pi r_j^2\, k(k-L)(1-c)^2}{c^2L^3},$$

where P is the required axial stiffness.

By the idea of superposition of displacements, the member can be released in the following procedure:

<u>Member Released at Initial End-i with P^i Applied at End-i</u>

$$P^i = \frac{E\pi r_j^2\, k(k-L)(1-c)^2}{c^2L^3} = k_{1,1}$$

where $k_{1,1}$ is the required axial stiffness coefficient at the coordinate 1 due to a unit axial displacement at the coordinate 1.

By the equilibrium condition of the member, $P^j = -P^i$, therefore

$$P^j = \frac{-E\pi r_j^2 k(k-L)(1-c)^2}{c^2 L^3} = k_{7,1}$$

where $k_{7,1}$ is the axial stiffness coefficient at the coordinate 7 due to a unit axial displacement at the coordinate 1.

<u>Member Released at Terminal End-j with P^j Applied at End-j</u>

$$P^j = \frac{E\pi r_j^2 k(k-L)(1-c)^2}{c^2 L^3} = k_{7,7} \;,$$

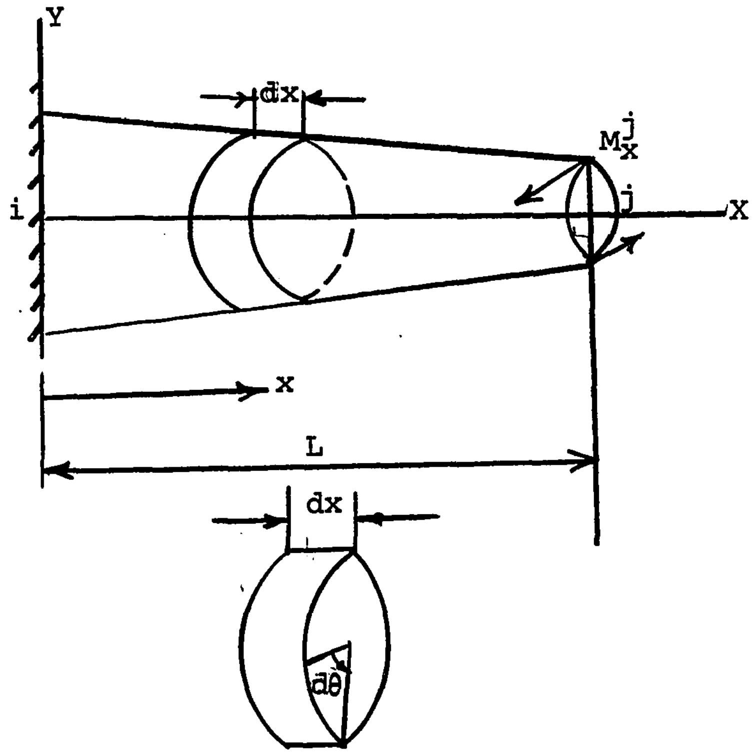
where $k_{7,7}$ is the axial stiffness coefficient at the coordinate 7 due to a unit axial displacement at the coordinate 7.

Since $P^i = -P^j$,

$$P^i = \frac{-E\pi r_j^2 \; k(k-L)(1-c)^2}{c^2 L^3} = k_{1,7}$$

where $k_{1,7}$ is the axial stiffness coefficient at the coordinate 1 due to a unit axial displacement at the coordinate 7.

Torsional Stiffness



Let the end-j be released first, and a couple M_x^j is applied at end-j. From Castigliano's theorem, it is known that the angle of twist at any cross-section of a shaft is given by

$$\theta = \frac{\partial U}{\partial M_x^j} ,$$

where M_x^j is a fictitious torsional couple applied at the section where the angle of twist is desired, U is the strain energy in the shaft as a result of the application of all the external forces, including the fictitious torsional couple M_x^j, to the member. The expression for the strain energy, U, due to torsional load M_x^j can be derived as follows:

Let M_x^j be the torsional load acting at the free end of the shaft. Take an element of the shaft of length dx which is x distance away from the fixed end i. The torsional load M_x in the element causes the right face of the element to rotate through a small angle $d\theta$ relative to the left face. Let the strain energy in the element be represented by dU. Since the material is constant and within the elastic limit of the material the angle of twist is directly proportional to the torsional load M_x, $dU=\frac{1}{2}M_x d\theta$.

If θ be the angle of twist of a cylindrical bar of length L subjected to a constant torsional load M_x, then

$$\theta = \frac{M_x L}{GJ} \quad \text{and} \quad d\theta = \frac{M_x\, dx}{GJ}$$

Therefore,

$$dU = \frac{1}{2}\left(\frac{M_x^2\, dx}{JG}\right) \quad \text{and} \quad U = \int_0^L \frac{M_x^2\, dx}{2GJ}$$

Substitution:

$$\theta = \int_0^L \frac{M_x}{JG}\left(\frac{\partial M_x}{\partial M_x^j}\right) dx$$

Since M_x^j is the torsional load applied at the free end of the shaft, M_x^j is real and $M_x^j = M_x$ = constant.

Therefore,

$$\theta = \int_0^L \frac{M_x}{JG}\,dx$$

Owing to the fact that the member is tapered of which J is given in terms of radius of a cross-section, J for each cross-section is a function of the coordinate of the cross-section along the member axis, i.e., $J_x = J(x)$, and is given by

$$J(x) = \frac{\pi r_j^4}{2c^4 L^4}\,(1-c)^4 (k-x)^4$$

where $k = \frac{L}{1-c}$, and $c = \frac{r_j}{r_i}$.

Therefore,

$$\theta_x^j = \frac{M_x}{G}\int_0^L \frac{2\,c^4\,L^4}{\pi r_j^4 (k-x)^4 (1-c)^4}\,dx$$

$$= \frac{2\,c^4\,L^4\,M_x}{G\pi r_j^4\,(1-c)^4}\int_0^L \frac{dx}{(k-x)^4}$$

$$= \frac{2\,c^4\,L^4\,M_x}{3G\pi r_j^4 (1-c)^4}\left[\frac{1}{(k-x)^3}\right]_0^L$$

$$= \frac{2\,c^4\,L^4\,M_x}{3G\pi (r_j^4)(1-c)^4}\left[\frac{1}{(k-L)^3} - \frac{1}{k^3}\right]$$

Let θ_x^j be the angle of twist of a unit value, then

$$M_x = \frac{3G\pi r_j^4\,k^3 (k-L)^3 (1-c)^4}{2c^4\,L^4\,k^3 - (k-L)^3} = M_x^j = k_{10,10}$$

where M_x^j is the required torsional stiffness, G is the shearing modulus of elasticity, L is the length of the member, $k_{10,10}$ is the torsional stiffness coefficient at the coordinate 10 due to a unit angle of twist at the coordinate 10.

By the equilibrium condition of the member, $M_x^i = -M_x^j$, therefore

$$M_x^i = \frac{-3G\pi r_j^4 k^3 (k-L)^3 (1-c)^4}{2c^4 L^4 [k^3 - (k-L)^3]} = k_{4,10}$$

where $k_{4,10}$ is the torsional stiffness coefficient at the coordinate 4 due to a unit angle of twist at the coordinate 10. Similarly, if the end i is freed and end j is fixed, then

$$M_x^i = \frac{3G\pi r_j^4 k^3 (k-L)^3 (1-c)^4}{2\ c^4\ L^4 [k^3 - (k-L)^3]} = k_{4,4}$$

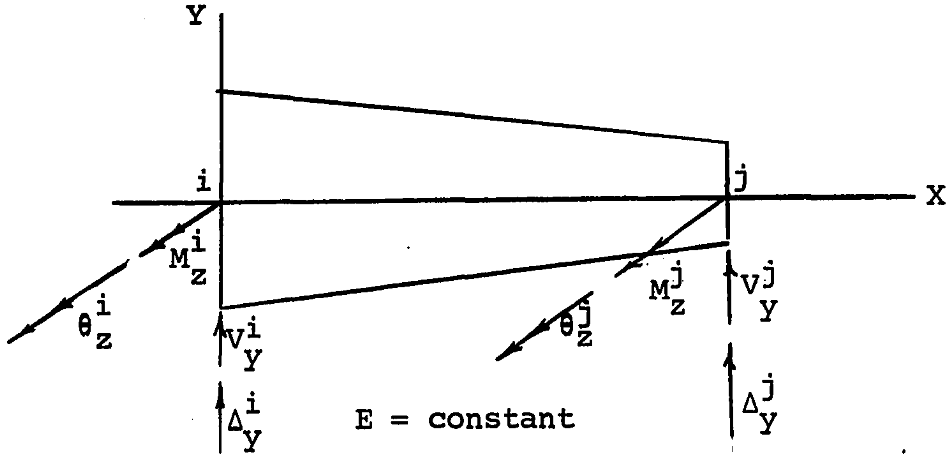
where $k_{4,4}$ is the torsional stiffness coefficient at the coordinate 4 due to a unit angle of twist at the coordinate 4.

By the equilibrium condition of the member, $M_x^j = -M_x^i$, therefore

$$M_x^j = \frac{-3G\pi r_j^4 k^3 (k-L)^3 (1-c)^4}{2c^4\ L^4\ [k^3 - (k-L)^3]} = k_{10,4}\ ,$$

where $k_{10,4}$ is the torsional stiffness coefficient at the coordinate 10 due to a unit angle of twist at the coordinate 4.

"Bending Stiffness due to End Rotation" (26)

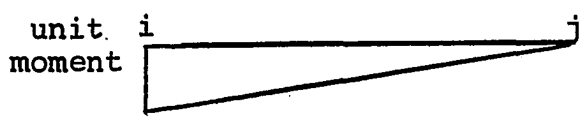


Since member i-j is tapered and of circular cross section, I_z varies along the member axis.

Suppose that the member is supported on hinges on both ends and a unit moment, in the counter clockwise direction, is applied at both ends separately, as follows:

<u>Unit Moment Applied at End i</u>

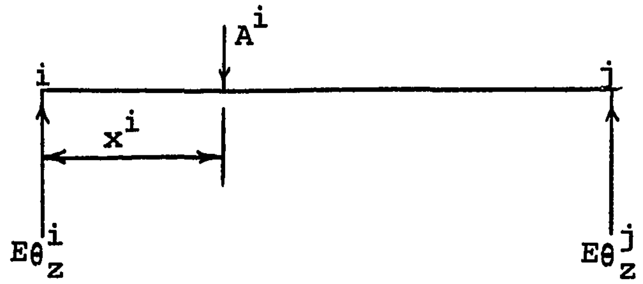
The diagrams of the moment and moment over the variable moment of inertia are given as follows:



moment-diagram

$\frac{\text{moment}}{I_z}$ - diagram

By the conjugate beam method, the loading of elastic weight and the end reactions are shown in the following sketch:



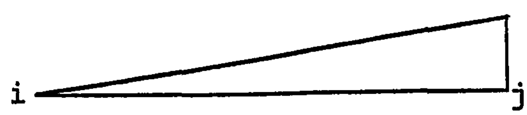
The end reactions of the conjugate beam loaded with the elastic weight, A^i, are given by

$$E\ \theta_z^i = \frac{L-x_i}{L}\ (A^i)$$

$$E\ \theta_z^j = -\ \frac{x_i}{L}\ (A^i)$$

Unit Moment Applied at End j

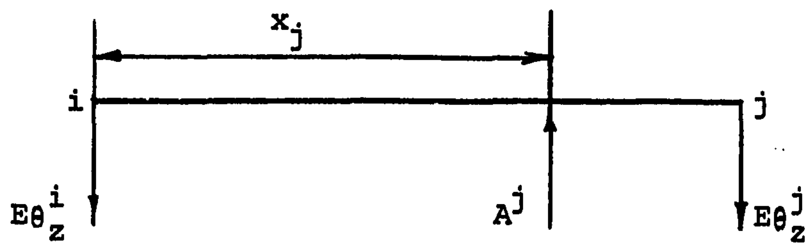
The diagrams of the moment and moment over the variable moment of inertia are given as follows:



moment diagram

$\frac{\text{moment}}{I_z}$ - diagram

By the conjugate beam method, the loading of elastic weight, A^j, and the end-reactions are shown in the following sketch:



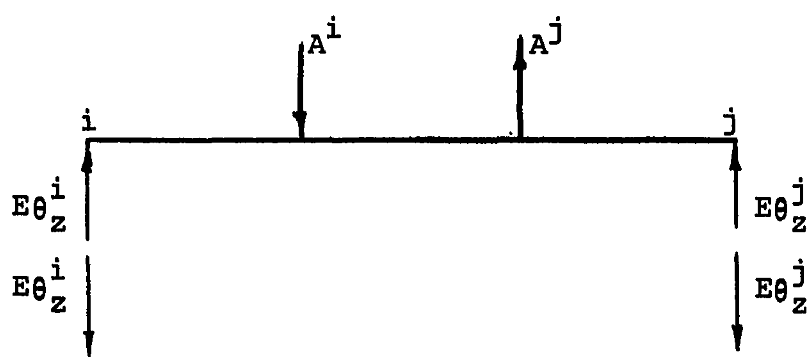
The end-reactions of the conjugate beam loaded with the elastic weight, A^j, are given by

$$E\theta_z^i = -\left[\frac{L-x_j}{L}\,(A^j)\right]$$

$$E\theta_z^j = +\left[\frac{x_j}{L}\,(A^j)\right]$$

Unit Moment Applied at Both Ends Simultaneously

The loading condition of the conjugate beam will be the combination of the previous two cases.



The final end reactions of the conjugate beam will be given by

$$E\theta_z^i = \frac{L-x_i}{L}(A^i) - \frac{L-x_j}{L}(A^j)$$

$$E\theta_z^j = \frac{-x_i}{L}(A^i) + \frac{x_j}{L}(A^j)$$

By the idea of superposition of displacements, there is established the state of displacement separately, as follows:

Fix End j

If end-j is fixed and end-i is left hinged, then only end-i will have end rotation about Z-axis, that is, $E\theta_z^i \neq 0 \quad E\theta_z^j = 0$.

Therefore,

$$E\theta_z^j = \frac{-A^i\ x_i}{L} + \frac{A^j\ x_j}{L} = 0 \quad \text{or} \quad -A^i\ x_i + A^j\ x_j = 0$$

If an end rotation θ_z^i is introduced at end-i, then there will be moments M_z^i and $C_{ij}M_z^i$ induced at the ends-i and j, respectively.

Then,

$$-A^i x_i M_z^i + C_{ij}\ M_z^i\ A^j x_j = 0$$

or

$$M_z^i\ A^i\ x_i = C_{ij}\ M_z^i A^j\ x_j$$

from which

$$C_{ij} = \frac{M_z^i\ A^i\ x_i}{M_z^i\ A^j\ x_j} = \frac{A^i\ x_i}{A^j\ x_j}\ .$$

where C_{ij} is the carry-over factor from end-i to end-j. Thus, the end reaction at end-i of the conjugate beam is given by

$$E\theta_z^i = \frac{1}{L}\left[-(L-x_j)A^j\left(\frac{A^i\ x_i}{A^j\ x_j}\right)\ M_z^i + (L-x_i)A^i\ M_z^i\right]$$

$$= \frac{A^i\ M_z^i}{L}\left[\frac{-x_i(L-x_j)}{x_j} + (L-x_i)\right] = \frac{A_i\ M_z^i}{x_j}\ (x_j-x_i)$$

from which

$$M_z^i = \frac{Ex_j\ \theta_z^i}{A^i(x_j-x_i)}$$

If the θ_z^i is of unit value, then

$$M_z^i = \frac{Ex_j}{A^i(x_j-x_i)} = k_{6,6}$$

where $k_{6,6}$ is the bending stiffness coefficient at the coordinate 6 due to a unit rotation at the coordinate 6.

The moment induced at end-j due to the introduced unit rotation θ_z^i is M_z^j.

$$M_z^j = C_{ij}M_z^i$$

$$= \frac{A^ix_i}{A^jx_j}\left[\frac{Ex_j}{A^i(x_j-x_i)}\right] = \frac{Ex_i}{A^j(x_j-x_i)} = k_{12,6}\ ,$$

where $k_{12,6}$ is the bending stiffness coefficient at the coordinate 12 due to a unit rotation at the coordinate 6.

Fix End i

In the like manner, only end-j has an end rotation about Z-axis, i.e., $E\theta_z^i = 0$.

Therefore,

$$\frac{L-x_i}{L} (A^i) - \frac{L-x_j}{L} (A^j) = 0$$

or

$$\frac{L-x_i}{L} (A^i) = \frac{L-x_j}{L} (A^j)$$

Similarly, if an end rotation θ_z^j is introduced at end-j, then there will be moments M_z^j and $C_{ji}\ M_z^j$ induced at the ends j and i respectively.

Then,

$$C_{ji}\ M_z^j \frac{(L-x_i)}{L} (A^i) - M_z^j \frac{(L-x_j)}{L} (A^j) = 0$$

or

$$C_{ji}\ M_z^j \frac{L-x_i}{L} (A^i) = M_z^j \left(\frac{L-x_j}{L}\right)(A^j)$$

from which

$$C_{ji} = \frac{A^j (L-x_j)}{A^i (L-x_i)}$$

Likewise, the end reaction at end-i of the conjugate beam is given by

$$E\theta_z^j = \frac{1}{L}\left[M_z^j\ A^j\ x_j - \frac{A^j (L-x_j)}{A^i (L-x_i)} M_z^j\ A^i\ x_i\right]$$

$$= \frac{M_z^j A^j}{L}\left[x_j - \frac{x_i(L-x_j)}{(L-x_i)}\right]$$

$$= M_z^j A^j \left[\frac{(x_j-x_i)}{(L-x_i)}\right]$$

from which, $M_z^j = \dfrac{E(L-x_i)\theta_z^j}{A^j(x_j-x_i)}$

If let θ_z^j be of unit value, then

$$M_z^j = \frac{E(L-x_i)}{A^j(x_j-x_i)} = k_{12,12}\ ,$$

where $k_{12,12}$ is the bending stiffness coefficient at the coordinate 12, due to a unit rotation at the coordinate 12.

The moment induced at end–i is M_z^i:

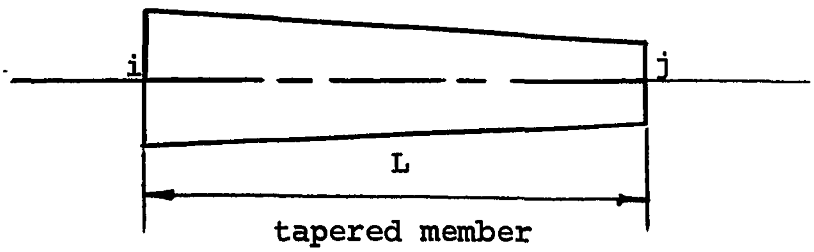
$$M_z^i = C_{ji}\ M_z^j$$

$$M_z^i = \frac{A^j(L-x_j)}{A^i(L-x_i)} = \frac{E(L-x_i)}{A^j(x_j-x_i)} = \frac{E(L-x_j)}{A^i(x_j-x_i)} = k_{6,12}\ ,$$

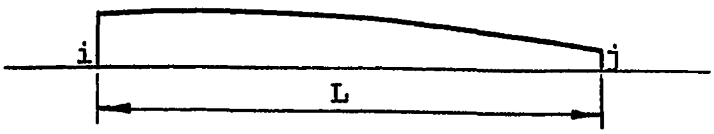
where $k_{6,12}$ is the bending stiffness coefficient at the coordinate 6 due to a unit rotation at the coordinate 12.

"Bending Stiffness due to End Deflection" (26)

The diagram of the moment of inertia of a tapered member of circular cross-section is shown in the following diagram:



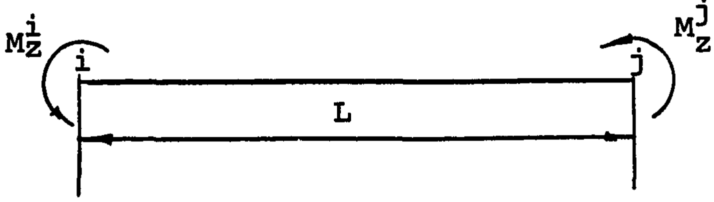
tapered member

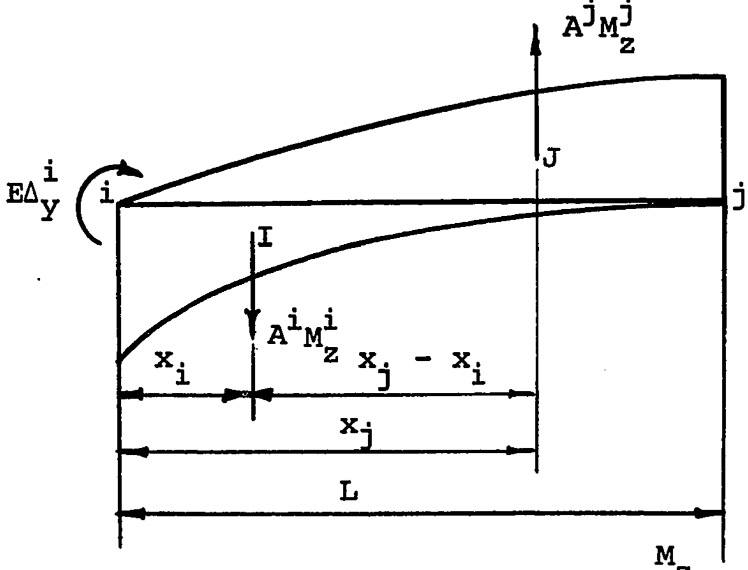


I_z - diagram

First let end-i of the member be released and have a deflection introduced in the positive direction of coordinate 2. By the conjugate beam method, the relations between the deflection Δ_y^i and the M_z^i and M_z^j can be developed as follows:

Since the deflection at the end i is introduced in the positive direction of the coordinate 2, i.e., Δ_y^i is upward, there must be fixed end moments M_z^i and M_z^j induced at the ends i and j respectively, as shown in the following sketch:





Conjugate beam loaded with $\frac{M_z}{I_z}$

Since the conjugate beam is in equilibrium, there must be a moment which is equivalent to Δ_y^i E to balance the effect of the two elastic loads $A^iM_z^i$ and $A^jM_z^j$. By $\Sigma M_J = 0$, it yields

$$E\Delta_y^i = A^iM_z^i(x_j - x_i)$$

from which,

$$M_z^i = \frac{E\Delta_y^i}{A^i(x_j - x_i)}$$

If Δ_y^i is of a unit value, then

$$M_z^i = \frac{E}{A^i(x_j - x_i)} = k_{6,2}$$

where $k_{6,2}$ is a bending stiffness coefficient at the coordinate 6, due to a unit deflection at the coordinate 2.

Similarly, by $\Sigma M_I = 0$, it yields $E\Delta_y^i = A^j M_z^j (x_j - x_i)$ from which

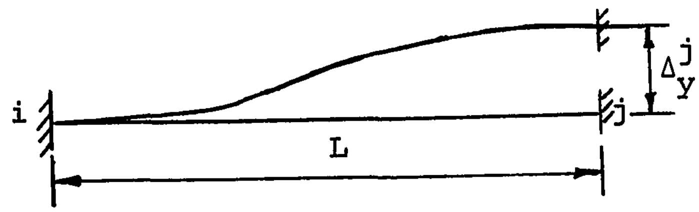
$$M_z^j = \frac{E\Delta_y^i}{A^j (x_j - x_i)}$$

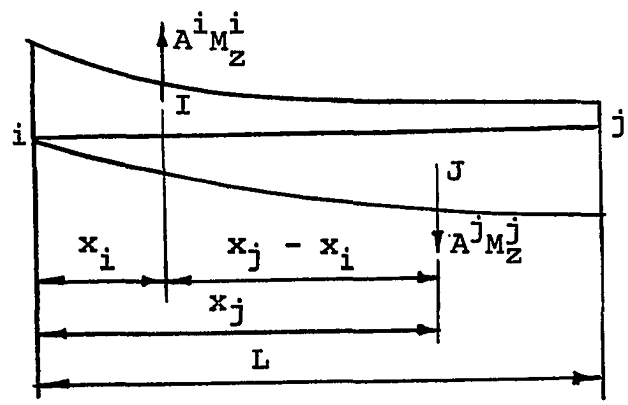
If Δ_y^i is allowed to deflect a unit, then

$$M_z^j = \frac{E}{A^j (x_j - x_i)} = k_{12,2}$$

where $k_{12,2}$ is a bending stiffness at the coordinate 12 due to a unit deflection at the coordinate 2.

Secondly, by the similar method if the deflection at end-j is introduced in the positive direction of the coordinate 8, then the fixed end moments M_z^i and M_z^j at the ends i and j are induced.





conjugate beam loaded with $\frac{M_z}{I_z}$

In order to keep the conjugate beam in an equilibrium condition, there must be a couple equivalent to $E\Delta_y^j$ applied at end-j in the counter clockwise direction. By $\Sigma M_I = 0$, it yields $E\Delta_y^j = A^j\, M_z^j(x_j - x_i)$

from which,

$$M_z^j = \frac{E\Delta_y^j}{A^j(x_j - x_i)} \quad \text{(in clockwise direction)}$$

If let Δ_y^j be a unit value, then

$$M_z^j = \frac{E}{A^j(x_j - x_i)} = k_{12,8}$$

where $k_{12,8}$ is a bending stiffness at the coordinate 12, due to a unit deflection at the coordinate 8.

Similarly, by $\Sigma M_J = 0$, it yields $E\Delta_y^j = A^i\, M_z^i(x_j - x_i)$

from which,

$$M_z^i = \frac{E\Delta_y^j}{A^i(x_j - x_i)} \quad \text{(in clockwise direction)}$$

If let Δ_y^j be a unit value, then

$$M_z^i = \frac{E}{A^i(x_j - x_i)} = k_{6,8}\ , \quad \text{(by frame-work convention, } M_z^i \text{ takes negative value)}$$

where $k_{6,8}$ is the bending stiffness coefficient at the coordinate 6, due to a unit deflection at the coordinate 8.

End Shearing Stiffness

The end shearing stiffness at both ends of the member due to either end rotation or end deflection can be obtained by applying the equilibrium condition of the forces acting on the member. The end shearing stiffness can be developed in two cases, as follows:

End Shearing Stiffness due to End Rotation

As it has been shown that the moments at both ends due to a unit rotation in the positive direction of coordinate 6 at end-i, about the Z-axis are

$$M_z^i = \frac{Ex_j}{A^i(x_j - x_i)} \quad \text{and} \quad M_z^j = \frac{Ex_i}{A^j(x_j - x_i)}$$

Therefore,

$$V_y^i = \left(\frac{1}{L}\right)\left[\frac{Ex_j}{A^i(x_j - x_i)} + \frac{Ex_i}{A^j(x_j - x_i)}\right]$$

$$= \frac{E}{LA^iA^j}\;\frac{A^jx_j + A^ix_i}{(x_j - x_i)} = k_{2,6}$$

where $k_{2,6}$ is the end shearing stiffness coefficient at the coordinate 2, due to a unit angular displacement at the coordinate 6. By the equilibrium condition of the member, $V_y^j = - V_y^i$, therefore,

$$V_y^j = -V_y^i$$

$$V_y^j = \frac{-E}{LA^iA^j}\left[\frac{A^ix_i + A^jx_j}{(x_j - x_i)}\right] = k_{8,6}$$

where $k_{8,6}$ is the end shearing stiffness coefficient at the coordinate 8 due to a unit angular displacement at the coordinate 6.

Similarly, the end shearing stiffnesses at both ends of the member due to a unit rotation of end-j about the Z-axis are

$$V_y^i = \frac{E}{LA^iA^j}\left[\frac{A^i(L-x_i) + A_j(L-x_j)}{x_i - x_j}\right] = k_{2,12}$$

where $k_{2,12}$ is the end shearing stiffness coefficient at the coordinate 2, due to a unit rotation at the coordinate 12, and

$$V_y^j = \frac{-E}{LA^iA^j}\left[\frac{A^i(L-x_i) + A^j(L-x_j)}{x_i - x_j}\right] = k_{8,12}$$

where $k_{8,12}$ is the end shearing stiffness coefficient at the coordinate 8, due to a unit rotation at the coordinate 12.

End Shearing Stiffness due to End Deflection

By following the same idea and procedure in the preceding case, one can obtain the end shearing stiffnesses at both ends due to the end deflections of both ends of the member as follows:

When end-i deflects in the positive direction of the coordinate 2, then

$$V_y^i = \frac{1}{L}\left[\frac{E}{A^i(x_j-x_i)} + \frac{E}{A^j(x_j-x_i)}\right] = \frac{E}{LA^iA^j}\left[\frac{A^j+A^i}{(x_j-x_i)}\right] = k_{2,2}$$

where $k_{2,2}$ is the end shearing stiffness coefficient at the coordinate 2, due to a unit deflection at the coordinate 2.

$$V_y^j = -V_y^i = \frac{-E}{LA^iA^j}\left[\frac{A^j+A^i}{(x_j-x_i)}\right] = k_{8,2}$$

where $k_{8,2}$ is the end shearing stiffness coefficient at the coordinate 8, due to a unit deflection at the coordinate 2. When end-j has a deflection in the positive direction of coordinate 8, then

$$V_y^j = \frac{E}{L}\left[\frac{1}{A^i(x_j-x_i)} + \frac{1}{A^j(x_j-x_i)}\right] = \frac{E}{LA^iA^j}\left[\frac{A^i+A^j}{(x_j-x_i)}\right] = k_{8,8}$$

where $k_{8,8}$ is the end shearing stiffness coefficient at the coordinate 8, due to a unit deflection at the coordinate 8.

$$V_y^i = -V_y^j = \frac{-E}{LA^iA^j}\left[\frac{A^i+A^j}{(x_j-x_i)}\right] = k_{2,8}$$

where $k_{2,8}$ is the end shearing stiffness coefficient at the coordinate 2, due to a unit deflection at the coordinate 8.

The stiffness matrix for the whole member is given in the following four tables:

TABLE 16

SUBMATRIX s^{i-i} OF A MEMBER STIFFNESS MATRIX OF A TAPERED MEMBER

$s^{i-i}=$

$\frac{E\pi r_j^2 k(k-L)(1-c)^2}{c^2 L^3}$					
	$\frac{E}{LA^iA^j}\left[\frac{A^i+A^j}{x_j-x_i}\right]$ (X-Y Plane)				$\frac{E}{LA^iA^j}\left[\frac{A^i x_i+A^j x_j}{x_j-x_i}\right]$ (X-Y Plane)
		$\frac{E}{LA^iA^j}\left[\frac{A^i+A^j}{x_j-x_i}\right]$ (X-Z Plane)		$\frac{-E}{LA^iA^j}\left[\frac{A^i x_i+A^j x_j}{x_j-x_i}\right]$ (X-Z Plane)	
			$\frac{3G\pi r_j^4 k^3(k-L)^3(1-c)^4}{2c^4L^4[k^3-(k-L)^3]}$		
		$\frac{-E}{A^i(x_j-x_i)}$ (X-Z Plane)		$\frac{E\,x_j}{A^i(x_j-x_i)}$ (X-Z Plane)	
	$\frac{E}{A^i(x_j-x_i)}$ (X-Y Plane)				$\frac{E\,x_j}{A^i(x_j-x_i)}$ (X-Y Plane)

TABLE 17

SUBMATRIX s^{i-j} OF A MEMBER STIFFNESS MATRIX OF A TAPERED MEMBER

$s^{i-j} =$

$\frac{-E\pi r_j^2 k(k-L)(1-c)^2}{c^2L^3}$					
	$\frac{-E}{LA^iA^j}\left[\frac{A^i+A^j}{x_j-x_i}\right]$ (X-Y Plane)				$\frac{E}{LA^iA^j}\left[\frac{A^i(L-x_i)+A^j(L-x_j)}{x_i-x_j}\right]$ (X-Y Plane)
		$\frac{-E}{LA^iA^j}\left[\frac{A^i+A^j}{x_j-x_i}\right]$ (X-Z Plane)		$\frac{-E}{LA^iA^j}\left[\frac{A^i(L-x_i)+A^j(L-x_j)}{x_i-x_j}\right]$ (X-Z Plane)	
			$\frac{3G\pi r_j^4 k^3(k-L)^3(1-c)^4}{2c^4L^4\left[k^3-(k-L)^3\right]}$		
		$\frac{E}{A^i(x_j-x_i)}$ (X-Z Plane		$\frac{E(L-x_j)}{A^i(x_j-x_i)}$ (X-Z Plane)	
	$\frac{-E}{A^j(x_j-x_i)}$ (X-Y Plane)				$\frac{E(L-x_j)}{A^i(x_j-x_i)}$ (X-Y Plane)

TABLE 18

SUBMATRIX s^{j-i} OF A MEMBER STIFFNESS MATRIX OF A TAPERED MEMBER

$s^{j-i}=$

$\frac{-E\pi r_j^2 k(k-L)(1-c)^2}{c^2 L^3}$					
	$\frac{-E}{LA^iA^j}\left[\frac{A^i+A^j}{x_j-x_i}\right]$ (X-Y Plane)				$\frac{-E}{LA^iA^j}\left[\frac{A^i x_i+A^j x_j}{x_j-x_i}\right]$ (X-Y Plane)
		$\frac{-E}{LA^iA^j}\left[\frac{A^i+A^j}{x_j-x_i}\right]$ (X-Z Plane)		$\frac{E}{LA^iA^j}\left[\frac{A^i x_i+A^j x_j}{x_j-x_i}\right]$ (X-Z Plane)	
			$\frac{-3G\pi r_j^4 k^3(k-L)^3(1-c)^4}{2c^4L^4[k^3-(k-L)^3]}$		
		$\frac{-E}{A^j(x_j-x_i)}$ (X-Z Plane)		$\frac{E x_i}{A^j(x_j-x_i)}$ (X-Z Plane)	
	$\frac{E}{A^j(x_j-x_i)}$ (X-Y Plane)				$\frac{E x_i}{A^j(x_j-x_i)}$ (X-Y Plane)

TABLE 19

SUBMATRIX s^{j-j} OF A MEMBER STIFFNESS MATRIX OF A TAPERED MEMBER

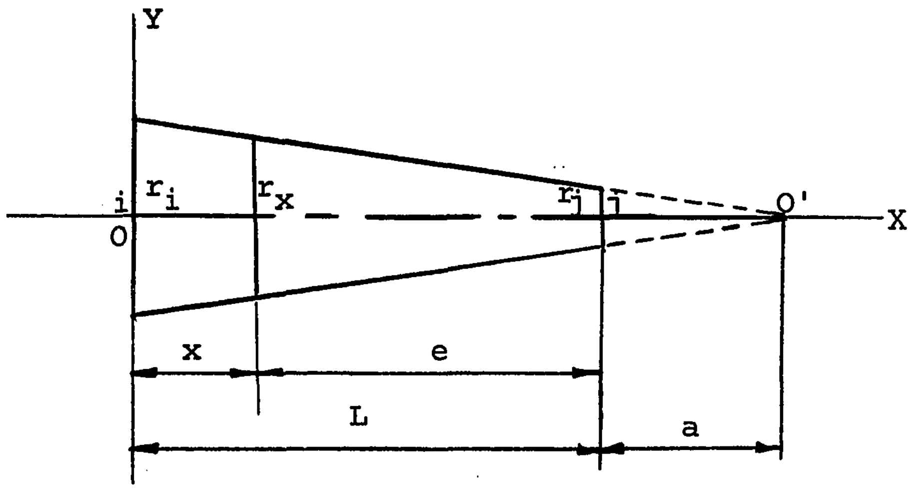
$s^{j-j} =$

$\frac{E\pi r_j^2 k(k-L)(1-c)^2}{c^2L^3}$					
	$\frac{E}{LA^iA^j}\left[\frac{A^i+A^j}{x_j-x_i}\right]$ (X-Y Plane)				$\frac{-E}{LA^iA^j}\left[\frac{A^i(L-x_i)+A^j(L-x_j)}{x_i-x_j}\right]$ (X-Y Plane)
		$\frac{E}{LA^iA^j}\left[\frac{A^i+A^j}{x_j-x_i}\right]$ (X-Z Plane)		$\frac{E}{LA^iA^j}\left[\frac{A^i(L-x_i)+A^j(L-x_j)}{x_i-x_j}\right]$ (X-Z Plane)	
			$\frac{3G\pi r_j^4 k^3(k-L)^3(1-c)^4}{2c^4L^4[k^3-(k-L)^3]}$		
		$\frac{E}{A^j(x_j-x_i)}$ (X-Z Plane)		$\frac{E(L-x_i)}{A^j(x_j-x_i)}$ (X-Z Plane)	
	$\frac{-E}{A^j(x_j-x_i)}$ (X-Y Plane)				$\frac{E(L-x_i)}{A^j(x_j-x_i)}$ (X-Y Plane)

To Define the Section Properties

When a member of which the cross-section is of a regular shape is nonprismatic, the section properties are variable along the length of the member. Therefore, when the different kinds of stiffness coefficients are desired, the section properties, which are related to the stiffness coefficients, have to be expressed in terms of the location of the cross-section along the member axis. The procedure to define the section properties in general expressions is as follows:

To Define the Radius of a Cross-Section



Front View of a Tapered Member

Let the origin be at joint-i, i.e., joint-i is taken as the initial joint. Let the member axis be the X-axis of the member-axes system. Extend both edges to the point 0'. Let $c = \frac{r_j}{r_i}$, in which r_i and r_j are the radii of the two end cross-sections. Then the unknown length a is given by

$$\frac{a}{L+a} = \frac{r_j}{r_i} = c$$

from which

$$a = \frac{cL}{1-c}$$

Therefore, the radius, r_x, of any cross-section, which is x distance away from the origin, can be given as follows: By two similar triangles,

$$\frac{r_j}{r_x} = \frac{a}{a+e}$$

in which $a = \frac{cL}{1-c}$, and $e = L-x$

therefore

$$\frac{r_j}{r_x} = \frac{\frac{cL}{1-c}}{\frac{cL}{1-c} + (L-x)} = \frac{cL}{L-(1-c)x}$$

If let $k = \frac{L}{1-c}$, then $r_x = r_j(\frac{k-x}{cL})(1-c)$

It means that the radius of any cross-section is a function of the location of the cross-section along the X-axis.

To Define the Area of a Cross-Section

Since the member under discussion is of a circular cross-section, the area, A_x, of any cross-section of the member, which is x distance away from the origin, can be given by

$$A_x = \pi r_x^2$$

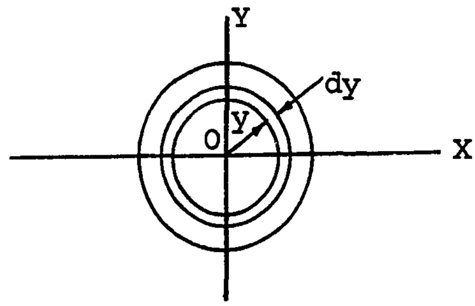
where $r_x = r_j\,(\frac{k-x}{cL})\,(1-c)$, $k = \frac{L}{1-c}$ and $c = \frac{r_j}{r_i}$

Therefore,

$$A_x = \pi \left[r_j\ (\frac{k-x}{cL}) \right]^2 (1-c)^2 = \frac{r_j^2}{(cL)^2}\ (k-x)^2 (1-c)^2$$

It means that A_x is a function of its location along the member axis.

To Define the Polar Moment of Inertia of a Cross-Section



For the purpose of finding the polar moment of inertia of a circular cross-section, "it is conceivable that the circular cross-section is divided into concentric elements of area" (24). Let dy be the width of one of these elements of area whose distance from the center of the circular cross-section is y. Let dA be the element area, then

$$dA = (2\pi y)\,dy = 2\pi y dy$$

$$J = \int_0^r y^2 dA$$

$$J = \int_o^r y^2 \ (2\pi y dy) = 2\pi \int_o^r y^3 dy = 2\pi \left(\frac{y^4}{4} \right)_o^r = \frac{\pi r^4}{2}$$

As has been found in the previous article, the radius, r_x, for any cross-section of this tapered member is

$$r_j \ (1-c)(\frac{k-x}{cL})$$

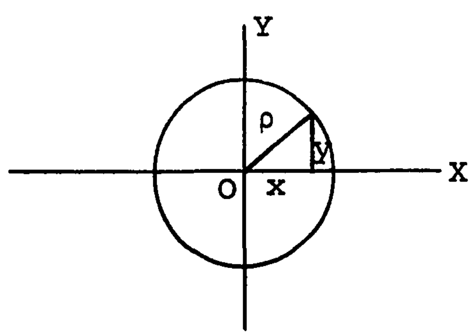
Therefore, if let the polar moment of inertia of any cross-section along the member axis is J_x, then J_x is given by

$$J_x = \frac{\pi}{2} \ (r_x)^4$$

$$J_x = \frac{\pi r_j^4}{2} \ (1-c)^4 \ \left(\frac{k-x}{cL} \right)^4$$

It means that the polar moment of inertia of any cross-section of this tapered member can be expressed in terms of the location of the cross-section along the member axis.

Define the Moment of Inertia of a Cross-Section
of a Nonprismatic Member i-j with
Circular Cross-Section of Length L



Cross Section of Member

For a circular cross-section, "all axes through the center of the cross-section are principle axes, and the moments of inertia about these axes are all equal" (24). And as it is known that the polar moment of inertia of an area w.r.t. the member axis is the sum of the moments of inertia about the two rectangular axes through the center of gravity of the area, i.e., $J_x = 2I_x$ or $I_x = \frac{1}{2}(J_x)$, where

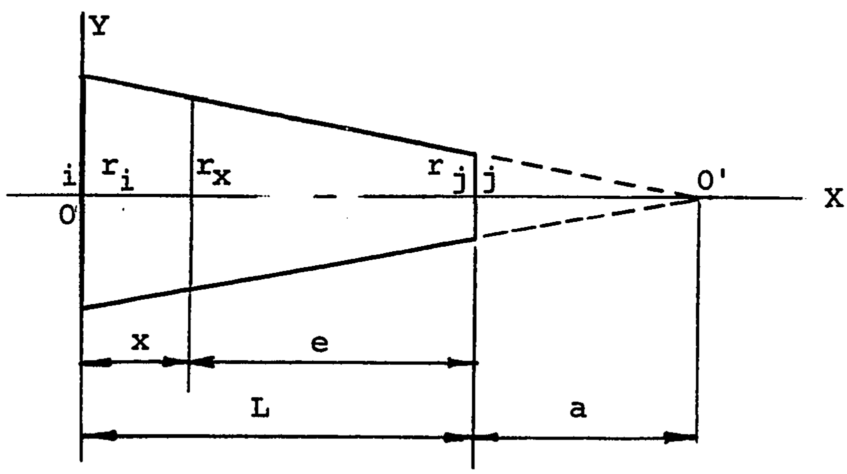
$$J_x = \frac{\pi r_j^4}{2} (1-c)^4 \left(\frac{k-x}{cL}\right)^4$$

Therefore,

$$I_x = \frac{\pi r_j^4}{4} (1-c) \left(\frac{k-x}{cL}\right)^4 .$$

It shows that the moment of inertia of a circular cross-section is also a function of the location of the section along the member axis.

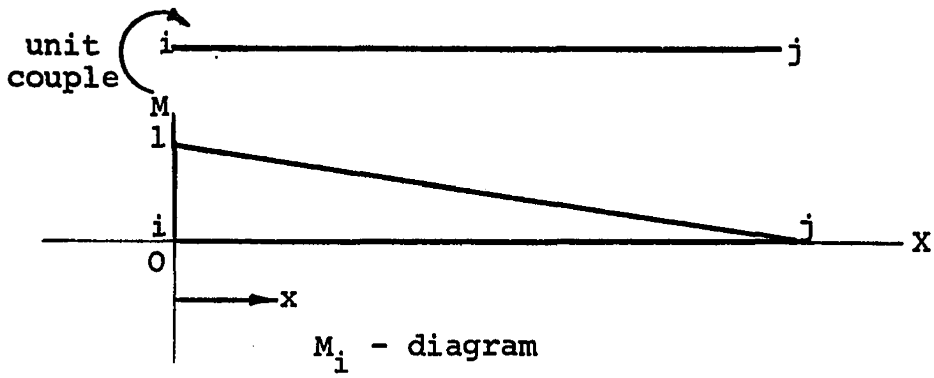
To Define the Curve of $\frac{M}{EI}$



Front View of a Tapered Member

The discussion is made on each case as follows:

Unit Couple Applied at Joint-i



M_i - diagram

The M_i curve can be written as a function of x coordinate of the section as follows:

$$M_{i-x} = M_i(x) = \frac{-1}{L}(x) + 1$$

$$I_x = I(x) = \frac{\pi r_j^4}{4}(1-c)^4 \left(\frac{k-x}{cL}\right)^4$$

If let $f_i(x) = \dfrac{M_i(x)}{EI(x)}$, then

$$f_i(x) = \frac{\frac{-1}{L}(x)+1}{(E)\ \frac{\pi r_j^4}{4}(1-c)^4\left(\frac{k-x}{cL}\right)^4}$$

$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left(\frac{L-x}{(k-x)^4}\right)$$

$$\frac{L-x}{(k-x)^4} = \frac{A}{(k-x)^4} + \frac{B}{(k-x)^3} + \frac{C}{(k-x)^2} + \frac{D}{(k-x)}$$

$$L-x = A+B(k-x)+C(k-x)^2+D(k-x)^3$$

$$L-x = A+Bk-Bx+Ck^2-2Ckx+Cx^2+Dk^3-3Dk^2x+3Dkx^2-Dx^3$$

$$= (A+Bk+Ck^2+Dk^3)-(B+2Ck+3Dk^2)x+(C+3Dk)x^2-Dx^3$$

$$(A+Bk+Ck^2+Dk^3) = L$$

$$-(B+2Ck+3Dk^2) = -1$$

$$C+3Dk = 0$$

$$-D = 0$$

$$C = 0$$

$$B = 1$$

$$A+k = L$$

$$A = L-k$$

$$\frac{L-x}{(k-x)^4} = \frac{L-k}{(k-x)^4} + \frac{1}{(k-x)^3}$$

Therefore,

$$f_i(x) = \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left[\frac{L-k}{(k-x)^4} + \frac{1}{(k-x)^3}\right]$$

If let A^i be the area of $f_i(x)$, then

$$A^i = \int_0^L f_i(x)dx = \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left[\int_0^L \frac{L-k}{(k-x)^4}dx + \int_0^L \frac{dx}{(k-x)^3}\right]$$

$$A^i = \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left\{\left[\frac{(L-k)}{3(k-x)^3}\right]_0^L + \left[\frac{1}{2(k-x)^2}\right]_0^L\right\}$$

$$A^i = \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left\{\frac{L-k}{3(k-L)^3} - \frac{(L-k)}{3k^3} + \frac{1}{2(k-L)^2} - \frac{1}{2k^2}\right\}$$

If let the distance of the centroid of A^i from the origin be $\bar{x}_i$, then

$$\bar{x}_i = \frac{1}{A^i}\int_0^L xf_i(x)\,dx$$

as derived previously,

$$f_i(x) = \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left(\frac{L-x}{(k-x)^4}\right),$$

then

$$xf_i(x) = \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left[\frac{Lx-x^2}{(k-x)^4}\right]$$

Referring to the previous integral,

$$Lx-x^2 = A+B(k-x)+C(k-x)^2+D(k-x)^3$$

$$Lx-x^2 = A+Bk-Bx+Ck^2-2Ckx+Cx^2+Dk^3-3Dk^2x+3Dkx^2-Dx^3$$

$$= (A+Bk+Ck^2+Dk^3)-(B+2Ck+3Dk^2)x+(C+3Dk)x^2-Dx^3$$

$$A+Bk+Ck^2+Dk^3 = 0$$

$$-(B+2Ck+3Dk^2) = L$$

$$(C+3Dk) = -1$$

$$-D = 0$$

$$C = -1$$

$$B = 2k-L$$

$$A = kL-k^2$$

Therefore,

$$xf_i(x) = \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left\{\frac{kL-k^2}{(k-x)^4} + \frac{2k-L}{(k-x)^3} - \frac{1}{(k-x)^2}\right\}$$

and

$$\int_0^L xf_i(x)dx = \int_0^L \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left\{\frac{kL-k^2}{(k-x)^4} + \frac{2k-L}{(k-x)^3} - \frac{1}{(k-x)^2}\right\} dx$$

$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left\{\int_0^L \frac{kL-k^2}{(k-x)^4}dx + \int_0^L \frac{2k-L}{(k-x)^3}dx - \int_0^L \frac{dx}{(k-x)^2}\right\}$$

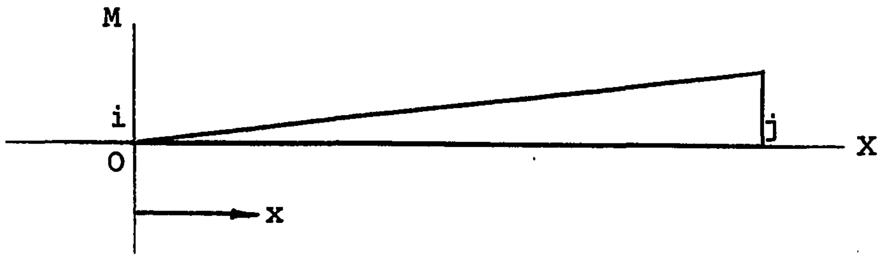
$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left\{\left[\frac{(kL-k^2)}{3(k-x)^3}\right]_0^L + \left[\frac{2k-L}{2(k-x)^2}\right]_0^L - \left[\frac{1}{(k-x)}\right]_0^L\right\}$$

$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left\{\frac{kL-k^2}{3(k-L)^3} - \frac{kL-k^2}{3k^3} + \frac{2k-L}{2(k-L)^2} - \frac{2k-L}{2k^2}\right.$$

$$\left. - \frac{1}{k-L} + \frac{1}{k}\right\}$$

$$\bar{x}_i = \frac{\dfrac{kL-k^2}{3(k-L)^3} - \dfrac{kL-k^2}{3k^3} + \dfrac{2k-L}{2(k-L)^2} - \dfrac{2k-L}{2k^2} - \dfrac{1}{k-L} + \dfrac{1}{k}}{\dfrac{L-k}{3(k-L)^3} - \dfrac{L-k}{3k^3} + \dfrac{1}{2(k-L)^2} - \dfrac{1}{2k^2}}$$

Unit Couple Applied at Joint-j



M_j - diagram

The M_j curve can be written as a function of x coordinate of the section as follows:

$$M_{j-x} = M_j(x) = \frac{1}{L}(x)$$

$$I_x = I(x) = \frac{\pi r_j^4}{4}(1-c)^4\left(\frac{k-x}{cL}\right)^4,$$

If let $f_j(x) = \dfrac{M_j(x)}{EI(x)}$, then

$$f_j(x) = \frac{\dfrac{x}{L}}{\dfrac{(E)\pi r_j^4}{4}(1-c)^4\left(\dfrac{k-x}{cL}\right)^4} = \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left[\frac{x}{(k-x)^4}\right]$$

$$\frac{x}{(k-x)^4} = \frac{A}{(k-x)^4} + \frac{B}{(k-x)^3} + \frac{C}{(k-x)^2} + \frac{D}{(k-x)}$$

$$x = A+B(k-x)+C(k-x)^2+D(k-x)^3$$

$$x = A+Bk-Bx+Ck^2-2Ckx+Cx^2+Dk^3-3Dk^2x+3Dkx^2-Dx^3$$

$$x = (A+Bk+Ck^2+Dk^3)-(B+2Ck+3Dk^2)x+(C+3Dk)x^2-Dx^3$$

$$A+Bk+Ck^2+Dk^3 = 0$$

$$-(B+2Ck+3Dk^2) = 1$$

$$+(C+3Dk) = 0$$

$$-D = 0$$

$$C = 0$$

$$B = -1$$

$$A = k$$

Therefore,

$$\frac{x}{(k-x)^4} = \frac{k}{(k-x)^4} - \frac{1}{(k-x)^3}$$

If let A^j be the area of $f_j(x)$, then

$$A^j = \int_0^L f_i(x)\,dx = \int_0^L \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left[\frac{x}{(k-x)^4}\right] dx$$

$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4} \int_0^L \frac{x}{(k-x)^4}\, dx$$

$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4} \left[\int_0^L \frac{k}{(k-x)^4}\, dx - \int_0^L \frac{dx}{(k-x)^3} \right]$$

$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4} \left[\left(\frac{k}{3(k-x)^3} \right)_0^L - \left(\frac{1}{2(k-x)^2} \right)_0^L \right]$$

$$A^j = \frac{4c^4L^3}{E\pi r_j^4(1-c)^4} \left\{ \left[\frac{k}{3(k-L)^3} - \frac{k}{3k^3} \right] - \left[\frac{1}{2(k-L)^2} - \frac{1}{2k^2} \right] \right\}$$

If let $\bar{x}_j$ be the distance of the centroid of A^j from the origin, then

$$\bar{x}_j = \frac{1}{A^j} \int_0^L x\, f_j(x)\, dx.$$

$$\int_0^L x f_j(x)\, dx = \int_0^L (x) \frac{M_j(x)}{EI(x)}\, dx$$

$$= \int_0^L \frac{4c^4L^3}{E\pi r_j^4(1-c)^4} \left[\frac{x^2}{(k-x)^4} \right] dx$$

$$\frac{x^2}{(k-x)^4} = \frac{A}{(k-x)^4} + \frac{B}{(k-x)^3} + \frac{C}{(k-x)^2} + \frac{D}{(k-x)}$$

Referring to the previous integral,

$$x^2 = (A+Bk+Ck^2+Dk^3)-(B+2Ck+3Dk^2)x+(C+3Dk)x^2-Dx^3$$

$$(A+Bk+Ck^2+Dk^3) = 0$$

$$-(B+2Ck+3Dk^2) = 0$$

$$(C+3Dk) = 1$$

$$-D = 0$$

$$C = 1$$

$$B = -2k$$

$$A = k^2$$

$$\frac{x^2}{(k-x)^4} = \frac{k^2}{(k-x)^4} - \frac{2k}{(k-x)^3} + \frac{1}{(k-x)^2}$$

Therefore,

$$\int_0^L x\, f_j(x)dx = \int_0^L \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left[\frac{x^2}{(k-x)^4}\right] dx$$

$$= \int_0^L \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left[\frac{k^2}{(k-x)^4} - \frac{2k}{(k-x)^3} + \frac{1}{(k-x)^2}\right] dx$$

$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left[\int_0^L \frac{k^2}{(k-x)^4}\,dx - \int_0^L \frac{2k}{(k-x)^3}\,dx + \int_0^L \frac{dx}{(k-x)^2}\right]$$

$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left[\left(\frac{k^2}{3(k-x)^3}\right)_0^L - \left(\frac{2k}{2(k-x)^2}\right)_0^L + \left(\frac{1}{k-x}\right)_0^L\right]$$

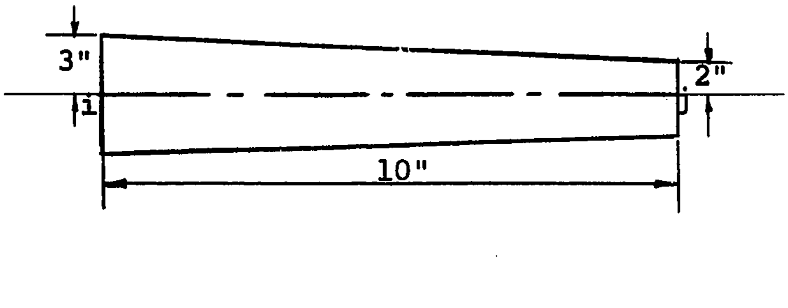
$$= \frac{4c^4L^3}{E\pi r_j^4(1-c)^4}\left\{\left[\frac{k^2}{3(k-L)^3} - \frac{k^2}{3k^3}\right] - \left[\frac{2k}{2(k-L)^2} - \frac{2k}{2(k)^2}\right] + \left[\left(\frac{1}{k-L}\right) - \frac{1}{k}\right]\right\}$$

and

$$\bar{x}_j = \frac{\dfrac{k^2}{3(k-L)^3} - \dfrac{k^2}{3k^3} - \dfrac{2k}{2(k-L)^2} + \dfrac{2k}{2(k)^2} + \dfrac{1}{k-L} - \dfrac{1}{k}}{\dfrac{k}{3(k-L)^3} - \dfrac{k}{3k^3} - \dfrac{1}{2(k-L)^2} + \dfrac{1}{2k^2}}$$

$$= \frac{\dfrac{k^2}{3(k-L)^3} - \dfrac{k}{(k-L)^2} + \dfrac{1}{k-L} - \dfrac{1}{3k}}{\dfrac{k}{3(k-L)^3} - \dfrac{1}{2(k-L)^2} + \dfrac{1}{6k^2}}$$

If the moment of inertia, I, of a structural segment is hard to express in terms of distance along the member length, x, then the author suggests that the semigraphical integration method be used to calculate area and centroid of the $\frac{M}{EI}$ diagram. A numerical example of semigraphical integration applied to a beam of circular cross-section varying in radius follows.



$$L = 10''$$

$$c = \frac{r_j}{r_i} = \frac{2}{3}$$

$$1\text{-}c = 1- \frac{2}{3} = \frac{1}{3}$$

$$k = \frac{L}{1-c} = 30$$

$$cL = \frac{20}{3}$$

$$I_x = \frac{\pi(2)^4}{4}\left[\frac{(30-x)^4}{(\frac{20}{3})^4}\ (\tfrac{1}{3})^4\right] = \frac{\pi}{4(10)^4}\ (30-x)^4$$

when $x = 0$, $I_x = \dfrac{\pi}{4(10)^4}\ (30)^4 = \dfrac{81\pi}{4} = 63.6$

when $x = 10$, $I_x = \dfrac{(20)^4\pi}{4(10)^4} = 4\pi = 12.6$

$$A_i = \frac{13.43}{10^2} = 0.1343\ (\text{in}^2)$$

$$\bar{x}_i = \frac{1^{st}\ \text{-Moment}}{A_i} = \frac{57.25}{13.43} = 4.27\ (\text{in})$$

(These results refer to Tables 20-A and 20-B on page 188.)

TABLE 20

SEMIGRAPHICAL INTEGRATION TABLE

(A)

Station	M	30-x	$\frac{(30-x)^4}{10^4}$	I_x	$\frac{M}{I_x}$
0	1.0	30	81.1	63.6	0.0157
2	0.8	28	61.5	48.3	0.0165
4	0.6	26	45.9	36.1	0.0166
6	0.4	24	33.2	26.1	0.0153
8	0.2	22	23.4	18.4	0.0109
10	0.0	20	16.0	12.6	0.0

(B)

Station	$(\frac{M}{I_x})(10)^2$	Sum-1	$\Delta\frac{x}{2}$	$\Delta A(10)^2$	$A(10)^2$	Sum-2	1^{st}-Moment$(10)^2$
0	1.57				13.43		
		3.22	1	3.22		23.64	23.64
2	1.65				10.21		
		3.31	1	3.31		17.11	17.11
4	1.66				6.90		
		3.19	1	3.19		10.61	10.61
6	1.53				3.71		
		2.62	1	2.62		4.80	4.80
8	1.09				1.09		
		1.09	1	1.09		1.09	1.09
10	0.0				0.0		
				13.43			57.25

CHAPTER VIII

CONCLUSION

It has been shown that the analysis of a framed structure, in either a plane or a space, can be performed by analyzing each member as a unit to simplify the process of establishing either the stiffness matrix or the flexibility matrix. In this dissertation, establishing the stiffness matrix of a member by the displacement method was the objective. The dissertation has demonstrated that by the advantage of partitioning a matrix into its submatrices, the overall stiffness matrix for a frame can be obtained by joining the stiffness matrix, considered as submatrix, of each of the members that compose the whole frame with the notation suggested by the author. Using the theory of the rotation of a vector in space, the stiffness matrix in structure-axes system of a member of any orientation in space can be obtained by transforming the stiffness matrix of the member to member-axes system. Especially, the λ system the author adopted for the transformation of the stiffness matrix of a member in member-axes system into structure-axes system can be done without undergoing the rotational transformation matrix equation.

The stiffness of a nonprismatic member in space and the idea, as well as the reference, of how to set a stiffness matrix of a curved member have been presented. With this information it should be possible to analyze most framed structures. Following this, how membrane structures, such as plates and shells, can be analyzed with the advantage and convenience of matrix theories and the support of digital computer should be a further milestone in the matrix method of structural analysis.

CHAPTER IX

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