GAIN OPTIMIZATION OF ANTENNA ARRAYS THROUGH MATRIX THEORY

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PREFACE

Many antennas are designed with the primary objective of obtaining maximum gain in specified directions. In almost every case they are designed by empirical methods based on a firm and comprehensive knowledge of theory describing similar classic antennas.

The properties of a function of a matrix vector may be used to obtain solutions to systems for which the parameters are significant only at discrete values of a chosen variable. The purpose of this study is to demonstrate that the gain of an antenna array can be considered as such a variable, and, having accomplished this, to demonstrate the technique by which a truly optimum design can be achieved. Some practical designs were developed in the effort and the results compared with known values.

I wish to express my appreciation to those who have aided me in the investigation and preparation of this thesis. I especially wish to thank Dr. K. R. Cook, Professor of Electrical Engineering, Oklahoma State University, for his interest and technical guidance in the subject of this thesis; and to Dr. D. K. Cheng, Electrical Engineering Department, Syracuse University, for his willing support in furnishing me with his paper introducing the subject of this thesis; and to C. E. Lewis, for his invaluable assistance in adapting the mathematics of the solution to a computer program; and to my dad, R. T. Moore, my wife, Ouida, my children, Leslie, Dorothy, and Brenda, for the many personal sacrifices without which this thesis would not have been written.

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CHAPTER I

INTRODUCTION

Fundamental Relations Pertaining to Gain Optimization of Antenna Arrays

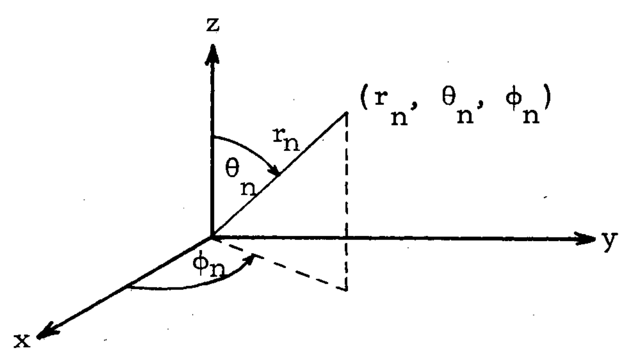
In almost every case antennas are designed by empirical methods based on a firm and comprehensive knowledge of theory describing similar classic antennas in idealized conditions. It is not feasible to use a purely theoretical approach to optimize the configuration of an antenna. Unrealistic simplifying assumptions must be made to reduce the resulting mathematical expressions into a form for which solutions can be obtained. Array antennas are almost always designed symmetrically to minimize the quantity of empirical variations which must be considered. The optimum configuration required to satisfy specific design goals may deviate from symmetrical dimensions or symmetrical element excitation but usually can not be obtained within a reasonable number of measurements.

The primary objective of many antenna designs is maximization of directive gain. Systems engineers have long since learned that a nominal increase in antenna gain significantly reduces the system transmitter power requirements or the system receiver sensitivity requirements. Increasing an electromagnetic circuit margin by increasing antenna gain has become so attractive that extensive efforts and facilities have been devoted to construction of antenna arrays.

The directive power gain, $G(\theta, \phi)$, of a given antenna array is taken with reference to a non-dissipative isotropic radiator. Neglecting heat losses in the array, the expression for gain as developed by Silver (1) is

$$G(\theta_o, \phi_o) = \frac{S(\theta_o, \phi_o)}{\frac{1}{4\pi} \int_o^{2\pi} \int_o^{\pi} S(\theta, \phi) \sin \theta d\theta d\phi} \tag{1}$$

where $S(\theta, \phi)$ is the power radiated per unit solid angle in the direction (θ, ϕ) corresponding to the spherical coordinate system shown in figure 1.



Location of Point n in Spherical Coordinates

Figure 1.

In equation (1),

$$S(\theta, \phi) = |E(\theta, \phi)|^2 \, g(\theta, \phi) \tag{2}$$

where $E(\theta, \phi)$ represents the field intensity as a function of the array and includes a phase factor corresponding to the space separation between elements in the direction of gain optimization. The function

$g(\theta, \phi)$ represents the power pattern of the reference element of the array and includes an amplitude and phase factor due to the element excitation. The element gain will be normalized to unity in the direction of optimization which is denoted as (θ_o, ϕ_o).

Recent Related Efforts

The problem of determining the maximum possible gain and how it can be achieved for a given number of discrete elements having finite separation has been approached by many investigators. Recent efforts include an article titled "A Mathematical Theory of Antenna Arrays with Randomly Spaced Elements," Y. T. Lo, IEEE Transactions on Antennas and Propagation, Vol. AP-12, May 1964, in which the gain degradation of a randomly spaced array was evaluated. In the same reference R. W. P. King and S. S. Sandler discussed broadside and endfire arrays and developed curves for specific curtain arrays. C. T. Tai wrote an article which appeared in IEEE Transactions on Antennas and Propagation, July, 1964 titled "The Optimum Directivity of Uniformly Spaced Broadside Arrays of Dipoles" in which he developed some gain curves for specific broadside arrays of N elements ranging from N = 3 to N = 20. The paper by Dr. Tai was limited to linear arrays. More recently D. K. Cheng (2) of Syracuse University wrote a paper titled "Gain Optimization for Arbitrary Antenna Arrays" which appeared in IEEE Transactions on Antennas and Propagation in

November, 1965. Dr. Cheng's paper demonstrated for linear arrays that systems in which certain system parameters have significance only at discrete values of a chosen variable may be resolved using a theorem on the properties of a function of a matrix vector.

Objective

The method introduced by Dr. Cheng is particularly interesting in that it can be extended to apply to planar and volumetric arrays as well as the linear arrays demonstrated by Dr. Cheng. This true optimization method is examined in detail in this thesis with generality and application to planar arrays as the goal. The resulting equations have been adapted to a computer program to yield the optimum gain of a given array in a specified direction and the amplitude and phase of excitation required at each element to yield that gain. The program has been utilized to design some basic array configurations and compared with results obtained utilizing classical procedures to establish advantages typical of the method developed in this thesis. The designs were also compared to the linear arrays reported by Dr. Cheng to establish confidence in the program.

Observations

If the amplitude of excitation for certain elements in the array is relatively small some array thinning may be achieved. This application is not investigated in this thesis but is considered an interesting aspect for future investigation. Another area that could benefit from further investigation is the effect and application for roots of the characteristic equation other than λ_1, which is considered in this text, should they be non-zero in a specific problem.

Discussion of Results

The results of this investigation demonstrate that the technique is a true optimization technique for determining the maximum gain in a particular direction that can be achieved with a given basic element and a fixed array configuration. Computation of the amplitude and phase distribution for the array which will result in that maximum gain is achieved within the gain calculation and the design can readily be applied to practical antenna array problems.

The computer program input requirements are minimal and, in most applications, already known for a specific design problem. The classical designs for which the technique described in this thesis was applied demonstrated the agreement of the results with known values. The impressive advantages of the optimization approach already demonstrated by Cheng for certain linear arrays should find direct application for planar and volumetric array designs.

One of the designs reported in Cheng's paper was repeated in this thesis. The results were identical and the program was considered verified for arrays of isotropic elements. Arrays of linear elements are handled in a slightly different way within the program so another comparison was made. One of the arrays reported by Tai was investigated and the results compared well with those obtained by Tai for an array of half-wave dipoles.

Some hypothetical planar and volumetric arrays were computed and the results indicate that the optimization technique can readily be applied to solve real array design problems.

CHAPTER II

ANALYTICAL DEVELOPMENT

Transformation to Matrix Notation

To solve equation (1) by application of the theory of matrices it is necessary to rewrite the equation for gain. The numerator will be examined first.

In equation (2) the field of the nth element of a volumetric array is developed by Stratton (3) as

$$E_n = -j60\, I_{o_n}\, F_o(\theta,\ \phi) \frac{e^{jkR - jwt - jk \vec{R}_o \cdot \vec{r}_n - j\beta_n}}{R}. \qquad (3)$$

*Note: The variable ϕ has been added to the reference element to accommodate elements having variation in both θ and ϕ.

The resultant field intensity is obtained by summing over the entire array.

$$E = -j60\, F_o(\theta,\ \phi) \frac{e^{jkR}}{R} \sum_{n=1}^{N} I_{o_n}\, e^{-j(k \vec{R}_o \cdot \vec{r}_n + \beta_n)}. \qquad (4)$$

R is the radius to the point of observation.

$\vec{R}_o$ is the unit vector in the direction of R.

F_o is the phase factor of the basic element.

I_{o_n} is the magnitude of the current in the nth element.

$k = 2\pi/\lambda$ where λ is the wavelength.

β_n is the phase of the nth element.

$\underset{\rightarrow}{r}_n$ locates the nth element.

In equation (4),

$$\underset{\rightarrow}{R}_o \cdot \underset{\rightarrow}{r}_n = \left|\underset{\rightarrow}{R}_o\right| \left|\underset{\rightarrow}{r}_n\right| \cos \Psi_n, \text{ where } \Psi_n \text{ is defined as the} \qquad (5)$$

angle between $\underset{\rightarrow}{r}_n$ and $\underset{\rightarrow}{R}_o$.

*Note: $\text{Cos } \Psi_n = \sin \theta_o \sin \theta_n \cos (\phi_o - \phi_n) + \cos \theta_o \cos \theta_n$. (6)

Since $\left|\underset{\rightarrow}{R}_o\right| = 1$ and $\left|\underset{\rightarrow}{r}_n\right|$ is just the distance from the origin to element n which will be defined as d_n, the following terms can be defined:

$$I_{o_n} e^{-j \beta_n} \text{ is the nth element excitation} \equiv a_n \qquad (7)$$

where $n = 1, 2, \ldots, N$ and a_n includes the magnitude and phase of element (r_n, θ_n, ϕ_n).

$$D_n \equiv k\, d_n, \text{ and } k \underset{\rightarrow}{R}_o \cdot \underset{\rightarrow}{r}_n = D_n \cos \Psi_n. \qquad (8)$$

$-j\, 60\, F_o(\theta, \phi)$ is the radiation field intensity of the array reference element which, multiplied by its complex conjugate, is the power pattern of the array reference element which will be defined as $g(\theta, \phi)$.

Now, equation (2) becomes

$$S(\theta, \phi) = g(\theta, \phi) \sum_{n=1}^{N} a_n e^{-j D_n \cos \Psi_n} \sum_{m=1}^{N} a_m^* e^{j D_m \cos \Psi_m}$$

where (*) denotes 'transpose conjugate'.

$$S(\theta, \phi) = g(\theta, \phi) \sum_{m=1}^{N} \sum_{n=1}^{N} a_m^* e^{-j(D_n \cos \Psi_n - D_m \cos \Psi_m)} a_n . \quad (9)$$

Power Patterns for Arrays

Substituting equation (6) for cos Ψ_n and cos Ψ_m demonstrates the variation in the power pattern function for various array configurations. The variation is given here for cos Ψ_n only since the variation with cos Ψ_m is identical.

(1) Volumetric Array,

$$\text{Cos } \Psi_n = \sin \theta_o \sin \theta_n \cos (\phi_o - \phi_n) + \cos \theta_o \cos \theta_n . \quad (10)$$

(2) Planar Array, $(\phi_n = \phi_m = 0)$

$$\text{Cos } \Psi_n = \sin \theta_o \sin \theta_n \cos \phi_o + \cos \theta_o \cos \theta_n . \quad (11)$$

(3) Linear Array, $(\theta_n = \theta_m = 0, \phi_n = \phi_m = 0)$

$$\text{Cos } \Psi_n = \cos \theta_o . \quad (12)$$

Since the element power will be normalized to unity in the direction of optimization

$g(\theta_o, \phi_o) = 1$ and, from equation (9), the power pattern of the array is

$$S(\theta_o, \phi_o) = \sum_{m=1}^{N} \sum_{n=1}^{N} a_m^* e^{-j(D_n \cos \Psi_n - D_m \cos \Psi_m)} a_n . \quad (13)$$

All of the constants in the exponent of equation (13) are known inputs for a given array design and α_{mn} can be arbitrarily defined as

$$\alpha_{mn} \equiv e^{-j(D_n \cos \Psi_n - D_m \cos \Psi_m)} . \quad (14)$$

The numerator of equation (1) can now be expressed as

$$S(\theta_o, \phi_o) = \sum_{m=1}^{N} \sum_{n=1}^{N} a_m^* \alpha_{mn} a_n. \quad (15)$$

The denominator of equation (1) will now be examined. Applying equation (9) to the denominator of (1) yields

$$\frac{1}{4\pi} \int_o^{2\pi} \int_o^{\pi} \sum_{m=1}^{N} \sum_{n=1}^{N} a_m^* e^{-j(D_n \cos \Psi_n - D_m \cos \Psi_m)} a_n \; g(\theta, \phi) \sin \theta d\theta d\phi. \quad (16)$$

Define $u \equiv \cos \theta$, $du = -\sin \theta d\theta$. (17)

Therefore, in equation (16),

$$\sin \theta = \sqrt{1 - u^2}$$

so that the exponent of (16) can now be written as

$$e^{-j\left\{\sqrt{1-u^2}\left[D_n \sin \theta_n \cos (\phi-\phi_n) - D_m \sin \theta_m \cos (\phi-\phi_m)\right] + u\left[D_n \cos \theta_n - D_m \cos \theta_m\right]\right\}}. \quad (18)$$

In (18) only the multiplier of $\sqrt{1 - u^2}$ contains a function of ϕ and the definitions

$$f_{mn}(\phi) \equiv \left[D_n \sin \theta_n \cos (\phi-\phi_n) - D_m \sin \theta_m \cos (\phi-\phi_m)\right] \quad (19)$$

and $C_{mn} \equiv D_n \cos \theta_n - D_m \cos \theta_m$ (20)

may be used to simplify (16) which becomes

$$\sum_{m=1}^{N} \sum_{n=1}^{N} a_m^* \left[\frac{1}{4\pi} \int_o^{2\pi} \int_{-1}^{1} e^{-j\left[\sqrt{1-u^2}\, f_{mn}(\phi) + u\, C_{mn}\right]} g(u, \phi)\, du\, d\phi\right] a_n. \quad (21)$$

Let

$$\beta_{mn} \equiv \frac{1}{4\pi}\int_{0}^{2\pi}\int_{-1}^{1} e^{-j\left[\sqrt{1-u^2}\, f_{mn}(\phi) + u\, C_{mn}\right]} g(u, \phi)\, du\, d\phi. \tag{22}$$

The denominator of equation (1) can now be expressed as

$$\frac{1}{4\pi}\int_{0}^{2\pi}\int_{0}^{\pi} S(\theta, \phi)\, d\theta d\phi = \sum_{m=1}^{N}\sum_{n=1}^{N} a^*_m\ \beta_{mn}\ a_n. \tag{23}$$

Combining equations (15) and (23) permits equation (1) to be rewritten as

$$G(\theta_o, \phi_o) = \frac{\sum_{m=1}^{N}\sum_{n=1}^{N} a^*_m\ \alpha_{mn}\ a_n}{\sum_{m=1}^{N}\sum_{n=1}^{N} a^*_m\ \beta_{mn}\ a_n}. \tag{24}$$

Definitions of Matrix Terms

Equation (24) can now be expressed as a product of matrices with the following definitions:

$$\text{The column vector } \underset{\rightarrow}{a} = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_N \end{bmatrix}. \tag{25}$$

$$\text{The row vector } \underset{\rightarrow}{a}^* = \begin{bmatrix} a^*_1 & a^*_2 & \dots & a^*_N \end{bmatrix}. \tag{26}$$

The Hermitian NXN square matrix

$$\bar{A} = \left[\alpha_{mn}\right] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1N} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{N1} & \alpha_{N2} & \cdots & \alpha_{NN} \end{bmatrix}. \quad (27)$$

*Note: Hermitian $\Longrightarrow \alpha_{mn} = \alpha^*_{nm}$.

The Hermitian NXN square matrix

$$\bar{B} = \left[\beta_{mn}\right] = \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1N} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{N1} & \beta_{N2} & \cdots & \beta_{NN} \end{bmatrix}. \quad (28)$$

$$\sum_{m=1}^{N} \sum_{n=1}^{N} a^*_m \, \alpha_{mn} \, a_n = \underset{\rightarrow}{a}^* \, \bar{A} \, \underset{\rightarrow}{a} . \quad (29)$$

$$\sum_{m=1}^{N} \sum_{n=1}^{N} a^*_m \, \beta_{mn} \, a_n = \underset{\rightarrow}{a}^* \, \bar{B} \, \underset{\rightarrow}{a} . \quad (30)$$

Now, $$G(\underset{\rightarrow}{a}) = \frac{\underset{\rightarrow}{a}^* \, \bar{A} \, \underset{\rightarrow}{a}}{\underset{\rightarrow}{a}^* \, \bar{B} \, \underset{\rightarrow}{a}} . \quad (31)$$

Matrix Solution

Matrix theory can be applied to equation (31) to determine the maximum gain that can be achieved with a given array in any specific direction, (θ_o, ϕ_o). Matrix theory will also be used to determine the excitation magnitude and phase required in each element of the array to yield that maximum gain.

Browne (4) and Gantmacher (5) have developed matrix theory applicable to equation (31). The characteristic equation of $G(\underset{\rightarrow}{a})$ is defined as

$$\left| \overline{A} - \lambda_n \overline{B} \right| = 0 \qquad (32)$$

where λ_n are the roots of the characteristic equation.

If $\overline{B}$, which appears in the denominator of (31) is positive definite, the following statements may be made:

(a) The roots ($\lambda_1, \lambda_2, \ldots, \lambda_N$) of the characteristic equation are all real.

(b) λ_1 and λ_N represent the bounds of the value of $G(\underrightarrow{a})$, i.e.

$$\lambda_1 \geq G(\underrightarrow{a}) \geq \lambda_N. \qquad (33)$$

(c) $\lambda_1 \geq G(\underrightarrow{a})$ is attained whenever $\underrightarrow{a}$ satisfies the equation

$$\overline{A}\, \underrightarrow{a} = \lambda_1 \overline{B}\, \underrightarrow{a}. \qquad (34)$$

Now, if λ_1 can be found to satisfy the equality of equation (33), the maximum possible gain for the antenna array under consideration would be determined. Likewise, if $\underrightarrow{a}$ can be found to satisfy equation (34) the current distribution for the array would be determined.

*Note: Statements (a), (b), and (c) above are contingent upon $\overline{B}$ being positive definite. It will not be attempted here to demonstrate, in general, that $\overline{B}$ is positive definite. It will be considered sufficient for the intent of this thesis to demonstrate that $\overline{B}$ is positive for each specific problem investigated. If $\overline{B}$ is not positive definite, the possibility of infinite gain would be introduced which is not reasonable for a finite number of discrete elements.

$\bar{B}$ positive definite implies

$$0 < \sum_{m=1}^{N} \sum_{n=1}^{N} \int_{o}^{2\pi} \int_{-1}^{1} \sin \left[\sqrt{1 - u^2}\, f_{mn}(\phi) + u\, C_{mn} \right] g(u, \phi)\, du\, d\phi. \quad (35)$$

Determination of Maximum Gain

The characteristic equation (32) can be expanded into its polynomial form utilizing the matrix theorem 27.1 on page 68 in Browne (4) for matrices where one is the identity matrix.

Theorem: Let $\bar{C}$ be an n-square matrix with elements in F. If the sum of all m-rowed principal minor determinants of $\bar{C}$ is denoted by σ_m, the characteristic function of $\bar{C}$ is

$$f(\lambda)' = \left| \bar{C} - \lambda I \right| = \sigma_o (-\lambda)^n + \sigma_1 (-\lambda)^{n-1} + \sigma_2 (-\lambda)^{n-2} + \ldots + \sigma_{n-1} (-\lambda)^1 + \sigma_n. \quad (36)$$

$$f(\lambda) = \sum_{m=o}^{n} (-\lambda)^{n-m} \sigma_m, \text{ where } \sigma_o = 1,\ \sigma_n = \left| \bar{C} \right|.$$

Applying the above theorem the characteristic equation

$$\left| \bar{C} - \lambda I \right| = 0, \quad (37)$$

where I is the identity matrix can be expanded to

$$\sigma_o(-\lambda)^n + \sigma_1(-\lambda)^{n-1} + \sigma_2(-\lambda)^{n-2} + \ldots + \sigma_{n-1} (-\lambda)^1 + \sigma_n = 0. \quad (38)$$

If $\bar{C}$ is defined as $\bar{B}^{-1} \bar{A}$ equation (32) can be written as

$\left| \bar{A} - \lambda \bar{B} \right| = \left| \bar{B} \right| \left(\left| \bar{C} - \lambda I \right| \right) = 0$ and the polynomial expansion of (32) would be

$$|\bar{B}|\left[\sigma_o(-\lambda)^n + \sigma_1(-\lambda)^{n-1} + \sigma_2(-\lambda)^{n-2} + \dots - \sigma_{n-1}\lambda\right] + |\bar{A}| = 0. \quad (39)$$

$\sigma_o = 1$ and σ_m, ($m=1, 2, \dots, n-1$), is given by Browne in the above theorem as follows:

σ_m consists of m columns of $\bar{C}$ and n-m columns of I chosen in all possible ways.

Example: Let N = 2.

$$\bar{C} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\sigma_1 = \begin{vmatrix} C_{11} & 0 \\ C_{21} & 1 \end{vmatrix} + \begin{vmatrix} 1 & C_{12} \\ 0 & C_{22} \end{vmatrix} = C_{11} + C_{22},$$

but, $\bar{C}$ was defined as $\bar{B}^{-1}\,\bar{A}$. Therefore,

$$\bar{A} = \bar{B}\;\bar{C}, \text{ or,}$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}.$$

$$\alpha_{11} = \beta_{11} C_{11} + \beta_{12} C_{21}. \quad (40)$$

$$\alpha_{12} = \beta_{11} C_{12} + \beta_{12} C_{22}. \quad (41)$$

$$\alpha_{21} = \beta_{21} C_{11} + \beta_{22} C_{21}. \quad (42)$$

$$\alpha_{22} = \beta_{21} C_{12} + \beta_{22} C_{22}. \quad (43)$$

Simultaneous solution of equations (40) and (42) yields

$$C_{11} = \frac{\begin{vmatrix} \alpha_{11} & \beta_{12} \\ \alpha_{21} & \beta_{22} \end{vmatrix}}{|\bar{B}|} \qquad \text{and}$$

equations (41) and (43) yield

$$C_{22} = \frac{\begin{vmatrix} \beta_{11} & \alpha_{12} \\ \beta_{21} & \alpha_{22} \end{vmatrix}}{|\bar{B}|}$$

The polynomial expansion of $|\bar{A} - \lambda\bar{B}| = 0$ for N = 2 can now be written as

$$\begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix} \lambda^2 - \left\{ \begin{vmatrix} \alpha_{11} & \beta_{12} \\ \alpha_{21} & \beta_{22} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \alpha_{12} \\ \beta_{21} & \alpha_{22} \end{vmatrix} \right\} \lambda + \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = 0. \tag{44}$$

Browne's statement for writing σ_m in connection with the theorem on page 14 can be elaborated on and, perhaps, clarified in light of the previous example.

σ_m is the sum of the determinants consisting of m columns of $\bar{A}$ and n-m columns of $\bar{B}$, (m=0, 1, 2, ... n), chosen in all possible combinations maintaining column correspondence between $\bar{A}$ and $\bar{B}$, i.e., the ith column of $\bar{B}$ is replaced with the ith column of $\bar{A}$.

Example: The coefficients, σ_m, for N=3 can be written directly from the above statement.

$\sigma_o \Rightarrow$ m = 0, n-m = n and the determinant consists only of n columns of $\overline{B}$ and since there is only one possible way to take the columns of $\overline{B}$ without violating column correspondence,

$$\sigma_o = \begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{vmatrix} = |\overline{B}| .$$

$\sigma_1 \Rightarrow$ m=1, n-m = 2 and the sum of the determinants is written as

$$\sigma_1 = \begin{vmatrix} \alpha_{11} & \beta_{12} & \beta_{13} \\ \alpha_{21} & \beta_{22} & \beta_{23} \\ \alpha_{31} & \beta_{32} & \beta_{33} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \alpha_{12} & \beta_{13} \\ \beta_{21} & \alpha_{22} & \beta_{23} \\ \beta_{31} & \alpha_{32} & \beta_{33} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \beta_{12} & \alpha_{13} \\ \beta_{21} & \beta_{22} & \alpha_{23} \\ \beta_{31} & \beta_{32} & \alpha_{33} \end{vmatrix} .$$

Likewise,

$$\sigma_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \beta_{13} \\ \alpha_{21} & \alpha_{22} & \beta_{23} \\ \alpha_{31} & \alpha_{32} & \beta_{33} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \beta_{12} & \alpha_{13} \\ \alpha_{21} & \beta_{22} & \alpha_{23} \\ \alpha_{31} & \beta_{32} & \alpha_{33} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \alpha_{12} & \alpha_{13} \\ \beta_{21} & \alpha_{22} & \alpha_{23} \\ \beta_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} ,$$

and, $\sigma_3 = |\overline{A}|$.

The polynomial can, therefore, be written for any number of terms and, in general, for N = n the coefficients are:

(a) The coefficient of $(-\lambda)^n$,

$$\sigma_o = |\bar{B}| = \begin{vmatrix} \beta_{11} & \beta_{12} \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots \\ \beta_{n1} & \beta_{n2} & \beta_{nn} \end{vmatrix} . \quad (45)$$

(b) The coefficient of $(-\lambda)^\circ$,

$$\sigma_n = |\bar{A}| = \begin{vmatrix} \alpha_{11} & \alpha_{12} \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} \cdots & \alpha_{nn} \end{vmatrix} . \quad (46)$$

(c) The coefficient of $(-\lambda)^{n-1}$,

$$\sigma_1 = \begin{vmatrix} \alpha_{11} & \beta_{12} & \beta_{13} \cdots & \beta_{1n} \\ \alpha_{21} & \beta_{22} & \beta_{23} \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \beta_{n2} & \beta_{n3} \cdots & \beta_{nn} \end{vmatrix} + \begin{vmatrix} \beta_{n} & \alpha_{12} & \beta_{13} \cdots & \beta_{1n} \\ \beta_{21} & \alpha_{22} & \beta_{23} \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \alpha_{n2} & \beta_{n3} \cdots & \beta_{nn} \end{vmatrix}$$

$$+ \ldots + \begin{vmatrix} \beta_{11} & \beta_{12} \cdots & \beta_{1,n-1} & \alpha_{1n} \\ \beta_{21} & \beta_{22} \cdots & \beta_{2,n-1} & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \beta_{22} \cdots & \beta_{n,n-1} & \alpha_{nn} \end{vmatrix} . \quad (47)$$

(d) The coefficient of $(-\lambda)^{n-2}$,

$$\sigma_2 = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \beta_{13} & \beta_{14} & \beta_{15} \cdots & \beta_{1n} \\ \alpha_{21} & \alpha_{22} & \beta_{23} & \beta_{24} & \beta_{25} \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \beta_{n3} & \beta_{n4} & \beta_{n5} \cdots & \beta_{nn} \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_{11} & \beta_{12} & \alpha_{13} & \beta_{14} & \beta_{15} \cdots & \beta_{1n} \\ \alpha_{21} & \beta_{22} & \alpha_{23} & \beta_{24} & \beta_{25} \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \beta_{n2} & \alpha_{n3} & \beta_{n4} & \beta_{n5} \cdots & \beta_{nn} \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_{11} & \beta_{12} & \beta_{13} & \alpha_{14} & \beta_{15} \cdots & \beta_{1n} \\ \alpha_{21} & \beta_{22} & \beta_{23} & \alpha_{24} & \beta_{25} \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \beta_{n2} & \beta_{n3} & \alpha_{n4} & \beta_{n5} \cdots & \beta_{nn} \end{vmatrix}$$

$$+ \ldots + \begin{vmatrix} \alpha_{11} & \beta_{12} & \beta_{13} \cdots & \beta_{1,n-1} & \alpha_{1n} \\ \alpha_{21} & \beta_{22} & \beta_{23} \cdots & \beta_{2,n-1} & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \beta_{n2} & \beta_{n3} \cdots & \beta_{n,n-1} & \alpha_{nn} \end{vmatrix}$$

$$+\begin{vmatrix} \beta_{11} & \alpha_{12} & \beta_{13} & \beta_{14} \cdots & \beta_{1,n-1} & \alpha_{1n} \\ \beta_{21} & \alpha_{22} & \beta_{23} & \beta_{24} \cdots & \beta_{2,n-1} & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \alpha_{n2} & \beta_{n3} & \beta_{n4} \cdots & \beta_{n,n-1} & \alpha_{nn} \end{vmatrix}$$

$$+\ldots+\begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} \cdots & \beta_{1,n-2} & \alpha_{1,n-1} & \alpha_{1n} \\ \beta_{21} & \beta_{22} & \beta_{23} \cdots & \beta_{2,n-2} & \alpha_{2,n-1} & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} \cdots & \beta_{n,n-2} & \alpha_{n,n-1} & \alpha_{nn} \end{vmatrix}. \quad (48)$$

(e) The coefficient of $(-\lambda)^{n-m}$,

$$\sigma_m = \begin{vmatrix} \alpha_{11} & \alpha_{12} \cdots & \alpha_{1m} & \beta_{1,m+1} & \beta_{1,m+2} \cdots & \beta_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} \cdots & \alpha_{nm} & \beta_{n,m+1} & \beta_{n,m+2} \cdots & \beta_{nn} \end{vmatrix}$$

$$+\begin{vmatrix} \alpha_{11} & \alpha_{12} \cdots & \alpha_{1,m-1} & \beta_{1m} & \alpha_{1,m+1} & \beta_{1,m+2} & \beta_{1,m+3} \cdots & \beta_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} \cdots & \alpha_{n,m-1} & \beta_{nm} & \alpha_{n,m+1} & \beta_{n,m+2} & \beta_{n,m+3} \cdots & \beta_{nn} \end{vmatrix}$$

$$+\ldots+\begin{vmatrix} \beta_{11} & \beta_{12} \cdots & \beta_{1,n-m} & \alpha_{1,n-m+1} & \alpha_{1,n-m+2} \cdots & \alpha_{1,n-1} & \alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \beta_{n2} \cdots & \beta_{n,n-m} & \alpha_{n,n-m+1} & \alpha_{n,n-m+2} \cdots & \alpha_{n,n-1} & \alpha_{nn} \end{vmatrix}.$$

(49)

The elements α_{mn} and β_{mn} are defined in equations (14) and (22).

It has been shown that the characteristic polynomial can be written in its general form. The roots of the polynomial, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \ldots \geq \lambda_n$, represent the solution for G, the gain of the array. A discussion of the roots of the characteristic equation can be found in Guillemin (6). Guillemin states that if the rank of the matrix $\overline{A}$ is r, the characteristic equation has exactly r non-zero roots.

Theorem: If $\overline{A}$ contains at least one r-rowed minor determinant that does not vanish, but no non-vanishing (r+1)-rowed minor determinant, $\overline{A}$ is said to be of rank r. If $\overline{A} = 0$, the rank is said to be zero.

The rank of $\overline{A}$ will be determined by application of the preceeding theorem to ascertain the number of non-zero roots.

From the definition of equation (27)

$$\overline{A} = \left[\alpha_{mn}\right] \qquad \text{and,}$$

inserting the values of α_{mn} given in equation (14) the matrix can be written in a form for which the rank can be determined.

Notice that for m=n,

$$\alpha_{mn} = 1 \tag{50}$$

and the off diagonal terms are the complex transpose of each other since $\overline{A}$ is Hermitian.

Laplace's method for expansion of a determinant about a column as developed in Browne (4) can be applied repeatedly to reduce the order of $\bar{A}$ until a sum of 2 x 2 determinants of the form

$$\begin{vmatrix} \alpha_{ii} & \alpha_{ij} \\ \alpha_{ji} & \alpha_{jj} \end{vmatrix} \quad \text{remain for which} \quad \alpha_{11} = \alpha_{22} = \ldots = \alpha_{ii} = \ldots = \alpha_{jj} = \ldots = \alpha_{nn} = 1$$

and,

$$\begin{vmatrix} \alpha_{ii} & \alpha_{ij} \\ \alpha_{ji} & \alpha_{jj} \end{vmatrix} = \begin{vmatrix} 1 & e^{-j(D_j \cos \Psi_j - D_i \cos \Psi_i)} \\ e^{-j(D_i \cos \Psi_i - D_j \cos \Psi_j} & 1 \end{vmatrix} \equiv 0,$$

and all 2-rowed minor determinants vanish. Since no element of $\bar{A}$ is zero there is at least one 1-rowed minor determinant that does not vanish and $\bar{A}$ is of rank 1. Therefore, for n = N, $\bar{A} = \left[\alpha_{mn}\right]$, and $\alpha_{mn} = e^{-j(D_n \cos \Psi_n - D_m \cos \Psi_m)}$ there exists only one non-zero root to equation (32) and it is λ_1, the upper bound of $G(\underrightarrow{a})$.

For $\bar{A}$ of rank 1 it is necessary to determine only the coefficients σ_o and σ_1 in equation (39) since σ_m, (m = 2, 3, ..., N), must necessarily be zero.

σ_o and σ_1 are defined in equations (45) and (47). The solution is greatly simplified by omission of σ_m, $m > 1$, and equation (32) is reduced to

$$|\bar{B}| \; \sigma_o \, (-\lambda)^n + |\bar{B}| \, \sigma_1 \, (-\lambda)^{n-1} = 0. \tag{51}$$

$|\bar{B}| \, (-\lambda)^{n-1}$ can be factored out leaving n-1 roots identically zero and

$$-\sigma_o \lambda_1 + \sigma_1 = 0, \text{ or,}$$

$$\lambda_1 = \frac{\sigma_1}{\sigma_o} . \tag{52}$$

Therefore $\lambda_1 =$

$$\frac{\begin{vmatrix} \alpha_{11} & \beta_{12} & \beta_{13} \cdots \beta_{1N} \\ \alpha_{21} & \beta_{22} & \beta_{23} \cdots \beta_{2N} \\ \cdots & \cdots & \cdots \\ \alpha_{N1} & \beta_{N2} & \beta_{N3} \cdots \beta_{NN} \end{vmatrix} + \begin{vmatrix} \beta_{11} & \alpha_{12} & \beta_{13} & \beta_{14} \cdots \beta_{1N} \\ & & & \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{N1} & \alpha_{N2} & \beta_{N3} \cdots \beta_{NN} & \end{vmatrix} + \ldots + \begin{vmatrix} \beta_{11} \cdots \beta_{1,N-1} & \alpha_{1N} \\ & \\ \cdots & \cdots \\ \beta_{N1} \cdots \beta_{N,N-1} & \alpha_{NN} \end{vmatrix}}{|\bar{B}|} \quad . \qquad (53)$$

The solution of equation (53) satisfies the equality of equation (33) and, therefore, is the upper bound for the gain of the array in the direction (θ_o, ϕ_o).

A more convenient expression for equation (53) may be obtained by defining two new terms.

Let,

$$\underset{\rightarrow}{\epsilon} = \begin{bmatrix} e^{j D_1 \cos \Psi_1} \\ e^{j D_2 \cos \Psi_2} \\ \cdot \\ \cdot \\ \cdot \\ e^{j D_N \cos \Psi_N} \end{bmatrix} . \qquad (54)$$

Let b_{mn} denote the cofactor of β_{mn} to facilitate writing the inverse of $\bar{B}$, ($\bar{B}^{-1}$).

$$\bar{B}^{-1} = \frac{1}{|\bar{B}|} \begin{bmatrix} b_{11} & b_{21} \cdots b_{N1} \\ b_{12} & b_{22} \cdots b_{N2} \\ \cdots & \cdots \\ b_{1N} & b_{2N} \cdots b_{NN} \end{bmatrix}. \qquad (55)$$

$$\bar{B}^{-1} \underset{\rightarrow}{\epsilon} = \frac{1}{|\bar{B}|} \begin{bmatrix} b_{11} \epsilon_1 + b_{21} \epsilon_2 + \cdots + b_{N1} \epsilon_N \\ b_{12} \epsilon_1 + b_{22} \epsilon_2 + \cdots + b_{N2} \epsilon_N \\ \cdots \\ b_{1N} \epsilon_1 + b_{2N} \epsilon_2 + \cdots + b_{NN} \epsilon_N \end{bmatrix}. \qquad (56)$$

$$\underset{\rightarrow}{\epsilon^*} \bar{B}^{-1} \underset{\rightarrow}{\epsilon} = \frac{1}{|\bar{B}|} \Big[b_{11} \epsilon_1 \epsilon^*_1 + b_{21} \epsilon_2 \epsilon^*_1 + \cdots + b_{N1} \epsilon_N \epsilon^*_1$$
$$+ b_{12} \epsilon_1 \epsilon^*_2 + b_{22} \epsilon_2 \epsilon^*_2 + \cdots + b_{N2} \epsilon_N \epsilon^*_2$$
$$+ \cdots + b_{1N} \epsilon_1 \epsilon^*_N + b_{2N} \epsilon_2 \epsilon^*_N + \cdots + b_{NN} \epsilon_N \epsilon^*_N \Big]. \qquad (57)$$

The product $\epsilon_m \epsilon^*_n$ is just α_{mn}.

Therefore, $\underset{\rightarrow}{\epsilon} \underset{\rightarrow}{\epsilon^*} = \left[\alpha_{mn}\right] = \bar{A}$. (58)

Equation (57) can now be written as

$$\underset{\rightarrow}{\epsilon^*} \bar{B}^{-1} \underset{\rightarrow}{\epsilon} = \frac{1}{|\bar{B}|} \Big[b_{11} \alpha_{11} + b_{21} \alpha_{21} + b_{31} \alpha_{31} + \cdots + b_{N1} \alpha_{N1}$$
$$+ b_{12} \alpha_{12} + b_{22} \alpha_{22} + b_{32} \alpha_{32} + \cdots + b_{N2} \alpha_{N2}$$
$$+ \cdots + b_{1N} \alpha_{1N} + b_{2N} \alpha_{2N} + b_{3N} \alpha_{3N} + \cdots + b_{NN} \alpha_{NN} \Big]. \qquad (59)$$

Each line within the brackets of the above equation generates one of the determinants in the numerator of equation (53).

Now, equation (53) becomes

$$\lambda_1 = \underset{\rightarrow}{\epsilon}^* \bar{B}^{-1} \underset{\rightarrow}{\epsilon}, \tag{60}$$

or,

$$G(\underset{\rightarrow}{a}) = \underset{\rightarrow}{\epsilon}^* \bar{B}^{-1} \underset{\rightarrow}{\epsilon}.$$

Determination of Element Excitation

If equation (60) is introduced into equation (34) an $\underset{\rightarrow}{a}$ may be chosen giving the magnitude and phase of excitation required for each element in the array to produce the maximum gain.

$\bar{A}\,\underset{\rightarrow}{a} = \lambda_1 \bar{B}\,\underset{\rightarrow}{a} = \bar{B}\,\underset{\rightarrow}{a}\,\lambda_1$, since λ_1 is a scalor.

Substituting equation (58) for $\bar{A}$ and (60) for λ_1 yields

$$\underset{\rightarrow}{\epsilon}\;\underset{\rightarrow}{\epsilon}^*\underset{\rightarrow}{a} = \bar{B}\,\underset{\rightarrow}{a}\;\underset{\rightarrow}{\epsilon}^* \bar{B}^{-1} \underset{\rightarrow}{\epsilon}.$$

The equation is satisfied if $\underset{\rightarrow}{a}$ is chosen to be $\bar{B}^{-1}\underset{\rightarrow}{\epsilon}$. The optimum excitations in the N elements of the array are, therefore,

$$\underset{\rightarrow}{a} = \bar{B}^{-1} \underset{\rightarrow}{\epsilon}, \tag{61}$$

which is the product shown in equation (56). $\underset{\rightarrow}{a}$ is defined as the column vector

$$\underset{\rightarrow}{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}. \tag{62}$$

a_n is the magnitude and phase of excitation required in element n to yield the maximum gain in the direction of optimization.

CHAPTER III

GENERAL SOLUTION FOR GAIN AND EXCITATION OF SPECIFIC ARRAYS

The solution to equation (60),

$\lambda_1 = \underset{\rightarrow}{\epsilon}^* \bar{B}^{-1} \underset{\rightarrow}{\epsilon}$, can best be obtained utilizing a computer program.

The program consists of a main program, a subroutine for determination of each β_{mn} in the Hermitian $\bar{B}$ matrix and a subroutine for determining the inverse of $\bar{B}$.

Main Program

The main program performs the operations necessary to determine equation (60). Equation (61) is included in the computation as seen by referring to equation (56) and is printed out in the process of computing the maximum gain. The subroutines described in this chapter are called out in the main program. The flow diagram for the main program is shown on page 43. A listing of the computer program and a set of sample problems computed with the program are included in the appendix.

Beta Subroutine

The subroutine for β_{mn} utilizes numerical integration of a double finite integral by application of Simpson's 1/3 rule as described in Salvadori and Baron (7) to solve equation (22).

For generality $g(u, \phi)$ in equation (22) should remain a variable. Solutions have been obtained for some typical power patterns of elements used most frequently.

(a) Isotropic Elements,

$$g(u, \phi) = 1 \quad . \tag{63}$$

(b) Linear Elements,

$$g(u, \phi) = \frac{\cos^2 (\frac{m\pi}{2} u)}{1 - u^2} \text{ , (m - odd),} \tag{64}$$

$$g(u, \phi) = \frac{\sin^2 (\frac{m\pi}{2} u)}{1 - u^2} \text{ , (m - even),} \tag{65}$$

where m is the number of half wavelengths in each element.

For more complex element designs the power pattern function becomes more complex, and it is apparent that a program for computing β_{mn} must be written after $g(u, \phi)$ has been determined.

The value of $g(u, \phi)$ in equations (64) and (65) can be shown by repeated application of l'Hospital's rule to be zero at the limits of integration with respect to u.

To accommodate computation, it is necessary to write the exponential of equation (22) in its trignometric form so that the real and imaginary parts may be integrated separately.

$$\beta_{mn} = \frac{1}{4\pi} \int_{o}^{2\pi} \int_{-1}^{1} e^{-j\left[\sqrt{1 - u^2}\, f_{mn}(\phi) + u\, C_{mn}\right]} g(u, \phi)\, du d\phi.$$

Let

$$f_i = e^{-j\left[\sqrt{1 - u^2}\, f_{mn}(\phi) + u\, C_{mn}\right]} g(u, \phi) .$$

Then,

$$f_i \; Re = \cos\left[\sqrt{1 - u^2}\, f_{mn}(\phi) + u\, C_{mn}\right] g(u, \phi), \text{ and}$$

$$f_i \; Im = - \sin\left[\sqrt{1 - u^2}\, f_{mn}(\phi) + u\, C_{mn}\right] g(u, \phi) .$$

The terms used in the above equation are as defined in the text.

Page 45 is the flow diagram for the β_{mn} subroutine. The functional values to be used in the double numerical integration process are computed in this routine. A separate set of values is computed for the real and imaginary parts for m either odd or even or for isotropic sources. The subroutine calls BINT, the double numerical integration subroutine shown on page 48. A listing for the β_{mn} subroutine and the integration subroutine is given in appendix B.

Inverse of $\overline{B}$ Matrix Subroutine

The Jordan elimination method described by Fox (8) was used to compute $\overline{B}^{-1}$ with complex elements in $\overline{B}$. Page 49 is the flow diagram for the matrix inversion, called CINV, and a listing for the subroutine is included in appendix B. A matrix multiply routine for complex numbers is also included in appendix B and is used both with the inversion subroutine and the main program.

Program Input

The following program array parameters are required to determine the gain and element excitation for the array. The terms defined below are the input terms found in the listing in appendix B. The computer program was written in Fortran IV.

Input Arrangement

Card A: N (Namelist format)

Card B_1: r_1, θ_1, ϕ_1, r_2, θ_2, ϕ_2 (6E12.6 format)

Card B_2: r_3, θ_3, ϕ_3, r_4, θ_4, ϕ_4

.
.
.

Card $B_{n/2}$: r_{n-1}, θ_{n-1}, ϕ_{n-1}, r_n, θ_n, ϕ_n

Card C: Title (12A6 format)

Card D: MKEY, nn, m, λ, θ, ϕ (Namelist format)

Definitions

N = number of elements in the array.

r_i = distance from origin to ith element.

θ_i = θ direction of ith element.

ϕ_i = ϕ direction of ith element.

Title = A description of the problem having an allowable length of 72 characters.

MKEY = 1 if the element lengths are odd multiples of half-wavelengths.

= 2 if the element lengths are even multiples of half-wavelengths.

= 3 if elements are isotropic sources.

nn = number of spaces desired for the double numerical integration routine. (nn must be even).

m = number of wavelengths in an element.

λ = wavelength, (dimension must be compatible with r_i).

θ = θ direction in which maximum gain is desired.

ϕ = ϕ direction in which maximum gain is desired.

Program Output

The program outputs for the array designs computed in connection with this thesis are compiled in appendix C.

The first sheet of the output for each problem is a printout of the input data. The second sheet of the output contains $\underrightarrow{a}$ which is composed of the magnitude and phase of each element in the array. Each term within the parenthesis for $\underrightarrow{a}$ has a real and imaginary part. The normalized amplitudes and phases are listed below $\underrightarrow{a}$ in the same order as the input data are listed so that correlation is maintained between a particular excitation and the specific element requiring that excitation. The second page also contains the solution for the maximum gain that can be achieved with the given array at the optimization angle specified in the input data.

CHAPTER IV

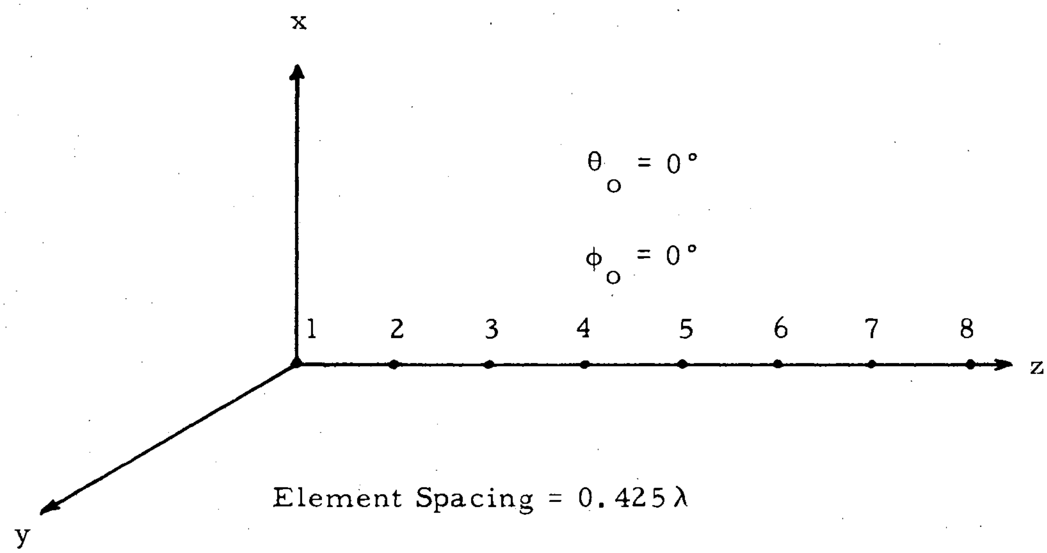
SAMPLE COMPUTATIONS

Comparison with Known Design Values

Initially, the program was used to determine the gain of simple classic arrays to demonstrate the theory and computer routines. A single isotropic element located at the origin, a single half-wave dipole at the origin, two isotropic elements located symmetrically about the origin and separated by a half-wavelength were all tried with complete success.

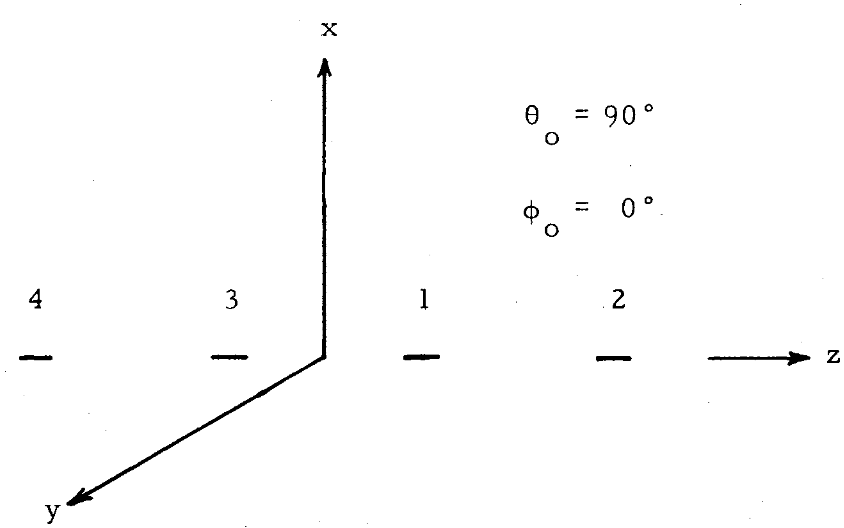
The linear array in figure 2 was examined by Cheng (2) and was evaluated in this program for comparison. The results are shown in appendix C on page 66. Dr. Cheng computed the maximum gain for the eight isotropic element end fire array spaced 0.425 λ apart to be 22. The results of this program show the gain to be 22.1. The gain for the same array with equal amplitude and phase element excitation is 12.5.

To demonstrate the application to arrays of linear elements a design examined by Tai (9) was investigated. The array of four col-linear half-wave dipoles spaced 0.8 wavelengths apart, shown in figure 3, was found by Tai to have a maximum gain of 6.4. The same



Eight Equally Spaced Linear Isotropic Elements

Figure 2.



Four Equally Spaced Half-Wavelength Dipoles

Figure 3.

design using the technique of this thesis yielded a maximum gain of 6.5 as shown on page 68 in appendix C.

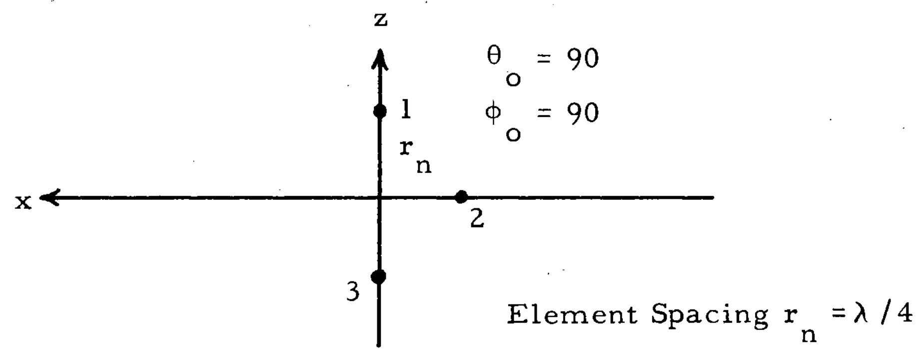
The question of computational errors in the numerical approach utilized in the program is of considerable interest. As in any integration process, the more nearly a function is evaluated at every point on a curve the more accurate will be the answer. In the evaluation of β_{mn} by Simpson's 1/3 rule the number of computations and, consequently, the accuracy is increased by increasing the input nn. The error in the technique for single integration is reported by Salvadori and Baron (7) to be on the order of (H^4). The effect of double integration on that figure could probably be determined in much the same way and should be determined for specific applications where the error control in the result warrants the computation.

Increasing nn significantly increases the computer time required for the routine as the accuracy is increased. It is necessary, therefore, to estimate a reasonable value for nn for which the computation time is acceptable and the error can be considered negligible.

Planar and Volumetric Arrays

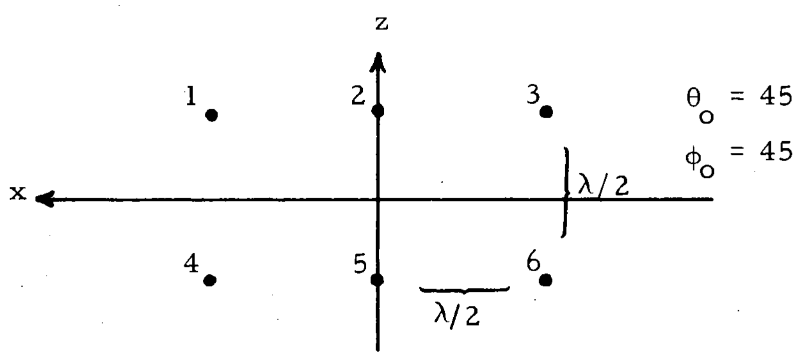
Several planar arrays have been optimized utilizing the technique developed in this thesis. The data are included in appendix C. The arrays reported include the three isotropic element array shown in figure 4, the six element planar array shown in figure 5, and a planar array of linear dipoles.

A volumetric array directive gain optimization was demonstrated for the configuration shown in figure 6. The isotropic elements are located at the corners of a cube. The results are shown in appendix C on page 78.



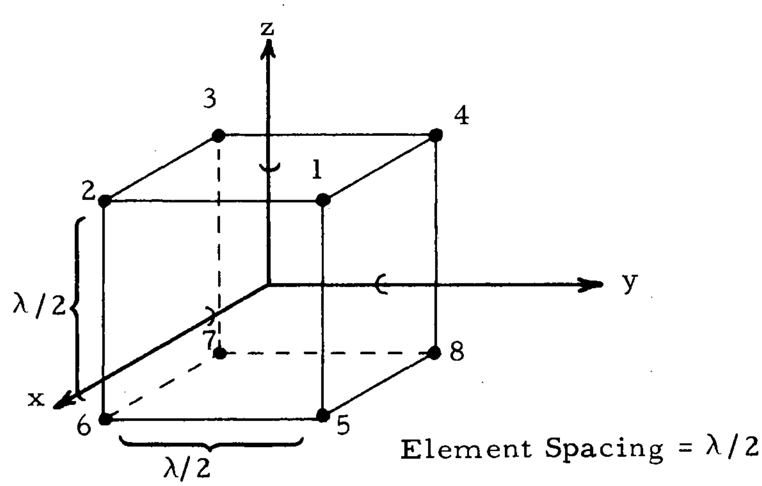
Three Planar Isotropic Elements

Figure 4.



Six Isotropic Element Planar Array

Figure 5.



Eight Isotropic Element Cubic Array

Figure 6.

CHAPTER V

CONCLUSIONS

The advantages of the ability to maximize the directive gain of an antenna array are obvious. Random design is too often unrewarding and empirical design is expensive and time consuming and often results in much less than optimum gain. With a true optimization technique a specific array configuration can be examined and immediately the decision can be made as to whether the design must be changed to meet system requirements.

The maximum gain in any particular direction for a specific array, and the element excitation required to produce that gain are readily determined by the technique described in this thesis. Arrays up to twenty elements can be optimized for basic elements for which the power pattern can be described as linear elements or isotropic elements with no changes in the program. More complex basic elements require a program modification. This limitation is not considered serious, since the large majority of arrays are composed of elements which approach those described.

The theory has been kept as general as possible and linear, planar, and volumetric arrays can be designed with equivalent facility.

The selection of the origin of the coordinate system is completely arbitrary as is the direction of optimization.

The sample computations checked quite satisfactorily with known values reflecting sufficient accuracy in the program. No negative or infinite gains were computed which indicates that for the arrays examined the Hermitian $\bar{B}$ matrix is positive definite.

The gain and current distribution obtained for a particular array in a specified direction gives no indication what effect that excitation would have on the gain in any other direction, nor is there any suggestion for altering various parameters of the array configuration if the gain is inadequate or excessive. Both of these additional design guides would be extremely useful for gain optimization, but their absence does not reflect upon the usefulness of the optimization technique described in this thesis. It merely indicates that this useful tool leaves, in some cases, certain other design problems to be solved in other ways.

The objectives of the investigation have been achieved and a useful gain optimization technique provided which should greatly enhance the solution of many array problems.

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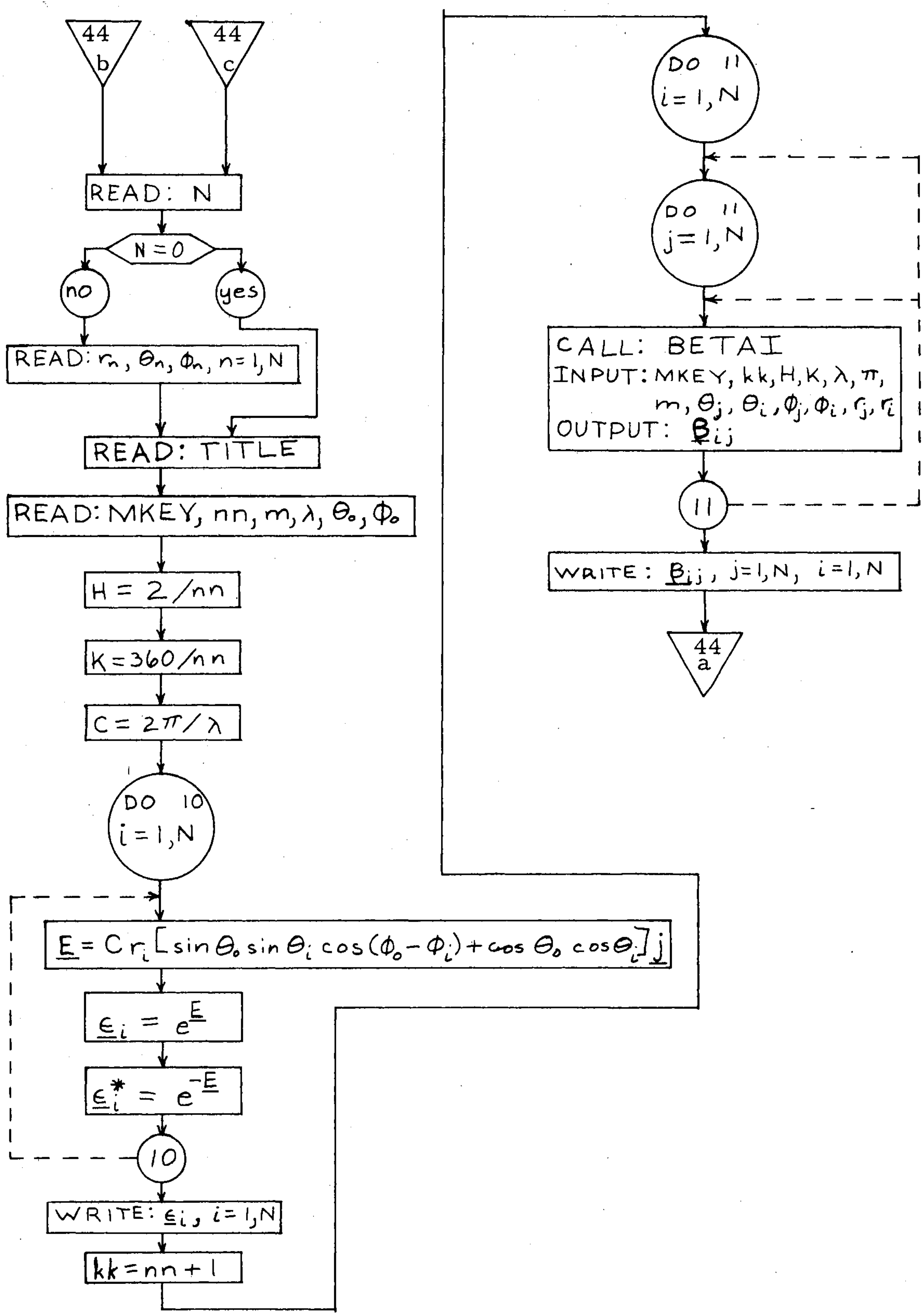
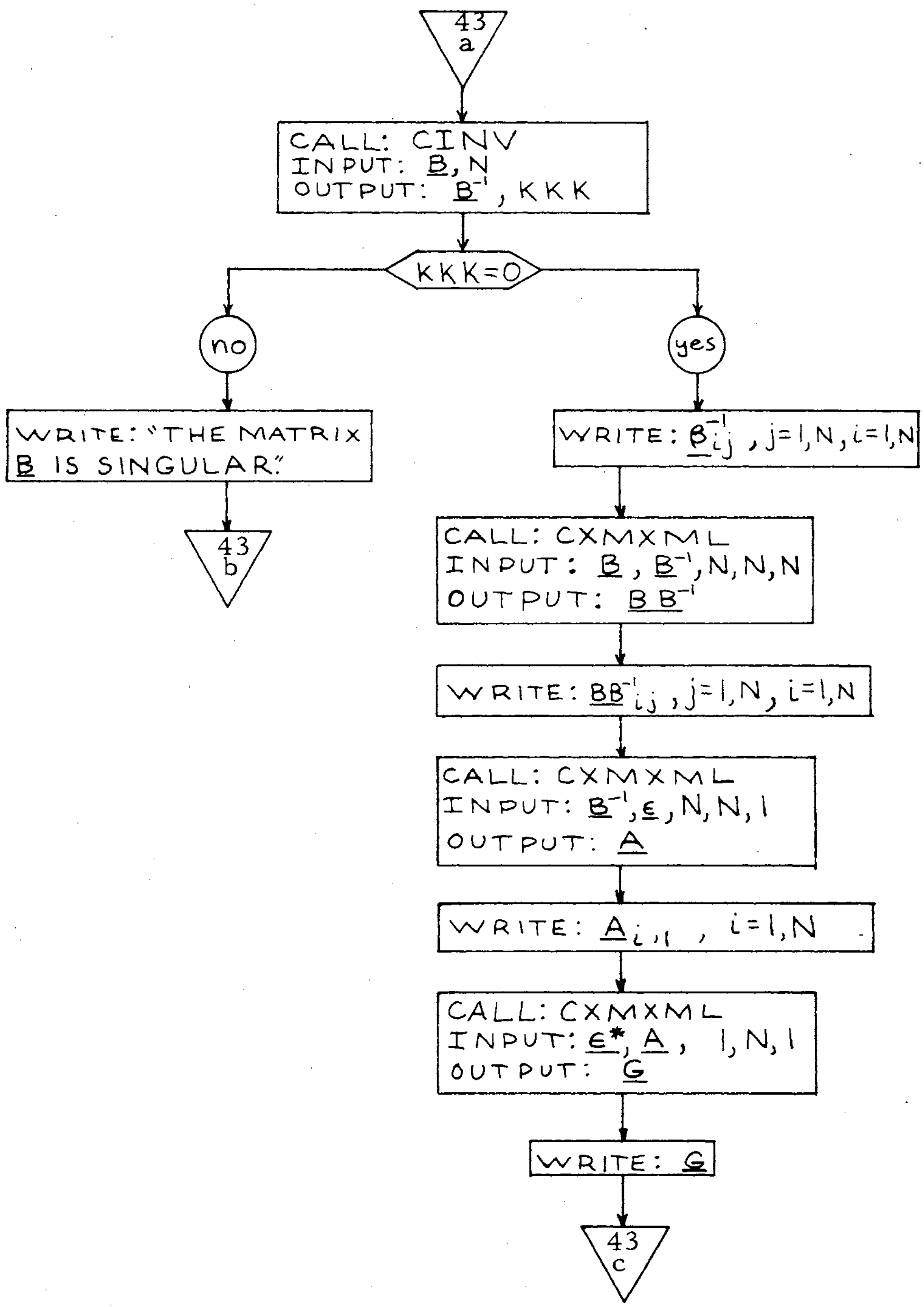
APPENDIX A

The flow diagrams for the various routines used in the computation of the maximum gain and the required element excitation are included in this appendix.

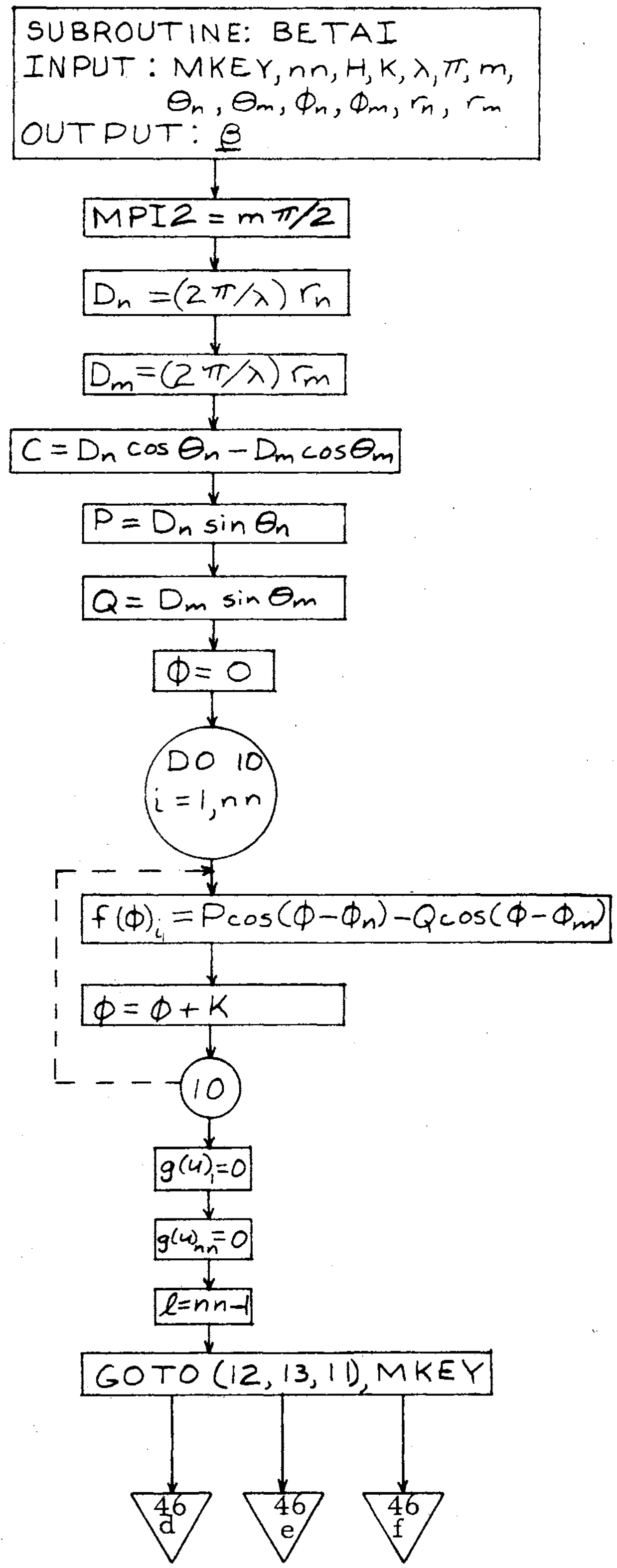
Note: All underscored symbols represent complex numbers.



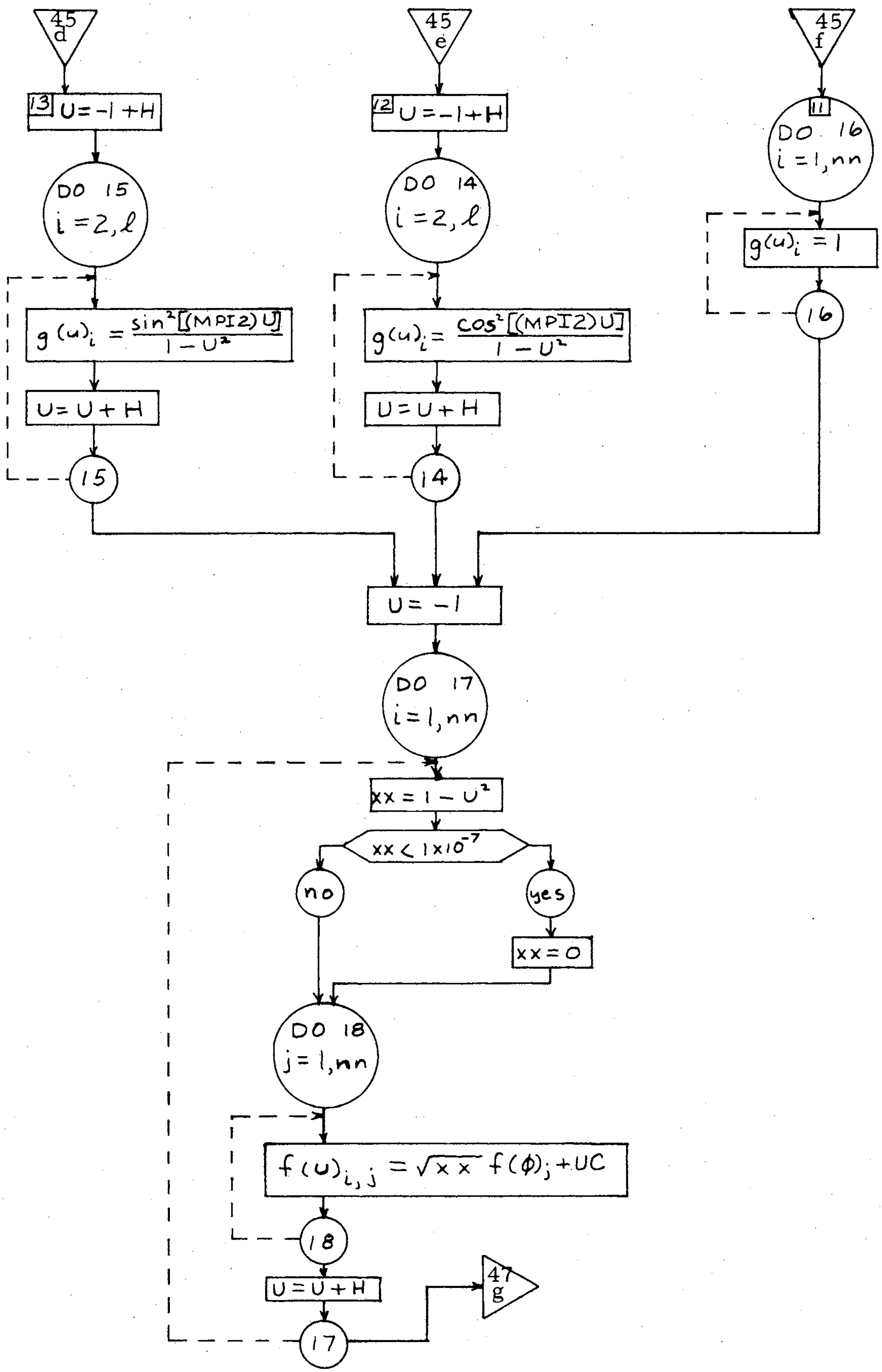
MAIN PROGRAM


43 a
CALL: CINV
INPUT: B, N
OUTPUT: B⁻¹, KKK
KKK=0
no
WRITE: "THE MATRIX B IS SINGULAR."
43 b
yes
WRITE: B⁻¹ij , j=1,N, i=1,N
CALL: CXMXML
INPUT: B, B⁻¹, N, N, N
OUTPUT: BB⁻¹
WRITE: BB⁻¹ij , j=1,N, i=1,N
CALL: CXMXML
INPUT: B⁻¹, ϵ, N, N, 1
OUTPUT: A
WRITE: Ai,1 , i=1,N
CALL: CXMXML
INPUT: ϵ*, A, 1, N, 1
OUTPUT: G
WRITE: G
43 c

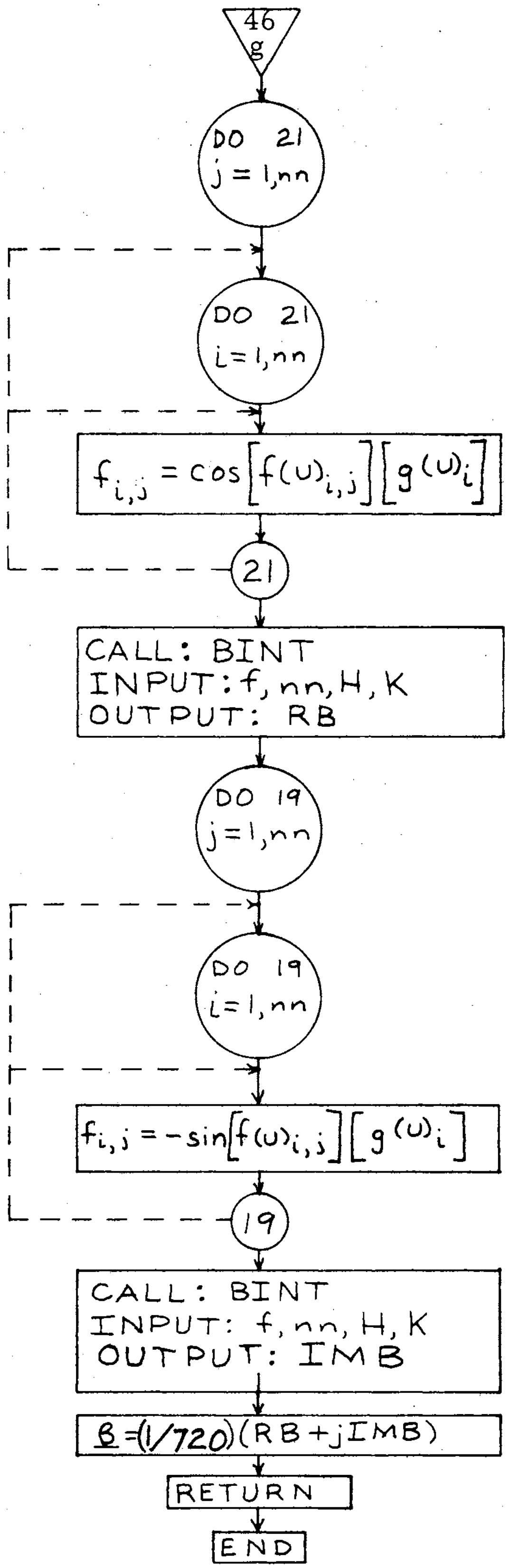
END

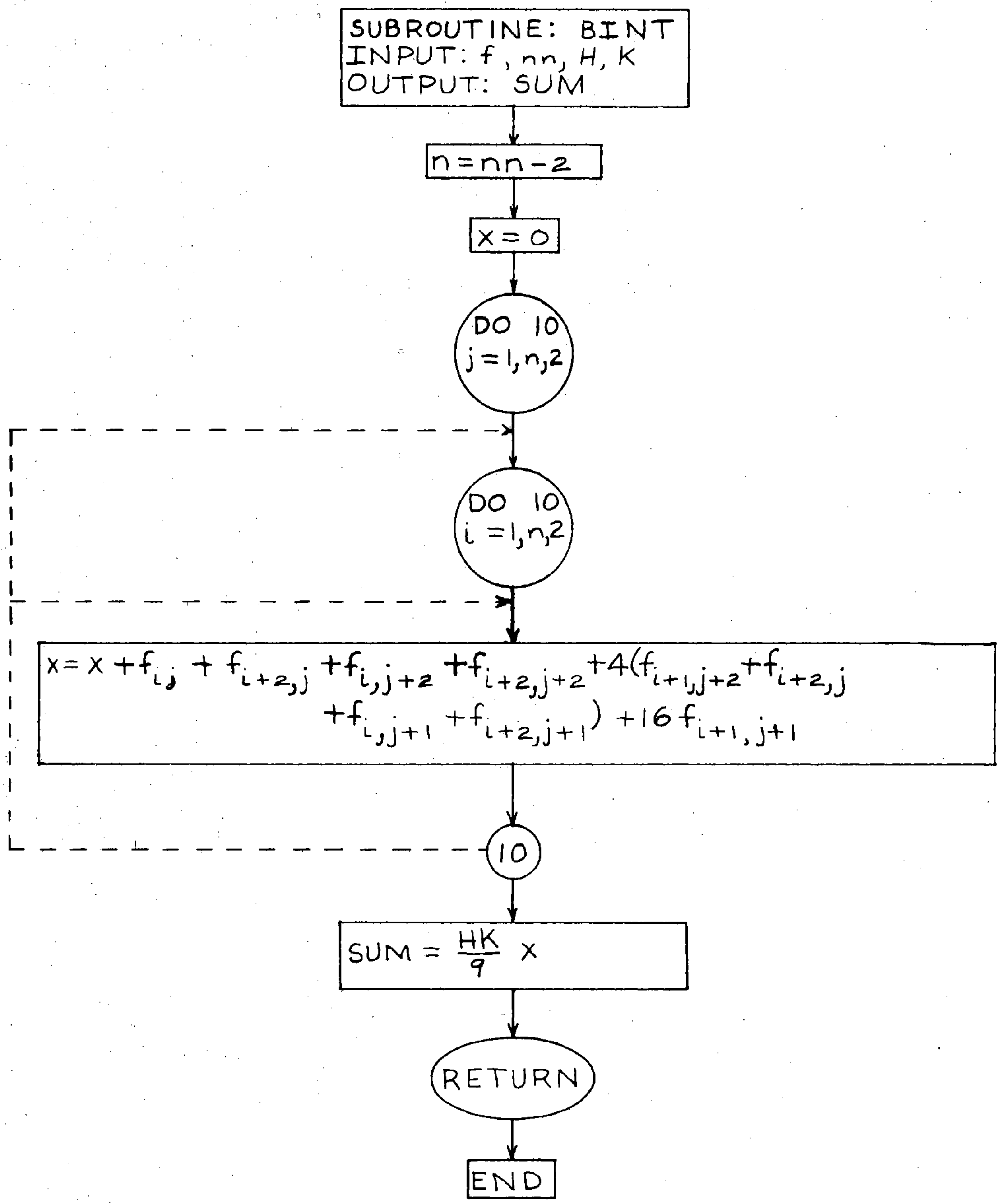


BETA COMPUTATION

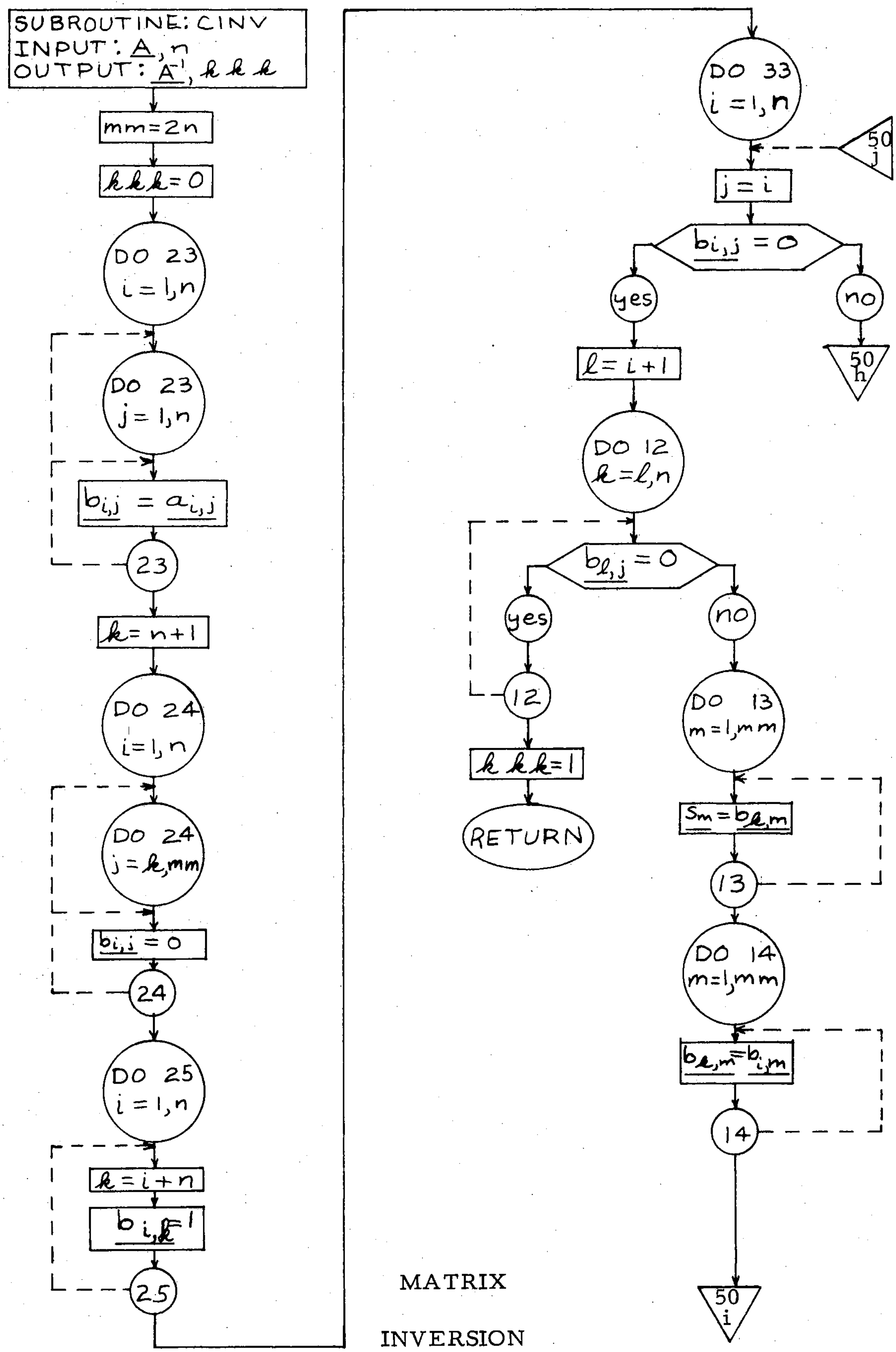


45 d
13 U=-1+H
DO 15 i=2,ℓ
g(u)_i = sin²[(MPI2)U] / 1-U²
U=U+H
15
45 e
12 U=-1+H
DO 14 i=2,ℓ
g(u)_i = cos²[(MPI2)U] / 1-U²
U=U+H
14
45 f
11 DO 16 i=1,nn
g(u)_i = 1
16
U= -1
DO 17 i=1,nn
xx = 1 - U²
xx < 1x10⁻⁷
no
yes
xx = 0
DO 18 j=1,nn
f(u)_i,j = √xx f(φ)_j + UC
18
U=U+H
17
47 g


46 g
DO 21 j = 1,nn
DO 21 i = 1,nn
f_{i,j} = cos[f(U)_{i,j}][g(U)_i]
21
CALL: BINT
INPUT: f, nn, H, K
OUTPUT: RB
DO 19 j = 1,nn
DO 19 i = 1,nn
f_{i,j} = -sin[f(U)_{i,j}][g(U)_i]
19
CALL: BINT
INPUT: f, nn, H, K
OUTPUT: IMB
B = (1/720)(RB + jIMB)
RETURN
END

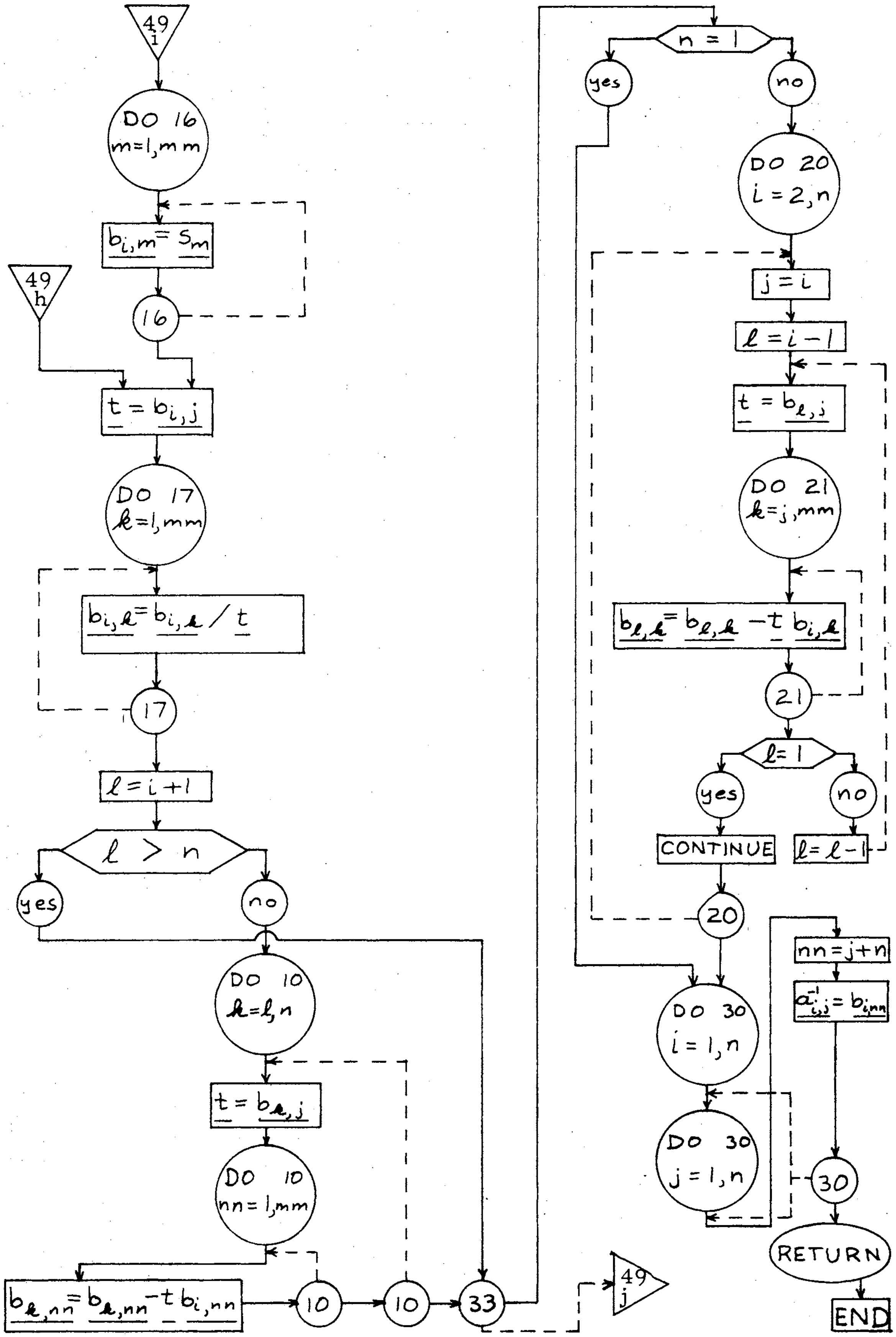


BETA INTEGRATION



MATRIX

INVERSION



49 i
DO 16 m=1,mm
b i,m = s m
49 h
16
t = b i,j
DO 17 k=1,mm
b i,k = b i,k / t
17
l = i+1
l > n
yes
no
DO 10 k=l,n
t = b k,j
DO 10 nn=1,mm
b k,nn = b k,nn − t b i,nn
10
10
33
49 j
n = 1
yes
no
DO 20 i = 2,n
j = i
l = i − 1
t = b l,j
DO 21 k=j,mm
b l,k = b l,k − t b i,k
21
l = 1
yes
no
CONTINUE
l = l − 1
20
DO 30 i = 1,n
DO 30 j = 1,n
nn=j+n
a⁻¹ i,j = b i,nn
30
RETURN
END

APPENDIX B

The listings for the various routines used in the computation of the maximum gain and the required element excitation are included in this appendix.

Note: All underscored symbols represent complex numbers.

MAIN PROGRAM

```
$IBFTC SETUP
      REAL K,M
      COMPLEX E,EP(20,20),EPS(20,20),B(20,20),BINV(20,20),BBINV(20,20),A
     1(20,20),G(1,1)
      DIMENSION R(20),TH(20),PH(20),HEAD(12)
      DIMENSION AA(20),AAT(20)
      NAMELIST/NAM1/N/NAM2/MKEY,NN,M,XLAM,THETA,PHI
1001  FORMAT(6E12.6)
1002  FORMAT(12A6)
1003  FORMAT(1H1//////12A6)
1004  FORMAT(//39X,9HEPSILON (,I3,3H,1))
1005  FORMAT(/(12X,1H(,1PE11.4,1H,,E11.4,1H),2X,1H(,E11.4,1H,,E11.4,1H),
     12X,1H(,E11.4,1H,,E11.4,1H)))
1006  FORMAT(//36X,16HEPSILON STAR (1,,I3,1H))
1007  FORMAT(//41X,3HB (,I3,1H,,I3,1H))
1008  FORMAT(//34X,25HTHE MATRIX B IS SINGULAR.)
1009  FORMAT(//36X,11HB INVERSE (,I3,1H,,I3,1H))
1010  FORMAT(//33X,16HB BY B INVERSE (,I3,1H,,I3,1H))
1011  FORMAT(//41X,3HA (,I3,3H,1))
1012  FORMAT(//40X,5HG = (,1PE11.4,1H,,E11.4,1H))
1013  FORMAT(//36X,1HN,4X,4HMKEY,4X,2HNN,6X,1HM,7X,6HLAMBDA/34X,I3,4X,I2
     1,5X,I3,3X,F6.1,2X,1PE11.4)
1014  FORMAT(//44X,5HTHETA,11X,3HPHI/41X,1PE11.4,5X,E11.4)
1015  FORMAT(//50X,8HELEMENTS//37X,1HR,14X,5HTHETA,11X,3HPHI)
1016  FORMAT(28X,1P3E16.4)
1017  FORMAT(//46X,17HNORMALIZED VALUES//42X,9HAMPLITUDE,9X,5HPHASE)
1018  FORMAT(36X,1P2E16.4)
      PI=3.1415927
13    READ(5,NAM1)
      IF(N.EQ.0) GO TO 63
      READ(5,1001)(R(I),TH(I),PH(I),I=1,N)
63    READ(5,1002) HEAD
      READ(5,NAM2)
```

```
      WRITE(6,1003) HEAD
      WRITE(6,1013) N,MKEY,NN,M,XLAM
      WRITE(6,1014) THETA,PHI
      WRITE(6,1015)
      WRITE(6,1016)(R(I),TH(I),PH(I),I=1,N)
      H=2.0/FLOAT(NN)
      K=360.0/FLOAT(NN)
      C=2.0*PI/XLAM
      DO 10 I=1,N
      EE=C*R(I)*(SIND(THETA)*SIND(TH(I))*COSD(PHI-PH(I))+COSD(THETA)*COS
     1D(TH(I)))
      E=EE*(0.0,1.0)
      EP(I,1)=CEXP(E)
10    EPS(1,I)=CEXP(-E)
      WRITE(6,1003) HEAD
      WRITE(6,1004) N
      WRITE(6,1005)(EP(I,1),I=1,N)
      WRITE(6,1006) N
      WRITE(6,1005)(EPS(1,I),I=1,N)
      KK=NN+1
      DO 11 I=1,N
      DO 11 J=1,N
11    CALL BETAI(MKEY,KK,H,K,XLAM,PI,M,TH(J),TH(I),PH(J),PH(I),R(J),R(I)
     1,B(I,J))
      WRITE(6,1003) HEAD
      WRITE(6,1007) N,N
      DO 14 I=1,N
14    WRITE(6,1005)(B(I,J),J=1,N)
      CALL CINV(B,N,BINV,KKK)
      IF(KKK.EQ.0) GO TO 12
      WRITE(6,1008)
```

```
      GO TO 13
12    WRITE(6,1003) HEAD
      WRITE(6,1009) N,N
      DO 15 I=1,N
15    WRITE(6,1005)(BINV(I,J),J=1,N)
      WRITE(6,1003) HEAD
      CALL CXMXML(B,BINV,BBINV,N,N,N)
      WRITE(6,1010) N,N
      DO 16 I=1,N
16    WRITE(6,1005)(BBINV(I,J),J=1,N)
      CALL CXMXML(BINV,EP,A,N,N,1)
      WRITE(6,1003) HEAD
      WRITE(6,1011) N
      WRITE(6,1005)(A(I,1),I=1,N)
      DO 81 I=1,N
81    AA(I)=CABS(A(I,1))
      XSA=AA(1)
      NKK=1
      DO 82 I=2,N
      IF(AA(I).GE.XSA) GO TO 82
      XSA=AA(I)
      NKK=I
82    CONTINUE
      DO 83 I=1,N
```

```
83    AA(I)=AA(I)/XSA
      DO 84 I=1,N
      YY=AIMAG(A(I,1))
      XX=REAL(A(I,1))
84    AAT(I)=ATAN2(YY,XX)*180.0/PI
      AATX=AAT(NKK)
      DO 85 I=1,N
85    AAT(I)=AAT(I)-AATX
      WRITE(6,1017)
      WRITE(6,1018)(AA(I),AAT(I),I=1,N)
      CALL CXMXML(EPS,A,G,1,N,1)
      WRITE(6,1012) G
      GO TO 13
      END
```

BETA SUBROUTINE

```
$IBFTC BETAIX
      SUBROUTINE BETAI(MKEY,NN,H,K,XLAM,PI ,M,THN,THM,PHN,PHM,RN,RM,BETA)
      DIMENSION F(51,51),FPH(51),FU(51,51),GU(51)
      REAL K,M,MPI2,IMB
      COMPLEX BETA,CN1,CN2,CN3
1000  FORMAT( 1P7E15.4)
      MPI2=M*PI/2.0
      DN=2.0*PI*RN/XLAM
      DM=2.0*PI*RM/XLAM
      C=DN*COSD(THN)-DM*COSD(THM)
      P=DN*SIND(THN)
      Q=DM*SIND(THM)
      PH=0.0
      DO 10 I=1,NN
      FPH(I)=P*COSD(PH-PHN)-Q*COSD(PH-PHM)
10    PH=PH+K
      GU(1)=0.0
      GU(NN)=0.0
      L=NN-1
      GO TO (12,13,11),MKEY
13    U=-1.0+H
      DO 15 I=2,L
      GU(I)=SIN(MPI2*U)**2/(1.0-U**2)
15    U=U+H
      GO TO 20
12    U=-1.0+H
      DO 14 I=2,L
      GU(I)=COS(MPI2*U)**2/(1.0-U**2)
14    U=U+H
      GO TO 20
11    DO 16 I=1,NN
16    GU(I)=1.0
20    U=-1.0
```

```
      DO 17 I=1,NN
      XX=1.0-U**2
      IF(XX.LT.1.0E-07) XX=0.0
      DO 18 J=1,NN
18    FU(I,J)=FPH(J)*SQRT(XX)+U*C
17    U=U+H
      DO 21 J=1,NN
      DO 21 I=1,NN
21    F(I,J)=GU(I)*COS(FU(I,J))
      CALL BINT(F,NN,H,K,RB)
      DO 19 J=1,NN
      DO 19 I=1,NN
19    F(I,J)=-GU(I)*SIN(FU(I,J))
      CALL BINT(F,NN,H,K,IMB)
      RB=RB/720.0
      IMB=IMB/720.0
      BETA=RB
      CN1=(0.0,1.0)
      CN2=IMB
      CN3=CN1*CN2
      BETA=BETA+CN3
      RETURN
      END
```

INTEGRATION SUBROUTINE

```
$IBFTC BINTX
      SUBROUTINE BINT(F,NN,H,K,SUM)
      DIMENSION F(51,51)
      REAL K
      N=NN-2
      X=0.0
      DO 10 J=1,N,2
      DO 10 I=1,N,2
10    X=X+F(I,J)+F(I+2,J)+F(I,J+2)+F(I+2,J+2)+4.0*(F(I+1,J+2)+F(I+1,J)+F
     1(I,J+1)+F(I+2,J+1))+16.0*F(I+1,J+1)
      SUM=H*K*X/9.0
      RETURN
      END
```

MATRIX INVERSION SUBROUTINE

```
$IBFTC INV
      SUBROUTINE CINV(A,N,AINV,KKK)
      COMPLEX A(20,20),AINV(20,20),S(40),B(20,40),T
1000  FORMAT(6E15.4)
      MM=2*N
      KKK=0
      DO 23 I=1,N
      DO 23 J=1,N
23    B(I,J)=A(I,J)
      K=N+1
      DO 24 I=1,N
      DO 24 J=K,MM
24    B(I,J)=(0.0,0.0)
      DO 25 I=1,N
      K=I+N
25    B(I,K)=(1.0,0.0)
      DO 33 I=1,N
      J=I
      IF(CABS(B(I,J)).GT.1.0E-08) GO TO26
      L=I+1
      DO 12 K=L,N
      IF(CABS(B(L,J)).GT.1.0E-08) GO TO27
12    CONTINUE
      KKK=1
      RETURN
27    DO 13 M=1,MM
13    S(M)=B(K,M)
      DO 14 M=1,MM
14    B(K,M)=B(I,M)
      DO 16 M=1,MM
16    B(I,M)=S(M)
26    T=B(I,J)
```

```
      DO 17 K=1,MM
17    B(I,K)=B(I,K)/T
      L=I+1
      IF(L.GT.N) GO TO 33
      DO 10 K=L,N
      T=B(K,J)
      DO 10 NN=1,MM
10    B(K,NN)=B(K,NN)-T*B(I,NN)
33    CONTINUE
      IF(N.EQ.1) GO TO 50
      DO 20 I=2,N
      J=I
      L=I-1
29    T=B(L,J)
      DO 21 K=J,MM
21    B(L,K)=B(L,K)-T*B(I,K)
      IF(L.EQ.1) GO TO 20
      L=L-1
      GO TO 29
20    CONTINUE
50    DO 30 I=1,N
      DO 30 J=1,N
      NN=J+N
30    AINV(I,J)=B(I,NN)
      RETURN
      END
```

COMPLEX MULTIPLICATION SUBROUTINE

```
$IBFTC MULLLL
      SUBROUTINE CXMXML(A,B,C,M,N,L)
      COMPLEX A(20,20),B(20,20),C(20,20)
1001  FORMAT( 1P6E15.4)
      DO 1 J=1,L
      DO 1 I=1,M
1     C(I,J)=(0.0,0.0)
      WRITE(6,1001) C(1,1)
      DO 2 J=1,L
      DO 2 I=1,M
      DO 2 K=1,N
      WRITE(6,1001) C(1,1)
2     C(I,J)=C(I,J)+A(I,K)*B(K,J)
      WRITE(6,1001) C(1,1)
      RETURN
      END
```

APPENDIX C

The program inputs and program outputs are printed in this appendix for various array configurations. The amplitude and phase distribution for the array is printed under A(N, 1) and normalized under the heading "NORMALIZED VALUES." The amplitude and phase order corresponds with the element order printed on the input page. The gain is printed below the distribution. It should be realized that the gain is a real number, and the imaginary part which is quite small compared to the real part is to be neglected.

8 LINEAR ISOTROPIC ELEMENTS

N	MKEY	$\bar{NN}$	M	LAMBDA
8	3	24	......	3.0000E 00

THETA	PHI
0.0000E-39	0.0000E-39

ELEMENTS

R	THETA	PHI
0.0000E-39	0.0000E-39	0.0000E-39
1.2750E 00	0.0000E-39	0.0000E-39
2.5500E 00	0.0000E-39	0.0000E-39
3.8250E 00	0.0000E-39	0.0000E-39
5.1000E 00	0.0000E-39	0.0000E-39
6.3750E 00	0.0000E-39	0.0000E-39
7.6500E 00	0.0000E-39	0.0000E-39
8.9250E 00	0.0000E-39	0.0000E-39

8 LINEAR ISOTROPIC ELEMENTS

A (8,1)

(1.0841E 00,-2.3102E 00)	(-8.7159E-01, 3.2457E 00)	(4.6493E-01,-3.9497E 00)
(6.2834E-02, 4.3091E 00)	(-6.1203E-01,-4.2659E 00)	(1.0771E 00, 3.8283E 00)
(-1.3686E 00,-3.0694E 00)	(1.4322E 00, 2.1122E 00)	(

NORMALIZED VALUES

AMPLITUDE	PHASE
1.0000E 00	-0.0000E-39
1.3169E 00	1.6989E 02
1.5584E 00	-1.8425E 01
1.6887E 00	1.5403E 02
1.6887E 00	-3.3303E 01
1.5584E 00	1.3915E 02
1.3169E 00	-4.9170E 01
1.0000E 00	1.2072E 02

G = (2.2098E 01,-1.1921E-07)

BROADSIDE ARRAY OF LINEAR DIPOLES

N	MKEY	NN	M	LAMBDA
4	1	24	1.0	3.0000E 00

THETA	PHI
9.0000E 01	0.0000E-39

ELEMENTS

R	THETA	PHI
1.2000E 00	0.0000E-39	0.0000E-39
3.6000E 00	0.0000E-39	0.0000E-39
1.2000E 00	1.8000E 02	0.0000E-39
3.6000E 00	1.8000E 02	0.0000E-39

BROADSIDE ARRAY OF LINEAR DIPOLES

A (4,1)

(1.6383E 00, 8.9167E-08) (1.6143E 00, 2.1163E-07) (1.6383E 00,-9.9019E-09)
(1.6143E 00, 1.2513E-08) (

NORMALIZED VALUES

AMPLITUDE	PHASE
1.0148E 00	-4.3927E-06
1.0000E 00	0.0000E-39
1.0148E 00	-7.8575E-06
1.0000E 00	-7.0571E-06

G = (6.5052E 00,-2.5535E-15)

10 ELEMENT BROADSIDE ARRAY

N	MKEY	NN	M	LAMBDA
10	1	24	1.0	3.0000E 00

THETA	PHI
9.0000E 01	0.0000E-39

ELEMENTS

R	THETA	PHI
1.2750E 00	0.0000E-39	0.0000E-39
3.8250E 00	0.0000E-39	0.0000E-39
6.3750E 00	0.0000E-39	0.0000E-39
8.9250E 00	0.0000E-39	0.0000E-39
1.1475E 01	0.0000E-39	0.0000E-39
1.2750E 00	1.8000E 02	0.0000E-39
3.8250E 00	1.8000E 02	0.0000E-39
6.3750E 00	1.8000E 02	0.0000E-39
8.9250E 00	1.8000E 02	0.0000E-39
1.1475E 01	1.8000E 02	0.0000E-39

IO ELEMENT BROADSIDE ARRAY

A (10,1)

(1.6764E 00, 6.2755E-08)	(1.6827E 00, 3.1312E-07)	(2.5454E 00, 9.0557E-07)
(2.6176E 00, 1.2129E-06)	(2.5221E 00, 1.3030E-05)	(1.6764E 00, 8.0613E-10)
(1.6827E 00,-7.7035E-08)	(2.5454E 00,-1.4568E-07)	(2.6176E 00,-2.7034E-07)
(2.5221E 00,-2.9728E-07)	(	

NORMALIZED VALUES

AMPLITUDE	PHASE
1.0000E 00	0.0000E-39
1.0037E 00	8.5172E-06
1.5184E 00	1.8239E-05
1.5615E 00	2.4405E-05
1.5045E 00	2.7457E-05
1.0000E 00	-2.1173E-06
1.0037E 00	-4.7579E-06
1.5184E 00	-5.4240E-06
1.5615E 00	-8.0623E-06
1.5045E 00	-8.8983E-06

G = (2.2088E 01,-0.0000E-39)

3 ELEMENT PLANAR ARRAY

N	MKEY	NN	M	LAMBDA
3	3	24	1.0	3.0000E 00

THETA	PHI
9.0000E 01	9.0000E 01

ELEMENTS

R	THETA	PHI
7.5000E-01	0.0000E-39	0.0000E-39
7.5000E-01	9.0000E 01	1.8000E 02
7.5000E-01	1.8000E 02	0.0000E-39

3 ELEMENT PLANAR ARRAY

A (3,1)

(8.6334E-01,-1.9181E-08) (3.8152E-01, 1.0977E-08) (8.6334E-01, 1.7134E-08)

NORMALIZED VALUES

AMPLITUDE	PHASE
2.2629E 00	-2.9215E-06
1.0000E 00	0.0000E-39
2.2629E 00	-5.1142E-07

G = (2.1082E 00, 8.8818E-16)

2 X 3 PLANAR ARRAY

N	MKEY	NN	M	LAMBDA
6	3	24	1.0	3.0000E 00

THETA	PHI
4.5000E 01	4.5000E 01

ELEMENTS

R	THETA	PHI
1.6771E 00	6.3420E 01	0.0000E-39
7.5000E-01	0.0000E-39	0.0000E-39
1.6771E 00	6.3420E 01	1.8000E 02
1.6771E 00	1.1658E 02	0.0000E-39
7.5000E-01	1.8000E 02	0.0000E-39
1.6771E 00	1.1658E 02	1.8000E 02

2 X 3 PLANAR ARRAY

A (6,1)

(-7.2151E-01, 2.5108E-01)	(4.8574E-01, 1.0001E 00)	(9.1397E-01,-7.3045E-01)
(9.1397E-01, 7.3045E-01)	(4.8574E-01,-1.0001E 00)	(-7.2151E-01,-2.5108E-01)

NORMALIZED VALUES

AMPLITUDE	PHASE
1.0000E 00	0.0000E-39
1.4553E 00	-9.6718E 01
1.5315E 00	-1.9944E 02
1.5315E 00	-1.2218E 02
1.4553E 00	-2.2491E 02
1.0000E 00	-3.2163E 02

G = (6.0257E 00,-4.4703E-08)

2 X 3 PLANAR ARRAY

N	MKEY	NN	M	LAMBDA
6	3	24	1.0	3.0000E 00

THETA	PHI
9.0000E 01	9.0000F 01

ELEMENTS

R	THETA	PHI
1.6771E 00	6.3420E 01	0.0000E-39
7.5000E-01	0.0000E-39	0.0000E-39
1.6771E 00	6.3420E 01	1.8000E 02
1.6771E 00	1.1658E 02	0.0000E-39
7.5000E-01	1.8000E 02	0.0000E-39
1.6771E 00	1.1658F 02	1.8000E 02

2 X 3 PLANAR ARRAY

A (6,1)

(1.2152E 00, 3.5801E-08)	(1.5269E 00, 3.0607E-09)	(1.2152E 00, 4.6050E-08)
(1.2152E 00, 4.1939E-08)	(1.5269E 00, 6.5602E-08)	(1.2152E 00, 6.3516E-08)

NORMALIZED VALUES

AMPLITUDE	PHASE
1.0000E 00	0.0000E-39
1.2565E 00	-1.5731E-06
1.0000E 00	4.8324E-07
1.0000E 00	2.8942E-07
1.2565E 00	7.7375E-07
1.0000E 00	1.3067E-06

G = (7.9148E 00, 4.8850E-15)

8 ELEMENTS-CUBIC ARRAY

N	MKEY	NN	M	LAMBDA
8	3	24	1.0	3.0000E 00

THETA	PHI
9.0000E 01	0.0000E-39

ELEMENTS

R	THETA	PHI
1.0607E 00	4.5000E 01	4.5000E 01
1.0607E 00	4.5000E 01	-4.5000E 01
1.0607E 00	4.5000E 01	-1.3500E 02
1.0607E 00	4.5000E 01	1.3500E 02
1.0607E 00	1.3500E 02	4.5000E 01
1.0607E 00	1.3500E 02	-4.5000E 01
1.0607E 00	1.3500E 02	-1.3500E 02
1.0607E 00	1.3500E 02	1.3500E 02

8 ELEMENTS-CUBIC ARRAY

A (8,1)

(3.8204E-01, 7.3630E-01) (3.8204E-01, 7.3630E-01) (3.8204E-01,-7.3630E-01)
(3.8204E-01,-7.3630E-01) (3.8204E-01, 7.3630E-01) (3.8204E-01, 7.3630E-01)
(3.8204E-01,-7.3630E-01) (3.8204E-01,-7.3630E-01) (

NORMALIZED VALUES

AMPLITUDE	PHASE
1.0000E 00	0.0000E-39
1.0000E 00	-6.8665E-05
1.0000E 00	-1.2515E 02
1.0000E 00	-1.2515E 02
1.0000E 00	4.2915E-06
1.0000E 00	-6.8188E-05
1.0000E 00	-1.2515E 02
1.0000E 00	-1.2515E 02

G = (6.6350E 00,-6.7055E-08)

VITA

Richard Love Moore

Candidate for the Degree of

Master of Science

Thesis: GAIN OPTIMIZATION OF ANTENNA ARRAYS THROUGH MATRIX THEORY

Major Field: Electrical Engineering

Biographical:

Personal Data: Born in Junction, Texas, March 22, 1936, the son of Richard T. and Mildred Irene Moore.

Education: Received the Bachelor of Science degree from the University of Oklahoma in June, 1960, with a major in Electrical Engineering; attended graduate courses at the University of Wichita, Kansas, the University of Tulsa, Oklahoma, and completed the requirements for the Master of Science degree at Oklahoma State University, with a major in Electrical Engineering, in July, 1966.

Professional Experience: Employed in The Boeing Company Antenna Systems Staff from June, 1960 to August, 1962 where basic duties included antenna and microwave measurements and antenna design; employed by North American Aviation, Inc., Space and Information Systems Division since August, 1962 where basic duties include advanced systems design and research pertaining to electromagnetics.

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