THE COUPLED TRANSMISSION LINES

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PREFACE

This thesis presents the theory of interaction or coupling between two parallel, closely spaced single transmission lines. This theory can also be applied to a pair of balanced transmission lines.

This study is part of the research project, "Probabilistic System Analysis," sponsored by Sandia Corporation, under Contract No. 50-7841. The original goal of this part of the project was to investigate the interference or induced voltage on a line among a bundle of lines of which pulses are past on one of the other lines.

In the course of this study, it was found that this investigation of coupled transmission line theory is not only of interest to Sandia Corporation for just knowing how much interference voltage is induced and how this voltage can be reduced, but the theory also has a great potential of application to a large class of transmission problems. For example, the analysis of multimode transmission systems and the analysis of interaction between drifting charged particles and propagating electromagnetic waves. In view of these more important applications, the study made here is beyond Sandia's interest.

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INTRODUCTION

The problem of interference or cross talk among a bundle of lines on which pulses or other signals are passed can be attacked by one of the two following approaches:

1. Field Theory: Starting with Maxwell's equations, two potentials associated with a signal on a line can be set up. Then the electric and magnetic field in that line, neighboring lines, and in the insulation dielectric between the lines can be derived from these two potentials. By using biaxial coordinates, this problem can be solved exactly for two parallel single lines. Unfortunately, this approach can not be applied to three or more lines owing to the complicated boundaries. However, numerical methods with the aid of a digital computer can be used to calculate the field at a finite number of points on a finite number of transverse planes spaced longitudinally along the lines.
2. Circuit Theory: By assuming self-line parameters, R_{ii}, L_{ii}, G_{ii}, C_{ii}, and mutual line parameters, R_{ij}, L_{ij}, G_{ij}, and C_{ij}, the telegraphist's equation can be formulated, then solved by Laplace transform technique.

The calculation of line parameters is a static field problem which is quite involved even for a bundle of a few lines. By introducing a few simplifying assumptions, the line parameters can be evaluated, and the result can be verified by measurements. It is also anticipated that the measurements are theoretically possible but may be very difficult to perform for a bundle of short lines.

The field theory approach is more general and rigorous but except for very simple cases it also is a formidable one. The circuit theory seems completely different from the field theory. By imposing a few restrictions, both King (1) and Pipe (2,3) have derived long line equations from Maxwell's equations. Schelkunoff (4,5) has studied the equivalence of these two approaches in detail for numerous cases. The infinite number of modes of Schelkunoff's generalized telegraphist's equations is due to the distributed coupling between dominant mode and other high order modes. Neglecting this coupling, Schelkunoff's generalized telegraphist's equations reduce to single mode transmission line equations.

The study of propagation of waves along several parallel wires can be traced as far back as 1920 (6,7). The early interest in this problem was in connection with cross-fire and cross-talk of telegraph and telephone circuits. The early work done in this area was directed toward calculating the far end and near end cross-talk coefficient and eliminating the coupling. For open wires a complete and systematic transpose scheme (8,9,10) has been developed. As for cables (11,12) twisted pairs and quads were successfully developed which effectively eliminate cross-talk among a cable with hundreds of thousands of pairs of wire. Matrix algebra (13,14) was introduced for a much simpler and more elegant solution in the late 1930's.

The coupling effect was put into use as coplane coupler (15) which works as filter or suppressor in a transmission line system through which an antenna is fed by two or more transmitters of different frequency simultaneously. The second use made of coupling effect is directional coupler which became a popular subject in the early 1950's. Although the major portion of the work was done on wareguide devices, the transmission line type (16,17) was also investigated.

The coupled mode theory (18,19) initiated by Pierce in 1954 is well appreciated in the study of directional coupler and traveling wave devices. But due to its weakly coupled nature, unfortunately, this well developed theory can not be adopted for this problem.

The approach used in this study is for tightly coupled system and for transient state. It can be easily applied to steady state problem.

Although the line parameters with the presence of all other lines are assumed. No attempt of evaluating these parameters is made. Only two symmetric lines are thoroughly investigated here but the extension to non-symmetric lines can be made with some mathematical difficulties.

The treatment of a transmission line by Laplace transform is quite extensive in the literature (20,21,22,23,24) but the most general case, lossy transmission line terminated at both ends is never treated due to mathematical complications. In Chapter II, the transmission line is studied systematically; use of the result is made in the following chapters. The most general case is also treated. The coupled lossless and lossy lines are studied in Chapter III and Chapter IV respectively. In Chapter V, a more practical problem is investigated.

CHAPTER II

ISOLATED TRANSMISSION LINE

Steady state transmission line theory is a topic in nearly every book dealing with circuit theory or linear systems. The transient state solution, using the Laplace transform approach, appears in many fine books of transient analysis (22-27). However, due to mathematical complication, the most general case, lossy line terminated at both ends, never has been treated before. In this chapter, the lossless line is treated, then treatment of lossy line follows.

Line Equations and Their Solution

A transmission line with line parameters L, C, R, and G is fully described by two first order differential equations:

$$-\frac{\partial V(z,t)}{\partial z} = (L\frac{\partial}{\partial t} + R)I(z,t)$$

$$-\frac{\partial I(z,t)}{\partial z} = (C\frac{\partial}{\partial t} + G)V(z,t) \quad . \tag{1}$$

If initial current and voltage all along the line are both assumed to be zero, i.e.

$$V(z,0) = 0$$

$$I(z,0) = 0 \tag{2}$$

then the Laplace transform with respect to t of Equation 1 is:

$$- \frac{\partial \overline{V}}{\partial z} = (Lp + R)\overline{I}$$

$$- \frac{\partial \overline{I}}{\partial z} = (Cp + G)\overline{V} \quad . \tag{3}$$

On differentiating with respect to z,

$$- \frac{\partial^2 \overline{V}}{\partial z^2} = (Lp + R) \frac{\partial \overline{I}}{\partial z}$$

$$- \frac{\partial^2 \overline{I}}{\partial z^2} = (Cp + G) \frac{\partial \overline{V}}{\partial z} \quad . \tag{4}$$

Then substitution of the values of $\partial V/\partial z$ and $\partial I/\partial z$ of Equation 3 into Equation 4 yields the following uncoupled equations:

$$\frac{\partial^2 \overline{V}}{\partial z^2} = (Lp + R)(Cp + G)\overline{V}$$

$$\frac{\partial^2 \overline{I}}{\partial z^2} = (Lp + R)(Cp + G)\overline{I} \quad . \tag{5}$$

The solutions of Equation 5 are:

$$\overline{V}(z,p) = Ae^{-rz} + Be^{rz}$$

$$\overline{I}(z,p) = Ce^{-rz} + De^{rz} \tag{6}$$

where

$$r = \sqrt{(Lp + R)(Cp + G)} \quad .$$

Substituting Equation 6 into Equation 3, the undetermined constants C and D are found in terms of A and B. The results are

$$C = \sqrt{\frac{Cp + G}{Lp + R}}\, A \quad , \tag{7}$$

$$D = -\sqrt{\frac{Cp + G}{Lp + R}}\,B \quad . \tag{8}$$

Therefore the voltage and current on the line in the "p" domain is

$$\overline{V}(z,p) = Ae^{-rz} + Be^{rz}$$

$$\overline{I}(z,p) = \sqrt{\frac{Cp + G}{Lp + R}}\,Ae^{-rz} - \sqrt{\frac{Cp + G}{Lp + R}}\,Be^{rz} \tag{9}$$

$$= \frac{A}{z_o}e^{-rz} - \frac{B}{z_o}e^{rz} \quad .$$

They may be denoted

$$Z_o(p) = \sqrt{\frac{Lp + R}{Cp + G}} = \sqrt{\frac{L}{C}}\sqrt{\frac{p + \frac{R}{L}}{C + \frac{G}{C}}}$$

(10)

$$r(p) = \sqrt{(Lp + R)(Cp + G)} = \sqrt{LC}\,\sqrt{(p + R/L)(p + G/C)}$$

as the transformed line impedance and propagation constant.

The constants A and B depend on the boundary conditions. If the line is of length ℓ, with a load Z_ℓ and a source of voltage $E_g(t)$ and output impedance Z_g as shown in Figure 1, then the boundary conditions in the "t" domain and "p" domain respectively are:

$$V(o,t) = E_g(t) - I(o,t)Z_g$$

(11)

$$V(\ell,t) = I(\ell,t)Z_\ell$$

$$\overline{V}(o,p) = \overline{E}_g(p) - \overline{I}(o,p)\overline{Z}_g$$

(12)

$$\overline{V}(\ell,p) = \overline{I}(\ell,p)\overline{Z}_\ell \quad .$$

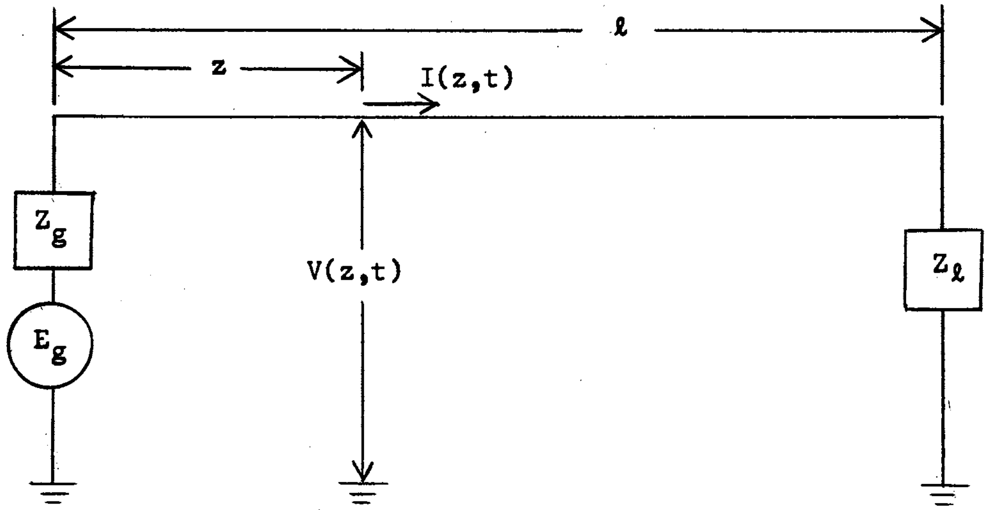

Figure 1. An Isolated Transmission Line With Both Ends Terminated

Substituting the solution in the "p" domain, Equation 9 into Equation 12, and solving for the constants A and B,

$$A = \frac{\bar{E}_g\left(1 + \frac{\bar{Z}_\ell}{\bar{Z}_o}\right)e^{r\ell}}{2\left[\frac{\bar{Z}_g + \bar{Z}_\ell}{\bar{Z}_o}\right]\text{Cosh}(r\ell) + 2\left[1 + \frac{\bar{Z}_g\,\bar{Z}_\ell}{\bar{Z}_o^2}\right]\text{Sinh}(r\ell)},$$

$$B = \frac{\bar{E}_g\left(1 - \frac{\bar{Z}_\ell}{\bar{Z}_o}\right)e^{-r\ell}}{2\left[\frac{\bar{Z}_g + \bar{Z}_\ell}{\bar{Z}_o}\right]\text{Cosh}(r\ell) + 2\left[1 + \frac{\bar{Z}_g\,\bar{Z}_\ell}{\bar{Z}_o^2}\right]\text{Sinh}(r\ell)}.$$

Inserting the above solution into Equation 9 yields the complete solution in the "p" domain.

$$\overline{V}(z,p) = \frac{\overline{E}_g(p)}{2} \frac{\left[1 + \frac{\overline{Z}_\ell}{\overline{Z}_o}\right] e^{r(\ell-z)} + \left[1 - \frac{\overline{Z}_\ell}{\overline{Z}_o}\right] e^{-r(\ell-z)}}{\left[\frac{\overline{Z}_g + \overline{Z}_\ell}{\overline{Z}_o}\right] \text{Cosh}(r\ell) + \left[1 + \frac{\overline{Z}_g \overline{Z}_\ell}{\overline{Z}_o^2}\right] \text{Sinh}(r\ell)}$$

(13)

$$\overline{I}(z,p) = \frac{\overline{E}_g(p)}{2\overline{Z}_o} \frac{\left[1 + \frac{\overline{Z}_\ell}{\overline{Z}_o}\right] e^{r(\ell-z)} - \left[1 - \frac{\overline{Z}_\ell}{\overline{Z}_o}\right] e^{-r(\ell-z)}}{\left[\frac{\overline{Z}_g + \overline{Z}_\ell}{\overline{Z}_o}\right] \text{Cosh}(r\ell) + \left[1 + \frac{\overline{Z}_g \overline{Z}_\ell}{\overline{Z}_o^2}\right] \text{Sinh}(r\ell)}$$

Equation 13 can be rewritten as:

$$\overline{V}(z,p) = \overline{E}_g(p) \frac{\text{Sinh}[r(\ell-z)] + \frac{\overline{Z}_\ell}{\overline{Z}_o} \text{Cosh}[r(\ell-z)]}{\left[\frac{\overline{Z}_g + \overline{Z}_\ell}{\overline{Z}_o}\right] \text{Cosh}(r\ell) + \left[1 + \frac{\overline{Z}_g \overline{Z}_\ell}{\overline{Z}_o}\right] \text{Sinh}(r\ell)}$$

$$= \frac{\overline{E}_g(p)}{\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_o}\right)^2\right]^{\frac{1}{2}}} \quad \frac{\text{Sinh}[r(\ell-z)+X_1]}{\text{Sinh}[r\ell + X_2]}$$

(14)

$$\overline{I}(z,p) = \frac{\overline{E}_g(p)}{\overline{Z}_o} \frac{\text{Cosh}[r(\ell-z)] + \frac{\overline{Z}_\ell}{\overline{Z}_o} \text{Sinh}[r(\ell-z)]}{\left[\frac{\overline{Z}_g + \overline{Z}_\ell}{\overline{Z}_o}\right] \text{Cosh}(r\ell) + \left[1 + \frac{\overline{Z}_g \overline{Z}_\ell}{\overline{Z}_o^2}\right] \text{Sinh}(r\ell)}$$

$$= \frac{\overline{E}_g(p)}{\overline{Z}_o\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_o}\right)^2\right]^{\frac{1}{2}}} \quad \frac{\text{Cosh}[r(\ell-z)+X_1]}{\text{Sinh}[r\ell + X_2]}$$

where

$$\operatorname{Tanh} X_1 = \frac{\overline{Z}_\ell}{\overline{Z}_o}$$

$$\operatorname{Tanh} X_2 = \frac{\dfrac{\overline{Z}_g + \overline{Z}_\ell}{\overline{Z}_o}}{1 + \dfrac{\overline{Z}_g \overline{Z}_\ell}{\overline{Z}_o^{\,2}}} \quad .$$

If $\overline{Z}_g/\overline{Z}_o = 1$, i.e., a complete match at the sending end, Equation 14 reduces to

$$\overline{V}(z,p) = \overline{E}_g(p)\left[\frac{\overline{Z}_o - \overline{Z}_\ell}{\overline{Z}_o + \overline{Z}_\ell}\right]^{\frac{1}{2}} e^{-r\ell} \operatorname{Sinh}[r(\ell - z) + X_1]$$

(14a)

$$\overline{I}(z,p) = \frac{\overline{E}_g(p)}{\overline{Z}_o}\left[\frac{\overline{Z}_o - \overline{Z}_\ell}{\overline{Z}_o + \overline{Z}_\ell}\right]^{\frac{1}{2}} e^{-r\ell} \operatorname{Cosh}[r(\ell - z) + X_1] \quad .$$

If $\overline{Z}_\ell/\overline{Z}_o = 1$, i.e., load impedance matches the line, Equation 14 reduces to

$$\overline{V}(z,p) = \overline{E}_g(p) \frac{\overline{Z}_o}{\overline{Z}_o + \overline{Z}_g} e^{-rz}$$

(14b)

$$\overline{I}(z,p) = \overline{E}_g(p) \frac{1}{\overline{Z}_o + \overline{Z}_g} e^{-rz} \quad .$$

If $\overline{Z}_\ell/\overline{Z}_o = 1 = \overline{Z}_g/\overline{Z}_o$, i.e., both ends of the line are matched, then

$$\overline{V}(z,p) = \frac{\overline{E}_g(p)}{2} e^{-rz}$$

(14c)

$$\overline{I}(z,p) = \frac{\overline{E}_g(p)}{2\overline{Z}_o} e^{-rz} \quad .$$

Using the inversion theorem, from Equation 14, the line voltage and current is:

$$V(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_g(p)}{\left[1-\left(\frac{\overline{Z}_g}{\overline{Z}_o}\right)^2\right]^{\frac{1}{2}}} \frac{\text{Sin}[r(\ell-z)+X_1]}{\text{Sinh}[r\ell + X_2]} e^{pt}\, dp \tag{15}$$

$$I(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_g(p)}{\overline{Z}_o\left[1-\left(\frac{\overline{Z}_g}{\overline{Z}_o}\right)^2\right]^{\frac{1}{2}}} \frac{\text{Cosh}[r(\ell-z)+X_1]}{\text{Sinh}[r\ell + X_2]} e^{pt}\, dp$$

where σ is a real number which is greater than the largest real part of the poles of the integrand.

Terminated Lossless Line

If the line is lossless and both source impedance and load impedance are pure resistive, then

$$Z_o = \sqrt{\frac{L}{C}} = R_o$$

$$V = \sqrt{LC}\, p = \frac{p}{v} \tag{16}$$

$$Z_g = R_g, \quad Z_\ell = R_\ell \quad .$$

For a step voltage:

$$\overline{E}_g(t) = E_o\, U_{-1}(t) = \begin{cases} E_o & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{17}$$

$$\overline{E}_g(p) = \frac{1}{p} \quad .$$

The inversion integrals of voltage assumes the form

$$V(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{E_o}{\left[1-\left(\frac{\overline{Z}_g}{\overline{Z}_o}\right)^2\right]^{\frac{1}{2}}} \frac{\text{Sinh}[r(\ell-z)+X_1]}{\text{Sinh}[r\ell + X_2]} \frac{e^{pt}}{p}\,dp \quad .$$

The integrand is a single value function of p with simple poles at $p = 0$ and $p = \frac{-X_2 + jn\pi}{\ell/v}$, $n = 0, \pm 1, \pm 2, \cdots$.

The residue at $p = 0$ is

$$\left[\frac{E_o \,\text{Sinh}[r(\ell-z) + X_1]e^{pt}}{\left[1-\left(\frac{\overline{Z}_g}{\overline{Z}_o}\right)^2\right]^{\frac{1}{2}} \text{Sinh}[r\ell + X_2]}\right]_{p=0} = \frac{E_o \,\text{Sinh}\, X_1}{\left[1-\left(\frac{\overline{Z}_g}{\overline{Z}_o}\right)^2\right]^{\frac{1}{2}} \text{Sinh}\, X_2}$$

$$= \frac{R_\ell}{R_\ell + R_g} E_o \quad .$$

The residue at $p = \frac{-X_2 + jn\pi}{\ell/v}$ is

$$\left[\frac{E_o \,\text{Sinh}[r(\ell-z) + X_1]e^{pt}}{\frac{d}{dp}\left\{\left[1-\left(\frac{\overline{Z}_g}{\overline{Z}_o}\right)^2\right]^{\frac{1}{2}} p\,\text{Sinh}[r\ell + X_2]\right\}}\right]_{p = \frac{-X_2+jn\pi}{\ell/v}}$$

$$= \frac{E_o\, e^{-X_2 \frac{t}{\ell/v}}}{\left[1-\left(\frac{R_g}{R_o}\right)^2\right]^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{X_2 + jn\pi}{X^2 + n^2\pi^2} \text{Sinh}\left[X_3 - \frac{z}{\ell} X_2 + j\frac{z}{\ell} n\pi\right] e^{jn\pi \frac{t}{\ell/v}} \quad .$$

The voltage along the line is

$$V(z,t) = \frac{R_\ell}{R_\ell + R_g} E_o$$

$$+ \frac{E_o e^{-X_2 \frac{t}{\ell/v}}}{\left[1 - \left(\frac{R_g}{R_o}\right)^2\right]^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{X_2 + jn\pi}{X_2^2 + n^2\pi^2} \operatorname{Sinh}\left[X_3 - \frac{z}{\ell} X_2 + j \frac{z}{\ell} 2n\pi\right] e^{jn\pi \frac{t}{\ell/v}}$$

where

$$X_1 = \operatorname{Tanh}^{-1} \frac{R_\ell}{R_o}$$

$$X_2 = \operatorname{Tanh}^{-1} \left[\frac{\frac{R_g + R_\ell}{R_o}}{1 + \frac{R_g R_\ell}{R_o^2}}\right]$$

$$X_3 = \operatorname{Tanh}^{-1} \left[\frac{R_g}{Z_o}\right] .$$

In a similar manner, evaluating the line current inversion integral yields

$$I(z,t) = \frac{1 + \frac{R_g R_\ell}{R_o^2}}{R_\ell} E_o$$

$$- \frac{E_o e^{-X_2 \frac{t}{\ell/v}}}{R_o\left[1 - \left(\frac{R_g}{R_o}\right)^2\right]^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{X_2 + jn\pi}{X_2^2 + n^2\pi^2} \operatorname{Cosh}\left[X_3 - \frac{z}{\ell} X_2 + j \frac{z}{\ell} n\pi\right] e^{jn\pi \frac{t}{\ell/v}} . \quad (18)$$

These final solutions of line voltage and current have been obtained as an infinite hyperbolic series. No physical interpretation can be made for each term of these series. An alternative method may be developed which gives a solution for which each term can be identified as successively reflected waves.

By substituting Equations 16 and 17 into Equation 13, then rewrite as

$$\overline{V}(z,p) = \frac{E_o}{1 + \frac{R_g}{R_o}} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n e^{-\frac{(2n\ell+z)p}{v}}$$

$$+ \frac{E_o}{1 + \frac{R_g}{R_o}} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n \rho_\ell e^{-\frac{(2n\ell+2\ell-z)p}{v}} \qquad (19)$$

$$\overline{I}(z,p) = \frac{E_o}{R_o + R_g} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n e^{-\frac{(2n\ell+z)p}{v}}$$

$$- \frac{E_o}{R_o + R_g} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n \rho_\ell e^{-\frac{(2n\ell+2\ -z)p}{v}}$$

Recalling the fundamental Laplace transform pair:

$$e^{-ap} \text{-----} U_{-1}(t-a) \quad ,$$

the inverse transforms of Equation 19 can be written as

$$V(z,t) = \frac{E_o}{1 + \frac{R_g}{R_o}} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n U_{-1}\left[t - \frac{2n\ell+z}{v}\right]$$

$$+ \frac{E_o}{1 + \frac{R_g}{R_o}} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n \rho_\ell U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v}\right] \qquad (20)$$

$$I(z,t) = \frac{E_o}{R_o + R_g} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n \, U_{-1}\left[t - \frac{2n\ell+z}{v}\right]$$

$$- \frac{E_o}{R_o + R_g} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n \, \rho_\ell \, U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v}\right]$$

where

$$\rho_\ell = \frac{R_o - R_\ell}{R_o + R_\ell}$$

$$\rho_g = \frac{R_o - R_g}{R_o + R_g}$$

are reflection coefficients at the receiving end and sending end respectively.

In Equation 20, the first series of voltage or current waves are forward waves of which the nth term represents the wave being reflected n times at both ends. The second series are backward waves of which the nth term represents the wave being reflected n times at the sending end and (n + 1) times at the receiving end.

If $\rho_g = \frac{R_o - R_g}{R_o + R_g} = 0$, this is the case for the source impedance equal to the line impedance, then only the first term of each series exist.

$$V(z,t) = \frac{E_o}{1 + \frac{R_g}{R_o}} U_{-1}\left[t - \frac{z}{v}\right] + \frac{E_o}{1 + \frac{R_g}{R_o}} \rho_\ell \, U_{-1}\left[t - \frac{2\ell-z}{v}\right]$$

(20a)

$$I(z,t) = \frac{E_o}{R_o + R_g} U_{-1}\left[t - \frac{z}{v}\right] - \frac{E_o}{R_o + R_g} \rho_\ell \, U_{-1}\left[t - \frac{2\ell-z}{v}\right] .$$

If $\rho_\ell = \frac{R_o - R_\ell}{R_o + R_\ell} = 0$, i.e. the load impedance matches the line, then only the first term of the first series exists.

$$V(z,t) = \frac{E_o}{1 + \frac{R_g}{R_o}} U_{-1}[t - \frac{z}{v}]$$

$$I(z,t) = \frac{E_o}{R_o + R_g} U_{-1}[t - \frac{z}{v}] \quad . \tag{20b}$$

Furthermore, if $\rho_\ell = \rho_g = 0$, then

$$V(z,t) = \frac{E_o}{2} U_{-1}[t - \frac{z}{v}]$$

$$I(z,t) = \frac{E_o}{2R_o} U_{-1}[t - \frac{z}{v}] \tag{20c}$$

The solutions of special cases, Equations 20a, 20b, and 20c, also can be easily obtained from Equations 14a, 14b, and 14c, respectively.

Instead of step voltage, a pulse of amplitude E_o and duration τ is applied to the line. Then $E_g(t)$ assumes the form

$$E_g(t) = E_o\left[U_{-1}(t) - U_{-1}(t-\tau)\right]$$

and (21)

$$\overline{E}_g(p) = E_o \frac{1}{p}\left[1 - e^{-\tau p}\right] \quad .$$

The general solution, Equation 20 assumes the form

$$V(z,t) = \frac{E_o}{1 + \frac{R_g}{R_o}} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v}\right] - U_{-1}\left[t - \frac{2n\ell+z}{v} - \tau\right]\right\}$$

$$+ \frac{E_o}{1 + \frac{R_g}{R_o}} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n \rho_\ell \left\{ U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v}\right] - U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v} - \tau\right]\right\}$$

$$I(z,t) = \frac{E_o}{R_o + R_g} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v}\right] - U_{-1}\left[t - \frac{2n\ell+z}{v} - \tau\right] \right\}$$

$$- \frac{E_o}{R_o + R_g} \sum_{n=0}^{\infty} (\rho_\ell \rho_g)^n \rho_\ell \left\{ U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v}\right] - U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v} - \tau\right] \right\}$$

Corresponding to each term of the solution of a step voltage, there is another term of the same amplitude but opposite polarity and delayed by a time interval τ for the solution of pulse voltage.

Terminated Lossy Line

For the most general case, a lossy line with general load impedance, $\overline{Z}_\ell = R_\ell + L_\ell p + \frac{1}{C_\ell p}$, and general source impedance, $\overline{Z}_g = R_g + L_g p + \frac{1}{C_g p}$, allows Equation 11 to be rewritten as

$$\overline{V}(z,p) = \overline{E}_g(p)\left[\frac{\overline{Z}_o}{\overline{Z}_o + \overline{Z}_g} \sum_{n=0}^{\infty} \left(\overline{\rho}_\ell \overline{\rho}_g\right)^n e^{-r(2n\ell+z)} + \frac{\overline{Z}_o}{\overline{Z}_o + \overline{Z}_g} \sum_{n=0}^{\infty} \left(\overline{\rho}_\ell \overline{\rho}_g\right)^n \overline{\rho}_\ell\, e^{-r(2n\ell+2\ell-z)}\right] \tag{23}$$

$$\overline{I}(z,p) = \overline{E}_g(p)\left[\frac{1}{\overline{Z}_o + \overline{Z}_g} \sum_{n=0}^{\infty} \left(\overline{\rho}_\ell \overline{\rho}_g\right)^n e^{-r(2n\ell-z)} - \frac{1}{\overline{Z}_o + \overline{Z}_g} \sum_{n=0}^{\infty} \left(\overline{\rho}_\ell \overline{\rho}_g\right)^n \overline{\rho}_\ell\, e^{-r(2n\ell+2\ell-z)}\right]$$

where $\overline{\rho}_\ell$ and $\overline{\rho}_g$ are not just a numerical reflection coefficient any more, but a function of p.

By the inversion theorem, the symbolic solutions of the line voltage and current are:

$$V(z,t) = \frac{1}{2\pi j} \sum_{n=0}^{\infty} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}(p)\overline{Z}_o}{\overline{Z}_o + \overline{Z}_g} (\overline{\rho}_\ell \overline{\rho}_g)^n \, e^{-r(2n\ell+z)} e^{pt} \, dp$$

$$+ \frac{1}{2\pi j} \sum_{n=0}^{\infty} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}(p)\overline{Z}_o}{\overline{Z}_o + \overline{Z}_g} (\overline{\rho}_\ell \overline{\rho}_g)^n \, \overline{\rho}_\ell \, e^{-r(2n\ell+2\ell-z)} e^{pt} \, dp \tag{24}$$

$$I(z,t) = \frac{1}{2\pi j} \sum_{n=0}^{\infty} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}(p)}{\overline{Z}_o + \overline{Z}_g} (\overline{\rho}_\ell \overline{\rho}_g)^n \, e^{-r(2n\ell+z)} e^{pt} \, dp$$

$$- \frac{1}{2\pi j} \sum_{n=0}^{\infty} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}(p)}{\overline{Z}_o + \overline{Z}_g} (\overline{\rho}_\ell \overline{\rho}_g)^n \, \overline{\rho}_\ell \, e^{-r(2n\ell+2\ell-z)} e^{pt} \, dp \quad .$$

There is no real difficulty in evaluating the inversion integrals although it is a rather complicated process and it is difficult to express the results explicitly.

All integrands of Equation 24 can be expressed in the following form.

$$\overline{E}_g(p)\left[\overline{U}(p) + \overline{Z}_o(p)\overline{W}(p)\right] e^{-y\overline{v}(p)} e^{pt}$$

where

$$y = (2n\ell+z) \quad \text{or} \quad (2n\ell+2\ell-z) \quad .$$

$\overline{U}(p)$ and $\overline{W}(p)$ are rational algebraic functions of p and can be written in the form

$$\overline{Q}(p) + \frac{\overline{R}(p)}{\overline{S}(p)} \quad .$$

$\overline{Q}(p)$, $\overline{R}(p)$, and $\overline{S}(p)$ are polynomials of p. $\overline{R}(p)$ is of a lower degree than $\overline{S}(p)$.

For example the integrand of the first integral of V(z,t) of Equation 24 is

$$\frac{\overline{E}(p)\overline{Z}_o}{\overline{Z}_o + \overline{Z}_g} \left[\frac{\overline{Z}_o - \overline{Z}_\ell}{\overline{Z}_o + \overline{Z}_\ell} \; \frac{\overline{Z}_o - \overline{Z}_g}{\overline{Z}_o + \overline{Z}_g} \right]^n e^{-\overline{r}(2n\ell+z)} e^{pt}$$

which can be written as

$$\overline{E}(p) \frac{\overline{Z}_o(Z_o - Z_g)}{\overline{Z}_o^2 - \overline{Z}_g^2} \left[\frac{(\overline{Z}_o - \overline{Z}_\ell)^2}{\overline{Z}_o^2 - \overline{Z}_\ell^2} \; \frac{(\overline{Z}_o - \overline{Z}_g)^2}{\overline{Z}_o^2 - \overline{Z}_g^2} \right]^n e^{-r(2n\ell+z)} e^{pt}$$

$$= \overline{E}(p)\left[\overline{U}(p) + \overline{Z}_o\overline{W}(p)\right] e^{-y\overline{r}(p)} e^{pt}$$

$$= \overline{E}(p)\overline{U}(p)e^{-y\overline{r}(p)} e^{pt} + \overline{E}(p)\overline{W}(p)\overline{Z}_o e^{-y\overline{r}(p)} e^{pt}$$

$$= \overline{E}_g(p)\overline{U}(p)e^{-y\sqrt{LC}\;\sqrt{(p+R/L)(p+G/C)}} e^{pt}$$

$$+ \overline{E}_g(p)\overline{W}(p)\sqrt{\frac{L}{C}}\;\sqrt{\frac{p+R/L}{p+G/C}}\; e^{-y\sqrt{LC}\;\sqrt{(p+R/L)(p+G/C)}} e^{pt} \quad .$$

The evaluation of integrals with integrands of the above type is a routine task once the closed contour, Figure 2, has been properly chosen. This takes care of the finite number of poles associated with $\overline{E}_g(p)\overline{U}(p)$ or $\overline{E}_g(p)\overline{W}(p)$ and the two branch points on the negative real axis, $Z_1 = -R/L$ and $Z_2 = -G/C$. The $\overline{E}_g(p)$, Laplace transform of time function of applied pulse, for most cases is quite simple and would not cause any difficulty.

An alternate way to write a general term of Equation 3 is

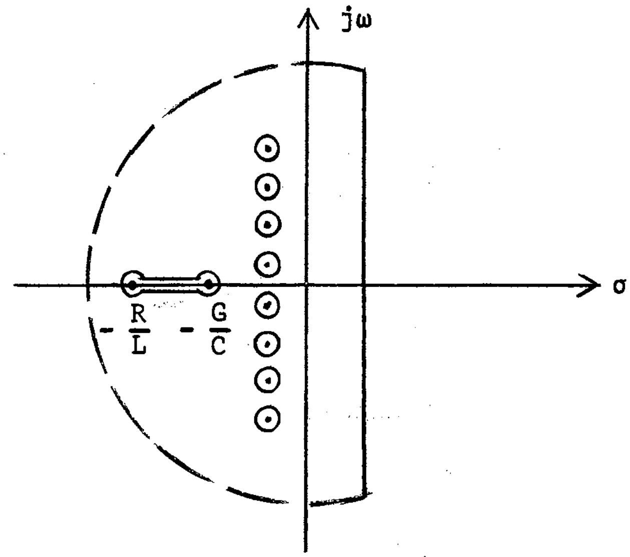

Figure 2. The Contour Used For the Special Type Integrands Discussed in the Text

$$\bar{E}_g(p)\bar{U}(p)e^{-y\sqrt{LC}\sqrt{(p+R/L)(p+G/C)}}$$

$$+ \bar{E}_g(p)\bar{W}(p)\sqrt{\frac{L}{C}}\sqrt{\frac{p+R/L}{p+G/C}}\, e^{-y\sqrt{LC}\sqrt{(p+R/L)(p+G/C)}}$$

$$= \bar{U}'(p)\,\frac{1}{p}\,e^{-y\sqrt{LC}\sqrt{(p+R/L)(p+G/C)}}$$

$$+ \bar{W}'(p)\,\frac{1}{p}\sqrt{\frac{p+R/L}{p+G/C}}\, e^{-y\sqrt{LC}\sqrt{(p+R/L)(p+G/C)}}$$

where

$$\bar{U}'(p) = p\,\bar{E}_g(p)\bar{U}(p)$$

$$\bar{W}'(p) = \sqrt{\frac{L}{C}}\;p\,\bar{E}_g(p)\bar{W}(p) \quad .$$

Since the inverse Laplace transform of

$$\overline{\phi}_1(p) = \frac{1}{p}\, e^{-y\sqrt{LC}\ \sqrt{(p+R/L)(p+G/C)}} ,$$

and

$$\overline{\phi}_2(p) = \frac{1}{p}\sqrt{\frac{p+R/L}{p+G/C}}\, e^{-y\sqrt{LC}\ \sqrt{(p+R/L)(p+G/C)}}$$

are known to be (24,27),

$$\phi_1(t) = e^{\frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right)y\sqrt{LC}}\ U_{-1}(t - y\sqrt{LC})$$

$$+ \frac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right)y\sqrt{LC}\int_{y\sqrt{LC}}^{t} e^{-\frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right)t}\ \frac{I_1\left[\frac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right)\sqrt{t^2 - y^2LC}\right]}{\sqrt{t^2 - y^2LC}}\,dt ,$$

$$\phi_2(t) = e^{-\frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right)}\ I_o\left[\frac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right)\sqrt{t^2 - y^2LC}\right]$$

$$+ \frac{G}{C}\int_{y\sqrt{LC}}^{t} e^{-\frac{1}{2}\left(\frac{R}{L} + \frac{G}{C}\right)}\ I_o\left[\frac{1}{2}\left(\frac{R}{L} - \frac{G}{C}\right)\sqrt{t^2 - y^2LC}\right]dt$$

$$\text{for } t > y\sqrt{LC} > 0$$

where I_o and I_1 are modified Bessel functions.

Therefore the Laplace inverse transform of this particular term is

$$\int_o^t U'(\xi)\phi_1(t - \xi)d\xi + \int_o^t W'(\xi)\phi_2(t - \xi)d\xi$$

or

$$\int_o^t U'(t - \xi)\phi_1(\xi)d\xi + \int_o^t W'(t - \xi)\phi_2(\xi)d\xi$$

where $U'(t)$ and $W'(t)$ are the inverse Laplace transforms of $\overline{U}'(p)$ and $\overline{W}'(p)$.

This gives the formal solution for the most general case. For practical problems, if the line parameters, L, C, R, G, and the terminal impedances, Z_ℓ and Z_g are given, by following the above scheme, one can solve these problems much easier than it appears at first sight. In particular, for transmission problems, where the resistive part of the terminal impedances R_ℓ and R_g are made equal to or compatible with the squared ratio $\sqrt{L/C}$ of the line, then only the first few terms of each series need to be calculated.

The condition, $R_\ell = R_g = \sqrt{L/C}$ represents the best matching condition for a lossy transmission line.

CHAPTER III

COUPLED LOSSLESS TRANSMISSION LINES

The so called telegraphist's equation in matrix form for n-parallel transmission lines as shown in Figure 3 are

$$
\begin{aligned}
-\frac{\partial}{\partial z}[V] &= [R][I] + [L]\frac{\partial}{\partial t}[I] = [Z][I] \\
-\frac{\partial}{\partial z}[I] &= [G][V] + [C]\frac{\partial}{\partial t}[V] = [Y][V]
\end{aligned}
\tag{25}
$$

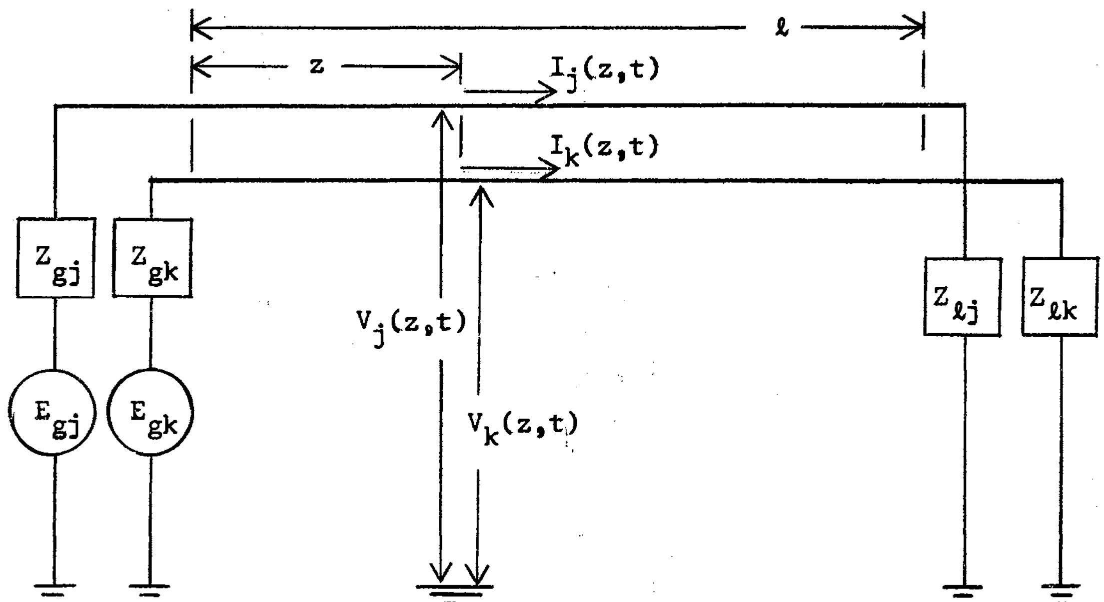

Figure 3. A N-Parallel Transmission Lines System

where [V] and [I] are column matrices of order n; their elements, $V_k(z,t)$

and $I_k(z,t)$ are the voltage and current of kth line at distance z from the sending end and at the time t. The matrices of [R], [L], [C], and [G] are non-singular, square matrices of order n; their elements are self or mutual line parameters.

R_{ii} = resistance of the ith line per unit length.

L_{ii} = inductance of the ith line per unit length.

C_{ii} = capacitance of the ith line per unit length.

G_{ii} = conductance of the ith line per unit length.

L_{ij} = coupling inductance between the ith and jth line per unit length.

C_{ij} = coupling capacitance between the ith and jth line per unit length.

All self and coupling parameters are distributed quantities. The lumped parameter analogy is shown in Figure 4.

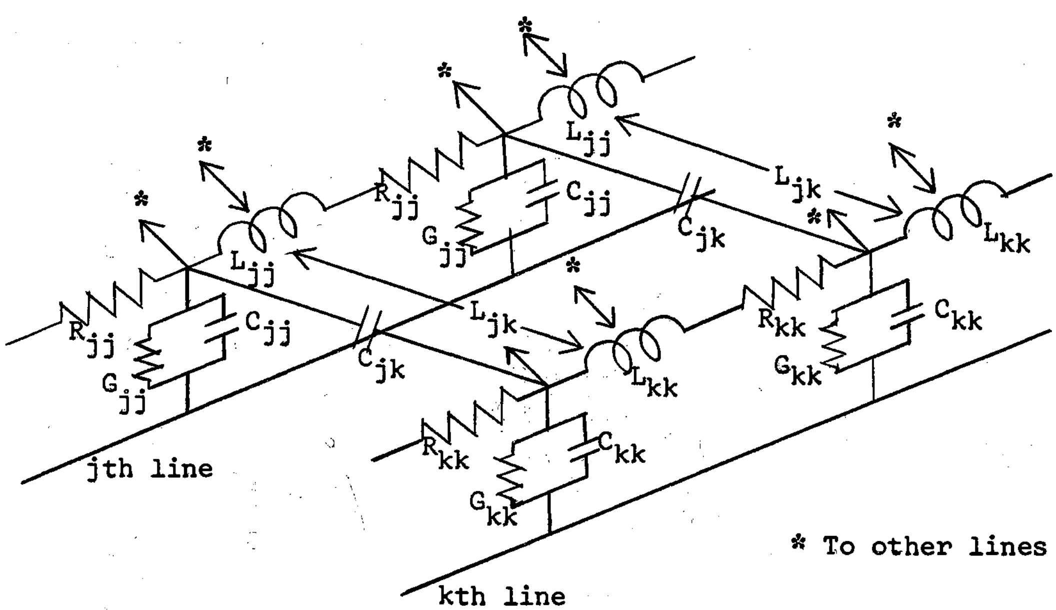

Figure 4. The Lumped Parameters Analogy of a N-Parallel Transmission Lines System

Two Symmetrical Lossless Lines

Grounded at Both Ends

In this thesis, only the two line system is investigated. For a two line system Equation 25 reduces to:

$$- \frac{\partial V_1}{\partial z} = Z_{11}I_1 + Z_{12}I_2$$

$$- \frac{\partial V_2}{\partial z} = Z_{21}I_1 + Z_{22}I_2$$

$$- \frac{\partial I_1}{\partial z} = Y_{11}I_1 - Y_{12}I_2 \qquad (26)$$

$$- \frac{\partial I_2}{\partial z} = -Y_{21}I_1 + Y_{22}I_{22}$$

Equation 26 is a set of simultaneous, partial differential equations. The transformed equations are:

$$\frac{d\overline{V}_1}{dz} + \overline{Z}_{11}\overline{I}_1 + \overline{Z}_{12}\overline{I}_2 = 0$$

$$\frac{d\overline{V}_2}{dz} + \overline{Z}_{12}\overline{I}_1 + \overline{Z}_{22}\overline{I}_2 = 0$$

$$\frac{d\overline{I}_1}{dz} + \overline{Y}_{11}\overline{V}_1 - \overline{Y}_{12}\overline{V}_2 = 0 \qquad (27)$$

$$\frac{d\overline{I}_2}{dz} - \overline{Y}_{12}\overline{V}_1 + \overline{Y}_{22}\overline{V}_2 = 0$$

where

$$\overline{V}_1 = \overline{V}_1(z,p),\ \overline{V}_2 = \overline{V}_2(z,p),\ \overline{I}_1 = \overline{I}_1(z,p),\ \overline{I}_2 = \overline{I}_2(z,p)$$

$$\overline{Z}_{11} = R_{11} + L_{11}p,\ \overline{Z}_{22} = R_{22} + L_{22}p,\ \overline{Z}_{12} = \overline{Z}_{21} = L_{12}p$$

$$\overline{Y}_{11} = G_{11} + C_{11}p,\ \overline{Y}_{22} = G_{22} + C_{22}p,\ \overline{Y}_{12} = \overline{Y}_{21} = C_{12}p\ .$$

By successive differentiation and some rather tedious algebraic operations, Equation 27 yields the following set of uncoupled differential equations.

$$\frac{d^4\overline{V}_1}{dz^4} + (2\overline{Z}_{12}\overline{Y}_{12}-\overline{Z}_{11}\overline{Y}_{11}-\overline{Z}_{22}\overline{Y}_{22})\,\frac{d^2\overline{V}_1}{dz^2} + (\overline{Z}_{11}\overline{Z}_{22}-\overline{Z}_{12}^{\,2})(\overline{Y}_{11}\overline{Y}_{22}-\overline{Y}_{12}^{\,2}) = 0$$

$$\frac{d^4\overline{V}_2}{dz^4} + (2\overline{Z}_{12}\overline{Y}_{12}-\overline{Z}_{11}\overline{Y}_{11}-\overline{Z}_{22}\overline{Y}_{22})\,\frac{d^2\overline{V}_2}{dz^2} + (\overline{Z}_{11}\overline{Z}_{22}-\overline{Z}_{12}^{\,2})(\overline{Y}_{11}\overline{Y}_{22}-\overline{Y}_{12}^{\,2}) = 0$$

$$\frac{d^4\overline{I}_1}{dz^4} + (2\overline{Z}_{12}\overline{Y}_{12}-\overline{Z}_{11}\overline{Y}_{11}-\overline{Z}_{22}\overline{Y}_{22})\,\frac{d^2\overline{I}_1}{dz^2} + (\overline{Z}_{11}\overline{Z}_{22}-\overline{Z}_{12}^{\,2})(\overline{Y}_{11}\overline{Y}_{22}-\overline{Y}_{12}^{\,2}) = 0 \qquad (28)$$

$$\frac{d^4\overline{I}_2}{dz^4} + (2\overline{Z}_{12}\overline{Y}_{12}-\overline{Z}_{11}\overline{Y}_{11}-\overline{Z}_{22}\overline{Y}_{22})\,\frac{d^2\overline{I}_2}{dz^4} + (\overline{Z}_{11}\overline{Z}_{22}-\overline{Z}_{12}^{\,2})(\overline{Y}_{11}\overline{Y}_{22}-\overline{Y}_{12}^{\,2}) = 0$$

It is quite interesting to note that the transformed voltage and current of each line, $\overline{V}_1(z,p)$, $\overline{V}_2(z,p)$, $\overline{I}_1(z,p)$ and $\overline{I}_2(z,p)$ all satisfy a differential equation of exactly the same form with the characteristic equation

$$r^4 + \alpha r^2 + \beta = 0 \qquad (29)$$

where

$$\alpha = 2\overline{Z}_{12}\overline{Y}_{12} - \overline{Z}_{11}\overline{Y}_{11} - \overline{Z}_{22}\overline{Y}_{22}$$

$$\beta = \left(\overline{Z}_{11}\overline{Z}_{22} - \overline{Z}_{12}^{\,2}\right)\left(\overline{Y}_{11}\overline{Y}_{22} - \overline{Y}_{12}^{\,2}\right) .$$

The four roots are:

$$r_1 = \sqrt{-\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \beta^2}}$$

$$r_2 = -\sqrt{-\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \beta^2}} = -r_1$$

$$r_3 = \sqrt{-\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \beta^2}} \qquad (30)$$

$$r_4 = -\sqrt{-\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \beta^2}} = -r_3 \quad .$$

Note that all square roots are assumed to be a positive quantity. This scheme is used throughout this thesis.

The solutions of these differential equations (Equation 28) are:

$$\begin{aligned}
\bar{V}_1 &= A_1 e^{r_1 z} + A_2 e^{-r_1 z} + A_3 e^{r_3 z} + A_4 e^{-r_3 z} \\
\bar{V}_2 &= B_1 e^{r_1 z} + B_2 e^{-r_1 z} + B_3 e^{r_3 z} + B_4 e^{-r_3 z} \\
\bar{I}_1 &= C_1 e^{r_1 z} + C_2 e^{-r_1 z} + C_3 e^{r_3 z} + C_4 e^{-r_3 z} \\
\bar{I}_2 &= D_1 e^{r_1 z} + D_2 e^{-r_1 z} + D_3 e^{r_3 z} + D_4 e^{-r_3 z} \quad .
\end{aligned} \tag{31}$$

Substituting Equation 31 into Equation 27 and solving for the coefficients B's, C's, and D's in terms of A's, then inserting the result into Equation 31

$$\bar{V}_1 = A_1 e^{r_1 z} + A_2 e^{-r_1 z} + A_3 e^{r_3 z} + A_4 e^{-r_3 z}$$

$$\bar{V}_2 = \frac{-r_1^2 \bar{Z}_{22} + \bar{Z}\,\bar{Y}_{11}}{-r_1^2 \bar{Z}_{12} + \bar{Z}\,\bar{Y}_{12}} \left(A_1 e^{r_1 z} + A_2 e^{-r_1 z}\right)$$

$$+ \frac{-r_3^2 \bar{Z}_{22} + \bar{Z}\,\bar{Y}_{11}}{-r_3^2 \bar{Z}_{12} + \bar{Z}\,\bar{Y}_{12}} \left(A_3 e^{r_3 z} + A_4 e^{-r_3 z}\right)$$

$$\bar{I}_1 = \frac{\bar{Z}_{12}\bar{Y}_{11} - \bar{Z}_{22}\bar{Y}_{12}}{-r_1^2 \bar{Z}_{12} + Z\,Y_{12}}\, r_1 \left(A_1 e^{r_1 z} - A_2 e^{-r_1 z}\right)$$

$$+ \frac{\bar{Z}_{12}\bar{Y}_{11} - \bar{Z}_{22}\bar{Y}_{22}}{-r_3^2 \bar{Z}_{12} + \bar{Z}\,\bar{Y}_{12}}\, r_3 \left(A_3 e^{r_3 z} - A_4 e^{-r_3 z}\right)$$

$$\overline{I}_2 = \frac{r_1^3 + \left(\overline{Z}_{12}\overline{Y}_{12} - \overline{Z}_{11}\overline{Y}_{11}\right)r_1}{-r_1^2\overline{Z}_{12} + \overline{Z}\,\overline{Y}_{12}}\left(A_1 e^{r_1 z} - A_2 e^{-r_1 z}\right)$$

$$+ \frac{r_3^2 + \left(\overline{Z}_{12}\overline{Y}_{12} - \overline{Z}_{11}\overline{Y}_{11}\right)r_3}{-r_3^2\overline{Z}_{12} + \overline{Z}\,\overline{Y}_{12}}\left(A_3 e^{r_3 z} - A_4 e^{-r_3 z}\right)$$

where

$$\overline{Z} = \overline{Z}_{11}\overline{Z}_{22} - \overline{Z}_{12}^{\,2} \quad .$$

Since the A's are only multiplication constants, the above equations can be rewritten as follows:

$$\overline{V}_1 = \left(-r_1^2\overline{Z}_{12}+\overline{Z}\,\overline{Y}_{12}\right)A_1' e^{r_1 z} + \left(-r_1^2\overline{Z}_{12}+\overline{Z}\,\overline{Y}_{12}\right)A_2' e^{-r_1 z}$$

$$+ \left(-r_3^2\overline{Z}_{12}+\overline{Z}\,\overline{Y}_{12}\right)A_3' e^{r_3 z} + \left(-r_3^2\overline{Z}_{12}+\overline{Z}\,\overline{Y}_{12}\right)A_4' e^{-r_3 z}$$

$$\overline{V}_2 = \left(-r_1^2\overline{Z}_{22}+\overline{Z}\,\overline{Y}_{11}\right)A_1' e^{r_1 z} + \left(-r_1^2\overline{Z}_{22}+\overline{Z}\,\overline{Y}_{11}\right)A_2' e^{-r_1 z}$$

$$+ \left(-r_3^2\overline{Z}_{22}+\overline{Z}\,\overline{Y}_{11}\right)A_3' e^{r_3 z} + \left(-r_3^2\overline{Z}_{22}+\overline{Z}\,\overline{Y}_{11}\right)A_4' e^{-r_3 z}$$

$$\overline{I}_1 = \left(\overline{Z}_{12}\overline{Y}_{11}-\overline{Z}_{22}\overline{Y}_{12}\right)r_1\ A_1' e^{r_1 z} + \left(\overline{Z}_{22}\overline{Y}_{12}-\overline{Z}_{12}\overline{Y}_{11}\right)r_1\ A_2' e^{-r_1 z}$$

$$+ \left(\overline{Z}_{12}\overline{Y}_{11}-\overline{Z}_{22}\overline{Y}_{22}\right)r_3\ A_3' e^{r_3 z} + \left(\overline{Z}_{22}\overline{Y}_{12}-\overline{Z}_{12}\overline{Y}_{11}\right)r_3\ A_4' e^{-r_3 z}$$

$$\overline{I}_2 = \left[r_1^3+\left(\overline{Z}_{12}\overline{Y}_{12}-\overline{Z}_{11}\overline{Y}_{11}\right)r_1\right]A_1' e^{r_1 z} + \left[-r_1^3+\left(\overline{Z}_{11}\overline{Y}_{11}-\overline{Z}_{12}\overline{Y}_{12}\right)r_1\right]A_2' e^{-r_1 z}$$

$$+ \left[r_3^3+\left(\overline{Z}_{12}\overline{Y}_{12}-\overline{Z}_{11}\overline{Y}_{11}\right)r_3\right]A_3' e^{r_3 z} + \left[-r_3^3+\left(\overline{Z}_{11}\overline{Y}_{11}-\overline{Z}_{12}\overline{Y}_{12}\right)r_3\right]A_4' e^{-r_3 z}$$

By dropping the prime of the A's and introducing four auxiliary functions of r, the above equations are further simplified.

$$\overline{V}_1 = A_1 f_1(r_1) e^{r_1 z} + A_2 f_1(-r_1) e^{-r_1 z} + A_3 f_1(-r_3) e^{r_3 z} + A_4 f_1(-r_3) e^{-r_3 z}$$

$$\overline{V}_2 = A_1 f_2(r_1) e^{r_1 z} + A_2 f_2(-r_1) e^{-r_1 z} + A_3 f_2(r_3) e^{r_3 z} + A_4 f_2(-r_3) e^{-r_3 z}$$

$$\overline{I}_1 = A_1 f_3(r_1) e^{r_1 z} + A_2 f_3(-r_1) e^{-r_1 z} + A_3 f_3(r_3) e^{r_3 z} + A_4 f_3(-r_3) e^{-r_3 z} \tag{32}$$

$$\overline{I}_2 = A_1 f_4(r_1) e^{r_1 z} + A_2 f_4(-r_1) e^{-r_1 z} + A_3 f_4(r_3) e^{r_3 z} + A_4 f_4(-r_3) e^{-r_3 z}$$

These four auxiliary functions are:

$$f_1(r) = -r^2 \overline{Z}_{12} - \overline{Y}_{12} \overline{Z}$$

$$f_2(r) = -r^2 \overline{Z}_{22} - \overline{Y}_{11} \overline{Z}$$

$$f_3(r) = \left(\overline{Z}_{12} \overline{Y}_{11} - \overline{Z}_{22} \overline{Y}_{12}\right) r \tag{33}$$

$$f_4(r) = r^3 + \left(\overline{Z}_{12} \overline{Y}_{12} - \overline{Z}_{11} \overline{Y}_{11}\right) \ .$$

If these two lines are symmetric and lossless, i.e.,

$$L_{11} = L_{22} = L \ , \qquad R_{11} = R_{22} = 0 \ ,$$

$$C_{11} = C_{22} = C \ , \qquad G_{11} = G_{22} = 0 \ ,$$

and

$$L_{12} = L_{21} \ , \qquad C_{12} = C_{21} \ , \tag{34}$$

then

$$\overline{Z}_{11} = \overline{Z}_{22} \ , \qquad \overline{Y}_{11} = \overline{Y}_{22} \ ,$$

and

$$r_1 = -r_2 = \sqrt{(L-L_{12})(C+C_{12})}\ p$$

$$r_3 = -r_4 = \sqrt{(L+L_{12})(C-C_{12})}\ p$$

Equation 32 reduces to the following.

$$
\begin{aligned}
\overline{V}_1 = {} & (L-L_{12})(C_{12}L-CL_{12})\left(A_1e^{r_1z} + A_2e^{-r_1z}\right)p^3 \\
& + (L+L_{12})(C_{12}L-CL_{12})\left(A_3e^{r_3z} + A_4e^{-r_3z}\right)p^3 \\
\overline{V}_2 = {} & (L-L_{12})(C_{12}L-CL_{12})\left(-A_1e^{r_1z} + A_2e^{-r_1z}\right) \\
& + (L+L_{12})(C_{12}L-CL_{12})\left(A_3e^{r_3z} + A_4e^{-r_3z}\right)p^3 \\
\overline{I}_1 = {} & (C_{12}L-CL_{12})\sqrt{(L-L_{12})(C+C_{12})}\left(-A_1e^{r_1z} + A_2e^{-r_1z}\right)p^3 \\
& + (C_{12}L-CL_{12})\sqrt{(L+L_{12})(C-C_{12})}\left(-A_3e^{r_3z} + A_4e^{-r_3z}\right)p^3 \\
\overline{I}_2 = {} & (C_{12}L-CL_{12})\sqrt{(L-L_{12})(C+C_{12})}\left(A_1e^{r_1z} - A_2e^{-r_1z}\right)p^3 \\
& + (C_{12}L-CL_{12})\sqrt{(L+L_{12})(C-C_{12})}\left(-A_3e^{r_3z} + A_4e^{-r_3z}\right)p^3 ,
\end{aligned}
\tag{35}
$$

where A_1, A_2, A_3, and A_4 are constants to be determined from the boundary conditions. Assume the sending end devices have output impedances Z_{g1} and Z_{g2}, the receiving end devices with input impedances $Z_{\ell 1}$ and $Z_{\ell 2}$, and the first line is fired with a voltage wave described by time function $E_{g1}(t)$; then the boundary conditions are:

$$
\begin{aligned}
V_1(o,t) &= E_{g1}(t) - I_1(o,t)Z_{g1} \\
V_2(o,t) &= -I_2(o,t)Z_{g2} \\
V_1(\ell,t) &= I_1(\ell,t)Z_{\ell 1} \\
V_2(\ell,t) &= I\,(\ell,t)Z_{\ell 2}
\end{aligned}
\tag{36}
$$

Their Laplace transforms are:

$$
\begin{aligned}
\overline{V}_1(o,p) &= \overline{E}_{g1}(p) - \overline{I}_1(o,p)\overline{Z}_{g1} \\
\overline{V}_2(o,p) &= -\overline{I}_2(o,p)\overline{Z}_{g2} \\
\overline{V}_1(\ell,p) &= \overline{I}_1(\ell,p)\overline{Z}_{\ell 1} \\
\overline{V}_2(\ell,p) &= \overline{I}_2(\ell,p)\overline{Z}_{\ell 2} \quad .
\end{aligned}
\tag{37}
$$

Substituting Equation 35 into Equation 37 and grouping the terms,

$$
\begin{aligned}
&(L-L_{12})(C_{12}L-CL_{12})\left[A_1\left(1-\sqrt{\frac{C+C_{12}}{L-L_{12}}}\,\overline{Z}_{g1}\right)+A_2\left(1+\sqrt{\frac{C+C_{12}}{L-L_{12}}}\,\overline{Z}_{g1}\right)\right] \\
&\quad + (L+L_{12})(C_{12}L-CL_{12})\left[A_3\left(1-\sqrt{\frac{C-C_{12}}{L+L_{12}}}\,\overline{Z}_{g1}\right)+A_4\left(1+\sqrt{\frac{C-C_{12}}{L+L_{12}}}\,\overline{Z}_{g1}\right)\right] = \frac{\overline{E}_{g1}}{p^3} \\
&(L-L_{12})(C_{12}L-CL_{12})\left[-A_1\left(1-\sqrt{\frac{C+C_{12}}{L-L_{12}}}\,\overline{Z}_{g2}\right)-A_2\left(1+\sqrt{\frac{C+C_{12}}{L-L_{12}}}\,\overline{Z}_{g2}\right)\right] \\
&\quad + (L+L_{12})(C_{12}L-CL_{12})\left[A_3\left(1-\sqrt{\frac{C-C_{12}}{L+L_{12}}}\,\overline{Z}_{g2}\right)+A_4\left(1+\sqrt{\frac{C-C_{12}}{L-L_{12}}}\,\overline{Z}_{g2}\right)\right] = 0 \\
&(L-L_{12})(C_{12}L-CL_{12})\left[A_1\left(1+\sqrt{\frac{C+C_{12}}{L-L_{12}}}\,\overline{Z}_{\ell 1}\right)e^{r_1\ell}+A_2\left(1-\sqrt{\frac{C+C_{12}}{L-L_{12}}}\,\overline{Z}_{\ell 1}\right)e^{-r_1\ell}\right] \\
&\quad + (L+L_{12})(C_{12}L-CL_{12})\left[A_3\left(1+\sqrt{\frac{C-C_{12}}{L+L_{12}}}\,\overline{Z}_{\ell}\right)e^{r_3\ell}+A_4\left(1-\sqrt{\frac{C-C_{12}}{L+L_{12}}}\right)e^{-r_3\ell}\right] = 0 \\
&(L-L_{12})(C_{12}L-CL_{12})\left[-A_1\left(1+\sqrt{\frac{C+C_{12}}{L-L_{12}}}\,\overline{Z}_{\ell 2}\right)e^{r_1\ell}-A_2\left(1-\sqrt{\frac{C+C_{12}}{L-L_{12}}}\,\overline{Z}_{\ell 2}\right)e^{-r_1\ell}\right] \\
&\quad + (L+L_{12})(C_{12}L-CL_{12})\left[A_3\left(1+\sqrt{\frac{C-C_{12}}{L+L_{12}}}\,\overline{Z}_{\ell 2}\right)e^{r_3\ell}+A\left(1-\sqrt{\frac{C-C_{12}}{L+L_{12}}}\,\overline{Z}_{\ell 2}\right)e^{-r_3\ell}\right] = 0 .
\end{aligned}
\tag{38}
$$

by assuming

$$Z_{g1} = Z_{g2} = 0$$

and

$$Z_{\ell 1} = Z_{\ell 2} = 0 ,$$

i.e., the output impedances of the sending end devices and the input impedances of the receiving end devices are all equal to zero. This is the case that these two lines are both grounded at two ends. Equation 38 is easily solved for the undetermined constant A's. They are:

$$\begin{aligned}
A_1 &= \frac{1}{(L-L_{12})(C_{12}L-CL_{12})} \frac{1}{\mathrm{Sinh}(r_1 \ell)} \frac{-\bar{E}_{g1} e^{-r_1 \ell}}{4p^3} \\
A_2 &= \frac{1}{(L-L_{12})(C_{12}L-CL_{12})} \frac{1}{\mathrm{Sinh}(r_1 \ell)} \frac{\bar{E}_{g1} e^{r_1 \ell}}{4p^3} \\
A_3 &= \frac{1}{(L-L_{12})(C_{12}L-CL_{12})} \frac{1}{\mathrm{Sinh}(r_3 \ell)} \frac{-\bar{E}_{g1} e^{-r_3 \ell}}{4p^3} \\
A_4 &= \frac{1}{(L-L_{12})(C_{12}L-CL_{12})} \frac{1}{\mathrm{Sinh}(r_3 \ell)} \frac{\bar{E}_{g1} e^{r_3 \ell}}{4p^3}
\end{aligned} \tag{39}$$

The complete solution in the p-domain under the assumed conditions are obtained by inserting the value of A's of Equation 39 into Equation 35.

$$\begin{aligned}
\bar{V}_1(z,p) &= \frac{\bar{E}_{g1}\ \mathrm{Sinh}[r_1(\ell-z)]}{2\ \mathrm{Sinh}(r_1 \ell)} + \frac{\bar{E}_{g1}\ \mathrm{Sinh}[r_3(\ell-z)]}{2\ \mathrm{Sinh}(r_3 \ell)} \\
\bar{V}_2(z,p) &= \frac{-E_{g1}\ \mathrm{Sinh}[r_1(\ell-z)]}{2\ \mathrm{Sinh}(r_1 \ell)} + \frac{\bar{E}_{g1}\ \mathrm{Sinh}[r_3(\ell-z)]}{2\ \mathrm{Sinh}(r_3 \ell)}
\end{aligned} \tag{40}$$

$$\overline{I}_1(z,p) = \frac{\overline{E}_{g1} \text{ Cosh}[r_1(\ell-z)]}{2 Z_{o1} \text{ Sinh}(r_1\ell)} + \frac{\overline{E}_{g1} \text{ Cosh}[r_3(\ell-z)]}{2 Z_{02} \text{ Sinh}(r_3\ell)}$$

$$\overline{I}_2(z,p) = \frac{-\overline{E}_{g1} \text{ Cosh}[r_1(\ell-z)]}{2 Z_{o1} \text{ Sinh}(r_1\ell)} + \frac{\overline{E}_{g1} \text{ Cosh}[r_3(\ell-z)]}{2 Z_{o2} \text{ Sinh}(r_3\ell)}$$

Using the inversion theorem, the formal solution of line voltages and currents can be written as

$$V_1(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1} \text{ Sinh}[r_1(\ell-z)]}{2 \text{ Sinh}(r_1\ell)} e^{pt}\, dp$$

$$+ \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1} \text{ Sinh}[r_3(\ell-z)]}{2 \text{ Sinh}(r_1\ell)} e^{pt}\, dp = V_a(z,t) + V_b(z,t)$$

$$V_2(z,t) = -V_a(z,t) + V_b(z,t) \tag{41}$$

$$I_1(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1} \text{ Cosh}[r_1(\ell-z)]}{2 Z_{o1} \text{ Sinh}(r_1\ell)} e^{pt}\, dp$$

$$+ \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1} \text{ Cosh}[r_3(\ell-z)]}{2 Z_{o2} \text{ Sinh}(r_1\ell)} e^{pt}\, dp = I_a(z,t) + I_b(z,t)$$

$$I_2(z,t) = -I_a(z,t) + I_b(z,t)$$

where σ is a real number which is greater than the real part of all poles of the complex integrand of that integral.

If a step voltage of amplitude E_o, as Equation 17, is applied on the first line, then the line voltages and currents are:

$$V_1(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\text{Sinh}[r_1(\ell-z)]}{2\ \text{Sinh}(r_1\ell)}\, e^{pt}\, \frac{dp}{p}$$

$$+ \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\text{Sinh}[r_3(\ell-z)]}{2\ \text{Sinh}(r_1\ell)}\, e^{pt}\, \frac{dp}{p} = V_{a1}(z,t) + V_{b1}(z,t)$$

$$V_2(z,t) = -V_{a1}(z,t) + V_{b1}(z,t)$$

$$I_1(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\text{Sinh}[r_1(\ell-z)]}{2\ Z_{o1}\ \text{Sinh}(r_1\ell)}\, e^{pt}\, \frac{dp}{p} \tag{42}$$

$$+ \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\text{Sinh}[r_3(\ell-z)]}{2\ Z_{o2}\ \text{Sinh}(r_3\ell)}\, e^{pt}\, \frac{dp}{p} = I_{a1}(z,t) + I_{b1}(z,t)$$

$$I_2(z,t) = -I_{a1}(z,t) + I_{b1}(z,t) \quad .$$

By comparing the Equations 41 and 42 of the coupled case with corresponding Equations 14 and 15 of the isolated case (since the isolated case discussed in Chapter II is the line with terminated impedance at both ends, it is better to let Z_g and Z_ℓ equate to zero, then $X_1=X_2=0$, for comparison), it is noted that:

(1) there are two sets of waves on each line of a coupled system and each set is of exactly the same form as the wave in the isolated case;

(2) the operational or transformed propagation constant and line impedance of these two sets are:

$$r_1 = \sqrt{(L-L_{12})(C+C_{12})}\ p = \frac{p}{v_1} \qquad Z_{o1} = \sqrt{\frac{L-L_{12}}{C-C_{12}}}$$

$$r_3 = \sqrt{(L+L_{12})(C-C_{12})}\ p = \frac{p}{v_2} \qquad Z_{o2} = \sqrt{\frac{L+L_{12}}{C-C_{12}}}\ ; \tag{43}$$

(3) these quantities are perturbations of the values under isolated conditions being

$$r = \sqrt{LC}\, p$$

$$Z_o = \sqrt{\frac{L}{C}} \quad .$$

The perturbed values usually are much different from the unperturbed ones for very closely spaced lines, in contrast to the usual perturbation theory in which only a very small amount of change is assumed. (Note that the value of a coupled line is that value of the line in the presence of the other line.)

The integration in a complex plane of Equation 42 can be performed by evaluating the residues of each integrand; much the same as was done for Equation 15 in Chapter II. The results are:

$$V_1(z,t) = \frac{\ell-z}{\ell} E_o + E_o \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \text{Sin}\left[\frac{\ell-z}{\ell} n\pi\right] \left\{ \text{Cos}\left[\frac{v_1 t}{\ell} n\pi\right] + \text{Cos}\left[\frac{v_2 t}{\ell} n\pi\right] \right\}$$

$$= \frac{\ell-z}{\ell} E_o + E_o \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \text{Sin}\left[\frac{\ell-z}{\ell} n\pi\right] \cdot$$

$$2\, \text{Cos}\left[\frac{1}{2} \frac{(v_1+v_2)t}{\ell} n\pi\right] \text{Cos}\left[\frac{1}{2} \frac{(v_1-v_2)t}{\ell} n\pi\right]$$

$$V_2(z,t) = E_o \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \text{Sin}\left[\frac{\ell-z}{\ell} n\pi\right] \left\{ \text{Cos}\left[\frac{v_2 t}{\ell} n\pi\right] - \text{Cos}\left[\frac{v_1 t}{\ell} n\pi\right] \right\}$$

$$= -E_o \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \text{Sin}\left[\frac{\ell-z}{\ell} n\pi\right] \cdot$$

$$2\, \text{Sin}\left[\frac{1}{2} \frac{(v_1+v_2)t}{\ell} n\pi\right] \text{Sin}\left[\frac{1}{2} \frac{(v_1-v_2)t}{\ell} n\pi\right] \qquad (44)$$

$$I_1(z,t) = \frac{E_o}{Z_{o1}} \sum_{n=1}^{\infty} (-j)\frac{(-1)^n}{n\pi} \operatorname{Cos}\left[\frac{\ell-z}{\ell} n\pi\right] \operatorname{Cos}\left[\frac{v_1 t}{\ell} n\pi\right]$$

$$+ \frac{E_o}{Z_{o2}} \sum_{n=1}^{\infty} (-j)\frac{(-1)^n}{n\pi} \operatorname{Cos}\left[\frac{\ell-z}{\ell} n\pi\right] \operatorname{Cos}\left[\frac{v_2 t}{\ell} n\pi\right]$$

$$= E_o \sum_{n=1}^{\infty} (-j)\frac{(-1)^n}{n\pi} \operatorname{Cos}\left[\frac{\ell-z}{\ell} n\pi\right] \cdot$$

$$\left\{\frac{1}{Z_{o1}} \operatorname{Cos}\left[\frac{v_1 t}{\ell} n\pi\right] + \frac{1}{Z_{o2}} \operatorname{Cos}\left[\frac{v_2 t}{\ell} n\pi\right]\right\}$$

$$I_2(z,t) = -\frac{E_o}{Z_{o1}} \sum_{n=1}^{\infty} (-j)\frac{(-1)^n}{n\pi} \operatorname{Cos}\left[\frac{\ell-z}{\ell} n\pi\right] \operatorname{Cos}\left[\frac{v_1 t}{\ell} n\pi\right]$$

$$+ \frac{E_o}{Z_{o2}} \sum_{n=1}^{\infty} (-j)\frac{(-1)^n}{n\pi} \operatorname{Cos}\left[\frac{\ell-z}{\ell} n\pi\right] \operatorname{Cos}\left[\frac{v_2 t}{\ell} n\pi\right]$$

$$= E_o \sum_{n=1}^{\infty} (-j)\frac{(-1)^n}{n\pi} \operatorname{Cos}\left[\frac{\ell-z}{\ell} n\pi\right] \cdot$$

$$\left\{-\frac{1}{Z_{o1}} \operatorname{Cos}\left[\frac{v_1 t}{\ell} n\pi\right] + \frac{1}{Z_{o2}} \operatorname{Cos}\left[\frac{v_2 t}{\ell} n\pi\right]\right\}$$

From the solution, Equation 43, the following conclusions can be drawn:

(1) the voltage at the sending end of line No. 1 is E_o and at the receiving end is zero; the voltage at both ends of line No. 2 is zero.

(2) the steady state currents on the first line and second line are zero all along the line; actually it is impossible to

reach a steady state or it takes an infinite time to reach the steady state in this case;

(3) The transient current and voltage are two infinite series of which each term is a cosine function of time that never dies out.

Any further interpretation is difficult to make. Therefore, following the method used in Chapter II, the alternate solution of the form of successive waves is obtained as follows.

$$V_1(z,t) = \frac{E_o}{2} \sum_{n=0}^{\infty} \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v_1}\right] - U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_1}\right]\right\}$$

$$+ \frac{E_o}{2} \sum_{n=0}^{\infty} \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v_2}\right] - U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_2}\right]\right\} = V_a(z,t)+V_b(z,t)$$

$$V_2(z,t) = -V_a(z,t) + V_b(z,t) \tag{45}$$

$$I_1(z,t) = \frac{E_o}{2\,Z_{o1}} \sum_{n=0}^{\infty} \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v_1}\right] + U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_1}\right]\right\}$$

$$+ \frac{E_o}{2\,Z_{o2}} \sum_{n=0}^{\infty} \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v_2}\right] + U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_2}\right]\right\}$$

$$= I_a(z,t) + I_b(z,t) = \frac{V_a(z,a)}{Z_{o1}} + \frac{V_b(z,t)}{Z_{o2}}$$

$$I_2(z,t) = -I_a(z,t) + I_b(z,t) = -\frac{V_a(z,t)}{Z_{o1}} + \frac{V_b(z,t)}{Z_{o2}}$$

Therefore, a few more conclusions are in order.

(4) The velocity associated with these two sets of waves are:

$$v_1 = \frac{1}{\sqrt{(L-L_{12})(C+C_{12})}} > v \qquad v = \frac{1}{\sqrt{LC}} \quad . \tag{46}$$

$$v_2 = \frac{1}{\sqrt{(L+L_{12})(C-C_{12})}} < v$$

One is greater, the other is smaller than that of an isolated line with parameters L and C. Each set of waves propagates on both lines with the same velocity.

(5) By comparing the corresponding voltage and current waves, it is found that the line impedance associated with each mode of the waves is

$$Z_{o1}^{+} = \sqrt{\frac{L-L_{12}}{C+C_{12}}} \qquad Z_{o1}^{-} = -\sqrt{\frac{L-L_{12}}{C+C_{12}}} \tag{47}$$

$$Z_{o2}^{+} = \sqrt{\frac{L+L_{12}}{C-C_{12}}} \qquad Z_{o2}^{-} = -\sqrt{\frac{L+L_{12}}{C-C_{12}}} \quad .$$

The superscripts indicate forward waves (+) or backward waves (-). The impedance of the backward waves is just the negative of that of the forward waves.

(6) For each mode, the relationship between line impedance and velocity is the same as that of the isolated lines.

Figure 5 shows the first forward and backward voltage wave on both line 1 and line 2. The corresponding current waves are shown in Figure 6.

If one rectangular pulse with amplitude E_o and duration τ is applied on the first line instead of step voltage, then

$$E_{g1}(t) = E_o\left[U_{-1}(t) - U_{-1}(t-\tau)\right] \tag{48}$$

$$\overline{E}_{g1}(p) = E_o\left[\frac{1}{p} - \frac{e^{p\tau}}{p}\right] \quad .$$

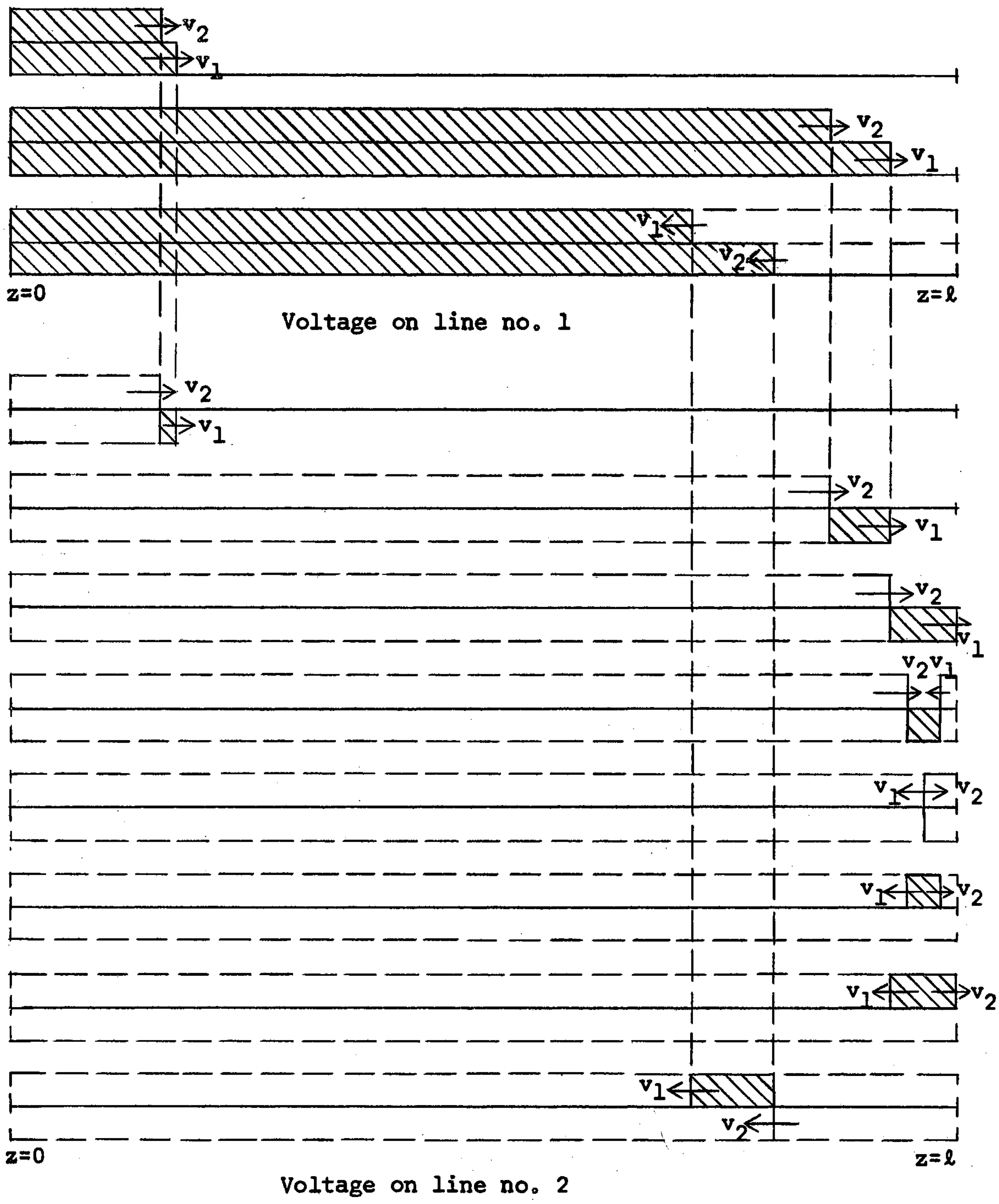

Figure 5. Voltages on a Two Coupled Transmission Lines, With a Step Voltage Applied on Line No. 1.

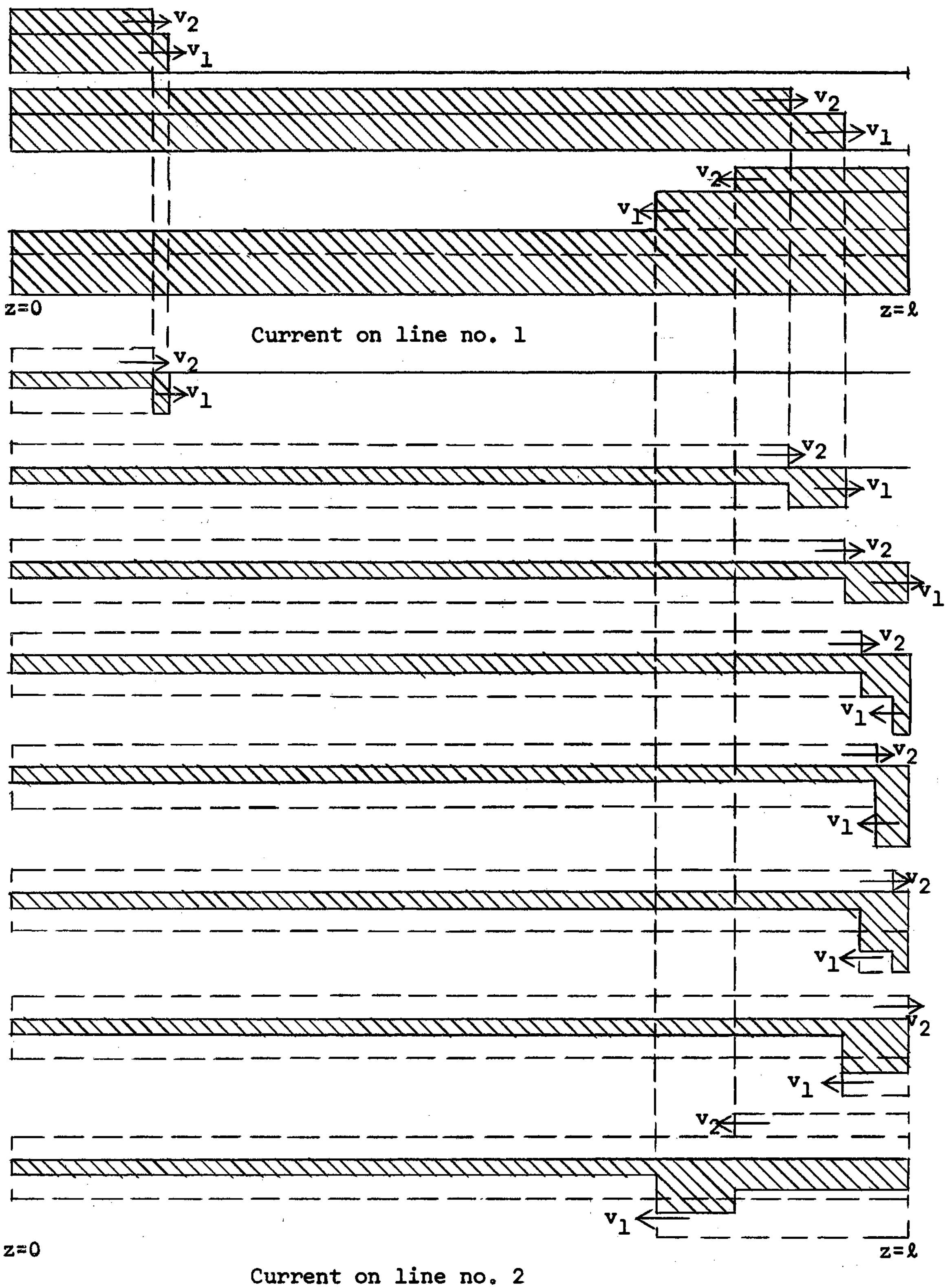

Figure 6. Currents on a Two Coupled Transmission Line, With a Step Voltage Applied on Line No. 1.

The complete series solutions assume the following forms.

$$V_1(z,t) = \frac{E_o}{2} \sum_{n=0}^{\infty} \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v_1}\right] - U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_1}\right]\right\}$$

$$- \frac{E_o}{2} \sum_{n=0}^{\infty} \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v_1} - \tau\right] - U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_1} - \tau\right]\right\}$$

$$+ \frac{E_o}{2} \sum_{n=0}^{\infty} \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v_2}\right] - U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_2}\right]\right\}$$

$$- \frac{E_o}{2} \sum_{n=0}^{\infty} \left\{ U_{-1}\left[t - \frac{2n\ell+z}{v_2} - \tau\right] - U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_2} - \tau\right]\right\}$$

$$= V_a(z,t) - V_a(z,t - \tau) + V_b(z,t) - V_b(z,t - \tau)$$

$$V_2(z,t) = -V_a(z,t) + V_a(z,t - \tau) + V_b(z,t) - V_b(z,t - \tau)$$

$$I_1(z,t) = \frac{1}{Z_{o1}} \left[V_a(z,t) - V_a(z,t - \tau)\right] + \frac{1}{Z_{o2}} \left[V_b(z,t) - V_b(z,t - \tau)\right] \tag{49}$$

$$I_2(z,t) = - \frac{1}{Z_{o1}} \left[V_a(z,t) - V_a(z,t - \tau)\right] + \frac{1}{Z_{o2}} \left[V_b(z,t) - V_b(z,t - \tau)\right] .$$

For a pulse input, to each line except for the voltage and current waves shown in Equation 45 and Figures 5 and 6 for step voltage input, there is another set of voltage and current waves with exactly the same amplitude, but of opposite polarity and delayed by a time interval, the duration of input pulse.

Two Symmetrical Lossless Lines Terminated at Both Ends

For two symmetrical lossless lines terminated at both ends such that

$$Z_{g1} = Z_{g2} = Z_g$$

$$Z_{\ell 1} = Z_{\ell 2} = Z_\ell \quad . \tag{50}$$

Substituting Equation 50 into Equation 38 and solving for A's,

$$A_1 = \frac{1}{4(L-L_{12})(C_{12}L-CL_{12})} \frac{-\overline{E}_{g1}\left(1 - \frac{\overline{Z}_\ell}{\overline{Z}_{o1}}\right)e^{-r_1\ell}}{p^3\left[\left(1 + \frac{\overline{Z}_\ell\overline{Z}_g}{\overline{Z}_{o1}^2}\right)\mathrm{Sinh}(r_1\ell) + \left(\frac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o1}}\right)\mathrm{Cosh}(r_1\ell)\right]}$$

$$A_2 = \frac{1}{4(L-L_{12})(C_{12}L-CL_{12})} \frac{\overline{E}_{g1}\left(1 + \frac{\overline{Z}_\ell}{\overline{Z}_{o1}}\right)e^{r_1\ell}}{p^3\left[\left(1 + \frac{\overline{Z}_\ell\overline{Z}_g}{\overline{Z}_{o1}^2}\right)\mathrm{Sinh}(r_1\ell) + \left(\frac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o1}}\right)\mathrm{Cosh}(r_1\ell)\right]}$$

$$A_3 = \frac{1}{4(L-L_{12})(C_{12}L-CL_{12})} \frac{-\overline{E}_{g1}\left(1 - \frac{\overline{Z}_\ell}{\overline{Z}_{o2}}\right)e^{-r_3\ell}}{p^3\left[\left(1 + \frac{\overline{Z}_\ell\overline{Z}_g}{\overline{Z}_{o2}^2}\right)\mathrm{Sinh}(r_3\ell) + \left(\frac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o2}}\right)\mathrm{Cosh}(r_3\ell)\right]} \tag{51}$$

$$A_4 = \frac{1}{4(L-L_{12})(C_{12}L-CL_{12})} \frac{\overline{E}_{g1}\left(1 + \frac{\overline{Z}_\ell}{\overline{Z}_{o2}}\right)e^{r_3\ell}}{p^3\left[\left(1 + \frac{\overline{Z}_\ell\overline{Z}_g}{\overline{Z}_{o2}^2}\right)\mathrm{Sinh}(r_3\ell) + \left(\frac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o2}}\right)\mathrm{Cosh}(r_3\ell)\right]} \; .$$

Inserting these A's into Equation 38, the complete solution of two terminated lines in the p-domain is:

$$\overline{V}_1(z,p) = \frac{\overline{E}_{g1}\left\{\text{Sinh}[r_1(\ell-z)] + \frac{\overline{Z}_\ell}{\overline{Z}_{o1}}\text{Cosh}[r_1(\ell-z)]\right\}}{2\left[\left(1+\frac{\overline{Z}_g\overline{Z}_\ell}{\overline{Z}_{o1}^2}\right)\text{Sinh}(r_1\ell) + \left(\frac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o1}}\right)\text{Cosh}(r_1\ell)\right]}$$

$$+ \frac{\overline{E}_{g1}\left\{\text{Sinh}[r_3(\ell-z)] + \frac{\overline{Z}_\ell}{\overline{Z}_{o1}}\text{Cosh}[r_3(\ell-z)]\right\}}{2\left[\left(1+\frac{\overline{Z}_g\overline{Z}_\ell}{\overline{Z}_{o2}^2}\right)\text{Sinh}(r_3\ell) + \left(\frac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o2}}\right)\text{Sinh}(r_3\ell)\right]} = \overline{V}_c(z,p) + \overline{V}_d(z,p)$$

$$\overline{V}_2(z,p) = -\overline{V}_c(z,p) + \overline{V}_d(z,p)$$

(52)

$$\overline{I}_1(z,p) = \frac{\overline{E}_{g1}\left\{\text{Cosh}[r_1(\ell-z)] + \frac{\overline{Z}_\ell}{\overline{Z}_{o1}}\text{Sinh}[r_1(\ell-z)]\right\}}{2\,\overline{Z}_{o1}\left[\left(1+\frac{\overline{Z}_g\overline{Z}_\ell}{\overline{Z}_{o1}^2}\right)\text{Sinh}(r_1\ell) + \left(\frac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o1}}\right)\text{Sinh}(r_1\ell)\right]}$$

$$+ \frac{\overline{E}_{g1}\left\{\text{Cosh}[r_3(\ell-z)] + \frac{\overline{Z}_\ell}{\overline{Z}_{o2}}\text{Sinh}[r_3(\ell-z)]\right\}}{2\,\overline{Z}_{o2}\left[\left(1+\frac{\overline{Z}_g\overline{Z}_\ell}{\overline{Z}_{o2}^2}\right)\text{Sinh}(r_3\ell) + \left(\frac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o2}}\right)\text{Sinh}(r_3\ell)\right]} = \overline{I}_c(z,p)+\overline{I}_d(z,p)$$

$$\overline{I}_2(z,p) = -\overline{I}_c(z,p) + \overline{I}_d(z,p) \quad .$$

$\overline{V}_c(z,p)$, $\overline{V}_d(z,p)$, $\overline{I}_c(z,p)$ and $\overline{I}_d(z,p)$ can be further simplified to a more compact form by assuming

$$Z_\ell \neq Z_{o1} \qquad Z_\ell \neq Z_{o2}$$

and

$$Z_g \neq Z_{o1} \qquad Z_g \neq Z_{o2} \quad .$$

$$\overline{V}_c(z,p) = \frac{\overline{E}_{g1}}{2\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_{o1}}\right)^2\right]^{\frac{1}{2}}} \frac{\mathrm{Sinh}[r_1(\ell-z) + X_4]}{\mathrm{Sinh}[r_1\ell + X_5]}$$

$$\overline{V}_d(z,p) = \frac{\overline{E}_{g1}}{2\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_{o2}}\right)^2\right]^{\frac{1}{2}}} \frac{\mathrm{Sinh}[r_3(\ell-z) + X_6]}{\mathrm{Sinh}[r_3\ell + X_7]}$$

$$\overline{I}_c(z,p) = \frac{\overline{E}_{g1}}{2\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_{o1}}\right)^2\right]^{\frac{1}{2}}} \frac{\mathrm{Cosh}[r_1(\ell-z) + X_4]}{\mathrm{Sinh}[r_1\ell + X_5]} \qquad (53)$$

$$\overline{I}_d(z,p) = \frac{\overline{E}_{g1}}{2\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_{o2}}\right)^2\right]^{\frac{1}{2}}} \frac{\mathrm{Cosh}[r_3(\ell-z) + X_6]}{\mathrm{Sinh}[r_3\ell + X_7]}$$

where

$$X_4 = \mathrm{Tanh}^{-1} \frac{\overline{Z}_\ell}{\overline{Z}_{o1}} \qquad X_5 = \mathrm{Tanh}^{-1} \frac{\dfrac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o1}}}{1 + \dfrac{\overline{Z}_g\overline{Z}_\ell}{\overline{Z}_{o1}^2}}$$

$$X_6 = \mathrm{Tanh}^{-1} \frac{\overline{Z}_\ell}{\overline{Z}_{o2}} \qquad X_7 = \mathrm{Tanh}^{-1} \frac{\dfrac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o1}}}{1 + \dfrac{\overline{Z}_g\overline{Z}_\ell}{\overline{Z}_{o2}^2}} .$$

These special cases of matching conditions, $\overline{Z}_\ell = \overline{Z}_{o1}$, $\overline{Z}_\ell = \overline{Z}_{o2}$, $\overline{Z}_g = \overline{Z}_{o1}$, or $\overline{Z}_g = \overline{Z}_{o2}$ are considered later.

Using the inversion theorem, the solution in the t-domain is:

$$V_1(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1}}{2\left[1-\left(\dfrac{\overline{Z}_g}{\overline{Z}_{o1}}\right)^2\right]^{\frac{1}{2}}} \; \frac{\text{Sinh}[r_1(\ell-z) + X_4]}{\text{Sinh}[r_1\ell + X_5]}\, e^{pt}\, dp$$

$$+ \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1}}{2\left[1-\left(\dfrac{\overline{Z}_g}{\overline{Z}_{o2}}\right)^2\right]^{\frac{1}{2}}} \; \frac{\text{Sinh}[r_3(\ell-z) + X_6]}{\text{Sinh}[r_3\ell + X_7]}\, e^{pt}\, dp$$

$$= V_c(z,t) + V_d(z,t)$$

$$V_2(z,t) = -V_c(z,t) + V_d(z,t) \tag{54}$$

$$I_1(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1}}{2\,\overline{Z}_{o1}\left[1-\left(\dfrac{\overline{Z}_g}{\overline{Z}_{o1}}\right)^2\right]^{\frac{1}{2}}} \; \frac{\text{Cosh}[r_1(\ell-z) + X_4]}{\text{Sinh}[r_1(\ell-z) + X_5]}\, e^{pt}\, dp$$

$$+ \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1}}{2\,\overline{Z}_{o2}\left[1-\left(\dfrac{\overline{Z}_g}{\overline{Z}_{o2}}\right)^2\right]^{\frac{1}{2}}} \; \frac{\text{Cosh}[r_3(\ell-z) + X_4]}{\text{Sinh}[r_3\ell + X_5]}\, e^{pt}\, dp$$

$$= I_c(z,t) + I_d(z,t)$$

$$I_2(z,t) = -I_c(z,t) + I_d(z,t) \;.$$

If the lines are lossless, and the source impedances and the load impedances are pure resistive, then

$$Z_{o1} = \sqrt{\frac{L-L_{12}}{C+C_{12}}} = R_{o1}\;, \qquad\qquad Z_{o2} = \sqrt{\frac{L+L_{12}}{C-C_{12}}} = R_{o2}\;,$$

$$r_1 = \sqrt{(L-L_{12})(C+C_{12})}\, p = \frac{p}{v_1} ,$$

$$r_2 = \sqrt{(L+L_{12})(C-C_{12})}\, p = \frac{p}{v_2} ,$$

$$Z_\ell = R_\ell , \quad Z_g = R_g .$$

The line voltages and currents due to a step voltage of amplitude E_o applied to the first line can be easily obtained by first replacing $\overline{E}_g(p)$ by 1/p and evaluating the residues of each integrand; much the same as done on Equation 15 in Chapter II.

$$V_c(z,t) = \frac{R_\ell}{2(R_g+R_\ell)} E_o + \frac{E_o\, e^{-X_5 \frac{v_1 t}{\ell}}}{2\left[1 - \left(\frac{R_g}{R_{o1}}\right)^2\right]^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{X_5 + jn\pi}{X_5 + n^2\pi^2}$$

$$\mathrm{Sinh}\left[X_8 - \frac{z}{\ell} X_5 + j \frac{z}{\ell} n\pi\right] e^{jn\pi \frac{v_1 t}{\ell}} .$$

$$V_d(z,t) = \frac{R_\ell}{2(R_g+R_\ell)} E_o + \frac{E_o\, e^{-X_7 \frac{v_2 t}{\ell}}}{2\left[1 - \left(\frac{R_g}{R_{o2}}\right)^2\right]^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{X_7 + jn\pi}{X_7^2 + n^2\pi^2}$$

$$\mathrm{Sinh}\left[X_8 - \frac{z}{\ell} X_7 + j \frac{z}{\ell} n\pi\right] e^{jn\pi \frac{v_2 t}{\ell}} . \tag{55}$$

$$I_c(z,t) = \frac{1 + \frac{R_g R_\ell}{R_{o1}^2}}{2R_\ell} E_o - \frac{E_o\, e^{-X_5 \frac{v_1 t}{\ell}}}{2\, R_{o1}\left[1 - \left(\frac{R_g}{R_{o1}}\right)^2\right]^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{X_5 + jn\pi}{X_5^2 + n^2\pi^2}$$

$$\mathrm{Cosh}\left[X_8 - \frac{z}{\ell} X_5 + j \frac{z}{\ell} n\pi\right] e^{jn\pi \frac{v_1 t}{\ell}}$$

$$I_d(z,t) = \frac{1 + \frac{R_g R_\ell}{R_{o2}^2}}{2R_\ell} E_o - \frac{E_o\, e^{-X_7 \frac{v_2 t}{\ell}}}{2\, R_{o2}\left[1 - \left(\frac{R_g}{R_{o2}}\right)^2\right]^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{X_7 + jn\pi}{X_7^2 + n^2\pi^2} \,\circ$$

$$\operatorname{Cosh}\left[X_9 - \frac{z}{\ell} X_7 + j\,\frac{z}{\ell}\, n\pi\right] e^{jn\pi \frac{v_2 t}{\ell}}$$

where

$$X_8 = \operatorname{Tanh}^{-1} \frac{\overline{Z}_\ell}{\overline{Z}_{o1}}\,, \qquad X_9 = \operatorname{Tanh}^{-1} \frac{\overline{Z}_\ell}{\overline{Z}_{o2}}\,.$$

It is noted that $\overline{V}_c(z,p)$ and etc. in Equation 52 can be rewritten as:

$$\overline{V}_c(z,p) = \frac{\overline{E}_{g1}}{1 + \frac{\overline{Z}_g}{\overline{Z}_{o1}}} \; \frac{e^{-r_1 z} - \overline{\rho}_{\ell 1} e^{-r_1(2\ell - z)}}{1 - \overline{\rho}_{g1}\overline{\rho}_{\ell 1} e^{-2r_1\ell}}$$

$$= \frac{\overline{Z}_{o1}}{\overline{Z}_g + \overline{Z}_{o1}}\, \overline{E}_{g1} \left[\sum_{n=0}^{\infty} (\overline{\rho}_{g1}\overline{\rho}_{\ell 1})^n\, e^{-(2n\ell + z) r_1}\right.$$

$$\left. - \sum_{n=0}^{\infty} (\overline{\rho}_{g1}\overline{\rho}_{\ell 1})^n\, \overline{\rho}_{\ell 1} e^{-(2n\ell + 2\ell - z) r_1}\right]$$

$$\overline{V}_d(z,p) = \frac{\overline{Z}_{o2}}{\overline{Z}_g + \overline{Z}_{o2}}\, \overline{E}_{g1} \left[\sum_{n=0}^{\infty} (\overline{\rho}_{g2}\overline{\rho}_{\ell 2})^n\, e^{-(2n\ell + 2\ell - z) r_3}\right.$$

$$\left. - \sum_{n=0}^{\infty} (\overline{\rho}_{g2}\overline{\rho}_{\ell 2})^n\, \overline{\rho}_{\ell 2}\, e^{-(2n\ell + 2\ell - z) r_3}\right]$$

$$\bar{I}_c(z,p) = \frac{\bar{E}_{g1}}{\bar{Z}_g+\bar{Z}_{o1}} \left[\sum_{n=0}^{\infty} (\bar{\rho}_{g1}\bar{\rho}_{\ell 1})^n e^{-(2n\ell+z)r_1} + \sum_{n=0}^{\infty} (\bar{\rho}_{g1}\bar{\rho}_{\ell 1})^n \bar{\rho}_{\ell 1} e^{-(2n\ell+2\ell-z)r_1} \right]$$

$$\bar{I}_d(z,p) = \frac{\bar{E}_{g1}}{\bar{Z}_g+\bar{Z}_{o2}} \left[\sum_{n=0}^{\infty} (\bar{\rho}_{g2}\bar{\rho}_{\ell 2})^n e^{-(2n\ell+z)r_3} + \sum_{n=0}^{\infty} (\bar{\rho}_{g2}\bar{\rho}_{\ell 2})^n \bar{\rho}_{\ell 2} e^{-(2n\ell+2\ell-z)r_3} \right] \tag{56}$$

The $\bar{\rho}_{g1}$ and etc. in Equation 56 denote the operational reflection coefficients at both ends for mode 1 and mode 2 waves;

$$\bar{\rho}_{g1} = \frac{\bar{Z}_g - \bar{Z}_{o1}}{\bar{Z}_g + \bar{Z}_{o1}} \qquad \bar{\rho}_{g2} = \frac{\bar{Z}_g - \bar{Z}_{o2}}{\bar{Z}_g + \bar{Z}_{o2}}$$

$$\bar{\rho}_{\ell 1} = \frac{\bar{Z}_\ell - \bar{Z}_{o1}}{\bar{Z}_\ell + \bar{Z}_{o1}} \qquad \bar{\rho}_{\ell 2} = \frac{\bar{Z}_\ell - \bar{Z}_{o2}}{\bar{Z}_\ell + \bar{Z}_{o2}}$$

Therefore the successive waves solutions in the p-domain are:

$$\begin{aligned} \bar{V}_1(z,p) &= \bar{V}_c(z,p) + \bar{V}_d(z,p) \\ \bar{V}_2(z,p) &= -\bar{V}_c(z,p) + \bar{V}_d(z,p) \\ \bar{I}_1(z,p) &= \bar{I}_c(z,p) + \bar{I}_d(z,p) \\ \bar{I}_2(z,p) &= -\bar{I}_c(z,p) + \bar{I}_d(z,p) \end{aligned} \tag{57}$$

where $\bar{V}_c(z,p)$ and etc. are given in Equation 56.

For lossless lines, pure resistive terminating impedances, and step voltage input, the successive waves solutions in the t-domain are:

$$V_1(z,t) = \frac{R_{o1}}{R_g+R_{o1}} E_o \left\{ \sum_{n=0}^{\infty} (\rho_{g1}\rho_{\ell 1})^n U_{-1}\left[t - \frac{2n\ell+z}{v_1}\right] - \sum_{n=0}^{\infty} (\rho_{g1}\rho_{\ell 1})^n \rho_{\ell 1} U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_1}\right]\right\}$$

$$+ \frac{R_{o2}}{R_g+R_{o2}} E_o \left\{ \sum_{n=0}^{\infty} (\rho_{g2}\rho_{\ell 2})^n U_{-1}\left[t - \frac{2n\ell+z}{v_2}\right] - \sum_{n=0}^{\infty} (\rho_{g2}\rho_{\ell 2})^n \rho_{\ell 2} U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_2}\right]\right\} = (S_1-S_2)+(S_3-S_4)$$

$$V_2(z,t) = -(S_1 - S_2) + (S_3 - S_4) \tag{58}$$

$$I_1(z,t) = \frac{1}{R_{o1}} (S_3 + S_4) + \frac{1}{R_{o2}} (S_3 + S_4)$$

$$I_2(z,t) = - \frac{1}{R_{o1}} (S_3 + S_4) + \frac{1}{R_{o2}} (S_3 + S_4)$$

where all reflection coefficients are just

$$\rho_{g1} = \frac{R_{o1} - R_g}{R_{o1} + R_g} \qquad R_{g2} = \frac{R_{o2} - R_g}{R_{o2} + R_g}$$

$$\rho_{\ell 1} = \frac{R_{o1} - R_\ell}{R_{o1} + R_\ell} \qquad R_{\ell 2} = \frac{R_{o2} - R_\ell}{R_{o2} + R_\ell} \ .$$

In the above solution, there are four sets of successive waves on each line of which S_1 and S_3 are forward waves of mode 1 and mode 2; S_2 and S_4 are backward waves of mode 1 and mode 2 respectively. The voltage and current waves of each mode, S_1 and S_2 or S_3 and S_4, on each line are very similar to the waves shown in Figures 5 and 6. The only modifications required are, that after each reflection takes place at the receiving end or sending end, the amplitude of the reflected wave is to be reduced by a factor $\rho_{\ell i}$ or ρ_{gi} $(i = 1, 2)$. Also it is easy to see that the wave front varies with time due to the difference in velocity and the difference in reflection coefficients associated with these two modes of waves. For both the line voltage and current, the smaller the absolute value of the reflection coefficients, the less time required for reaching steady state. Theoretically it takes an infinitely long time to reach the steady state; but for practical purposes, it is said that the line is at steady state when the amplitude of reflected voltage or current wave is a small fraction of that quantity already on the line.

Since the receiving end voltages are of most interest, in Equation 58 let $z = \ell$.

$$V_1(z,\ell) = \frac{R_{o1}}{R_g+R_{o1}} E_o \sum_{n=0}^{\infty} (\rho_{g1}\rho_{\ell 1})^n (1 - \rho_{\ell 1}) U_{-1}\left[t - \frac{(2n+1)\ell}{v_1}\right] + \frac{R_{o2}}{R_g+R_{o2}} E_o \sum_{n=0}^{\infty} (\rho_{g2}\rho_{\ell 2})^n (1 - \rho_{\ell 2}) U_{-1}\left[t - \frac{(2n+1)\ell}{v_2}\right]$$

$$V_2(z,\ell) = -\frac{R_{o1}}{R_g+R_{o1}} E_o \sum_{n=0}^{\infty} (\rho_{g1}\rho_{\ell 1})^n (1 - \rho_{\ell 1}) U_{-1}\left[t - \frac{(2n+1)\ell}{v_1}\right] + \frac{R_{o2}}{R_g+R_{o2}} E_o \sum_{n=0}^{\infty} (\rho_{g2}\rho_{\ell 2})^n (1 - \rho_{\ell 2}) U_{-1}\left[t - \frac{(2n+1)\ell}{v_2}\right] \tag{59}$$

The variation of receiving end voltages with time are shown in Figure 7. Of course no numerical values can be given unless the line parameters and the terminating impedances are known.

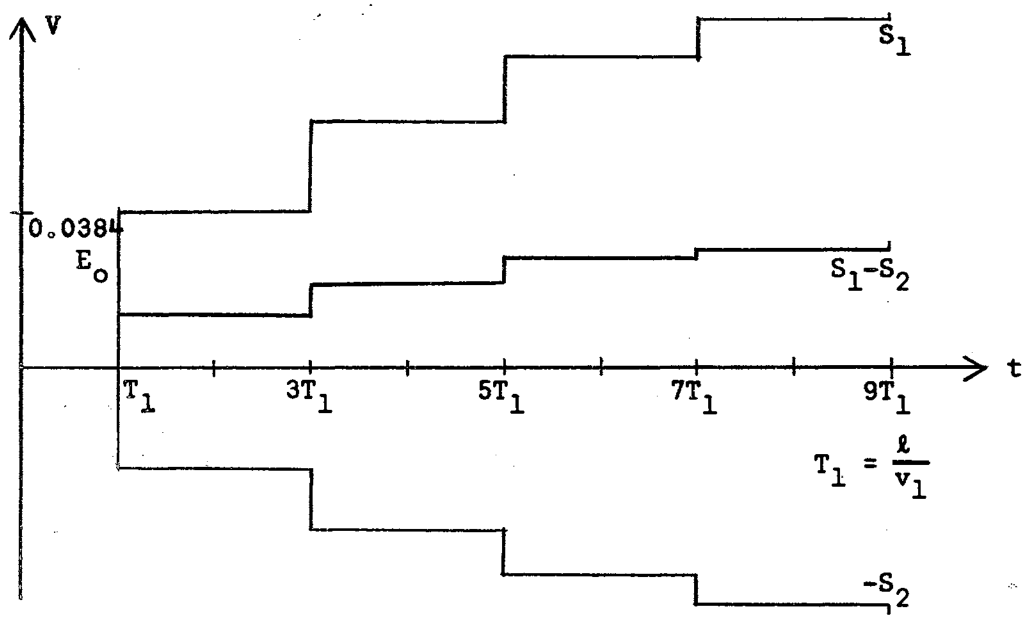

Figure 7a. Receiving End Voltages, Successive Waves of Mode 1.

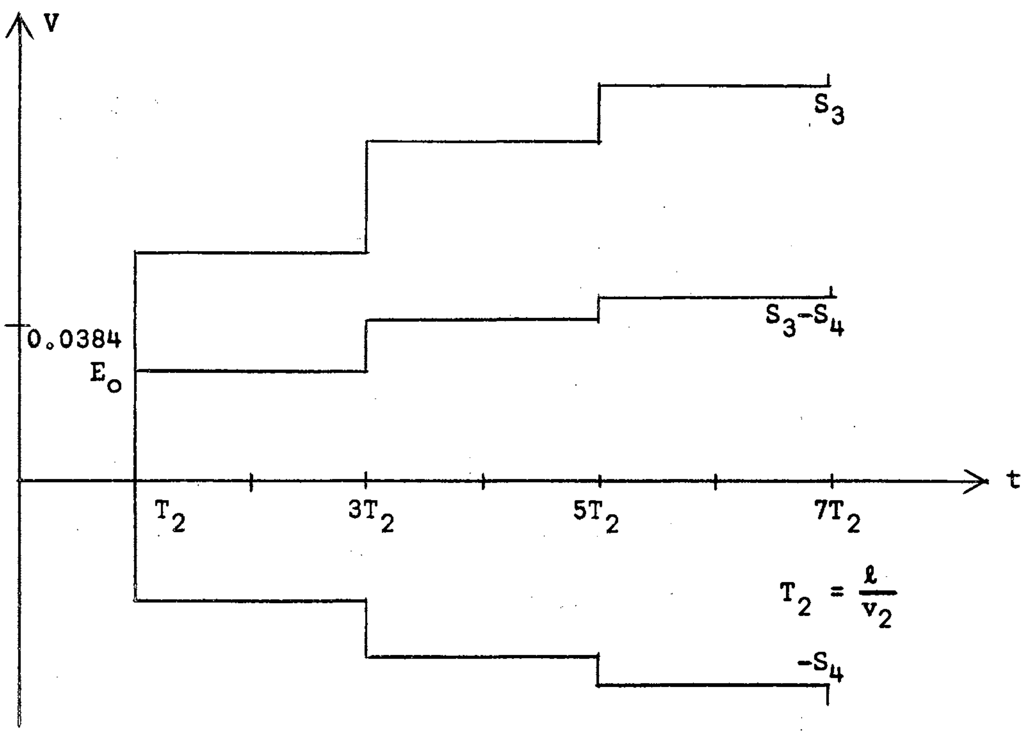

Figure 7b. Receiving End Voltages, Successive Waves of Mode 2.

Figure 7. Receiving End Voltages of a Coupled Transmission Lines.

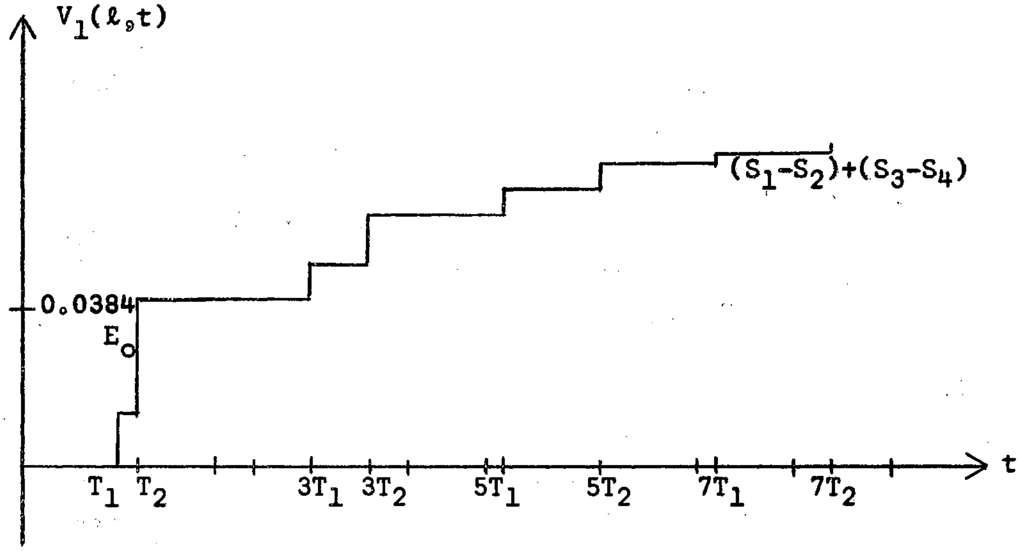

Figure 7c. Receiving End Voltage, Line No. 1.

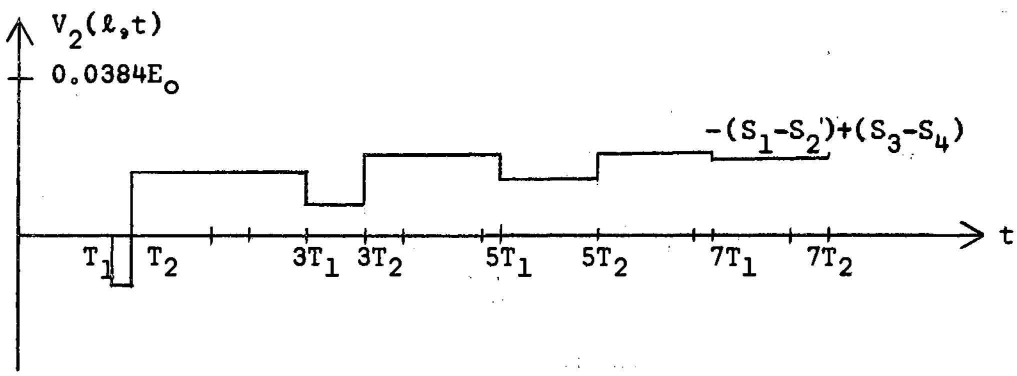

Figure 7d. Receiving End Voltage, Line No. 2.

Assumptions:

Z_{o1} = 40 ohms, Z_{o2} = 60 ohms,

$Z_{g1} = Z_{g2}$ = 1000 ohms,

$Z_{\ell 1} = Z_{\ell 2}$ = 200 ohms,

$v_2 = \frac{5}{6} v_1$ (arbitrary assumption)

Step voltage E_o applied on line no. 1

Figure 7. Receiving End Voltages of a Coupled Transmission Lines.

There are two line impedances for each line corresponding to the two modes of waves. At each end the terminating impedances R_ℓ and R_g may be chosen to match one of these two line impedances. By combination, there are six special cases.

(1) $Z_g = Z_{o1}$ i.e. $\rho_{g1} = 0$

(2) $Z_g = Z_{o2}$ i.e. $\rho_{g2} = 0$

(3) $Z_\ell = Z_{o1}$ i.e. $\rho_{\ell 1} = 0$

(4) $Z_\ell = Z_{o2}$ i.e. $\rho_{\ell 2} = 0$

(5) $Z_\ell = Z_{o1}$ and $Z_g = Z_{o2}$ i.e. $\rho_{\ell 1} = \rho_{g2} = 0$

(6) $Z_\ell = Z_{o2}$ and $Z_g = Z_{o1}$ i.e. $\rho_{\ell 2} = \rho_{g1} = 0$

For the first two cases, the voltage and current waves of one mode (mode 1 for case 1, mode 2 for case 2) propagate from the sending end to the receiving end; being reflected at the receiving end, they propagate back to the sending end, then stop there without any further reflection. The voltage and current waves of the other mode bounce back and forth between the two ends.

For the next two cases, the voltage and current waves of one mode (mode 1 for case 3, mode 2 for case 4) propagate from the sending end to the receiving end, then stop there without any reflection.

Cases 5 and 6 are optimum cases. The waves of one mode (mode 1 for 5, mode 2 for case 6) propagate to the receiving end without reflection; the other mode, being reflected from the receiving end, propagates back to the sending end and then stops there without further reflection.

CHAPTER IV

COUPLED LOSSY TRANSMISSION LINES

Two coupled, symmetrical and lossy transmission lines, grounded at both ends or terminated at both ends have been considered in the preceding chapter.

The solution in the p-domain for both ends grounded is:

$$\overline{V}_1(z,p) = \frac{\overline{E}_{g1}(p)\ \text{Sinh}[r_1(\ell-z)]}{2\text{Sinh}(r_1\ell)} + \frac{\overline{E}_{g1}(p)\ \text{Sinh}[r_3(\ell-z)]}{2\text{Sinh}(r_3\ell)}$$

$$= \overline{V}'_a(z,p) + \overline{V}'_b(z,p)$$

$$\overline{V}_2(z,p) = -\overline{V}'_a(z,p) + \overline{V}'_b(z,p) \tag{60}$$

$$\overline{I}_1(z,p) = \frac{\overline{E}_{g1}(p)\ \text{Cosh}[r_1(\ell-z)]}{2\ \overline{Z}_{o1}\ \text{Sinh}(r_1\ell)} + \frac{\overline{E}_{g1}(p)\ \text{Cosh}[r_3(\ell-z)]}{2\ \overline{Z}_{o2}\ \text{Sinh}(r_3\ell)}$$

$$= \overline{I}'_a(z,p) + \overline{I}'_b(z,p)$$

$$\overline{I}_2(z,p) = -\overline{I}'_a(z,p) + \overline{I}'_b(z,p) .$$

In the t-domain, the solution is:

$$V_1(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1}(p)\ \text{Sinh}[r_1(\ell-z)]}{2\text{Sinh}(r_1\ell)}\, e^{pt}\, dp$$

$$+ \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1}(p)\ \text{Sinh}[r_3(\ell-z)]}{2\text{Sinh}(r_3\ell)}\, e^{pt}\, dp = V'_a(z,t) + V'_b(z,t) \tag{61}$$

$$V_2(z,t) = -V_a'(z,t) + V_b'(z,t)$$

$$I_1(z,t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1}(p)\ \mathrm{Cosh}[r_1(\ell-z)]}{2\ \overline{Z}_{o1}\ \mathrm{Sinh}(r_1\ell)}\ e^{pt}\ dp$$

$$+ \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \frac{\overline{E}_{g1}(p)\ \mathrm{Cosh}[r_3(\ell-z)]}{2\ \overline{Z}_{o2}\ \mathrm{Sinh}(r_3\ell)}\ e^{pt}\ dp = I_a'(z,t) + I_b'(z,t)$$

$$I_2(z,t) = -I_a'(z,t) + I_b'(z,t) \quad .$$

They are of the exact same form as Equations 40 and 41 of the lossless case. The difference is that the parameters r_1, r_2, $\overline{Z}_{o1}$, $\overline{Z}_{o2}$, are functions of p, not just pure numbers as for the lossless case.

$$r_1 = \sqrt{(\overline{Z}-\overline{Z}_{12})(\overline{Y}+\overline{Y}_{12})} = \sqrt{(L-L_{12})(C+C_{12})}\ \sqrt{(p+2\alpha)(p+2\beta)} = \frac{1}{v_1}\sqrt{(p+2\alpha)(p+2\beta)}$$

$$r_2 = \sqrt{(\overline{Z}+\overline{Z}_{12})(\overline{Y}-\overline{Y}_{12})} = \sqrt{(L+L_{12})(C+C_{12})}\ \sqrt{(p+2\gamma)(p+2\delta)} = \frac{1}{v_2}\sqrt{(p+2\alpha)(p+2\beta)}$$

(62)

$$\overline{Z}_{o1} = \sqrt{\frac{\overline{Z}-\overline{Z}_{12}}{\overline{Y}+\overline{Y}_{12}}} = \sqrt{\frac{L-L_{12}}{C+C_{12}}}\sqrt{\frac{p+2\alpha}{p+2\beta}}$$

$$\overline{Z}_{o2} = \sqrt{\frac{\overline{Z}+\overline{Z}_{12}}{\overline{Y}-\overline{Y}_{12}}} = \sqrt{\frac{L+L_{12}}{C-C_{12}}}\sqrt{\frac{p+2\gamma}{p+2\delta}}$$

where

$$2\alpha = \frac{R_1-R_2}{L-L_{12}}$$

$$2\beta = \frac{G+G_{12}}{C+C_{12}}$$

$$2\gamma = \frac{R_1+R_2}{L+L_{12}}$$

$$2\delta = \frac{G-G_{12}}{C-C_{12}} \quad .$$

The complex integrals in Equation 61 can be evaluated by finding the residues but the presence of the doubled-value function r_1, r_3, and $\overline{Z}_{o1}$, $\overline{Z}_{o2}$, complicates the process of evaluation. The alternate method to be used here is by expressing the hyperbolic functions in Equation 60 in an exponential series and from the known transform pairs, obtain the solution in the t-domain.

For a step input voltage of amplitude E_o, $\overline{V}'_a(z,p)$ in Equation 60 can be rewritten as:

$$\overline{V}'_a(z,p) = \frac{E_o}{2p} \frac{\mathrm{Sin}[r_1(\ell - z)]}{\mathrm{Sin}(r_1 \ell)}$$

$$= \frac{E_o}{2} \sum_{n=0}^{\infty} \left[e^{-\frac{2n\ell+z}{v_1}\sqrt{(p+2\alpha)(p+2\beta)}} - e^{-\frac{2n\ell+2\ell-z}{2}\sqrt{(p+2\alpha)(p+2\beta)}} \right].$$

From the known Laplace pair,

$$V_a(z,t) = \frac{E_o}{2} \sum_{n=0}^{\infty} \Bigg\{ e^{-(\alpha+\beta)\frac{(2n\ell+z)}{v_1}} U_{-1}\left[t - \frac{2n\ell+z}{v_1}\right]$$

$$+ (\alpha-\beta)\frac{(2n\ell+z)}{v_1} \int_{\frac{2n\ell+z}{v_1}}^{t} e^{-(\alpha+\beta)t} \frac{I_1\left[(\alpha-\beta)\sqrt{t^2 - \left(\frac{2n\ell+z}{v_1}\right)^2}\right]dt}{\sqrt{t^2 - \left(\frac{2n\ell+z}{v_1}\right)^2}}$$

$$- e^{-(\alpha+\beta)(2n\ell+2\ell-z)} U_{-1}\left[t - \frac{2n\ell+2\ell-z}{v_1}\right] \tag{63}$$

$$+ (\alpha-\beta)\frac{2n\ell+2\ell-z}{v_1} \int_{\frac{2n\ell+2\ell-z}{v_1}}^{t} e^{-(\alpha+\beta)t} \frac{I_1\left[(\alpha-\beta)\sqrt{t^2-\left(\frac{2n\ell+2\ell-z}{v_1}\right)^2}\right]dt}{\sqrt{t^2 - \left(\frac{2n\ell+2\ell-z}{v_1}\right)^2}}.$$

The voltage wave of mode 2, $V_b(z,t)$, is of the exact same form if α, β, and v_1 in $V_a(z,t)$ are replaced by the corresponding quantities γ, δ, and v_2.

The current wave of mode 1, $\overline{I}'_a(z,p)$, may be rewritten as:

$$\overline{I}'_a(z,p) = \frac{E_o}{2}\sqrt{\frac{C+C_{12}}{L-L_{12}}}\sqrt{\frac{p+2\beta}{p+2\alpha}}\sum_{n=0}^{\infty}\left[e^{-\frac{2n\ell+z}{v_1}\sqrt{(p+2\alpha)(p+2\beta)}} + e^{-\frac{2n\ell+2\ell-z}{v_1}\sqrt{(p+2\alpha)(p+2\beta)}}\right].$$

The inverse Laplace transform of $I'_a(z,p)$ is also known.

$$I'_a(z,t) = \frac{E_o}{2}\sqrt{\frac{C+C_{12}}{L-L_{12}}}\sum_{n=0}^{\infty}\left\{e^{-(\alpha+\beta)t}\, I_o\left[(\alpha-\beta)\sqrt{t^2-\left(\frac{2n\ell+z}{v_1}\right)^2}\right]\right.$$

$$+ \alpha\int_{\frac{2n\ell+z}{v_1}}^{t} e^{-(\alpha+\beta)}\, I_o\left[(\alpha-\beta)\sqrt{t^2-\left(\frac{2n\ell+z}{v_1}\right)^2}\right]dt$$

$$+ e^{-(\alpha+\beta)}\, I_o\left[(\alpha-\beta)\sqrt{t^2-\left(\frac{2n\ell+2\ell-z}{v_1}\right)^2}\right] \tag{64}$$

$$\left. + \alpha\int_{\frac{2n\ell+2\ell-z}{v_1}}^{t} e^{-(\alpha+\beta)}\, I_o\left[(\alpha-\beta)\sqrt{t^2-\left(\frac{2n\ell+2\ell-z}{v_1}\right)^2}\right]dt\right\}.$$

The current wave of mode 2, $I'_b(z,t)$, is obtained by replacing α, β, v_1 and the factor $\sqrt{C+C_{12}/L-L_{12}}$ by γ, δ, v_2 and the factor $\sqrt{C-C_{12}/L+L_{12}}$.

As for the coupled lossy transmission lines, terminated at both ends, the solution in the p-domain is:

$$\begin{aligned}
\overline{V}_1(z,p) &= \overline{V}_c'(z,p) + \overline{V}_d'(z,p) \\
\overline{V}_2(z,p) &= -\overline{V}_c'(z,p) + \overline{V}_d'(z,p) \\
\overline{I}_1(z,p) &= \overline{I}_c'(z,p) + \overline{I}_d'(z,p) \\
\overline{I}_2(z,p) &= -\overline{I}_c'(z,p) + \overline{I}_d'(z,p) \quad ,
\end{aligned} \tag{65}$$

where $\overline{V}_c'(z,p)$ and etc. are given by

$$\overline{V}_c'(z,p) = \frac{\overline{E}_{g1}(p)}{2\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_{o1}}\right)^2\right]^{\frac{1}{2}}} \frac{\mathrm{Sinh}[r_1(\ell-z) + X_4']}{\mathrm{Sinh}[r_1\ell + X_5']}$$

$$\overline{V}_d'(z,p) = \frac{\overline{E}_{g1}(p)}{2\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_{o2}}\right)^2\right]^{\frac{1}{2}}} \frac{\mathrm{Sinh}[r_3(\ell-z) + X_6']}{\mathrm{Sinh}[r_3\ell + X_7']}$$

$$\overline{I}_c'(z,p) = \frac{\overline{E}_{g1}(p)}{2\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_{o1}}\right)^2\right]^{\frac{1}{2}}} \frac{\mathrm{Cosh}[r_3(\ell-z) + X_4']}{\mathrm{Sinh}[r_3\ell + X_5']}$$

$$\overline{I}_d'(z,p) = \frac{\overline{E}_{g1}(p)}{2\left[1 - \left(\frac{\overline{Z}_g}{\overline{Z}_{o2}}\right)^2\right]^{\frac{1}{2}}} \frac{\mathrm{Cosh}[r_3(\ell-z) + X_6']}{\mathrm{Sinh}[r_3\ell + X_7']} \quad ,$$

where

$$X_4' = \mathrm{Tanh}^{-1} \frac{\overline{Z}_\ell}{\overline{Z}_{o1}} \quad ,$$

$$X_5' = \mathrm{Tanh}^{-1} \frac{\dfrac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o1}}}{1 + \dfrac{\overline{Z}_g\overline{Z}_\ell}{\overline{Z}_{o1}^2}}$$

$$X_6' = \mathrm{Tanh}^{-1} \frac{\overline{Z}_\ell}{\overline{Z}_{o2}}$$

$$X_7' = \mathrm{Tanh}^{-1} \frac{\dfrac{\overline{Z}_g+\overline{Z}_\ell}{\overline{Z}_{o2}}}{1 + \dfrac{\overline{Z}_g\overline{Z}_\ell}{\overline{Z}_{o2}^2}} \quad .$$

By the inversion theorem, the formal solution is:

$$\begin{aligned} V_1(z,t) &= V_c'(z,t) + V_d'(z,t) \\ V_2(z,t) &= -V_c'(z,t) + V_d'(z,t) \\ I_1(z,t) &= I_c'(z,t) + I_d'(z,t) \\ I_2(z,t) &= -I_c'(z,t) + I_d'(z,t) \quad . \end{aligned} \tag{66}$$

The evaluation of each complex integral can be done much the same as for the isolated transmission line in the end of Chapter II, since the voltage or current waves of each mode is of the exact same form as the waves on an isolated line.

CHAPTER V

A PRACTICAL EXAMPLE - COUPLED LOSSLESS TRANSMISSION LINES WITH CAPACITIVE TERMINATIONS

In this chapter, a practical example, but a less general case, is considered to demonstrate the theory presented in this thesis.

A symmetrical coupled lossless transmissional line, with capacitive termination, is chosen because the lossless line is a good approximation for most practical cases (especially for the physically short line). The capacitive termination is assumed to represent the most solid state devices, which in general have capacitive input and output impedance.

The operational line impedances and propagation constants are:

$$\overline{Z}_{o1} = R_{o1}, \qquad \overline{Z}_{o2} = R_{o2},$$

$$r_1 = \sqrt{(L-L_{12})(C+C_{12})}\, p, \qquad r_2 = \sqrt{(L+L_{12})(C-C_{12})}\, p,$$

and the operational load impedances and source impedances are:

$$\begin{aligned} \overline{Z}_{g1} &= \overline{Z}_{g2} = R_g + \frac{1}{C_g p}, \\ \overline{Z}_{\ell 1} &= \overline{Z}_{\ell 2} = R_\ell + \frac{1}{C_\ell p}. \end{aligned} \tag{67}$$

Then the operational reflection coefficients are:

$$\overline{\rho}_{g1} = \rho_{g1}\frac{p+a_1}{p+b_1}, \quad \rho_{g1} = \frac{R_g-R_{o1}}{R_g+R_{o1}}, \quad a_1 = \frac{1}{(R_g-R_{o1})C_g}, \quad b_1 = \frac{1}{(R_g+R_{o1})C_g}, \tag{68}$$

$$\overline{\rho}_{g2} = \rho_{g2}\frac{p+a_2}{p+b_2}, \quad \rho_{g2} = \frac{R_g-R_{o2}}{R_g+R_{o1}}, \quad a_2 = \frac{1}{(R_g-R_{o2})C_g}, \quad b_2 = \frac{1}{(R_g+R_{o2})C_g}, \tag{68}$$

$$\overline{\rho}_{\ell 1} = \rho_{\ell 1}\frac{p+c_1}{p+d_1}, \quad \rho_{\ell 1} = \frac{R_\ell-R_{o1}}{R\ +R_{o2}}, \quad c_1 = \frac{1}{(R_\ell-R_{o1})C_g}, \quad d_1 = \frac{1}{(R_\ell+R_{o1})C_g},$$

$$\overline{\rho}_{\ell 2} = \rho_{\ell 2}\frac{p+c_2}{p+d_2}, \quad \rho_{\ell 2} = \frac{R_\ell-R_{o2}}{R_\ell+R_{o2}}, \quad c_2 = \frac{1}{(R_\ell-R_{o2})C_g}, \quad d_2 = \frac{1}{(R_\ell+R_{o2})C_g}.$$

Inserting these values into Equation 54, the transformed voltage on the two lines for a step voltage applied on the first line is:

$$\overline{V}_1(z,p) = \overline{V}_c(z,p) + \overline{V}_d(z,p)$$
$$\overline{V}_2(z,p) = -\overline{V}_c(z,p) + \overline{V}_d(z,p) \tag{69}$$

where $\overline{V}_c(z,p)$ and $\overline{V}_d(z,p)$ are given by:

$$\overline{V}_c(z,p) = E_o\frac{R_{o1}}{R_g+R_{o1}}\Bigg[\sum_{n=0}^{\infty}(\rho_{g1}\rho_{\ell 1})^n\frac{(p+a_1)^n}{(p+b_1)^{n+1}}\left(\frac{p+c_1}{p+d_1}\right)^n e^{-\frac{2n\ell+z}{v_1}p}$$
$$-\sum_{n=1}^{\infty}\rho_{g1}^{n}\rho_{\ell 1}^{n+1}\frac{(p+a_1)^n}{(p+b_1)^{n+1}}\left(\frac{p+c_1}{p+d_1}\right)^{n+1} e^{-\frac{2n\ell+2\ell-z}{v_1}p}\Bigg] \tag{70}$$

$$\overline{V}_d(z,p) = E_o\frac{R_{o2}}{R_g+R_{o2}}\Bigg[\sum_{n=0}^{\infty}(\rho_{g2}\rho_{\ell 2})^n\frac{(p+a_2)^n}{(p+b_2)^{n+1}}\left(\frac{p+c_1}{p+d_1}\right)^n e^{-\frac{2n\ell+z}{v_2}p}$$
$$-\sum_{n=0}^{\infty}\rho_{g2}^{n}\rho_{\ell 2}^{n+1}\frac{(p+a_2)^n}{(p+b_2)^{n+1}}\left(\frac{p+c_1}{p+d_1}\right)^{n+1} e^{-\frac{2n\ell+2\ell-z}{v_2}p}\Bigg].$$

By expanding in partial fractions and taking the inverse Laplace transform term by term,

$$\mathcal{L}^{-1}\left[\frac{(p+a_1)^n}{(p+b_1)^{n+1}}\left(\frac{p+c_1}{p+d_1}\right)^n\right]$$

$$= \sum_{i=1}^{n+1} \frac{K_{1i}}{(n+1-i)!} t^{(n+1-i)} e^{-b_1 t} + \sum_{i=1}^{n} \frac{K_{2i}}{(n-i)!} t^{(n-i)} e^{-d_1 t}.$$

Then by the shifting theorem,

$$\mathcal{L}^{-1}\left[\frac{(p+a_1)^n}{(p+b_1)^{n+1}}\left(\frac{p+c_1}{p+d_1}\right)^n e^{-\frac{2n\ell+z}{v_1}p}\right]$$

$$= \sum_{i=1}^{n+1} \frac{K_{1i}}{(n+1-i)!}\left(t-\frac{2n\ell+z}{v_1}\right)^{(n+1-i)} e^{-b_1\left(t-\frac{2n\ell+z}{v_1}\right)} U_{-1}\left[t-\frac{2n\ell+z}{v_1}\right]$$

$$\sum_{i=1}^{n} \frac{K_{2i}}{(n-i)!}\left(t-\frac{2n\ell+z}{v_1}\right)^{(n-i)} e^{-d_1\left(t-\frac{2n\ell+z}{v_1}\right)} U_{-1}\left[t-\frac{2n\ell+z}{v_1}\right].$$

Therefore the line voltage for waves of mode 1 is:

$$V_c(z,t) = E_o \frac{R_{o1}}{R_g+R_{o1}} \sum_{n=0}^{\infty} \cdot$$

$$\left\{ \sum_{i=1}^{n+1} (\rho_{g1}\rho_{\ell 1})^n \frac{K_{1i}}{(n+1-i)!}\left(t-\frac{2n\ell+z}{v_1}\right)^{(n+1-i)} e^{-b_1\left(t-\frac{2n\ell+z}{v_1}\right)} U_{-1}\left[t-\frac{2n\ell+z}{v_1}\right]\right. \tag{71}$$

$$+ \sum_{i=1}^{n} (\rho_{g1}\rho_{\ell 1})^n \frac{K_{2i}}{(n-i)!}\left(t-\frac{2n\ell+z}{v_1}\right)^{(n-i)} e^{-d_1\left(t-\frac{2n\ell+z}{v_1}\right)} U_{-1}\left[t-\frac{2n\ell+z}{v_1}\right]$$

$$- \sum_{i=1}^{n+1} \rho_{g1}^{\,n}\rho_{\ell 1}^{\,n+1} \frac{K_{3i}}{(n+1-i)!}\left(t-\frac{2n\ell+2\ell-z}{v_1}\right)^{(n+1-i)} e^{-b_1\left(t-\frac{2n\ell+2\ell-z}{v_1}\right)} U_{-1}\left[t-\frac{2n\ell+2\ell-z}{v_1}\right]$$

$$\left.- \sum_{i=1}^{n+1} \rho_{g1}^{\,n}\rho_{\ell 1}^{\,n+1} \frac{K_{4i}}{(n+1-i)!}\left(t-\frac{2n\ell+2\ell-z}{v_1}\right)^{(n+1-i)} e^{-b_1\left(t-\frac{2n\ell+2\ell-z}{v_1}\right)} U_{-1}\left[t-\frac{2n\ell+2\ell-z}{v_1}\right]\right\}$$

where K_{ji} (j = 1, 2, 3, 4) are coefficients of each term of the partial fractional expressions

$$K_{1i} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dp^{i-1}}(p+a_1)^n\left(\frac{p+c_1}{p+d_1}\right)^n\right]_{p=b_1}$$

$$K_{2i} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dp^{i-1}}(p+c_1)^n\ \frac{(p+a_1)^n}{(p+b_1)^{n+1}}\right]_{p=d_1}$$

$$K_{3i} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dp^{i-1}}(p+a_1)^n\left(\frac{p+c_1}{p+d_1}\right)^{n+1}\right]_{p=b_1} \tag{72}$$

$$K_{4i} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dp^{i-1}}(p+c_1)^{n+1}\ \frac{(p+a_1)^n}{(p+b_1)^{n+1}}\right]_{p=d_1}$$

By a similar process, it is found that the line voltage for waves of mode 2 is:

$$V_d(z,t) = E_o\,\frac{R_{o2}}{R_g+R_{o2}}\sum_{n=0}^{\infty}\ \cdot$$

$$\left\{\sum_{i=1}^{n+1}(\rho_{g2}\rho_{\ell 2})^n\,\frac{K'_{1i}}{(n+1-i)!}\left(t-\frac{2n\ell+z}{v_2}\right)^{(n+1-i)} e^{-b_2\left(t-\frac{2n\ell+z}{v_1}\right)}U_{-1}\left[t-\frac{2n\ell+z}{v_1}\right]\right.$$

$$+\sum_{i=1}^{n}(\rho_{g2}\rho_{\ell 2})^n\,\frac{K'_{2i}}{(n-i)!}\left(t-\frac{2n\ell+z}{v_2}\right)^{(n-i)} e^{-d_2\left(t-\frac{2n\ell+z}{v_1}\right)}U_{-1}\left[t-\frac{2n\ell+z}{v_2}\right] \tag{73}$$

$$-\sum_{i=1}^{n+1}\rho_{g2}^{\,n}\rho_{\ell 2}^{\,n+1}\,\frac{K'_{3i}}{(n+1-i)!}\left(t-\frac{2n\ell+2\ell-z}{v_2}\right)^{(n+1-i)} e^{-b_2\left(t-\frac{2n\ell+2\ell-z}{v_2}\right)}U_{-1}\left[t-\frac{2n\ell+2\ell-z}{v_2}\right]$$

$$\left.-\sum_{i=1}^{n+1}\rho_{g2}^{\,n}\rho_{\ell 2}^{\,n+1}\,\frac{K'_{4i}}{(n+1-i)!}\left(t-\frac{2n\ell+2\ell-z}{v_2}\right)^{(n+1-i)} e^{-d_2\left(t-\frac{2n\ell+2\ell-z}{v_2}\right)}U_{-1}\left[t-\frac{2n\ell+2\ell-z}{v_2}\right]\right\}$$

and

$$K'_{1i} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dp^{i-1}}\ (p+a_2)^n\left(\frac{p+c_2}{p+d_2}\right)^n\right]_{p=b_2}$$

$$K'_{2i} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dp^{i-1}}\ (p+c_2)^n\ \frac{(p+a_2)^n}{(p+b_2)^{n+1}}\right]_{p=d_2}$$

$$K'_{3i} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dp^{i-1}}\ (p+a_2)^n\left(\frac{p+c_2}{p+d_2}\right)^{n+1}\right]_{p=b_2} \qquad (74)$$

$$K'_{4i} = \frac{1}{(i-1)!}\left[\frac{d^{i-1}}{dp^{i-1}}\ (p+c_2)^{n+1}\ \frac{(p+a_2)^n}{(p+b_2)^{n+1}}\right]_{p=d_2} \ .$$

This formal solution for line voltage for each line which consists of eight double infinite series looks very complicated. Fortunately, for a practical transmission problem, only the first few terms of each series are required for an approximate solution.

CHAPTER VI

SUMMARY

This thesis presents the theory of an insolated transmission line with both ends terminated, and then the theory of two coupled symmetrical lines. It is shown that there are two modes of waves on two coupled transmission lines and that each of these two modes of waves is of the exact same form as the waves existing on an isolated transmission line under the same conditions. The parameters, line impedances, and propagation constants of these two modes of waves of two coupled lines are the perturbed quantities of an isolated line.

The future work suggested in this area is:

(1) the theory of a coupled system of two non-symmetrical lines;

(2) the theory of a coupled system of multi-transmission lines, symmetrical or non-symmetrical;

(3) the evaluation of line parameters, L, C, R, G, $\overline{\overline{Z}}_{ij}$, and r_{ij}, of a practical multi-line system.

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