UNIFORMIZABLE SPACES

By

DONALD FRANK SHULT

Bachelor of Science
Northwestern University
Evanston, Illinois
1952

Master of Science
Southern Illinois University
Carbondale, Illinois
1962

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
July, 1970

UNIFORMIZABLE SPACES

Thesis Approved:

Forrest D. Whitfield
Thesis Adviser

John Jobe

E. K. McLachlan

W. Ware Marsden

D. Durham
Dean of the Graduate College

PREFACE

Uniform spaces were introduced in 1937 by A. Weil (18) in an attempt to generalize the idea of a metric space to a space for which uniform continuity and completeness could be defined without using a metric or a distance function. It was generally believed until this time that a distance function was needed to define these concepts (4).

Briefly stated, a uniform space consists of a set X and a non-empty collection U^* of relations on X which satisfy certain properties in such a way that U^* can be used to define a topology T_U for X, uniform continuity of a function, and completeness. The collection U^* is called a uniformity for X and T_U is said to be the uniform topology for X induced by U^*. A topological space is uniformizable if its topology is induced by some uniformity for X. Chapter I is an introduction to uniformizable spaces.

There are many different characterizations of uniformizability. Perhaps the most widely known of these is the property of being completely regular. Completely regular spaces, first introduced by Tychonoff (17) in 1929, are defined in terms of the family of all continuous real-valued functions on the space. This characterization is investigated in Chapter II. Other characterizations of uniformizability which also use this family of functions are studied in Chapter III. Chapter IV discusses various characterizations of uniformizability which are in terms of a family of pseudometrics or pseudometric spaces. Characterizations of uniformizable T_1 spaces are investigated in Chapter V. In Chapter VI a sufficient condition and then a necessary condition will be given in order that a uniformizable space will have a unique uniformity which induces the topology.

I am very grateful for the guidance and encouragement given to me by my adviser Dr. Forrest Whitfield, during the preparation of this thesis. I also wish to thank Dr. E. K. McLachlan, Dr. John Jobe, and Dr. W. Ware Marsden for serving as members of my advisory committee.

I am especially thankful for the encouragement and support of my wife Ruth, who sacrificed much that I might finish this degree.

TABLE OF CONTENTS

CHAPTER I

BASIC CONCEPTS AND THEOREMS OF UNIFORMIZABLE SPACES

Before defining what a uniform space is, some basic definitions and notations must be introduced. If U is a relation on X, then U^{-1} is the set $\{(y, x) : (x, y) \in U\}$. U is said to be symmetric if $U = U^{-1}$. If U and V are relations on X, then

$$U \circ V = \{(x, z) : (x, y) \in V \text{ and } (y, z) \in U \text{ for some } y \text{ in } X\}.$$

The diagonal of X is the set $\{(x, x) : x \in X\}$ and is denoted by D_X. If U is a relation on X and A is a subset of X, then

$$U[A] = \{y \in X : (x, y) \in U \text{ for some } x \in A\}.$$

Although there are several equivalent definitions of a uniform space, the following definition in Kelly (12) will be used.

<u>Definition 1.1</u> A uniform space (X, U^*) is a set X and a non-empty collection U^* of relations on X such that

(a) For any U in U^*, D_X is a subset of U.

(b) For any U in U^*, U^{-1} is in U^*.

(c) For any U in U^*, there exists a V in U^* such that $V \circ V \subset U$.

(d) If U and V are in U^*, then $U \cap V$ is in U^*.

(e) If U is in U^* and $U \subset V \subset X \times X$, then V is in U^*.

Definition 1.2 If (X, U^*) is a uniform space, then U^* is called a uniformity for X or a uniform structure for X. Members of U^* are called entourages of U^*.

Definition 1.3 If (X, U^*) is a uniform space, then the uniform topology T_U for X is the collection of all subsets O of X with the property that for any x in O there exists a U in U^* such that $U[x] \subset O$. T_U is said to be induced by U^*.

It is easy to verify that T_U in the preceding definition is indeed a topology for X. For certainly $\emptyset$ and X are in T_U. For any i in I, an index set, let O_i be in T_U and let x be in $\bigcup_{i \in I} O_i$. Then for some j in I, x is in O_j. This implies that there exists a U in U^* such that $U[x] \subset O_j$ and hence $U[x] \subset \bigcup_{i \in I} O_i$. Therefore $\bigcup_{i \in I} O_i$ is a member of T_U. If G and H are in T_U and x is in $G \cap H$, then there are entourages U and V in U^* such that $U[x] \subset G$ and $V[x] \subset H$. Therefore $U[x] \cap V[x] \subset G \cap H$. Since

$$U[x] \cap V[x] = (U \cap V)[x]$$

and $U \cap V$ is in U^*, then $G \cap H$ is in T_U.

The main concern of this paper is uniformizable spaces which will now be defined.

Definition 1.4 A topological space (X, T) is uniformizable if and only if there exists a uniformity U^* for X such that the uniform topology induced by U^* is the topology T. U^* is called an admissible uniformity for T or U^* is said to be compatible with T.

Example 1.5 For a given set X, define $U^* = \{X \times X\}$. It is easy to verify that U^* is a uniformity for X. For any x in X and U in U^*, $U[x] = X$, and consequently T_U is the indiscrete topology for X.

Example 1.6 For a given set X, define U^* to be the family of all relations on X which contain the diagonal of X. If U is in U^*, then U contains the diagonal and hence U^{-1} does also. Therefore U in U^* implies that U^{-1} is in U^*. If V is in U^*, then D_X is in U^* and $D_X \circ D_X = D_X \subset V$. The other three properties of a uniformity are obviously true for U^* and so U^* is a uniformity for X. For any $x \in X$, $D_X[x] = \{x\}$, and hence T_U is the discrete topology for X.

As the preceding examples show, any topological space with the discrete or indiscrete topology is uniformizable. However, not every topological space is uniformizable as the following example shows.

Example 1.7 A topological space which is not uniformizable. Let $X = \{a, b\}$ and let $T = \{\emptyset, \{a\}, X\}$. If U^* is a uniformity for X and D_X is in U^*, then by property (e) of a uniformity any relation on X containing the diagonal is in U^*, and hence U^* is the uniformity of Example 1.6. Suppose D_X is not in U^*, U is in U^*, and U is not $X \times X$. Then without loss of generality $U = D_X \cup \{(a,b)\}$. Because of property (b) of a uniformity $U^{-1} = D_X \cup \{(b,a)\}$ is in U^*. Therefore, by property (d) of a uniformity, $U \cap U^{-1} = D_X$ is in U^*. But D_X is not in U^*. Therefore if D_X is not in U^*, $U^* = \{X \times X\}$, the uniformity of Example 1.5. Since the only uniformities which can be defined on X are compatible with the discrete and indiscrete topologies for X, then (X, T) is not uniformizable.

The two following definitions are often used in defining a uniformity as will be shown later.

Definition 1.8 A subfamily B^* of a uniformity U^* is a base for U^* if and only if each entourage in U^* contains an entourage of B^*.

Definition 1.9 A subfamily S^* of a uniformity U^* is a subbase for U^* if and only if the collection of all intersections of finite subfamilies of S^* is a base for U^*.

Three other basic definitions which will be used in this paper are the following.

Definition 1.10 If (X, U^*) and (Y, V^*) are uniform spaces and if f is a function from X into Y, then f is uniformly continuous relative to U^* and V^* if and only if for each V in V^*, $f_2^{-1}(V)$ is in U^*. f_2 is the function from $X \times X$ into $Y \times Y$ such that

$$f_2(x, y) = (f(x), f(y))$$

for any (x, y) in $X \times X$.

An equivalent condition for uniform continuity is that for any V in V^*, there is a U in U^* such that $f_2(U) \subset V$.

Definition 1.11 If (X, T) is a topological space, define $C(X)$ to be the set of all real-valued continuous functions defined on X, where the real numbers have their usual topology.

Definition 1.12 The topological space (X, T) is said to be completely regular if and only if for any closed subset F of X and any x in $X - F$, there exists an f in $C(X)$ such that $f(x) = 0$, $f(F) \subset \{1\}$, and $f(X) \subset [0, 1]$.

An equivalent condition for (X, T) to be completely regular is that for any closed subset F of X and any $x \in X - F$, there exists an f in $C(X)$ such that $f(x) = 1$, $f(F) \subset \{0\}$, and $f(X) \subset [0, 1]$. Another equivalent condition for (X, T) to be completely regular is that for any x in X and any neighborhood N of x, there is an f in $C(X)$ such that $f(x) = 0$, $f(X - N) \subset \{1\}$, and $f(X) \subset [0, 1]$.

The following theorems and corollaries will be frequently referred to in various proofs in this paper, since they are often used to define a uniformity for a set.

Theorem 1.13 A non-empty family S^* of subsets of $X \times X$ is a subbase for a uniformity for X if

(a) For any S in S^*, $S \supset D_X$.

(b) For any S in S^*, S^{-1} contains a member of S^*.

(c) For any S in S^*, there is a V in S^* such that $V \circ V \subset S$.

Proof: Let U^* be the collection of all relations on X which contain the intersection of a finite subfamily of S^*. Since U^* contains S^* and S^* is non-empty, U^* is non-empty. Let U be a member of U^*. Then there exist sets $S_1, S_2, \ldots, S_n$ in S^* such that $U \supset \bigcap_1^n S_i$. Since $S_i \supset D_X$ for any i, $\bigcap_1^n S_i \supset D_X$. Therefore $U \supset D_X$. For any i, there exists a T_i in S^* such that $S_i^{-1} \supset T_i$.

This implies that

$$\bigcap_1^n S_i^{-1} \supset \bigcap_1^n T_i.$$

Since U contains $\bigcap_1^n S_i$,

$$U^{-1} \supset (\bigcap_1^n S_i)^{-1}.$$

But

$$(\bigcap_1^n S_i)^{-1} = \bigcap_1^n S_i^{-1}$$

and consequently $U^{-1} \supset \bigcap_1^n T_i$. Therefore U^{-1} is in U^*.

For any i, there exists a T_i in S^* such that $T_i \circ T_i \subset S_i$. This implies that

$$\bigcap_1^n (T_i \circ T_i) \subset \bigcap_1^n S_i.$$

Since

$$(\bigcap_1^n T_i) \circ (\bigcap_1^n T_i) \subset \bigcap_1^n (T_i \circ T_i),$$

then

$$(\bigcap_1^n T_i) \circ (\bigcap_1^n T_i) \subset \bigcap_1^n S_i \subset U.$$

Since $\bigcap_1^n T_i$ is in U^*, then for any U in U^*, there is a V in U^* such that $V \circ V \subset U$.

If U and V are members of U^*, then each contains the intersection of a finite subfamily of S^* and hence their intersection contains the intersection of a finite subfamily of S^*. Therefore $U \cap V$

is in U^*.

If U is in U^* and $U \subset V \subset X \times X$, then by definition of U^*, V contains the intersection of a finite subfamily of S^*. This implies that V is in U^*.

Since the five properties of a uniformity are true and since S^* is a subfamily of U^*, then S^* is a subbase for U^*.

In the next theorem and the remainder of this paper the notion of a pseudometric will be the same as that defined in Kelley (12).

Theorem 1.14 If F is a non-empty family of pseudometrics on X and for any p in F and $r > 0$, $V_{p,r} = \{(x, y) : p(x, y) < r\}$, then $A^* = \{V_{p,r} : p \in F \text{ and } r > 0\}$ is a subbase for a uniformity for X.

Proof: Let $V_{p,r}$ be a member of A^*. If $x \in X$, then $p(x, x) = 0$ and hence (x,x) is in $V_{p,r}$. Therefore $D_X \subset V_{p,r}$.

For any (x, y), $p(x, y) < r$ if and only if $p(y, x) < r$. Therefore $V_{p,r} = V^{-1}_{p,r}$ and hence $V^{-1}_{p,r}$ contains a member of A^*.

Now

$$V_{p,\frac{r}{2}} \circ V_{p,\frac{r}{2}} \subset V_{p,r}.$$

For if (x, z) is in $V_{p,\frac{r}{2}} \circ V_{p,\frac{r}{2}}$, then there exists a y such that (x, y) is in $V_{p,\frac{r}{2}}$ and (y, z) is in $V_{p,\frac{r}{2}}$. This implies that

$$p(x, y) < \frac{r}{2} \quad \text{and} \quad p(y, z) < \frac{r}{2},$$

which in turn implies that

$$p(x, z) \leq p(x, y) + p(y, z) < r.$$

Since $V_{p, \frac{r}{2}}$ is in A^*, property (c) of Theorem 1.13 is true. Therefore by Theorem 1.13, A^* is a subbase for a uniformity U^* for X, which consists of all relations on X which contain the intersection of a finite subfamily of A^*.

Corollary 1.15 If p is a pseudometric on X, then

$$A^* = \{V_{p, r} : r > 0\}$$

is a base for a uniformity for X which is compatible with the pseudometric topology T for X. Therefore any pseudometric space is uniformizable.

Proof: By Theorem 1.14, A^* is a subbase for a uniformity U^* for X, where U^* consists of all relations on X which contain the intersection of a finite subfamily of A^*. Let U be in U^*. Then there are positive numbers $r_1, r_2, \ldots, r_n$ such that $U \supset \bigcap_1^n V_{p, r_i}$. If r is the minimum of $\{r_1, r_2, \ldots, r_n\}$, then

$$\bigcap_1^n V_{p, r_i} \supset V_{p, r}$$

and so $U \supset V_{p, r}$. Since $V_{p, r}$ is in A^* and $A^* \subset U^*$, then A^* is a base for U^*. Actually U^* is the collection of all relations on X which contain a member of A^*.

For any $r > 0$ and any $x \in X$,

$$V_{p, r}[x] = \{y : (x, y) \in V_{p, r}\} = \{y : p(x, y) < r\} = S_r(x),$$

an open sphere about x of radius r. If O is in T_U and x is in O, then there exists a U in U^* such that $U[x] \subset O$. Since A^* is a base for U^*, there is an $r > 0$ such that $V_{p,r} \subset U$. This implies that

$$V_{p,r}[x] \subset U[x]$$

and hence that $S_r(x) \subset O$. Therefore O is in the pseudometric topology for X. Conversely, if O is in the pseudometric topology for X and x is in O, then there is an open sphere $S_r(x)$ such that $S_r(x) \subset O$. This implies that $V_{p,r}[x] \subset O$ and consequently O is in T_U. Therefore T_U is equal to the pseudometric topology for X and so the pseudometric space induced by p is uniformizable.

Example 1.16 The space E^1 is uniformizable. The collection of sets of the form $V_r = \{(x, y) : |x - y| < r\}$, r a positive number, is a base for a uniformity U^* for E^1, called the usual uniformity for E^1. U^* is compatible with the open interval topology for E^1. This is a direct result of Corollary 1.15, because if the function p from $E^1 \times E^1$ into E^1 is defined so that $p(x, y) = |x - y|$ for any x and y in E^1, then p is a pseudometric on E^1 and for any $r > 0$, $V_{p,r} = V_r$.

Theorem 1.17 Let F be a non-empty family of functions such that each f in F maps X into a uniform space (Y_f, U_f^*). Then

$$A^* = \{f_2^{-1}(U) : f \in F \text{ and } U \in U_f^*\}$$

is a subbase for a uniformity for X.

Proof: Let f be in F and U be in U_f^*. For any $x \in X$, $f_2(x, x) = (f(x), f(x))$ which is in the diagonal of Y_f. Since the diagonal of Y_f is a subset of U, $f_2(x, x)$ is in U. Consequently (x, x) is in $f_2^{-1}(U)$ and hence D_X is a subset of $f_2^{-1}(U)$.

There is a V in U_f^* such that $V = V^{-1}$ and $V \subset U$ since $U \cap U^{-1}$ is a member of U_f^* with this property. Since

$$f_2^{-1}(V) \subset f_2^{-1}(U),$$

then

$$[f_2^{-1}(V)]^{-1} \subset [f_2^{-1}(U)]^{-1}.$$

Now

$$f_2^{-1}(V) \subset [f_2^{-1}(V)]^{-1}.$$

For if (x, y) is in $f_2^{-1}(V)$, then $f_2(x, y) = (f(x), f(y))$ is in V. This implies that $(f(y), f(x)) = f_2(y, x)$ is in V which implies that (y, x) is in $f_2^{-1}(V)$. Therefore (x, y) is in $[f_2^{-1}(V)]^{-1}$. Therefore $[f_2^{-1}(U)]^{-1}$ contains $f_2^{-1}(V)$ which is in A^*.

There is a V in U_f^* such that $V \circ V \subset U$ and this implies that

$$f_2^{-1}(V \circ V) \subset f_2^{-1}(U).$$

Now

$$f_2^{-1}(V) \circ f_2^{-1}(V) \subset f_2^{-1}(V \circ V).$$

For if (x, y) is in $f_2^{-1}(V)$ and (y, z) is in $f_2^{-1}(V)$, then $(f(x), f(y))$ is in V and $(f(y), f(z))$ is in V. But this implies that $(f(x), f(z)) = f_2(x,z)$ is in $V \circ V$ which implies that (x, z) is in $f_2^{-1}(V \circ V)$. Therefore there exists a member $f_2^{-1}(V)$ of A^* such that $f_2^{-1}(V) \circ f_2^{-1}(V) \subset f_2^{-1}(U)$. Therefore by Theorem 1.13, A^* is a subbase for a uniformity for X.

Theorem 1.17 can be used to define a uniformity for the cartesian product of uniform spaces as the next corollary shows.

Corollary 1.18 For any a in an index set A, let (X_a, U_a^*) be a uniform space and let $Z = \underset{a \in A}{\times} X_a$. Then the family of all sets of the form

$$\{(x, y) \in Z \times Z : (x_a, y_a) \in U\},$$

where a is in A and U is a member of U_a^*, is a subbase for a uniformity for Z. This uniformity is called the product uniformity for Z.

Proof: For any $a \in A$, let P_a be the projection of Z into X_a. By Theorem 1.17,

$$\{P_{a2}^{-1}(U) : a \in A \text{ and } U \in U_a^*\}$$

is a subbase for a uniformity for Z. For any $a \in A$ and $U \in U_a^*$,

$P_{a2}^{-1}(U)$

$$= \{(x, y) \in Z \times Z : (P_a(x), P_a(y)) \in U\} = \{(x, y) \in Z \times Z : (x_a, y_a) \in U\}.$$

CHAPTER II

THE EQUIVALENCE OF UNIFORMIZABLE, COMPLETELY REGULAR, AND (T) SPACES

The main objective of Chapter II is to show the equivalence of completely regular spaces, (T) spaces, and uniformizable spaces. It will first be proved that if a space is completely regular then it is uniformizable. The converse of this theorem is also true but is more difficult to prove. In this chapter the converse will be proved by first proving that all uniformizable spaces are (T) spaces and then proving that all (T) spaces are completely regular. Some examples of spaces will be given where it can easily be shown whether they are completely regular or not and hence whether they are uniformizable.

Theorem 2.1 If the topological space (X, T) is completely regular, then (X, T) is uniformizable.

Proof: Let V^* be the usual uniformity for E^1. By Theorem 1.17,

$$B^* = \{f_2^{-1}(V) : f \in C(X) \text{ and } V \in V^*\}$$

is a subbase for a uniformity U^* for X. By Theorem 1.16, all relations of the form $V_r = \{(x, y) : |x - y| < r\}$, where $r > 0$, form a base for V^*. Therefore

$$S^* = \{f_2^{-1}(V_r) : f \in C(X) \text{ and } r > 0\}$$

is also a subbase for U^*. The remainder of this proof will verify that U^* is compatible with T.

Let O be in T_U and let x be in O. Then there is a U in U^* such that $U[x] \subset O$. Since S^* is a subbase for U^*, there exist relations $S_1, S_2, \ldots, S_k$ in S^* such that $\bigcap_1^k S_i \subset U$. Since

$$(\bigcap_1^k S_i)[x] \subset U[x]$$

and

$$(\bigcap_1^k S_i)[x] = \bigcap_1^k (S_i[x]),$$

then

$$\bigcap_1^k (S_i[x]) \subset O.$$

For any i,

$$S_i[x] = (f_2^{-1}(V_r))[x],$$

for some $f \in C(X)$ and some $r > 0$.

$$(f_2^{-1}(V_r))[x] = \{y : |f(y) - f(x)| < r\} = f^{-1}(f(x) - r,\ f(x) + r).$$

Therefore, for any i, x is in $S_i[x]$ and $S_i[x]$ is in T. Consequently $x \in \bigcap_1^k (S_i[x]) \subset O$, where $\bigcap_1^k (S_i[x])$ is in T. Therefore O is in T.

Now let O be in T and let x be in O. Then, since (X, T) is completely regular, there is an f in C(X) such that $f(x) = 0$ and $f(X - O) \subset \{1\}$. It follows that $(f_2^{-1}(V_1))[x] \subset O$. For if y is in $(f_2^{-1}(V_1))[x]$, then (x, y) is in $f_2^{-1}(V_1)$ and hence (f(x), f(y) is in V_1.

This implies that $|f(y)| = |f(x) - f(y)| < 1$. Hence, since

$$f(X - O) \subset \{1\},$$

y is in O. Therefore, since $f_2^{-1}(V_1)$ is in U^*, O is in T_U.

Consequently there exists a uniformity U^* for X which is compatible with T.

This theorem and its proof shows an interesting property of completely regular spaces concerning uniform continuity. For any completely regular space (X, T), there is a uniformity U^* for X compatible with T such that for any f in $C(X)$, f is uniformly continuous relative to U^* and V^*, the usual uniformity for E^1.

The next major theorem to be proved is that if a space is uniformizable then it is a (T) space. Before this can be done though, several definitions must be given and several lemmas will be proved.

Definition 2.2 The collection of sets $\{O_d : d \in D\}$ is said to be a scale of open sets if and only if for any d in D, O_d is open, D is a dense subset of the closed unit interval $[0, 1]$, and for any d_1 and d_2 in D for which $d_1 < d_2$, it is true that $\overline{O}_{d_1} \subset O_{d_2}$.

Definition 2.3 A topological space (X, T) is called a (T) space if and only if for any closed set C and any x in $X - C$, there exists a scale $\{O_d : d \in D\}$ of open sets such that for any d in D, x is in O_d and O_d does not intersect C.

The term (T) space for the space described in the previous definition is not in common usage but is used by Gaal (9).

Lemma 2.4 If U and V are relations on a set X and A is a subset of X, then $U\left[V[A]\right] \subset (U \circ V)[A]$.

Proof: Let z be in $U\left[V[A]\right]$. Then there is a y in $V[A]$ such that (y, z) is in U. Since y is in $V[A]$, there is an x in A such that (x, y) is in V. Therefore (x, z) is in $U \circ V$ and hence z is in $(U \circ V)[A]$.

Lemma 2.5 If (X, U^*) is a uniform space, $A \subset X$, and T_U is the uniform topology for X, then the set

$$B = \{x : U[x] \subset A \text{ for some } U \in U^*\}$$

is the T_U-interior of A.

Proof: Let x be in the T_U-interior of A. Then there is an O in T_U such that $x \in O \subset A$. By definition of T_U, there exists a U in U^* such that $U[x] \subset O$. This implies that $U[x] \subset A$, which in turn implies that x is in B. Therefore the T_U-interior of A is a subset of B.

Now let x be in B. By the definition of B, there is a U in U^* such that $U[x] \subset A$. By the definition of a uniformity, there is a V in U^* such that $V \circ V \subset U$. Assume y is in $V[x]$ and let u be in $V[y]$. Then (x, y) is in V and (y, u) is in V, which implies that (x, u) is in $V \circ V$. Since $V \circ V \subset U$, (x, u) is in U, which implies that u is in $U[x]$. Since $U[x] \subset A$, u is in A. Therefore if y is in $V[x]$, then $V[y] \subset A$. Consequently, by the definition of B, $V[x] \subset B$. It has now been shown that for any x in B, there exists a V in U^* such that $V[x] \subset B$. Consequently B is T_U-open. Since for any U in U^*, x is in $U[x]$, then B is a subset of A. The T_U-interior of A is the union of all subsets of A which are T_U-open. Therefore B is a subset of the T_U-interior of A. Therefore B is the T_U-interior of A.

Corollary 2.6 If (X, U^*) is a uniform space, x is in X, and U is a member of U^*, then $U[x]$ is a T_U-neighborhood of x.

Proof: By Lemma 2.5, x is in the T_U-interior of $U[x]$. The T_U-interior of $U[x]$ is T_U-open and is contained in $U[x]$. Therefore $U[x]$ is a T_U-neighborhood of x.

Lemma 2.7 If U^* is a uniformity for X, U is in U^*, A and B are subsets of X, and $U[A] \subset B$, then $\overline{A} \subset B^i$, where $\overline{A}$ is the T_U-closure of A and B^i is the T_U-interior of B.

Proof: There is a V in U^* such that $V \circ V \subset U$. This implies that $(V \circ V)[A] \subset U[A]$. Because of the hypothesis and Lemma 2.4, $V[V[A]] \subset B$. If x is in $V[A]$, then $V[x] \subset V[V[A]]$. But this implies that $V[x] \subset B$, which in turn implies by Lemma 2.5 that x is in the T_U-interior of B. Therefore

$$V[A] \subset B^i. \tag{1}$$

Now let $W = V \cap V^{-1}$. Since $W \subset V$,

$$W[A] \subset V[A]. \tag{2}$$

Let x be in $\overline{A}$. Because of Corollary 2.6 and since W is in U^*, $W[x]$ is a T_U-neighborhood of x. Therefore there is an a in A such that a is also in $W[x]$. Since a is in $W[x]$, (x, a) is in W and, since W is symmetric, (a, x) is in W. Hence x is in $W[A]$. Because of (1) and (2), x is in B^i. Therefore $\overline{A} \subset B^i$.

Lemma 2.8 If U^* is a uniformity for X, $A \subset X$, if U, V, and W are in U^*, and if $U \circ V \subset W$, then $\overline{V[A]} \subset W[A]^i$, where $\overline{V[A]}$ is the T_U-closure of $V[A]$ and $W[A]^i$ is the T_U-interior of $W[A]$.

Proof: By Lemma 2.4, $U\left[V[A]\right] \subset (U \circ V)[A]$. Since $U \circ V \subset W$, $(U \circ V)[A] \subset W[A]$. Hence $U\left[V[A]\right] \subset W[A]$. Therefore by Lemma 2.7, $\overline{V[A]} \subset W[A]^i$.

Lemma 2.9 Let $\{U_n\}$ be a sequence of entourages in a uniformity U^* for X such that for any non-negative integer n,

$$U_{n+1} \circ U_{n+1} \subset U_n.$$

Then:

(i) If m and n are integers such that $m \geq n \geq 0$, then $U_m \subset U_n$.

(ii) If $n_1, n_2, \ldots, n_k$ are integers such that $0 \leq n_1 < n_2 < \ldots < n_k$, then

$$U_{n_k} \circ U_{n_{k-1}} \circ \ldots \circ U_{n_2} \subset U_{n_1}.$$

Proof: Certainly $U_m \subset U_n$ if $m = n$ so consider $m > n$. Let $n \geq 0$. Then

$$U_{n+1} \circ U_{n+1} \subset U_n$$

by the hypothesis. Assume $U_{n+k} \subset U_n$ for some positive integer k. Then by the hypothesis of the lemma,

$$U_{(n+k)+1} \circ U_{(n+k)+1} \subset U_{n+k}.$$

Since

$$U_{(n+k)+1} \subset U_{(n+k)+1} \circ U_{(n+k)+1},$$

then $U_{n+(k+1)} \subset U_n$. Therefore it is true by mathematical induction that for any $n \geq 0$ and any $j \geq 1$,

$$U_{n+j} \subset U_n. \tag{3}$$

If $m > n \geq 0$ then $m = n+j$, where $j \geq 1$, and hence by (3) $U_m \subset U_n$. Therefore part (i) of the conclusion is true.

Let n_1 and n_2 be integers such that $0 \leq n_1 < n_2$. Then by part (i) of the lemma, $U_{n_2} \subset U_{n_1}$. Assume that

$$U_{n_k} \circ U_{n_{k-1}} \circ \ldots \circ U_2 \subset U_1$$

for any integers $n_1, n_2, \ldots, n_k$ such that $0 \leq n_1 < n_2 < \ldots < n_k$. Let $n_1, n_2, \ldots, n_k, n_{k+1}$ be integers such that

$$0 \leq n_1 < n_2 < \ldots < n_k < n_{k+1}.$$

Then because of the assumption,

$$U_{n_{k+1}} \circ U_{n_k} \circ \ldots \circ U_{n_3} \subset U_{n_2}.$$

This implies that

$$(U_{n_{k+1}} \circ U_{n_k} \circ \ldots \circ U_{n_3}) \circ U_{n_2} \subset U_{n_2} \circ U_{n_2}. \tag{4}$$

By the hypothesis of this lemma,

$$U_{n_2} \circ U_{n_2} \subset U_{n_2 - 1}. \tag{5}$$

Since $n_2 - 1 \geq n_1$ and because of part (i) of the conclusion of the lemma,

$$U_{n_2 - 1} \subset U_{n_1}. \tag{6}$$

Because of the set inclusions in (4), (5), and (6),

$$U_{n_{k+1}} \circ U_{n_k} \circ \ldots \circ U_{n_2} \subset U_{n_1}.$$

Therefore part (ii) of the conclusion if true by mathematical induction.

Theorem 2, 10 If a topological space (X, T) is uniformizable then it is a (T) space.

Proof: There is a uniformity U^* for X such that $T_U = T$ since (X, T) is uniformizable. Let C be a closed subset of X and let x be in $X - C$. Since $X - C$ is T_U-open, there is a U_0 in U^* such that

$$x \in U_0[x] \subset X - C.$$

A sequence $\{U_i\}$ of entourages in U^* can be defined in the following way. Let U_1 be an entourage in U^* such that $U_1 \circ U_1 \subset U_0$. If U_n is in the sequence, let U_{n+1} be an entourage in U^* such that

$$U_{n+1} \circ U_{n+1} \subset U_n.$$

Now let D be the collection of all diadic rationals in the open interval $(0, 1)$, that is any number equal to $m/2^n$, where $n = 1, 2, \ldots$ and $m = 1, 2, \ldots, 2^n - 1$.

For any d in D, d can be uniquely represented as a finite sum $2^{-n_1} + 2^{-n_2} + \ldots + 2^{-n_k}$, where each n_i is an integer and $0 < n_1 < \ldots < n_k$. For any d in D, define

$$U_d = U_{n_k} \circ U_{n_{k-1}} \circ \ldots \circ U_{n_1}$$

and define O_d to be $(U_d[x])^i$, the interior of $U_d[x]$. Since U_{n_1} is in U^* and since $U_d \supset U_{n_1}$, U_d is in U^*. Therefore by Lemma 2.5, x is in O_d, for each $d \in D$. Also $\{O_d : d \in D\}$ is a collection of open subsets of X indexed by a dense subset of $[0, 1]$.

In order to show that (X, T) is a (T) space, it must finally be proved that for any d in D, O_d does not intersect C, and that for any d_1 and d_2 in D for which $d_1 < d_2$, $\overline{O}_{d_1} \subset O_{d_2}$.

For any $d \in D$,

$$U_d = U_{n_k} \circ U_{n_{k-1}} \circ \ldots \circ U_{n_1},$$

where $0 < n_1 < \ldots < n_k$. By Lemma 2.9, part (ii),

$$U_{n_k} \circ \ldots \circ U_{n_2} \subset U_{n_1}.$$

This implies that

$$U_d = (U_{n_k} \circ \ldots \circ U_{n_2}) \circ U_{n_1} \subset U_{n_1} \circ U_{n_1}.$$

By Lemma 2.9, part (i), $U_{n_1} \subset U_1$. Therefore

$$U_{n_1} \circ U_{n_1} \subset U_1 \circ U_1.$$

By definition of the sequence $\{U_n\}$, $U_1 \circ U_1 \subset U_0$. Therefore $U_d \subset U_0$.

This implies that $U_d[x] \subset U_0[x]$ which in turn implies that

$$O_d = (U_d[x])^i \subset U_0[x].$$

Since $U_0[x]$ does not intersect C, O_d does not intersect C.

Let d_1 and d_2 be in D and $d_1 < d_2$. Then there exist unique sets of integers $\{n_1, n_2, \ldots, n_{k-1}\}$ and $\{m_1, m_2, \ldots, m_j\}$ such that

$$0 < n_1 < n_2 < \ldots < n_{k-1},$$

$$0 < m_1 < m_2 < \ldots < m_j,$$

$$d_1 = 2^{-n_1} + 2^{-n_2} + \ldots + 2^{-n_{k-1}}, \quad \text{and}$$

$$d_2 = 2^{-m_1} + 2^{-m_2} + \ldots + 2^{-m_j}.$$

Let n_k be an integer such that $n_k > n_i$, for any $i \leq k-1$, and $n_k > m_i$, for any $i \leq j$. It then is true that there is an integer $i \leq j$ such that $m_i \neq n_i$. For if $k - 1 < j$, then $k \leq j$, and hence, by the definition of n_k, $m_k \neq n_k$. Now if $k - 1 \geq j$ and $m_i = n_i$ for all $i \leq j$, then

$$d_2 = \sum_1^j 2^{-m_i} = \sum_1^j 2^{-n_i} \leq \sum_1^{k-1} 2^{-n_i} = d_1,$$

which contradicts the assumption that $d_1 < d_2$. Therefore if $k - 1 \geq j$, then $m_i \neq n_i$ for some $i \leq j$.

Consequently there is a smallest positive integer h such that $m_h \neq n_h$. Now

$$\text{for any } i < h,\ m_i = n_i. \tag{7}$$

Now $h \leq k$. For otherwise, (7) implies that $m_k = n_k$, which contradicts the definition of n_k.

First consider $h = k$. Then $U_{n_i} = U_{m_i}$ for $i \leq k-1$, because of (7). Therefore

$$U_{n_{k-1}} \circ U_{n_{k-2}} \circ \ldots \circ U_{n_1} = U_{m_{h-1}} \circ U_{m_{h-2}} \circ \ldots \circ U_{m_1}. \tag{8}$$

By definition of n_k, $n_k > m_h$. Hence, by Lemma 2.9, $U_{n_k} \subset U_{m_h}$.

$$U_{m_h} \subset U_{m_j} \circ U_{m_{j-1}} \circ \ldots \circ U_{m_h},$$

since for any i U_i is in U^*. Therefore

$$U_{n_k} \subset U_{m_j} \circ U_{m_{j-1}} \circ \ldots \circ U_{m_h}. \tag{9}$$

Because of (8) and (9),

$$U_{n_k} \circ U_{n_{k-1}} \circ \ldots \circ U_{n_1} \subset U_{m_j} \circ U_{m_{j-1}} \circ \ldots \circ U_{m_1}.$$

Now consider $h < k$ and suppose that $n_h < m_h$. Then $n_h \leq m_h - 1$, which implies that $2^{-n_h} \geq 2^{-m_h+1}$.

$$\sum_h^j 2^{-m_i} \leq \sum_{i=m_h}^{\infty} 2^{-i} = 2^{-m_h+1}.$$

Therefore

$$2^{-n_h} \geq \sum_h^j 2^{-m_i}. \tag{10}$$

Because of (7),

$$\sum_{1}^{h-1} 2^{-n_i} = \sum_{1}^{h-1} 2^{-m_i}. \qquad (11)$$

Therefore, by (10) and (11),

$$\sum_{1}^{h} 2^{-n_i} \geq \sum_{1}^{j} 2^{-m_i} = d_2,$$

Hence $d_1 \geq d_2$, since

$$d_1 = \sum_{1}^{k-1} 2^{-n_i} \geq \sum_{1}^{h} 2^{-n_i}.$$

But this is a contradiction since $d_1 < d_2$. Therefore when $h < k$, $n_h \geq m_h$. Since $n_h \neq m_h$, $n_h > m_h$. Since $h < k$, $n_k > n_h$. Since $n_k > n_{k-1} > \ldots > n_1$ and since $n_k > n_h > m_h$, then by Lemma 2, 9 (ii)

$$U_{n_k} \circ U_{n_{k-1}} \circ \ldots \circ U_{n_h} \subset U_{m_h}. \qquad (12)$$

Because of the definition of h, $U_{m_i} = U_{n_i}$ for any $i < h$. This implies that

$$U_{n_{h-1}} \circ U_{n_{h-2}} \circ \ldots \circ U_{n_1} = U_{m_{h-1}} \circ U_{m_{h-2}} \circ \ldots \circ U_{m_1}. \qquad (13)$$

Because of (12) and (13),

$$U_{n_k} \circ U_{n_{k-1}} \circ \ldots \circ U_{n_1} \subset U_{m_h} \circ U_{m_{h-1}} \circ \ldots \circ U_{m_1}.$$

Since $h \leq j$,

$$U_{m_h} \circ U_{m_{h-1}} \circ \ldots \circ U_{m_1} \subset U_{m_j} \circ U_{m_{j-1}} \circ \ldots \circ U_{m_1}.$$

Therefore in all cases

$$U_{n_k} \circ U_{n_{k-1}} \circ \cdots \circ U_{n_1} \subset U_{m_j} \circ U_{m_{j-1}} \circ \cdots \circ U_{m_1},$$

By definition of U_d for d in D,

$$U_{d_1} = U_{n_{k-1}} \circ U_{n_{k-2}} \circ \cdots \circ U_{n_1}$$

and

$$U_{d_2} = U_{m_j} \circ U_{m_{j-1}} \circ \cdots \circ U_{m_1}.$$

Therefore $U_{n_k} \circ U_{d_1} \subset U_{d_2}$. Since

$$U_{d_1} \supset U_{n_1} \text{ and } U_{n_1} \in U^*,$$

then $U_{d_1} \in U^*$. Likewise, since

$$U_{d_2} \supset U_{m_1} \text{ and } U_{m_1} \in U^*,$$

then $U_{d_2} \in U^*$. Therefore, by Lemma 2.8, $\overline{U_{d_1}[x]} \subset (U_{d_2}[x])^i$. Since

$$\overline{(U_{d_1}[x])^i} \subset \overline{U_{d_1}[x]},$$

then

$$\overline{(U_{d_1}[x])^i} \subset (U_{d_2}[x])^i.$$

Hence, by the definition of O_d, for d in D, $\overline{O_{d_1}} \subset O_{d_2}$. Therefore (X, T) is a (T) space.

The next lemma will be used to prove that if a topological space (X, T) is a (T) space then it is a completely regular space.

Lemma 2.11 If $\{O_d : d \in D\}$ is a scale of open sets in a topological space (X, T) and $\bigcup_{d \in D} O_d = X$, then the function f from X into E^1, where

$$f(x) = \inf \{d \in D : x \in O_d\}$$

for any $x \in X$, is a continuous real-valued function.

Proof: It is sufficient to prove that for any real number s, $\{x : f(x) < s\}$ and $\{x : f(x) > s\}$ are T-open subsets of X. For then the inverse image with respect to f of any open interval in E^1 will be open in X and hence f will be continuous. Let w be in $\{x : f(x) < s\}$. Then $f(w) = \inf \{d \in D : w \in O_d\} < s$, which implies that s is not a lower bound of $\{d \in D : w \in O_d\}$. Consequently there exist a d' in D such that w is in $O_{d'}$ and $d' < s$. This implies that w is in

$$\bigcup \{O_d : d \in D \text{ and } d < s\}.$$

Now let w be in

$$\bigcup \{O_d : d \in D \text{ and } d < s\}.$$

Then there exists a d' in D such that $d' < s$ and $w \in O_{d'}$. This implies that

$$f(w) = \inf\{d \in D : w \in O_d\} \leq d'.$$

Therefore $f(w) < s$ which implies that $w \in \{x : f(x) < s\}$.

Therefore $\{x : f(x) < s\} = \bigcup \{O_d : d \in D \text{ and } d < s\}$. For any

$d \in D$, O_d is open and therefore $\{x : f(x) < s\}$ is open.

In order to show that $\{x : f(x) > s\}$ is open, it is sufficient to show that $\{x : f(x) \leq s\}$ is closed. If $s \geq 1$, then $\{x : f(x) \leq s\} = X$, because the definition of f and $D \subset [0, 1]$ imply that $f(x) \leq 1$ for any $x \in X$. Since X is closed, $\{x : f(x) \leq s\}$ is closed.

Now suppose that $s < 1$ and that u is in $\{x : f(x) \leq s\}$. Then $f(u) \leq s$, which implies that $\inf\{d \in D : u \in O_d\} \leq s$. Let t be a member of D such that $t > s$. Note that since D is a dense subset of $[0,1]$ and since $s < 1$, such a t exists. Since $\inf\{d \in D : u \in O_d\} < t$, t is not a lower bound of $\{d \in D : u \in O_d\}$. Therefore there is a d' in D such that u is in $O_{d'}$ and $d' < t$. By the definition of a scale of open sets, $\bar{O}_{d'} \subset O_t$. This implies that u is in O_t. Therefore u is in

$$\bigcap \{O_d : d \in D \text{ and } d > s\},$$

Now let u be in

$$\bigcap \{O_d : d \in D \text{ and } d > s\}.$$

Since $s < 1$ and since D is dense in $[0, 1]$, then for any $r > 0$ there exists a $t \in D$ such that $s < t < s + r$. Therefore u is in O_t which implies that $\inf\{d \in D : u \in O_d\} \leq t$. Consequently $f(u) \leq s + r$. Since r was arbitrarily chosen, $f(u) \leq s$, which implies that u is in $\{x : f(x) \leq s\}$. Therefore

$$\{x : f(x) \leq s\} = \bigcap \{O_d : d \in D \text{ and } d > s\}. \tag{14}$$

It will next be shown that

$$\bigcap \{O_d : d \in D \text{ and } d > s\} = \bigcap \{\bar{O}_d : d \in D \text{ and } d > s\}$$

from which it will follow by (14) that $\{x : f(x) \leq s\}$ is closed. For any $d \in D$, $O_d \subset \overline{O}_d$, and hence

$$\cap \{O_d : d \in D,\ d > s\} \subset \cap \{\overline{O}_d : d \in D,\ d > s\}.$$

Let w be a member of $\cap \{\overline{O}_d : d \in D$ and $d > s\}$. Since $s < 1$ and since D is dense in $[0, 1]$, if d is in D and $d > s$ then there exists a $t \in D$ such that $s < t < d$. $\overline{O}_t \subset O_d$ by definition of a scale of open sets. Because of the choice of w, w is in $\overline{O}_t$, and this implies that w is in O_d. Consequently w is in $\cap \{O_d : d \in D$ and $d > s\}$. Therefore

$$\cap \{O_d : d \in D \text{ and } d > s\} = \cap \{\overline{O}_d : d \in D \text{ and } d > s\}.$$

Therefore $\{x : f(x) \leq s\}$ is the intersection of a family of closed sets and hence is itself a closed set. Since

$$X - \{x : f(x) \leq s\} = \{x : f(x) > s\},$$

then $\{x : f(x) > s\}$ is an open set.

Theorem 2.12 If (X, T) is a (T) space then (X, T) is completely regular.

Proof: Let C be a closed subset of X and let x be in $X - C$. Since (X, T) is a (T) space, there is a scale of open sets $\{O_d : d \in D\}$ such that, for any d in D, x is in O_d and O_d does not intersect C. First a scale of open sets will be defined such that the union of the open sets is X, and then an f in $C(X)$ will be defined using this scale and Lemma 2.11.

First consider the case where 1 is in D. For any d in D for which $d < 1$, define $G_d = O_d$; and for $d = 1$, define $G_d = X$. For any d in D for which $d < 1$, $\overline{G}_d \subset G_1 = X$. For any d_1 and d_2 in $D - \{1\}$ for which $d_1 < d_2$, $\overline{G}_{d_1} = \overline{O}_{d_1} \subset O_{d_2} = G_{d_2}$. Therefore $\{G_d : d \text{ in } D\}$ is a scale of open sets.

Now suppose that $1 \notin D$. Let $D' = D \cup \{1\}$. For any d in D, define $G_d = O_d$ and for $d = 1$, define $G_d = X$. Since $[0, 1] \subset \overline{D}$ and $\overline{D} \subset \overline{D'}$, $[0, 1] \subset \overline{D'}$. Therefore D' is dense in $[0, 1]$. Then, as was done when $1 \in D$, it can be shown that $\{G_d : d \in D'\}$ is a scale of open sets.

Therefore in either case there exist a scale of open sets $\{F_d : d \in \Delta\}$ such that

$$\bigcup_{d \in \Delta} F_d = X$$

and such that for any $d \in \Delta$ for which $d < 1$, x is in F_d and F_d does not intersect C. By Lemma 2.11 there exist a function f in $C(X)$ such that for any x in X, $f(x) = \inf\{d \in \Delta : x \in F_d\}$. Since $\Delta \subset [0, 1]$, then $0 \leq f(x) \leq 1$ for all x in X, which implies that $f(X) \subset [0, 1]$. If a is in C then a is not in F_d for any d in Δ less than 1. Since

$$a \in \bigcup_{d \in \Delta} F_d,$$

then a is in F_d for $d = 1$. Therefore $f(a) = 1$ and hence $f(C) \subset \{1\}$. Since x is in F_d for all d in Δ and since Δ is a dense subset of $[0, 1]$, then $f(x) = \inf \Delta = 0$.

Theorems 2.1, 2.10, and 2.12 show that uniformizable spaces, (T) spaces, and completely regular spaces are equivalent. These three theorems show that if a space is completely regular then it is a (T) space; but this depends on first showing that a completely regular space is uniformizable. It can be proved rather easily, without mentioning uniformizable spaces, that if a space is completely regular then it is a (T) space.

Theorem 2.13 If a space (X, T) is completely regular then it is a (T) space.

Proof: Let C be a closed subset of X and let x be in $X - C$. Then, by definition of completely regular, there exists a function f in $C(X)$ such that $f(x) = 0$, $f(C) \subset \{1\}$, and $f(X) \subset [0, 1]$. Let $D = [0,1]$. For any d in D, define

$$O_d = \{x \in X : f(x) < \frac{d}{2} + \frac{1}{2}\} = f^{-1}\left((-\infty, \frac{d}{2} + \frac{1}{2})\right).$$

Since for any d, $(-\infty, \frac{d}{2} + \frac{1}{2})$ is open in E^1 and f is in $C(X)$, then O_d is an open subset of X.

Let a and b be in D such that $a < b$. Since f is continuous,

$$\overline{O}_a = \overline{f^{-1}\left((-\infty, \frac{a}{2} + \frac{1}{2})\right)} \subset f^{-1}\left(\overline{(-\infty, \frac{a}{2} + \frac{1}{2})}\right) \subset f^{-1}\left((-\infty, \frac{a}{2} + \frac{1}{2}]\right),$$

Since

$$(-\infty, \frac{a}{2} + \frac{1}{2}] \subset (-\infty, \frac{b}{2} + \frac{1}{2}),$$

then

$$f^{-1}\left((-\infty, \frac{a}{2} + \frac{1}{2}]\right) \subset f^{-1}\left((-\infty, \frac{b}{2} + \frac{1}{2})\right) = O_b.$$

Therefore $\overline{O}_a \subset O_b$ and hence $\{O_d : d \in D\}$ is a scale of open sets, For any d in D, $d/2 + 1/2 \geq 1/2$. Therefore for any d in D, $f(x) = 0 < d/2 + 1/2$; which implies that x is in O_d.

For any d in D, let u be in O_d. Then u is in X and $f(u) < d/2 + 1/2$. Since $d \leq 1$, $1/2 + d/2 \leq 1$. This implies that $f(u) < 1$. Since $f(C) \subset \{1\}$, u is not in C. Therefore for any d in D, O_d does not intersect C. Consequently (X, T) is a (T) space.

For some spaces it is difficult to show that there is or is not an admissible uniformity. Since uniformizable spaces and completely regular spaces are equivalent, it is sometimes easier to show that a space is uniformizable or not by showing that it is or is not completely regular. The next theorem will show that a completely regular space is regular. An example of a space will then be given which is easily shown to be not regular and hence not uniformizable.

<u>Definition 2.14</u> A topological space (X, T) is said to be regular if, and only if, for any closed subset C of X and any x in $X - C$, there exist disjoint open subsets O_x and O_c such that x is in O_x and $C \subset O_c$.

<u>Theorem 2.15</u> If a topological space (X, T) is completely regular then it is regular.

<u>Proof:</u> Let C be a closed subset of X and let x be in $X - C$. Since (X, T) is completely regular there exists an f in $C(X)$ such that $f(x) = 0$ and $f(C) \subset \{1\}$. Hence $f^{-1}((-1, 1/2))$ and $f^{-1}((1/2, 2))$ are disjoint open subsets of X containing respectively x and C.

<u>Example 2.16</u> Let X be an infinite set and let T be the cofinite topology for X. That is, O is in T if and only if O is the empty set

or $X - O$ is finite. Then it is easy to show that (X, T) is not regular. For let x and y be in X such that $x \neq y$. Then $\{y\}$ is closed and $x \notin \{y\}$. If there exist disjoint open sets O_x and O_y such that x is in O_x and $\{y\} \subset O_y$, then $X - O_x$ is finite. This implies that O_y is finite. Since O_y is open and non-empty, $X - O_y$ is finite. Therefore $X = O_y \cup (X - O_y)$ is finite; which is a contradiction since X was given to be an infinite set. Consequently no two disjoint open sets separate x and $\{y\}$. So (X, T) is not regular. By Theorem 2.15 (X, T) is not completely regular. Therefore (X, T) is not uniformizable.

CHAPTER III

VARIOUS CHARACTERIZATIONS OF UNIFORMIZABLE SPACES USING C(X)

The property of being a completely regular space is a characterization of uniformizable spaces which involves the family of functions $C(X)$. There are several other characterizations of uniformizable spaces which make use of $C(X)$. This chapter will investigate these various characterizations.

Definition 3.1 A family F of functions defined on a space (X, T) is said to distinguish points and closed sets if and only if for any closed subset A of X and any x in $X - A$, there is an f in F such that $f(x)$ is not in $\overline{f(A)}$.

Theorem 3.2 A space (X, T) is completely regular if and only if $C(X)$ distinguishes points and closed sets.

Proof: Let (X, T) be completely regular, A be a closed subset of X, and x be in $X - A$. Then there exists an f in $C(X)$ such that $f(x) = 0$ and $f(A) \subset \{1\}$. Since $f(A)$ is either the empty set or is $\{1\}$, $\overline{f(A)}$ is either the empty set or is $\{1\}$. Therefore $f(x)$ is not in $\overline{f(A)}$.

Suppose that $C(X)$ distinguishes points and closed sets. Let A be a closed subset of X and let x be in $X - A$. Then there is an f in

$C(X)$ such that $f(x)$ is not in $\overline{f(A)}$. Because of Example 1.16, E^1 is uniformizable. As was shown in Chapter II, any uniformizable space is completely regular. Therefore E^1 is completely regular. $\overline{f(A)}$ is a closed subset of E^1 and $f(x)$ is not in $\overline{f(A)}$. Therefore there exists a g in $C(E^1)$ such that $g(f(x)) = 0$, $g(\overline{f(A)}) \subset \{1\}$ and $g(E^1) \subset [0, 1]$. $g \circ f$ is in $C(X)$ and $(g \circ f)(x) = 0$. Since

$$(g \circ f)(A) = g(f(A)) \subset g(\overline{f(A)}),$$

then $(g \circ f)(A) \subset \{1\}$. Since $f(X) \subset E^1$ and since $g(E^1) \subset [0, 1]$,

$$(g \circ f)(X) = g(f(X)) \subset [0, 1].$$

Therefore (X, T) is completely regular.

Definition 3.3 If (X, T) is a topological space, then the weak topology induced by $C(X)$ is the topology for X which has as a subbase the set $\{f^{-1}(G) : f \in C(X) \text{ and } G \text{ open in } E^1\}$.

Theorem 3.4 If (X, T) is completely regular, then the weak topology T' induced by $C(X)$ is equal to T.

Proof: Let O be in T and let p be in O. Then there exists an f in $C(X)$ such that $f(p) = 0$ and $f(X - O) \subset \{1\}$. Therefore p is in $f^{-1}((-1, 1))$. If x is in $f^{-1}((-1, 1))$, $f(x) \neq 1$. Hence x is in O. Therefore $f^{-1}((-1, 1)) \subset O$. Since $(-1, 1)$ is open in E^1 and f is in $C(X)$, $f^{-1}((-1, 1))$ is in T'. Therefore O is in T'.

Now let O be in T' and let p be in O. By definition of T', there exist open sets $O_1, O_2, \ldots, O_n$ in E^1 and functions $f_1, f_2, \ldots, f_n$ in $C(X)$ such that p is in

$$\bigcap_{1}^{n} f_i^{-1}(O_i) \subset O.$$

For any i, $f_i^{-1}(O_i)$ is in T, and hence $\bigcap_{1}^{n} f_i^{-1}(O_i)$ is in T. Consequently O is in T. Therefore $T' = T$.

Definition 3.5 If (X, T) is a topological space then a subset A of X is a zero set in X if and only if $A = \{x : f(x) = 0\}$ where f is in C(X). The collection of all zero sets in X is denoted by Z(X).

Definition 3.6 A collection B^* of subsets of a topological space (X, T) is a base for the closed sets in X if and only if each set in B^* is closed and each closed subset of X is the intersection of sets in B^*.

Lemma 3.7 If (X, T) is a topological space, f is in C(X), and r is a real number, then $\{x : f(x) \geq r\}$ and $\{x : f(x) \leq r\}$ are zero sets in X.

Proof: $\{x : f(x) \geq r\} = \{x : ((f - r) \wedge 0)(x) = 0\}$, where

$$((f - r) \wedge 0)(x) = \min\{(f - r)(x), 0\}.$$

Also

$$\{x : f(x) \leq r\} = \{x : ((f - r) \vee 0)(x) = 0\},$$

where

$$((f - r) \vee 0)(x) = \max\{(f - r)(x), 0\}.$$

Since the functions $(f - r) \wedge 0$ and $(f - r) \vee 0$ are in C(X), then, by definition of zero sets, $\{x : f(x) \geq r\}$ and $\{x : f(x) \leq r\}$ are zero sets in X.

Lemma 3.8 If (X, T) is a topological space and if A and B are zero sets in X, then $A \cup B$ is a zero set in X.

Proof: There is an f in $C(X)$ such that $A = \{x : f(x) = 0\}$ and there is a g in $C(X)$ such that $B = \{x : g(x) = 0\}$. Now

$$A \cup B = \{x : (fg)(x) = 0\}$$

and fg is in $C(X)$. Therefore $A \cup B$ is a zero set in X.

These two lemmas will now be used to prove the next theorem.

Theorem 3.9 If (X, T) is a topological space with the property that the weak topology induced by $C(X)$ is equal to T, then $Z(X)$ is a base for the closed sets in X.

Proof: For any A in $Z(X)$ there is an f in $C(X)$ such that $A = f^{-1}(\{0\})$. Since $\{0\}$ is closed in E^1, $f^{-1}(\{0\})$ is closed in X. Therefore each set in $Z(X)$ is closed.

It must now be shown that each closed subset of X is the intersection of sets in $Z(X)$. Let C be a closed subset of X. Then $X - C$ is open. By the hypothesis of the theorem

$$X - C = \bigcup_{j \in M} \left(\bigcap_{1}^{K_j} f_i^{-1}(I_i) \right),$$

where each I_i is an open interval in E^1 and each f_i is in $C(X)$. Therefore

$$C = X - \bigcup_{j \in M} \left(\bigcap_{1}^{K_j} f_i^{-1}(I_i) \right) = \bigcap_{j \in M} \left(\bigcup_{1}^{K_j} f_i^{-1}(X - I_i) \right).$$

For any i,

$$f_i^{-1}(X - I_i) = f_i^{-1}\Big((-\infty, a] \cup [b, +\infty)\Big),$$

for some real numbers a and b. Now

$$f_i^{-1}\Big((-\infty, a] \cup [b, +\infty)\Big) = \{x : f_i(x) \leq a\} \cup \{x : f_i(x) \geq b\}.$$

Therefore, by Lemma 3.7 and the generalization of Lemma 3.8 to a finite number of zero sets,

$$\bigcup_{1}^{K_j} f_i^{-1}(X - I_i)$$

is a zero set in X for any j in M. Therefore C is the intersection of zero sets in X. Therefore Z(X) is a base for the closed sets in X.

<u>Theorem 3.10</u> If Z(X) is a base for the closed sets in a topological space (X, T), then (X, T) is completely regular.

<u>Proof</u>: Let F be a closed subset of X and let p be in X - F. By the hypothesis, there is a subfamily B^* of Z(X) such that

$$F = \bigcap \{B : B \in B^*\}.$$

Since p is not in F, there is a B' in B^* such that p is not in B'. By definition of Z(X), there is an f in C(X) such that $B' = f^{-1}(\{0\})$. This implies that $f(B') \subset \{0\}$. Since $F \subset B'$, $f(F) \subset f(B')$, which implies that $f(F) \subset \{0\}$. Since p is not in B', f(p) = a and $a \neq 0$.

Define a function g from X into E^1 such that for any x in X, $g(x) = \frac{f(x)}{a}$. Then $g \in C(X)$, g(p) = 1, and $g(F) \subset \{0\}$.

Define a function h from X into E^1 such that h(x) = 0 if $g(x) \leq 0$, h(x) = g(x) if $0 \leq g(x) \leq 1$, and h(x) = 1 if $g(x) \geq 1$. Then

$h(F) \subset \{0\}$, $h(p) = 1$, and $h(X) \subset [0, 1]$. The function h is in $C(X)$ since $h(x) = \min \{\max \{g(x), 0\}, 1\}$ for any x in X.

Theorems 3.2, 3.4, 3.9, and 3.10 give three more characterizations of completely regular spaces or, equivalently, uniformizable spaces. Theorems 3.4, 3.9, and 3.10 show that if a space (X, T) is completely regular, then $Z(X)$ is a base for the closed sets in X. The proof of the following theorem shows the same thing without any mention of the weak topology for X.

<u>Theorem 3.11</u> If a topological space (X, T) is completely regular, then $Z(X)$ is a base for the closed sets in X.

<u>Proof:</u> Let F be a closed non-empty subset of X such that $F \neq X$. Let p be in $X - F$. Since (X, T) is completely regular, there exists an f in $C(X)$ such that $f(p) = 0$, $f(F) = \{1\}$, and $f(X) \subset [0, 1]$. Define a function g from E^1 into E^1 such that $g(x) = 1 - x$ for any x in E^1. Since g is continuous, $g \circ f$ is in $C(X)$. Also

$$(g \circ f)(p) = g(0) = 1.$$

Now, $(g \circ f)(F) = \{0\}$, since for any x in F, $(g \circ f)(x) = g(1) = 0$. Therefore p is not in $(g \circ f)^{-1}(\{0\})$ and $F \subset (g \circ f)^{-1}(\{0\})$. $(g \circ f)^{-1}(\{0\})$ is a zero set. Therefore

$$\text{if } p \in X - F \text{, there is an } A \in Z(X) \text{ such that } p \notin A \text{ and } F \subset A. \quad (15)$$

Define $A^* = \{A : A \in Z(X) \text{ and } F \subset A\}$. Clearly

$$F \subset \bigcap \{A : A \in A^*\}.$$

If p is not in F then, because of (15), there exists an A in A^* such that p is not in A. This implies that p is not in $\bigcap \{A : A \in A^*\}$. Consequently $\bigcap \{A : A \in A^*\} \subset F$ and therefore $F = \bigcap \{A : A \in A^*\}$. That is to say, F is the intersection of zero sets in X.

The function h from X into E^1 such that $h(x) = 0$ for all x in X is in $C(X)$ and $\{x : h(x) = 0\} = X$. Therefore X is in $Z(X)$. The function h' from X into E^1 such that $h'(x) = 1$ for all x in X is in $C(X)$ and $\{x : h(x) = 0\} = \emptyset$. Therefore $\emptyset$ is in $Z(X)$.

Therefore any closed subset of X is the intersection of zero sets in X. Each set in $Z(X)$ is a closed subset of X. Therefore $Z(X)$ is a base for the closed sets in X.

The next characterization of completely regular or uniformizable spaces makes use of the family of all continuous functions from a space into the extended real line and the concepts of lower semi-continuous functions and the upper envelope of a family of functions. Consequently some definitions and lemmas concerning these concepts will have to be introduced before proving the characterization.

Definition 3.12 A real-valued function f defined on a topological space (X, T) is said to be lower semi-continuous at a in X if and only if for any $h < f(a)$, there exists a neighborhood V of a such that $f(V) \subset (h, +\infty)$. The function f is lower semi-continuous on X if and only if f is lower semi-continuous at each point of X.

Lemma 3.13 Let (X, T) be a topological space. A real-valued function f is lower semi-continuous on X if and only if for any real number K, $f^{-1}((K, +\infty))$ is open in X.

Proof: Assume f is lower semi-continuous on X and let K be a real number. Let a be in $f^{-1}((K, +\infty))$. Then $f(a) > K$. By definition of lower semi-continuous, there is a neighborhood V of a such that $f(V) \subset (K, +\infty)$. This implies that $V \subset f^{-1}((K, +\infty))$. Therefore $f^{-1}((K, +\infty))$ is an open subset of X.

Suppose that for any real number K, $f^{-1}((K, +\infty))$ is open in X. Let a be in X and let $h < f(a)$. Then $f^{-1}((h, +\infty))$ is open in X. Since a is in $f^{-1}((h, +\infty))$, $V = f^{-1}((h, +\infty))$ is a neighborhood of a. Clearly $f(V) \subset (h, +\infty)$. Therefore f is lower semi-continuous at a. Since a was arbitrarily chosen, f is lower semi-continuous on X.

<u>Lemma 3.14</u> If O is an open subset of the topological space (X, T), then the characteristic function ϕ_O is lower semi-continuous on X.

<u>Proof</u>: If $K \geq 1$, then $\phi_O^{-1}((K, +\infty))$ is the empty set. If $0 \leq K < 1$, then $\phi_O^{-1}((K, +\infty)) = O$. If $K < 0$, then $\phi_O^{-1}((K, +\infty)) = X$. Therefore by Lemma 3.13, ϕ_O is lower semi-continuous on X.

<u>Definition 3.15</u> A real-valued function g defined on the space (X, T) is said to be the upper envelope of $\{f_i : i \in I\}$, a family of functions from X into the extended real line, if and only if

$$g(x) = \sup \{f_i(x) : i \in I\}$$

for each x in X.

<u>Theorem 3.16</u> Let (X, T) be a topological space. If every real-valued lower semi-continuous function f defined on X is the upper envelope of the family G^* of all continuous functions g from X into the extended real line $\overline{R}$ for which $g \leq f$, then (X, T) is

completely regular.

Proof: Let C be a closed subset of X and let p be in $X - C$. Then $X - C$ is an open set and thus, by Lemma 3.14, the characteristic function ϕ_{X-C} is lower semi-continuous on X. Therefore ϕ_{X-C} is the upper envelope of the family G^* of all continuous functions g from X into $\overline{R}$ for which $g \leq \phi_{X-C}$. Since $\phi_{X-C}(p) = 1$, $\sup \{g(p) : g \in G^*\} = 1$. Therefore there exists a g in G^* such that $0 < g(p)$. This implies that $g(p) = a$, where $a > 0$.

Define g^+ to be a function from X into $\overline{R}$ such that

$$g^+(x) = \max \{0, g(x)\}$$

for any x in X. Since g^+ is continuous, the function $\frac{g^+}{a}$ is continuous.

Define the function h from X into $\overline{R}$ such that

$$h(x) = \min \{1, \frac{g^+(x)}{a}\}.$$

This function h is also continuous. For any x in X, $h(x) \leq 1$ and $g^+(x) \geq 0$. This implies that $\frac{g^+(x)}{a} \geq 0$ which implies that $h(x) \geq 0$. Therefore $h(X) \subset [0, 1]$. Now $g^+(p) = \max \{0, a\} = a$, since $a > 0$. Therefore $h(p) = \min \{1, 1\} = 1$.

Let x be in C. Then $\phi_{X-C}(x) = 0$. Since $g \leq \phi_{X-C}$, then $g(x) \leq 0$. This implies that $h(x) = \min \{1, 0\} = 0$. Therefore $h(C) \subset \{0\}$. Therefore (X, T) is completely regular.

The next theorem will be the converse of Theorem 3.16. These two theorems will then give a new characterization of completely regular or uniformizable spaces. A lemma will first be proved.

<u>Lemma 3.17</u> If (X, T) is a completely regular space and f is a real-valued lower semi-continuous function defined on X such that $f(X) \subset [-1, 1]$, then for any p in X and any real number $a < f(p)$, there exists a g in $C(X)$ such that $g \leq f$, $g(p) \geq a$, and $g(X) \subset [-1, 1]$.

<u>Proof</u>: Let (X, T) be completely regular and let f be a lower semi-continuous real valued function defined on X such that $f(X) \subset [-1, 1]$. Let p be in X and let $a < f(p)$. Either $a \leq -1$ or $a > -1$.

If $a \leq -1$, define a function g in $C(X)$ such that $g(x) = -1$ for all x in X. Then $g \leq f$, $g(p) = -1 \geq a$, and $g(X) \subset [-1, 1]$.

Now let $a > -1$. Since f is lower semi-continuous at p, there exists a neighborhood V of p such that $f(V) \subset (a, +\infty)$. Since (X, T) is completely regular, there is a function h in $C(X)$ such that $h(p) = 0$, $h(X - V) \subset \{1\}$, and $h(X) \subset [0, 1]$.

Define a function g from X into E^1 such that

$$g(x) = a - (a+1) \cdot h(x).$$

Since h is in $C(X)$, g is in $C(X)$. If x is not in V, then $h(x) = 1$, and hence $g(x) = -1$. Since $f(X) \subset [-1, 1]$, $g(x) \leq f(x)$. If x is in V, then $f(x) > a$, because $f(V) \subset (a, +\infty)$. Now since $h(X) \subset [0, 1]$ and $a + 1 > 0$, $0 \leq (a+1) \cdot h(x) \leq a+1$. This implies that

$$-1 \leq g(x) = a - (a+1) \cdot h(x) \leq a.$$

Since $g(x) \leq a < f(x)$, $g(x) < f(x)$. Therefore $g \leq f$. Now

$$g(p) = a - (a+1) \cdot h(p) = a.$$

Therefore $g(p) \geq a$. Since $g \leq f$ and since $f(X) \subset [-1, 1]$, then

$g(x) \leq 1$ for any x in X. Since $g(x) \geq -1$ for all x in X, then $g(X) \subset [-1, 1]$.

Theorem 3.18 If (X, T) is completely regular, then every real-valued lower semi-continuous function f defined on X is the upper envelope of the family G^* of all continuous functions g from X into the extended real line $\overline{R}$ for which $g \leq f$.

Proof: Let f be a real-valued lower semi-continuous function defined on X. The arctan function is a strictly increasing homeomorphism from R onto $(-\pi/2, \pi/2)$. Therefore the function $g = \frac{\arctan}{\pi/2}$ is a strictly increasing homeomorphism from R onto $(-1, 1)$. Now let the extended real line $\overline{R}$ have the usual order topology induced by the less than relation on $\overline{R}$. Then there exists a continuous extension h of g to $\overline{R}$, where $h(-\infty) = -1$ and $h(+\infty) = 1$. The function h is a strictly increasing homeomorphism from $\overline{R}$ onto $[-1, 1]$. Therefore $h \circ f$ is a real-valued function defined on X such that

$$(h \circ f)(X) \subset [-1, 1].$$

The function $h \circ f$ is lower semi-continuous. For if $k \geq 1$, then $(h \circ f)^{-1}((k, +\infty)) = \emptyset$ and if $k < -1$, $(h \circ f)^{-1}((k, +\infty)) = X$. If $-1 \leq k < 1$, there is an r in $\overline{R}$ such that $k = h(r)$. Therefore, since h is strictly increasing, $h^{-1}((k, +\infty)) = (r, +\infty]$. This implies that

$$(h \circ f)^{-1}((k, +\infty)) = f^{-1}((r, +\infty]).$$

By Lemma 3.13, $f^{-1}((r, +\infty])$ is open in X. Therefore for any real number k, $(h \circ f)^{-1}((k, +\infty))$ is open in X. Hence by Lemma 3.13, $h \circ f$ is lower semi-continuous.

In order to show that f is the upper envelope of G^*, it must be shown that for any p in X and any $a < f(p)$, there exists a g' in G^* such that $a \leq g'(p)$. Therefore let p be in X and $a < f(p)$. Since h is strictly increasing $h(a) < (h \circ f)(p)$. Because $h \circ f$ satisfies the hypothesis of Lemma 3.17 and $h(a) < (h \circ f)(p)$, there exists a g in C(X) such that $g \leq h \circ f$, $h(a) \leq g(p)$, and $g(X) \subset [-1, 1]$. Therefore, since the function h^{-1} is strictly increasing, $h^{-1} \circ g \leq f$. Since h^{-1} and g are continuous functions, $h^{-1} \circ g$ is a continuous function from X into $\overline{R}$. Thus $h^{-1} \circ g$ is in G^*. Since h^{-1} is strictly increasing and since $h(a) \leq g(p)$, then $a = h^{-1}(h(a)) \leq h^{-1}(g(p)) = (h^{-1} \circ g)(p)$. Therefore f is the upper envelope of G^*.

CHAPTER IV

CHARACTERIZATIONS OF UNIFORMIZABLE SPACES BY MEANS OF PSEUDOMETRICS

Uniformizable spaces may be characterized in terms of pseudometric spaces or families of pseudometrics (2). It will be proved in this chapter that the family of spaces which can be embedded in a product of pseudometric spaces is the same as the family of uniformizable spaces. This can be done either by using the definition of completely regular spaces or the definition of uniformizable spaces. In this chapter it will also be shown that a space is uniformizable if and only if there is a non-empty family of pseudometrics defined on the space for which the collection of open spheres determined by this family is a subbase for the topology of the space.

<u>Theorem 4.1</u> If (X, T) is a completely regular space, then X is homeomorphic to a subspace of a product of pseudometric spaces.

<u>Proof</u>: The family $C(X)$ can be used to define a family of pseudometrics on X. For any f in $C(X)$ define d_f to be a function from X into $E^1 \times E^1$ such that

$$d_f(x, y) = |f(x) - f(y)|$$

for any x and y in X. It can quickly be verified that d_f is a pseudometric on X.

Let (X, d_f) be the associated pseudometric space. Now define a function g from X into

$$Z = \times \{(X, d_f) : f \in C(X)\}$$

such that $g(x)_f = x$ for any x in X and any f in $C(X)$. Let Z have the product topology. For any f in $C(X)$, let P_f be the projection of Z into (X, d_f). For any f in $C(X)$, $P_f \circ g$ is the identity map, since

$$(P_f \circ g)(x) = P_f(g(x)) = g(x)_f = x.$$

Let $S_r(a)$ be an open sphere in (X, d_f), where f is in $C(X)$. Then

$$(P_f \circ g)^{-1}(S_r(a)) = S_r(a).$$

$$S_r(a) = \{x : d_f(x, a) < r\} = \{x : |f(x) - f(a)| < r\} = f^{-1}((f(a) - r, f(a) + r)).$$

Since f is continuous $f^{-1}((f(a) - r, f(a) + r))$ is an open set in (X, T), and therefore $P_f \circ g$ is continuous. Since $P_f \circ g$ is continuous for any f in $C(X)$, g is continuous (12, p. 91).

The function g is a 1-1 mapping. For let x and y be in X such that $x \neq y$ and let f be in $C(X)$. Then $g(x)_f = x$ and $g(y)_f = y$, which implies that $g(x) \neq g(y)$.

It will next be shown that g is an open mapping. For any open set O in (X, T) it must be shown that $g(O)$ is open in $g(X)$. If y is in $g(O)$, then $y = g(x)$ for some x in O. Since (X, T) is completely regular there exists an h in $C(X)$ such that $h(x) = 0$ and $h(X - O) \subset \{1\}$. Let $O_h = h^{-1}((-1, 1))$. Clearly

$$x \text{ is in } O_h \text{ and } O_h \subset O. \tag{16}$$

Now $O_h = \{u \in X : |h(u)| < 1\} = \{u \in X : |h(u) - h(x)| < 1\}$ since $h(x) = 0$. Also $\{u \in X : |h(u) - h(x)| < 1\} = \{u \in X : d_h(u, x) < 1\}$. Therefore

$$O_h \text{ is an open sphere in the pseudometric space } (X, d_h). \qquad (17)$$

Because of (16), y is in $g(O_h)$ and $g(O_h) \subset g(O)$.

If it can now be shown that $g(O_h)$ is open in $g(X)$, then $g(O)$ will be open in $g(X)$. This will be accomplished by using the fact that P_h is continuous. Because of (17) and the previous statement, $P_h^{-1}(O_h)$ is open in Z. This implies that $P_h^{-1}(O_h) \cap g(X)$ is open in $g(X)$. But

$$\begin{aligned} P_h^{-1}(O_h) \cap g(X) &= \{g(x) \in Z : x \in X \text{ and } g(x)_h \in O_h\} \\ &= \{g(x) \in Z : x \in X \text{ and } x \in O_h\} \\ &= g(O_h). \end{aligned}$$

Consequently $g(O_h)$ is open in $g(X)$ and hence $g(O)$ is open in $g(X)$. Therefore g is an open mapping. It is therefore true that g is a homeomorphism from (X, T) onto a subspace, $g(X)$, of a product Z of pseudometric spaces.

<u>Lemma 4.2</u> Any pseudometric space is completely regular.

<u>Proof</u>: Let (X, p) be a pseudometric space. Let C be a closed subset of X and let y be in $X - C$. If $C = \phi$, then define a function f from X into E^1 such that $f(x) = 0$ for all x in X. Then f is in $C(X)$, $f(y) = 0$, $f(C) \subset \{1\}$, and $f(X) \subset [0, 1]$. So assume that C is not empty. Define F from X into E^1 such that

$$F(x) = D(C, x) = \inf \{p(x, y) : y \in C\}.$$

F is a continuous function (12, p. 120). Since y is not in C and C is closed, then $D(C, y) = r > 0$.

Define the function g from X into E^1 such that

$$g(x) = \frac{(F \wedge D(C, y))(x)}{D(C, y)}$$

for any x in X. For any two functions h_1 and h_2 in $C(X)$, $h_1 \wedge h_2$ is in $C(X)$ (5, p. 133). Therefore g is in $C(X)$. Let x be in X. Then

$$(F \wedge D(C, y))(x) = \min\{F(x), D(C, y)\},$$

where $F(x) \geq 0$ and $D(C, y) > 0$. Therefore $(F \wedge D(C, y))(x) \geq 0$ which implies that $g(x) \geq 0$. Also $(F \wedge D(C, y))(x) \leq D(C, y)$ which implies that $g(x) \leq 1$. Therefore $g(X) \subset [0, 1]$.

$$g(y) = \frac{\min\{F(y), D(C, y)\}}{D(C, y)} = 1,$$

since $F(y) = D(C, y)$. If x is in C then

$$g(x) = \frac{\min\{F(x), D(C, y)\}}{D(C, y)} = 0,$$

since $F(x) = 0$ when x is in C. This implies that $g(C) \subset \{0\}$. Therefore (X, p) is completely regular.

Lemma 4.3 If, for any i in an index set I, X_i is completely regular, then the product space $Z = \underset{i \in I}{\times} X_i$ is completely regular.

Proof: Let C be a closed subset of Z and let y be in $Z - C$. Then $Z - C$ is open and hence there is a finite subset F of I such that

$$y \in \bigcap_{i \in F} (P_i^{-1}(G_i)) \subset Z - C, \tag{18}$$

where for each i in F, G_i is open in X_i and P_i is the projection of Z onto X_i. Let i be in F. Then $P_i(y)$ is in G_i and X_i is completely regular. Therefore there exists an f_i in $C(X_i)$ such that

$$f_i(P_i(y)) = 0, \quad f_i(X - G_i) \subset \{1\}, \text{ and } \quad f_i(X_i) \subset [0, 1], \tag{19}$$

Since both f_i and P_i are continuous, $f_i \circ P_i$ is in C(Z). Define a function g from Z into E^1 such that

$$g(x) = \max \{(f_i \circ P_i)(x) : i \in F\}.$$

For any functions $f_1, f_2, \ldots, f_k$ in C(Z), the function h defined on Z, such that

$$h(x) = \max \{f_1(x), f_2(x), \ldots, f_k(x)\},$$

is in C(Z) (5, p. 133). Therefore g is in C(Z). For any i in F, $(f_i \circ P_i)(y) = 0$, by (19). Thus $g(y) = 0$. If x is in C then, because of (18), there is a j in F such that x is not in $P_j^{-1}(G_j)$. This implies that $P_j(x)$ is in $X_j - G_j$. Because of (19), $f_j(P_j(x)) = 1$ and hence $g(x) \geq 1$. Since $f_i(X_i) \subset [0, 1]$ for each i in F, $g(Z) \subset [0, 1]$. Consequently $g(x) = 1$. Therefore $g(C) \subset \{1\}$. The product space Z is therefore completely regular.

<u>Lemma 4.4</u> A non-empty subspace of a completely regular space is completely regular.

<u>Proof</u>: Let (X, T) be completely regular and let Y be a non-empty subspace of X. Let F be a closed subset of the subspace Y

and let y be in $Y - F$. Then $F = G \cap Y$, where G is a closed subset of the space X. Since y is not in $G \cap Y$, y is not in G. Since the space X is completely regular, there exists an f in $C(X)$ such that $f(y) = 0$, $f(G) \subset \{1\}$, and $f(X) \subset [0, 1]$. Now let $g = f|Y$. Since the restriction of f to any non-empty subset of X is continuous, g is in $C(Y)$. Since y is in Y, $g(y) = f(y) = 0$. For any x in F, x is in Y, and hence $g(x) = f(x)$. Since $F \subset G$ and $f(G) \subset \{1\}$, then $f(F) \subset \{1\}$. Therefore for any x in F, $g(x) = 1$. Consequently $g(F) \subset \{1\}$. Since $f(X) \subset [0, 1]$ and $g(Y) = f(Y) \subset f(X)$, then $g(Y) \subset [0, 1]$.

Lemma 4.5 If the space (X, T) is homeomorphic to a completely regular space, (Y, T'), then (X, T) is completely regular.

Proof: Let f be a homeomorphism from (X, T) onto (Y, T'). Let F be a closed subset of X and let y be in $X - F$. Then $f(F)$ is a closed subset of Y. Since y is not in F, $f(y)$ is not in $f(F)$. Since (Y, T') is a completely regular space, there exists an h in $C(Y)$ such that

$$h(f(y)) = 0, \quad h(f(F)) \subset \{1\}, \text{ and } h(Y) \subset [0, 1]. \tag{20}$$

Since f and h are continuous $g = h \circ f$ is in $C(X)$ and $g(y) = h(f(y) = 0$. For any x in F, $g(x) = h(f(x))$, where $f(x)$ is in $f(F)$. Therefore, because of (20), $g(x) = 1$. Hence $g(F) \subset \{1\}$. Since $h(Y) \subset [0, 1]$, then $g(X) = h(f(X)) \subset [0, 1]$. Therefore (X, T) is a completely regular space.

Theorem 4.6 If the space (X, T) is homeomorphic to a subspace of a product of pseudometric spaces, then (X, T) is completely regular.

Proof: Let f be a homeomorphism from (X, T) onto a subspace of $Z = \underset{i \in I}{\times} X_i$, where X_i is a pseudometric space for each i in I. By Lemma 4.2, X_i is completely regular for each i in I. Because of Lemma 4.3, Z is completely regular. Therefore, by Lemma 4.4, f(X) is completely regular. Hence by Lemma 4.5, (X, T) is completely regular.

Theorems 4.1 and 4.6 show that the family of completely regular spaces is the same as the family of spaces which can be embedded in a product of pseudometric spaces. As shown in Chapter II, the family of completely regular spaces is the same as the family of uniformizable spaces. Therefore Theorems 4.1 and 4.6 can be proved by using the concept of a uniform space instead of using the concept of a completely regular space. The next two theorems will do this. Several lemmas will first be proved.

Lemma 4.7 A product of uniformizable spaces is uniformizable.

Proof: For any a in A, let (X_a, T_a) be a uniformizable space. Then, for any a in A, there is a uniformity U_a^* for X_a which is compatible with T_a. Let (Z, T) be the product space determined by $\{(X_a, T_a) : a \in A\}$. By Corollary 1.18, the collection of all sets of the form $\{(x, y) \in Z \times Z ; (x_a, y_a) \in U_a\}$, where a is in A and U_a is in U_a^*, is a subbase for a uniformity U^* for Z called the product uniformity. Let T_U be the uniform topology for Z induced by U^*. If $T_U = T$, then (Z, T) is uniformizable.

So let O be in T_U and let u be in O. Then there exists a U in U^* such that $U[u] \subset O$. By definition of U^*, there is a finite subset F of A such that

$$B = \bigcap_{a \in F} \{(x, y) \in Z \times Z : (x_a, y_a) \in U_a\} \subset U,$$

where, for each a in F, U_a is in U_a^*. Therefore $B[u] \subset U[u] \subset O$, and

$$B[u] = \{v : (u, v) \in B\} = \bigcap_{a \in F} \{v : (u_a, v_a) \in U_a\}$$

$$= \bigcap_{a \in F} \{v : v_a \in U_a[u_a]\}.$$

Therefore $\bigcap_{a \in F} \{v : v_a \in U_a[u_a]\} \subset O$. By Corollary 2.6, for each a in F, $U_a[u_a]$ is a T_a-neighborhood of u_a. Consequently, for any a in F, there exists an O_a in T_a such that $u_a \in O_a \subset U_a[u_a]$. Therefore

$$u \in \bigcap_{a \in F} \{v \in Z : v_a \in O_a\} \subset \bigcap_{a \in F} \{v \in Z : v_a \in U_a[u_a]\} \subset O.$$

By the definition of the product topology for Z, $\bigcap_{a \in F} \{v \in Z : v_a \in O_a\}$ is in T. Therefore O is in T.

Now let O be in T and let u be in O. There exists a finite subset F of A such that $u \in \bigcap_{a \in F} \{w \in Z : w_a \in O_a\} \subset O$, where for each a in F, O_a is in T_a. Consequently for each a in F, there exists a U_a in U_a^* such that $U_a[u_a] \subset O_a$. Therefore

$$u \in \bigcap_{a \in F} \{w \in Z : w_a \in U_a[u_a]\} \subset \bigcap_{a \in F} \{w \in Z : w_a \in O_a\}.$$

But $\bigcap_{a \in F} \{w \in Z : w_a \in U_a[u_a]\} = N[u]$, where

$N = \bigcap_{a \in F} \{(x, y) : (x_a, y_a) \in U_a\}$. Since N is in U^* and because of Corollary 2.6, $N[u]$ is a T_U-neighborhood of u. Since $N[u] \subset O$, O is a T_U-neighborhood of u. Consequently O is in T_U. Therefore $T = T_U$ and (Z, T) is uniformizable.

Lemma 4.8 A subspace of a uniformizable space is uniformizable.

Proof: Let (X, T) be a uniformizable space and let Y be a subspace of X. Then there exists a uniformity U^* for X compatible with T. Define U_Y^* to be $\{U \cap (Y \times Y) : U \in U^*\}$ and let A be in U_Y^*. Then $A = U \cap (Y \times Y)$ for some U in U^*. Since $D_Y \subset D_X \subset U$, then $D_Y \subset U \cap (Y \times Y) = A$.

Also $\qquad A^{-1} = [U \cap (Y \times Y)]^{-1} = U^{-1} \cap (Y \times Y)$. Since

$U^{-1} \in U^*$, A^{-1} is in U_Y^*. Now there is a V in U^* such that $V \circ V \subset U$. This implies that $V \cap (Y \times Y)$ is in U_Y^* and

$$[V \cap (Y \times Y)] \circ [V \cap (Y \times Y)] \subset U \cap (Y \times Y) = A.$$

Let C and B be in U_Y^*. Then $C = U \cap (Y \times Y)$ and $B = V \cap (Y \times Y)$, where U and V are in U^*. Hence $C \cap B = (U \cap V) \cap (Y \times Y)$, where $U \cap V$ is in U^*. Therefore $C \cap B$ is in U_Y^*. Suppose $D = U \cap (Y \times Y) \subset B \subset Y \times Y$, where U is in U^*. Then

$$(U \cup B) \cap (Y \times Y) = [U \cap (Y \times Y)] \cup B = B.$$

Since $U \cup B$ is in U^*, B is in U_Y^*. Therefore U_Y^* is a uniformity for Y.

In order to show that the subspace Y is uniformizable it must be proved that the topology T' induced by U_Y^* is equal to the relative topology T_Y for Y. Therefore let O be in T_Y and let x be in O. Then $O = G \cap Y$, where G is in T. Since U^* is compatible with T, there exists a U in U^* such that $U[x] \subset G$. Therefore $U[x] \cap Y \subset G \cap Y$. Since $(U \cap (Y \times Y))[x] = U[x] \cap Y$, then $(U \cap (Y \times Y))[x] \subset O$. Because $U \cap (Y \times Y)$ is in U_Y^*, O is in T'.

Now let O be in T' and let x be in O. Then there exist a U in U^* such that $(U \cap (Y \times Y))[x] \subset O$. Hence $U[x] \cap Y \subset O$. By Corollary 2.6, $U[x]$ is a T-neighborhood of x. Therefore there exists a T-open set G such that $x \in G \subset U[x]$. This implies that $x \in G \cap Y \subset U[x] \cap Y \subset O$. Since $G \cap Y$ is in T_Y, O is in T_Y. Therefore $T_Y = T$ and so the subspace Y is uniformizable.

<u>Definition 4.9</u> If (X, U^*) is a uniform space and $Y \subset X$, then

$$\{U \cap (Y \times Y) : U \in U^*\}$$

is called the relative uniformity for Y or the relativization of U^* to Y (12, p. 182).

<u>Lemma 4.10</u> If a space (X, T_1) is homeomorphic to a uniformizable space (Y, T_2), then (X, T_1) is also uniformizable.

<u>Proof</u>: Let U^* be a uniformity for Y which is compatible with T_2 and let f be a homeomorphism from X onto Y. By Theorem 1.17, $\{f_2^{-1}(U) : U \in U^*\}$ is a subbase for a uniformity V^* for X. Actually $\{f_2^{-1}(U) : U \in U^*\}$ is a base for V^*. Let T_V be the topology for X which is induced by V^*. It must be shown that $T_V = T_1$.

Let O be in T_V and let x be in O. There exists a V in V^* such that $V[x] \subset O$. There is a U in U^* such that $f_2^{-1}(U) \subset V$, which implies that

$$(f_2^{-1}(U))[x] \subset V[x] \subset O. \tag{21}$$

By Lemma 2.5, $f(x)$ is in $G = \text{int}(U[f(x)])$, the T_2-interior of $U[f(x)]$. Clearly x is in $f^{-1}(G)$. Let u be in $f^{-1}(G)$. Then $f(u)$ is in G which implies that $f(u)$ is in $U[f(x)]$. Hence $f_2(x, u) = (f(x), f(u))$ is in U, which implies that (x, u) is in $f_2^{-1}(U)$. Thus u is in $(f_2^{-1}(U))[x]$, and hence, by (21), u is in O. Therefore $f^{-1}(G) \subset O$. Since G is in T_2 and f is in $C(X)$, $f^{-1}(G)$ is T_1-open. It has been shown then that $x \in f^{-1}(G) \subset O$, where $f^{-1}(G)$ is T_1-open. Therefore O is in T_1.

Now let O be in T_1 and let x be in O. Since f is a homeomorphism, $f(O)$ is in T_2 and $f(x)$ is in $f(O)$. There is a U in U^* such that

$$U[f(x)] \subset f(O). \tag{22}$$

Let y be in $(f_2^{-1}(U))[x]$. Then (x, y) is in $f_2^{-1}(U)$, which implies that $(f(x), f(y))$ is in U. This implies that $f(y)$ is in $U[f(x)]$. Because of (22), $f(y)$ is in $f(O)$. Since f is one-to-one, y is in O. Therefore $(f_2^{-1}(U))[x] \subset O$. Since $f_2^{-1}(U)$ is in V^* and since x is in $(f_2^{-1}(U))[x]$, O is in T_V. Therefore $T_1 = T_V$, and hence (X, T_1) is uniformizable.

<u>Theorem 4.11</u> If a space (X, T) is homeomorphic to a subspace of a product of pseudometric spaces, then (X, T) is uniformizable.

<u>Proof:</u> Let f be a homeomorphism from the space (X, T) onto

a subspace of $Z = \underset{i \in I}{\times} X_i$, a product of pseudometric spaces. Because of Corollary 1.5, X_i is uniformizable for each i in I. Then, because of Lemma 4.7, Z is uniformizable. f(X) is uniformizable by Lemma 4.8. Therefore by Lemma 4.10, (X, T) is uniformizable.

The following lemmas are needed to prove the converse of Theorem 4.11.

<u>Lemma 4.12</u> If (X, U^*) is a uniform space, then for any U in U^* there is a symmetric V in U^* such that $V \circ V \circ V \subset U$.

<u>Proof</u>: Let U be in U^*. Then there is a W in U^* such that $W \circ W \subset U$. If $V = W \cap W^{-1}$ then, by properties of uniformity, V is in U^*, V is symmetric, and $V \circ V \subset U$. Therefore for any U in U^*, there is a symmetric V in U^* such that $V \circ V \subset U$. Therefore, for any U in U^*, there exist symmetric members V_1 and V_2 of U^* such that $V_1 \circ V_1 \subset U$ and $V_2 \circ V_2 \subset V_1$. Since $V_2 \subset V_2 \circ V_2$, $V_2 \subset V_1$. Therefore $V_2 \circ (V_2 \circ V_2) \subset V_1 \circ V_1$ and hence $V_2 \circ V_2 \circ V_2 \subset U$.

<u>Lemma 4.13</u> For each a in A, let (X_a, U_a^*) be a uniform space. Let f be a function from the uniform space (Y, V^*) into (Z, U^*) where $Z = \underset{a \in A}{\times} X_a$ and U^* is the product uniformity for Z. If, for any a in A, $P_a \circ f$ is uniformly continuous relative to V^* and U_a^*, then f is uniformly continuous relative to V^* and U^*.

<u>Proof</u>: Let V be a member of the subbase of the product uniformity. Then $V = (P_a^{-1})_2 (U_a) = \{(x, y) : (x_a, y_a) \in U\}$, where U is in U_a^* and a is in A. Hence

$$f_2^{-1}(V) = f_2^{-1}\{(x, y) : (x_a, y_a) \in U\} = (P_a \circ f)_2^{-1}(U).$$

Since $P_a \circ f$ is uniformly continuous relative to V^* and U_a^*, then $(P_a \circ f)_2^{-1}(U)$ is in V^*. Therefore

$$f_2^{-1}(V) \text{ is in } V^* \text{ for any } V \text{ in the subbase of } U^*. \tag{23}$$

For any W in U^*,

$$f_2^{-1}(W) \supset f_2^{-1}\left(\bigcap_1^n V_i\right) = \bigcap_1^n f_2^{-1}(V_i)$$

where each V_i is in the subbase of U^*. Because of (23), $\bigcap_1^n f_2^{-1}(V_i)$ is in V^*. Therefore $f_2^{-1}(W)$ is in V^* since it contains a member of V^*. Therefore f is uniformly continuous relative to V^* and U^*.

<u>Lemma 4.14</u> Let (X, U^*) be a uniform space and let d be a pseudometric on X. Then d is uniformly continuous relative to the product uniformity for $X \times X$ and the usual uniformity for E^1 if, and only if, for any $r > 0$, $V_{d,r}$ is in U^*.

<u>Proof:</u> Suppose d is uniformly continuous relative to the product uniformity for $X \times X$ and the usual uniformity for E^1. Let $r > 0$ and $U_r = \{(x, y) : |x - y| < r\}$. Then $d_2^{-1}(U_r)$ is in the product uniformity for $X \times X$. By Corollary 1.18, the product uniformity has a subbase consisting of sets of the form

$$\{((x, y), (u, v)) : (x, u) \in U\}$$

where $U \in U^*$ or $\{((x, y), (u, v)) : (y, v) \in V\}$, where $V \in U^*$. Therefore $d_2^{-1}(U_r)$ contains a finite intersection of sets of this type. It can easily be shown that a finite intersection of sets in the subbase of the product uniformity contains a set

$$\{((x, y), (u, v)) : (x, u) \in V \text{ and } (y, v) \in V\},$$

where V is in U^*. Therefore

$$d_2^{-1}(U_r) \supset \{((x, y), (u, v)) : (x, u) \in V \text{ and } (y, v) \in V\}. \qquad (24)$$

Let (x, y) be in V. Then $((x, x), (x, y))$ is in $d_2^{-1}(U_r)$ because $(x, x) \in V$ and because of (24). Therefore $d_2((x, x), (x, y))$ is in U_r, which implies that $|d(x, x) - d(x, y)| = d(x, y) < r$. Hence (x, y) is in $V_{d, r}$. Therefore $V \subset V_{d, r}$ and, since V is in U^*, $V_{d, r}$ is in U^*.

Suppose now that for any $r > 0$, $V_{d, r}$ is in U^*. Let U be in the usual uniformity for E^1. Then there exists an $r > 0$ such that $U_r = \{(x, y) : |x - y| < r\} \subset U$. Let

$$M = \{((x, y), (u, v)) : (x, u) \in V_{d, \frac{r}{2}} \text{ and } (y, v) \in V_{d, \frac{r}{2}}\}.$$

By the supposition $V_{d, \frac{r}{2}}$ is in U^*, and hence M is in the product uniformity for $X \times X$. It will now be shown that $M \subset d_2^{-1}(U_r)$. Let $((x, y)(u, v))$ be in M. Then

$$d(x, u) < \frac{r}{2} \text{ and } d(y, v) < \frac{r}{2}. \qquad (25)$$

Now $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$. Therefore by (25), $d(x, y) - d(u, v) < r$. Also $d(u, v) \leq d(u, x) + d(x, y) + d(y, v)$. Therefore by (25), $d(u, v) - d(x, y) < r$. Because of these last two inequalities, $|d(x, y) - d(u, v)| < r$. Hence $d_2((x, y), (u, v))$ is in U_r, and consequently $((x, y), (u, v))$ is in $d_2^{-1}(U_r)$. Therefore $M \subset d_2^{-1}(U_r)$, which implies that $M \subset d_2^{-1}(U)$. Since M is in the product uniformity for $X \times X$, $d_2^{-1}(U)$ also is in the product uniformity. By the definition of uniformly continuous, d is uniformly continuous relative to the product uniformity for $X \times X$ and the usual uniformity for E^1.

The next lemma is often called The Metrization Lemma. It will be needed later in this chapter to prove that any uniformity U^* for a set X can be generated by the family of all pseudometrics on X which are uniformly continuous relative to U^* and the usual uniformity for E^1. The Metrization Lemma is also used to prove that a uniformity can be generated by a single pseudometric if and only if the uniformity has a countable base (12,p. 186).

Lemma 4.15 Let $\{U_n, n \geq 0\}$ be a sequence of subsets of $X \times X$ such that

(i) $U_0 = X \times X$.

(ii) $U_n \supset D_X$ for any n.

(iii) $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$ for any n.

(iv) Each U_n is symmetric.

Then there is a pseudometric d on X such that for any $n \geq 1$,

$$U_n \subset \{(x, y) : d(x, y) < 2^{-n}\} \subset U_{n-1}.$$

Proof: Define a function f from $X \times X$ into E^1 such that $f(x, y) = 2^{-n}$ if there exists a least positive integer n for which (x, y) is not in U_n, and $f(x, y) = 0$ if (x, y) is in each U_n. Define a function d from $X \times X$ into E^1 such that $d(x, y)$ is the infimum of the set of all sums $\sum_0^n f(x_i, x_{i+1})$, where $\{x_i : 0 \leq i \leq n + 1\}$ is a sequence of points in X, $x_0 = x$, and $x_{n+1} = y$. For the sake of convenience call any finite sequence $x_0, x_1, \ldots, x_{n+1}$ in X a chain of $n+2$ points and call $\sum_{0}^{n} f(x_i, x_{i+1})$ the length of the chain from x_0 to x_{n+1}. It follows from the definition of d and properties of the infimum that d is a pseudometric on X (15, p. 130).

It will first be shown that for any $n \geq 1$,

$$U_n \subset \{(x, y) : d(x, y) < 2^{-n}\}.$$

Let (x, y) be in U_n. Because of (ii) and (iii), $\{U_n\}$ is a monotonically decreasing sequence of sets. Therefore for each $i \leq n$, (x, y) is in U_i. Hence, by definition of f, $f(x, y) < 2^{-n}$. Since $f(x, y)$ is the length of a chain from x to y and since $d(x, y)$ is the infimum of the lengths of all chains from x to y, $d(x, y) \leq f(x, y)$. Therefore $d(x, y) < 2^{-n}$.

In order to prove the other set inclusion in the conclusion of the lemma it will first be shown that

$$f(x_0, x_{n+1}) \leq 2 \sum_{0}^{n} f(x_i, x_{i+1}),$$

for any chain of two or more points. If $n = 0$ or equivalently if the chain has two points, then $\sum_{0}^{n} f(x_i, x_{i+1}) = 0$ implies that $f(x_0, x_{n+1}) = 0$. Assume, for any chain of $n + 2$ points, where $n \geq 0$, that

$$\sum_{0}^{n} f(x_i, x_{i+1}) = 0 \text{ implies that } f(x_0, x_{n+1}) = 0,$$

Consider a chain of $n + 3$ points such that $\sum_{0}^{n+1} f(x_i, x_{i+1}) = 0$. Then $\sum_{0}^{n} f(x_i, x_{i+1}) = 0$ and $f(x_{n+1}, x_{n+2}) = 0$, since $f \geq 0$. By the assumption, $f(x_0, x_{n+1}) = 0$. Consequently, by definition of f, (x_0, x_{n+1}) and (x_{n+1}, x_{n+2}) are in U_m for any m. Therefore (x_0, x_{n+2}) is in

$U_m \circ U_m$ for any m. Since

$$U_m \circ U_m \subset U_m \circ U_m \circ U_m \subset U_{m-1}$$

for any $m \geq 1$, then (x_0, x_{n+2}) is in U_m for any m. Consequently, by the definition of f, $f(x_0, x_{n+2}) = 0$. It has been proved by mathematical induction, for any chain of two or more points, that

$$\text{if } \sum_0^n f(x_i, x_{i+1}) = 0 \text{ then } f(x_0, x_{n+1}) = 0.$$

Therefore for any chain of two or more points,

$$\sum_0^n f(x_i, x_{i+1}) = 0 \text{ implies that } f(x_0, x_{n+1}) \leq 2 \sum_0^n f(x_i, x_{i+1}). \qquad (26)$$

It will now be proved by mathematical induction that, for any chain of two or more points,

$$\text{if } \sum_0^n f(x_i, x_{i+1}) \neq 0, \text{ then } f(x_0, x_{n+1}) \leq 2 \sum_0^n f(x_i, x_{i+1}). \qquad (27)$$

Assume in the following argument that for any chain of two or more points,

$$\sum_0^n f(x_i, x_{i+1}) \neq 0.$$

If $n = 0$ or equivalently if the chain has two points, then

$$f(x_0, x_{n+1}) \leq 2 \sum_0^n f(x_i, x_{i+1}).$$

Assume the inequality to be true for any $n \leq q - 1$, where $q \geq 1$, and for any chain of $n + 2$ points. Consider a chain of $q + 2$ points and let

$$a = \sum_{0}^{q} f(x_i, x_{i+1}).$$

It must be shown that $f(x_0, x_{q+1}) \leq 2a$. Now if $a \geq 1/4$ then $f(x_0, x_{q+1}) \leq 2a$, since $f(x_0, x_{q+1}) \leq 1/2$. Therefore assume that $a < 1/4$. If there exists a largest integer j such that

$$\sum_{0}^{j-1} f(x_i, x_{i+1}) \leq a/2,$$

let $k = j$. Otherwise let $k = 0$. Note that $k \leq q$ since a is positive. If $k = 0$, $f(x_0, x_k) = 0 \leq a$. If $k \geq 1$ then, by the induction hypothesis and the definition of k,

$$f(x_0, x_k) \leq 2 \sum_{0}^{k-1} f(x_i, x_{i+1}) \leq 2\,(a/2) = a,$$

Therefore

$$\text{for } k \geq 0, \ f(x_0, x_k) \leq a. \tag{28}$$

If $k = q$, then $f(x_k, x_{k+1}) = f(x_q, x_{q+1}) \leq a$. If $k < q$ and $\sum_{k+1}^{q} f(x_i, x_{i+1}) > a/2$, then $\sum_{0}^{k} f(x_i, x_{i+1}) \leq a/2$. This contradicts the definition of k. Therefore if $k < q$, then $\sum_{k+1}^{q} f(x_i, x_{i+1}) \leq a/2$. By the induction hypothesis

$$f(x_{k+1}, x_{q+1}) \leq 2 \sum_{k+1}^{q} f(x_i, x_{i+1}).$$

Hence $f(x_{k+1}, x_{q+1}) \leq 2\,(a/2) = a$ if $k < q$. Define m to be the smallest positive integer such that

$$2^{-m} \leq a. \tag{29}$$

Note that since $a < 1/4$, $m \geq 3$. Suppose that (x_0, x_k) is not in U_{m-1}. Then let p be the smallest positive integer such that (x_0, x_k) is not in U_p. Then $p < m$ which implies that $2^{-p} > 2^{-m}$. By definition of f, $2^{-p} = f(x_0, x_k)$. By (28), $2^{-p} \leq a$. But, since $p < m$, this contradicts the definition of m. Therefore (x_0, x_k) is in U_{m-1}. By the same reasoning (x_k, x_{k+1}) and (x_{k+1}, x_{q+1}) are in U_{m-1}. Therefore, if $k = q$, then (x_0, x_{q+1}) is in $U_{m-1} \circ U_{m-1}$, and, if $k < q$, then (x_0, x_{q+1}) is in $U_{m-1} \circ U_{m-1} \circ U_{m-1}$. Now, by (i) and (ii),

$$U_{m-1} \circ U_{m-1} \subset U_{m-1} \circ U_{m-1} \circ U_{m-1} \subset U_{m-2}.$$

Therefore (x_0, x_{q+1}) is in U_{m-2} in either case. Now if (x_0, x_{q+1}) is in U_i for each i, then $f(x_0, x_{q+1}) = 0 \leq 2a$. If (x_0, x_{q+1}) is not in each U_i, then let j be the least integer such that (x_0, x_{q+1}) is not in U_j. Then, by definition of f, $f(x_0, x_{q+1}) = 2^{-j}$. Since (x_0, x_{q+1}) is in U_{m-2} and since $\{U_n : n \geq 0\}$ is a monotonically decreasing sequence, then $j > m - 2$. This implies that $j \geq m - 1$ and hence that $f(x_0, x_{q+1}) \leq 2^{-m+1}$. Consequently, because of (29), $f(x_0, x_{q+1}) \leq 2 \cdot 2^{-m} \leq 2a$. Therefore, by mathematical induction,

$$f(x_0, x_{n+1}) \leq 2 \sum_0^n f(x_i, x_{i+1}), \text{ for any chain of two or more points.} \quad (30)$$

Let $d(x, y) < 2^{-n}$, where $n \geq 1$. If $n = 1$, then $U_{n-1} = X \times X$ and hence (x, y) is in U_{n-1}. Let $n > 1$. Because of (30),

$$1/2 \cdot f(x, y) \leq \sum_0^n f(x_i, x_{i+1})$$

for any chain from $x_0 = x$ to $x_{n+1} = y$. Therefore, by definition of d, $1/2 \cdot f(x, y) \leq d(x, y)$, and hence

$$f(x, y) \leq 2\,d(x, y) < 2^{-n+1}. \tag{31}$$

If $f(x, y) = 0$, then, by the definition of f, (x, y) is in U_{n-1}. Assume $f(x, y) \neq 0$ and (x, y) is not in U_{n-1}. Then there is a least positive integer r such that (x, y) is not in U_r and $f(x, y) = 2^{-r}$. Then, by the assumption, $r \leq n - 1$. This implies that $2^{-r} \geq 2^{-n+1}$, which implies that $f(x, y) \geq 2^{-n+1}$. But this contradicts (31). Therefore, if $f(x, y) \neq 0$, then (x, y) is in U_{n-1}. Therefore, for any $n \geq 1$,

$$\{(x, y) : d(x, y) < 2^{-n}\} \subset U_{n-1}.$$

Lemma 4.16 If (X, U^*) is a uniform space, then U^* is generated by the family P of all pseudometrics on X which are uniformly continuous relative to the product uniformity for $X \times X$ and the usual uniformity for E^1.

Proof: P is non-empty. For if p is a function from $X \times X$ into E^1 such that $p(x, y) = 0$ for any (x, y), then p is a pseudometric on X. For any $r > 0$, $V_{p, r} = X \times X$, which is in U^*. Therefore, by Lemma 4.14, p is in P. By Theorem 1.14, $\{V_{p, r} : p \in P \text{ and } r > 0\}$ is a subbase for a uniformity U_p^* for X. It will be now be shown that $U_p^* = U^*$.

Let U be in U_p^*. Then $U \supset \bigcap_1^n V_{p_i, r_i}$, where for each i, p_i is in P and $r_i > 0$. By the definition of P and by Lemma 4.14, V_{p_i, r_i} is in U^* for each i. Therefore, by properties of a uniformity, U is in U^*.

Let U be in U^*. Define a sequence of entourages in U^* in the following way. Let $U_0 = X \times X$. Define U_1 to be a symmetric entourage in U^* such that $U_1 \circ U_1 \circ U_1 \subset U$. For any $i \geq 1$, define U_{i+1} to be a symmetric entourage in U^* such that $U_{i+1} \circ U_{i+1} \circ U_{i+1} \subset U_i$. It is possible to do this because of Lemma 4.12. This sequence satisfies the hypothesis of Lemma 4.15. Therefore there exists a pseudometric d on X such that

$$U_n \subset \{(x, y) : d(x, y) < 2^{-n}\} \subset U_{n-1} \text{ for any } n \geq 1. \tag{32}$$

Let $r > 0$. Then there is a positive integer m such that $2^{-m} < r$. Therefore, by (32), $U_m \subset V_{d,r}$. Since U_m is in U^*, $V_{d,r}$ is in U^*. Hence by Lemma 4.14, d is in P. This implies that $V_{d,1/4}$ is in U_p^*. Because of (32), $V_{d,1/4} \subset U_1$. Since

$$U_1 \subset U_1 \circ U_1 \circ U_1 \subset U,$$

then $V_{d,1/4} \subset U$. Hence U is in U_p^*. Therefore $U^* = U_p^*$.

Lemma 4.17 If f is a function from the uniform space (X, U^*) into the uniform space (Y, V^*) and f is uniformly continuous relative to U^* and V^*, then f is continuous relative to the uniform topologies of these spaces.

Proof: Let O be in the uniform topology T_V induced by V^* and let x be in $f^{-1}(O)$. Then $f(x)$ is in O and hence there is a V in V^* such that

$$V[f(x)] \subset O. \tag{33}$$

Since f is uniformly continuous, $f_2^{-1}(V) = U$, for some U in U^*. Therefore $f_2(U) \subset V$. Let y be in $U[x]$. This implies that (x, y) is in U. Then $f_2(x, y) = (f(x), f(y))$ is in V which implies that f(y) is in $V[f(x)]$. By (33), f(y) is in O and hence y is in $f^{-1}(O)$. Therefore $U[x] \subset f^{-1}(O)$. Hence $f^{-1}(O)$ is in the uniform topology T_U induced by U^*. Therefore f is continuous relative to T_U and T_V.

<u>Theorem 4.18</u> If a space (X, T) is uniformizable, then it is homeomorphic to a subspace of a product of pseudometric spaces.

<u>Proof</u>: Since (X, T) is uniformizable there is a uniformity U^* for X which is compatible with T. Let F be the family of pseudometrics on X which are uniformly continuous relative to the product uniformity for $X \times X$ and the usual uniformity for E^1. As shown in the proof of Lemma 4.16, F is non-empty. For any d in F, let (X, d) be the associated pseudometric space. Let Z be the product space determined by these pseudometric spaces. For any d in F, let U_d^* be the uniformity for X generated by d. By Corollary 1.15, U_d^* is compatible with the pseudometric topology of (X, d). Let V^* be the product uniformity for Z determined by the family $\{U_d^* : d \in F\}$. Define a function f from X into Z such that $f(x)_d = x$, for any x in X and any d in F. Note that f is defined in the same way that the function g is Theorem 4.1 was defined, except that now the index set for Z is different. As was done for Theorem 4.1, it can be shown that for any d in F, $P_d \circ f$ is an identity mapping. It follows directly that for any d in F, $(P_d \circ f)_2$ is an identity mapping. Therefore for any U in U_d^*, $(P_d \circ f)_2^{-1}(U) = U$. There is an $r > 0$ such that $U \supset V_{d, r}$. By Lemma 4.16, F generates U^*. Hence U is in U^*. Consequently for any d in F, $P_d \circ f$ is uniformly continuous relative to U^* and U_d^*. Therefore, by Lemma 4.13, f is uniformly continuous

relative to U^* and V^*. Because of Lemma 4.17, f is continuous relative to T and the topology T_V for Z which is induced by V^*. As shown in the proof of Lemma 4.7, T_V is the product topology for Z. Therefore f is continuous relative to T and the product topology for Z. It can easily be verified, as was done for the function g in Theorem 4.1, that f is a one-to-one mapping. Define the function h from X onto $f(X) = Y$ such that $h(x) = f(x)$ for any x in X. Then h is one-to-one and is continuous relative to T and the relative topology T_Y for Y. Let g be the inverse of h and let V_Y^* be the relativization of V^* to Y. It will now be shown that g is uniformly continuous relative to V_Y^* and U^*.

Because $g_2^{-1}(U) = h_2(U) = f_2(U)$ for any U in U^*, g is uniformly continuous provided that $f_2(U)$ is in V_Y^* for every U in U^*. So let U be in U^*. Since F generates U^*, U contains

$$\bigcap_1^n V_{d_i, r_i},$$

where, for each i, d_i is in F and $r_i > 0$. Therefore

$$f_2(U) \supset f_2\left(\bigcap_1^n V_{d_i, r_i}\right).$$

Since f is one-to-one, f_2 is also one-to-one, and hence

$$f_2(U) \supset \bigcap_1^n f_2(V_{d_i, r_i}). \tag{34}$$

For any d in F and $r > 0$, the set

$$W = \{(x, y) \in Z \times Z : (x_d, y_d) \in V_{d, r}\} \cap [Y \times Y]$$

is in V_Y^*. Now let (x, y) be in W. Then (x_d, y_d) is in $V_{d,r}$, and there exist a and b in X such that $x = f(a)$ and $y = f(b)$. By definition of f, $x_d = f(a)_d = a$ and $y_d = f(b)_d = b$. Hence (a, b) is in $V_{d,r}$ and $(x, y) = f_2(a, b)$. Consequently (x, y) is in $f_2(V_{d,r})$. Therefore $W \subset f_2(V_{d,r})$, and hence $f_2(V_{d,r})$ is in V_Y^*. Because of (34) and the properties of a uniformity, $f_2(U)$ is in V_Y^*. Therefore g is uniformly continuous relative to V_Y^* and U^*. As shown in the proof of Lemma 4.8, the topology induced on Y by V_Y^* is the relative topology T_Y of the subspace Y. Consequently, because of Lemma 4.17, g is continuous relative to T_Y and T. Therefore h is a homeomorphism from X onto Y, a subspace of Z.

Lemma 4.16 and Theorem 1.14 result in the following characterization for a uniformity or uniform structure for a set X. If U^* is a collection of relations on X, then U^* is a uniformity for X if and only if there exists a non-empty family P of pseudometrics defined on X such that $\{V_{d,r} : d \in P,\ r > 0\}$ is a subbase for U^*. This definition is given by Bourbaki (3, p. 139). Sometimes a uniform structure is defined to be a non-empty family of pseudometrics with certain stated properties (11, p. 217).

The last theorem in this chapter is a characterization of uniformizable spaces in terms of a family of pseudometrics. It states that the topology of any uniformizable space can be generated by the open spheres associated with a family of pseudometrics.

<u>Theorem 4.19</u> A topological space (X, T) is uniformizable if and only if there exists a non-empty family F of pseudometrics on X such that the collection of open spheres

$$\{S_{d,r}(x) : d \in F, \ r > 0, \ \text{and} \ \ x \in X\}$$

is a subbase for T.

Proof: Suppose (X, T) is uniformizable. As was done in Theorem 4.1, a family of pseudometrics $F = \{d_f : f \in C(X)\}$ can be defined, where $d_f(x, y) = |f(x) - f(y)|$, for any f in C(X) and any x and y in X. Since any constant function is continuous, F is non-empty. For any f in C(X) and any $r > 0$

$$\begin{aligned} S_{d_f, r}(x) &= \{y : d_f(x, y) < r\} \\ &= \{y : |f(x) - f(y)| < r\} \\ &= f^{-1}((f(x) - r, \ f(x) + r)). \end{aligned} \tag{35}$$

Therefore, since f is continuous, $S_{d_f, r}(x)$ is in T. Consequently

$$\{S_{d,r}(x) : d \in F, \ r > 0\} \subset T.$$

Let O be in T and let x be in O. Since (X, T) is completely regular, then, by Theorem 3.4, T is the weak topology induced by C(X). Therefore

$$x \in \bigcap_{1}^{n} f_i^{-1}(G_i) \subset O,$$

where each f_i is in C(X) and G_i is open in E^1. For any i, $f_i(x) \in G_i$ and hence there is an $r_i > 0$ such that

$$f_i(x) \in (f_i(x) - r_i, \ f_i(x) + r_i) \subset G_i.$$

Hence

$$x \in f_i^{-1}((f_i(x) - r_i, \ f_i(x) + r_i)) \subset f_i^{-1}(G_i).$$

Therefore

$$x \in \bigcap_1^n f_i^{-1}((f_i(x) - r_i, f_i(x) + r_i)) \subset \bigcap_1^n f_i^{-1}(G_i) \subset O.$$

Because of (35),

$$x \in \bigcap_1^n S_{d_{f_i}, r_i}(x) \subset O.$$

Therefore

$$\{S_{d,r}(x) : d \in F,\ r > 0,\ \text{and}\ x \in X\}$$

is a subbase for T.

Now suppose that there is a non-empty family P of pseudo-metrics on X such that

$$S^* = \{S_{d,r}(x) ; d \in P,\ r > 0\}$$

is a subbase for T. By Theorem 1.14,

$$\{V_{d,r} : d \in P \text{ and } r > 0\} = A^*$$

is a subbase for a uniformity U^* for X. Let T_U be the topology induced by U^*. It will now be shown that $T_U = T$ and hence that (X, T) is uniformizable.

Let O be in T_U and let x be in O. Since finite intersections of relations in A^* for a base for U^*, then

$$O \supset (\bigcap_1^n V_{d_i, r_i})[x],$$

where, for each i, V_{d_i, r_i} is in A^*. Now

$$(\bigcap_1^n V_{d_i, r_i})[x] = \bigcap_1^n (V_{d_i, r_i}[x]) = \bigcap_1^n (S_{d_i, r_i}(x))$$

and the last set contains x. Since

$$\bigcap_1^n (S_{d_i, r_i}(x))$$

is in T, O is in T.

Now let O be in T and let x be in O. Since S^* is a subbase for T, then

$$x \in \bigcap_1^n (S_{d_i, r_i}(x_i)) \subset O, \text{ where for each } i, \; S_{d_i, r_i}(x_i) \in S^*$$

For each i, there is a t_i such that $S_{d_i, t_i}(x) \subset S_{d_i, r_i}(x_i)$. Therefore

$$\bigcap_1^n S_{d_i, t_i}(x) \subset \bigcap_1^n S_{d_i, r_i}(x_i),$$

which implies that

$$\bigcap_1^n S_{d_i, t_i}(x) \subset O.$$

Since

$$\bigcap_1^n S_{d_i, t_i}(x) = (\bigcap_1^n V_{d_i, t_i})[x],$$

then

$$(\bigcap_1^n V_{d_i, t_i})[x] \subset O.$$

Since A is a subbase for U^*, then $\bigcap_1^n V_{d_i, t_i}$ is in U^*. Therefore O is in T_U. Therefore $T = T_U$ and (X, T) is uniformizable.

CHAPTER V

CHARACTERIZATIONS OF TYCHONOFF SPACES

If a completely regular space is also a T_1 space, then it has some interesting characterizations besides those mentioned in the previous chapters. These various characterizations will be examined in this chapter. All of the characterizations will be in terms of embeddings, except for the first which will be in terms of $C(X)$. Several definitions must first be stated.

Definition 5.1 A topological space is a Tychonoff space if and only if it is completely regular and T_1.

There is no general agreement about the name of this type of space. The definition given above agrees with that of Kelley (12). Cullen (5) calls this space a completely regular space. Since a completely regular space is also a regular space, the T_1 property and the T_2 property are equivalent properties for a completely regular space. Therefore the T_1 property in this definition can be replaced by the T_2 property.

Definition 5.2 A family F of functions on X distinguishes points if and only if, for any two distinct points x and y in X, there exists an f in F such that $f(x) \neq f(y)$.

Theorem 5.3 A topological space (X, T) is a Tychonoff space if and only if $C(X)$ distinguishes points and distinguishes points and closed sets.

Proof: By Theorem 3.2 (X, T) is completely regular if and only if $C(X)$ distinguishes points and closed sets.

Suppose first that (X, T) is a Tychonoff space and let x and y be in X such that $x \neq y$. Since (X, T) is T_1, $\{y\}$ is a closed set and $x \in \{y\}$. Therefore, since (X, T) is completely regular, there is an f in $C(X)$ such that $f(x) = 0$ and $f(y) = 1$. Hence $f(x) \neq f(y)$. Therefore if (X, T) is a Tychonoff space, then $C(X)$ distinguishes points.

Suppose now that $C(X)$ distinguishes points. Let x and y be in X such that $x \neq y$. Then there exists an f in $C(X)$ such that $f(x) \neq f(y)$. Since E^1 is Hausdorff, there exists two disjoint open subsets G and H of E^1 such that $f(x)$ is in G and $f(y)$ is in H. Then x is in $f^{-1}(G)$ and y is in $f^{-1}(H)$. Since f is continuous $f^{-1}(G)$ and $f^{-1}(H)$ are open subsets of X. Also $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint. Therefore if $C(X)$ distinguishes points, then (X, T) is T_1. Consequently (X, T) is a Tychonoff space if and only if $C(X)$ distinguishes points and distinguishes points and closed sets.

Definition 5.4 If for any i in an indexing set Q, $X_i = [0, 1]$, and $[0, 1]$ has the usual topology, then $\underset{i \in Q}{\times} X_i$, with the product topology, is called a cube.

Theorem 5.5 A topological space (X, T) is a Tychonoff space if and only if it can be embedded in a cube.

Proof: Assume (X, T) is a Tychonoff space. Let

$$F = \{f \in C(X) : f(X) \subset [0, 1]\}$$

and let I^F be the cube indexed by F. Define a function e from X into I^F such that, for any x in X and any f in F, $e(x)_f = f(x)$. For any f in F, let P_f be the projection of I^F onto $[0, 1]$. Then, for any f in F and any x in X, $(P_f \circ e)(x) = e(x)_f = f(x)$. Therefore, for any f in F, $P_f \circ e = f$, which implies that $P_f \circ e$ is continuous. Therefore e is a continuous function (12, p. 91).

Let x and y be in X such that $x \neq y$. Since (X, T) is completely regular and T_1, there exists an f in F such that $f(x) = 0$ and $f(y) = 1$. Hence $e(x)_f \neq e(y)_f$ and consequently $e(x) \neq e(y)$. Therefore e is one-to-one.

Finally it must be shown that e is an open mapping. So let O be an open subset of X and let y be in e(O). Then there is an x in O such that $e(x) = y$. Since (X, T) is completely regular, there exists an f in F such that

$$f(x) = 0 \text{ and } f(X - O) \subset \{1\}. \tag{36}$$

Let

$$N = \{w \in I^F : w_f \in [0, 1)\} = P_f^{-1}([0, 1)).$$

The interval $[0, 1)$ is an open subset of the space $[0, 1]$ and hence N is open in I^F. Therefore $N \cap e(X)$ is open in the subspace e(X) of I^F. Because of (36) $y_f = e(x)_f = f(x) = 0$. Therefore by definition of N, y is in N. Since y is in e(X), y is in $N \cap e(X)$. Now $N \cap e(X) \subset e(O)$. For, u in $N \cap e(X)$ implies that u is in N and $u = e(w)$, where w is in X. Therefore $u_f = e(w)_f = f(w)$ is in $[0, 1)$, and hence $f(w) \neq 1$. Because of (36), w is in O. Consequently u is in $e(O)$. Therefore $y \in N \cap e(X) \subset e(O)$, where $N \cap e(X)$ is open in the subspace $e(X)$. This implies that $e(O)$ is open in the subspace $e(X)$. Therefore the space (X, T) can be embedded in the cube I^F.

Now suppose that (X, T) can be embedded in a cube I^Q. Then since the space $[0, 1]$ is a pseudometric space, (X, T) can be embedded in a product of pseudometric spaces. Therefore, by Theorem 4.6, (X, T) is completely regular. The space $[0, 1]$ is a T_1 space. Since a product of T_1 spaces is a T_1 space, I^Q is a T_1 space. Let f be a homeomorphism from X onto a subspace of I^Q and let x be in X. Then $\{f(x)\}$ is closed in I^Q. Since f is continuous, $f^{-1}\{f(x)\}$ is closed in X. Since f is one-to-one, $f^{-1}\{f(x)\} = \{x\}$ and hence $\{x\}$ is closed in X. Therefore (X, T) is a T_1 space. Consequently (X, T) is a Tychonoff space.

<u>Theorem 5.6</u> A topological space (X, T) is a Tychonoff space if and only if it can be embedded in a product of metric spaces.

<u>Proof</u>: If (X, T) is a Tychonoff space, then by Theorem 5.5 (X, T) can be embedded in a cube. Since a cube is a product of metric spaces, (X, T) can be embedded in a product of metric spaces.

Suppose (X, T) can be embedded in a product of metric spaces. Since a metric space is a pseudometric space, (X, T) can be embedded in a product of pseudometric spaces. Hence, by Theorem 4.6, (X, T) is completely regular. Since a metric space is a T_1 space, (X, T) can be embedded in a product of T_1 spaces. It was shown in the last part of the proof of Theorem 5.5 that any space which can be embedded in a product of T_1 spaces is also a T_1 space. Thus (X, T) is a T_1 space. Therefore (X, T) is a Tychonoff space.

Theorem 5.7 A topological space is a Tychonoff space if and only if it can be embedded in a compact Hausdorff space.

Proof: Suppose (X, T) is a Tychonoff space. By Theorem 5.5, (X, T) can be embedded in a cube. The closed unit interval with its usual topology is compact and Hausdorff. Tychonoff's product theorem states that a product of compact spaces is compact. Also a product of Hausdorff spaces is a Hausdorff space. Therefore the cube is a compact Hausdorff space, and hence (X, T) can be embedded in a compact Hausdorff space.

Suppose (X, T_1) can be embedded in a compact Hausdorff space (Y, T_2). A compact Hausdorff space is a normal space. Therefore (Y, T_2) is a normal space. Let C be a closed subset of Y and let x be in $Y - C$. Since (Y, T_2) is a Hausdorff space it is a T_1 space, and hence $\{x\}$ is a closed subset of Y. According to Urysohn's lemma, if A and B are disjoint closed subsets of a normal space X, then there exists an f in $C(X)$ such that $f(X) \subset [0, 1]$, $f(A) \subset \{0\}$, and $f(B) \subset \{1\}$. Therefore there exists an f in $C(Y)$ such that $f(Y) \subset [0, 1]$, $f(x) = 0$, and $f(C) \subset \{1\}$. Therefore (Y, T_2) is a completely regular space. Therefore, by Lemma 4.4 and Lemma 4.5, (X, T_1) is completely regular. As was shown in the last part of the proof of Theorem 5.5, a space which can be embedded in a T_1 space is also a T_1 space. Hence (X, T_1) is a T_1 space. Therefore (X, T_1) is a Tychonoff space.

Definition 5.8 A topological space (X, T_1) has a Hausdorff compactification if and only if there exists a compact Hausdorff space (Y, T_2) and a homeomorphism f from X onto a dense subspace of Y.

Theorem 5.9 A topological space (X, T_1) is a Tychonoff space if and only if it has a Hausdorff compactification.

Proof: If the space (X, T_1) has a Hausdorff compactification then it can be embedded in a compact Hausdorff space. Therefore, by Theorem 5.7, (X, T_1) is a Tychonoff space.

Suppose (X, T_1) is a Tychonoff space. By Theorem 5.7, (X, T_1) can be embedded in a compact Hausdorff space (Y, T_2). Hence there exists a homeomorphism f from X onto M, where M is a subspace of Y. $\overline{M}$, the T_2-closure of M, is a closed subset of Y. Any closed subset of a compact space is compact. Therefore $\overline{M}$ is compact. Any subspace of a Hausdorff space is Hausdorff. Hence $\overline{M}$ is Hausdorff. Also M is a dense subspace of $\overline{M}$, since the closure of M relative to the subspace $\overline{M}$ is $\overline{M}$. Therefore f is a homeomorphism from X onto M, where M is a dense subspace of the compact Hausdorff space $\overline{M}$. Therefore (X, T_1) has a Hausdorff compactification.

CHAPTER VI

THE UNIQUENESS OF ADMISSIBLE UNIFORMITIES

In the previous chapters, conditions under which a topological space has an admissible uniformity were investigated. It is a natural outcome of this to ask when a uniformizable space has a unique admissible uniformity. A sufficient condition for a uniformizable space to have a unique admissible uniformity is that the space be compact. This was first proved by A. Weil (18). Doss (7) proved that a necessary and sufficient condition for a Hausdorff uniformizable space to have a unique admissible uniformity is that for any two normally separable sets at least one must be compact. The problem of characterizing spaces with a unique admissible uniformity has also been studied by Newns (13) and Gál (9). In this chapter it will be proved that if the space is uniformizable and compact then it has a unique admissible uniformity. It will also be proved that the criterion given by Doss is a necessary condition for a uniformizable space to have a unique admissible uniformity. Then this necessary condition will be used to show why various uniformizable spaces have more than one admissible uniformity.

In the following two lemmas, subsets of $X \times X$ which are said to be closed, open, neighborhoods of points, or closures of sets will be so in terms of the product topology for $X \times X$ determined by a uniform topology for X.

Lemma 6.1 The family of closed members of a uniformity is a base for the uniformity.

Proof: Let U^* be a uniformity for X and let U be in U^*. By Lemma 4.12, there is a symmetric V in U^* such that $V \circ V \circ V \subset U$. It will be shown that $\overline{V} \subset V \circ V \circ V$. Let (a, b) be in $\overline{V}$. By Corollary 2.6, V[a] and V[b] are T_U-neighborhoods of a and b respectively. Then there exist sets G and H in T_U such that

$$(a, b) \in G \times H \subset V[a] \times V[b].$$

Therefore $V[a] \times V[b]$ is a neighborhood of (a, b). Hence $V[a] \times V[b]$ intersects V. So let (x, y) be a member of V which is also in $V[a] \times V[b]$. Then (a, x) and (b, y) are in V. Since V is symmetric, (y, b) is in V. Because (a, x), (x, y), and (y, b) are in V, (a, b) is in $V \circ V \circ V$. Therefore $\overline{V} \subset V \circ V \circ V$ which implies that $\overline{V} \subset U$. $\overline{V}$ is in U^* since $V \subset \overline{V}$ and V is in U^*. $\overline{V}$ is closed. Consequently each member of U^* contains a closed member of U^*. Therefore the closed members of U^* form a base for U^*.

Lemma 6.2 The interior of any member of a uniformity is also in the uniformity.

Proof: Let U be in U^*, a uniformity for X. By Lemma 4.12 there is a symmetric V in U^* such that $V \circ V \circ V \subset U$. It will be shown that $V \subset \text{int}\, U$, the interior of U relative to the product topology for $X \times X$ determined by T_U. So let (x, y) be in V. As was shown in the proof of Lemma 6.1, $V[x] \times V[y]$ is a neighborhood of

(x, y) relative to the product topology. Now $V[x] \times V[y] \subset U$. For if $(u, v) \in V[x] \times V[y]$, then (x, u) and (y, v) are in V. Since V is symmetric, (u, x) is in V. Hence (u, v) is in $V \circ V \circ V$, which implies that (u, v) is in U. Because of the last set inclusion U is a neighborhood of (x, y). This implies that (x, y) is in $\operatorname{int} U$. Therefore $V \subset \operatorname{int} U$. Since V is in U^*, $\operatorname{int} U$ is in U^*.

Theorem 6.3 If (X, T) is a compact uniformizable space then there is a unique uniformity U^* for X which is compatible with T. In fact U^* is the set of all neighborhoods of D_X.

Proof: Let U^* be a uniformity for X such that $T_U = T$. Let U be in U^*. By Lemma 6.2, $\operatorname{int} U$ is in U^*. This implies that $D_X \subset \operatorname{int} U$. Since $\operatorname{int} U$ is open in the product space $X \times X$ and since $\operatorname{int} U \subset U$, then U is a neighborhood of D_X.

Now let N be a neighborhood of D_X. Then there is an open subset G of the product space such that $D_X \subset G \subset N$. By Lemma 6.1, there exists a family A^* of closed members of U^* which is a base for U^*. Let $B = \bigcap \{U : U \in A^*\}$ and let (x, y) be in B. Since (x, x) is in G, there are T-open sets O_1 and O_2 such that $(x, x) \in O_1 \times O_2 \subset G$. There is a U in U^* such that $U[x] \subset O_2$, since U^* is compatible with T. There exists a V in A^* such that $V \subset U$. Since (x, y) is in B, then (x, y) is in U. This implies that y is in $U[x]$ and hence that y is in O_2. This implies that (x, y) is in $O_1 \times O_2$ and hence that (x, y) is in G. Consequently $B \subset G$. Therefore $X \times X - G \subset X \times X - B$. Consequently

$$X \times X - B = \bigcup \{X \times X - U : U \in A^*\} \supset X \times X - G.$$

Because (X, T) is compact and because of Tychonoff's product theorem, the product space $X \times X$ is compact. Since any closed subset of a compact space is compact, $X \times X - G$ is compact. For any U in A^*, $X \times X - U$ is open. Therefore $\{X \times X - U : U \in A^*\}$ is an open covering of $X \times X - G$. Hence there is a finite subfamily $\{U_1, U_2, \ldots, U_n\}$ of A^* such that

$$\bigcup_1^n \{X \times X - U_i\} \supset X \times X - G.$$

Hence

$$X \times X - \bigcap_1^n U_i \supset X \times X - G,$$

which implies that $\bigcap_1^n U_i \subset G$. Since $G \subset N$, then $\bigcap_1^n U_i \subset N$. Since each U_i is in U^*, $\bigcap_1^n U_i$ is in U^*. Thus N is in U^*. Therefore U^* is the set of all neighborhoods of the diagonal in the product space.

Compactness is not necessary in order that a uniformizable space have a unique admissible uniformity. An example of a uniformizable space with a unique admissible uniformity which is not a compact space has been given by Dieudonné (6). If (X, T) is a uniformizable space for which X is finite, then (X, T) is a compact space, and hence, by Theorem 6.3, (X, T) will have a unique admissible uniformity. Because of this there is a one-to-one correspondence between the topologies for a finite set X for which the resulting spaces are completely regular and the uniformities which may be defined for X.

Theorem 6.3 can be used to prove a theorem which is analogous to a theorem in analysis.

Theorem 6.4 If (X, U^*) and (Y, V^*) are uniform spaces, (X, T_U) is compact, and f is a continuous function from X into Y relative to T_U and T_V, then f is uniformly continuous relative to U^* and V^*.

Proof: Define P_a and P_b to be the projections of $Y \times Y$ into Y such that $P_a(u, v) = u$ and $P_b(u, v) = v$ for any u and v in Y. Let O be in T_V. Then

$$(P_a \circ f_2)^{-1}(O) = f_2^{-1}(P_a^{-1}(O)) = f_2^{-1}(O \times Y)$$

$$= \{(x, y) : (f(x), f(y)) \in O \times Y\}$$

But this last set is $f^{-1}(O) \times X$, which is open in the product space $X \times X$. Therefore $P_a \circ f_2$ is continuous relative to the product topology for $X \times X$ and T_V. Similarly it can be shown that $P_b \circ f_2$ is also continuous relative to the same topologies. Therefore f_2 is continuous relative to the product topology for $X \times X$ and the product topology for $Y \times Y$. Now let V be in V^*. By Lemma 6.2, int V is in V^*, which implies that $D_Y \subset \text{int} V$. For any (x, x) in D_X, $f_2(x, x) = (f(x), f(x))$ is in D_Y and hence is in int V. Therefore $D_X \subset f_2^{-1}(\text{int} V)$. Since int V is open in $Y \times Y$ and since f_2 is continuous, $f_2^{-1}(\text{int} V)$ is open in $X \times X$. Hence $f_2^{-1}(\text{int} V)$ is a neighborhood of D_X in the product space $X \times X$. Since

$$f_2^{-1}(\text{int} V) \subset f_2^{-1}(V),$$

then $f_2^{-1}(V)$ is a neighborhood of D_X. Since (X, T_U) is compact and because of Theorem 6.3, $f_2^{-1}(V)$ is in U^*. Therefore f is uniformly continuous relative to U^* and V^*.

The next theorem will give a necessary condition for a space to have a unique admissible uniformity. It is part of the previously mentioned theorem by Doss (7). Before it is proved, a number of definitions and lemma will be presented.

Definition 6.5 A filter on a set X is a non-empty collection F^* of subsets of X which have the following properties.

(i) The empty set is not a member of F^*.

(ii) The intersection of the sets in any finite subfamily of F^* is also a member of F^*.

(iii) Any subset of X which contains a member of F^* is a member of F^*.

Definition 6.6 B^* is said to be a base of the filter F^* on the set X if and only if $B^* \subset F^*$ and each member of F^* contains a member of B^*.

Definition 6.7 If F^* is a filter on (X, T) and x is in X, then x is a cluster point of F^* if and only if x is in the T-closure of each set in F^*.

Definition 6.8 If F^* is a filter on (X, T) and x is in X, then x is a limit point of F^* or, F^* converges to x, if and only if every T-neighborhood of x is a member of F^*.

Definition 6.9 A filter F^* on a uniform space (X, U^*) is said to be a Cauchy filter on (X, U^*) or relative to U^* if and only if for any U in U^*, there is an N in F^* such that $N \times N \subset U$.

Definition 6.10 Two subsets A and B of (X, T) are said to be normally separable if and only if A and B are closed, disjoint, and there is an f in $C(X)$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$.

Lemma 6.11 If G^* is a non-empty collection of subsets of X with the property that the intersection of the sets in any finite subfamily of G^* is non-empty, then there is a filter F^* on X such that $G^* \subset F^*$.

Proof: Let F^* be the collection of all subsets of X such that each contains the intersection of the sets in a finite subfamily of G^*. Then F^* is a non-empty collection of non-empty sets and $G^* \subset F^*$. Obviously any subset of X which contains a member of F^* will be a member of F^*. Let H^* be the collection of all sets which are the intersection of the sets in a finite subfamily of G^* and let $F_1, F_2, \ldots, F_n$ be in F^*. Then for any i $F_i \supset G_i$, where G_i is in H^*. Since

$$\bigcap_1^n F_i \supset \bigcap_1^n G_i$$

and since $\bigcap_1^n G_i$ is the intersection of a finite number of sets in G^*, $\bigcap_1^n F_i$ is in F^*. Therefore F^* is a filter on X.

Lemma 6.12 If F^* is a Cauchy filter on (X, U^*), a uniform space, then any cluster point of F^* relative to T_U is a limit point of F^* relative to T_U.

Proof: Let x be a cluster point of F^* relative to T_U and let

U be in $\mathcal{U}^*$. There is a V in $\mathcal{U}^*$ such that $V \circ V \subset U$. Since $\mathcal{F}^*$ is a Cauchy filter, there exists an N in $\mathcal{F}^*$ such that $N \times N \subset V$. By Corollary 2.6, $V[x]$ is a $T_{\mathcal{U}}$-neighborhood of x. Since x is a cluster point of $\mathcal{F}^*$ relative to $T_{\mathcal{U}}$, then x is in the $T_{\mathcal{U}}$-closure of N and consequently $V[x] \cap N \neq \emptyset$. Let a be in $V[x] \cap N$. Then (x, a) is in V and a is in N. Now $N \subset U[x]$. For if b is in N, then

$$(a, b) \in N \times N \subset V.$$

Since (x, a) and (a, b) are in V,

$$(x, b) \in V \circ V \subset U.$$

This implies that b is in $U[x]$. By property (iii) of a filter, $U[x]$ is in $\mathcal{F}^*$. Since $\{U[x]: U \in \mathcal{U}^*\}$ is a base for the $T_{\mathcal{U}}$-neighborhood system of x, then any $T_{\mathcal{U}}$-neighborhood of x is in $\mathcal{F}^*$. Therefore x is a limit point of $\mathcal{F}^*$ relative to $T_{\mathcal{U}}$.

Lemma 6.13 If (X, T) is a Hausdorff space, then no filter on X can have more than one limit point.

Proof: Let $\mathcal{F}^*$ be a filter on X and let x and y be limit points of $\mathcal{F}^*$ such that $x \neq y$. Since (X, T) is Hausdorff there are two disjoint neighborhoods O_x and O_y of x and y respectively. Since x is a limit point of $\mathcal{F}^*$, O_x is a member of $\mathcal{F}^*$. Likewise O_y is a member of $\mathcal{F}^*$. Hence, by properties of a filter, $O_x \cap O_y \neq \emptyset$. But O_x and O_y are disjoint. Therefore $x = y$.

Lemma 6.14 If (X, T) has the property that every filter on X has at least one cluster point, then (X, T) is compact.

Proof: Suppose (X, T) is not compact. Then there is a family G^* of closed subsets of X which has an empty intersection but which has the property that the intersection of each finite subfamily of G^* is non-empty. Since the intersection of the sets in G^* is empty, G^* is non-empty. By Lemma 6.11, there is a filter F^* on X such that $G^* \subset F^*$. By the hypothesis, F^* has a cluster point x. Since x belongs to the closure of each set in F^*, then x belongs to the closure of each set in G^*. Since the sets in G^* are closed, x is in the intersection of the sets in G^*. But the intersection of the sets in G^* is empty. Therefore (X, T) is compact.

Lemma 6.15 If $\{U_i^*, i \in I\}$ is a family of uniformities for X, then $S^* = \bigcup_{i \in I} U_i^*$ is a subbase for a uniformity for X.

Proof: Clearly, each member of S^* contains the diagonal of X. For any U in S^*, there is a j in I such that U is in U_j^*. Since U_j^* is a uniformity, U^{-1} is in U_j^* and there exists a V in U_j^* such that $V \circ V \subset U$. Since $U_j^* \subset S^*$, U^{-1} and V are in S^*. Therefore, by Theorem 1.13, S^* is a subbase for a uniformity for X.

Lemma 6.16 If (X, T) is a uniformizable space and F is the family of all uniformities for X which are compatible with T, then

$$S^* = \bigcup \{U^* : U^* \in F\}$$

is a subbase for a uniformity W^* on X. Moreover W^* is compatible with T and, for any f in $C(X)$, f is uniformly continuous relative to W^* and the usual uniformity V^* for E^1.

Proof: By Lemma 6.15, S^* is a subbase for a uniformity W^*

for X. It must be shown that $T_W = T$. Let O be a T-open set and let x be in O. Since (X, T) is uniformizable there is a uniformity U^* for X compatible with T. Hence there is a U in U^* such that $U[x] \subset O$. Since $U^* \subset S^*$, $U^* \subset W^*$. Therefore O is T_W-open. Now let O be T_W-open and let x be in O. Then there is a U in W^* such that $U[x] \subset O$. Since S^* is a subbase for W^*, there exist relations $U_1, U_2, \ldots, U_n$ in S^* such that $\bigcap_1^n U_i \subset U$. This implies that

$$\bigcap_1^n (U_i[x]) = (\bigcap_1^n U_i)[x] \subset U[x].$$

Therefore

$$\bigcap_1^n (U_i[x]) \subset O.$$

Because of Corollary 2.6 and because each U_i is a member of a uniformity compatible with T, $U_i[x]$ is a T-neighborhood of x for each i. Therefore O is a T-neighborhood of x. Thus O is a T-open set. Since $T = T_W$, W^* is compatible with T. Since (X, T) is uniformizable, it is completely regular. As was shown in the proof of Theorem 2.1,

$$P = \{f_2^{-1}(V) : f \in C(X) \text{ and } V \in V^*\}$$

is a subbase for a uniformity U_C^* for X which is compatible with T. Since $P \subset U_C^* \subset S^* \subset W^*$, $f_2^{-1}(V)$ is in W^* for each f in C(X) and each V in V^*. Therefore, for any f in C(X), f is uniformly continuous relative to W^* and V^*.

<u>Lemma 6.17</u> If (X, T) is a uniformizable space and F^* is a filter on X without a cluster point in X, then there is a uniformity

U^* for X compatible with T such that F^* is a Cauchy filter relative to U^*.

Proof: Let $a \in X$ and let N_a be a neighborhood of a. Since a is not a cluster point of F^*, there is an F in F^* and a neighborhood N_1 of a such that $N_1 \cap F$ is empty. Let $N = N_a \cap N_1$. Then N is a neighborhood of a, $N \subset N_a$, and $N \cap F$ is empty. Since (X, T) is uniformizable, (X, T) is completely regular. Therefore there exists an f in $C(X)$ such that $f(a) = 0$ and $f(X - N) = \{1\}$. Since $f \subset X - N$, $f(F) = \{1\}$.

Now let $r > 0$. Since f is continuous, then for any x in X there exists a neighborhood M_x^r of x such that $|f(x) - f(y)| < r$ for any y in M_x^r. Let N_x^r be the set of all points y in X for which $|f(x) - f(y)| < r$. Since $M_x^r \subset N_x^r$, then N_x^r is a neighborhood of x. Let

$$V_r = \bigcup_{x \in X} (N_x^r \times N_x^r).$$

It is true that $F \times F \subset V_r$. For let b be in F. Then $f(b) = 1$ and b is in N_b^r. Since $f(y) = 1$ for any y in F, $|f(b) - f(y)| = 0 < r$ for any y in F. This implies that y is in N_b^r for any y in F. Thus $F \subset N_b^r$ and hence $F \times F \subset N_b^r \times N_b^r$. Since $N_b^r \times N_b^r \subset V_r$,

$$F \times F \subset V_r. \tag{37}$$

It will now be shown that $\{V_r : r > 0\}$ is a base for a uniformity for X. Let $r > 0$ and let (x, x) be in D_X. Since $|f(x) - f(x)| < r$, x is in N_x^r. Hence (x, x) is in $N_x^r \times N_x^r$ which implies that (x, x) is in V_r. Therefore, for any $r > 0$, $D_X \subset V_r$.

Let (u, v) be in V_r. Then for some x in X, (u, v) is in $N_x^r \times N_x^r$. This implies that (v, u) is in $N_x^r \times N_x^r$ which implies that (v, u) is in V_r. Therefore for any $r > 0$, V_r is symmetric.

Let (x, z) be in $V_{r/4} \circ V_{r/4}$. Then there is a y such that (x, y) and (y, z) are in $V_{r/4}$. Therefore, there is a u in X such that (x, y) is in $N_u^{r/4} \times N_u^{r/4}$ and there is a v in X such that (y, z) is in $N_v^{r/4} \times N_v^{r/4}$. This implies that

$$|f(u) - f(x)| < r/4, \quad |f(u) - f(y)| < r/4, \quad |f(v) - f(y)| < r/4,$$

and $|f(v) - f(z)| < r/4$. The first two inequalities imply that $|f(x) - f(y)| < r/2$ and the last two inequalities imply that $|f(y) - f(z)| < r/2$. Now these latter two inequalities imply that $|f(x) - f(z)| < r$. Therefore z is in N_x^r. Since x is in N_x^r, (x, z) is in $N_x^r \times N_x^r$. This implies that (x, z) is in V_r. Consequently $V_{r/4} \circ V_{r/4} \subset V_r$. Therefore, by Theorem 1.13, $\{V_r : r > 0\}$ is a subbase for a uniformity $U^*_{a,N}$ for X. Now let U be in $U^*_{a,N}$. Then there exist positive numbers $r_1, r_2, \ldots, r_n$ such that

$$U \supset \bigcap_1^n V_{r_i}.$$

Let r_k be the minimum of $\{r_1, r_2, \ldots, r_n\}$. It is a straightforward exercise to show that

$$\bigcap_1^n V_{r_i} = V_{r_k}.$$

Therefore $\{V_r : r > 0\}$ is a base for $U^*_{a,N}$.

Suppose that $r < 1/2$ and that x is in $V_r[a]$. Then (a, x) is in V_r which implies that (a, x) is in $N_u^r \times N_u^r$, for some u in X.

This implies that $|f(u) - f(a)| < r$ and $|f(u) - f(x)| < r$. Therefore $|f(a) - f(x)| < 2r < 1$. Since $f(a) = 0$, $|f(x)| < 1$. Since $f(X - N) = \{1\}$, x is in N, which is a subset of N_a. Therefore

$$\text{if } r < 1/2, \text{ then } V_r[a] \subset N_a. \tag{38}$$

Now let x be in X, $r > 0$, and y be in N_x^r. Since x is in N_x^r, (x, y) is in $N_x^r \times N_x^r$. So (x, y) is in V_r and hence y is in $V_r[x]$. Therefore $N_x^r \subset V_r[x]$. Since N_x^r is a neighborhood of x, $V_r[x]$ is a T-neighborhood of x. Therefore,

$$\text{for any } x \text{ in } X \text{ and } r > 0, \ V_r[x] \text{ is a T-neighborhood of } x. \tag{39}$$

The results of the last two paragraphs will be used later in this proof. Presently though, it has been shown that, for any a in X and for any neighborhood N_a of a, the associated collection $\{V_r : r > 0\}$ is a base for a uniformity $U^*_{a,N}$ for X. By Lemma 6.15,

$$S^* = \bigcup \{U^*_{a,N}, \ a \in X \text{ and } N \text{ a neighborhood of } a\}$$

is a subbase for a uniformity U^* for X. It will now be shown that U^* is compatible with T.

Let O be a T_U-open set and let x be in O. Then there is a U in U^* such that $U[x] \subset O$. Then there exist uniformities U_1, U_2, ..., U_n in S^* such that

$$U \supset \bigcap_1^n U_i.$$

For each U_i, there is a set V_{r_i} such that $U_i \supset V_{r_i}$. Therefore $U \supset \bigcap_1^n V_{r_i}$ and hence $(\bigcap_1^n V_{r_i})[x] \subset U[x]$. Since

$$(\bigcap_{1}^{n} V_{r_i})[x] = \bigcap_{1}^{n} (V_{r_i}[x]),$$

then $\bigcap_{1}^{n} (V_{r_i}[x]) \subset O$. By statement (39), $V_{r_i}[x]$ is a T-neighborhood of x for each i. Therefore O is a T-neighborhood of x. Hence O is a T-open set.

Let O be a T-open set and let x be in O. Since O is a T-neighborhood of x and because of (38), there exists an $r > 0$ such that $V_r[x] \subset O$. Since V_r is in U^*, O is a T_U-open set.

Finally it must be shown that F^* is a Cauchy filter relative to U^*. Let V be in U^*. Then there exist sets $V_{r_1}, V_{r_2}, \ldots, V_{r_n}$ such that $V \supset \bigcap_{1}^{n} V_{r_i}$. Because of (37), there exists an F_i in F^*, for each i, such that $F_i \times F_i \subset V_{r_i}$. Let $F = \bigcap_{1}^{n} F_i$. Then F is in F^* and $F \times F \subset \bigcap_{1}^{n} V_{r_i}$. Therefore $F \times F \subset V$. Hence F^* is a Cauchy filter relative to U^*.

<u>Theorem 6.18</u> If (X, T) is a space with a unique uniformity compatible with T, then, for any two normally separable subsets of X, at least one is compact.

<u>Proof:</u> Let (X, T) be a space with a unique uniformity U^* compatible with T and suppose A and B are two normally separable subsets of X such that neither is compact. By the definition of normally separable, there is an f in C(X) such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. Let U_w^* be the uniformity for X referred to in Lemma 6.16. Then $U^* = U_w$ and hence f is uniformly continuous relative to U^* and the usual uniformity V^* for E^1. By Lemma 6.14, there is a filter F^* on A such that no point of A is a cluster point of F^* relative to T_A, the relativization of T to A. Let F be in F^*. The

closure $\overline{F}$ of F relative to T is a subset of A, since $f \subset A$ and since A is T-closed. The closure of F relative to T_A is equal to $A \cap \overline{F}$, which, by the previous statement, is equal to $\overline{F}$. Therefore no point in A is a cluster point of F^* relative to T. For any x in $X - A$ and any F in F^*, x is not in $\overline{F}$, since $\overline{F} \subset A$. Therefore no point of X is a cluster point of F^* relative to T. In a similar manner it can be shown that there is a filter G^* on B such that no point of X is a cluster point of G^*. Let $S^* = \{F \cup G : F \in F^* \text{ and } G \in G^*\}$. Clearly S^* is non-empty. For any finite subfamily $\{F_i \cup G_i : 1 \leq i \leq n\}$ of S^*,

$$\bigcap_1^n (F_i \cup G_i) \supset \bigcap_1^n F_i.$$

Since F^* is a filter, $\bigcap_1^n F_i$ and $\bigcap_1^n (F_i \cup G_i)$ are non-empty. Because of Lemma 6.11, there is a filter H^* on X such that $S^* \subset H^*$. As explained in the proof of Lemma 6.11, H^* is the collection of sets such that each contains the intersection of a finite number of sets in S. Let $x \in X$. Then x is not a cluster point of either F^* or G^*. Therefore, there is an F in F^* and a G in G^* such that x is not in $\overline{F}$ and x is not in $\overline{G}$. Hence x is not in $\overline{F} \cup \overline{G} = \overline{F \cup G}$. Since $F \cup G$ is in H^*, x is not a cluster point of H^*. Thus no point of X is a cluster point of H^*. Because of Lemma 6.17 and because of the fact that the uniformity for X is unique, H^* is a Cauchy filter on X relative to U^*.

Let $f(H^*) = \{f(H) : H \in H^*\}$. Since H^* is non-empty, $f(H^*)$ is non-empty. For any H in H^*, $f(H)$ is non-empty. If

$$\{f(H_i) : 1 \leq i \leq n\}$$

is a subfamily of $f(H^*)$, then

$$\bigcap_1^n f(H_i) \supset f(\bigcap_1^n H_i).$$

Since $\bigcap_1^n H_i$ is in H^*, $f(\bigcap_1^n H_i)$ is in $f(H^*)$. Let K^* be the collection of all subsets of E^1 which contain a member of $f(H^*)$. Then K^* is a filter on E^1 because of the previously stated properties of $f(H^*)$. Let V be in V^*, the usual uniformity for E^1. Since f is uniformly continuous relative to U^* and V^*, then $f_2^{-1}(V)$ is in U^*. Since H^* is a Cauchy filter on X relative to U^*, there is an H in H^* such that $H \times H \subset f_2^{-1}(V)$. This implies that $f(H) \times f(H) \subset V$. Since f(H) is in K^*, K^* is a Cauchy filter on E^1 relative to V^*.

Let H be in H^*. Then

$$H \supset \bigcap_1^n (F_i \cup G_i),$$

where, for each i, F_i is in F^* and G_i is in G^*. There is an x in $\bigcap_1^n F_i$ and a y in $\bigcap_1^n G_i$, since F^* and G^* are filters. Since $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$, $f(x) = 0$ and $f(y) = 1$. Now $\{0, 1\} \subset f(H)$, because x and y are in H. Since $f(H^*)$ is a base for K^*, 0 and 1 are in K, for any K in K^*. Therefore 0 and 1 are cluster points of K^*. By Lemma 6.12, 0 and 1 are limit points of K^*. But E^1 is a Hausdorff space and, by Lemma 6.13, K^* can have no more than one limit point. Because of this contradiction the original supposition in the proof is not true. Therefore both A and B are compact.

Example 6.19 Let X be the set of positive real numbers and let T be the open interval topology for X. If $A = \{1, 3, 5, \ldots\}$ and $B = \{2, 4, 6, \ldots\}$ then A and B are closed disjoint sets. Define a function f from X into E^1 such that $f(x) = x$ for $0 < x \leq 1$, $f(x) = x - 2n$ for any integer $n \geq 1$ and $2n \leq x \leq 2n + 1$, and such that $f(x) = -x + 2n$ for any integer $n \geq 1$ and $2n - 1 \leq x \leq 2n$. Then f is in C(X), $f(A) \subset \{1\}$, and $f(B) \subset \{0\}$. Therefore A and B are normally separable. Since A and B are unbounded, neither set is compact. Therefore, by Theorem 6.18, (X, T) does not have a unique admissible uniformity. One admissible uniformity for the space is the uniformity which has as a base sets of the type $\{(x, y) \in X \times X : |x - y| < r\}$, where $r > 0$. A different admissible uniformity for the space is the uniformity which has as a base sets of the form $\{(x, y) \in X \times X : |x - y| < rx\}$, where $r > 0$ and x is in X.

<u>Example 6.20</u> Let X be an infinite set and let T be the discrete topology for X. There are two disjoint infinite subsets A and B of X and these sets are closed. Define a function f from X into E^1 such that $f(x) = 1$ for x in B and $f(x) = 0$ for x in X - B. Then f is in C(X), $f(A) \subset \{0\}$, and $f(B) \subset \{1\}$. Therefore A and B are normally separable. The set A is not compact since $\{\{x\} : x \in A\}$ is an open covering of A which doesn't contain a finite sub-covering of A. Likewise the set B is not compact. Therefore by Theorem 6.18, (X, T) does not have a unique admissible uniformity.

Theorem 6.18 can be used to show that any Euclidean space E^n, where $n \geq 2$, does not have a unique uniformity compatible with its topology. Let $A = \{(x_1, x_2, \ldots, x_n) : x_1 \geq 1 \text{ and } x_i = 0 \text{ for } i > 1\}$

and $B = \{(x_1, x_2, \ldots, x_n) ; x_n \geq 1$ and $x_i = 0$ for $i < n\}$. Then A and B are disjoint closed subsets of E^n. E^n is a normal space. Therefore, by Urysohn's lemma, there is an f in C(X) such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. Consequently A and B are normally separable. Neither A nor B is compact. Therefore, by Theorem 6.18, the space E^n, for $n \geq 2$, does not have a unique admissible uniformity.

If a metric space (X, d) has two disjoint, closed, unbounded subsets A and B, then it does not have a unique admissible uniformity. A and B are normally separable because (X, d) is a normal space and because of Urysohn's lemma. Neither A nor B is compact since a compact subset of a metric space must be bounded. Therefore by Theorem 6.18, (X, d) does not have a unique admissible uniformity.

BIBLIOGRAPHY

1. Behrend, F. A. "Uniformizability and Compactifiability of Topological Spaces." Mathematische Zeitschrift, 67(1957), 203-210.

2. Bourbaki, Nicolas. Elements of Mathematics, General Topology, Part I. Paris: Herman, 1966.

3. Bourbaki, Nicolas. Elements of Mathematics, General Topology, Part II. Paris: Herman, 1966.

4. Bushaw, D. Elements of General Topology. New York: John Wiley and Sons, Inc., 1963.

5. Cullen, Helen F. Introduction to General Topology. Boston: D. C. Heath and Company, 1968.

6. Dieudonné J. "Un exemple d'espace normal non susceptible l'une structure uniforme d'espace complet." Comptes Rendus, 209 (1939), 145-147.

7. Doss, Raouf. "On Uniform Spaces with a Unique Structure." American Journal of Mathematics, 71 (1949), 19-23.

8. Engelking, R. Outline of General Topology. Amsterdam: North-Holland Publishing Company, 1968.

9. Gaal, S. A. Point Set Topology. New York: Academic Press, 1964.

10. Gál, I. S. "Uniformizable Spaces with a Unique Structure." Pacific Journal of Mathematics, 9 (1959), 1053-1060.

11. Gillman, L. and Jerison, M. Rings of Continuous Functions. Princeton: D. Van Nostrand Company, Inc., 1960.

12. Kelley, John L. General Topology. New York: D. Van Nostrand Company, Inc., 1955.

13. Newns, W. F. "Uniform Spaces with Unique Structure." American Journal of Mathematics, 79 (1957), 48-52.

14. Samuel, P. "Ultrafilters and Compactification of Uniform Spaces." Transactions of the American Mathematical Society, 64 (1948), 100-131.

15. Schubert, Horst. Topology. Boston: Allyn and Bacon, Inc., 1968.

16. Stone, M. H. "Applications of the Theory of Boolean Rings to General Topology." Transactions of the American Mathematical Society, 41 (1937), 375-481.

17. Tychonoff, A. "Über die topologische Erweiterung von Räumen." Mathematische Annalen, 102 (1929), 544-561.

18. Weil, A. "Sur les espaces á structure uniforme et sur la topologie générale." Actualities Scientifiques et Industrielles, 551 (1937).

19. Zeighami, Elaine. Uniform Spaces, Thesis, Oklahoma State University, 1969.

VITA

Donald Frank Shult

Candidate for the Degree of

Doctor of Education

Thesis: UNIFORMIZABLE SPACES

Major Field: Higher Education

Biographical:

Personal Data: Born in Kewanee, Illinois, August 8, 1929, the son of Carroll and Doris Shult.

Education: Graduated from Fairbury Township High School, Fairbury, Illinois, in May 1947; received the Bachelor of Science degree from Northwestern University, Evanston, Illinois, in June, 1952, with a major in mathematics; received the Master of Science degree from Southern Illinois University, Carbondale, Illinois, in August, 1962, with a major in mathematics; completed requirements for the Doctor of Education degree at Oklahoma State University in July, 1970.

Professional Experience: Graduate assistant in the Department of Mathematics, Southern Illinois University, 1960-1962; instructor in the Department of Mathematics, Elmhurst College, 1962-1963; assistant professor of mathematics, Ohio Northern University, 1963-1966; graduate assistant in the Department of Mathematics and Statistics, Oklahoma State University, 1966-1970.

Professional Organizations: Member of the Mathematical Association of America and Pi Mu Epsilon.