EXISTENCE CRITERIA OF OVERCONSTRAINED MECHANISMS WITH TWO PASSIVE COUPLINGS

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WITH TWO PASSIVE COUPLINGS

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CHAPTER I

INTRODUCTION

One of the most interesting and important areas of research in the science of mechanisms is the mobility of space mechanisms.[1] This involves the examination of the conditions under which a spatial kinematic chain with a specified number and type of links and joints can have constrained motion. Such a study is essential in order to fully utilize the capabilities of space mechanisms.

The mobility of mechanisms has been the subject of numerous investigations in the past many years. These studies have resulted in a number of mobility criteria. An excellent account of such criteria has been given by Harrisberger and Soni [23].[2]

An examination of the various mobility criteria reveals certain important features. For instance, some of the criteria, like those of Grübler, Malytcheff and Kutzbach, are concerned only with the number and the type of kinematic links and joints in a mechanism. Such an approach is quite satisfactory in the case of plane and spherical mechanisms because of the special orientation of the axes, but fails to explain the existence of many well-known space mechanisms. These criteria thus give rise to the so-called paradoxical mechanisms, the term paradoxical only emphasizing lack of complete knowledge of the nature

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[1]See items 1 and 2 in Appendix A.

[2]Numbers in brackets denote references given in the Bibliography.

of the motion of mechanisms. It would appear that, in general, any mobility criterion for space mechanisms must take into account not only the number and the type of kinematic links and joints in a mechanism, but also its constant kinematic parameters.[3] This is perhaps best illustrated by the well-known Bennett [4] and Goldberg [20] mechanisms in which there are definite conditions imposed on the constant kinematic parameters. This is also borne out by certain recent studies on space mechanisms [40,44,45].

Some of the mobility criteria explain the existence of paradoxical mechanisms by taking into account the presence of additional constraints in a mechanism. Thus, Artobolevskii and Dobrovol'ski have introduced the concept of "general constraints." According to Kolchin, mechanisms can also have "passive constraints" in addition to general constraints. The exact nature of these constraints is, however, not known nor is there any definite procedure given to identify their existence.

The conventional mobility criteria predict the existence of hundreds of mechanisms with mobility one [22,23,17]. They do not, however, give any information as to how to go about building these mechanisms. Further, the criteria also predict many mechanisms that are not valid either because they have too many degrees of freedom or because they do not have "true mobility;"[4] that is, they result in certain joints that remain locked. An example for the former case is the space P-P-P-P-P-P (P: Prismatic Pair) mechanism which has three degrees of freedom. An example for the latter case is the space R-P-P-P-P (R: Revolute Pair) mechanism in which the revolute pair remains permanently locked. The

[3]See item 3 in Appendix A.

[4]See item 4 in Appendix A.

conventional mobility criteria are thus quite inadequate and unsuitable for obtaining the existence criteria of space mechanisms. This has prompted investigators in recent years to adopt alternate approaches to the study of mechanism mobility.

The approach of Moroshkin [29] is based on the number of closed loops in a mechanism. In this method, transformation equations are used to describe the basic geometry of a mechanism. The number of independent transformation equations, which is also the rank of the system of equations, is determined by the configuration of the mechanism. The mobility of the mechanism is related to the number of degrees of freedom in all the joints and the rank of the system of transformation equations.

Another important approach to the problem is based on the classical theory of screws. This theory was developed during the last century and is based on two fundamental theorems proposed by Chasles [10] and Poinsot [35]. A detailed account of the theory has been given by Ball in his monumental work published in 1900 [2]. An excellent review of the theory has also been given by Henrici [24]. In recent years, Sharikov [39], Voinea and Atanasiu [49], Waldron [50,51,52,53,54] and Hunt [25,26,27] have employed this theory to examine the mobility of mechanisms. In this approach, a mechanism is regarded as a group or a collection of screws in space. The screws define a screw system whose order is determined by the configuration of the mechanism and the pitch values of the screws. The mobility of the mechanism is related to the total number of screws in the mechanism and the order of the screw system formed by them.

The existence conditions of mechanisms can also be examined by using a mathematical theory recently developed by Soni [40] and by Soni and Harrisberger [42]. In this method, the total geometry of a mechanism is described by a matrix called the residual coefficient matrix (RCM) by using (3x3) matrices with dual-number elements. The rank of this matrix is related to the mobility of the mechanism. The procedure for obtaining this matrix is iterative in nature. The existence criteria of mechanisms can be obtained by a systematic investigation of the properties of this matrix. The procedure has been employed in an investigation of the existence criteria of a six-link, six-revolute mechanism [44]. The properties of the RCM also permit it to be used as a basis for the classification of mechanisms [43].

Yet another approach to the study of mechanism mobility is based on the use of vector algebra. A general method for obtaining the compatibility conditions of mechanisms by using this method has recently been proposed by Pelecudi and Soni [33,46].

The various methods described above for examining the mobility of mechanisms are important contributions to the study of mobility and represent significant improvement over the conventional mobility criteria. These methods have contributed considerably to a better understanding of the nature of space mechanisms. Nevertheless, all these approaches suffer from one serious shortcoming, and this is that they are all essentially concerned only with instantaneous or transitory mobility and not with finite mobility.[5] This feature makes them generally unsuitable particularly for examining the existence criteria of mechanisms in which there are conditions imposed not only on the twist angles, but also on the other constant kinematic parameters. This

[5] See item 4 Appendix A.

drawback is, however, overcome by the passive coupling method developed by Dimentberg and first introduced by him in 1948 [13,14,15]. In this method, the existence criteria of an overconstrained mechanism[6] are obtained from the displacement relationships of an appropriate zero family mechanism [22][6] by imposing suitable passive coupling conditions[6] on the latter; that is, by making some of the joints passive. The method not only assures finite mobility, but is also capable of yielding the necessary conditions for the existence of the derived mechanisms.

The purpose of the present study is to obtain the existence criteria[7] of overconstrained mechanisms with two passive couplings and consisting of revolute and prismatic pairs. A systematic investigation of these mechanisms has been greatly hindered so far by the non-availability of closed-form displacement relationships of spatial five-link mechanisms. However, the results recently obtained by Yang [56,41] make it possible to obtain the existence criteria of these mechanisms by using Dimentberg's passive coupling technique.

Specifically, the objectives of the present investigation are:

1. To obtain the existence criteria of the five-link, five-revolute (R-R-R-R-R) space mechanism. This is the primary objective of the present study. The derived criteria should not only justify the existence of known five-revolute mechanisms [20,30], but should also make it possible to investigate the existence of other five-revolute mechanisms.

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[6]See item 5 in Appendix A.

[7]See item 6 in Appendix A.

2. To obtain the existence criteria of the five-link R-R-R-P-R space mechanism. The derived criteria should facilitate the investigation of the existence of such mechanisms.

3. To obtain the existence criteria of the five-link 3R+2P and 2R+3P space mechanisms. Besides explaining the existence of known mechanisms [25, 52, 16], the derived criteria should also reveal the existence of other mechanisms.

In the next chapter, the Dimentberg passive coupling method employed for the above purpose is discussed in great detail. Chapter III is devoted to a general discussion of mechanisms with two passive couplings. In the remaining chapters, the results of the objectives mentioned above are presented.

CHAPTER II

DIMENTBERG'S PASSIVE COUPLING METHOD

The Dimentberg passive coupling method can be used to obtain the existence criteria of overconstrained mechanisms. Dimentberg first introduced this method in 1948 and has used it to obtain the existence criteria of a number of overconstrained four-link mechanisms [13, 14, 15].[1]

Nature of Dimentberg's Method

In Dimentberg's method, an overconstrained mechanism is obtained by imposing suitable passive coupling conditions on an appropriate zero family mechanism. The zero family mechanism so chosen is referred to here as the parent mechanism.

The use of Dimentberg's method for obtaining the existence criteria of an overconstrained mechanism involves the following three distinct steps:

1. The first step is to select the parent mechanism. It is, in general, possible to derive an overconstrained mechanism from more than one parent mechanism. Thus, for instance, the R-C-R-C[2] mechanism can be derived from either the R-C-C-C

[1]Ogino and Watanabe [31] have recently used dual-number quaternion algebra to study the mobility of a spatial four-link chain with four cylinder pairs and have come up with certain overconstrained four-link mechanisms. They are, however, apparently unaware of the work of Dimentberg [13,14] in which similar results were obtained many years ago.

[2]Throughout this study, R, P, H and C are used to denote the revolute, prismatic, helical and cylinder pairs respectively.

mechanism or the R-C-R-C-R mechanism.

2. The next step is to obtain the displacement relationships of the parent mechanism.[3] If the parent mechanism has no helical pairs, the displacement relationships are algebraic in nature. If, however, the parent mechanism has helical pairs, the relationships involving only the rotations at the helical pairs still remain algebraic in nature, but the relationships involving the translations at the helical pairs become non-algebraic in nature.

3. The third and final step in Dimentberg's method is to impose the required passive coupling conditions on the parent mechanism so as to obtain the desired overconstrained mechanism. When the displacement relationships involved are algebraic in nature, this step very often involves examination of the conditions for common roots between two algebraic polynomials or between successive sets of two polynomials. The results obtained lead to conditions on the constant kinematic parameters of the parent mechanism and provide the necessary conditions for the existence of the desired overconstrained mechanism.

Example

The Dimentberg method described above can be best illustrated by an example.

Let it be required to obtain the existence criteria of an R-C-C-R mechanism. This can be done by considering an R-C-C-C mechanism as the

[3]As a rule, the complexity of the displacement relationships increases as the number of links in the parent mechanism increases. See, for instance, references [55] and [56].

parent mechanism.

Consider the R-C-C-C space mechanism shown schematically in Fig. 1. This mechanism reduces to an R-C-C-R mechanism if the translation y at the output cylinder pair remains constant at all positions of the mechanism (Fig. 2).

The relationships between the input variable ϕ and the output variables ψ and y of the mechanism in Fig. 1 are given by [55]

$$A_2(\Phi)\Psi^2 + A_1(\Phi)\Psi + A_0(\Phi) = 0 \qquad (2\text{-}1)$$

and $$y[B_2(\Phi)\Psi^2 + B_1(\Phi)\Psi + B_0(\Phi)] + C_2(\Phi)\Psi^2 + C_1(\Phi)\Psi + C_0(\Phi) = 0 \qquad (2\text{-}2)$$

where $\Phi = \tan(\phi/2)$

$\Psi = \tan(\psi/2)$

and
$$\begin{aligned} A_i(\Phi) &= A_{i2}\Phi^2 + A_{i1}\Phi + A_{i0} \\ B_i(\Phi) &= B_{i2}\Phi^2 + B_{i1}\Phi + B_{i0} \\ C_i(\Phi) &= C_{i2}\Phi^2 + C_{i1}\Phi + C_{i0}\ , \quad i = 0,\ 1,\ 2 \end{aligned} \qquad (2\text{-}3)$$

The constants in Eqs. (2-3) involve only the constant kinematic parameters of the mechanism in Fig. 1.

Let the translation y at the output cylinder pair be now held constant at all positions of the mechanism. Denoting this constant value by y_k, Eq. (2-2) becomes

$$y_k[B_2(\Phi)\Psi^2 + B_1(\Phi)\Psi + B_0(\Phi)] + C_2(\Phi)\Psi^2 + C_1(\Phi)\Psi + C_0(\Phi) = 0 \qquad (2\text{-}4)$$

Since y_k is a constant, the above equation can be rewritten as

$$D_2(\Phi)\Psi^2 + D_1(\Phi)\Psi + D_0(\Phi) = 0 \qquad (2\text{-}5)$$

where $D_i(\Phi) = D_{i2}\Phi^2 + D_{i1}\Phi + D_{i0}\Phi\ , \quad i = 0,\ 1,\ 2 \qquad (2\text{-}6)$

If an R-C-C-R mechanism is to exist, it is necessary for the

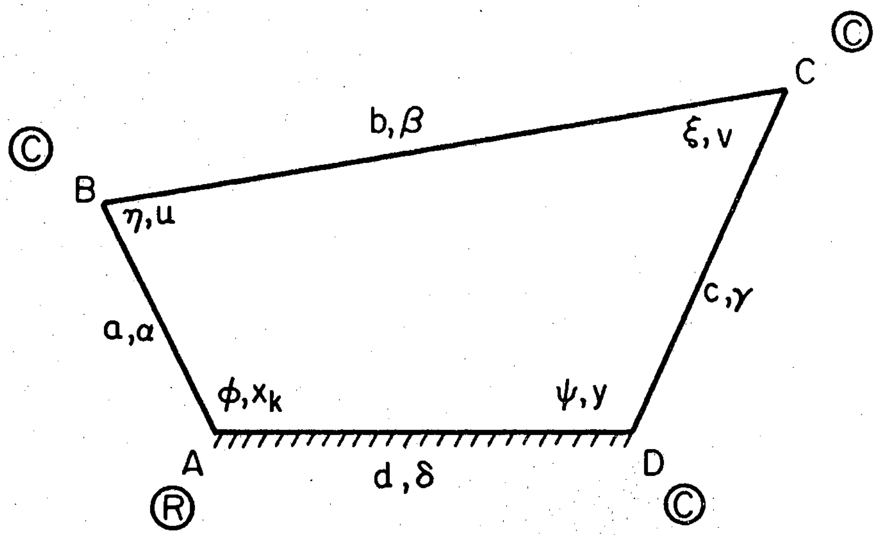


Figure 1. R-C-C-C Space Mechanism

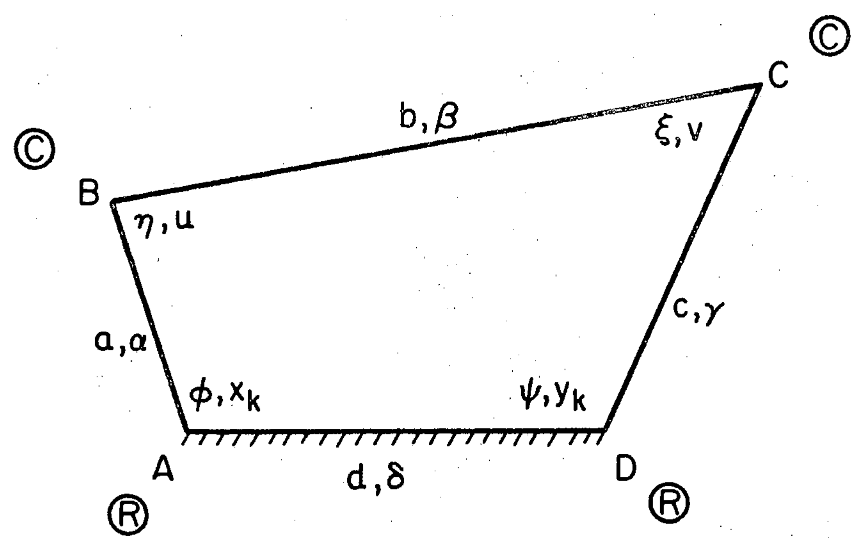


Figure 2. R-C-C-R Space Mechanism Obtained from the Mechanism in Fig. 1 by Making $y = y_k =$ a Constant

quadratic equations (2-1) and (2-5) to have at least one common root. This gives the condition (Appendix B)

$$\begin{vmatrix} A_2(\Phi) & A_1(\Phi) & A_0(\Phi) & 0 \\ 0 & A_2(\Phi) & A_1(\Phi) & A_0(\Phi) \\ D_2(\Phi) & D_1(\Phi) & D_0(\Phi) & 0 \\ 0 & D_2(\Phi) & D_1(\Phi) & D_0(\Phi) \end{vmatrix} = 0 \qquad (2\text{-}7)$$

Expanding and simplifying Eq. (2-7), we get

$$E_8\Phi^8 + E_7\Phi^7 + \ldots\ldots + E_1\Phi + E_0 = 0$$

or, in short,

$$\sum_{i=0}^{8} E_i\Phi^i = 0 \qquad (2\text{-}8)$$

The constants in the above equation involve only the constant kinematic parameters of the mechanism in Fig. 2.

Eq. (2-8) consists of only the variable Φ describing the position of the mechanism in Fig. 2 and must be satisfied at all positions of that mechanism. It must, therefore, hold good at all values of the variable Φ. Its coefficients must, therefore vanish [5]. This gives

$$E_i = 0,\ i = 0, 1, 2, \ldots, 8 \qquad (2\text{-}9)$$

Condition (2-9) represents nine equations among the ten constant kinematic parameters of the mechanism in Fig. 2 (namely, the four link lengths a, b, c and d, the four twist angles α, β, γ and δ and the two constant offset distances x_k and y_k at the input and output revolute pairs). These nine equations provide the necessary conditions for the existence of an R-C-C-R mechanism.

Scope of Dimentberg's Method

In his investigations, Dimentberg has employed his method in those cases in which the translational freedom of a cylinder pair is made passive [13, 14, 15]. The method is, however, equally applicable to the case in which the rotational freedom of a cylinder pair is made passive. This is illustrated by the example in Appendix C in which the existence criteria of R-P-C-P and R-C-P-P mechanisms have been obtained by imposing passive coupling condition on the rotational freedom of the output cylinder pair of an R-C-C-C mechanism. This approach appears to be more convenient and efficient than the one adopted by Dimentberg and Yoslovich [16].

The Dimentberg method is also valid for the case in which the entire freedom at a joint is made passive. The joint thus becomes locked and no motion is possible at that joint. This is illustrated by the example in Appendix D in which the existence criteria of an R-C-R-C mechanism have been obtained by imposing passive coupling condition on the rotational freedom of the output revolute pair of an R-C-R-C-R mechanism. The results obtained agree with those obtained by Dimentberg who derived them by imposing passive coupling conditions on a parent R-C-C-C mechanism [13, 14]. This example also shows that it is possible to derive an overconstrained mechanism from more than one parent mechanism.

The extensions to Dimentberg's method as illustrated by the examples in Appendices C and D demonstrate the immense scope of the method and show that the method can be employed to handle a variety of passive coupling conditions.

Passive Coupling Conditions Considered in This Study

The passive coupling conditions considered in the present study are confined to those cases in which the required displacement relationships are algebraic in nature. The cases considered are summarized in Table I and fall into the following five categories:

1. Passive couplings in two cylinder pairs to obtain two revolute pairs (see Case 1 in Table I).
2. Passive couplings in two cylinder pairs to obtain one revolute pair and one prismatic pair (see Case 2 in Table I).
3. Passive couplings in two cylinder pairs to obtain two prismatic pairs (see Case 3 in Table I).
4. Passive coupling in a cylinder pair to obtain a prismatic pair (see Case 4 in Table I).
5. Passive coupling in a revolute pair to prevent it from executing rotational motion (see Case 5 in Table I).

TABLE I

PASSIVE COUPLING CONDITIONS CONSIDERED IN THE PRESENT STUDY
(R: Revolute Pair, P: Prismatic Pair, C: Cylinder Pair)

Case	Kinematic Pair(s) Selected for Inducing Passive Coupling Condition(s)	Kinematic Pair(s) Obtained Because of Passive Coupling Condition(s)	Parent Mechanism Examined for Inducing Passive Coupling Condition(s)	Overconstrained Mechanism Obtained Because of Passive Coupling Condition(s)	Considered in
1	C-C	R-R	R-C-R-C-R or R-R-C-C-R	R-R-R-R-R	Chapter IV
2	C-C	R-P or P-R	R-C-R-C-R or R-R-C-C-R	R-R-R-P-R	Chapter V
3	C-C	P-P	R-C-R-C-R R-R-C-C-R R-C-P-C-R R-C-R-C-P	R-P-R-P-R R-R-P-P-R R-P-P-P-R R-P-R-P-P	Chapter VI
4	C	P	R-C-C-C	R-P-C-P R-C-P-P	Appendix C
5	R	Passive coupling is introduced to prevent the revolute pair from executing rotational motion.	R-C-R-C-R	R-C-R-C	Appendix D

CHAPTER III

MECHANISMS WITH TWO PASSIVE COUPLINGS

Mechanisms with two passive couplings are overconstrained mechanisms in which the sum of the degrees of freedom in all the joints is equal to five. The number of links may be equal to or less than five.

Several examples of mechanisms with two passive couplings and mobility one have been recorded by investigators in the past many years. A five-link R-R-H-R-R mechanism proposed by Reuleaux [36] is shown in Fig. 3. In this mechanism, the axis of the helical pair and the axis of one of the revolute pairs are coaxial; the axes of the remaining revolute pairs are parallel to one another and normal to the common direction of the coaxial axes. A five-link R-R-R-H-P mechanism and a five-link H-R-R-P-P mechanism proposed by Artobolevskii [1] are shown in Figs. 4 and 5. The mechanism in Fig. 4 can be obtained by replacing the output revolute pair in a Hooke's coupling by a helical pair and a prismatic pair with coincident axes. In the mechanism in Fig. 5, the axes of the helical pair and the two revolute pairs are parallel to one another.

A five-link H-H-H-H-H mechanism proposed by Voinea and Atanasiu [49] is shown in Fig. 6. In this mechanism, the pitch values of the helical pairs are randomly selected and the pair axes are all parallel to one another. The mobility of this mechanism is unaffected even if a maximum of three of the helical pairs have the same pitch values [25].

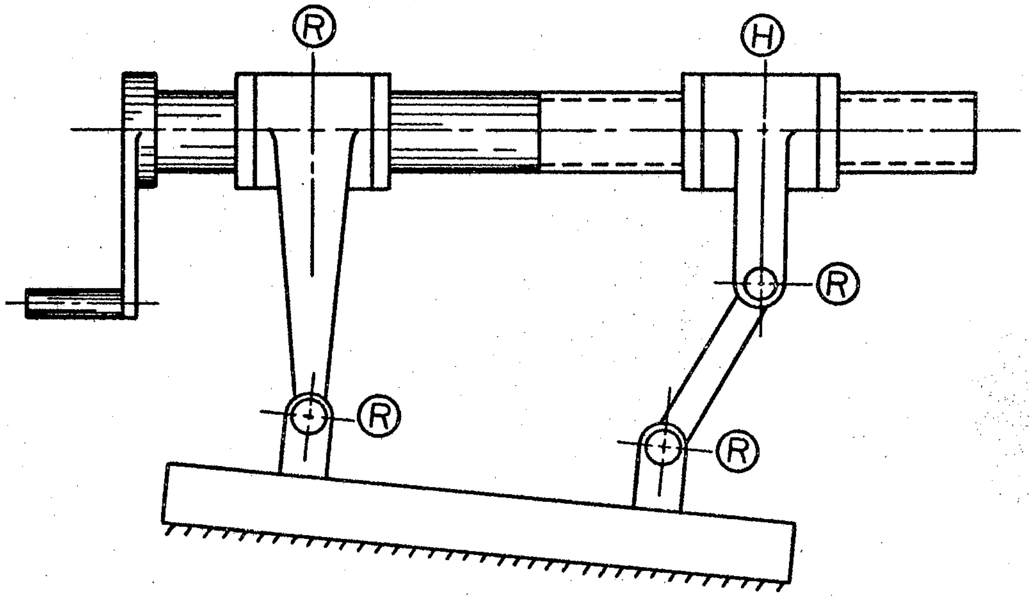


Figure 3. R-R-H-R-R Space Mechanism [36]

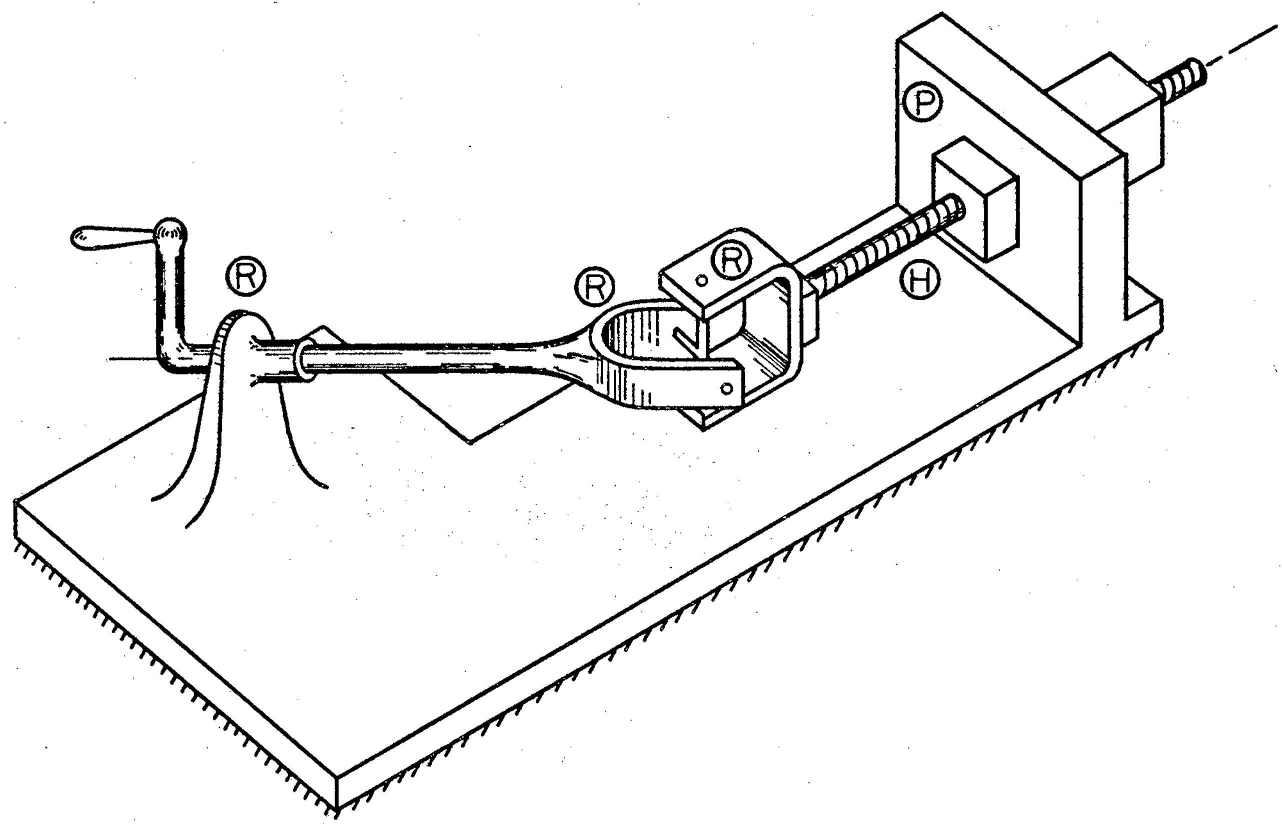


Figure 4. R-R-R-H-P Space Mechanism [1]

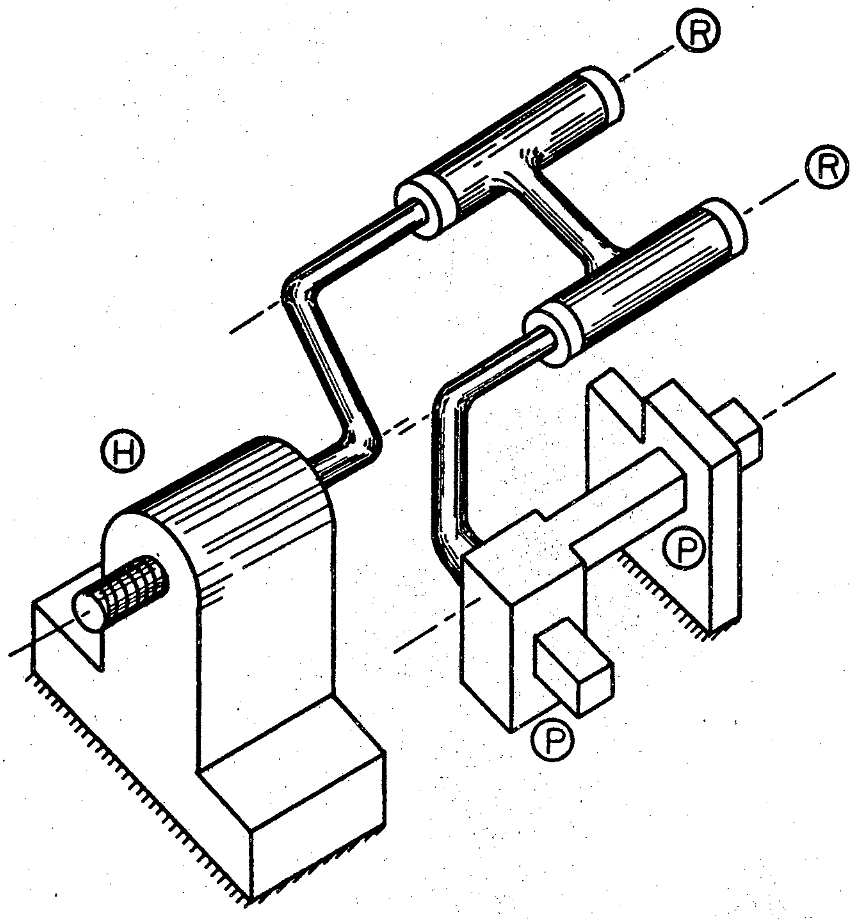


Figure 5. H-R-R-P-P Space Mechanism [1]

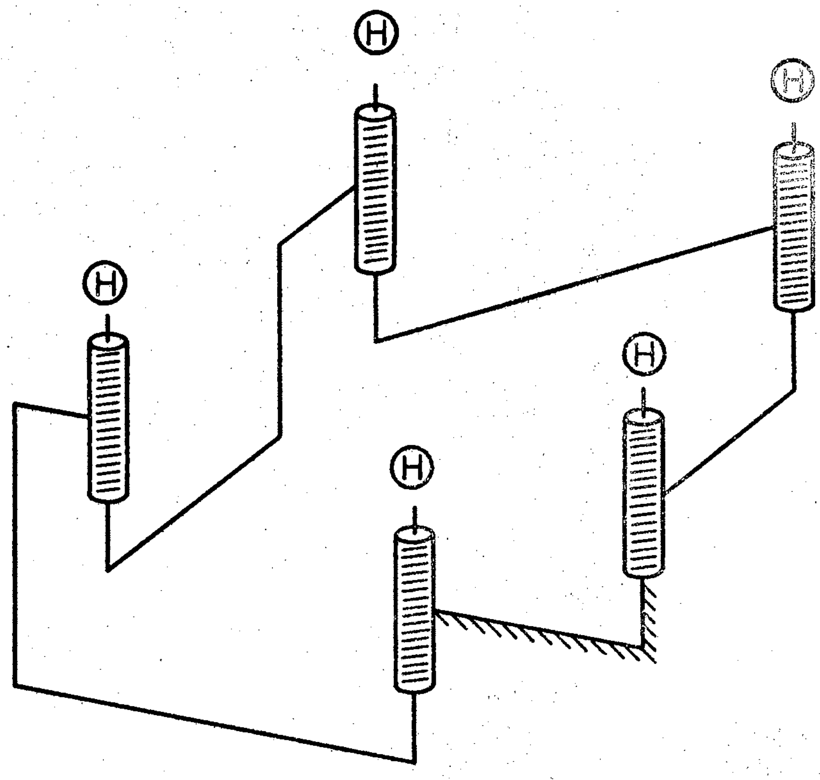


Figure 6. H-H-H-H-H Space Mechanism [49]

Four-link mechanisms with two passive couplings consisting of one revolute pair, one cylinder pair and two prismatic pairs have been derived by Dimentberg and Yoslovich [16] by means of screw calculus and dual numbers. These are shown in Fig. 7. In these mechanisms, the axes of the revolute pair and the cylinder pair are parallel to each other. Using the H-H-H-H-H mechanism (Fig. 6) proposed by Voinea and Atanasiu as a basis, Hunt [25] and Waldron [52] have recently obtained a whole class of mechanisms with prismatic pairs. Two examples are shown in Fig. 8. The Dimentberg-Yoslovich mechanisms shown in Fig. 7 and the Artobolevskii mechanism shown in Fig. 5 are special cases of these mechanisms. A five-link H-H-P-H-H mechanism with plane symmetry has recently been isolated by Waldron [54]. In this mechanism, the prismatic pair is normal to the plane of symmetry and the axes of the two helical pairs on each side of the plane of symmetry are parallel to each other. However, the axes of the symmetrical helical pairs themselves need not be parallel.

There are not many five-link, five-revolute mechanisms. Myard [30] has proposed a five-revolute mechanism by considering a rectangular Bennett mechanism (that is, a mechanism with one twist angle equal to 90°). The most interesting five-revolute mechanisms known, however, are those proposed by Goldberg [20]. These are shown in Fig. 9 and are obtained by the combination (addition or subtraction) of two Bennett mechanisms. The mechanism proposed by Myard is symmetrical about a plane and is a special case of the Goldberg mechanisms [54].

The mechanisms described above have been obtained as a result of useful, but essentially isolated, attempts. In the present study, a systematic investigation of mechanisms with two passive couplings and

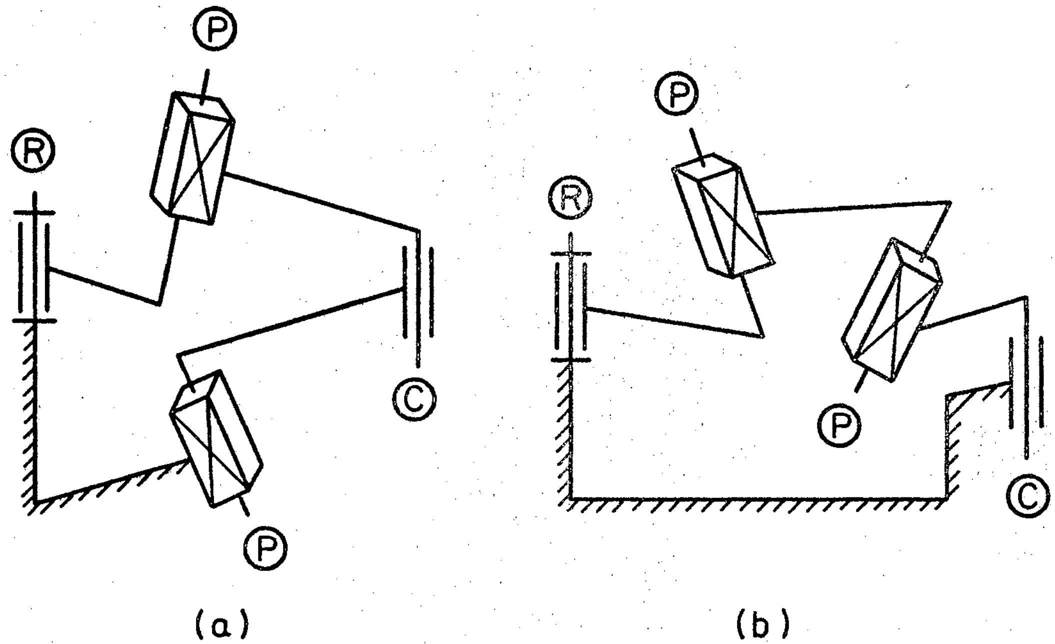


Figure 7. R-P-C-P and R-P-P-C Space Mechanisms [16]

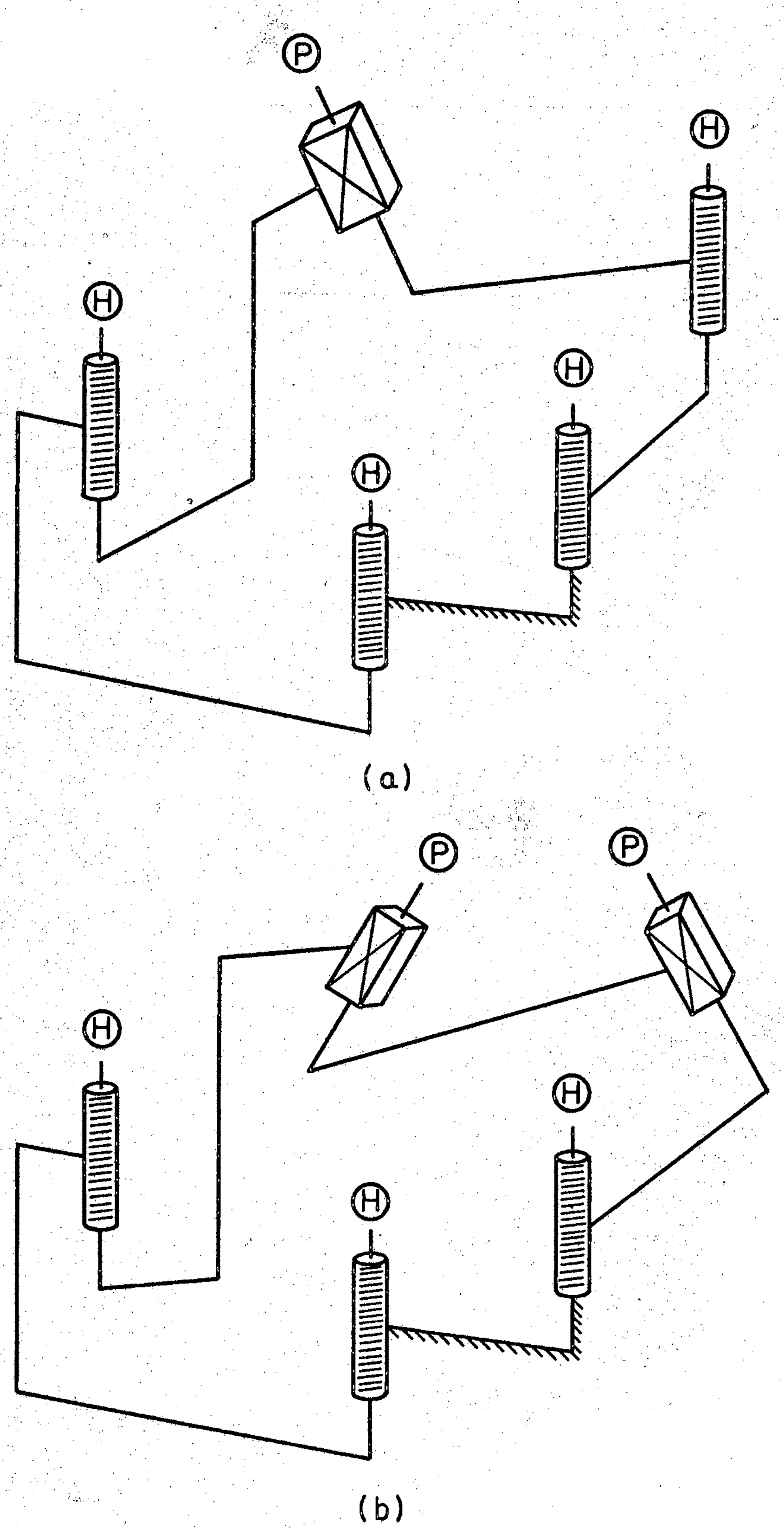
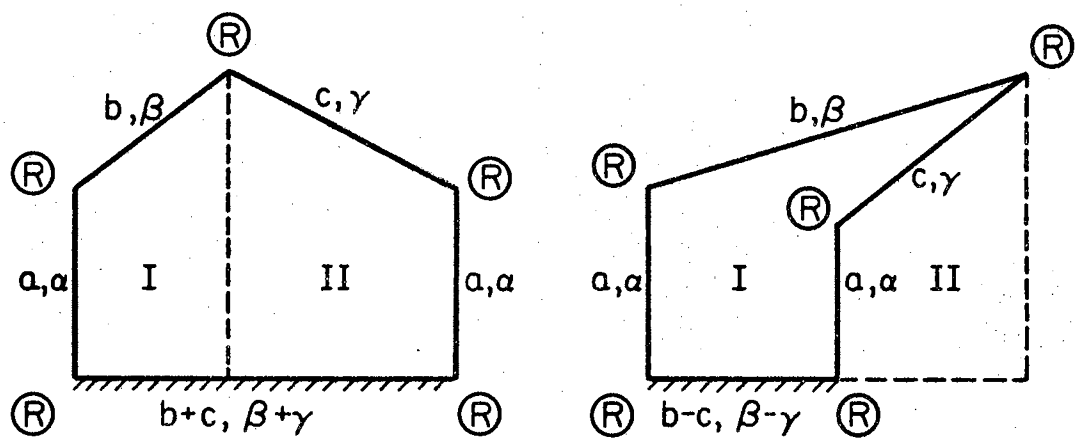


Figure 8. H-H-P-H-H and H-H-P-P-H Space Mechanisms [25, 52]



(a) ADDITION OF TWO BENNETT MECHANISMS

(b) SUBTRACTION OF TWO BENNETT MECHANISMS

$$\frac{a}{\sin\alpha} = \pm\frac{b}{\sin\beta} = \pm\frac{c}{\sin\gamma}$$

Figure 9. Goldberg Five-Revolute Space Mechanisms

consisting of revolute and prismatic pairs has been conducted by using Dimentberg's passive coupling method.

Displacement Relationships for Obtaining Existence Criteria

The use of Dimentberg's method for obtaining the existence criteria of overconstrained mechanisms requires the displacement relationships of the appropriate parent mechanisms. The required relationships can always be obtained by suitably arranging the loop-closure condition of the parent mechanism.

Consider a general five-link mechanism consisting of helical, revolute, prismatic and cylinder pairs combined in such a way that the sum of the degrees of freedom in all the joints is equal to seven (Fig. 10). Such a mechanism would necessarily have to have two cylinder pairs. If the type of the remaining three pairs and the location of all the five pairs in the mechanism are properly chosen, this mechanism will serve as a parent mechanism for any overconstrained mechanism with two passive couplings.

The mechanism in Fig. 10 is completely defined by the following two sets of dual angles:

i) Between adjacent pair axes

$$\begin{aligned}\hat{\alpha} &= \alpha + \varepsilon a\\ \hat{\beta} &= \beta + \varepsilon b\\ \hat{\gamma} &= \gamma + \varepsilon c\\ \hat{\delta} &= \delta + \varepsilon d\\ \hat{\lambda} &= \lambda + \varepsilon e\end{aligned} \tag{3-1}$$

where α, β, γ, δ and λ are the twist angles and a, b, c, d and e are the

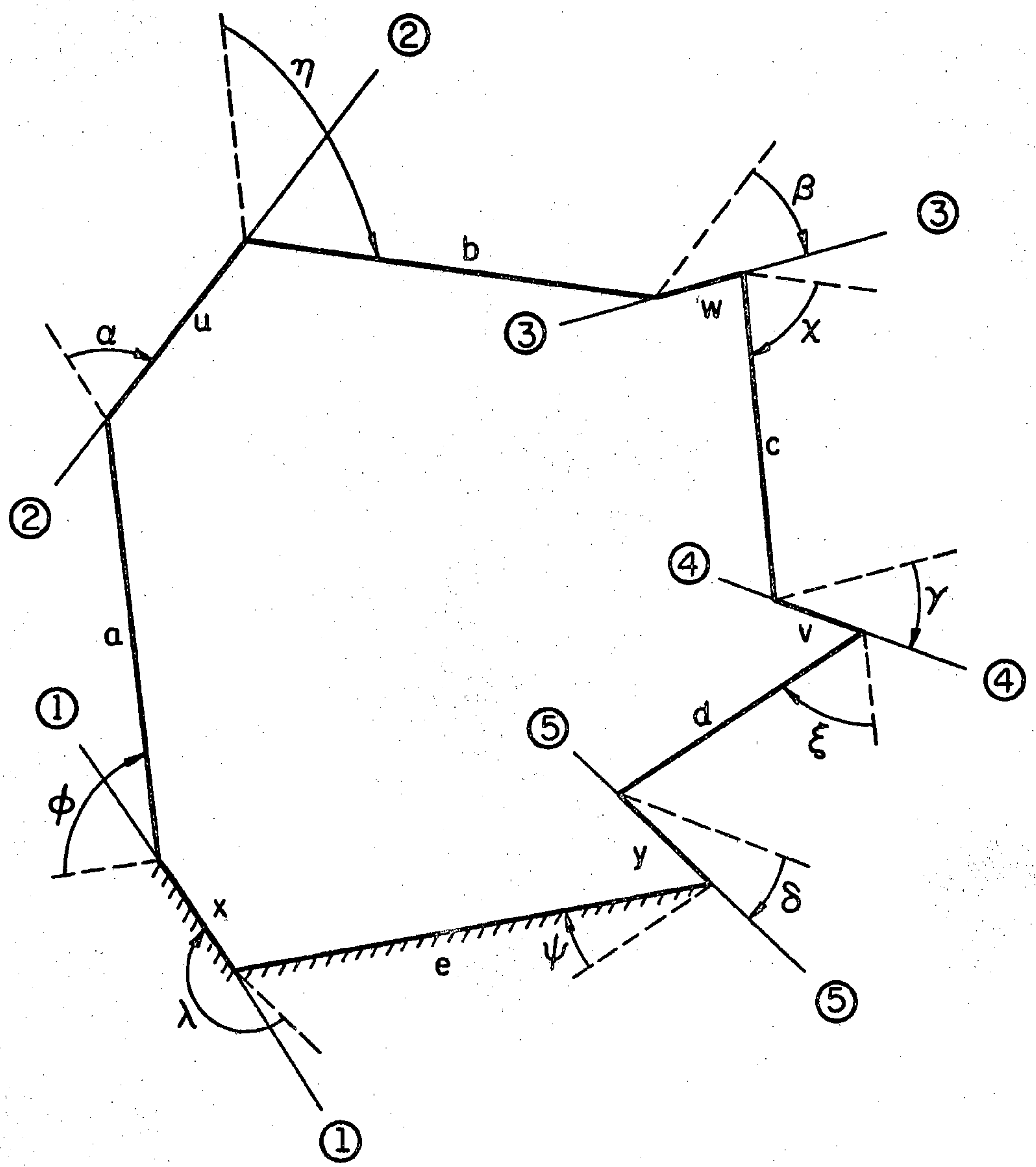


Figure 10. General Five-Link Space Mechanism with Helical, Revolute, Prismatic and Cylinder Pairs [$\sum f_i = 7$]

link lengths. These ten quantities are constant for any given mechanism. Note also that, by definition, $\varepsilon^2 = 0$.

ii) Between adjacent common perpendiculars

$$\begin{aligned}\hat{\phi} &= \phi + \varepsilon x \\ \hat{\eta} &= \eta + \varepsilon u \\ \hat{\chi} &= \chi + \varepsilon w \\ \hat{\xi} &= \xi + \varepsilon v \\ \hat{\psi} &= \psi + \varepsilon y\end{aligned} \tag{3-2}$$

where ϕ, η, χ, ξ and ψ are the angular displacements at the kinematic pairs and x, u, w, v and y are the translations along the kinematic axes. These quantities may be variable or remain constant depending upon the type of kinematic pairs used in the mechanism.[1]

The loop-closure condition of the mechanism in Fig. 10 is given by [56]

$$[\hat{\alpha}]_1[\hat{\phi}]_3[\hat{\lambda}]_1[\hat{\psi}]_3[\hat{\delta}]_1[\hat{\xi}]_3[\hat{\gamma}]_1[\hat{\chi}]_3[\hat{\beta}]_1[\hat{\eta}]_3 = [I] \tag{3-3}$$

where

$$[\hat{\alpha}]_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\hat{\alpha} & S\hat{\alpha} \\ 0 & -S\hat{\alpha} & C\hat{\alpha} \end{bmatrix} \qquad [\hat{\phi}]_3 = \begin{bmatrix} C\hat{\phi} & S\hat{\phi} & 0 \\ -S\hat{\phi} & C\hat{\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\hat{\beta}]_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\hat{\beta} & S\hat{\beta} \\ 0 & -S\hat{\beta} & C\hat{\beta} \end{bmatrix} \qquad [\hat{\eta}]_3 = \begin{bmatrix} C\hat{\eta} & S\hat{\eta} & 0 \\ -S\hat{\eta} & C\hat{\eta} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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[1]In a prismatic pair, the angular displacement remains constant while, in a revolute pair, the translation along the axis is constant. In a helical pair, the translation along the axis and the angular displacement both vary in such a way that their ratio is always constant and equal to the pitch. In a cylinder pair, the translation along the axis and the angular displacement both vary and are independent of each other.

$$[\hat{\gamma}]_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\hat{\gamma} & C\hat{\gamma} \\ 0 & -S\hat{\gamma} & C\hat{\gamma} \end{bmatrix} \qquad [\hat{\chi}]_3 = \begin{bmatrix} C\hat{\chi} & S\hat{\chi} & 0 \\ -S\hat{\chi} & C\hat{\chi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3\text{-}4)^2$$

$$[\hat{\delta}]_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\hat{\delta} & S\hat{\delta} \\ 0 & -S\hat{\delta} & C\hat{\delta} \end{bmatrix} \qquad [\hat{\xi}]_3 = \begin{bmatrix} C\hat{\xi} & S\hat{\xi} & 0 \\ -S\hat{\xi} & C\hat{\xi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\hat{\lambda}]_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\hat{\lambda} & S\hat{\lambda} \\ 0 & -S\hat{\lambda} & C\hat{\lambda} \end{bmatrix} \qquad \text{and} \quad [\hat{\psi}]_3 = \begin{bmatrix} C\hat{\psi} & S\hat{\psi} & 0 \\ -S\hat{\psi} & C\hat{\psi} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Two arrangements of Eq. (3-3) are useful in the study of existence criteria.

1) Relationship involving two adjacent dual displacement angles and the dual displacement angle opposite to both of them.

In this arrangement of Eq. (3-3), five matrices are used on either side of the equality sign. Thus, we have, for instance,

$$[\hat{\alpha}]_1[\hat{\phi}]_3[\hat{\lambda}]_1[\hat{\psi}]_3[\hat{\delta}]_1 = [\hat{\eta}]_3^{-1}[\hat{\beta}]_1^{-1}[\hat{\chi}]_3^{-1}[\hat{\gamma}]_1^{-1}[\hat{\xi}]_3^{-1} \qquad (3\text{-}5)$$

Simplifying the above equation by using relations (3-4) and equating the "33" elements of the resultant matrix equation, we get

$$\begin{aligned} F(\hat{\phi},\hat{\psi},\hat{\chi}) &= (S\hat{\alpha}S\hat{\delta}S\hat{\psi})S\hat{\phi} - S\hat{\alpha}(C\hat{\delta}S\hat{\lambda}+S\hat{\delta}C\hat{\lambda}C\hat{\psi})C\hat{\phi} \\ &\quad + C\hat{\alpha}(C\hat{\delta}C\hat{\lambda}-S\hat{\delta}S\hat{\lambda}C\hat{\psi}) - (C\hat{\beta}C\hat{\gamma}-S\hat{\beta}S\hat{\gamma}C\hat{\chi}) = 0 \qquad (3\text{-}6) \end{aligned}$$

Note that Eq. (3-6) involves the adjacent displacement angles $\hat{\phi}$ and $\hat{\psi}$ and the displacement angle $\hat{\chi}$ opposite to both of them.

---

[2] In this equation and in all the subsequent equations and tables, C and S denote the cosine and sine of the respective angles.

Cyclic permutation permits Eq. (3-5) to be written in five different ways. It is, therefore, possible to get five equations of the form (3-6) involving different combinations of two adjacent angles and the angle opposite to both of them.

2) Relationship involving three adjacent dual displacement angles.

In this arrangement of Eq. (3-3), seven matrices are used on one side of the equality sign and three matrices on the other. The important point to note is that the central matrix on the side containing three matrices involves only the constant kinematic parameters of the mechanism. Thus, we have, for instance,

$$[\hat{\beta}]_1[\hat{\eta}]_3[\hat{\alpha}]_1[\hat{\phi}]_3[\hat{\lambda}]_1[\hat{\psi}]_3[\hat{\delta}]_1 = [\hat{\chi}]_3^{-1}[\hat{\gamma}]_1^{-1}[\hat{\xi}]_3^{-1} \tag{3-7}$$

Note that the central matrix $[\hat{\gamma}]_1^{-1}$ on the right hand side involves only the constant kinematic parameters of the mechanism.

Simplifying Eq. (3-7) by using relations (3-4) and equating the "33" elements of the resultant matrix equation, we get

$$\begin{aligned} f(\hat{\eta},\hat{\phi},\hat{\psi}) = {} & [(S\hat{\alpha}C\hat{\beta}+C\hat{\alpha}S\hat{\beta}C\hat{\eta})S\hat{\phi} + S\hat{\beta}S\hat{\eta}C\hat{\phi}](S\hat{\delta}S\hat{\psi}) \\ & + [S\hat{\beta}S\hat{\eta}S\hat{\phi} - (S\hat{\alpha}C\hat{\beta}+C\hat{\alpha}S\hat{\beta}C\hat{\eta})C\hat{\phi}](C\hat{\delta}S\hat{\lambda}+S\hat{\delta}C\hat{\lambda}C\hat{\psi}) \\ & + (C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta})(C\hat{\delta}C\hat{\lambda}-S\hat{\delta}S\hat{\lambda}C\hat{\psi}) - C\hat{\gamma} = 0 \end{aligned} \tag{3-8}$$

Note that Eq. (3-8) involves the three adjacent displacement angles $\hat{\eta}$, $\hat{\phi}$ and $\hat{\psi}$.

Cyclic permutation allows Eq. (3-7) to be written in five different ways. It is, therefore, possible to obtain five equations of the form (3-8) involving different combinations of three adjacent angles.

Observe that Eqs. (3-6) and (3-8) are both dual equations. Each of them, therefore, represents two scalar equations. Since five equations of the form (3-6) and five equations of the form (3-8) are possible, a total of twenty scalar equations are available. These twenty equations make it possible to obtain the existence criteria of all mechanisms with two passive couplings (and also many mechanisms with one passive coupling with number of links equal to or less than five).

CHAPTER IV

EXISTENCE CRITERIA OF THE FIVE-LINK, FIVE-REVOLUTE MECHANISM

The five-link, five-revolute mechanism can be derived from either the R-C-R-C-R mechanism or the R-R-C-C-R mechanism. In this chapter, the Dimentberg method has been used to obtain the existence criteria of a five-revolute mechanism with zero offset distances along its pair axes from the displacement relationships of an R-C-R-C-R mechanism. An attempt to derive these criteria has also been made by Dimentberg [14]. His results, though incomplete, indicate that they lead to two sixty-fourth degree polynomials. The results obtained in this chapter, however, lead to two polynomials of only the twenty-fourth degree.

Derivation of the Existence Criteria

Consider the R-C-R-C-R space mechanism shown schematically in Fig. 11. Note that the constant offset distances at the three revolute pairs are taken to be zero. If the translations u and v at the two cylinder pairs are reduced to zero at all positions of this mechanism, it reduces to a five-revolute mechanism with zero offset distances at its pairs (Fig. 12).

By considering the loop-closure condition of the mechanism in Fig. 11 in three different ways, the following displacement relationships can be obtained:

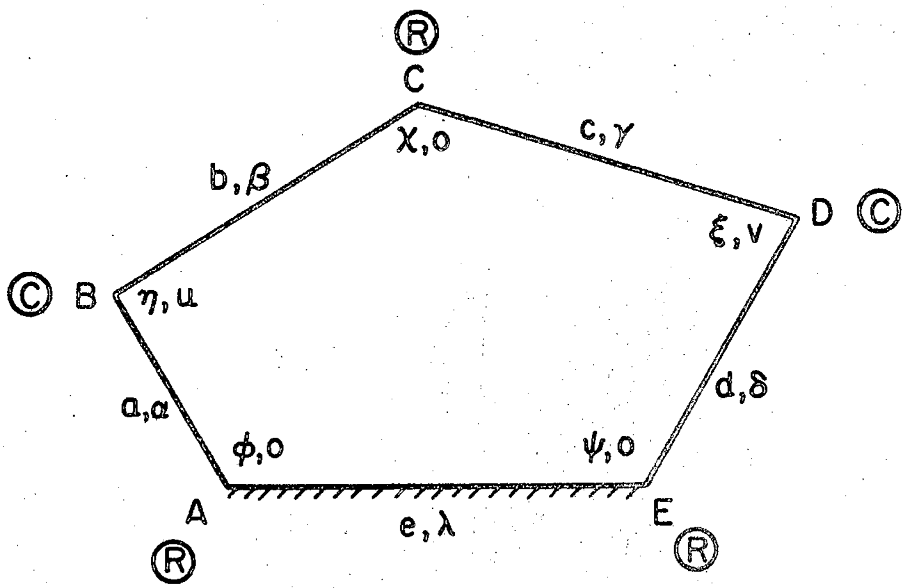

Figure 11. R-C-R-C-R Space Mechanism

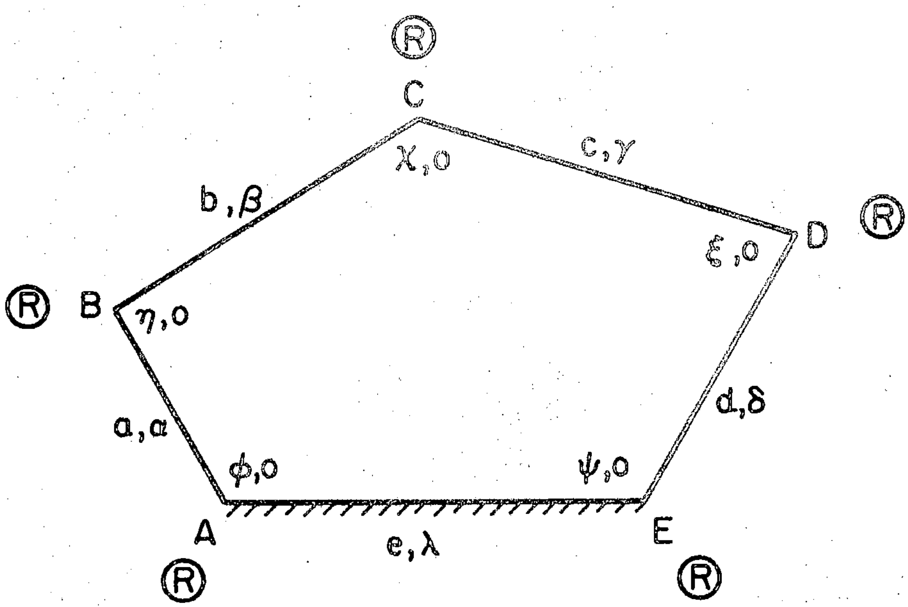

Figure 12. R-R-R-R-R Space Mechanism Obtained from the Mechanism in Fig. 11 by Making u = 0 and v = 0

$$F(\hat{\phi},\hat{\psi},\hat{\chi}) = (S\hat{\alpha}S\hat{\delta}S\hat{\psi})S\hat{\phi} - S\hat{\alpha}(C\hat{\delta}S\hat{\lambda}+S\hat{\delta}C\hat{\lambda}C\hat{\psi})C\hat{\phi}$$
$$+ C\hat{\alpha}(C\hat{\delta}C\hat{\lambda}-S\hat{\delta}S\hat{\lambda}C\hat{\psi}) - (C\hat{\beta}C\hat{\gamma}-S\hat{\beta}S\hat{\gamma}C\hat{\chi}) = 0 \tag{4-1}$$

$$f(\hat{\eta},\hat{\phi},\hat{\psi}) = [(S\hat{\alpha}C\hat{\beta}+C\hat{\alpha}S\hat{\beta}C\hat{\eta})S\hat{\phi} + S\hat{\beta}S\hat{\eta}C\hat{\phi}](S\hat{\delta}S\hat{\psi})$$
$$+ [S\hat{\beta}S\hat{\eta}S\hat{\phi} - (S\hat{\alpha}C\hat{\beta}+C\hat{\alpha}S\hat{\beta}C\hat{\eta})C\hat{\phi}](C\hat{\delta}S\hat{\lambda}+S\hat{\delta}C\hat{\lambda}C\hat{\psi})$$
$$+ (C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta})(C\hat{\delta}C\hat{\lambda}-S\hat{\delta}S\hat{\lambda}C\hat{\psi}) - C\hat{\gamma} = 0 \tag{4-2}$$

$$F(\hat{\eta},\hat{\phi},\hat{\xi}) = (S\hat{\beta}S\hat{\lambda}S\hat{\phi})S\hat{\eta} - S\hat{\beta}(S\hat{\alpha}C\hat{\lambda}+C\hat{\alpha}S\hat{\lambda}C\hat{\phi})C\hat{\eta}$$
$$+ C\hat{\beta}(C\hat{\alpha}C\hat{\lambda}-S\hat{\alpha}S\hat{\lambda}C\hat{\phi}) - (C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi}) = 0 \tag{4-3}$$

Observe that Eqs. (4-1) and (4-3) are similar in form to Eq. (3-6) and Eq. (4-2) is similar to Eq. (3-8).

Eliminating the angle χ from the primary and dual parts of Eq. (4-1), we get

$$A_2(\Phi)\Psi^2 + A_1(\Phi)\Psi + A_0(\Phi) = 0 \tag{4-4}$$

where $\Phi = \tan(\phi/2)$

$\Psi = \tan(\psi/2)$

and
$$\begin{aligned} A_2(\Phi) &= A_{22}\Phi^2 + A_{20} \\ A_1(\Phi) &= A_{11}\Phi \\ A_0(\Phi) &= A_{02}\Phi^2 + A_{00} \end{aligned} \tag{4-5}$$

The constants in Eqs. (4-5) depend only on the constant kinematic parameters of the mechanism in Fig. 11 and are defined in Table II. It may also be noted here that Eq. (4-4) above corresponds to Eq. (16) in reference [56].

The primary part of Eq. (4-2) can be written as

TABLE II

CONSTANTS FOR USE IN EQS. (4-5) AND (4-7) AND TABLE III

$$A_{22} = U_1 - U_2C(\delta-\lambda+\alpha) - (d-e+a)S\beta S\gamma S(\delta-\lambda+\alpha)$$

$$A_{20} = U_1 - U_2C(\delta-\lambda-\alpha) - (d-e-a)S\beta S\gamma S(\delta-\lambda-\alpha)$$

$$A_{11} = 4(aC\alpha S\beta S\gamma S\delta - bS\alpha C\beta S\gamma S\delta - cS\alpha S\beta C\gamma S\delta + dS\alpha S\beta S\gamma C\delta)$$

$$A_{02} = U_1 - U_2C(\delta+\lambda-\alpha) - (d+e-a)S\beta S\gamma S(\delta+\lambda-\alpha)$$

$$A_{00} = U_1 - U_2C(\delta+\lambda+\alpha) - (d+e+a)S\beta S\gamma S(\delta+\lambda+\alpha)$$

where $U_1 = bS\gamma C\gamma + cS\beta C\beta$

and $U_2 = bC\beta S\gamma + cS\beta C\gamma$

$$B_{222} = C(\delta-\lambda+\alpha-\beta) - C\gamma$$

$$B_{220} = C(\delta-\lambda-\alpha+\beta) - C\gamma$$

$$B_{211} = -4S\beta S(\delta-\lambda)$$

$$B_{202} = C(\delta-\lambda+\alpha+\beta) - C\gamma$$

$$B_{200} = C(\delta-\lambda-\alpha-\beta) - C\gamma$$

$$B_{121} = 4S\delta S(\alpha-\beta)$$

$$B_{112} = -4S\beta S\delta$$

$$B_{110} = 4S\beta S\delta$$

$$B_{101} = 4S\delta S(\alpha+\beta)$$

$$B_{022} = C(\delta+\lambda-\alpha+\beta) - C\gamma$$

$$B_{020} = C(\delta+\lambda+\alpha-\beta) - C\gamma$$

$$B_{011} = 4S\beta S(\delta+\lambda)$$

$$B_{002} = C(\delta+\lambda-\alpha-\beta) - C\gamma$$

$$B_{000} = C(\delta+\lambda+\alpha+\beta) - C\gamma$$

$$B_2(\Phi,H)\Psi^2 + B_1(\Phi,H)\Psi + B_0(\Phi,H) = 0 \tag{4-6}$$

where $H = \tan(\eta/2)$

and

$$\begin{aligned}
B_2(\Phi,H) &= (B_{222}\Phi^2+B_{220})H^2 + B_{211}\Phi H + (B_{202}\Phi^2+B_{200}) \\
B_1(\Phi,H) &= B_{121}\Phi H^2 + (B_{112}\Phi^2+B_{110})H + B_{101}\Phi \\
B_0(\Phi,H) &= (B_{022}\Phi^2+B_{020})H^2 + B_{011}\Phi H + (B_{002}\Phi^2+B_{000})
\end{aligned} \tag{4-7}$$

The constants in Eqs. (4-7) are defined in Table II.

The quadratic equations (4-4) and (4-6) represent two different forms of displacement relationships for the same mechanism. They should, therefore, have at least one root in common between them. This gives the condition (Appendix B)

$$\begin{vmatrix} A_2(\Phi) & A_1(\Phi) & A_0(\Phi) & 0 \\ 0 & A_2(\Phi) & A_1(\Phi) & A_0(\Phi) \\ B_2(\Phi,H) & B_1(\Phi,H) & B_0(\Phi,H) & 0 \\ 0 & B_2(\Phi,H) & B_1(\Phi,H) & B_0(\Phi,H) \end{vmatrix} = 0 \tag{4-8}$$

Expanding and simplifying the above equation, we get

$$C_4(\Phi)H^4 + C_3(\Phi)H^3 + C_2(\Phi)H^2 + C_1(\Phi)H + C_0(\Phi) = 0 \tag{4-9}$$

The coefficients in Eq. (4-9) are polynomials in the variable Φ and are as follows:

$$\begin{aligned}
C_4(\Phi) &= C_{48}\Phi^8 + C_{46}\Phi^6 + C_{44}\Phi^4 + C_{42}\Phi^2 + C_{40} \\
C_3(\Phi) &= C_{37}\Phi^7 + C_{35}\Phi^5 + C_{33}\Phi^3 + C_{31}\Phi \\
C_2(\Phi) &= C_{28}\Phi^8 + C_{26}\Phi^6 + C_{24}\Phi^4 + C_{22}\Phi^2 + C_{20} \\
C_1(\Phi) &= C_{17}\Phi^7 + C_{15}\Phi^5 + C_{13}\Phi^3 + C_{11}\Phi \\
C_0(\Phi) &= C_{08}\Phi^8 + C_{06}\Phi^6 + C_{04}\Phi^4 + C_{02}\Phi^2 + C_{00}
\end{aligned} \tag{4-10}$$

The constants in Eqs. (4-10) are defined in Table III.

Note that no conditions have so far been imposed on the R-C-R-C-R mechanism under consideration. Eq. (4-10) is, therefore, valid for any R-C-R-C-R mechanism with zero offset distances at its revolute pairs.

Let the translations u and v at the two cylinder pairs be now reduced to zero at all positions of the mechanism.

Eliminating the angle ξ from the primary and dual parts of Eq. (4-3), we get

$$D_2(\Phi)H^2 + D_1(\Phi)H + D_0(\Phi) = 0 \tag{4-11}$$

$$\text{where}\quad \begin{aligned} D_2(\Phi) &= D_{22}\Phi^2 + D_{20} \\ D_1(\Phi) &= D_{11}\Phi \\ D_0(\Phi) &= D_{02}\Phi^2 + D_{00} \end{aligned} \tag{4-12}$$

The constants in Eqs. (4-12) are defined in Table IV. Observe also that Eq. (4-11) is similar in form to Eq. (4-4).

If a five-revolute mechanism of the type under consideration is to exist, the polynomial equations (4-9) and (4-11) must have at least one common root. This gives the condition (Appendix B)

$$\begin{vmatrix} C_4(\Phi) & C_3(\Phi) & C_2(\Phi) & C_1(\Phi) & C_0(\Phi) & 0 \\ 0 & C_4(\Phi) & C_3(\Phi) & C_2(\Phi) & C_1(\Phi) & C_0(\Phi) \\ D_2(\Phi) & D_1(\Phi) & D_0(\Phi) & 0 & 0 & 0 \\ 0 & D_2(\Phi) & D_1(\Phi) & D_0(\Phi) & 0 & 0 \\ 0 & 0 & D_2(\Phi) & D_1(\Phi) & D_0(\Phi) & 0 \\ 0 & 0 & 0 & D_2(\Phi) & D_1(\Phi) & D_0(\Phi) \end{vmatrix} = 0 \tag{4-13}$$

Eq. (4-13) is a function of only the variable Φ. Expanding and simplifying it, we get

TABLE III

CONSTANTS FOR USE IN EQS. (4-10) AND TABLE V

$$C_{48} = -(A_{22}B_{022}-A_{02}B_{222})^2$$

$$\begin{aligned} C_{46} = {} & A_{02}A_{22}[2(B_{020}B_{222}+B_{022}B_{220}) - B_{121}^2] \\ & - 2A_{22}B_{022}(A_{22}B_{020}+A_{20}B_{022}-A_{00}B_{222}) \\ & - 2A_{02}B_{222}(A_{02}B_{220}+A_{00}B_{222}-A_{20}B_{022}) \\ & + A_{11}[B_{121}(A_{22}B_{022}+A_{02}B_{222}) - A_{11}B_{022}B_{222}] \end{aligned}$$

$$\begin{aligned} C_{44} = {} & -(A_{22}B_{020}-A_{02}B_{220})^2 - (A_{20}B_{022}-A_{00}B_{222})^2 \\ & + A_{11}B_{121}(A_{22}B_{020}+A_{20}B_{022}+A_{02}B_{220}+A_{00}B_{222}) \\ & + (A_{00}A_{22}+A_{02}A_{20})[2(B_{020}B_{222}+B_{022}B_{220}) - B_{121}^2] \\ & - A_{11}^2(B_{020}B_{222}+B_{022}B_{220}) \\ & - 4(A_{20}A_{22}B_{020}B_{022}+A_{00}A_{02}B_{220}B_{222}) \end{aligned}$$

$$\begin{aligned} C_{42} = {} & A_{00}A_{20}[2(B_{020}B_{222}+B_{022}B_{220}) - B_{121}^2] \\ & - 2A_{20}B_{020}(A_{22}B_{020}+A_{20}B_{022}-A_{02}B_{220}) \\ & - 2A_{00}B_{220}(A_{02}B_{220}+A_{00}B_{222}-A_{22}B_{020}) \\ & + A_{11}[B_{121}(A_{20}B_{020}+A_{00}B_{220}) - A_{11}B_{020}B_{220}] \end{aligned}$$

$$C_{40} = -(A_{20}B_{020}-A_{00}B_{220})^2$$

TABLE III (CONTINUED)

$$C_{37} = 2A_{02}A_{22}(B_{011}B_{222}+B_{022}B_{211}-B_{112}B_{121})$$
$$+ A_{22}B_{022}(A_{11}B_{112}-2A_{22}B_{011})$$
$$+ A_{02}B_{222}(A_{11}B_{112}-2A_{02}B_{211})$$

$$C_{35} = 2A_{02}A_{22}(B_{011}B_{220}+B_{020}B_{211}-B_{110}B_{121})$$
$$+ A_{11}A_{22}(B_{011}B_{121}+B_{020}B_{112}+B_{022}B_{110})$$
$$+ A_{02}A_{11}(B_{110}B_{222}+B_{112}B_{220}+B_{121}B_{211})$$
$$+ 2(A_{00}A_{22}+A_{02}A_{20})(B_{011}B_{222}+B_{022}B_{211}-B_{112}B_{121})$$
$$+ A_{11}[B_{112}(A_{20}B_{022}+A_{00}B_{222}) - A_{11}(B_{011}B_{222}+B_{022}B_{211})]$$
$$- 2A_{22}B_{011}(A_{22}B_{020}+2A_{20}B_{022})$$
$$- 2A_{02}B_{211}(A_{02}B_{220}+2A_{00}B_{222})$$

$$C_{33} = 2A_{00}A_{20}(B_{011}B_{222}+B_{022}B_{211}-B_{112}B_{121})$$
$$+ A_{11}A_{20}(B_{011}B_{121}+B_{020}B_{112}+B_{022}B_{110})$$
$$+ A_{00}A_{11}(B_{110}B_{222}+B_{112}B_{220}+B_{121}B_{211})$$
$$+ 2(A_{00}A_{22}+A_{02}A_{20})(B_{011}B_{220}+B_{020}B_{211}-B_{110}B_{121})$$
$$+ A_{11}[B_{110}(A_{22}B_{020}+A_{02}B_{220}) - A_{11}(B_{011}B_{220}+B_{020}B_{211})]$$
$$- 2A_{20}B_{011}(A_{20}B_{022}+2A_{22}B_{020})$$
$$- 2A_{00}B_{211}(A_{00}B_{222}+2A_{02}B_{220})$$

$$C_{31} = 2A_{00}A_{20}(B_{011}B_{220}+B_{020}B_{211}-B_{110}B_{121})$$
$$+ A_{20}B_{020}(A_{11}B_{110}-2A_{20}B_{011})$$
$$+ A_{00}B_{220}(A_{11}B_{110}-2A_{00}B_{211})$$

TABLE III (CONTINUED)

$$C_{28} = 2A_{22}B_{002}(A_{02}B_{222}-A_{22}B_{022}) - A_{02}A_{22}B_{112}^{2}$$
$$+ 2A_{02}B_{202}(A_{22}B_{022}-A_{02}B_{222})$$

$$C_{26} = 2A_{02}A_{22}(B_{000}B_{222}+B_{002}B_{220}+B_{011}B_{211}+B_{020}B_{202}$$
$$+ B_{022}B_{200}-B_{101}B_{121}-B_{110}B_{112})$$
$$+ (A_{00}A_{22}+A_{02}A_{20})[2(B_{002}B_{222}+B_{022}B_{202}) - B_{112}^{2}]$$
$$- A_{11}^{2}(B_{002}B_{222}+B_{022}B_{202})$$
$$- A_{22}^{2}[2(B_{000}B_{022}+B_{002}B_{020}) + B_{011}^{2}]$$
$$- A_{02}^{2}[2(B_{200}B_{222}+B_{202}B_{220}) + B_{211}^{2}]$$
$$+ A_{11}[A_{22}(B_{002}B_{121}+B_{011}B_{112}+B_{022}B_{101})$$
$$+ A_{02}(B_{101}B_{222}+B_{112}B_{211}+B_{121}B_{202})]$$
$$- 4(A_{20}A_{22}B_{002}B_{022}+A_{00}A_{02}B_{202}B_{222})$$

TABLE III (CONTINUED)

$$
\begin{aligned}
C_{24} = \; & A_{02}A_{22}[2(B_{000}B_{220}+B_{020}B_{200}) - B_{110}^{2}] \\
& + A_{00}A_{20}[2(B_{002}B_{222}+B_{022}B_{202}) - B_{112}^{2}] \\
& + 2(A_{00}A_{22}+A_{02}A_{20})(B_{000}B_{222}+B_{002}B_{220}+B_{011}B_{211} \\
& \qquad + B_{020}B_{202}+B_{022}B_{200}-B_{101}B_{121}-B_{110}B_{112}) \\
& - A_{11}^{2}(B_{000}B_{222}+B_{002}B_{220}+B_{011}B_{211}+B_{020}B_{202}+B_{022}B_{200}) \\
& + A_{22}[A_{11}(B_{000}B_{121}+B_{011}B_{110}+B_{020}B_{101}) - 2A_{22}B_{000}B_{020}] \\
& + A_{20}[A_{11}(B_{002}B_{121}+B_{011}B_{112}+B_{022}B_{101}) - 2A_{20}B_{002}B_{022}] \\
& + A_{02}[A_{11}(B_{101}B_{220}+B_{110}B_{211}+B_{121}B_{200}) - 2A_{02}B_{200}B_{220}] \\
& + A_{00}[A_{11}(B_{101}B_{222}+B_{112}B_{211}+B_{121}B_{202}) - 2A_{00}B_{202}B_{222}] \\
& - 2A_{20}A_{22}[2(B_{000}B_{022}+B_{002}B_{020}) + B_{011}^{2}] \\
& - 2A_{00}A_{02}[2(B_{200}B_{222}+B_{202}B_{220}) + B_{211}^{2}]
\end{aligned}
$$

TABLE III (CONTINUED)

$$
\begin{aligned}
C_{22} \;=\;& 2A_{00}A_{20}(B_{000}B_{222}+B_{002}B_{220}+B_{011}B_{211}+B_{020}B_{202} \\
&\qquad + B_{022}B_{200}-B_{101}B_{121}-B_{110}B_{112}) \\
&+ (A_{00}A_{22}+A_{02}A_{20})[2(B_{000}B_{220}+B_{020}B_{200}) - B_{110}^2] \\
&- A_{11}^2(B_{000}B_{220}+B_{020}B_{200}) \\
&- A_{20}^2[2(B_{000}B_{022}+B_{002}B_{020}) + B_{011}^2] \\
&- A_{00}^2[2(B_{200}B_{222}+B_{202}B_{220}) + B_{211}^2] \\
&+ A_{11}[A_{20}(B_{000}B_{121}+B_{011}B_{110}+B_{020}B_{101}) \\
&\qquad + A_{00}(B_{101}B_{220}+B_{110}B_{211}+B_{121}B_{200})] \\
&- 4(A_{20}A_{22}B_{000}B_{020}+A_{00}A_{02}B_{200}B_{220}) \\
C_{20} \;=\;& 2A_{20}B_{000}(A_{00}B_{220}-A_{20}B_{020}) - A_{00}A_{20}B_{110}^2 \\
&+ 2A_{00}B_{200}(A_{20}B_{020}-A_{00}B_{220})
\end{aligned}
$$

TABLE III (CONTINUED)

$$C_{17} = 2A_{02}A_{22}(B_{002}B_{211}+B_{011}B_{202}-B_{101}B_{112})$$
$$+ A_{22}B_{002}(A_{11}B_{112}-2A_{22}B_{011})$$
$$+ A_{02}B_{202}(A_{11}B_{112}-2A_{02}B_{211})$$

$$C_{15} = 2A_{02}A_{22}(B_{000}B_{211}+B_{011}B_{200}-B_{101}B_{110})$$
$$+ A_{11}A_{22}(B_{000}B_{112}+B_{002}B_{110}+B_{011}B_{101})$$
$$+ A_{02}A_{11}(B_{101}B_{211}+B_{110}B_{202}+B_{112}B_{200})$$
$$+ 2(A_{00}A_{22}+A_{02}A_{20})(B_{002}B_{211}+B_{011}B_{202}-B_{101}B_{112})$$
$$+ A_{11}[B_{112}(A_{20}B_{002}+A_{00}B_{202}) - A_{11}(B_{002}B_{211}+B_{011}B_{202})]$$
$$- 2A_{22}B_{011}(A_{22}B_{000}+2A_{20}B_{002})$$
$$- 2A_{02}B_{211}(A_{02}B_{200}+2A_{00}B_{202})$$

$$C_{13} = 2A_{00}A_{20}(B_{002}B_{211}+B_{011}B_{202}-B_{101}B_{112})$$
$$+ A_{11}A_{20}(B_{000}B_{112}+B_{002}B_{110}+B_{011}B_{101})$$
$$+ A_{00}A_{11}(B_{101}B_{211}+B_{110}B_{202}+B_{112}B_{200})$$
$$+ 2(A_{00}A_{22}+A_{02}A_{20})(B_{000}B_{211}+B_{011}B_{200}-B_{101}B_{110})$$
$$+ A_{11}[B_{110}(A_{22}B_{000}+A_{02}B_{200}) - A_{11}(B_{000}B_{211}+B_{011}B_{200})]$$
$$- 2A_{20}B_{011}(A_{20}B_{002}+2A_{22}B_{000})$$
$$- 2A_{00}B_{211}(A_{00}B_{202}+2A_{02}B_{200})$$

$$C_{11} = 2A_{00}A_{20}(B_{000}B_{211}+B_{011}B_{200}-B_{101}B_{110})$$
$$+ A_{20}B_{000}(A_{11}B_{110}-2A_{20}B_{011})$$
$$+ A_{00}B_{200}(A_{11}B_{110}-2A_{00}B_{211})$$

TABLE III (CONTINUED)

$$
\begin{aligned}
C_{08} &= -(A_{22}B_{002}-A_{02}B_{202})^2 \\
C_{06} &= A_{02}A_{22}[2(B_{000}B_{202}+B_{002}B_{200}) - B_{101}^2] \\
&\quad - 2A_{22}B_{002}(A_{22}B_{000}+A_{20}B_{002}-A_{00}B_{202}) \\
&\quad - 2A_{02}B_{202}(A_{02}B_{200}+A_{00}B_{202}-A_{20}B_{002}) \\
&\quad + A_{11}[B_{101}(A_{22}B_{002}+A_{02}B_{202}) - A_{11}B_{002}B_{202}] \\
C_{04} &= -(A_{22}B_{000}-A_{02}B_{200})^2 - (A_{20}B_{002}-A_{00}B_{202})^2 \\
&\quad + A_{11}B_{101}(A_{22}B_{000}+A_{20}B_{002}+A_{02}B_{200}+A_{00}B_{202}) \\
&\quad + (A_{00}A_{22}+A_{02}A_{20})[2(B_{000}B_{202}+B_{002}B_{200}) - B_{101}^2] \\
&\quad - A_{11}^2(B_{000}B_{202}+B_{002}B_{200}) \\
&\quad - 4(A_{20}A_{22}B_{000}B_{002}+A_{00}A_{02}B_{200}B_{202}) \\
C_{02} &= A_{00}A_{20}[2(B_{000}B_{202}+B_{002}B_{200}) - B_{101}^2] \\
&\quad - 2A_{20}B_{000}(A_{22}B_{000}+A_{20}B_{002}-A_{02}B_{200}) \\
&\quad - 2A_{00}B_{200}(A_{02}B_{200}+A_{00}B_{202}-A_{22}B_{000}) \\
&\quad + A_{11}[B_{101}(A_{20}B_{000}+A_{00}B_{200}) - A_{11}B_{000}B_{200}] \\
C_{00} &= -(A_{20}B_{000}-A_{00}B_{200})^2
\end{aligned}
$$

TABLE IV

CONSTANTS FOR USE IN EQS. (4-12)
AND TABLE VI

$$D_{22} = V_1 - V_2C(\lambda-\alpha+\beta) - (e-a+b)S\gamma S\delta S(\lambda-\alpha+\beta)$$

$$D_{20} = V_1 - V_2C(\lambda+\alpha-\beta) - (e+a-b)S\gamma S\delta S(\lambda+\alpha-\beta)$$

$$D_{11} = 4(bC\beta S\gamma S\delta S\lambda - cS\beta C\gamma S\delta S\lambda - dS\beta S\gamma C\delta S\lambda + eS\beta S\gamma S\delta C\lambda)$$

$$D_{02} = V_1 - V_2C(\lambda-\alpha-\beta) - (e-a-b)S\gamma S\delta S(\lambda-\alpha-\beta)$$

$$D_{00} = V_1 - V_2C(\lambda+\alpha+\beta) - (e+a+b)S\gamma S\delta S(\lambda+\alpha+\beta)$$

where $V_1 = cS\delta C\delta + dS\gamma C\gamma$

and $V_2 = cC\gamma S\delta + dS\gamma C\delta$

$$R_{24}\Phi^{24} + R_{22}\Phi^{22} + \ldots + R_2\Phi^2 + R_0 = 0$$

or, in short,

$$\sum_{i=0}^{12} R_{(2i)}\Phi^{(2i)} = 0 \qquad (4\text{-}14)$$

The constants in Eq. (4-14) are defined in Table VII. The terms used in Table VII are defined in Tables V and VI.

Now, by considering the relationship $f(\hat{\phi},\hat{\psi},\hat{\xi}) = 0$ instead of Eq. (4-2) and the relationship $F(\hat{\psi},\hat{\xi},\hat{\eta}) = 0$ instead of Eq. (4-3), and by following a procedure similar to that described above, we can get the equation

$$\sum_{i=0}^{12} S_{(2i)}\Psi^{(2i)} = 0 \qquad (4\text{-}15)$$

Eq. (4-15) is exactly similar in form to Eq. (4-14). Its coefficients can be obtained from the coefficients of Eq. (4-14) by replacing the parameters a, b, c, d, α, β, γ and δ by the parameters d, c, b, a, δ, γ, β and α respectively. In other words, the coefficients of Eq. (4-15) can be regarded as "mirror images" of the coefficients of Eq. (4-14) and can be obtained from the latter by the transformations $a \leftrightarrow d$, $b \leftrightarrow c$, $\alpha \leftrightarrow \delta$ and $\beta \leftrightarrow \gamma$.

Observe that each of the equations (4-14) and (4-15) consists of only one variable. These two equations must hold good at all values of the variables involved. Their coefficients must, therefore, vanish [5]. This gives

$$R_{(2i)} = 0, \quad i = 0, 1, 2, \ldots, 12 \qquad (4\text{-}16)$$

$$S_{(2i)} = 0, \quad i = 0, 1, 2, \ldots, 12 \qquad (4\text{-}17)$$

Conditions (4-16) and (4-17) together represent 26 equations among the ten constant kinematic parameters of the five-revolute mechanism in Fig. 12 (namely, the five link lengths a, b, c, d and e and the five

TABLE V

CONSTANTS FOR USE IN TABLE VII

when $i = 1, 2, 3,$ $j = 0, 2, 4$

$$P_{i16} = C^2_{j8}$$
$$P_{i14} = 2C_{j8}C_{j6}$$
$$P_{i12} = 2C_{j8}C_{j4} + C^2_{j6}$$
$$P_{i10} = 2C_{j8}C_{j2} + 2C_{j6}C_{j4}$$
$$P_{i08} = 2C_{j8}C_{j0} + 2C_{j6}C_{j2} + C^2_{j4}$$
$$P_{i06} = 2C_{j6}C_{j0} + 2C_{j4}C_{j2}$$
$$P_{i04} = 2C_{j4}C_{j0} + C^2_{j2}$$
$$P_{i02} = 2C_{j2}C_{j0}$$
$$P_{i00} = C^2_{j0}$$

when $i = 4, 5, 6,$ $j = 0, 0, 2$ and $k = 2, 4, 4$

$$P_{i16} = C_{j8}C_{k8}$$
$$P_{i14} = C_{j8}C_{k6} + C_{j6}C_{k8}$$
$$P_{i12} = C_{j8}C_{k4} + C_{j6}C_{k6} + C_{j4}C_{k8}$$
$$P_{i10} = C_{j8}C_{k2} + C_{j6}C_{k4} + C_{j4}C_{k6} + C_{j2}C_{k8}$$
$$P_{i08} = C_{j8}C_{k0} + C_{j6}C_{k2} + C_{j4}C_{k4} + C_{j2}C_{k6} + C_{j0}C_{k8}$$
$$P_{i06} = C_{j6}C_{k0} + C_{j4}C_{k2} + C_{j2}C_{k4} + C_{j0}C_{k6}$$
$$P_{i04} = C_{j4}C_{k0} + C_{j2}C_{k2} + C_{j0}C_{k4}$$
$$P_{i02} = C_{j2}C_{k0} + C_{j0}C_{k2}$$
$$P_{i00} = C_{j0}C_{k0}$$

TABLE V (CONTINUED)

$$P_{i15} = C_{j8}C_{k7}$$
$$P_{i13} = C_{j8}C_{k5} + C_{j6}C_{k7}$$
$$P_{i11} = C_{j8}C_{k3} + C_{j6}C_{k5} + C_{j4}C_{k7}$$
$$P_{i09} = C_{j8}C_{k1} + C_{j6}C_{k3} + C_{j4}C_{k5} + C_{j2}C_{k7}$$
$$P_{i07} = C_{j6}C_{k1} + C_{j4}C_{k3} + C_{j2}C_{k5} + C_{j0}C_{k7}$$
$$P_{i05} = C_{j4}C_{k1} + C_{j2}C_{k3} + C_{j0}C_{k5}$$
$$P_{i03} = C_{j2}C_{k1} + C_{j0}C_{k3}$$
$$P_{i01} = C_{j0}C_{k1}$$

when i = 7, 8, 9, 10, 11, 12,
j = 0, 2, 0, 2, 4, 4
and k = 1, 1, 3, 3, 1, 3

$$P_{i14} = C_{j7}^2$$
$$P_{i12} = 2C_{j7}C_{j5}$$
$$P_{i10} = 2C_{j7}C_{j3} + C_{j5}^2$$
$$P_{i08} = 2C_{j7}C_{j1} + 2C_{j5}C_{j3}$$
$$P_{i06} = 2C_{j5}C_{j1} + C_{j3}^2$$
$$P_{i04} = 2C_{j3}C_{j1}$$
$$P_{i02} = C_{j1}^2$$

when i = 13, 14,
j = 1, 3

$$P_{1514} = C_{17}C_{37}$$
$$P_{1512} = C_{17}C_{35} + C_{15}C_{37}$$
$$P_{1510} = C_{17}C_{33} + C_{15}C_{35} + C_{13}C_{37}$$
$$P_{1508} = C_{17}C_{31} + C_{15}C_{33} + C_{13}C_{35} + C_{11}C_{37}$$
$$P_{1506} = C_{15}C_{31} + C_{13}C_{33} + C_{11}C_{35}$$
$$P_{1504} = C_{13}C_{31} + C_{11}C_{33}$$
$$P_{1502} = C_{11}C_{31}$$

TABLE VI

CONSTANTS FOR USE IN TABLE VII

$$Q_{108} = D_{22}^4$$

$$Q_{106} = 4D_{22}^3 D_{20}$$

$$Q_{104} = 6D_{22}^2 D_{20}^2$$

$$Q_{102} = 4D_{22} D_{20}^3$$

$$Q_{100} = D_{20}^4$$

$$Q_{208} = D_{02}^2 D_{22}^2$$

$$Q_{206} = 2D_{02} D_{22} (D_{02} D_{20} + D_{22} D_{00})$$

$$Q_{204} = D_{02}^2 D_{20}^2 + 4D_{02} D_{00} D_{22} D_{20} + D_{22}^2 D_{00}^2$$

$$Q_{202} = 2D_{00} D_{20} (D_{02} D_{20} + D_{00} D_{22})$$

$$Q_{200} = D_{00}^2 D_{20}^2$$

$$Q_{308} = D_{02}^4$$

$$Q_{306} = 4D_{02}^3 D_{00}$$

$$Q_{304} = 6D_{02}^2 D_{00}^2$$

$$Q_{302} = 4D_{02} D_{00}^3$$

$$Q_{300} = D_{00}^4$$

TABLE VI (CONTINUED)

$$Q_{408} = -2D_{02}D_{22}^{3}$$

$$Q_{406} = D_{22}^{2}[D_{11}^{2} - 2(3D_{02}D_{20}+D_{00}D_{22})]$$

$$Q_{404} = 2D_{22}D_{20}[D_{11}^{2} - 3(D_{02}D_{20}+D_{00}D_{22})]$$

$$Q_{402} = D_{20}^{2}[D_{11}^{2} - 2(D_{02}D_{20}+3D_{00}D_{22})]$$

$$Q_{400} = -2D_{00}D_{20}^{3}$$

$$Q_{508} = 2Q_{208}$$

$$Q_{506} = 2(Q_{206}-2D_{11}^{2}D_{02}D_{22})$$

$$Q_{504} = D_{11}^{2}[D_{11}^{2} - 4(D_{00}D_{22}+D_{02}D_{20})] + 2Q_{204}$$

$$Q_{502} = 2(Q_{202}-2D_{11}^{2}D_{00}D_{20})$$

$$Q_{500} = 2Q_{200}$$

$$Q_{608} = -2D_{02}^{3}D_{22}$$

$$Q_{606} = D_{02}^{2}[D_{11}^{2} - 2(D_{02}D_{20}+3D_{00}D_{22})]$$

$$Q_{604} = 2D_{02}D_{00}[D_{11}^{2} - 3(D_{02}D_{20}+D_{00}D_{22})]$$

$$Q_{602} = D_{00}^{2}[D_{11}^{2} - 2(3D_{02}D_{20}+D_{00}D_{22})]$$

$$Q_{600} = -2D_{00}^{3}D_{20}$$

TABLE VI (CONTINUED)

$$Q_{707} = -D_{11}D_{22}^3$$
$$Q_{705} = -3D_{11}D_{22}^2D_{20}$$
$$Q_{703} = -3D_{11}D_{22}D_{20}^2$$
$$Q_{701} = -D_{11}D_{20}^3$$

$$Q_{807} = -D_{11}D_{02}D_{22}^2$$
$$Q_{805} = -D_{11}D_{22}(2D_{02}D_{20}+D_{00}D_{22})$$
$$Q_{803} = -D_{11}D_{20}(D_{02}D_{20}+2D_{00}D_{22})$$
$$Q_{801} = -D_{11}D_{00}D_{20}^2$$

$$Q_{907} = -3Q_{807}$$
$$Q_{905} = -D_{11}^3D_{22}-3Q_{805}$$
$$Q_{903} = -D_{11}^3D_{20}-3Q_{803}$$
$$Q_{901} = -3Q_{801}$$

TABLE VI (CONTINUED)

$$Q_{1007} = -D_{11}D_{22}D_{02}^2$$

$$Q_{1005} = -D_{11}D_{02}(2D_{22}D_{00}+D_{20}D_{02})$$

$$Q_{1003} = -D_{11}D_{00}(D_{22}D_{00}+2D_{20}D_{02})$$

$$Q_{1001} = -D_{11}D_{20}D_{00}^2$$

$$Q_{1107} = -3Q_{1007}$$

$$Q_{1105} = -D_{11}^3D_{02} - 3Q_{1005}$$

$$Q_{1103} = -D_{11}^3D_{00} - 3Q_{1003}$$

$$Q_{1101} = -3Q_{1001}$$

$$Q_{1207} = -D_{11}D_{02}^3$$

$$Q_{1205} = -3D_{11}D_{02}^2D_{00}$$

$$Q_{1203} = -3D_{11}D_{02}D_{00}^2$$

$$Q_{1201} = -D_{11}D_{00}^3$$

TABLE VI (CONTINUED)

$$Q_{1308} = D_{02}D_{22}^3$$

$$Q_{1306} = D_{22}^2(3D_{02}D_{20}+D_{00}D_{22})$$

$$Q_{1304} = 3D_{22}D_{20}(D_{02}D_{20}+D_{00}D_{22})$$

$$Q_{1302} = D_{20}^2(D_{02}D_{20}+3D_{00}D_{22})$$

$$Q_{1300} = D_{00}D_{20}^3$$

$$Q_{1408} = D_{02}^3D_{22}$$

$$Q_{1406} = D_{02}^2(D_{02}D_{20}+3D_{00}D_{22})$$

$$Q_{1404} = 3D_{02}D_{00}(D_{02}D_{20}+D_{00}D_{22})$$

$$Q_{1402} = D_{00}^2(3D_{02}D_{20}+D_{00}D_{22})$$

$$Q_{1400} = D_{00}^3D_{20}$$

$$Q_{1508} = -2Q_{208}$$

$$Q_{1506} = D_{11}^2D_{02}D_{22} - 2Q_{206}$$

$$Q_{1504} = D_{11}^2(D_{00}D_{22}+D_{02}D_{20}) - 2Q_{204}$$

$$Q_{1502} = D_{11}^2D_{00}D_{20} - 2Q_{202}$$

$$Q_{1500} = -2Q_{200}$$

TABLE VII

CONSTANTS FOR USE IN EQS. (4-14) AND (4-16)

$$R_{24} = \sum_{i=1}^{6} [P_{i16}Q_{i08}]$$

$$R_{22} = \sum_{i=1}^{6} [P_{i16}Q_{i06} + P_{i14}Q_{i08}]$$

$$+ \sum_{i=7}^{12} [P_{i15}Q_{i07}]$$

$$+ \sum_{i=13}^{15} [P_{i14}Q_{i08}]$$

$$R_{20} = \sum_{i=1}^{6} [P_{i16}Q_{i04} + P_{i14}Q_{i06} + P_{i12}Q_{i08}]$$

$$+ \sum_{i=7}^{12} [P_{i15}Q_{i05} + P_{i13}Q_{i07}]$$

$$+ \sum_{i=13}^{15} [P_{i14}Q_{i06} + P_{i12}Q_{i08}]$$

$$R_{18} = \sum_{i=1}^{6} [P_{i16}Q_{i02} + P_{i14}Q_{i04} + P_{i12}Q_{i06} + P_{i10}Q_{i08}]$$

$$+ \sum_{i=7}^{12} [P_{i15}Q_{i03} + P_{i13}Q_{i05} + P_{i11}Q_{i07}]$$

$$+ \sum_{i=13}^{15} [P_{i14}Q_{i04} + P_{i12}Q_{i06} + P_{i10}Q_{i08}]$$

$$R_{16} = \sum_{i=1}^{6} [P_{i16}Q_{i00} + P_{i14}Q_{i02} + P_{i12}Q_{i04} + P_{i10}Q_{i06} + P_{i08}Q_{i08}]$$

$$+ \sum_{i=7}^{12} [P_{i15}Q_{i01} + P_{i13}Q_{i03} + P_{i11}Q_{i05} + P_{i09}Q_{i07}]$$

$$+ \sum_{i=13}^{15} [P_{i14}Q_{i02} + P_{i12}Q_{i04} + P_{i10}Q_{i06} + P_{i08}Q_{i08}]$$

TABLE VII (CONTINUED)

$$R_{14} = \sum_{i=1}^{6}[P_{i14}Q_{i00} + P_{i12}Q_{i02} + P_{i10}Q_{i04} + P_{i08}Q_{i06} + P_{i06}Q_{i08}]$$

$$+ \sum_{i=7}^{12}[P_{i13}Q_{i01} + P_{i11}Q_{i03} + P_{i09}Q_{i05} + P_{i07}Q_{i07}]$$

$$+ \sum_{i=13}^{15}[P_{i14}Q_{i00} + P_{i12}Q_{i02} + P_{i10}Q_{i04} + P_{i08}Q_{i06} + P_{i06}Q_{i08}]$$

$$R_{12} = \sum_{i=1}^{6}[P_{i12}Q_{i00} + P_{i10}Q_{i02} + P_{i08}Q_{i04} + P_{i06}Q_{i06} + P_{i04}Q_{i08}]$$

$$+ \sum_{i=7}^{12}[P_{i11}Q_{i01} + P_{i09}Q_{i03} + P_{i07}Q_{i05} + P_{i05}Q_{i07}]$$

$$+ \sum_{i=13}^{15}[P_{i12}Q_{i00} + P_{i10}Q_{i02} + P_{i08}Q_{i04} + P_{i06}Q_{i06} + P_{i04}Q_{i08}]$$

$$R_{10} = \sum_{i=1}^{6}[P_{i10}Q_{i00} + P_{i08}Q_{i02} + P_{i06}Q_{i04} + P_{i04}Q_{i06} + P_{i02}Q_{i08}]$$

$$+ \sum_{i=7}^{12}[P_{i09}Q_{i01} + P_{i07}Q_{i03} + P_{i05}Q_{i05} + P_{i03}Q_{i07}]$$

$$+ \sum_{i=13}^{15}[P_{i10}Q_{i00} + P_{i08}Q_{i02} + P_{i06}Q_{i04} + P_{i04}Q_{i06} + P_{i02}Q_{i08}]$$

$$R_{8} = \sum_{i=1}^{6}[P_{i08}Q_{i00} + P_{i06}Q_{i02} + P_{i04}Q_{i04} + P_{i02}Q_{i06} + P_{i00}Q_{i08}]$$

$$+ \sum_{i=7}^{12}[P_{i07}Q_{i01} + P_{i05}Q_{i03} + P_{i03}Q_{i05} + P_{i01}Q_{i07}]$$

$$+ \sum_{i=13}^{15}[P_{i08}Q_{i00} + P_{i06}Q_{i02} + P_{i04}Q_{i04} + P_{i02}Q_{i06}]$$

TABLE VII (CONTINUED)

$$R_6 = \sum_{i=1}^{6} [P_{i06}Q_{i00} + P_{i04}Q_{i02} + P_{i02}Q_{i04} + P_{i00}Q_{i06}]$$

$$+ \sum_{i=7}^{12} [P_{i05}Q_{i01} + P_{i03}Q_{i03} + P_{i01}Q_{i05}]$$

$$+ \sum_{i=13}^{15} [P_{i06}Q_{i00} + P_{i04}Q_{i02} + P_{i02}Q_{i04}]$$

$$R_4 = \sum_{i=1}^{6} [P_{i04}Q_{i00} + P_{i02}Q_{i02} + P_{i00}Q_{i04}]$$

$$+ \sum_{i=7}^{12} [P_{i03}Q_{i01} + P_{i01}Q_{i03}]$$

$$+ \sum_{i=13}^{15} [P_{i04}Q_{i00} + P_{i02}Q_{i02}]$$

$$R_2 = \sum_{i=1}^{6} [P_{i02}Q_{i00} + P_{i00}Q_{i02}]$$

$$+ \sum_{i=7}^{12} [P_{i01}Q_{i01}]$$

$$+ \sum_{i=13}^{15} [P_{i02}Q_{i00}]$$

$$R_0 = \sum_{i=1}^{6} [P_{i00}Q_{i00}]$$

twist angles α, β, γ, δ and λ). These 26 equations provide the necessary conditions for the existence of a five-link, five-revolute (R-R-R-R-R) mechanism with zero offset distances along its pair axes.

The Goldberg Five-Revolute Mechanisms

The Goldberg five-revolute mechanisms [20] (Fig. 9) are obtained by the combination (addition or subtraction) of two Bennett mechanisms [4]. Referring to the five-revolute mechanism in Fig. 12, the Goldberg mechanisms satisfy the following relationships:

$$\begin{aligned} a &= d \\ \alpha &= \delta \\ e &= b \pm c \\ \lambda &= \beta \pm \gamma \\ \frac{a}{S\alpha} &= \pm \frac{b}{S\beta} = \pm \frac{c}{S\gamma} \end{aligned} \tag{4-18}$$

When the relationships (4-18) are used, the 26 equations given by the conditions (4-16) and (4-17) are identically satisfied. This confirms the correctness and validity of the derived existence criteria.

On Obtaining Five-Revolute Mechanisms from the Derived Criteria

The existence criteria derived above can be used to obtain the constant kinematic parameters of a five-revolute mechanism with zero offset distances.

If the constant kinematic parameters are regarded as unknowns, it is possible to write each of the 26 equations given by conditions (4-16) and (4-17) as a polynomial equation in several variables. The entire set of 26 equations can, therefore, be regarded as a system of non-linear, simultaneous algebraic equations and can be represented as

$$F_i(a,b,c,d,e,\alpha,\beta,\gamma,\delta,\lambda) = 0, \quad i = 1, 2, 3, \ldots, 26 \qquad (4\text{-}19)$$

Eqs. (4-19) represent a system of 26 equations in the ten unknown constant kinematic parameters. They are of eighth degree in each of the link lengths and of twenty-fourth degree in each of the twist angles.

It is important to note that the equations given by (4-19) represent only necessary conditions for the existence of a five-revolute mechanism. The conditions are not sufficient because satisfaction of the criteria does not by itself guarantee a five-revolute mechanism with a true mobility[1] of one. This is because Eqs. (4-19) also have solutions that correspond to five-revolute mechanisms without a true mobility of one. Such solutions are called here trivial solutions.

There are two types of trivial solutions to be considered. The first type gives an overconstrained mechanism with mobility greater than one. Thus, for instance, Eqs. (4-19) are satisfied identically when $a = b = c = d = e = 0$ and α, β, γ, δ and λ have arbitrary values. This, however, represents a spherical five-revolute mechanism with mobility two.

The second type of trivial solution gives an overconstrained mechanism without true mobility. Thus, for example, Eqs. (4-19) are satisfied identically when any three adjacent twist angles are zero and the other constant kinematic parameters have arbitrary values. This gives a configuration in which the axes of four of the revolute pairs are parallel to one another. However, with this arrangement, no motion is

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[1]See item 4 in Appendix A.

possible at the remaining revolute pair which, therefore, remains permanently locked. The mechanism behaves, in effect, like a plane four-link mechanism. The solution thus represents a five-revolute mechanism without true mobility.

A non-trivial solution of Eqs. (4-19) yields a five-revolute mechanism with a true mobility of one. The Goldberg five-revolute mechanisms are examples of such non-trivial solutions.

The triviality or non-triviality of the solutions of Eq. (4-19) can be checked by substituting the values of the constant kinematic parameters in the original displacement relationships of the parent R-C-R-C-R mechanism [56]. A non-trivial solution will give zero offset distances at the two cylinder pairs at all positions of the parent mechanism without, at the same time, affecting its true mobility. A trivial solution will not meet these requirements.

Since Eqs. (4-19) have trivial solutions and are also satisfied by the Goldberg mechanisms, it is clear that they represent a set of consistent equations. The complexity of the equations, however, makes it very difficult to examine their relationship analytically. Since the equations involve ten unknowns, a maximum of only ten of the 26 equations can be expected to be independent. However, since Eqs. (4-18) satisfy Eqs. (4-19) identically and since four of the parameters in Eqs. (4-18) can be chosen arbitrarily,[2] it is clear that there is an infinite number of non-trivial solutions to the system (4-19). This indicates that the actual number of independent equations in the system

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[2]Eqs. (4-18) show that it is not possible to choose <u>any</u> four of the parameters arbitrarily, but only certain combinations of four parameters. Thus, for example, a, α, β and γ can all be chosen arbitrarily, but a, d, α and δ cannot all be chosen arbitrarily.

is less than ten. Eqs. (4-18) suggest that this number may be six.

The solution of the independent equations in (4-19) can be attempted by numerical means (Appendix E). The extremely high nonlinearity of the equations, however, indicates that the potential number of solutions to such a system is, as is evident from Bezout's theorem [38],[3] in the millions. The solution of the system, therefore, poses many problems [28, 37, 34].

The points discussed above clearly show that the investigation of the existence of new five-revolute mechanisms by using the derived criteria is a problem in its own right. It is, therefore, considered beyond the scope of the present work.

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[3]According to Bezout's theorem, the number of roots of a system of polynomial equations is equal to the product of the degrees of the individual polynomial equations.

CHAPTER V

EXISTENCE CRITERIA OF THE FIVE-LINK R-R-R-P-R MECHANISM

The five-link R-R-R-P-R mechanism can be derived, like the five-link, five-revolute mechanism, from either the R-C-R-C-R mechanism or the R-R-C-C-R mechanism. In this chapter, the Dimentberg passive coupling method has been used to obtain the existence criteria of an R-R-R-P-R mechanism with zero offset distances at its revolute pairs from the displacement relationships of an R-C-R-C-R mechanism.

Derivation of the Existence Criteria

Consider the five-link R-C-R-C-R space mechanism shown schematically in Fig. 13.[1] Note that the constant offset distances at the three revolute pairs are taken to be zero. If the translation u at the cylinder pair at B reduces to zero and the angular displacement ξ at the cylinder pair at D remains constant at all positions of this mechanism, then it reduces to an R-R-R-P-R mechanism with zero offset distances at its revolute pairs (Fig. 14).

By considering the loop-closure condition of the mechanism in Fig. 13 in two different ways, the following relationships can be obtained:

$$\begin{aligned} F(\hat{\chi},\hat{\eta},\hat{\psi}) \quad = \quad & (S\hat{\alpha}S\hat{\gamma}S\hat{\eta})S\hat{\chi} - S\hat{\gamma}(C\hat{\alpha}S\hat{\beta}+S\hat{\alpha}C\hat{\beta}C\hat{\eta})C\hat{\chi} \\ & + C\hat{\gamma}(C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta}) - (C\hat{\delta}C\hat{\lambda} \;\; S\hat{\delta}S\hat{\lambda}C\hat{\psi}) \quad = \quad 0 \qquad (5\text{-}1) \end{aligned}$$

[1]Fig. 13 is the same as Fig. 11.

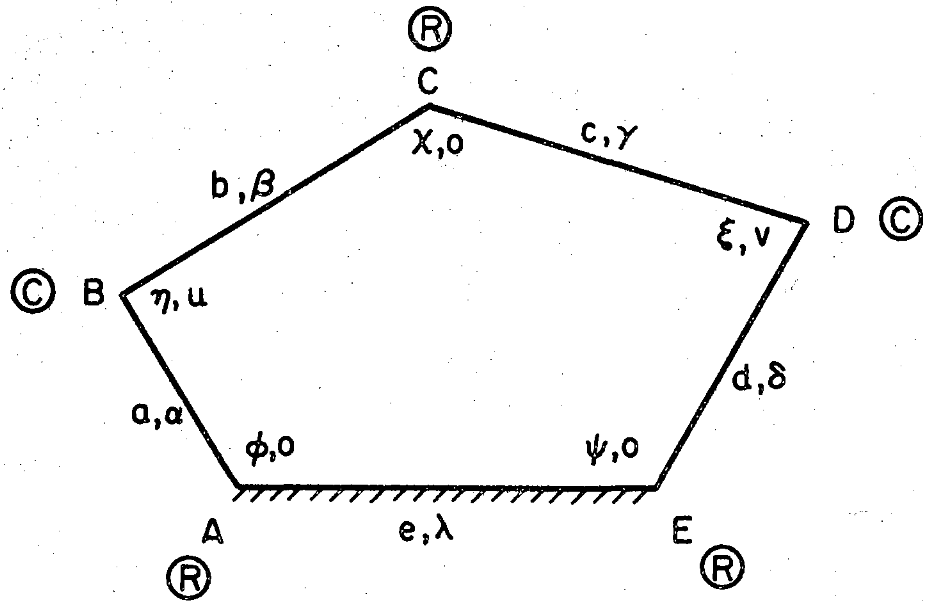

Figure 13. R–C–R–C–R Space Mechanism

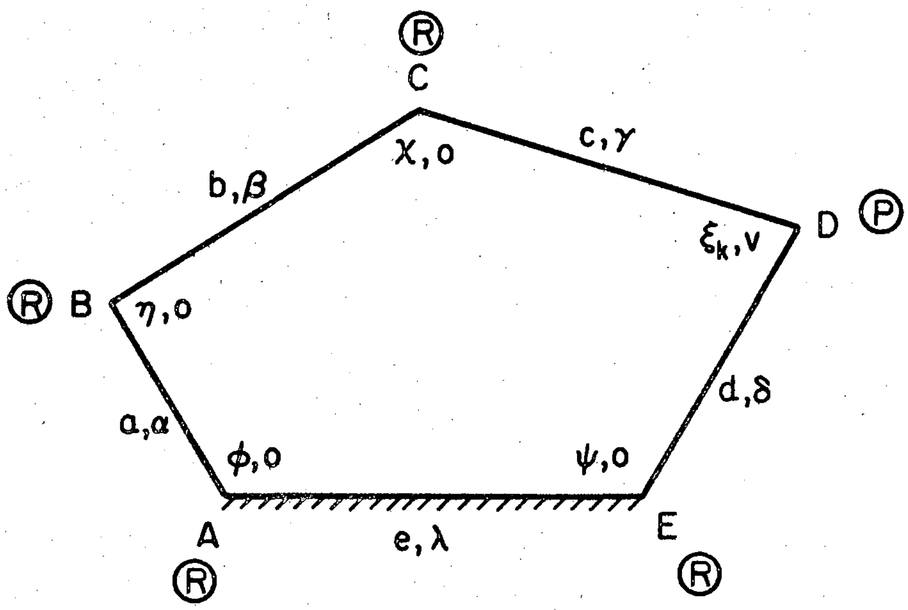

Figure 14. R–R–R–P–R Space Mechanism Obtained from the Mechanism in Fig. 13 by Making u = 0 and $\xi = \xi_k$ = a Constant

$$f(\hat{\xi},\hat{\chi},\hat{\eta}) = [(S\hat{\gamma}C\hat{\delta}+C\hat{\gamma}S\hat{\delta}C\hat{\xi})S\hat{\chi} + S\hat{\delta}S\hat{\xi}C\hat{\chi}](S\hat{\alpha}S\hat{\eta})$$
$$+ [S\hat{\delta}S\hat{\xi}S\hat{\chi} - (S\hat{\gamma}C\hat{\delta}+C\hat{\gamma}S\hat{\delta}C\hat{\xi})C\hat{\chi}](C\hat{\alpha}S\hat{\beta}+S\hat{\alpha}C\hat{\beta}C\hat{\eta})$$
$$+ (C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi})(C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta}) - C\hat{\lambda} = 0 \qquad (5\text{-}2)$$

Note that Eq. (5-1) is similar in form to Eq. (3-6) and Eq. (5-2) is similar to Eq. (3-8).

Now, let the translation u become zero and the angle ξ a constant at all positions of the mechanism.

Eliminating the angle ψ from the primary and dual parts of Eq. (5-1), we get

$$L_2(X)H^2 + L_1(X)H + L_0(X) = 0 \qquad (5\text{-}3)$$

where $X = \tan(\chi/2)$

$H = \tan(\eta/2)$

and
$$\begin{aligned} L_2(X) &= L_{22}X^2 + L_{20} \\ L_1(X) &= L_{11}X \\ L_0(X) &= L_{02}X^2 + L_{00} \end{aligned} \qquad (5\text{-}4)$$

The constants in Eqs. (5-4) involve only the constant kinematic parameters of the mechanism and are defined in Table VIII.

Denoting the constant value of the angle ξ by ξ_k, the primary part of Eq. (5-2) becomes

$$M_2(X)H^2 + M_1(X)H + M_0(X) = 0 \qquad (5\text{-}5)$$

where
$$\begin{aligned} M_2(X) &= M_{22}X^2 + M_{21}X + M_{20} \\ M_1(X) &= M_{12}X^2 + M_{11}X + M_{10} \\ M_0(X) &= M_{02}X^2 + M_{01}X + M_{00} \end{aligned} \qquad (5\text{-}6)$$

TABLE VIII

CONSTANTS FOR USE IN EQS. (5-4) AND (5-6) AND TABLE IX

$$L_{22} = W_1 - W_2C(\alpha-\beta+\gamma) - (a-b+c)S\delta S\lambda S(\alpha-\beta+\gamma)$$

$$L_{20} = W_1 - W_2C(\alpha-\beta-\gamma) - (a-b-c)S\delta S\lambda S(\alpha-\beta-\gamma)$$

$$L_{11} = 4(cC\gamma S\delta S\lambda S\alpha - dS\gamma C\delta S\lambda S\alpha - eS\gamma S\delta C\lambda S\alpha + aS\gamma S\delta S\lambda C\alpha)$$

$$L_{02} = W_1 - W_2C(\alpha+\beta-\gamma) - (a+b-c)S\delta S\lambda S(\alpha+\beta-\gamma)$$

$$L_{00} = W_1 - W_2C(\alpha+\beta+\gamma) - (a+b+c)S\delta S\lambda S(\alpha+\beta+\gamma)$$

where $W_1 = dS\lambda C\lambda + eS\delta C\delta$

and $W_2 = dC\delta S\lambda + eS\delta C\lambda$

$$M_{22} = \Xi_k^2[C(\alpha-\beta+\gamma-\delta)-C\lambda] + [C(\alpha-\beta+\gamma+\delta)-C\lambda]$$

$$M_{21} = -4\Xi_k S\delta S(\alpha-\beta)$$

$$M_{20} = \Xi_k^2[C(\alpha-\beta-\gamma+\delta)-C\lambda] + [C(\alpha-\beta-\gamma-\delta)-C\lambda]$$

$$M_{12} = -4\Xi_k S\alpha S\delta$$

$$M_{11} = 4S\alpha[\Xi_k^2 S(\gamma-\delta) + S(\gamma+\delta)]$$

$$M_{10} = 4\Xi_k S\alpha S\delta$$

$$M_{02} = \Xi_k^2[C(\alpha+\beta-\gamma+\delta)-C\lambda] + [C(\alpha+\beta-\gamma-\delta)-C\lambda]$$

$$M_{01} = 4\Xi_k S\delta S(\alpha+\beta)$$

$$M_{00} = \Xi_k^2[C(\alpha+\beta+\gamma-\delta)-C\lambda] + [C(\alpha+\beta+\gamma+\delta)-C\lambda]$$

The constants in Eqs. (5-6) are defined in Table VIII.[2]

If an R-R-R-P-R mechanism of the type under consideration is to exist, the quadratic equations (5-3) and (5-5) must have at least one common root. This gives the condition (Appendix B)

$$\begin{vmatrix} L_2(X) & L_1(X) & L_0(X) & 0 \\ 0 & L_2(X) & L_1(X) & L_0(X) \\ M_2(X) & M_1(X) & M_0(X) & 0 \\ 0 & M_2(X) & M_1(X) & M_0(X) \end{vmatrix} = 0 \tag{5-7}$$

Eq. (5-7) is a function of only the variable X. Expanding and simplifying it, we get

$$N_8X^8 + N_7X^7 + \ldots + N_1X + N_0 = 0$$

or, in short,

$$\sum_{i=0}^{8} N_iX^i = 0, \quad i = 0, 1, 2, \ldots, 8 \tag{5-8}$$

The constants in the above equation are defined in Table IX.

Eq. (5-8) must hold good at all values of the variable X. Its coefficients must, therefore, vanish. Thus, we have

$$N_i = 0, \quad i = 0, 1, 2, \ldots, 8 \tag{5-9}$$

Condition (5-9) represents nine equations among the 11 constant kinematic parameters of the R-R-R-P-R mechanism in Fig. 14 (namely, the five link lengths a, b, c, d and e, the five twist angles α, β, γ, δ and λ and the constant displacement angle ξ_k at the prismatic pair at joint D). These nine equations provide the necessary conditions for the existence of a five-link R-R-R-P-R mechanism with zero offset distances

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[2]In Table VIII, the constant $\tan(\xi_k/2)$ is denoted by Ξ_k.

TABLE IX

CONSTANTS FOR USE IN EQS. (5-8) AND (5-9)

$$N_8 = -(L_{22}M_{02}-L_{02}M_{22})^2 - L_{02}L_{22}M_{12}^2$$

$$\begin{aligned} N_7 = {} & 2L_{02}L_{22}(M_{01}M_{22}+M_{02}M_{21}-M_{11}M_{12}) \\ & + L_{22}M_{02}(L_{11}M_{12}-2L_{22}M_{01}) \\ & + L_{02}M_{22}(L_{11}M_{12}-2L_{02}M_{21}) \end{aligned}$$

$$\begin{aligned} N_6 = {} & L_{02}L_{22}[2(M_{00}M_{22}+M_{01}M_{21}+M_{02}M_{20}-M_{10}M_{12}) - M_{11}^2] \\ & + (L_{00}L_{22}+L_{02}L_{20})(2M_{02}M_{22}-M_{12}^2) - L_{11}^2M_{02}M_{22} \\ & - L_{22}M_{02}[2(L_{22}M_{00}+L_{20}M_{02}) - L_{11}M_{11}] \\ & - L_{02}M_{22}[2(L_{02}M_{20}+L_{00}M_{22}) - L_{11}M_{11}] \\ & + L_{22}M_{01}(L_{11}M_{12}-L_{22}M_{01}) \\ & + L_{02}M_{21}(L_{11}M_{12}-L_{02}M_{21}) \end{aligned}$$

$$\begin{aligned} N_5 = {} & 2L_{02}L_{22}(M_{00}M_{21}+M_{01}M_{20}-M_{10}M_{11}) \\ & + 2(L_{00}L_{22}+L_{02}L_{20})(M_{01}M_{22}+M_{02}M_{21}-M_{11}M_{12}) \\ & - L_{11}^2(M_{01}M_{22}+M_{02}M_{21}) \\ & - L_{22}M_{01}[2(L_{22}M_{00}+2L_{20}M_{02}) - L_{11}M_{11}] \\ & - L_{02}M_{21}[2(L_{02}M_{20}+2L_{00}M_{22}) - L_{11}M_{11}] \\ & + L_{11}[M_{12}(L_{20}M_{02}+L_{22}M_{00}+L_{00}M_{22}+L_{02}M_{20}) \\ & \qquad + M_{10}(L_{22}M_{02}+L_{02}M_{22})] \end{aligned}$$

TABLE IX (CONTINUED)

$$\begin{aligned}
N_4 = & -(L_{22}M_{00}-L_{20}M_{02})^2 - (L_{02}M_{20}-L_{00}M_{22})^2 \\
& + (L_{00}L_{22}+L_{02}L_{20})[2(M_{00}M_{22}+M_{01}M_{21}+M_{02}M_{20} \\
& \qquad -M_{10}M_{12}) - M_{11}^2] \\
& - L_{11}^2(M_{00}M_{22}+M_{01}M_{21}+M_{02}M_{20}) \\
& + L_{11}[L_{22}(M_{00}M_{11}+M_{01}M_{10}) + L_{20}(M_{01}M_{12}+M_{02}M_{11}) \\
& \qquad + L_{02}(M_{10}M_{21}+M_{11}M_{20}) + L_{00}(M_{11}M_{22}+M_{12}M_{21})] \\
& - L_{22}[L_{02}M_{10}^2 + 2L_{20}(2M_{00}M_{02}+M_{01}^2)] \\
& - L_{00}[L_{20}M_{12}^2 + 2L_{02}(2M_{20}M_{22}+M_{21}^2)]
\end{aligned}$$

$$\begin{aligned}
N_3 = & \; 2L_{00}L_{20}(M_{01}M_{22}+M_{02}M_{21}-M_{11}M_{12}) \\
& + 2(L_{00}L_{22}+L_{02}L_{20})(M_{00}M_{21}+M_{01}M_{20}-M_{10}M_{11}) \\
& - L_{11}^2(M_{00}M_{21}+M_{01}M_{20}) \\
& - L_{20}M_{01}[2(L_{20}M_{02}+2L_{22}M_{00}) - L_{11}M_{11}] \\
& - L_{00}M_{21}[2(L_{00}M_{22}+2L_{02}M_{20}) - L_{11}M_{11}] \\
& + L_{11}[M_{10}(L_{22}M_{00}+L_{20}M_{02}+L_{02}M_{20}+L_{00}M_{22}) \\
& \qquad + M_{12}(L_{20}M_{00}+L_{00}M_{20})]
\end{aligned}$$

$$\begin{aligned}
N_2 = & \; L_{00}L_{20}[2(M_{00}M_{22}+M_{01}M_{21}+M_{02}M_{20}-M_{10}M_{12}) - M_{11}^2] \\
& + (L_{00}L_{22}+L_{02}L_{20})(2M_{00}M_{20}-M_{10}^2) - L_{11}^2M_{00}M_{20} \\
& - L_{20}M_{00}[2(L_{22}M_{00}+L_{20}M_{02}) - L_{11}M_{11}] \\
& - L_{00}M_{20}[2(L_{02}M_{20}+L_{00}M_{22}) - L_{11}M_{11}] \\
& + L_{20}M_{01}(L_{11}M_{10}-L_{20}M_{01}) \\
& + L_{00}M_{21}(L_{11}M_{10}-L_{00}M_{21})
\end{aligned}$$

TABLE IX (CONTINUED)

$$N_1 = 2L_{00}L_{20}(M_{00}M_{21}+M_{01}M_{20}-M_{10}M_{11})$$
$$+ L_{20}M_{00}(L_{11}M_{10}-2L_{20}M_{01})$$
$$+ L_{00}M_{20}(L_{11}M_{10}-2L_{00}M_{21})$$

$$N_0 = -(L_{20}M_{00}-L_{00}M_{20})^2 - L_{00}L_{20}M_{10}^2$$

at its revolute pairs.

On Obtaining R-R-R-P-R Mechanisms from the Derived Criteria

The existence criteria obtained above can be utilized to obtain the constant kinematic parameters of an R-R-R-P-R mechanism with zero offset distances at its revolute pairs.

Considering the constant kinematic parameters as unknowns, the nine equations given by condition (5-9) can be represented as

$$G_i(a,b,c,d,e,\alpha,\beta,\gamma,\delta,\lambda,\xi_k) = 0, \quad i = 1, 2, \ldots, 9 \qquad (5\text{-}10)$$

Eqs. (5-10) represent a system of nine nonlinear equations in the 11 unknown constant kinematic parameters. They are of second degree in each of the link lengths, of eighth degree in each of the twist angles and of fourth degree in the constant displacement angle ξ_k.

Like Eqs. (4-19), Eqs. (5-10) also have trivial solutions. Thus, for instance, Eqs. (5-10) are satisfied identically when the axes of the four revolute pairs are parallel to one another and the axis of the prismatic pair is normal to the other axes. This, however, yields a configuration with mobility two. Similarly, Eqs. (5-10) are also satisfied identically when the axes of the four revolute pairs are parallel to one another and the axis of the prismatic pair is obliquely oriented with respect to the other axes. With this arrangement, no motion is possible at the prismatic pair which, therefore, remains permanently locked. The configuration behaves, in effect, like a plane four-link mechanism. The solution thus gives a mechanism without true mobility.

A non-trivial solution of Eqs. (5-10) yields an R-R-R-P-R mechanism with a true mobility of one. As in the case of the five-revolute mechanism in Chapter IV, the triviality or non-triviality of a solution of Eqs. (5-10) can be checked by substituting the values of the constant kinematic parameters in the original displacement relationships of the parent R-C-R-C-R mechanism [56].

Since Eqs. (5-10) represent a system of nine equations among the 11 unknown constant kinematic parameters, two of the parameters can be assigned arbitrary values and the solution of the system can be attempted by numerical means (Appendix E) for the remaining nine parameters. The high nonlinearity of the equations once again emphasizes the complexity of the problem [28, 37, 34]. As in the case of the five-revolute mechanism, the investigation of the existence of R-R-R-P-R mechanisms by using the criteria derived in this chapter is thus a problem in its own right and is considered beyond the scope of the present investigation.

CHAPTER VI

EXISTENCE CRITERIA OF THE FIVE-LINK 3R+2P AND 2R+3P MECHANISMS

In this chapter, the Dimentberg passive coupling technique has been employed to obtain the existence criteria of the five-link 3R+2P and 2R+3P mechanisms. These criteria are obtained by considering only the primary parts of the displacement relationships of the appropriate parent mechanisms. They, therefore, lead to conditions on only the twist angles and constant displacement angles of the mechanisms considered and are independent of their link lengths and constant offset distances.

In a 3R+2P mechanism, the two prismatic pairs may either be separated by a revolute pair or be adjacent to each other. Similarly, in a 2R+3P mechanism, the two revolute pairs may be either adjacent to each other or be separated by a prismatic pair. All possible types of 3R+2P and 2R+3P mechanisms are, therefore, represented by the following mechanisms:

i) R-P-R-P-R Mechanism

ii) R-R-P-P-R Mechanism

iii) R-P-P-P-R Mechanism

iv) R-P-R-P-P Mechanism

Existence Criteria of the Five-Link R-P-R-P-R Mechanism

The existence criteria of an R-P-R-P-R mechanism can be obtained from the displacement relationships of an R-C-R-C-R mechanism.

Consider the R-C-R-C-R space mechanism shown schematically in Fig.

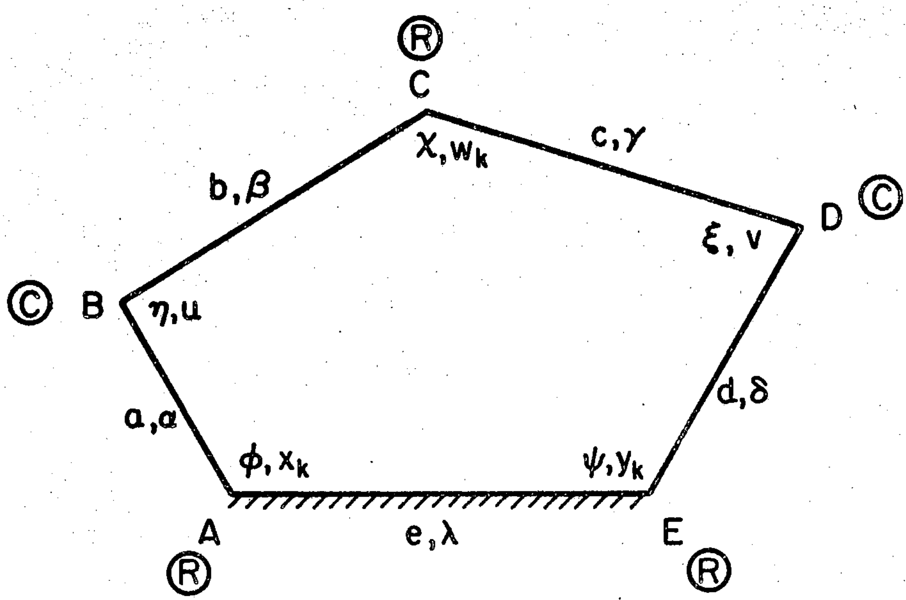

Figure 15. R-C-R-C-R Space Mechanism

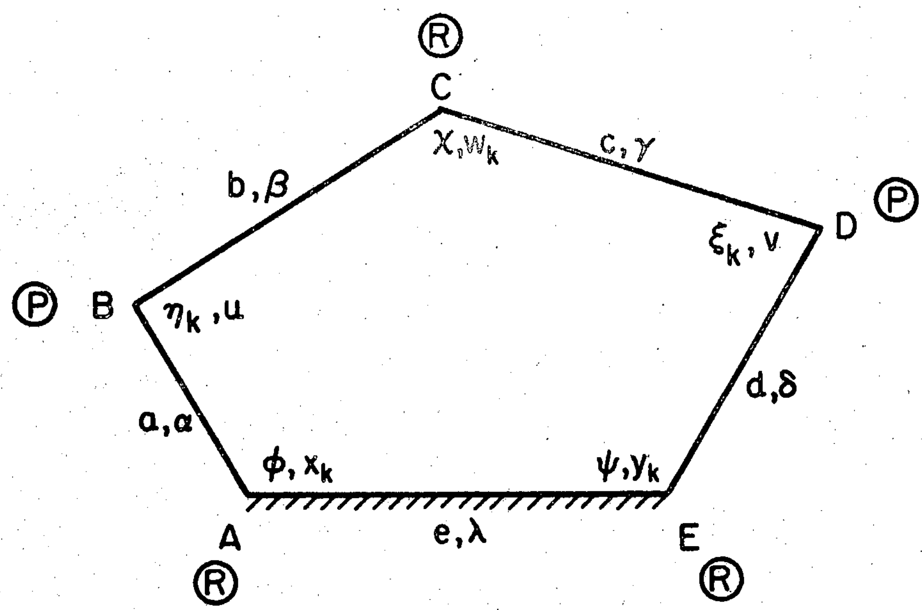

Figure 16. R-P-R-P-R Space Mechanism Obtained from the Mechanism in Fig. 15 by Making $\eta = \eta_k$ = a Constant and $\xi = \xi_k$ = a Constant

15. This mechanism reduces to an R-P-R-P-R mechanism if the displacement angles η and ξ at the two cylinder pairs remain constant at all positions of the mechanism (Fig. 16).

By considering the loop-closure condition of the mechanism in Fig. 15 in three different ways, the following relationships can be obtained:

$$F(\hat{\eta},\hat{\phi},\hat{\xi}) = (S\hat{\beta}S\hat{\lambda}S\hat{\phi})S\hat{\eta} - S\hat{\beta}(S\hat{\alpha}C\hat{\lambda}+C\hat{\alpha}S\hat{\lambda}C\hat{\phi})C\hat{\eta} + C\hat{\beta}(C\hat{\alpha}C\hat{\lambda}-S\hat{\alpha}S\hat{\lambda}C\hat{\phi}) - (C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi}) = 0 \quad (6\text{-}1)$$

$$F(\hat{\psi},\hat{\xi},\hat{\eta}) = (S\hat{\gamma}S\hat{\lambda}S\hat{\xi})S\hat{\psi} - S\hat{\lambda}(C\hat{\gamma}S\hat{\delta}+S\hat{\gamma}C\hat{\delta}C\hat{\xi})C\hat{\psi} + C\hat{\lambda}(C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi}) - (C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta}) = 0 \quad (6\text{-}2)$$

$$f(\hat{\xi},\hat{\chi},\hat{\eta}) = [(S\hat{\gamma}C\hat{\delta}+C\hat{\gamma}S\hat{\delta}C\hat{\xi})S\hat{\chi} + S\hat{\delta}S\hat{\xi}C\hat{\chi}](S\hat{\alpha}S\hat{\eta}) + [S\hat{\delta}S\hat{\xi}S\hat{\chi} - (S\hat{\gamma}C\hat{\delta}+C\hat{\gamma}S\hat{\delta}C\hat{\xi})C\hat{\chi}](C\hat{\alpha}S\hat{\beta}+S\hat{\alpha}C\hat{\beta}C\hat{\eta}) + (C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi})(C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta}) - C\hat{\lambda} = 0 \quad (6\text{-}3)$$

Observe that Eqs. (6-1) and (6-2) are similar in form to Eq. (3-6) and Eq. (6-3) is similar to Eq. (3-8). Note also that each of the above equations relates the dual displacement angles $\hat{\eta}$ and $\hat{\xi}$ at the two cylinder pairs to a third dual displacement angle.

Let the displacement angles η and ξ at the two cylinder pairs be now made constant at all positions of the mechanism. Denoting these constant values by η_k and ξ_k respectively, the primary parts of Eqs. (6-1), (6-2) and (6-3) give

$$A_s S\phi + A_c C\phi + A_n = 0 \quad (6\text{-}4)$$

$$B_s S\psi + B_c C\psi + B_n = 0 \quad (6\text{-}5)$$

$$C_s S\chi + C_c C\chi + C_n = 0 \quad (6\text{-}6)$$

The constants in the above equations involve the constant kinematic parameters and are defined in Table X.

TABLE X

CONSTANTS FOR USE IN EQS. (6-4) THROUGH (6-7)

$$A_s = S\beta S\lambda S\eta_k$$

$$A_c = -S\lambda(S\alpha C\beta + C\alpha S\beta C\eta_k)$$

$$A_n = C\lambda(C\alpha C\beta - S\alpha S\beta C\eta_k) - (C\gamma C\delta - S\gamma S\delta C\xi_k)$$

$$B_s = S\gamma S\lambda S\xi_k$$

$$B_c = -S\lambda(C\gamma S\delta + S\gamma C\delta C\xi_k)$$

$$B_n = C\lambda(C\gamma C\delta - S\gamma S\delta C\xi_k) - (C\alpha C\beta - S\alpha S\beta C\eta_k)$$

$$C_s = S\alpha(S\gamma C\delta + C\gamma S\delta C\xi_k)S\eta_k + S\delta(C\alpha S\beta + S\alpha C\beta C\eta_k)S\xi_k$$

$$C_c = S\alpha S\delta S\eta_k S\xi_k - (C\alpha S\beta + S\alpha C\beta C\eta_k)(S\gamma C\delta + C\gamma S\delta C\xi_k)$$

$$C_n = (C\alpha C\beta - S\alpha S\beta C\eta_k)(C\gamma C\delta - S\gamma S\delta C\xi_k) - C\lambda$$

Observe that each of the equations (6-4), (6-5) and (6-6) consists of only one variable and must be valid at varying values of that variable. This is possible only if their coefficients vanish. This gives

$$\begin{aligned} A_s &= A_c = A_n = 0 \\ B_s &= B_c = B_n = 0 \\ C_s &= C_c = C_n = 0 \end{aligned} \tag{6-7}$$

Examination of Eqs. (6-7) shows that the following cases are possible:

1. $C\eta_k < |1|$, $C\xi_k < |1|$ (That is, $\eta_k \neq m\pi$, $\xi_k \neq m\pi$, $m = 0,1,2,\ldots$)

   The only real solution possible in this case is given by

$$\alpha = \beta = \gamma = \delta = \lambda = 0 \tag{6-8}$$

Eq. (6-8) shows that the kinematic axes are all parallel to one another. An R-P-R-P-R mechanism satisfying this condition, however, represents only a trivial solution since it yields a configuration in which the three revolute pairs remain locked and in which the only motion possible is a "trombone-like" translation [25] at the two prismatic pairs.

2. $C\eta_k < |1|$, $C\xi_k = |1|$ (That is, $\eta_k \neq m\pi$, $\xi_k = m\pi$, $m = 0,1,2,\ldots$)

   This gives

$$\begin{aligned} &\alpha = \beta = \lambda = 0 \\ \text{and } &\gamma \pm \delta = m\pi, \qquad m = 0,1,2,\ldots \end{aligned} \tag{6-9}$$

3. $C\eta_k = |1|$, $C\xi_k < |1|$ (That is, $\eta_k = m\pi$, $\xi_k \neq m\pi$, $m = 0,1,2,\ldots$)

   This gives

$$\begin{aligned} &\gamma = \delta = \lambda = 0 \\ \text{and } &\alpha \pm \beta = m\pi, \qquad m = 0,1,2,\ldots \end{aligned} \tag{6-10}$$

4. $C\eta_k = |1|$, $C\xi_k = |1|$ (That is, $\eta_k = m\pi$, $\xi_k = m\pi$, $m = 0,1,2,\ldots$)

This gives

$$\lambda = 0$$
$$\text{and} \quad \alpha \pm \beta = m\pi, \ \gamma \pm \delta = m\pi, \ m = 0,1,2,\ldots \qquad (6\text{-}11)$$

Eqs. (6-9), (6-10) and (6-11) give the necessary conditions for the existence of an R-P-R-P-R mechanism. All these conditions show that the axes of the three revolute pairs are parallel to one another.

Existence Criteria of the Five-Link R-R-P-P-R Mechanism

The existence criteria of an R-R-P-P-R mechanism can be obtained from the displacement relationships of an R-R-C-C-R mechanism.

Consider the R-R-C-C-R space mechanism shown schematically in Fig. 17. This mechanism reduces to an R-R-P-P-R mechanism if the displacement angles χ and ξ at the two cylinder pairs remain constant at all positions of the mechanism (Fig. 18).

By considering the loop-closure condition of the mechanism in Fig. 17 in three different ways, the following relationships can be obtained:

$$\begin{aligned} F(\hat{\xi},\hat{\chi},\hat{\phi}) &= (S\hat{\beta}S\hat{\delta}S\hat{\chi})S\hat{\xi} - S\hat{\delta}(C\hat{\beta}S\hat{\gamma}+S\hat{\beta}C\hat{\gamma}C\hat{\chi})C\hat{\xi} \\ &\quad + C\hat{\delta}(C\hat{\beta}C\hat{\gamma}-S\hat{\beta}S\hat{\gamma}C\hat{\chi}) - (C\hat{\alpha}C\hat{\lambda}-S\hat{\alpha}S\hat{\lambda}C\hat{\phi}) = 0 \qquad (6\text{-}12) \end{aligned}$$

$$\begin{aligned} f(\hat{\psi},\hat{\xi},\hat{\chi}) &= [(S\hat{\delta}C\hat{\lambda}+C\hat{\delta}S\hat{\lambda}C\hat{\psi})S\hat{\xi} + S\hat{\lambda}S\hat{\psi}C\hat{\xi}](S\hat{\beta}S\hat{\chi}) \\ &\quad + [S\hat{\lambda}S\hat{\psi}S\hat{\xi} - (S\hat{\delta}C\hat{\lambda}+C\hat{\delta}S\hat{\lambda}C\hat{\psi})C\hat{\xi}](C\hat{\beta}S\hat{\gamma}+S\hat{\beta}C\hat{\gamma}C\hat{\chi}) \\ &\quad + (C\hat{\delta}C\hat{\lambda}-S\hat{\delta}S\hat{\lambda}C\hat{\psi})(C\hat{\beta}C\hat{\gamma}-S\hat{\beta}S\hat{\gamma}C\hat{\chi}) - C\hat{\alpha} = 0 \qquad (6\text{-}13) \end{aligned}$$

$$\begin{aligned} f(\hat{\xi},\hat{\chi},\hat{\eta}) &= [(S\hat{\gamma}C\hat{\delta}+C\hat{\gamma}S\hat{\delta}C\hat{\xi})S\hat{\chi} + S\hat{\delta}S\hat{\xi}C\hat{\chi}](S\hat{\alpha}S\hat{\eta}) \\ &\quad + [S\hat{\delta}S\hat{\xi}S\hat{\chi} - (S\hat{\gamma}C\hat{\delta}+C\hat{\gamma}S\hat{\delta}C\hat{\xi})C\hat{\chi}](C\hat{\alpha}S\hat{\beta}+S\hat{\alpha}C\hat{\beta}C\hat{\eta}) \\ &\quad + (C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi})(C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta}) - C\hat{\lambda} = 0 \qquad (6\text{-}14) \end{aligned}$$

Note that Eq. (6-12) is of the same form as Eq. (3-6) and Eqs. (6-13) and

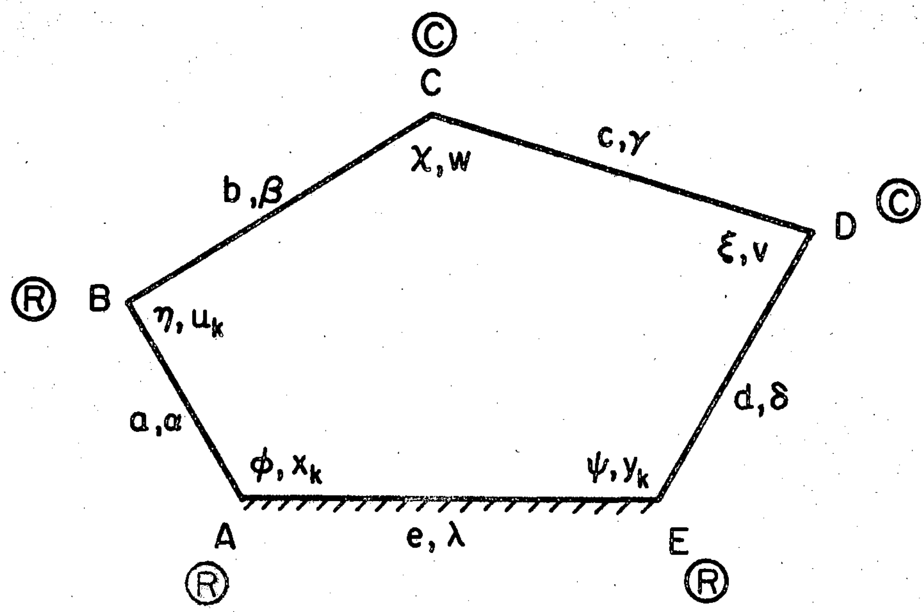

Figure 17. R-R-C-C-R Space Mechanism

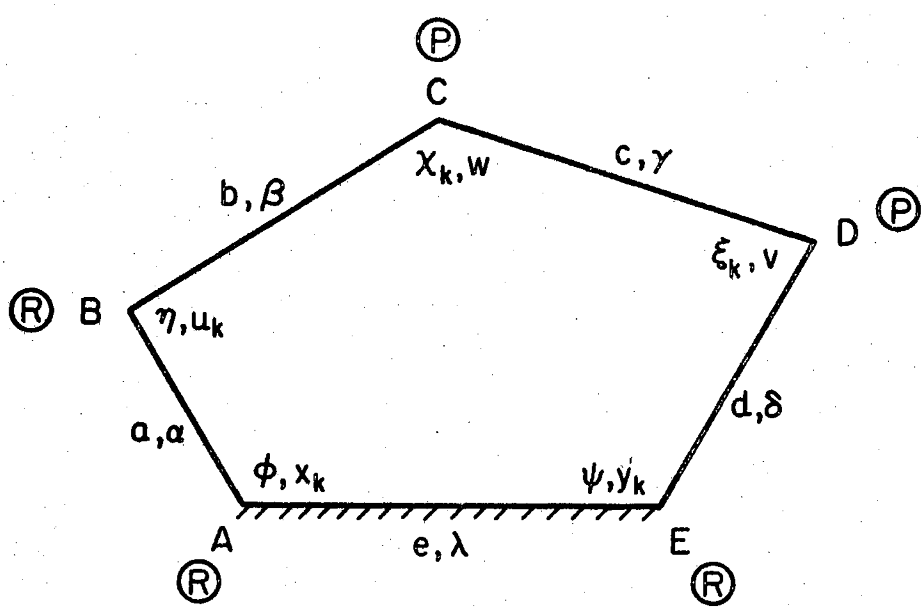

Figure 18. R-R-P-P-R Space Mechanism Obtained from the Mechanism in Fig. 17 by Making $\chi = \chi_k$ = a Constant and $\xi = \xi_k$ = a Constant

(6-14) are similar to Eq. (3-8). Observe also that each of the above equations relates the dual displacement angles $\hat{\chi}$ and $\hat{\xi}$ at the two cylinder pairs to a third dual displacement angle.

Let the displacement angles χ and ξ at the two cylinder pairs be now held constant at all positions of the mechanism. Denoting these constant values by χ_k and ξ_k respectively, the primary parts of Eqs. (6-12), (6-13) and (6-14) give

$$D_c C\phi + D_n = 0 \tag{6-15}$$

$$E_s S\psi + E_c C\psi + E_n = 0 \tag{6-16}$$

$$\text{and} \quad F_s S\eta + F_c C\eta + F_n = 0 \tag{6-17}$$

The constants used in the above equations are defined in Table XI.

Note that each of the equations (6-15), (6-16) and (6-17) contains only one variable and must hold good at varying values of that variable. Their coefficients must, therefore, vanish. This gives

$$\begin{aligned} D_c &= D_n = 0 \\ E_s = E_c &= E_n = 0 \\ \text{and} \quad F_s = F_c &= F_n = 0 \end{aligned} \tag{6-18}$$

Examination of Eqs. (6-18) yields the following relationships:

$$\alpha = \lambda = 0 \tag{6-19}$$

$$\begin{aligned} S\gamma(S\beta C\delta C\chi_k + C\beta S\delta C\xi_k) - C\gamma(C\beta C\delta - S\beta S\delta C\chi_k C\xi_k) \\ - S\beta S\delta S\chi_k S\xi_k + 1 = 0 \end{aligned} \tag{6-20}$$

The above equations provide the necessary conditions for the existence of an R-R-P-P-R mechanism. Condition (6-19) shows that the axes of the three revolute pairs are parallel to one another. Eq. (6-20) is

TABLE XI

CONSTANTS FOR USE IN EQS. (6-15) THROUGH (6-18)

$$D_c = S\alpha S\lambda$$

$$D_n = S\beta[S\delta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - S\gamma C\delta C\chi_k] + C\beta(C\gamma C\delta - S\gamma S\delta C\xi_k) - C\alpha C\lambda$$

$$E_s = S\lambda[S\beta(S\chi_k C\xi_k + C\gamma C\chi_k S\xi_k) + C\beta S\gamma S\xi_k]$$

$$E_c = S\lambda\{C\delta[S\beta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - C\beta S\gamma C\xi_k] - S\delta(C\beta C\gamma - S\beta S\gamma C\chi_k)\}$$

$$E_n = C\lambda\{S\delta[S\beta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - C\beta S\gamma C\xi_k] + C\delta(C\beta C\gamma - S\beta S\gamma C\chi_k)\} - C\alpha$$

$$F_s = S\alpha[S\delta(C\chi_k S\xi_k + C\gamma S\chi_k C\xi_k) + S\gamma C\delta S\chi_k]$$

$$F_c = S\alpha\{C\beta[S\delta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - S\gamma C\delta C\chi_k] - S\beta(C\gamma C\delta - S\gamma S\delta C\xi_k)\}$$

$$F_n = C\alpha\{S\beta[S\delta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - S\gamma C\delta C\chi_k] + C\beta(C\gamma C\delta - S\gamma S\delta C\xi_k)\} - C\lambda$$

a closure condition relating the twist angles β, γ and δ of the mechanism with the constant displacement angles χ_k and ξ_k at the two prismatic pairs (Fig. 18).

Existence Criteria of the Five-Link R-P-P-P-R Mechanism

The existence criteria of an R-P-P-P-R mechanism can be obtained from the displacement relationships of an R-C-P-C-R mechanism.

Consider the R-C-P-C-R space mechanism shown schematically in Fig. 19. Note that the displacement angle χ_k at the prismatic pair is constant. This mechanism reduces to an R-P-P-P-R mechanism if the displacement angles η and ξ at the two cylinder pairs remain constant at all positions of the mechanism (Fig. 20).

By considering the loop-closure condition of the mechanism in Fig. 19 in seven different ways, the following relationships can be obtained.

$$F(\hat{\eta},\hat{\phi},\hat{\xi}) = (S\hat{\beta}S\hat{\lambda}S\hat{\phi})S\hat{\eta} - S\hat{\beta}(S\hat{\alpha}C\hat{\lambda}+C\hat{\alpha}S\hat{\lambda}C\hat{\phi})C\hat{\eta} + C\hat{\beta}(C\hat{\alpha}C\hat{\lambda}-S\hat{\alpha}S\hat{\lambda}C\hat{\phi}) - (C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi}) = 0 \qquad (6\text{-}21)$$

$$F(\hat{\psi},\hat{\xi},\hat{\eta}) = (S\hat{\gamma}S\hat{\lambda}S\hat{\xi})S\hat{\psi} - S\hat{\lambda}(C\hat{\gamma}S\hat{\delta}+S\hat{\gamma}C\hat{\delta}C\hat{\xi})C\hat{\psi} + C\hat{\lambda}(C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi}) - (C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta}) = 0 \qquad (6\text{-}22)$$

$$F(\hat{\chi},\hat{\eta},\hat{\psi}) = (S\hat{\alpha}S\hat{\gamma}S\hat{\eta})S\hat{\chi} - S\hat{\gamma}(C\hat{\alpha}S\hat{\beta}+S\hat{\alpha}C\hat{\beta}C\hat{\eta})C\hat{\chi} + C\hat{\gamma}(C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta}) - (C\hat{\delta}C\hat{\lambda}-S\hat{\delta}S\hat{\lambda}C\hat{\psi}) = 0 \qquad (6\text{-}23)$$

$$F(\hat{\xi},\hat{\chi},\hat{\phi}) = (S\hat{\beta}S\hat{\delta}S\hat{\chi})S\hat{\xi} - S\hat{\delta}(C\hat{\beta}S\hat{\gamma}+S\hat{\beta}C\hat{\gamma}C\hat{\chi})C\hat{\xi} + C\hat{\delta}(C\hat{\beta}C\hat{\gamma}-S\hat{\beta}S\hat{\gamma}C\hat{\chi}) - (C\hat{\alpha}C\hat{\lambda}-S\hat{\alpha}S\hat{\lambda}S\hat{\phi}) = 0 \qquad (6\text{-}24)$$

$$f(\hat{\chi},\hat{\eta},\hat{\phi}) = [(S\hat{\beta}C\hat{\gamma}+C\hat{\beta}S\hat{\gamma}C\hat{\chi})S\hat{\eta} + S\hat{\gamma}S\hat{\chi}C\hat{\eta}](S\hat{\lambda}S\hat{\phi}) + [S\hat{\gamma}S\hat{\chi}S\hat{\eta} - (S\hat{\beta}C\hat{\gamma}+C\hat{\beta}S\hat{\gamma}C\hat{\chi})C\hat{\eta}](S\hat{\alpha}C\hat{\lambda}+C\hat{\alpha}S\hat{\lambda}C\hat{\phi}) + (C\hat{\beta}C\hat{\gamma}-S\hat{\beta}S\hat{\gamma}C\hat{\chi})(C\hat{\alpha}C\hat{\lambda}-S\hat{\alpha}S\hat{\lambda}C\hat{\phi}) - C\hat{\delta} = 0 \qquad (6\text{-}25)$$

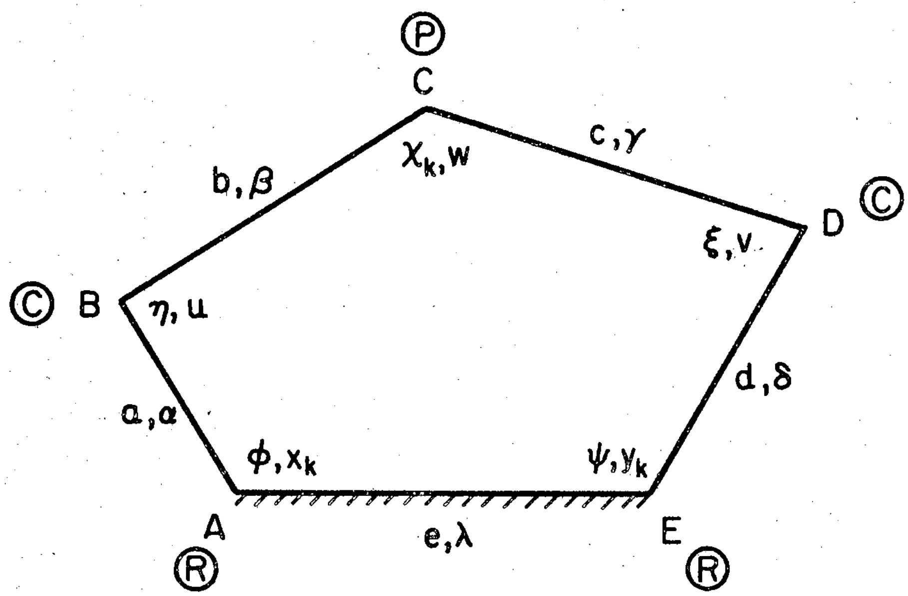

Figure 19. R-C-P-C-R Space Mechanism

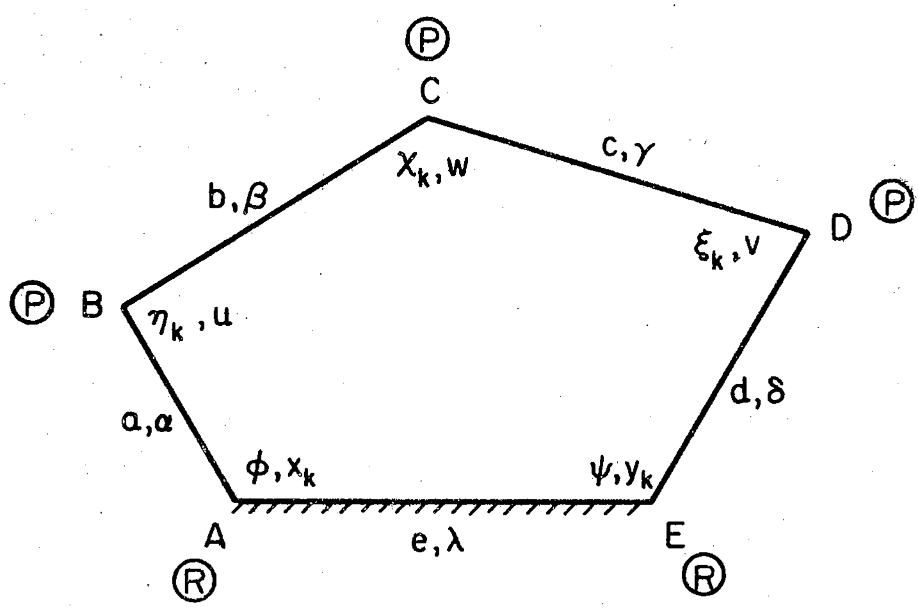

Figure 20. R-P-P-P-R Space Mechanism Obtained from the Mechanism in Fig. 19 by Making $\eta = \eta_k$ = a Constant and $\xi = \xi_k$ = a Constant

$$f(\hat{\psi},\hat{\xi},\hat{\chi}) = [(S\hat{\delta}C\hat{\lambda}+C\hat{\delta}S\hat{\lambda}C\hat{\psi})S\hat{\xi} + S\hat{\lambda}S\hat{\psi}C\hat{\xi}](S\hat{\beta}S\hat{\chi})$$
$$+ [S\hat{\lambda}S\hat{\psi}S\hat{\xi} - (S\hat{\delta}C\hat{\lambda}+C\hat{\delta}S\hat{\lambda}C\hat{\psi})C\hat{\xi}](S\hat{\gamma}C\hat{\beta}+C\hat{\gamma}S\hat{\beta}C\hat{\chi})$$
$$+ (C\hat{\delta}C\hat{\lambda}-S\hat{\delta}S\hat{\lambda}C\hat{\psi})(C\hat{\beta}C\hat{\gamma}-S\hat{\beta}S\hat{\gamma}C\hat{\chi}) - C\hat{\alpha} = 0 \tag{6-26}$$

$$f(\hat{\xi},\hat{\chi},\hat{\eta}) = [(S\hat{\gamma}C\hat{\delta}+C\hat{\gamma}S\hat{\delta}C\hat{\xi})S\hat{\chi} + S\hat{\delta}S\hat{\xi}C\hat{\chi}](S\hat{\alpha}S\hat{\eta})$$
$$+ [S\hat{\delta}S\hat{\xi}S\hat{\chi} - (S\hat{\gamma}C\hat{\delta}+C\hat{\gamma}S\hat{\delta}C\hat{\xi})C\hat{\chi}](C\hat{\alpha}S\hat{\beta}+S\hat{\alpha}C\hat{\beta}C\hat{\eta})$$
$$+ (C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi})(C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta}) - C\hat{\lambda} = 0 \tag{6-27}$$

Note that Eqs. (6-21) through (6-24) are similar in form to Eq. (3-6) and Eqs. (6-25), (6-26) and (6-27) are similar to Eq. (3-8). Observe also that each of the above equations contains the dual displacement angle at at least one of the two cylinder pairs.

Let the displacement angles η and ξ at the two cylinder pairs be now held constant at all positions of the mechanism. Denoting these constant values by η_k and ξ_k respectively, the primary parts of Eqs. (6-21) through (6-27) give

$$G_sS\phi + G_cC\phi + G_n = 0 \tag{6-28}$$
$$H_sS\psi + H_cC\psi + H_n = 0 \tag{6-29}$$
$$I_cC\psi + I_n = 0 \tag{6-30}$$
$$J_cC\phi + J_n = 0 \tag{6-31}$$
$$K_sS\phi + K_cC\phi + K_n = 0 \tag{6-32}$$
$$L_sS\psi + L_cC\psi + L_n = 0 \tag{6-33}$$
and
$$M_n = 0 \tag{6-34}$$

The constants used in the above equations are defined in Table XII.

Observe that each of the equations (6-28) through (6-33) contains only one variable and must hold good at varying values of that variable. This is possible only if their coefficients vanish. This gives

TABLE XII

CONSTANTS FOR USE IN EQS. (6-28) THROUGH (6-35)

$$G_s = S\beta S\lambda S\eta_k$$

$$G_c = -S\lambda(S\alpha C\beta + C\alpha S\beta C\eta_k)$$

$$G_n = C\lambda(C\alpha C\beta - S\alpha S\beta C\eta_k) - (C\gamma C\delta - S\gamma S\delta C\xi_k)$$

$$H_s = S\gamma S\lambda S\xi_k$$

$$H_c = -S\lambda(C\gamma S\delta + S\gamma C\delta C\xi_k)$$

$$H_n = C\lambda(C\gamma C\delta - S\gamma S\delta C\xi_k) - (C\alpha C\beta - S\alpha S\beta C\eta_k)$$

$$I_c = S\delta S\lambda$$

$$I_n = S\gamma[S\alpha(S\eta_k S\chi_k - C\beta C\eta_k C\chi_k) - C\alpha S\beta C\chi_k] + C\gamma(C\alpha C\beta - S\alpha S\beta C\eta_k) - C\delta C\lambda$$

$$J_c = S\alpha S\lambda$$

$$J_n = S\beta[S\delta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - S\gamma C\delta C\chi_k] + C\beta(C\gamma C\delta - S\gamma S\delta C\xi_k) - C\alpha C\lambda$$

$$K_s = S\lambda[S\gamma(C\eta_k S\chi_k + C\beta S\eta_k C\chi_k) + S\beta C\gamma S\eta_k]$$

$$K_c = S\lambda\{C\alpha[S\gamma(S\eta_k S\chi_k - C\beta C\eta_k C\chi_k) - S\beta C\gamma C\eta_k] - S\alpha(C\beta C\gamma - S\beta S\gamma C\chi_k)\}$$

$$K_n = C\lambda\{S\alpha[S\gamma(S\eta_k S\chi_k - C\beta C\eta_k C\chi_k) - S\beta C\gamma C\eta_k] + C\alpha(C\beta C\gamma - S\beta S\gamma C\chi_k)\} - C\delta$$

TABLE XII (CONTINUED)

$$L_s = S\lambda[S\beta(S\chi_k C\xi_k + C\gamma C\chi_k S\xi_k) + C\beta S\gamma S\xi_k]$$

$$L_c = S\lambda\{C\delta[S\beta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - C\beta S\gamma C\xi_k] - S\delta(C\beta C\gamma - S\beta S\gamma C\chi_k)\}$$

$$L_n = C\lambda\{S\delta[S\beta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - C\beta S\gamma C\xi_k] + C\delta(C\beta C\gamma - S\beta S\gamma C\chi_k)\} - C\alpha$$

$$\begin{aligned} M_n = {} & S\alpha[S\delta(C\chi_k S\xi_k + C\gamma S\chi_k C\xi_k) + S\gamma C\delta S\chi_k]S\eta_k \\ & + S\alpha\{C\beta[S\delta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - S\gamma C\delta C\chi_k] \\ & - S\beta(C\gamma C\delta - S\gamma S\delta C\xi_k)\}C\eta_k \\ & + C\alpha\{S\beta[S\delta(S\chi_k S\xi_k - C\gamma C\chi_k C\xi_k) - S\gamma C\delta C\chi_k] \\ & + C\beta(C\gamma C\delta - S\gamma S\delta C\xi_k)\} - C\lambda \end{aligned}$$

$$\begin{aligned} G_s &= G_c = G_n = 0 \\ H_s &= H_c = H_n = 0 \\ I_c &= I_n = 0 \\ J_c &= J_n = 0 \\ K_s &= K_c = K_n = 0 \\ \text{and } L_s &= L_c = L_n = 0 \end{aligned} \tag{6-35}$$

Examination of Eqs. (6-35) along with Eq. (6-34) gives the following relationships:

$$\lambda = 0 \tag{6-36}$$

$$(C\alpha C\beta - S\alpha S\beta C\eta_k) - (C\gamma C\delta - S\gamma S\delta C\xi_k) = 0 \tag{6-37}$$

$$\begin{aligned} &C\gamma(C\alpha C\beta - S\alpha S\beta C\eta_k) + S\gamma(C\alpha S\beta - S\alpha C\beta C\eta_k)C\chi_k \\ &\quad + S\alpha S\gamma S\eta_k S\chi_k - C\delta = 0 \end{aligned} \tag{6-38}$$

$$\begin{aligned} &C\beta(C\gamma C\delta - S\gamma S\delta C\xi_k) + S\beta(S\gamma C\delta - C\gamma S\delta C\xi_k)C\chi_k \\ &\quad + S\beta S\delta S\xi_k S\chi_k - C\alpha = 0 \end{aligned} \tag{6-39}$$

$$\begin{aligned} &C\alpha C\delta(2 - C\beta C\gamma + S\beta S\gamma C\chi_k) + S\alpha S\delta[S\eta_k C\chi_k S\xi_k + (C\gamma S\eta_k C\xi_k + C\beta C\eta_k S\xi_k)S\chi_k \\ &\quad - (S\beta S\gamma - C\beta C\gamma C\chi_k)C\eta_k C\xi_k] - 1 = 0 \end{aligned} \tag{6-40}$$

The above relationships provide the necessary conditions for the existence of an R-P-P-P-R mechanism. Eq. (6-36) shows that the axes of the two revolute pairs are parallel to each other. Eqs. (6-37) through (6-40) are closure conditions relating the twist angles α, β, γ and δ of the mechanism with the constant angles η_k, χ_k and ξ_k at the three prismatic pairs (Fig. 20).

Existence Criteria of the Five-Link R-P-R-P-P Mechanism

The existence criteria of an R-P-R-P-P mechanism can be obtained from the displacement relationships of an R-C-R-C-P mechanism.

Consider the R-C-R-C-P space mechanism shown schematically in Fig. 21. Note that the displacement angle ψ_k at the prismatic pair is constant. This mechanism reduces to an R-P-R-P-P mechanism if the displacement angles η and ξ at the two cylinder pairs remain constant at all positions of the mechanism (Fig. 22).

By considering the loop-closure condition of the mechanism in Fig. 21 in seven different ways, the following relationships can be obtained:

$$\begin{aligned} F(\hat\eta,\hat\phi,\hat\xi) = {} & (S\hat\beta S\hat\lambda S\hat\phi)S\hat\eta - S\hat\beta(S\hat\alpha C\hat\lambda + C\hat\alpha S\hat\lambda C\hat\phi)C\hat\eta \\ & + C\hat\beta(C\hat\alpha C\hat\lambda - S\hat\alpha S\hat\lambda C\hat\phi) - (C\hat\gamma C\hat\delta - S\hat\gamma S\hat\delta C\hat\xi) = 0 \end{aligned} \qquad (6\text{-}41)$$

$$\begin{aligned} f(\hat\xi,\hat\chi,\hat\eta) = {} & [(S\hat\gamma C\hat\delta + C\hat\gamma S\hat\delta C\hat\xi)S\hat\chi + S\hat\delta S\hat\xi C\hat\chi](S\hat\alpha S\hat\eta) \\ & + [S\hat\delta S\hat\xi S\hat\chi - (S\hat\gamma C\hat\delta + C\hat\gamma S\hat\delta C\hat\xi)C\hat\chi](C\hat\alpha S\hat\beta + S\hat\alpha C\hat\beta C\hat\eta) \\ & + (C\hat\gamma C\hat\delta - S\hat\gamma S\hat\delta C\hat\xi)(C\hat\alpha C\hat\beta - S\hat\alpha S\hat\beta C\hat\eta) - C\hat\lambda = 0 \end{aligned} \qquad (6\text{-}42)$$

$$\begin{aligned} f(\hat\eta,\hat\phi,\hat\psi) = {} & [(S\hat\alpha C\hat\beta + C\hat\alpha S\hat\beta C\hat\eta)S\hat\phi + S\hat\beta S\hat\eta C\hat\phi](S\hat\delta S\hat\psi) \\ & + [S\hat\beta S\hat\eta S\hat\phi - (S\hat\alpha C\hat\beta + C\hat\alpha S\hat\beta C\hat\eta)C\hat\phi](C\hat\delta S\hat\lambda + S\hat\delta C\hat\lambda C\hat\psi) \\ & + (C\hat\alpha C\hat\beta - S\hat\alpha S\hat\beta C\hat\eta)(C\hat\delta C\hat\lambda - S\hat\delta S\hat\lambda C\hat\psi) - C\hat\gamma = 0 \end{aligned} \qquad (6\text{-}43)$$

$$\begin{aligned} F(\hat\chi,\hat\eta,\hat\psi) = {} & (S\hat\alpha S\hat\gamma S\hat\eta)S\hat\chi - S\hat\gamma(C\hat\alpha S\hat\beta + S\hat\alpha C\hat\beta C\hat\eta)C\hat\chi \\ & + C\hat\gamma(C\hat\alpha C\hat\beta + S\hat\alpha S\hat\beta C\hat\eta) - (C\hat\delta C\hat\lambda - S\hat\delta S\hat\lambda C\hat\psi) = 0 \end{aligned} \qquad (6\text{-}44)$$

$$\begin{aligned} f(\hat\phi,\hat\psi,\hat\xi) = {} & [(C\hat\alpha S\hat\lambda + S\hat\alpha C\hat\lambda C\hat\phi)S\hat\psi + S\hat\alpha S\hat\phi C\hat\psi](S\hat\gamma S\hat\xi) \\ & + [S\hat\alpha S\hat\phi S\hat\psi - (C\hat\alpha S\hat\lambda + S\hat\alpha C\hat\lambda C\hat\phi)C\hat\psi](C\hat\gamma S\hat\delta + S\hat\gamma C\hat\delta C\hat\xi) \\ & + (C\hat\alpha C\hat\lambda - S\hat\alpha S\hat\lambda C\hat\phi)(C\hat\gamma C\hat\delta - S\hat\gamma S\hat\delta C\hat\xi) - C\hat\beta = 0 \end{aligned} \qquad (6\text{-}45)$$

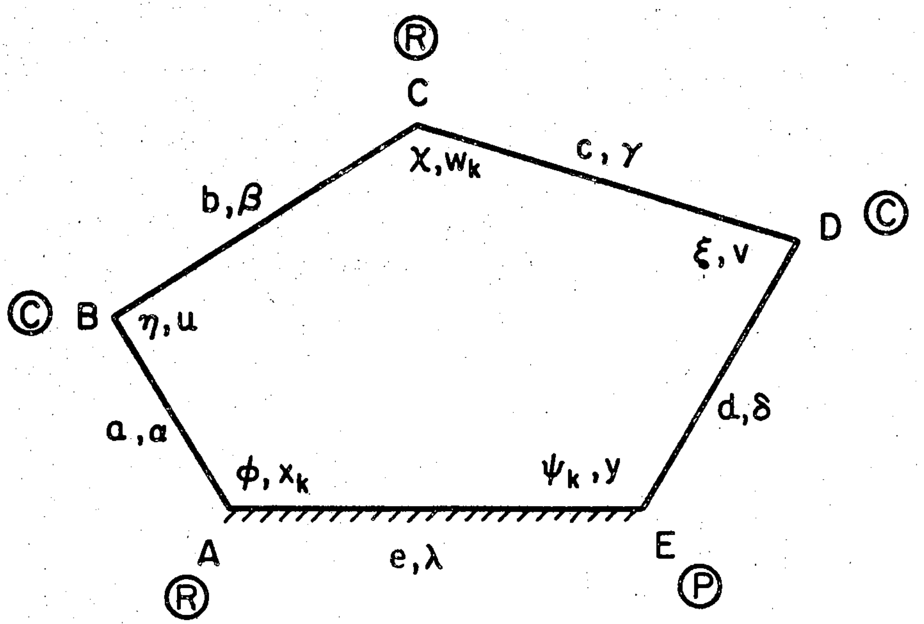

Figure 21. R-C-R-C-P Space Mechanism

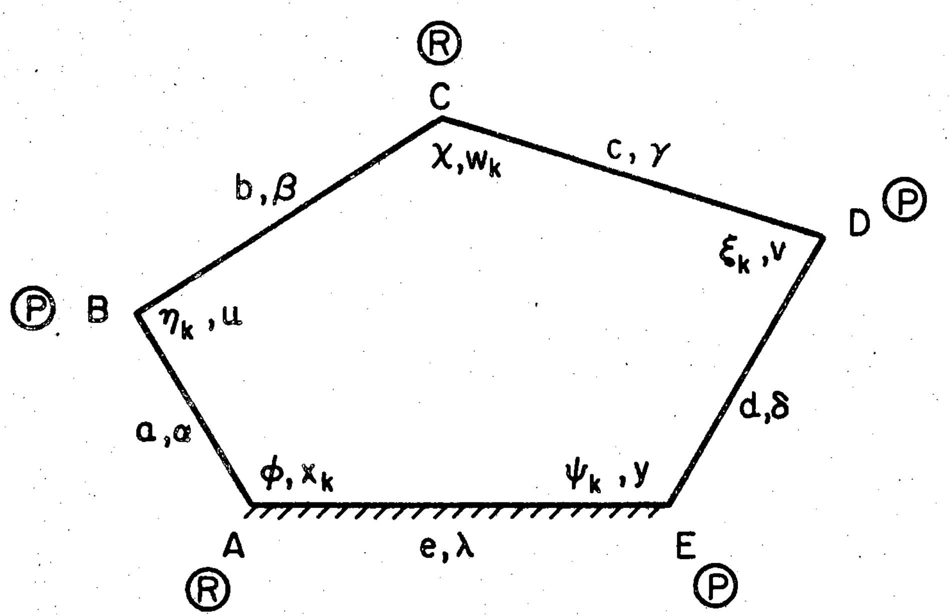

Figure 22. R-P-R-P-P Space Mechanism Obtained from the Mechanism in Fig. 21 by Making $\eta = \eta_k$ = a Constant and $\xi = \xi_k$ = a Constant

$$f(\hat{\psi},\hat{\xi},\hat{\chi}) = [(S\hat{\delta}C\hat{\lambda}+C\hat{\delta}S\hat{\lambda}C\hat{\psi})S\hat{\xi} + S\hat{\lambda}S\hat{\psi}C\hat{\xi}](S\hat{\beta}S\hat{\chi})$$
$$+ [S\hat{\lambda}S\hat{\psi}S\hat{\xi} - (S\hat{\delta}C\hat{\lambda}+C\hat{\delta}S\hat{\lambda}C\hat{\psi})C\hat{\xi}](C\hat{\beta}S\hat{\gamma}+S\hat{\beta}C\hat{\gamma}C\hat{\chi})$$
$$+ (C\hat{\delta}C\hat{\lambda}-S\hat{\delta}S\hat{\lambda}C\hat{\psi})(C\hat{\beta}C\hat{\gamma}-S\hat{\beta}S\hat{\gamma}C\hat{\chi}) - C\hat{\alpha} = 0 \quad (6\text{-}46)$$

$$F(\hat{\psi},\hat{\xi},\hat{\eta}) = (S\hat{\gamma}S\hat{\lambda}S\hat{\xi})S\hat{\psi} - S\hat{\lambda}(C\hat{\gamma}S\hat{\delta}+S\hat{\gamma}C\hat{\delta}C\hat{\xi})C\hat{\psi}$$
$$+ C\hat{\lambda}(C\hat{\gamma}C\hat{\delta}-S\hat{\gamma}S\hat{\delta}C\hat{\xi}) - (C\hat{\alpha}C\hat{\beta}-S\hat{\alpha}S\hat{\beta}C\hat{\eta}) = 0 \quad (6\text{-}47)$$

Note that Eqs. (6-41), (6-44) and (6-47) are similar in form to Eq. (3-6) and Eqs. (6-42), (6-43), (6-45) and (6-46) are of the same form as Eq. (3-8). Note also that each of the above equations contains the dual displacement angle at at least one of the two cylinder pairs.

Let the displacement angles η and ξ at the two cylinder pairs be now made constant at all positions of the mechanism. Denoting these constant values by η_k and ξ_k respectively, the primary parts of Eqs. (6-41) through (6-47) give

$$N_sS\phi + N_cC\phi + N_n = 0 \quad (6\text{-}48)$$
$$P_sS\chi + P_cC\chi + P_n = 0 \quad (6\text{-}49)$$
$$Q_sS\phi + Q_cC\phi + Q_n = 0 \quad (6\text{-}50)$$
$$R_sS\chi + R_cC\chi + R_n = 0 \quad (6\text{-}51)$$
$$S_sS\phi + S_cC\phi + S_n = 0 \quad (6\text{-}52)$$
$$T_sS\chi + T_cC\chi + T_n = 0 \quad (6\text{-}53)$$
and
$$U_n = 0 \quad (6\text{-}54)$$

The constants used in the above equations are defined in Table XIII.

Observe that each of the equations (6-48) through (6-53) consists of only one variable and must hold good at varying values of that variable. Their coefficients should, therefore, vanish. This gives

TABLE XIII

CONSTANTS FOR USE IN EQS. (6-48) THROUGH (6-55)

$$N_s = S\beta S\lambda S\eta_k$$

$$N_c = -S\lambda(S\alpha C\beta + C\alpha S\beta C\eta_k)$$

$$N_n = C\lambda(C\alpha C\beta - S\alpha S\beta C\eta_k) - (C\gamma C\delta - S\gamma S\delta C\xi_k)$$

$$P_s = S\alpha(S\gamma C\delta + C\gamma S\delta C\xi_k)S\eta_k + S\delta(C\alpha S\beta + S\alpha C\beta C\eta_k)S\xi_k$$

$$P_c = S\alpha S\delta S\eta_k S\xi_k - (C\alpha S\beta + S\alpha C\beta C\eta_k)(S\gamma C\delta + C\gamma S\delta C\xi_k)$$

$$P_n = (C\alpha C\beta - S\alpha S\beta C\eta_k)(C\gamma C\delta - S\gamma S\delta C\xi_k) - C\lambda$$

$$Q_s = S\beta(C\delta S\lambda + S\delta C\lambda C\psi_k)S\eta_k + S\delta(S\alpha C\beta + C\alpha S\beta C\eta_k)S\psi_k$$

$$Q_c = S\beta S\delta S\eta_k S\psi_k - (S\alpha C\beta + C\alpha S\beta C\eta_k)(C\delta S\lambda + S\delta C\lambda C\psi_k)$$

$$Q_n = (C\alpha C\beta - S\alpha S\beta C\eta_k)(C\delta C\lambda - S\delta S\lambda C\psi_k) - C\gamma$$

$$R_s = S\alpha S\gamma S\eta_k$$

$$R_c = -S\gamma(C\alpha S\beta + S\alpha C\beta C\eta_k)$$

$$R_n = C\gamma(C\alpha C\beta - S\alpha S\beta C\eta_k) - (C\delta C\lambda - S\delta S\lambda C\psi_k)$$

$$S_s = S\alpha[S\gamma(S\xi_k C\psi_k + C\delta C\xi_k S\psi_k) + C\gamma S\delta S\psi_k]$$

$$S_c = S\alpha\{C\lambda[S\gamma(S\xi_k S\psi_k - C\delta C\xi_k C\psi_k) - C\gamma S\delta C\psi_k] - S\lambda(C\gamma C\delta - S\gamma S\delta C\xi_k)\}$$

$$S_n = C\alpha\{S\lambda[S\gamma(S\xi_k S\psi_k - C\delta C\xi_k C\psi_k) - C\gamma S\delta C\psi_k] + C\lambda(C\gamma C\delta - S\gamma S\delta C\xi_k)\} - C\beta$$

TABLE XIII (CONTINUED)

$$T_s = S\beta[S\lambda(C\xi_k S\psi_k + C\delta S\xi_k C\psi_k) + S\delta C\lambda S\xi_k]$$

$$T_c = S\beta\{C\gamma[S\lambda(S\xi_k S\psi_k - C\delta C\xi_k C\psi_k) - S\delta C\lambda C\xi_k]$$
$$- S\gamma(C\delta C\lambda - S\delta S\lambda C\psi_k)\}$$

$$T_n = C\beta\{S\gamma[S\lambda(S\xi_k S\psi_k - C\delta C\xi_k C\psi_k) - S\delta C\lambda C\xi_k]$$
$$+ C\gamma(C\delta C\lambda - S\delta S\lambda C\psi_k)\} - C\alpha$$

$$U_n = S\gamma S\lambda S\xi_k S\psi_k - S\lambda(C\gamma S\delta + S\gamma C\delta C\xi_k)C\psi_k$$
$$+ C\lambda(C\gamma C\delta - S\gamma S\delta C\xi_k) - (C\alpha C\beta - S\alpha S\beta C\eta_k)$$

$$\begin{aligned} N_s &= N_c = N_n = 0 \\ P_s &= P_c = P_n = 0 \\ Q_s &= Q_c = Q_n = 0 \\ R_s &= R_c = R_n = 0 \\ S_s &= S_c = S_n = 0 \\ \text{and} \quad T_s &= T_c = T_n = 0 \end{aligned} \tag{6-55}$$

Examination of Eqs. (6-55) along with Eq. (6-54) shows that the following cases are possible:

1. $S\eta_k \neq 0$ (That is, $\eta_k \neq m\pi$, $m = 0, 1, 2, \ldots$)

   This gives

$$\alpha = \beta = 0 \tag{6-56}$$

$$\begin{aligned} &C\gamma C\delta - S\gamma S\delta C\xi_k - C\lambda = 0 \\ &C\delta C\lambda - S\delta S\lambda C\psi_k - C\gamma = 0 \\ \text{and} \quad &S\gamma(S\xi_k S\psi_k - C\delta C\xi_k C\psi_k) \\ &\quad - C\gamma S\delta C\psi_k - S\lambda = 0 \end{aligned} \tag{6-57}$$

2. $S\eta_k = 0$ (That is, $\eta_k = m\pi$, $m = 0, 1, 2, \ldots$)

   This gives

$$\alpha \pm \beta = p\pi \tag{6-58}$$

$$\begin{aligned} &C\gamma C\delta - S\gamma S\delta C\xi_k - (-1)^p C\lambda = 0 \\ &C\delta C\lambda - S\delta S\lambda C\psi_k - (-1)^p C\gamma = 0 \\ \text{and} \quad &S\lambda S\psi_k - (-1)^p S\gamma S\xi_k = 0, \quad p = 0, 1, 2, \ldots \end{aligned} \tag{6-59}$$

Conditions (6-56) through (6-59) provide the necessary conditions for the existence of an R-P-R-P-P mechanism. Eqs. (6-56) and (6-58) show that the axes of the two revolute pairs are parallel to each other. Eqs. (6-57) and (6-59) are closure conditions relating the twist angles γ, δ and λ of the mechanism with the constant displacement angles ξ_k

and ψ_k at two of the three prismatic pairs (Fig. 22).

Extension of the Results to Other Mechanisms

The existence criteria derived in the above sections clearly show that the five-link 3R+2P and 2R+3P mechanisms can exist only when the axes of the revolute pairs are parallel to one another. Note that the results have been obtained by considering only the primary parts of the displacement relationships of the respective parent mechanisms. Hence, the results will remain unaffected even if one or more of the revolute pairs are replaced by helical pairs of finite pitch values. The results are, therefore, equally valid for the five-link 3H+2P, 2H+1R+2P, 1H+2R+2P, 2H+3P and 1H+1R+3P mechanisms.

Note further that the results obtained are independent of the link lengths involved. Hence, if one of the link lengths is taken to be zero, the results will apply with equal validity to four-link mechanisms derivable from the above five-link mechanisms.[1]

The results obtained in this chapter also confirm the results obtained by Hunt [25] and Waldron [52] by using the theory of screws. It is, however, important to note one significant point. The results of Hunt and Waldron were obtained by considering the H-H-H-H-H mechanism of Voinea and Atanasiu [49] which is itself an overconstrained mechanism. The results in this chapter have, on the other hand, been obtained by considering the more general zero family mechanisms. The present results, therefore, go beyond those of Hunt and Waldron and show that there are no mechanisms with two passive couplings consisting of two or three prismatic pairs other than those derived by them and

[1]See, for instance, reference [16].

confirmed in this study. Further, in addition to the parallelism of the axes, the present results also give the definite closure conditions to be satisfied by the constant kinematic parameters of the respective mechanisms.

CHAPTER VII

SUMMARY AND CONCLUSIONS

The present study is another step in the continuous search at unraveling the mysteries of space mechanisms. In this study, the existence criteria of overconstrained mechanisms with two passive couplings and consisting of revolute and prismatic pairs have been obtained by using Dimentberg's passive coupling method. This represents the first attempt in using this method after its usefulness in the case of four-link mechanisms was first demonstrated by Dimentberg.

The mechanisms considered in this study are the five-link, five-revolute (R-R-R-R-R) mechanism, the five-link R-R-R-P-R mechanism and the five-link 3R+2P and 2R+3P mechanisms. The existence criteria of the five-revolute mechanism and the R-R-R-P-R mechanism obtained in the study are new. The results obtained in the case of 3R+2P and 2R+3P mechanisms confirm the findings of other investigators.

The principal results of the investigation are as follows:

1. The existence criteria of the five-link, five-revolute mechanism with zero offset distances are obtained as two sets of 13 nonlinear algebraic equations in the ten constant kinematic parameters of the mechanism. The number of independent equations, however, appears to be less than ten. The derived criteria are satisfied identically by the Goldberg five-revolute mechanisms. This acts as a check on the correctness and

validity of the results. The derived criteria also make it possible to investigate the existence of additional five-revolute mechanisms. However, the extremely high nonlinearity and complexity of the criteria indicate that this aspect of the investigation is a problem in its own right. It is, therefore, considered beyond the scope of the present study.

2. The existence criteria of the five-link R-R-R-P-R mechanism with zero offset distances at its revolute pairs are obtained as a set of nine nonlinear algebraic equations in the 11 constant kinematic parameters of the mechanism. These equations make it possible to investigate the existence of R-R-R-P-R mechanisms by assigning arbitrary values to two of the 11 constant kinematic parameters. However, the high nonlinearity of the equations once again emphasizes the complex nature of the investigation and shows that it is a problem by itself. Hence, it is considered beyond the scope of the present study.

3. The existence criteria of the five-link 3R+2P and 2R+3P mechanisms obtained in the study show that these mechanisms (and others obtained by extending the results) exist if and only if the axes of the revolute (and/or helical) pairs are parallel to one another. This confirms the results that were obtained by Hunt and Waldron by considering the H-H-H-H-H mechanism of Voinea and Atanasiu. The results of the present study have, however, been obtained by considering the more general zero family mechanisms and give, besides the parallelism of the axes, the definite closure conditions to be satisfied by the constant kinematic parameters of the mechanisms concerned.

The present study clearly shows that the derived criteria represent only necessary conditions for existence. This is a direct consequence of the nature of Dimentberg's method. The conditions are not sufficient because satisfaction of the criteria does not by itself guarantee an overconstrained mechanism of the desired type. This is because the criteria also have trivial solutions that give mechanisms without a true mobility of one.

As indicated in Chapters IV and V, trivial solutions can be one of two types:

1. A solution becomes trivial if the constant kinematic parameters yield an overconstrained mechanism with mobility greater than one.
2. A solution becomes trivial if the constant kinematic parameters yield an overconstrained mechanism of a higher family, that is, an overconstrained mechanism having more than the required number of passive couplings. In such cases, one or more of the joints remain permanently locked, thus resulting in a mechanism without true mobility.

The triviality or non-triviality of a solution can be examined by substituting the values of the constant kinematic parameters in the original displacement relationships of the parent mechanism. If the mobility is two or more, the variable kinematic parameters in the parent mechanism become indeterminate unless two or more variables are specified. A locked joint is indicated by the fact that the variable kinematic parameter corresponding to that joint becomes constant. If neither of the above conditions is present, the solution represents a non-trivial solution and yields an overconstrained mechanism of the desired type with a true mobility of one.

Since trivial solutions always exist, the existence criteria obtained by Dimentberg's method always represent a set of consistent equations. However, all the equations in the system may not, in general, be independent. This is particularly so when the number of unknowns in the equations is less than the number of equations. It may, however, not be possible to examine the relationship between the equations analytically in all cases. When the existence criteria involve only twist angles and constant displacement angles, they can generally be expected to be comparatively simple. It may then be possible to investigate the relationship between the equations analytically. This is illustrated in the present study by the existence criteria of the 3R+2P and 2R+3P mechanisms obtained in Chapter VI where they have been examined fairly thoroughly. When, however, the existence criteria involve link lengths and constant offset distances in addition to twist angles and constant displacement angles, they can generally be expected to be complicated. It may then become very difficult to examine the relationship between the equations analytically. This is illustrated in the present study by the existence criteria of the five-revolute and R-R-R-P-R mechanisms obtained in Chapters IV and V respectively.

The present study has also shown that, while using Dimentberg's method, it is possible to get useful results and often avoid unnecessary analytical work if certain important points are borne in mind. These are discussed below:

1. When the displacement relationships involved are algebraic in nature, the Dimentberg method ultimately leads to one or more polynomial equations. The complexity and the order of these

polynomials can be appreciably reduced by considering the entire spectrum of equations available by arranging the loop-closure condition in various ways rather than by considering just a few of the available equations. This is well illustrated by the existence criteria of the five-revolute mechanism. Thus, since only some of the available equations have been considered, the results of Dimentberg lead to two sixty-fourth degree polynomials. On the other hand, since all of the available equations have been considered, the results of the present study lead to two polynomials of only the twenty-fourth degree.

2. The primary part of a dual equation contains only the primary parts of its component terms. The dual part of a dual equation, however, involves both the primary and the dual parts of its component terms. The dual part of any dual equation is, therefore, always more complicated than its primary part. Now, when passive coupling is imposed on a cylinder pair to reduce it to a prismatic pair, restrictions are put on only the rotation at the cylinder pair and thus one has to deal with the primary parts of the concerned displacement relationships. However, when passive coupling is imposed on a cylinder pair to reduce it to a revolute pair, restrictions are put on only the translation at the cylinder pair and thus one has to deal with the dual parts of the concerned displacement relationships. It, therefore, follows that the analytical work involved in reducing a cylinder pair to a prismatic pair is always much less complicated than in reducing that cylinder pair to a revolute pair. This is well illustrated in the present study

in which it can be seen that the existence criteria of the R-R-R-P-R mechanism obtained in Chapter V are much less complicated than those of the five-revolute mechanism obtained in Chapter IV.

3. When the displacement relationships involved are algebraic in nature, the Dimentberg method often involves examination of the common roots between two polynomials or successive sets of two polynomials. In all cases, it is necessary to consider only one common root between the equations involved. This leads to the most general results. It is, however, theoretically possible to consider more than one common root between the equations involved. This results in more severe conditions on the coefficients of the equations involved. The resultant conditions, however, represent only special cases of the more general case obtained by considering only one common root. This, of course, is to be expected because when two equations have more than one common root, it certainly implies that they have one common root.

4. If the parent mechanism has no helical pairs, the existence criteria of the derived overconstrained mechanisms are algebraic in nature. If the parent mechanism contains helical pairs, the derived existence criteria remain algebraic in nature if only the rotations at the helical pairs are involved. The results, however, become non-algebraic if both the rotations and the translations at the helical pairs are involved. Thus, in the present study, the existence criteria obtained are all algebraic in nature because the parent mechanisms consid-

ered do not have any helical pairs. Further, it is possible to extend the results obtained in Chapter VI to the indicated mechanisms with helical pairs because only the rotations at the appropriate pairs are considered. On the other hand, the existence criteria of a mechanism like the general H-H-H-H-H mechanism are expected to be non-algebraic in nature since they are expected to involve both the rotations and the translations at the helical pairs.

The present study has also demonstrated the general usefulness, applicability and scope of Dimentberg's method for obtaining the existence criteria of overconstrained mechanisms. Even though a high level of algebra is often required, the method has the following distinct points in its favor:

1. The most important feature of the method is the assurance of the finite mobility of the derived overconstrained mechanisms. Since one starts with a parent mechanism of assured finite mobility, the finite mobility of the derived mechanisms is assured.
2. The method is capable of yielding the necessary conditions for the existence of an overconstrained mechanism. These include all possible solutions. The method thus has the feature of uniqueness and completeness.
3. The method clearly shows that, in general, the mobility of an overconstrained mechanism is a function of all of its constant kinematic parameters. The so-called paradoxical mechanisms, therefore, no longer remain paradoxical since it can be shown that they exist only because their constant kinematic parame-

ters satisfy certain definite mathematical relationships.

4. The derived criteria permit the computation of the constant kinematic parameters of an overconstrained mechanism.

The results of the present study have clearly demonstrated that the investigation of the conditions for the existence of an overconstrained mechanism consists of two distinct steps:

1. The first step is to obtain or derive the existence criteria. These criteria are in reality a set (or sets) of equations relating the constant kinematic parameters of the overconstrained mechanism. The derived criteria provide necessary, but not sufficient, conditions for existence. Further, the criteria represent a consistent system of equations.
2. The second step is to obtain a compatible set of constant kinematic parameters of the overconstrained mechanism satisfying the derived criteria. When the derived criteria are comparatively simple, it may be possible to examine them analytically and obtain simple functional relationships between the constant kinematic parameters. However, when the derived criteria are very complicated, it may not be possible to examine them analytically. In such cases, numerical methods have to be resorted to in order to obtain a compatible set of constant kinematic parameters satisfying the derived criteria.

Except in very simple cases, each of the above two steps can be regarded as a problem by itself. Thus, for instance, the existence criteria of a five-link, five-revolute mechanism with non-zero offset distances are expected to lead to two sixty-fourth degree polynomials which, in turn, lead to 130 conditions on its constant kinematic parameters. It can be seen that errors are apt to be introduced if such high order polynomials and such a large number of equations are not carefully handled. Again, the examination of the resultant conditions in order to obtain a compatible set of constant kinematic parameters presents a task of formidable proportions.

The present study shows that the mobility of space mechanisms is a field of continued interest and challenge. In the coming years, the following important areas of research appear to offer great promise:

1. The derivation of the existence criteria of the H-H-H-H-H-H, H-H-H-H-H and H-H-H-H space mechanisms. These mechanismss represent the most general overconstrained mechanisms with one, two and three passive couplings respectively. The existence criteria of all other overconstrained mechanisms within these families can be obtained as special cases of the existence criteria of these mechanisms by proper selection of pitch values and constant kinematic parameters.
2. Development of suitable mathematical methods to obtain the constant kinematic parameters of overconstrained mechanisms from their existence criteria. It should be possible to utilize the derived criteria to generate new mechanisms.
3. Investigation of the type of motion provided by overconstrained mechanisms. This is required in order to fully utilize the capabilities of space mechanisms.

The present study represents another attempt in understanding the nature of space mechanisms. Similar studies in future can be expected to lead to a greater insight into their nature and thus make it possible to unravel the mysteries of space mechanisms.

BIBLIOGRAPHY

1. Artobolevskii, I. I., Theory of Mechanisms and Machines, Moscow, 1953.

2. Ball, Sir R. S., A Treatise on the Theory of Screws, Cambridge University Press, Cambridge, England, 1900.

3. Barnes, J. G. P., "An Algorithm for Solving Non-linear Equations Based on the Secant Method," The Computer Journal, Vol. 8, No. 1, April 1965, pp. 66-72.

4. Bennett, G. T., "A New Mechanism," Engineering, Vol. 76, Dec. 4, 1903, pp. 777-778.

5. Bocher, M., Introduction to Higher Algebra, The Macmillan Company, New York, 1907.

6. Borofsky, S., Elementary Theory of Equations, The Macmillan Company, New York, 1950.

7. Broyden, C. G., "A Class of Methods for Solving Nonlinear Simultaneous Equations," Mathematics of Computation, Vol. 19, No. 92, Oct. 1965, pp. 577-593.

8. Broyden, C. G., "Quasi-Newton Methods and Their Application to Function Minimisation," Mathematics of Computation, Vol. 21, No. 99, July 1967, pp. 368-381.

9. Broyden, C. G., "A New Method of Solving Nonlinear Simultaneous Equations," The Computer Journal, Vol. 12, No. 1, Feb. 1969, pp. 94-99.

10. Chasles, M., "Note sur les propriétés générales du système de deux corps . . . ," Bull. Sci. mathématiques Ferussac, Vol. 14, 1830, pp. 321-326.

11. Davidenko, D. F., "On A New Method of Numerical Solution of Systems of Nonlinear Equations," Doklady Akad. Nauk SSSR (N.S.), Vol. 88, 1953, pp. 601-602.

12. Deist, F. H., and Sefor, L., "Solution of Systems of Non-linear Equations by Parameter Variation," The Computer Journal, Vol. 10, No. 1, May 1967, pp. 78-82.

13. Dimentberg, F. M., "A General Method for the Investigation of Finite Displacements of Spatial Mechanisms and Certain Cases of Passive Joints," Akad. Nauk. SSSR Trudi Sem. Teorii Mash. Mekh., Vol. 5, No. 17, 1948, pp. 5-39.

14. Dimentberg, F. M., "The Determination of the Positions of Spatial Mechanisms," Izd. Akad. Nauk SSSR, Moscow, 1950.

15. Dimentberg, F. M., The Screw Calculus and Its Applications in Mechanics, Izdatel'stvo "Nauka," Glavnaya Redaktsiya, Fiziko-Matematicheskoy Literatury, Moscow, 1965.

16. Dimentberg, F. M., and Yoslovich, I. V., "A Spatial Four-link Mechanism Having Two Prismatic Pairs," Journal of Mechanisms, Vol. 1, 1966, pp. 291-300.

17. Dobrjanskyj, L., and Freudenstein, F., "Some Applications of Graph Theory to the Structural Analysis of Mechanisms," Journal of Engineering for Industry, February 1967, pp. 153-158.

18. Freudenstein, F., and Roth, B., "Numerical Solution of Systems of Nonlinear Equations," Journal of the Association for Computing Machinery, Vol. 10, 1963, pp. 550-556.

19. Gilmartin, M. J., and Duffy, J., "Limit Positions of Four-Link Spatial Mechanisms--2. Mechanisms Having Revolute, Cylindric and Prismatic Pairs," Journal of Mechanisms, Vol. 4, 1969, pp. 273-281.

20. Goldberg, M., "New Five-Bar and Six-Bar Linkages in Three Dimensions," Trans. ASME, Vol. 65, No. 6, August 1943, pp. 649-661.

21. Goldstein, A. A., Constructive Real Analysis, Harper and Row, New York, 1967.

22. Harrisberger, L., "A Number Synthesis Survey of Three-Dimensional Mechanisms," Journal of Engineering for Industry, May 1965, pp. 213-220.

23. Harrisberger, L., and Soni, A. H., "A Survey of Three-Dimensional Mechanisms with One General Constraint," ASME 9th Mechanisms Conference, 1966, Paper No. 66-MECH-44.

24. Henrici, O., "The Theory of Screws," Nature, Vol. 42, No. 1075, June 5, 1890, pp. 127-132.

25. Hunt, K. H., "Prismatic Pairs in Spatial Linkages," Journal of Mechanisms, Vol. 2, 1967, pp. 213-230.

26. Hunt, K. H., "Screw Axes and Mobility in Spatial Mechanisms via the Linear Complex," Journal of Mechanisms, Vol. 3, 1967, pp. 307-327.

27. Hunt, K. H., "Note on Complexes and Mobility," Journal of Mechanisms, Vol. 3, 1968, pp. 199-202

28. McLarnan, C. W., "Synthesis of Six-Link Plane Mechanisms by Numerical Analysis," Journal of Engineering for Industry, February 1963, pp. 5-11.

29. Moroshkin, Iu. F., "On the Geometry of Compound Kinematic Chains," Soviet Phys.-Doklady 3, 2, 1958, pp. 269-272 (Translation of Doklady Akad. Nauk SSSR (N.S.) 119, 1, 38-41, Mar.-Apr. 1958, by Amer. Inst. Phys., Inc., New York).

30. Myard, F. E., "Contribution a la géométrie des systèmes articulés, Bull. Soc. math. France, Vol. 59, 1931, pp. 183-210.

31. Ogino, S., and Watanabe, K., "Study on Spatial Four-Link Chain Having Four Cylindrical Pairs," Bulletin of JSME, Vol. 12, No. 49, February 1969, pp. 97-113.

32. Pamidi, P. R., Communication on the research note: "Solution of Kinematic Equations by Newton's Method," The Journal of Mechanical Engineering Science, Vol. 11, No. 3, 1969, pp. 346.

33. Pelecudi, C., and Soni, A. H., "Compatibility Conditions as the Existence Criteria of Three Dimensional Mechanisms," Rev. Roum. Sci. Techn. Mec. Appl., 1970. In press.

34. Philipp, R. E., and Freudenstein, F., "Synthesis of Two-Degree-of-Freedom Linkages--A Feasibility Study of Numerical Methods of Synthesis of Bivariate Function Generators," Journal of Mechanisms, Vol. 1, 1966, pp. 9-21.

35. Poinsot, L., "Sur la composition . . . des aires," Journal de l'Ecole Polytechnique, Vol. 6, 1806, pp. 182-205.

36. Reuleaux, F., The Kinematics of Machinery, Dover Publications, Inc., New York, 1963.

37. Roth, B., and Freudenstein, F., "Synthesis of Path-Generating Mechanisms by Numerical Methods," Journal of Engineering for Industry, August 1963, pp. 298-306.

38. Semple, J. G., and Roth, L., Algebraic Geometry, Oxford University Press, London, England, 1949.

39. Sharikov, V. I., "Theory of Screws in the Structural and Kinematic Analysis of Mechanisms," Trudi Inst. Maschinoved. Akad. Nauk SSSR, Vol. 22, 85/86, 1961, pp. 108-136.

40. Soni, A. H., The Existence Criteria of One-General Constraint Mechanisms, Doctoral Dissertation, Oklahoma State University, Stillwater, Oklahoma, May 1967.

41. Soni, A. H., Discussion on the paper: "Displacement Analysis of Spatial Five-Link Mechanisms Using (3x3) Matrices With Dual-Number Elements," Journal of Engineering for Industry, August 1969, pp. 921-923.

42. Soni, A. H., and Harrisberger, L., "Theory of Identifying the Existence of General Constraints," VDI Mechanisms Conference, West Germany, April 3-4, 1968.

43. Soni, A. H., and Harrisberger, L., "Existence Criteria of Mechanisms," Rev. Roum. Sci. Techn.--Mec. Appl., Vol. 14, No. 2, 1969, pp. 373-394.

44. Soni, A. H., and Harrisberger, L., "Existence Criteria of Six-Link, Six-Revolute Space Mechanism," Proceedings of the Second International Congress on Theory of Machines and Mechanisms, Warsaw, Poland, Sept. 1969.

45. Soni, A. H., and Harrisberger, L., "Design of a Spatial Four-Link Mechanism with or without Passive Constraint," Journal of Mechanisms, Vol. 4, 1969, pp. 337-348.

46. Soni, A. H., and Pelecudi, C., "Détermination vectorielle des équations générales de mouvement des mécanismes tridimensionnels," Conférence Nationale de Mecanique Appliquée, Bucarest, June, 1969.

47. Todd, J., ed., Survey of Numerical Analysis, McGraw-Hill Book Company, Inc., New York, 1962.

48. Turner, L. R., "Solution of Nonlinear Systems," Ann. New York Acad. Sci., Vol. 86, 1960, pp. 817-827.

49. Voinea, R. P., and Atanasiu, M. C., "Geometrical Theory of Screws and Some Applications to the Theory of Mechanisms," Revue de Mécanique Appliquée, Vol. 7, No. 4, 1962, pp. 845-860.

50. Waldron, K. J., "The Constraint Analysis of Mechanisms," Journal of Mechanisms, Vol. 1, 1966, pp. 101-114.

51. Waldron, K. J., "Application of the Theory of Screw Axes to Linkages Which Disobey the Kutzbach-Grübler Constraint Criterion," ASME 9th Mechanisms Conference, 1966, Paper No. 66-MECH-36.

52. Waldron, K. J., "A Family of Overconstrained Linkages," Journal of Mechanisms, Vol. 2, 1967, pp. 201-211.

53. Waldron, K. J., "Hybrid Overconstrained Linkages," Journal of Mechanisms, Vol. 3, 1968, pp. 73-78.

54. Waldron, K. J., "Symmetric Overconstrained Linkages," Journal of Engineering for Industry, February 1969, pp. 158-164.

55. Yang, A. T., and Freudenstein, F., "Application of Dual-Number Quaternion Algebra to the Analysis of Spatial Mechanisms," Journal of Applied Mechanics, June 1964, pp. 300-308.

56. Yang, A. T., "Displacement Analysis of Spatial Five-Link Mechanisms Using (3x3) Matrices With Dual-Number Elements," Journal of Engineering for Industry, February 1969, pp. 152-157.

APPENDIX A

DEFINITIONS AND EXPLANATION OF TERMS USED

1. Mechanism. A closed kinematic chain in which one of the links is fixed is called a mechanism.
2. Mobility of a Mechanism. The mobility of a mechanism is the number of independent quantities required to specify its motion completely. A mechanism with mobility one is said to have constrained motion.
3. Constant and Variable Kinematic Parameters of a Mechanism. The constant kinematic parameters of a mechanism are the link lengths, the twist angles, the constant offset distances (or kink links, as they are sometimes called) and the constant displacement angles. These parameters are constant for a given mechanism and remain unchanged during its motion.

   The variable kinematic parameters of a mechanism are the variable offset distances (or translations) along its pair axes and the variable displacement angles. These parameters are not constant for a given mechanism, but vary during its motion.

   In the present study, the link lengths are denoted by a, b, c, d and e, the twist angles by α, β, γ, δ and λ, the variable offset distances by x, u, w, v and y, the constant offset distances by x_k, u_k, w_k, v_k and y_k, the variable displacement angles by ϕ, η, χ, ξ and ψ and the constant displacement angles by ϕ_k, η_k, χ_k, ξ_k and ψ_k.

4. Finite Mobility, Transitory Mobility and True Mobility of a Mechanism. A mechanism is said to have finite mobility when it is capable of executing motion over a finite range. A mechanism is said to have transitory or instantaneous mobility when it is capable of executing motion over only an infinitesimal range. Thus, for example, a spherical four-link, four-revolute mechanism has a finite mobility of one. However, if the revolute pairs are replaced by helical pairs of equal pitch values, the resulting configuration will not have finite mobility, but only a transitory mobility of one [26]. It may also be noted that instantaneous mobility at all instants may often lead to finite mobility [25, 52].

A mechanism is said to have true mobility when it has finite mobility with all the freedoms in all of its joints active. A mechanism does not have true mobility when it has finite mobility with some of the freedoms in some of its joints not active. If this effect occurs only at certain discrete positions, then those configurations of the mechanism represent its locking positions (limit positions or dead center positions) [25, 52]. Thus, for instance, a plane four-link four-revolute mechanism has, except at its locking positions, a true mobility of one, but a five-link H-P-P-P-P space mechanism does not have true mobility since its helical pair remains permanently locked.

In the context of the present study, a mechanism is said to "exist" when it has a true mobility of one.

5. Zero family Mechanisms, Overconstrained Mechanisms and Passive Couplings. Consider a single-loop space mechanism. Let f_i denote the number of degrees of freedom permitted at the i^{th} joint. $\sum f_i$

then denotes the total number of degrees of freedom permitted at all the joints.

When $\sum f_i = 7$, any random combination of constant kinematic parameters will, in general, yield a mechanism with mobility one.[1] Such mechanisms in which there are, therefore, no conditions imposed on the constant kinematic parameters are called zero family mechanisms [22].[2] The R-C-C-C mechanism, the R-C-R-C-R mechanism and the R-R-R-R-C-R mechanism are some examples of zero family mechanisms.

When $\sum f_i < 7$, a random combination of constant kinematic parameters will, in general, yield a configuration which is a structure.[3] Mechanisms with $\sum f_i < 7$ can exist with mobility one only when their constant kinematic parameters satisfy certain definite mathematical relationships. Such mechanisms in which there are, therefore, conditions imposed on the constant kinematic parameters are called overconstrained mechanisms. The plane and spherical four-link mechanisms, the Bennett mechanism [4] and the Goldberg mechanisms [20] are some examples of overconstrained mechanisms. Note that in all of these mechanisms, the constant kinematic parameters satisfy certain definite relationships.

A zero family mechanism will function as an overconstrained mechanism if its constant kinematic parameters are so chosen as to

---

[1]This is not always true. Thus, for instance, a seven-link P-P-P-P-P-P-P space mechanism ($\sum f_i = 7$) with a random combination of constant kinematic parameters yields a configuration with mobility four.

[2]The mechanism "series" mentioned in reference [22] is referred to here as the mechanism "family." See also reference [40].

[3]This is not always true. Thus, for example, a four-link P-P-P-P space mechanism ($\sum f_i = 4 < 7$) with a random combination of constant kinematic parameters yields a configuration with mobility one.

make the freedoms in some of its joints passive. Thus, for example, if the kinematic axes in an R-C-C-C mechanism are taken to be parallel to one another, the translational freedoms in all the cylinder pairs will become passive and the mechanism will, in effect, function as a plane four-link mechanism. Overconstrained mechanisms can, therefore, be considered to have "passive couplings" and all overconstrained mechanisms can be regarded as special cases of appropriate zero family mechanisms on which suitable "passive coupling" conditions have been imposed. The passive couplings are, in effect, conditions imposed on the constant kinematic parameters.

The number of passive couplings C_p in an overconstrained mechanism is given by the simple relationship

$$C_p = 7 - \sum f_i \tag{A-1}$$

where $\sum f_i$ denotes the total number of degrees of freedom permitted at all the joints of the overconstrained mechanism. Observe that the value of C_p given by the above relationship also gives the family number of the overconstrained mechanism [22]. Zero family mechanisms thus do not have any passive couplings.

6. <u>Existence criteria of an Overconstrained Mechanism</u>. In the context of the present study, the existence criteria of an overconstrained mechanism denote a set (or sets) of conditions that are necessary for its existence. These conditions are equations relating the constant kinematic parameters of the mechanism. An overconstrained mechanism of the prescribed type satisfies all of the conditions forming the existence criteria simultaneously.

APPENDIX B

CONDITION FOR COMMON ROOTS

The number of common roots of two polynomials is decided entirely by their coefficients. In particular, certain matrices formed from the coefficients play an important role in the examination of the number of common roots.

Consider the following two equations:

$$F_m(x) = a_m x^m + a_{m-1}x^{m-1} + \dots + a_2x^2 + a_1x + a_0 = 0 \qquad \text{(B-1)}$$

$$f_n(x) = b_n x^n + b_{n-1}x^{n-1} + \dots + b_2x^2 + b_1x + b_0 = 0 \qquad \text{(B-2)}$$

By using the coefficients of the two polynomials, we first form the matrix Δ_1 shown on the next page. This is a square matrix with (m+n) rows and (m+n) columns. It consists of two groups of rows. The first group consists of n rows and is formed from the coefficients of $F_m(x)$; the second group consists of m rows and is formed from the coefficients of $f_n(x)$.

Matrices Δ_2, Δ_3, . . . , Δ_k, . . . are obtained from the matrix Δ_1 by the deletion of suitable rows and columns. Thus, matrix Δ_2 is obtained by deleting the first column of Δ_1 and the first row from each of the two groups of rows in Δ_1. Matrix Δ_3 is obtained by deleting the first two columns of Δ_1 and the first two rows from each of the two groups of rows in Δ_1. In general, the matrix Δ_k is obtained by deleting the first (k-1) columns of Δ_1 and the first (k-1) rows from each of the

$$\Delta_1 = \left[\begin{array}{ccccccccccc}
a_m & a_{m-1} & a_{m-2} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\
0 & a_m & a_{m-1} & \cdots & a_3 & a_2 & a_1 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & a_m & a_{m-1} & a_{m-2} & \cdots & a_1 & a_0 & 0 \\
0 & 0 & 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_2 & a_1 & a_0 \\
b_n & b_{n-1} & b_{n-2} & \cdots & b_2 & b_1 & b_0 & \cdots & 0 & 0 & 0 \\
0 & b_n & b_{n-1} & \cdots & b_3 & b_2 & b_1 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & b_n & b_{n-1} & b_{n-2} & \cdots & b_1 & b_0 & 0 \\
0 & 0 & 0 & \cdots & 0 & b_n & b_{n-1} & \cdots & b_2 & b_1 & b_0
\end{array}\right]
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} n \text{ rows} \\
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} m \text{ rows}
\end{array}
\qquad (B - 3)$$

two groups of rows in Δ_1.

The following two important theorems deal with the number of common roots [6].

Theorem 1. If $F_m(x)$ and $f_n(x)$ have p or more common roots, then rank of Δ_p is less than or equal to (m+n-2p+1).

Theorem 2. If a_m or b_n is non-zero and the rank of Δ_p is less than or equal to (m+n-2p+1), then $F_m(x)$ and $f_n(x)$ have at least p common roots.

For the particular case of one common root, the following corollary follows directly from Theorem 1:

Corollary. If $F_m(x)$ and $f_n(x)$ have one common root, the determinant of Δ_1 vanishes.

The above theorems can be best illustrated by examples.

Example 1. Find the value of q if the two equations

$$\begin{aligned} F_3(x) &= x^3 - 7x + q = 0 \\ \text{and} \quad f_2(x) &= x^2 + x - 2 = 0 \end{aligned}$$

have at least one common root.

Solution. Here we have m = 3, n = 2 and p = 1. From Theorem 1, it follows that the rank of Δ_1 must be less than or equal to (3+2-2+1), that is, 4. We have

$$\Delta_1 = \begin{bmatrix} 1 & 0 & -7 & q & 0 \\ 0 & 1 & 0 & -7 & q \\ 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}$$

The rank of the above matrix should not exceed 4. Its determinant must, therefore, vanish. This gives the condition $q = \pm 6$.

If $q = 6$, the polynomial $F_3(x)$ is $x^3 - 7x + 6 = 0$. $F_3(x)$ and $f_2(x)$ then have 1 as their common root.

If $q = -6$, the polynomial $F_3(x)$ is $x^3 - 7x - 6 = 0$. $F_3(x)$ and $f_2(x)$ then have -2 as their common root.

Example 2. Examine the number of common roots between the following two equations:

$$F_3(x) = x^3 - 2x^2 - x + 2 = 0$$

$$f_3(x) = x^3 + 3x^2 - x - 3 = 0$$

Solution. Here we have $m = 3$ and $n = 3$. We also have

$$\Delta_1 = \begin{bmatrix} 1 & -2 & -1 & 2 & 0 & 0 \\ 0 & 1 & -2 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 \\ 1 & 3 & -1 & -3 & 0 & 0 \\ 0 & 1 & 3 & -1 & -3 & 0 \\ 0 & 0 & 1 & 3 & -1 & -3 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} 1 & -2 & -1 & 2 & 0 \\ 0 & 1 & -2 & -1 & 2 \\ 1 & 3 & -1 & -3 & 0 \\ 0 & 1 & 3 & -1 & -3 \end{bmatrix}$$

$$\text{and} \quad \Delta_3 = \begin{bmatrix} 1 & -2 & -1 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$$

The rank of Δ_1 is less than 6. Hence, from Theorem 2, there is at least one common root.

The rank of Δ_2 is less than 4. Hence, there are at least two common roots.

The rank of Δ_3 is 2. Hence, there cannot be three common roots.

It, therefore, follows that the two given equations have two common roots. It can be seen that these two common roots are 1 and -1.

APPENDIX C

DERIVATION OF THE R-P-C-P AND R-C-P-P MECHANISMS FROM THE R-C-C-C MECHANISM

The existence criteria of the R-P-C-P and R-C-P-P space mechanisms can be obtained from the displacement relationships of an R-C-C-C space mechanism.

Consider the R-C-C-C space mechanism shown schematically in Fig. 23. By suppressing the rotational freedom of the cylinder pair at the output joint D, it is possible to examine the conditions for the existence of a prismatic pair in this mechanism.

The relationship between the input angle ϕ and the output angle ψ of the mechanism in Fig. 23 is given by [55]

$$(F_{22}\Phi^2+F_{20})\Psi^2 + F_{11}\Phi\Psi + (F_{02}\Phi^2+F_{00}) = 0 \qquad \text{(C-1)}$$

where $\Phi = \tan(\phi/2)$

$\Psi = \tan(\psi/2)$

and

$$\begin{aligned} F_{22} &= C(\delta-\alpha-\gamma) - C\beta \\ F_{20} &= C(\delta+\alpha-\gamma) - C\beta \\ F_{11} &= 4S\alpha S\gamma \\ F_{02} &= C(\delta-\alpha+\gamma) - C\beta \\ F_{00} &= C(\delta+\alpha+\gamma) - C\beta \end{aligned} \qquad \text{(C-2)}$$

Let the angle ψ be now made constant for varying values of the

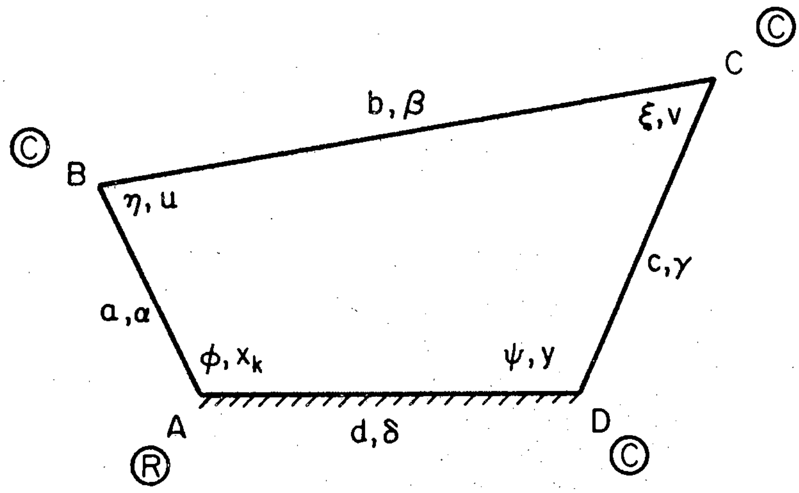

Figure 23. R-C-C-C Space Mechanism

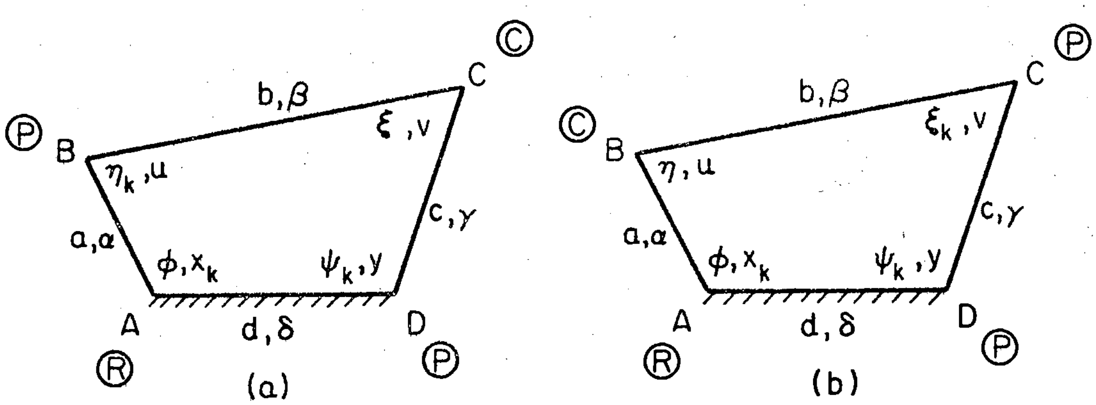

Figure 24. R-P-C-P and R-C-P-P Space Mechanisms Obtained from the Mechanism in Fig. 23 by Making $\psi = \psi_k$ = a Constant

angle ϕ. The cylinder pair at joint D (Fig. 23) then reduces to a prismatic pair. Denoting the constant value of ψ by ψ_k and the corresponding value of Ψ by Ψ_k, Eq. (C-1) becomes

$$(\Psi_k^2 F_{22}+F_{02})\Phi^2 + \Psi_k F_{11}\Phi + (\Psi_k^2 F_{20}+F_{00}) = 0 \qquad \text{(C-3)}$$

The above equation must hold good at varying values of the variable Φ. Its coefficients must, therefore, vanish. This gives

$$\begin{aligned} \Psi_k^2 F_{22} + F_{02} &= 0 \\ \Psi_k F_{11} &= 0 \\ \text{and} \quad \Psi_k^2 F_{20} + F_{00} &= 0 \end{aligned} \qquad \text{(C-4)}$$

Examination of the above equations shows that the following cases are possible:

(a) $\Psi_k = 0$ (That is, $\psi_k = 2n\pi$, n = 0, 1, 2, . . .)

This gives

$$\begin{aligned} F_{02} &= 0 \\ \text{and} \quad F_{00} &= 0 \end{aligned} \qquad \text{(C-5)}$$

(b) $\Psi_k = \infty$ (That is, $\psi_k = (2n+1)\pi$, n = 0, 1, 2, . . .)

This gives

$$\begin{aligned} F_{22} &= 0 \\ \text{and} \quad F_{20} &= 0 \end{aligned} \qquad \text{(C-6)}$$

(c) $\Psi_k \neq 0$ and $\Psi_k \neq \infty$ (That is $\psi_k \neq n\pi$, n = 0, 1, 2, . . .)

This gives

$$\begin{aligned} \Psi_k^2 F_{22} + F_{02} &= 0 \\ F_{11} &= 0 \\ \text{and} \quad \Psi_k^2 F_{20} + F_{00} &= 0 \end{aligned} \qquad \text{(C-7)}$$

Substitution of relations (C-2) in Eqs. (C-5), (C-6) and (C-7) and examination of the resultant equations show that the above cases together give the following three independent sets of solutions:

Solution 1

$$\begin{aligned} \delta + \gamma &= n\pi \\ \alpha \pm \beta &= n\pi \\ \text{and} \quad \psi_k &= 2n\pi \\ \text{for } n &= 0, 1, 2, \ldots \end{aligned} \tag{C-8}$$

Solution 2

$$\begin{aligned} \delta - \gamma &= n\pi \\ \alpha \pm \beta &= n\pi \\ \text{and} \quad \psi_k &= (2n+1)\pi \\ \text{for } n &= 0, 1, 2, \ldots \end{aligned} \tag{C-9}$$

Solution 3

$$\begin{aligned} \alpha &= 0 \text{ or } \pi \\ \text{and} \quad S\delta S\gamma C\psi_k - C\delta C\gamma \pm C\beta &= 0 \end{aligned} \tag{C-10}$$

Substitution of the above conditions in the displacement relationships of the parent R-C-C-C mechanism [55] show that Solutions 1 and 2 give a prismatic pair at joint B in addition to a prismatic pair at joint D. These solutions, therefore, give an R-P-C-P mechanism [Fig. 24(a)]. They also show that the axes of the revolute pair at joint A and the cylinder pair at joint C are parallel to each other.

Solution 3 gives a prismatic pair at joint C in addition to a prismatic pair at joint D. It, therefore, gives an R-C-P-P mechanism [Fig. 24(b)]. It also shows that the axes of the revolute pair at joint A and the cylinder pair at joint B are parallel to each other.

The above results thus lead to the conclusion that, in an R-C-C-C mechanism, when one cylinder pair is reduced to a prismatic pair, another cylinder pair is also reduced to a prismatic pair. Further, the axes of the revolute pair and the remaining cylinder pair are then parallel to each other. These results agree with those obtained by Dimentberg and Yoslovich [16].[1]

[1]See also reference [19].

APPENDIX D

DERIVATION OF THE R-C-R-C MECHANISM FROM THE R-C-R-C-R MECHANISM

The existence criteria of an R-C-R-C space mechanism can be obtained from the displacement relationships of an R-C-R-C-R space mechanism.

Consider the R-C-R-C-R space mechanism shown schematically in Fig. 25. Note that the constant offset distances at the three revolute pairs are taken to be zero. This mechanism reduces to an R-C-R-C mechanism if the output angle ψ is forced to be constant at all positions of the mechanism. If, in addition, we take $e = 0$ and $\lambda = 0$, the resulting R-C-R-C mechanism reduces to the conventional form shown in Fig. 26.

With $e = 0$ and $\lambda = 0$, the input-output relationship of the mechanism in Fig. 25 is given by [56]

$$(G_0+G_2\Phi^2)\Psi^2 + G_1\Phi\Psi + (G_2+G_0\Phi^2) = 0 \tag{D-1}$$

where
$$\Phi = \tan(\phi/2)$$
$$\Psi = \tan(\psi/2)$$

and
$$\begin{aligned} G_0 &= bS\gamma C\gamma + cS\beta C\beta - (bC\beta S\gamma+cS\beta C\gamma)C(\delta-\alpha) \\ &\quad - (d-a)S(\delta-\alpha)S\beta S\gamma \\ G_2 &= bS\gamma C\gamma + cS\beta C\beta - (bC\beta S\gamma+cS\beta C\gamma)C(\delta+\alpha) \\ &\quad - (d+a)S(\delta+\alpha)S\beta S\gamma \end{aligned} \tag{D-2}$$

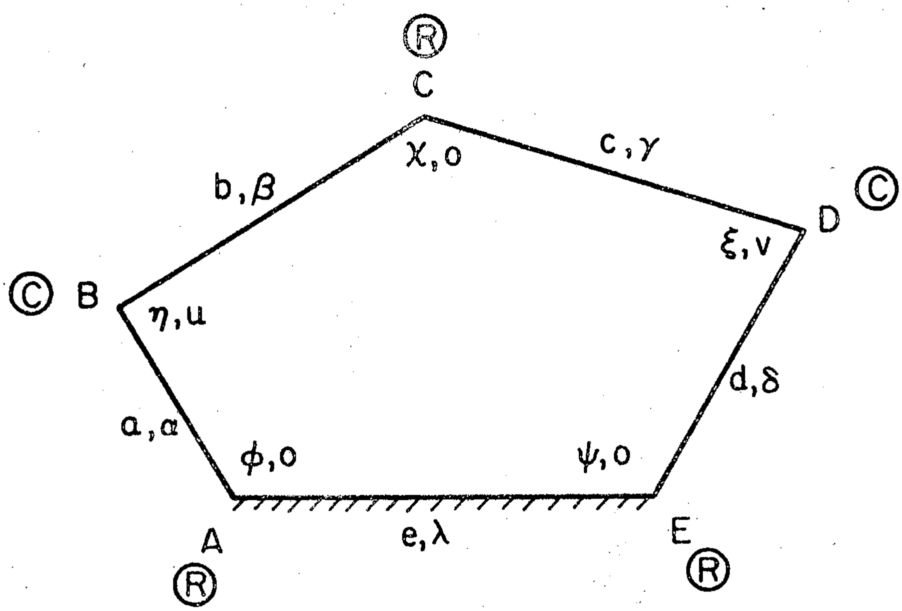

Figure 25. R-C-R-C-R Space Mechanism

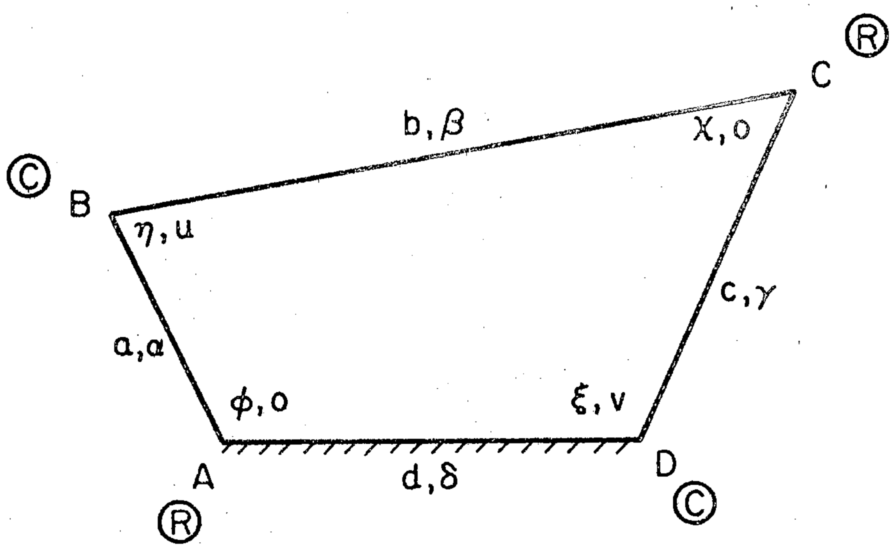

Figure 26. R-C-R-C Space Mechanism Obtained from the Mechanism in Fig. 25 by Taking $e = 0$ and $\lambda = 0$ and by Making ψ = a Constant

and $G_1 = 4(aC\alpha S\beta S\gamma S\delta - bS\alpha C\beta S\gamma S\delta - cS\alpha S\beta C\gamma S\delta + dS\alpha S\beta S\gamma C\delta)$

Let the angle ψ be now held constant for varying values of the angle ϕ. Denoting the constant value of ψ by ψ_k and the corresponding value of Ψ by Ψ_k, Eq. (D-1) becomes

$$(\Psi_k^2 G_2 + G_0)\Phi^2 + \Psi_k G_1 \Phi + (\Psi_k^2 G_0 + G_2) = 0 \qquad \text{(D-3)}$$

The above equation must be valid for varying values of the variable Φ. Its coefficients must, therefore, vanish. This gives

$$\begin{aligned} \Psi_k^2 G_2 + G_0 &= 0 \\ \Psi_k G_1 &= 0 \\ \text{and} \quad \Psi_k^2 G_0 + G_2 &= 0 \end{aligned} \qquad \text{(D-4)}$$

Examination of the above equations shows that the following cases are possible:

(a) $\Psi_k = 0$ or ∞ (That is, $\psi_k = n\pi$, n = 0, 1, 2, . . .)

This gives

$$\begin{aligned} G_0 &= 0 \\ \text{and} \quad G_2 &= 0 \end{aligned} \qquad \text{(D-5)}$$

(b) $\Psi_k \neq 0$ and $\Psi_k \neq \infty$ (That is, $\psi_k \neq n\pi$, n = 0, 1, 2, ...)

This gives either

$$\begin{aligned} G_0 &= 0 \\ G_2 &= 0 \\ \text{and} \quad G_1 &= 0 \end{aligned} \qquad \text{(D-6)}$$

or

$$\begin{aligned} G_0 + G_2 &= 0 \\ \text{and} \quad G_1 &= 0 \end{aligned} \qquad \text{(D-7)}$$

Substitution of relations (D-2) in Eqs. (D-5), (D-6) and (D-7) and examination of the resultant equations show that they all lead to the same results which, after simplification, can be written as follows:

$$S\alpha S\delta(bC\beta S\gamma+cS\beta C\gamma) = S\beta S\gamma(aC\alpha S\delta+dS\alpha C\delta) \tag{D-8}$$

$$S\alpha S\delta(bS\gamma C\gamma+cS\beta C\beta) = S\beta S\gamma(aS\delta C\delta+dS\alpha C\alpha) \tag{D-9}$$

The above equations represent the necessary conditions for the existence of an R-C-R-C space mechanism with zero offset distances at its revolute pairs and are identical with the results that were obtained by Dimentberg from the displacement relationships of an R-C-C-C space mechanism [13, 14].

APPENDIX E

NUMERICAL SOLUTION OF NONLINEAR SIMULTANEOUS EQUATIONS

Except in very simple cases, the solution of systems of nonlinear equations can be attempted only by numerical means. Of the various methods that are available, Newton's method, which is a second-order iterative process [47], is generally preferable [48]. This method, like other functional iterative methods, requires the selection of an initial approximation to the solution of the problem. The approximation is then continuously improved until there is convergence or it is clear that there is no convergence.

Let

$$f_i(x_1, x_2, \ldots, x_n) = 0, \quad i = 1, 2, \ldots, n \qquad \text{(E-1)}$$

represent a system of n nonlinear equations in n unknowns. Using the initial approximation to the solution, the functions f_i and their partial derivatives are first evaluated. The "corrections" Δx_j are then calculated by solving the set of linear equations

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\Delta x_j) = -f_i, \quad i = 1, 2, \ldots, n \qquad \text{(E-2)}$$

The corrections are then added to original solution and the procedure repeated until there is convergence or it is obvious that there is no convergence.

The Newton method described above is characterized by two important features. One is the need for a good initial approximation to the final solution. The other is the necessity of evaluating and inverting the Jacobian [21] (The coefficient matrix in Eq. (E-2)) at every stage of the iteration.

The need to choose a good initial approximation to the solution may not be a severe restriction in many cases, but it becomes quite important when one is dealing with a large number of highly nonlinear equations [32]. It may then become very difficult to choose initial approximations that eventually lead to convergence. This restriction can be removed by employing methods that involve "parameter perturbation" [37, 18] or "parameter variation" [12]. These methods are based essentially on an idea originally proposed by Davidenko [11] and consist, in effect, in reducing the main problem into a number of subsidary problems that are more readily solvable.

When the equations in the system (E-1) are very complicated, explicit evaluation and subsequent inversion of the Jacobian may become very difficult or even impossible. In such cases, the difficulty can be overcome by using the so-called quasi-Newton methods [3, 7, 8]. These methods involve the use of some form of approximation to the inverse Jacobian and modification of this approximate matrix at every stage of the iteration.

The methods proposed so far for the solution of complicated nonlinear simultaneous equations have employed either the Davidenko approach or the approach of the quasi-Newton methods. A new method combining the features of both these approaches has recently been proposed by Broyden [9] who has shown it to be extremely useful in the solution of many difficult problems.

VITA

3

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