THE NONEXISTENCE OF A CONVEX CURVE WITH TWO EQUICHORDAL POINTS

By

GLENDA KAY HASTING OWENS

Bachelor of Science
Southeastern State College
Durant, Oklahoma
1963

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1965

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF PHILOSOPHY
July, 1970

THE NONEXISTENCE OF A CONVEX CURVE

WITH TWO EQUICHORDAL POINTS

Thesis Approved:

E. K. McLachlan
Thesis Adviser

John Jewett

John G. Hoffman

John W. Shelton

D. Durham
Dean of the Graduate College

PREFACE

The purpose of this thesis is to prove that no closed convex curve has two equichordal points. A point S in the interior of a Jordan curve C is an equichordal point of C if and only if each line L through S meets the curve C in exactly two points A_L and B_L such that the length of the segment $A_L B_L$ is a constant. Helfenstein [7] (numbers in square brackets refer to the bibliography at the end of the paper) and Kelly [8] have shown the existence of infinitely many curves with one equichordal point. In 1916, Fujiwara [6] proved the nonexistence of a closed convex curve with three equichordal points, and first proposed the question of the existence of a closed convex curve with two equichordal points. During the past fifty years, several authors have conjectured that no closed convex curve has two equichordal points. However, according to Klee [9], no one has proved this conjecture and the question remains open. This thesis does provide an answer to Fujiwara's question.

I wish to express my appreciation to all those who assisted me in pursuing my graduate studies and in the preparation of this thesis. In particular, I would like to thank Professor E. K. McLachlan who gave so generously of his time and whose suggestions and directions were of great value. My thanks go to Professors John Jewett, John Hoffman, and John Shelton for their interest and encouragement. Finally, my deepest thanks go to my husband, David, without whose confidence, encouragement, and assistance I could never have completed my graduate studies.

THE NONEXISTENCE OF A CONVEX CURVE

WITH TWO EQUICHORDAL POINTS

Let C be a closed Jordan curve in the Euclidean plane and let S be an interior point of C. Then S is an equichordal point of C if and only if each line L through the point S meets the curve C in exactly two points A_L and B_L such that the length of the segment A_LB_L is a constant. If S is an equichordal point of C then S is called an e-point, each of the segments A_LB_L is called an e-chord, and the constant length of the segments A_LB_L is called the e-length of the curve C. If C has n, $n \geq 1$, distinct equichordal points and C is the boundary of a convex body in the plane, then C is an n-e-curve.

Helfenstein [7] and Kelly [8] have shown that there are infinitely many 1-e-curves. However, the nonexistence of n-e-curves for $n \geq 3$ was proved by Fujiwara [6] and later by Dirac [2]. This result lead to the question of the existence of a 2-e-curve. According to Klee [9], this question was first asked by Fujiwara [6] in 1916 and independently by Blaschke, Rothe, and Weitzenböck [1] in 1917.

During the past fifty years, several authors have assumed the existence of a plane convex curve C with two equichordal points R and S, and have established a number of necessary conditions for such curves. Süss [11] proved that C is symmetric with respect to the line RS and with respect to the midpoint of the segment RS. Dirac [2] and Ehrhart [5] obtained quantitative results on the chord containing both R and S. Dirac [2] proved that C is differentiable and Dulmage [4] studied the tangents of C. Wirsing [12] proved, without assuming the curve to be convex, that C is analytic at every point. Previous to Wirsing's result, Linis [10] had claimed to prove that C is not twice differentiable and Helfenstein [7] that C is not six times differentiable. Either of these would, with Wirsing's result, answer the question by showing that a 2-e-curve C does not exist. However, Dirac [3] and Wirsing [12] have indicated mistakes in the work of Linis and Helfenstein. The question was revived in 1969 by Klee [9], who remarked that it remains an open question.

The purpose of this thesis is to answer Fujiwara's question by showing that there does not exist a closed convex curve with two equichordal points. This will be done by using a construction process to obtain a set of points $\{P_n\}$ which must be points of a 2-e-curve if such a curve exists, and then showing by some of the results of Dirac [2] and Wirsing [12] that it is impossible for all of the points $\{P_n\}$ to be points of a 2-e-curve. The following notation will be used in the proof.

If P is a point of the 2-e-curve C, with equichordal points R and S, denote the angle RSP by θ and the distance PS by $r(\theta)$ as illustrated in Figure 1. Let 2a denote the length of the segment RS and suppose the e-length of C is 2 units. Then a, the ratio of the length of the segment RS to the e-length of the curve, is called the <u>eccentricity</u> of the curve C. In accordance with the following proposition which Ehrhart [5] and Dirac [2] have shown, let $0 < a < ½$.

<u>Proposition 1</u> (Ehrhart [5]): There does not exist a 2-e-curve C with eccentricity greater than or equal to ½.

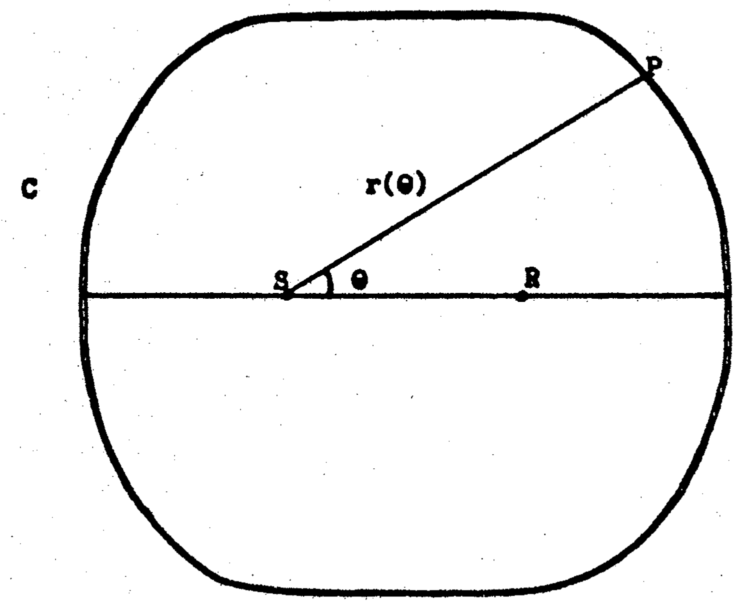

Figure 1.

The following are pertinent results of Dirac [2] and Wirsing [12]:

Proposition 2 (Dirac [2]): A 2-e-curve with e-points R and S is symmetric with respect to the line RS and with respect to the midpoint of the segment RS. Also, $r(\theta)$ is strictly decreasing if $0 \leq \theta \leq \pi$.

Proposition 3 (Wirsing [12]): If a closed curve has two e-points, then it is analytic at every point.

The following is a preliminary result that will prove useful later in the discussion.

Proposition 4: If a closed curve C has an equichordal point R and a center of symmetry O, distinct from R, then the symmetric image point of R with respect to O is also an equichordal point of C and therefore, C has two equichordal points.

Proof: Let the point R be an e-point of the curve C, and suppose all e-chords through R have length 2. Let S be the symmetric image of R with respect to O, as illustrated in Figure 2. Suppose the line L passes through S, then L', the symmetric image of L with respect to O, passes through R. Since R is an e-point of C, it follows that L' intersects C in exactly two points, P' and Q'. Moreover, the length of the segment P'Q' is 2 units. The curve C is symmetric with respect to O; therefore, the line L intersects the curve C in exactly two points P and Q which are the symmetric images of P' and Q', respectively. Therefore, PQ = P'Q' = 2, and it follows that S is an equichordal point of C. Clearly, the points R and S are distinct. Therefore, the curve C has two equichordal points.

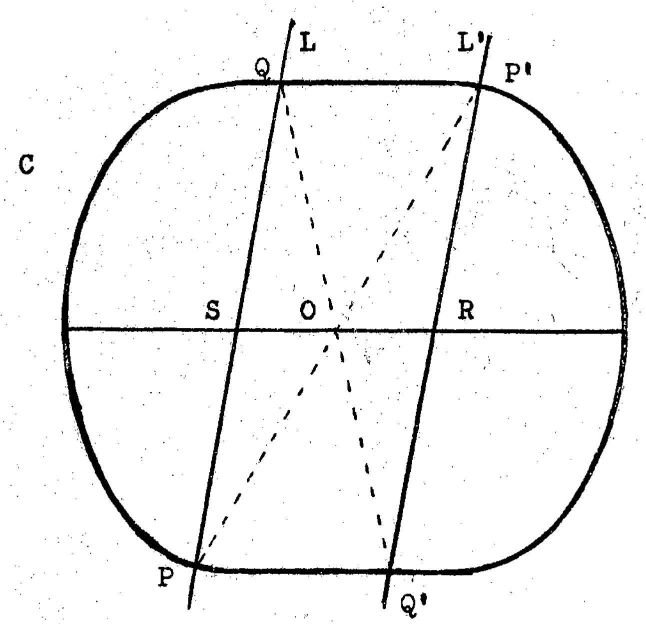

Figure 2.

In order to determine an infinite sequence of points, fix $a < \frac{1}{2}$. Let $R(2a, 0)$ and $S(0, 0)$ be points in the Euclidean plane and suppose the point P_1 has the rectangular coordinates (x_1, y_1), with respect to the origin S, such that $x_1 \geq 0$, $y_1 > 0$. Let E_1 and E_2 denote the open half-planes, upper and lower, respectively, determined by the line RS. Then the point P_1 is in the half-plane E_1. Let P_2' be the point of the line P_1S such that $P_1P_2' = 2$ and P_2' is a point in E_2. Let P_2 denote the symmetric image of P_2' with respect to O, the midpoint of the segment RS. Then P_2 is a point in the half-plane E_1. Let P_3' denote the point of the line P_2S such that $P_2P_3' = 2$ and P_3' is a point of the half-plane E_2. Let P_3 denote the symmetric image of P_3' with respect to O. Now, if the points $P_1, \cdots, P_k$ have been determined in this manner, let P_{k+1}' be the point of the line P_kS such that $P_kP_{k+1}' = 2$ and P_{k+1}' is a point in the half-plane E_2. Let P_{k+1} denote the symmetric image of P_{k+1}' with respect to O. Then P_{k+1} is a point in the half-plane E_1. Hence, the points $P_1, \cdots, P_k, P_{k+1}$ are in the half-plane E_1 and the points $P_2', \cdots, P_k', P_{k+1}'$ are in the half-plane E_2. In this manner, the infinite sequences $\{P_k\}$ and $\{P_k'\}$ are obtained as illustrated in Figure 3.

If the point P_n is written in complex notation

$$z_n = x_n + iy_n = r_n e^{i\theta_n}$$

with respect to the origin S, then the point P_{n+1}' becomes

$$z_{n+1}' = (2 - r_n)e^{i(\pi + \theta_n)} = (r_n - 2)e^{i\theta_n}$$

as illustrated in Figure 4.

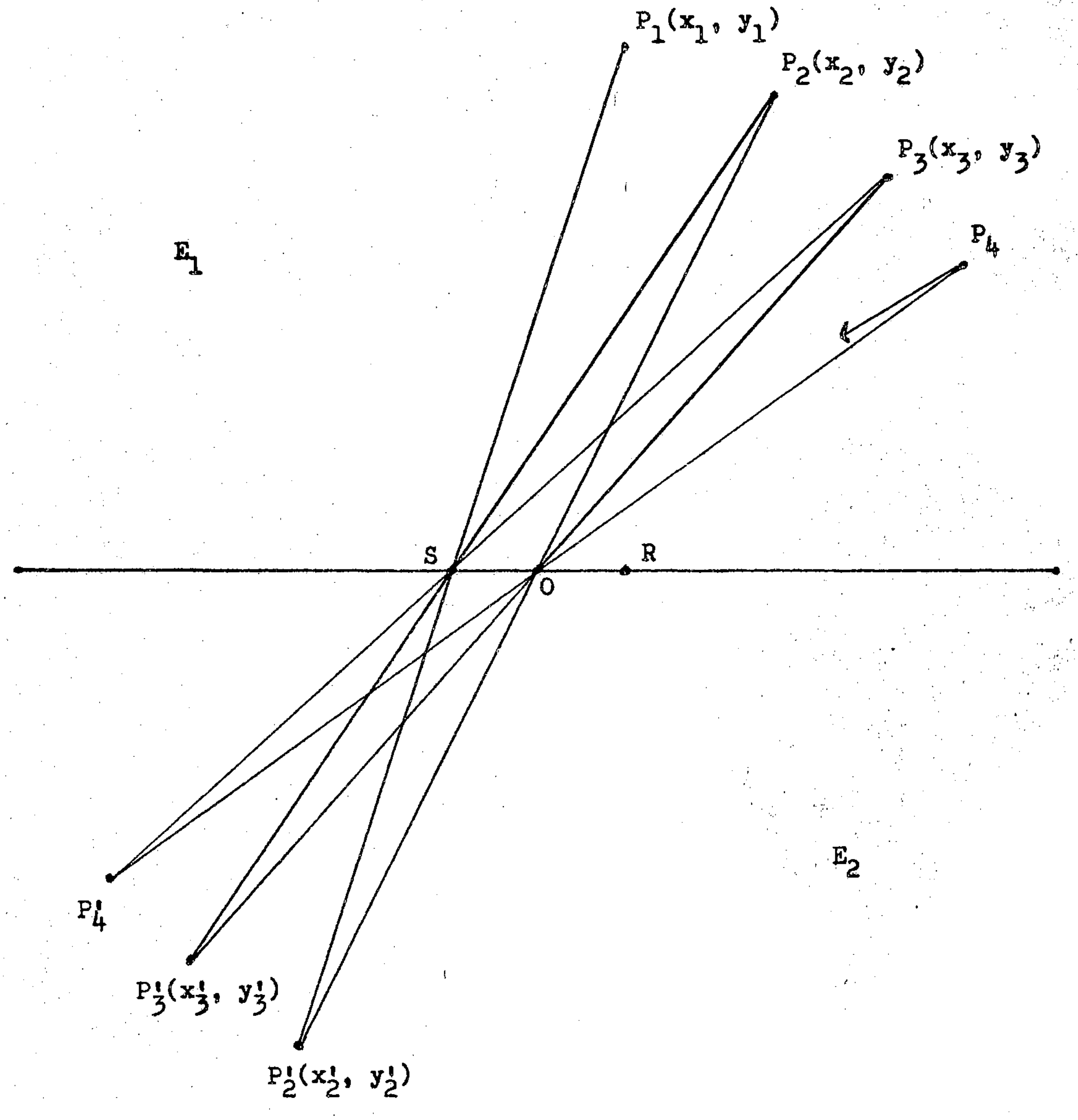

Figure 3.

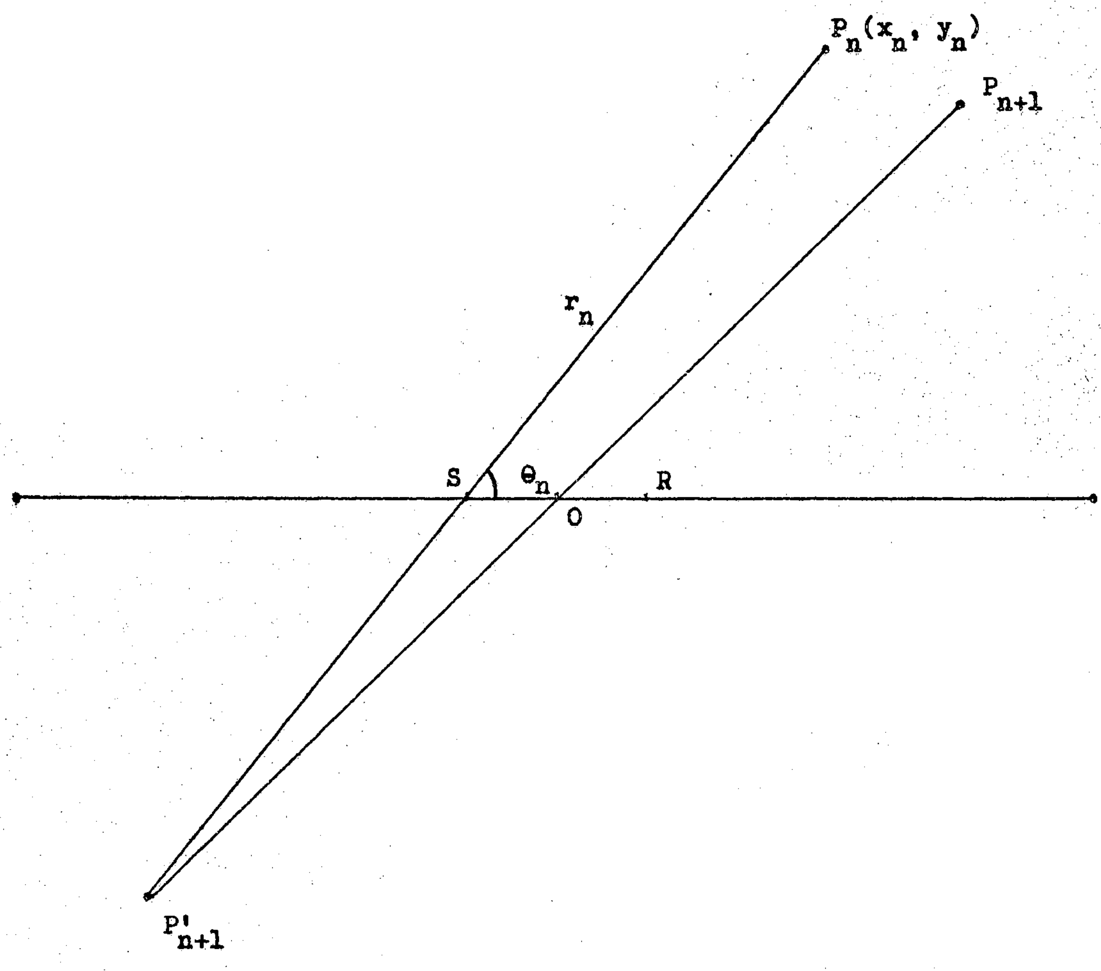

Figure 4.

It follows that the point P_{n+1} may be written in complex notation

$$z_{n+1} = x_{n+1} + iy_{n+1} = 2a + (-z'_{n+1}) = 2a + (-r_n + 2)e^{i\theta_n}$$

$$= 2a + x_n\left[\frac{2}{\sqrt{x_n^2 + y_n^2}} - 1\right] + iy_n\left[\frac{2}{\sqrt{x_n^2 + y_n^2}} - 1\right].$$

Therefore,

$$x_{n+1} = 2a + x_n\left[\frac{2}{\sqrt{x_n^2 + y_n^2}} - 1\right],$$

$$y_{n+1} = y_n\left[\frac{2}{\sqrt{x_n^2 + y_n^2}} - 1\right],$$

and

$$r_{n+1}^2 = (2 - r_n)^2 + 4a^2 + 4a(2 - r_n)\cos\theta_n.$$

This computation leads directly to the following property.

<u>Lemma 1</u>: If $2a < r_n < 2$, then $2a < r_{n+1} < 2$.

Proof: If $r_n < 2$, then

$$r_{n+1}^2 = (2 - r_n)^2 + 4a^2 + 4a(2 - r_n)\cos\theta_n > 4a^2.$$

Consequently, $2a < r_{n+1}$.

If $2a < r_n < 2$, assume $r_{n+1} \geq 2$. Then

$$(2 - r_n)^2 + 4a^2 + 4a(2 - r_n) \geq (2 - r_n)^2 + 4a^2 + 4a(2 - r_n)\cos \theta_n \geq 4$$

so that $2 - r_n + 2a \geq 2$ and $2a \geq r_n$, which contradicts the hypothesis. Therefore, $r_{n+1} < 2$ and the proof is complete.

In particular, let $x_1 = 2a$ and $y_1 = 1$. Throughout the remainder of this paper, let $\{P_k\}$ and $\{P'_k\}$ denote the sequences determined by the preceding construction with this particular initial point $P_1(x_1, y_1)$. It follows from Proposition 2 that if there is a 2-e-curve C with equichordal points R and S, then C must pass through the point $P_1(2a, 1)$. Moreover, as a result of the construction, each of the points P_n, $n = 1, 2, \cdots$, must be a point of C. However, the following lemmas will be used to prove that not all of the points P_n, $n = 1, 2, \cdots$, can be points of a 2-e-curve. The proof of Lemma 2 follows exactly the work of Wirsing [12].

Lemma 2: If y_n is the ordinate of the point P_n, as indicated above, then $\lim_{n\to\infty} y_n = 0$.

Proof: In Figure 3, it is easy to see from the construction that

$$\frac{x_n - a}{y_n} = \frac{a - x'_n}{-y_n} = \frac{x'_n - a}{y'_n} = \frac{x'_n}{y'_n} - \frac{a}{y'_n}$$

$$= \frac{x_{n-1}}{y_{n-1}} + \frac{a}{y_n}$$

$$= \frac{x_{n-1} - a}{y_{n-1}} + \frac{a}{y_{n-1}} + \frac{a}{y_n}$$

$$= \frac{x_{n-2} - a}{y_{n-2}} + \frac{a}{y_{n-2}} + \frac{a}{y_{n-1}} + \frac{a}{y_n}$$

$$= \frac{x_0 - a}{y_0} + \frac{a}{y_0} + \frac{a}{y_1} + \cdots + \frac{a}{y_n}.$$

Therefore,

$$\frac{x_n - a}{y_n} = \frac{x_0 - a}{y_0} + a \sum_{k=0}^{n} \frac{1}{y_k},$$

and

$$\lim_{n\to\infty} \frac{x_n - a}{y_n} = \lim_{n\to\infty} \left[\frac{x_0 - a}{y_0} + a \sum_{k=0}^{n} \frac{1}{y_k} \right] = \frac{x_0 - a}{y_0} + a \sum_{k=0}^{\infty} \frac{1}{y_k} .$$

But $y_k \leq r_k < 2$, and consequently, $\frac{1}{y_k} > \frac{1}{2}$, for every k.

Therefore,

$$\lim_{n\to\infty} \frac{x_n - a}{y_n} = \frac{x_0 - a}{y_0} + a(\tfrac{1}{2} + \tfrac{1}{2} + \cdots) = \infty.$$

Consequently, $\lim_{n\to\infty} y_n = 0$.

The following lemma is a direct result of the construction.

Lemma 3: The sequence $\{\theta_n\}$ is strictly monotone decreasing.

This result may be used to prove the following lemma.

Lemma 4: For every $n = 1, 2, \cdots, x_n < x_{n+2} < 2$.

Proof: For every n, $x_n < 2$ since $x_n \leq r_n < 2$. Also,

$$x_n + x_{n+1} = 2a + 2\cos\theta_n < 2a + 2\cos\theta_{n+1} = x_{n+2} + x_{n+1},$$

and it follows that $x_n < x_{n+2}$, for every n.

Therefore, the sequences $\{x_{2k}\}$ and $\{x_{2k+1}\}$ are monotonic increasing and bounded above, and hence convergent. Suppose $x_{2k} \rightarrow T_2$ and $x_{2k+1} \rightarrow T_1$. Since

$$x_{2k+1} = 2a + x_{2k}\left[\frac{2}{\sqrt{x_{2k}^2 + y_{2k}^2}} - 1\right],$$

it follows that

$$T_1 = \lim_{k\to\infty} x_{2k+1} = 2a + T_2\left[\frac{2}{\sqrt{T_2^2 + 0}} - 1\right]$$

$$= 2a + 2 - T_2.$$

Therefore,

$$T_1 + T_2 = 2 + 2a.$$

Finally, the following lemma shows that the points P_n, n = 1, 2, 3, · · ·, cannot all be points of a 2-e-curve.

<u>Lemma 5</u>: There exists at least one N such that $r_N \geq r_{N+1}$.

Proof: Suppose $r_n < r_{n+1}$, for every n. Then

$$x_n = r_n \cos \theta_n < r_{n+1} \cos \theta_n < r_{n+1} \cos \theta_{n+1} = x_{n+1}$$

and $x_n < x_{n+1}$, for every n. Therefore, $T_1 = T_2 = 1 + a$. Let A denote the point (1+a, 0) and let B denote the point (-1+a, 0). Then $P_k \longrightarrow A$ and $P_k' \longrightarrow B$. Let $P_0(x_0, y_0)$ denote the point (0, 1), Q denote the point $(a, (1+a^2)^{1/2})$, and A_{01} be the union of the line segments P_0Q and P_1Q. Note that the arc A_{01} is symmetric with respect to the perpendicular bisector of the segment RS and that the arc A_{01} fails to have a tangent at the point Q. In order to complete the proof of this lemma, it will be shown that the assumption $r_n < r_{n+1}$, for every n, permits the construction of a closed curve C with two equichordal points such that the arc A_{01} is a subset of the curve C. Therefore, C is not analytic at the point Q and this contradicts Wirsing's Proposition 3. Now, suppose MM' is a segment of length 2 pivoted at S. As the segment MM' moves such that M traverses A_{01} from P_0 to P_1, the point M' determines the arc A_{12}' connecting the points P_1' and P_2'. Let A_{12} denote the symmetric image of A_{12}' with respect to O, then $A_{01} \cup A_{12}$ connects P_0 to P_2 via P_1 as illustrated in Figure 5. As the segment MM' pivoted at S moves such that M traverses the arc A_{12}, the point M' determines the arc A_{23}' connecting the points P_2' and P_3'. Let A_{23} denote the symmetric image of A_{23}' with respect to the point O. Then $A_{01} \cup A_{12} \cup A_{23}$ connects P_0 to P_3 via P_1 and P_2. Now, suppose the arcs $A_{01}, A_{12}, \cdots, A_{k-1,k}$ have been determined such that $A_{01} \cup A_{12} \cup \cdots \cup A_{k-1,k}$ is an arc containing all of the points $P_0, P_1, \cdots, P_k$. Then, as the segment MM' continues to move such that the point M traverses the arc $A_{k-1,k}$, the point M' determines the arc $A_{k,k+1}'$ connecting the points P_k' and P_{k+1}'. If $A_{k,k+1}$

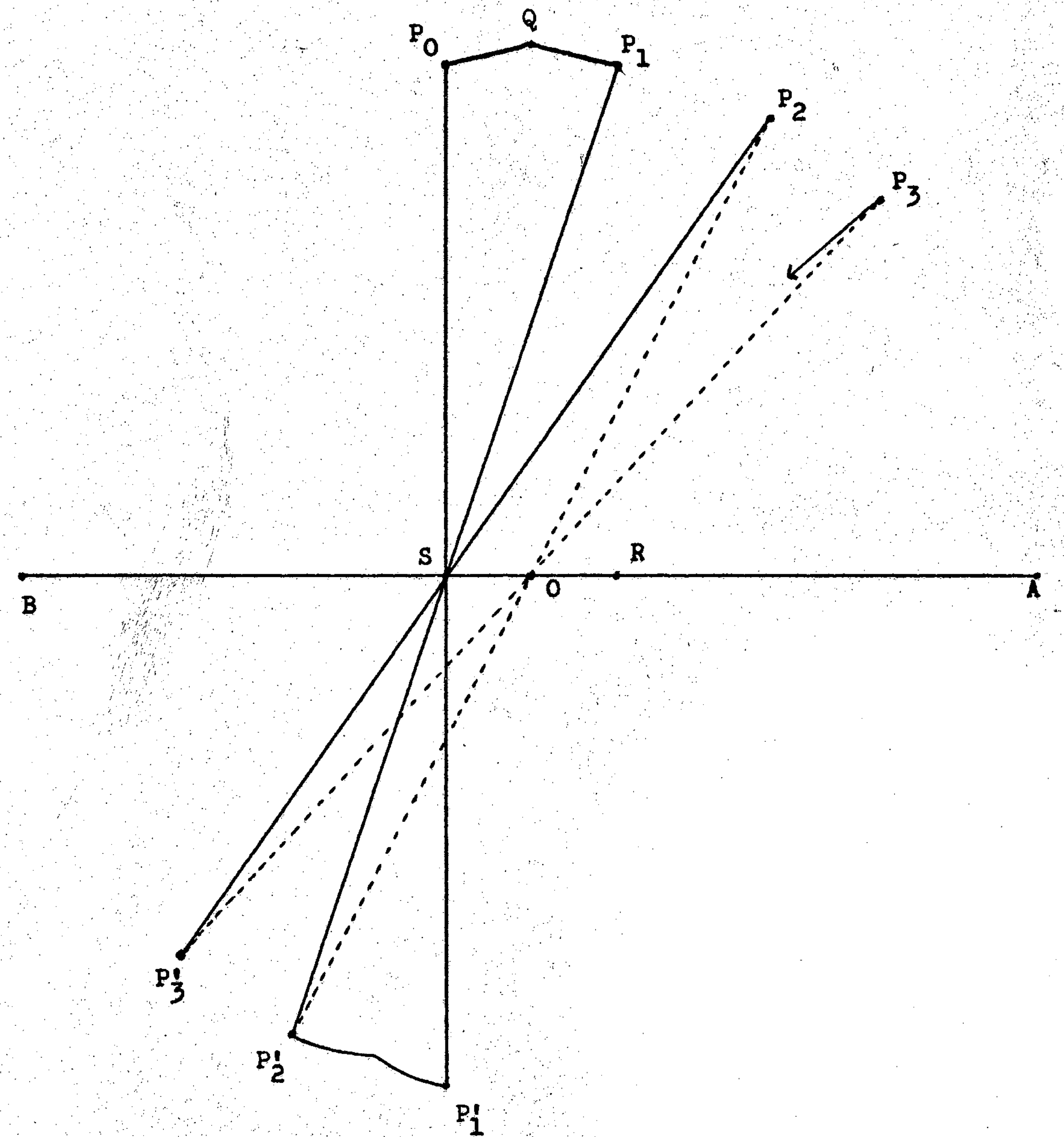

Figure 5.

is the symmetric image of $A'_{k,k+1}$ with respect to O, then $A_{01} \cup A_{12} \cup \cdots \cup A_{k-1,k} \cup A_{k,k+1}$ is an arc containing all of the points $P_0, P_1, \cdots, P_k, P_{k+1}$. The fact that $P_n \longrightarrow A$ indicates that

$$C_1 = \bigcup_{k=0}^{\infty} A_{k,k+1}$$

is an arc connecting all of the points P_n. Also,

$$C_1' = \bigcup_{k=0}^{\infty} A'_{k,k+1}$$

is the symmetric image of C_1 with respect to the point O, and C_1' connects the points P_n', $n = 0, 1, \cdots$. Let C_2' denote the symmetric image of C_1 with respect to the line RS and let C_2 denote the symmetric image of C_2' with respect to O. Then consider the set of points

$$C = C_1 \cup C_1' \cup C_2 \cup C_2'.$$

Let V_1 be a point of A_{01} and let $\{V_k\}$ denote the sequence of points obtained, as illustrated in Figure 3, if V_1 is the initial point in the construction. Since $a < ½$, it follows that $2a < SV_1 < 2$. Then, by Proposition 1 and induction on k, $2a < SV_k < 2$, for every k. Therefore, C is a closed curve with equichordal point S and center of symmetry O, distinct from S. It follows from Proposition 4 that C has two e-points, namely R and S. However, the fact that C is not analytic at the point Q contradicts Wirsing's Proposition 3. Therefore, there exists at least one N such that $r_N \geq r_{N+1}$.

These results may now be used to prove the following proposition which provides an answer to Fujiwara's question.

Proposition 5: No closed convex curve has two equichordal points.

Proof: Suppose there exists a closed convex curve C with two equichordal points R and S and e-length 2. Then, according to Proposition 2, the point $P_1(2a, 1)$, and consequently, each of the points P_n, $n = 1, 2, 3, \cdots$, must be points of C. Furthermore, it follows from Proposition 2 that $r_n < r_{n+1}$, for every n. But this clearly contradicts Lemma 5. Hence, there does not exist a closed convex curve with two equichordal points.

Therefore, the question asked by Fujiwara [6] in 1916 is answered: There is no 2-e-curve. Consequently, a closed convex curve has at most one equichordal point.

The result of this thesis leads quite naturally to a question which seems to have been avoided in the literature:

Is there a closed curve with two equichordal points?

More generally,

Is there a closed curve with n, $n > 1$, equichordal points?

Note that if C is a Jordan curve, then an interior point P of C is an equichordal point if and only if the sum $\|X - P\| + \|Y - P\|$ is constant for all chords XY through P. Analogously, P is an equireciprocal point of C if and only if the sum $\|X - P\|^{-1} + \|Y - P\|^{-1}$ is constant for all chords XY through P. According to Klee [9], the following question is open:

Is there a closed convex curve with two equireciprocal points?

BIBLIOGRAPHY

1. W. Blaschke, H. Rothe, and R. Weitzenböck, Aufgabe 552, Arch. Math. u. Phys., 27 (1917) 82.

2. G. A. Dirac, "Ovals With Equichordal Points." J. London Math. Soc., 27 (1952) 429-437.

3. ________, Review of [10], Math. Rev., 19 (1958) 446.

4. L. Dulmage, "Tangents to Ovals With Two Equichordal Points." Trans. Roy. Soc. Canada, 48 (1954) 7-10.

5. E. Ehrhart, "Un Ovale À Deux Points Isocordes?" Enseignement Math., 13 (1967) 119-124.

6. M. Fujiwara, "Über Die Mittelkurve Zweier Geschlossenen Konvexen Kurven in Bezug Auf Einen Punkt." Tôhoku Math. J., 10 (1916) 99-103.

7. H. G. Helfenstein, "Ovals With Equichordal Points." J. London Math. Soc., 31 (1956) 54-57.

8. P. J. Kelly, "Curves With A Kind Of Constant Width." Am. Math. Mon., 64 (1957) 333-336.

9. V. Klee, "Can A Plane Convex Body Have Two Equichordal Points?" Am. Math. Mon., 76 (1969) 54-55.

10. V. Linis, "Ovals With Equichordal Points." Am. Math. Mon., 64 (1957) 420-422.

11. W. Süss, "Einbereiche Mit Ausgezeichneten Punkten, Inhalts, und Umfangspunkt." Tôhoku Math. J., 25 (1925) 86-98.

12. E. Wirsing, "Zur Analytizität Von Doppelspeichkurven." Arch. Math., 9 (1958) 300-307.

VITA

Glenda Kay Hasting Owens

Candidate for the Degree of

Doctor of Philosophy

Thesis: THE NONEXISTENCE OF A CONVEX CURVE WITH TWO EQUICHORDAL POINTS

Major Field: Mathematics

Biographical:

Personal Data: Born in Durant, Oklahoma, December 4, 1942, the daughter of Mr. and Mrs. Horace M. Hasting.

Education: Attended grade and high school in Caddo, Oklahoma, and was graduated from Caddo High School in 1960; received the Bachelor of Science degree from Southeastern State College, Durant, Oklahoma, with a major in mathematics in 1963; received the Master of Science degree in mathematics from Oklahoma State University in 1965; completed requirements for the Doctor of Philosophy degree in mathematics from Oklahoma State University in July, 1970.

Professional Experience: Graduate Assistant in the Department of Mathematics and Statistics, Oklahoma State University 1963-1965 and 1966-1967; Instructor in the Department of Mathematics, Central State College, Edmond, Oklahoma 1965-1966 and 1967-1970.