AN APPROXIMATE ANALYSIS OF OPEN NONCIRCULAR CYLINDRICAL SHELLS

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NOMENCLATURE

$A_{\bar{I}}$	$\bar{I}^{th}$ constant for nondimensional curvature
b_i, c_i, d_i	i^{th} constant of the series representing displacement in axial, circumferential, and radial directions, respectively
[BB], [CC], [DD], [BC], [BD], [CD]	Matrices associated with coefficients of displacements b_i only, c_i only, d_i only, $b_i c_i$, $b_i d_i$ and $c_i d_i$, respectively
c	$12\left(\frac{r}{h}\right)^2 \left\{ \frac{r}{h} \text{Log} \left[\frac{1 + \frac{h}{2r}}{1 - \frac{h}{2r}} \right] - 1 \right\}$
D	Bending rigidity
[DISP]	Resultant Displacement Matrix
E	Modulus of elasticity
e_x, e_s, e_{xs}	Axial, circumferential, and shear strain components, respectively
e_{xo}, e_{so}, e_{xso}	Axial, circumferential, and shear strain components at median surface, respectively
G	Shear modulus
h	Thickness of shell
i	Index of summation
$\bar{I}$, $\bar{II}$, $\bar{J}$, $\bar{JJ}$	Indices associated with curvature constants
K	$\frac{Eh}{1-\nu^2}$
$\bar{k}$	Number of terms less one necessary to express the curvature

k	Number of terms less one in the displacement function along the curved edge of the shell
L_x, L_s	Length of the shell in x- and s-directions
m	Index of displacement summation along straight edges of the shell
n, $\bar{n}$	Indices of displacement summation along curved edges of the shell
M_η, M_ξ, M_x, M_s	Bending moment resultants
$M_{\eta\xi}$, $M_{\xi\eta}$, M_{xs}, M_{sx}	Twisting moment resultants
N_η, N_ξ, N_x, N_s	Membrane stress resultants
$N_{\eta\xi}$, $N_{\xi\eta}$, N_{xs}, N_{sx}	Shearing stress resultants
P_x, P_y, P_z	Point load on the shell surface in axial, circumferential, and radial direction, respectively
p	Number of terms in the displacement function along the straight edges of the shell
Q_x, Q_s	Transverse shear stress resultants
q_x, q_y, q_z	Total line load on shell surface in axial, circumferential, and radial direction, respectively
r	Radius of curvature
s, x, z	Spatial coordinates
u, v, w	Displacements in x, s, and z direction, respectively
U	Strain energy in the shell
V	Potential of external loads
X_L, Y_L, Z_L	Load per unit area of shell surface in axial, circumferential, and radial direction, respectively
$[X]$	Resultant strain energy matrix
$[Y]$	Resultant load matrix

$\alpha, \beta, \gamma, \delta, \Omega, \theta, \Phi$	Constants defining the boundary conditions on the straight edges
ϵ	Noncircularity parameter
η, ξ	Nondimensional x and s coordinates $\frac{x}{L_x}, \frac{s}{L_s}$
$\kappa_x, \kappa_s, \kappa_{xs}$	Axial and circumferential curvature changes and twist of element of median surface of shell, respectively
ν	Poisson's ratio
Π	Total Potential Energy
ρ	Nondimensional radius of curvature
ρ_o	Uniform pressure per unit area
σ_x, σ_s	Normal stresses
τ_{xs}, τ_{sx}	Shearing stresses
$\omega_x, \omega_s, \omega_z$	Median surface rotations
$\sum$	Summation

CHAPTER I

INTRODUCTION

1.1 Discussion

As a result of considerable technical development, cylindrical shells find wide industrial application in the design of certain class of structures. A problem encountered by the structural engineer is to find the stresses and displacements in a noncircular cylindrical shell subjected to arbitrary loading and restrained in some manner along its boundaries. Practical cases of this problem include open shell roofs, submarine hulls and aircraft structures. Furthermore, noncircularity of the cross section may be introduced during construction of circular cylindrical shells.

It is well-known that the classical, exact methods of analytical solution cannot be applied to the above problem with the desired degree of generality and that approximate techniques must be used to obtain answers to realistic problems.

It was the objective of this research to study, using the energy principles and approximate methods, the displacements in an open noncircular cylindrical shell where the curvature can be expressed as a power series.

1.2 Background

For the discussion to follow, reference should be made to the geometry and nomenclature of Figure 1. The quantities appearing in Figure 1 are as follows: x, s, and z are orthogonal coordinates; u, v, and w are corresponding displacement components; r is the variable radius of curvature; h is the shell thickness; L_x is the length of the shell in the axial direction; and L_s is the arc length of the cylindrical shell measured along the circumferential coordinate axis.

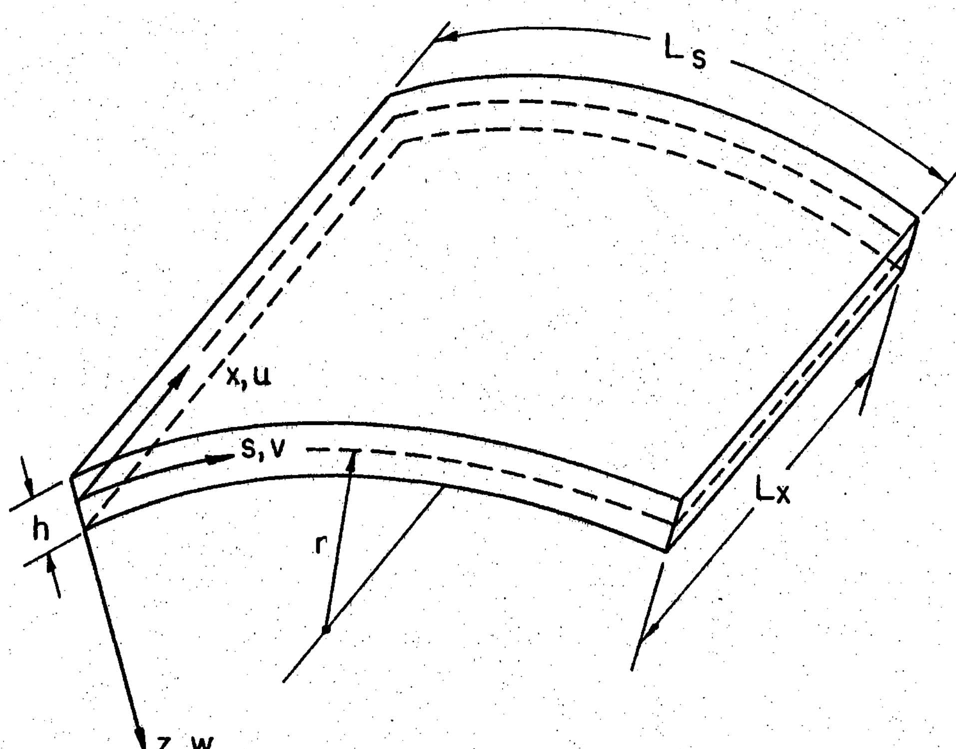


Figure 1. Shell Geometry

A number of analyses have been performed on circular cylindrical shell structures. These are discussed in any classical text on shells (e. g., books by Flugge (1), Timoshenko and Woinowsky-Krieger (2), Harry Kraus (3) or Lundgren (4)). Relatively little work has been done on noncircular cylinders.

Probably the first attempt to solve noncircular cylindrical shell problems was made by Timoshenko. Lundgren too solved shell problems for a few specific noncircular cases. Kempner and his associates (5) and (6) performed a series of investigations into a class of closed oval cylinders having a variable radius of curvature used by Marguerre (7), which contains an "eccentricity" parameter. This curvature expression is given by equation 1.1. That is,

$$\frac{1}{r} = \frac{1}{r_o}\left[1 + \epsilon \cos\left(\frac{4\pi s}{L_o}\right)\right] \tag{1.1}$$

where

r = local radius of curvature of cross section;

r_o = radius of a circle whose circumference is equal to that of the oval (L_o), i.e., $L_o = 2\pi r_o$;

ϵ = parameter measuring eccentricity of the oval and obeying the inequality $|\epsilon| \leq 1$.

Kempner's approach has basic limitations in that the curvature expression contains only one "degree of freedom" (i.e., the eccentricity parameter) and consequently it cannot be applied to any sufficiently general shell segment.

Boyd (8) expressed the curvature in general nondimensional parameters as a finite power series. This expression is

$$\frac{1}{\rho} = \sum_{\bar{i}=o}^{\bar{k}} A_{\bar{i}} \, \xi^{\bar{i}} \qquad (1.2)$$

where

$\frac{1}{\rho}$ = nondimensional shell curvature, which is related to the actual shell curvature $\frac{1}{r}$ by the relation $\rho = \frac{r}{L_s}$;

$\xi = \frac{s}{L_s}$;

$A_{\bar{i}}$ = unitless constants dependent upon $\bar{i}$;

$\bar{k}$ = number of terms necessary to express accurately the curvature.

The curvature series can be obtained for any general cross section using a suitable best-fit technique such as Lagrange interpolation or the least-square method (9).

For the special case of a circular cross-section with a constant radius, equation 1.2 reduces to

$$\frac{1}{\rho} = A_o$$

Another special case of equation 1.2 is the flat plate for which all $A_{\bar{i}}$ are identically zero. Physically, this represents the case of an infinite radius of curvature.

In his paper Boyd used Donnell equations[1] to solve the

---

[1]Donnell (10), in 1933, derived a simplified set of equilibrium equations for circular cylinders. In deriving this set of equations, Donnell made two simplifying assumptions. He first assumed that the transverse shear force makes a negligible contribution to the equilibrium of forces in the circumferential direction. As the ratio of the radius to the thickness of the shell increases, this assumption can be expected to improve in accuracy (3). In addition, he assumed that the circumferential displacements result in negligible contributions to the changes in the curvature and twist.

noncircular cylindrical shell simply ... curved edges. He assumed the displace- ... in the form of a doubly infinite series; ... ries in the x-direction and a power series ..., and reduced the set of partial differ- ... o a set of three recurrence formulas. ... nce formulas and the eight equations ... ry conditions along the straight edges, ... ual displacements in the shell segment were calculated.

he method of Boyd, however, has some basic limitations.

... it was applied to Donnell-type equations, ... r short shells with large radii of curvature;

2. Since the loading function must be expanded as a Fourier series, it is difficult to take into account any general loading function;

3. though theoretically possible to solve any problem, it sometimes becomes impractical to solve problems where convergence of the assumed power series does not occur after taking into account a limited number of terms.

The purpose of this thesis, which is the extension of the ideas expressed by Boyd in his work, is to find an approximate solution which would satisfactorily overcome some of the above limitations and hopefully provide a more realistic approach to the problem.

1.3 Approach

Instead of following Boyd's approach of using the governing differential equations and Donnell's simplified assumptions, the principle of stationary potential energy of the system is used.

The energy method used here, commonly known as the Rayleigh-Ritz method, is an approximate procedure by which a continuous system with an infinite number of degrees of freedom is reduced to a system with a finite number of degrees of freedom. The method is based on the principle of stationary potential energy; i.e., "of all displacements satisfying the given boundary conditions those which satisfy the equilibrium equations make the total potential energy stationary." In this method the independent parameters of the assumed displacement functions are determined by minimizing the total potential energy with respect to each of these independent parameters.

One important feature of the above method is that the assumed trial function for displacements need not satisfy all boundary conditions, but only the "essential" (i.e., "displacement") boundary conditions. The additional "natural" (i.e., "force") boundary conditions are automatically satisfied simultaneously with the equilibrium equations through the use of the principle of stationary potential energy. An extensive discussion of some of the subtleties of this method is given in books by Oden (11), Kantorovich and Krylov (12), and Langhaar (13).

CHAPTER II

FORMULATION OF THE SOLUTION

2.1 The Strain Energy Expression

The expression for strain energy U (equation 2.1) for a noncircular cylindrical shell is given by Kempner (4). For completeness a summary of the derivation is given in Appendix A.

$$U = \frac{Eh}{2(1-\nu^2)} \int_{L_s}\int_{L_x} \Big\langle (u,_x)^2 + \left[v,_s - \left(\frac{w}{r}\right)\right]^2 + 2\nu\, u,_x \left[v,_s - \left(\frac{w}{r}\right)\right] + (\tfrac{1}{2})(1-\nu)(u,_s + v,_x)^2 + \frac{h^2}{12}\Big\{(w,_{xx})^2 + c\left[w,_{ss} + \left(\frac{w}{r^2}\right) - \left(\frac{r,_s}{r^2}\right)v\right]^2 + 2\nu\, w,_{xx}\left[w,_{ss} + \left(\frac{1}{r}\right)v,_s - \left(\frac{r,_s}{r^2}\right)v\right] + (1-\nu)\frac{c}{2}\left[w,_{xs} - \left(\frac{1}{r}\right)u,_s\right]^2 + \left(\frac{3}{2}\right)(1-\nu)\left[w,_{xs} + \left(\frac{1}{r}\right)v,_x\right]^2 + \left(\frac{2}{r}\right)w,_{xx}\,u,_x\Big\}\Big\rangle dx\, ds \tag{2.1}$$

where the subscripts following a comma indicate differentiation. From Figure 1

x, s, z = longitudinal, circumferential, and radial coordinates, respectively;

u, v, w = displacements in the x, s, and z directions, respectively;

r = variable radius of curvature;

ν = Poisson's ratio;

h = shell thickness (assumed constant);

E = Young's modulus of elasticity;

$$c = 12 \left(\frac{r}{h}\right)^2 \left\{ \frac{r}{h} \operatorname{Log} \left[\frac{1 + h/2r}{1 - h/2r} \right] - 1 \right\}$$

$$= 1 + \frac{3}{20} \left(\frac{h}{r}\right)^2 + \frac{3}{112} \left(\frac{h}{r}\right)^4 + \ldots$$

c represents a rapidly converging power series in terms of the ratio $\frac{h}{r}$. Since the thickness of the shell is assumed to be very small compared to the radius of curvature, all second order and higher powers of $\frac{h}{r}$ can be neglected and c then becomes equal to unity.

Equation 2.1 may be written in nondimensional form by using the following nondimensional parameters;

$$\eta = \frac{x}{L_x}$$

$$\xi = \frac{s}{L_s} \qquad (2.2)$$

$$\rho = \frac{r}{L_s}$$

$$U = \frac{Eh\, L_x L_s}{2(1 - \nu^2)} \int_{\eta=0}^{1} \int_{\xi=0}^{1} \sum_{i=1}^{25} U_i \, d\xi \, d\eta \qquad (2.3)$$

where

$$U_1 = \frac{1}{L_x^{\,2}} (u,_\eta)^2$$

$$U_2 = \frac{1}{L_s^{\,2}} (v,_\xi)^2$$

$$U_3 = \frac{1}{L_s^{\,2}} \left(\frac{w}{\rho}\right)^2$$

$$U_4 = -\frac{2}{L_s^{\,2}} (v,_\xi)\left(\frac{w}{\rho}\right)$$

$$U_5 = \frac{2\nu}{L_x L_s} (u,_\eta)(v,_\xi)$$

$$U_6 = -\frac{2\nu}{L_x L_s} (u,_\eta) \left(\frac{w}{\rho}\right)$$

$$U_7 = \frac{(1-\nu)}{2\ L_s^2} (u,_\xi)^2$$

$$U_8 = \frac{(1-\nu)}{2\ L_x^2} (v,_\eta)^2$$

$$U_9 = \frac{(1-\nu)}{L_x\ L_s} (u,_\xi)(v,_\eta)$$

$$U_{10} = \frac{h^2}{12\ L_x^4} (w,_{\eta\eta})^2$$

$$U_{11} = \frac{h^2}{12\ L_s^4} (w,_{\xi\xi})^2$$

$$U_{12} = \frac{h^2}{12\ L_s^4} \left(\frac{w}{\rho^2}\right)^2$$

$$U_{13} = \frac{h^2}{12\ L_s^4} (\rho,_\xi)^2 \left(\frac{v}{\rho^2}\right)^2$$

$$U_{14} = \frac{h^2}{6\ L_s^4} (w,_{\xi\xi}) \left(\frac{w}{\rho^2}\right)$$

$$U_{15} = -\frac{h^2}{6\ L_s^4} \left(\frac{w}{\rho^2}\right)(\rho,_\xi) \left(\frac{v}{\rho^2}\right)$$

$$U_{16} = -\frac{h^2}{6\ L_s^4} (w,_{\xi\xi})(\rho,_\xi) \left(\frac{v}{\rho^2}\right)$$

$$U_{17} = \frac{h^2\ \nu}{6\ L_x^2\ L_s^2} (w,_{\eta\eta})(w,_{\xi\xi})$$

$$U_{18} = \frac{h^2\ \nu}{6\ L_x^2\ L_s^2} (w,_{\eta\eta})\ \frac{1}{\rho}\ (v,_\xi)$$

$$U_{19} = -\frac{h^2\ \nu}{6\ L_x^2\ L_s^2} (w,_{\eta\eta})(\rho,_\xi) \left(\frac{v}{\rho^2}\right)$$

$$U_{20} = \frac{h^2(1-\nu)}{6\ L_x^{\ 2}\ L_s^{\ 2}}\ (w,_{\eta\xi})^2$$

$$U_{21} = \frac{h^2(1-\nu)}{24\ L_s^{\ 4}}\ (u,_{\xi})^2\ \left(\frac{1}{\rho}\right)^2$$

$$U_{22} = -\ \frac{h^2(1-\nu)}{12\ L_x\ L_s^{\ 3}}\ (w,_{\eta\xi})(u,_{\xi})\left(\frac{1}{\rho}\right)$$

$$U_{23} = \frac{h^2(1-\nu)}{8\ L_x^{\ 2}\ L_s^{\ 2}}\ (v,_{\eta})^2\ \left(\frac{1}{\rho}\right)^2$$

$$U_{24} = \frac{h^2(1-\nu)}{4\ L_x^{\ 2}\ L_s^{\ 2}}\ (w,_{\eta\xi})(v,_{\eta})\ \left(\frac{1}{\rho}\right)$$

$$U_{25} = \frac{h^2}{6\ L_x^{\ 3}\ L_s}\ \left(\frac{1}{\rho}\right)\ (w,_{\eta\eta})(u,_{\eta}) \qquad (2.4)$$

2.2 Potential of External Loads

In keeping with the basic Kirchhoff-Love assumptions of the classical theory of thin shells, all applied loads are considered to remain fixed in direction and magnitude during any deformation of the shell. Thus, with reference to Figure 2, the potential of surface loads is

$$V = -\int_{L_x}\int_{L_s}(X_L\ u + Y_L\ v + Z_L\ w)\ ds\ dx \qquad (2.5)$$

where X_L, Y_L, and Z_L are the surface loads per unit area of the shell surface applied in the axial, circumferential, and radial directions, respectively.

Introducing the nondimensional form of equation 2.2 into equation 2.5 we obtain

$$V = - L_x L_s \int_{\eta=\eta_1}^{\eta=\eta_2} \int_{\xi=\xi_1}^{\xi=\xi_2} (X_L\ u + Y_L\ v + Z_L\ w) d\xi\, d\eta \qquad (2.6)$$

It should be noted here that in equation 2.6, integration is to be performed only over the area actually covered by the load.

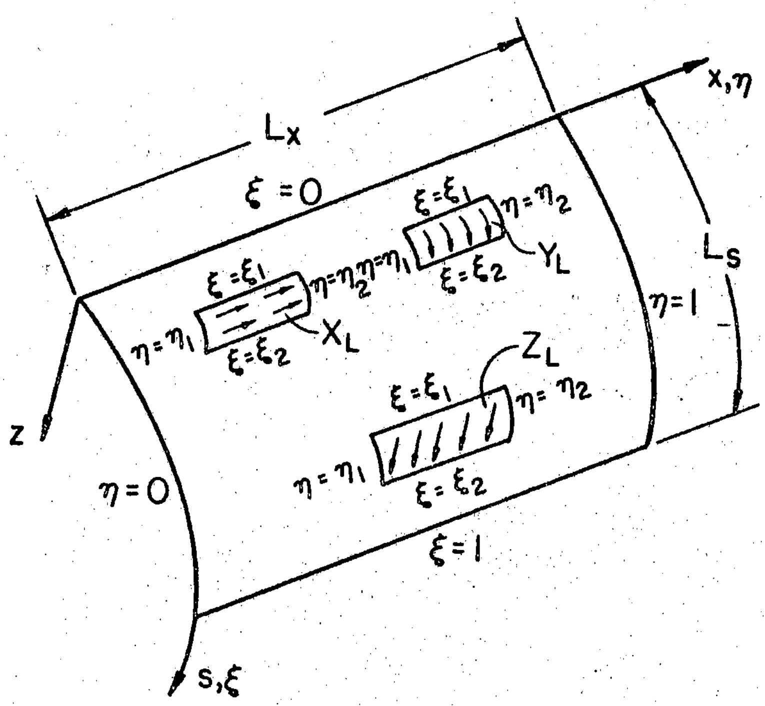


Figure 2. Shell Subjected to General Loading in the Axial, Circumferential, and Radial Directions

2.3 Boundary Conditions

An examination of the problem reveals that sixteen independent boundary conditions are needed for complete solution of the problem, there being four along each of the four edges. In the method used here, as stated earlier, only the displacement boundary conditions need to be satisfied along each edge, the natural boundary conditions being automatically satisfied. In general, the boundary conditions to be satisfied are:

$$\left.\begin{array}{lll} u = 0 & \text{or} & N_{\xi\eta} = 0 \\ v = 0 & \text{or} & N_{\xi} = 0 \\ w = 0 & \text{or} & V_{\xi} = 0 \\ \beta_{\xi} = 0 & \text{or} & M_{\xi} = 0 \end{array}\right\} \quad \text{along } \xi = 0,\ 1 \qquad (2.7)$$

and

$$\left.\begin{array}{lll} u = 0 & \text{or} & N_{\eta} = 0 \\ v = 0 & \text{or} & N_{\eta\xi} = 0 \\ w = 0 & \text{or} & V_{\eta} = 0 \\ \beta_{\eta} = 0 & \text{or} & M_{\eta} = 0 \end{array}\right\} \quad \text{along } \eta = 0,\ 1 \qquad (2.8)$$

where

$N_{\xi\eta}$, $N_{\eta\xi}$ = Shear stress resultant

N_{ξ} , N_{η} = Normal stress resultant

V_{ξ} , V_{η} = Effective transverse shear resultant

β_{ξ} , β_{η} = Slope of the normal to the middle surface

M_{ξ} , M_{η} = Moment stress resultant

Reference should be made to Appendix A for the notations used above.

For the problem here it is assumed that the boundary conditions along the curved edges always remain the same, i.e., at $\eta = 0$ and 1, $v = 0$, $w = 0$, $N_\eta = 0$, and $M_\eta = 0$. This is the case of simply supported edges which are free to move in the axial direction but not in the other two directions. It is necessary to satisfy only $v = 0$ and $w = 0$ at $\eta = 0$ and 1. Consequently, u, v, and w are assumed in the following form:

$$u = \sum_{m=1}^{p} f_{um}(\xi) \cos m\pi\eta$$
$$v = \sum_{m=1}^{p} f_{vm}(\xi) \sin m\pi\eta \qquad (2.9)$$
$$w = \sum_{m=1}^{p} f_{wm}(\xi) \sin m\pi\eta$$

where f_{um}, f_{vm}, and f_{wm} are undetermined functions of ξ. However, the other two conditions $N_\eta = 0$ and $M_\eta = 0$ along the curved edges $\eta = 0$ and 1, are also satisfied.

The boundary conditions along the straight edges ($\xi = 0$ and 1) expressed in terms of displacements and non-dimensional coordinates, are given in reference (14) by equation 2.10

$$N_{\xi\eta} = \frac{Gh}{L_s}\left[\frac{L_s}{L_x} v,_{\eta} + u,_{\xi}\right]$$
$$N_\xi = \frac{K}{L_s}\left[v,_{\xi} + \frac{L_s}{r} w + \nu\left(\frac{L_s}{L_x}\right) u,_{\eta}\right] \qquad (2.10)$$
$$V_\xi = -\frac{D}{L_s^3}\left[w,_{\xi\xi\xi} + (2-\nu)\left(\frac{L_s}{L_x}\right)^2 w,_{\eta\eta\xi}\right]$$

$$\beta_\xi = \frac{1}{L_s}\left[\frac{L_s}{r} v - w,_\xi\right]$$

$$M_\xi = -\frac{D}{L_s^2}\left[w,_{\xi\xi} + \nu\left(\frac{L_s}{L_x}\right)^2 w,_{\eta\eta}\right] \qquad (2.10)$$

where

$$\begin{Bmatrix} K \\ D \end{Bmatrix} = \begin{Bmatrix} h \\ h^3/12 \end{Bmatrix} \frac{E}{1-\nu^2}$$

For the straight edges, the following conditions are true:

Free edge: $N_\xi = V_\xi = M_\xi = N_{\xi\eta} = 0$

Clamped edge: $u = v = w = \beta_\xi = 0$

Simply supported edge, not free to move:

$$u = v = w = M_\xi = 0$$

Simply supported edge, free to move in axial direction:

$$v = w = N_{\xi\eta} = M_\xi = 0$$

Free edge, constrained from moving in the axial direction: $u = N_\xi = V_\xi = M_\xi = 0$

Other combinations can of course be conceived, but, in any event, one must be certain to select only one condition from each of the pairs given in equation 2.7.

In order to have perfectly general boundary conditions along the straight edges, seven parameters, θ, Φ, Ω, δ, α, β, and γ, have been incorporated into the assumed displacement functions. For each value of m;

$$f_{um} = \xi^{\theta}(1-\Phi\xi)(b_{m0} + b_{m1}\xi + b_{m2}\xi^2 + \ldots + b_{mk}\xi^k)$$

$$f_{vm} = \xi^{\Omega}(1-\delta\xi)(c_{m0} + c_{m1}\xi + c_{m2}\xi^2 + \ldots + c_{mk}\xi^k)$$

$$f_{wm} = \xi^{\alpha}\,(1 - \beta\xi + \gamma\xi^{\beta})(d_{m0} + d_{m1}\xi + d_{m2}\xi^{2} + \ldots + d_{mk}\xi^{k}) \qquad (2.11)$$

or

$$f_{um} = \sum_{n=0}^{k} b_{mn}[\xi^{n+\theta} - \phi\xi^{n+\theta+1}]$$

$$f_{vm} = \sum_{n=0}^{k} c_{mn}[\xi^{n+\Omega} - \delta\xi^{n+\Omega+1}] \qquad (2.12)$$

$$f_{wm} = \sum_{n=0}^{k} d_{mn}[\xi^{n+\alpha} - \beta\xi^{n+\alpha+1} + \gamma\xi^{n+\alpha+\beta}]$$

And, using equation 2.9, u, v, and w can be written as

$$u = \sum_{m=1}^{p}\sum_{n=0}^{k} b_{mn}[\xi^{n+\theta} - \phi\xi^{n+\theta+1}] \cos m\pi\eta$$

$$v = \sum_{m=1}^{p}\sum_{n=0}^{k} c_{mn}[\xi^{n+\Omega} - \delta\xi^{n+\Omega+1}] \sin m\pi\eta \qquad (2.13)$$

$$w = \sum_{m=1}^{p}\sum_{n=0}^{k} d_{mn}[\xi^{n+\alpha} - \beta\xi^{n+\alpha+1} + \gamma\xi^{n+\alpha+\beta}] \sin m\pi\eta$$

θ, Ω, and α express the boundary conditions at edge $\xi = 0$ and ϕ, δ, β, and γ express the boundary conditions at edge $\xi = 1$. It should be noted here that the boundary conditions actually considered are:

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \\ w &= 0 \\ w,_{\xi} &= 0 \end{aligned}\right\} \quad \text{along} \quad \xi = 0 \quad \text{or} \quad \xi = 1 \qquad (2.14)$$

For the various boundary conditions stated earlier the values of the parameters θ, Ω, α and Φ, δ, β, γ are shown in Table I.

TABLE I

PARAMETERS FOR VARIOUS BOUNDARY CONDITIONS

BOUNDARY CONDITIONS	Conditions along $\xi = 0$			Conditions along $\xi = 1$			
	θ	Ω	α	Φ	δ	β	γ
Free	0	0	0	0	0	0	0
Clamped	1	1	2	1	1	2	1
Simply supported, not free to move	1	1	1	1	1	1	0
Simply supported, free to move in axial direction	0	1	1	0	1	1	0
Free, constrained from moving in the axial direction	1	0	0	1	0	0	0

With the help of the above table, any boundary conditions can be introduced along the straight edges. Other boundary conditions can also be treated by choosing the appropriate values of the parameters.

2.4 Loads

Referring to Figure 2, it is apparent that any arbitrary load acting anywhere on the shell can be expressed in terms of X_L, Y_L, and Z_L. The coordinates η_1, η_2 and ξ_1, ξ_2 indicate the range over which these loads act.

If the shell is subjected to an internal pressure p_o only, then $X_L = Y_L = 0$ and $Z_L = -p_o$ and $\xi_1 = 0$, $\xi_2 = 1$, $\eta_1 = 0$, $\eta_2 = 1$. For a point load P applied at the center of the shell surface and directed towards the center of curvature of the shell, $X_L = Y_L = 0$ and $Z_L = \frac{P}{L_x L_s}$ and $\xi_1 = \xi_2 = \eta_1 = \eta_2 = 0.5$. It is also possible to include line loads. For example, if the shell is subjected to a shear loading as shown in Figure 3, then $X_L = \frac{q}{L_x}$ and $\xi_1 = \xi_2 = \xi^*$ and $\eta_1 = 0$, $\eta_2 = 1$ and $Y_L = Z_L = 0$.

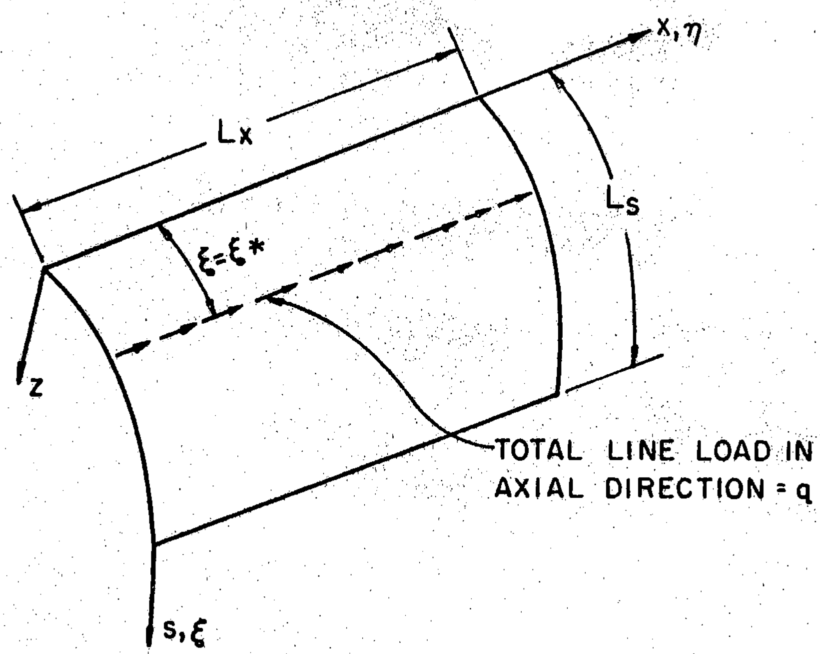


Figure 3. Line Load Applied Parallel to the Straight Edges in the Axial Direction

Similarly other line loads, either in the s or z direction or line loads parallel to the other major axis of the shell can be prescribed.

It is also possible to consider any combination of loadings which may result on resolving a particular load in its components along the principal directions of the shell.

2.5 The Total Potential Energy

The total potential energy Π is the sum of the strain energy U and the potential of external loads V

$$\Pi = U + V$$
$$= \frac{EhL_x L_s}{2(1-\nu^2)} \int_{\eta=0}^{1} \int_{\xi=0}^{1} \sum_{i=1}^{25} U_i \, d\xi \, d\eta$$
$$- L_s L_x \int_{\eta=\eta_1}^{\eta_2} \int_{\xi=\xi_1}^{\xi_2} (X_L u + Y_L v + Z_L w) \, d\xi \, d\eta \tag{2.15}$$

Substituting equations 2.13 into 2.15 and integrating we obtain

$$\Pi = \sum_{m=1}^{p} \left\{ \frac{Eh L_x L_s}{2(1-\nu^2)} \left[\sum_{i=1}^{25} \Pi_i \right] - L_s L_x \left[\Pi_u + \Pi_v + \Pi_w \right] \right\} \tag{2.16}$$

where

$$\Pi_1 = \frac{\pi^2}{2L_x^2} m^2 \left[\sum_{n=0}^{k} \sum_{\bar{n}=0}^{k} b_{mn} b_{m\bar{n}} \left\{ \frac{1}{n+\bar{n}+2\theta+1} + \frac{\phi^2}{n+\bar{n}+2\theta+3} - \frac{2\phi}{n+\bar{n}+2\theta+2} \right\} \right]$$

$$\Pi_2 = \frac{1}{2L_s^2}\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k} c_{mn}\, c_{m\bar{n}} \left\{\frac{(n+\Omega)(\bar{n}+\Omega)}{n+\bar{n}+2\Omega-1} + \frac{\delta^2(n+\Omega+1)(\bar{n}+\Omega+1)}{n+\bar{n}+2\Omega+1}\right.\right.$$

$$\left.\left. - \frac{\delta\langle(n+\Omega)(\bar{n}+\Omega+1)+(n+\Omega+1)(\bar{n}+\Omega)\rangle}{n+\bar{n}+2\Omega}\right\}\right]$$

$$\Pi_3 = \frac{1}{2L_s^2}\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k}\sum_{\bar{i}=0}^{\bar{k}}\sum_{\bar{j}=0}^{\bar{k}} d_{mn}\, d_{m\bar{n}}\, A_{\bar{i}}\, A_{\bar{j}} \left\{\frac{1}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+1} +\right.\right.$$

$$\frac{\beta^2}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+3} + \frac{\gamma^2}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+2\beta+1} - \frac{2\beta}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+2}$$

$$\left.\left. + \frac{2\gamma}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+\beta+1} - \frac{2\beta\gamma}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+\beta+2}\right\}\right]$$

$$\Pi_4 = -\frac{1}{L_s^2}\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k}\sum_{\bar{i}=0}^{\bar{k}} c_{mn}\, d_{m\bar{n}}\, A_{\bar{i}} \left\{\frac{(n+\Omega)}{n+\bar{n}+\bar{i}+\alpha+\Omega}\right.\right.$$

$$- \frac{\delta(n+\Omega+1)}{n+\bar{n}+\bar{i}+\alpha+\Omega+1} - \frac{\beta(n+\Omega)}{n+\bar{n}+\bar{i}+\alpha+\Omega+1} + \frac{\beta\delta(n+\Omega+1)}{n+\bar{n}+\bar{i}+\alpha+\Omega+2}$$

$$\left.\left. + \frac{\gamma(n+\Omega)}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta} - \frac{\delta\gamma(n+\Omega+1)}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta+1}\right\}\right]$$

$$\Pi_5 = -\frac{\nu\pi}{L_x L_s}\, m\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k} b_{mn}\, c_{m\bar{n}} \left\{\frac{(\bar{n}+\Omega)}{n+\bar{n}+\theta+\Omega} + \frac{\delta\Phi(\bar{n}+\Omega+1)}{n+\bar{n}+\theta+\Omega+2}\right.\right.$$

$$\left.\left. - \frac{\Phi(\bar{n}+\Omega)+\delta(\bar{n}+\Omega+1)}{n+\bar{n}+\theta+\Omega+1}\right\}\right]$$

$$\Pi_6 = \frac{\nu\pi}{L_x L_s}\, m\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k}\sum_{\bar{i}=0}^{\bar{k}} b_{mn}\, d_{m\bar{n}}\, A_{\bar{i}} \left\{\frac{1}{n+\bar{n}+\bar{i}+\alpha+\theta+1}\right.\right.$$

$$- \frac{\beta}{n+\bar{n}+\bar{i}+\alpha+\theta+2} + \frac{\gamma}{n+\bar{n}+\bar{i}+\alpha+\theta+\beta+1} - \frac{\Phi}{n+\bar{n}+\bar{i}+\alpha+\theta+2}$$

$$+ \frac{\beta\Phi}{n+\bar{n}+\bar{1}+\alpha+\theta+3} - \frac{\gamma\Phi}{n+\bar{n}+\bar{1}+\alpha+\theta+\beta+2}\Bigg\}\Bigg]$$

$$\Pi_7 = \frac{(1-\nu)}{4L_s^2}\Bigg[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k} b_{mn}\, b_{m\bar{n}} \Bigg\{\frac{(n+\theta)(\bar{n}+\theta)}{n+\bar{n}+2\theta-1} + \frac{\Phi^2(n+\theta+1)(\bar{n}+\theta+1)}{n+\bar{n}+2\theta+1}$$

$$- \frac{\Phi\langle(n+\theta)(\bar{n}+\theta+1)+(n+\theta+1)(\bar{n}+\theta)\rangle}{n+\bar{n}+2\theta}\Bigg\}\Bigg]$$

$$\Pi_8 = \frac{(1-\nu)\pi^2}{4L_x^2} m^2\Bigg[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k} c_{mn}\, c_{m\bar{n}} \Bigg\{\frac{1}{n+\bar{n}+2\Omega+1}$$

$$+ \frac{\delta^2}{n+\bar{n}+2\Omega+3} - \frac{2\delta}{n+\bar{n}+2\theta+2}\Bigg\}\Bigg]$$

$$\Pi_9 = \frac{(1-\nu)\pi}{2L_x L_s} m\Bigg[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k} b_{mn}\, c_{m\bar{n}} \Bigg\{\frac{(n+\theta)}{n+\bar{n}+\theta+\Omega}$$

$$+ \frac{\delta\Phi(n+\theta+1)}{n+\bar{n}+\theta+\Omega+2} - \frac{\delta(n+\theta)+\Phi(n+\theta+1)}{n+\bar{n}+\theta+\Omega+1}\Bigg\}\Bigg]$$

$$\Pi_{10} = \frac{h^2\pi^2}{24\, L_x^4} m^4\Bigg[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k} d_{mn}\, d_{m\bar{n}} \Bigg\{\frac{1}{n+\bar{n}+2\alpha+1}$$

$$+ \frac{\beta^2}{n+\bar{n}+2\alpha+3} + \frac{\gamma^2}{n+\bar{n}+2\alpha+2\beta+1} - \frac{2\beta}{n+\bar{n}+2\alpha+2}$$

$$+ \frac{2\gamma}{n+\bar{n}+2\alpha+\beta+1} - \frac{2\beta\gamma}{n+\bar{n}+2\alpha+\beta+2}\Bigg\}\Bigg]$$

$$\Pi_{11} = \frac{h^2}{24\, L_s^4}\Bigg[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k} d_{mn}\, d_{m\bar{n}} \Bigg\{\frac{(n+\alpha)(n+\alpha-1)(\bar{n}+\alpha)(\bar{n}+\alpha-1)}{n+\bar{n}+2\alpha-3}$$

$$+ \frac{\beta^2(n+\alpha+1)(n+\alpha)(\bar{n}+\alpha+1)(\bar{n}+\alpha)}{n+\bar{n}+2\alpha-1}$$

$$+\frac{\gamma^2(n+\alpha+\beta)(n+\alpha+\beta-1)(\bar{n}+\alpha+\beta)(\bar{n}+\alpha+\beta-1)}{n+\bar{n}+2\alpha+2\beta-3}$$

$$-\frac{\beta\langle(n+\alpha)(n+\alpha-1)(\bar{n}+\alpha+1)(\bar{n}+\alpha)+(\bar{n}+\alpha)(\bar{n}+\alpha-1)(n+\alpha+1)(n+\alpha)\rangle}{n+\bar{n}+2\alpha-2}$$

$$+\frac{\gamma\langle(n+\alpha)(n+\alpha-1)(\bar{n}+\alpha+\beta)(\bar{n}+\alpha+\beta-1)\rangle}{n+\bar{n}+2\alpha+\beta-3}$$

$$+\frac{\gamma\langle(\bar{n}+\alpha)(\bar{n}+\alpha-1)(n+\alpha+\beta)(n+\alpha+\beta-1)\rangle}{n+\bar{n}+2\alpha+\beta-3}$$

$$-\frac{\beta\gamma\langle(n+\alpha+1)(n+\alpha)(\bar{n}+\alpha+\beta)(\bar{n}+\alpha+\beta-1)\rangle}{n+\bar{n}+2\alpha+\beta-2}$$

$$\left.\left.-\frac{\beta\gamma\langle(\bar{n}+\alpha+1)(\bar{n}+\alpha)(n+\alpha+\beta)(n+\alpha+\beta-1)\rangle}{n+\bar{n}+2\alpha+\beta-2}\right\rangle\right]$$

$$\Pi_{12}=\frac{h^2}{24\,L_s^4}\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k}\sum_{\bar{i}=0}^{\bar{k}}\sum_{\bar{j}=0}^{\bar{k}}\sum_{\bar{i}\bar{i}=0}^{\bar{k}}\sum_{\bar{j}\bar{j}=0}^{\bar{k}} d_{mn}\, d_{m\bar{n}}\, A_{\bar{i}}\, A_{\bar{j}}\, A_{\bar{i}\bar{i}}\, A_{\bar{j}\bar{j}}\right.$$

$$\left\{\frac{1}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\bar{j}\bar{j}+2\alpha+1}+\frac{\beta^2}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\bar{j}\bar{j}+2\alpha+3}\right.$$

$$+\frac{\gamma^2}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\bar{j}\bar{j}+2\alpha+2\beta+1}-\frac{2\beta}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\bar{j}\bar{j}+2\alpha+2}$$

$$\left.\left.+\frac{2\gamma}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\bar{j}\bar{j}+2\alpha+\beta+1}-\frac{2\beta\gamma}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\bar{j}\bar{j}+2\alpha+\beta+2}\right\}\right]$$

$$\Pi_{13}=\frac{h^2}{24\,L_s^4}\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k}\sum_{\bar{i}=0}^{\bar{k}}\sum_{\bar{j}=0}^{\bar{k}} c_{mn}\, c_{m\bar{n}}\, A_{\bar{i}}\, A_{\bar{j}}\, \bar{i}\, \bar{j}\right.$$

$$\left.\left\{\frac{1}{n+\bar{n}+\bar{i}+\bar{j}+2\Omega+1}+\frac{\delta^2}{n+\bar{n}+\bar{i}+\bar{j}+2\Omega+3}+\frac{2\delta}{n+\bar{n}+\bar{i}+\bar{j}+2\Omega+2}\right\}\right]$$

$$\Pi_{14}=\frac{h^2}{12\,L_s^4}\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k}\sum_{\bar{i}=0}^{\bar{k}}\sum_{\bar{j}=0}^{\bar{k}} d_{mn}\, d_{m\bar{n}}\, A_{\bar{i}}\, A_{\bar{j}}\right.$$

$$\left\{(\bar{n}+\alpha)(\bar{n}+\alpha+1)\left\langle\frac{1}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha-1}+\frac{\beta}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha}+\frac{\gamma}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+\beta-1}\right\rangle\right.$$

$$-\beta(\bar{n}+\alpha+1)(\bar{n}+\alpha)\left\langle\frac{1}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha}-\frac{\beta}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+1}+\frac{\gamma}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+\beta}\right\rangle$$

$$+\gamma(\bar{n}+\alpha+\beta)(\bar{n}+\alpha+\beta-1)\left\langle\frac{1}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+\beta-1}-\frac{\beta}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+\beta}\right.$$

$$\left.\left.\left.+\frac{\gamma}{n+\bar{n}+\bar{i}+\bar{j}+2\alpha+2\beta-1}\right\rangle\right\}\right]$$

$$\Pi_{15}=\frac{h^2}{12\ L_s^{\ 4}}\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k}\sum_{\bar{i}=0}^{\bar{k}}\sum_{\bar{j}=0}^{\bar{k}}\sum_{\bar{i}\bar{i}=0}^{\bar{k}} c_{mn}\ d_{m\bar{n}}\ A_{\bar{i}}\ A_{\bar{j}}\ A_{\bar{i}\bar{i}}\ \bar{i}\right.$$

$$\left\{\frac{1}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\alpha+\Omega+1}-\frac{\beta}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\alpha+\Omega+2}\right.$$

$$+\frac{\gamma}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\alpha+\Omega+\beta+1}-\frac{\delta}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\alpha+\Omega+2}$$

$$\left.\left.+\frac{\delta\beta}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\alpha+\Omega+3}-\frac{\delta\gamma}{n+\bar{n}+\bar{i}+\bar{j}+\bar{i}\bar{i}+\alpha+\Omega+\beta+2}\right\}\right]$$

$$\Pi_{16}=-\frac{h^2}{12\ L_s^{\ 4}}\left[\sum_{n=0}^{k}\sum_{\bar{n}=0}^{k}\sum_{\bar{i}=0}^{\bar{k}} c_{mn}\ d_{m\bar{n}}\ A_{\bar{i}}\ \bar{i}\right.$$

$$\left\{(\bar{n}+\alpha)(\bar{n}+\alpha-1)\left\langle\frac{1}{n+\bar{n}+\bar{i}+\alpha+\Omega-1}-\frac{\delta}{n+\bar{n}+\bar{i}+\alpha+\Omega}\right\rangle\right.$$

$$-\beta(\bar{n}+\alpha+1)(\bar{n}+\alpha)\left\langle\frac{1}{n+\bar{n}+\bar{i}+\alpha+\Omega}-\frac{\delta}{n+\bar{n}+\bar{i}+\alpha+\Omega+1}\right\rangle$$

$$\left.\left.+\gamma(\bar{n}+\alpha+\beta)(\bar{n}+\alpha+\beta-1)\left\langle\frac{1}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta-1}-\frac{\delta}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta}\right\rangle\right\}\right]$$

$$\Pi_{17} = -\frac{h^2 \nu \pi^2}{12 L_x^2 L_s^2} m^2 \Bigg[\sum_{n=0}^{k} \sum_{\bar{n}=0}^{k} d_{mn} d_{m\bar{n}}$$

$$\Bigg\{(\bar{n}+\alpha)(\bar{n}+\alpha-1) \left\langle \frac{1}{n+\bar{n}+2\alpha-1} - \frac{\beta}{n+\bar{n}+2\alpha} + \frac{\gamma}{n+\bar{n}+2\alpha+\beta-1} \right\rangle$$

$$- \beta(\bar{n}+\alpha+1)(\bar{n}+\alpha) \left\langle \frac{1}{n+\bar{n}+2\alpha} - \frac{\beta}{n+\bar{n}+2\alpha+1} + \frac{\gamma}{n+\bar{n}+2\alpha+\beta} \right\rangle$$

$$+ \gamma(\bar{n}+\alpha+\beta)(\bar{n}+\alpha+\beta-1) \left\langle \frac{1}{n+\bar{n}+2\alpha+\beta-1} - \frac{\beta}{n+\bar{n}+2\alpha+\beta} + \frac{\gamma}{n+\bar{n}+2\alpha+2\beta-1} \right\rangle \Bigg\} \Bigg]$$

$$\Pi_{18} = -\frac{h^2 \nu \pi^2}{12 L_x^2 L_s^2} m^2 \Bigg[\sum_{n=0}^{k} \sum_{\bar{n}=0}^{k} \sum_{\bar{i}=0}^{\bar{k}} c_{mn} d_{m\bar{n}} A_{\bar{i}} \Bigg\{ \frac{(n+\alpha)}{n+\bar{n}+\bar{i}+\alpha+\Omega}$$

$$- \frac{\delta(n+\Omega+1)}{n+\bar{n}+\bar{i}+\alpha+\Omega+1} - \frac{\beta(n+\Omega)}{n+\bar{n}+\bar{i}+\alpha+\Omega+1} + \frac{\beta\delta(n+\Omega+1)}{n+\bar{n}+\bar{i}+\alpha+\Omega+2}$$

$$+ \frac{\gamma(n+\Omega)}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta} - \frac{\delta\gamma(n+\Omega+1)}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta+1} \Bigg\} \Bigg]$$

$$\Pi_{19} = \frac{-h^2 \nu \pi^2}{12 L_x^2 L_s^2} m^2 \Bigg[\sum_{n=0}^{k} \sum_{\bar{n}=0}^{k} \sum_{\bar{i}=0}^{\bar{k}} c_{mn} d_{m\bar{n}} A_{\bar{i}} \bar{i} \Bigg\{ \frac{1}{n+\bar{n}+\bar{i}+\alpha+\Omega+1}$$

$$- \frac{\beta}{n+\bar{n}+\bar{i}+\alpha+\Omega+2} + \frac{\gamma}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta+1} - \frac{\delta}{n+\bar{n}+\bar{i}+\alpha+\Omega+2}$$

$$+ \frac{\delta\beta}{n+\bar{n}+\bar{i}+\alpha+\Omega+3} - \frac{\delta\gamma}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta+2} \Bigg\} \Bigg]$$

$$\Pi_{20} = \frac{h^2(1-\nu)\pi^2}{12 L_x^2 L_s^2} m^2 \Bigg[\sum_{n=0}^{k} \sum_{\bar{n}=0}^{k} d_{mn} d_{m\bar{n}} \Bigg\{ \frac{(n+\alpha)(\bar{n}+\alpha)}{n+\bar{n}+2\alpha-1}$$

$$+ \frac{\beta^2(n+\alpha+1)(\bar{n}+\alpha+1)}{n+\bar{n}+2\alpha+1} + \frac{\gamma^2(n+\alpha+\beta)(\bar{n}+\alpha+\beta)}{n+\bar{n}+2\alpha+2\beta-1}$$

$$- \frac{\beta \left\langle (n+\alpha)(\bar{n}+\alpha+1) + (n+\alpha+1)(\bar{n}+\alpha) \right\rangle}{n+\bar{n}+2\alpha}$$

$$+ \frac{\gamma\langle(n+\alpha)(\bar{n}+\alpha+\beta) + (n+\alpha+\beta)(\bar{n}+\alpha)\rangle}{n+\bar{n}+2\alpha+\beta-1}$$

$$- \frac{\beta\gamma\langle(n+\alpha+1)(\bar{n}+\alpha+\beta) + (n+\alpha+\beta)(\bar{n}+\alpha+1)\rangle}{n+\bar{n}+2\alpha+\beta}\Bigg\}\Bigg]$$

$$\Pi_{21} = \frac{h^2(1-\nu)}{48\ L_s^{\ 4}}\left[\sum_{i=0}^{k}\ \sum_{j=0}^{k}\ \sum_{\bar{i}=0}^{\bar{k}}\ \sum_{\bar{j}=0}^{\bar{k}}\ b_{mn}\ b_{m\bar{n}}\ A_{\bar{i}}\ A_{\bar{j}}\left\{\frac{(n+\theta)(\bar{n}+\theta)}{n+\bar{n}+\bar{i}+\bar{j}+2\theta-1}\right.\right.$$

$$\left.\left. + \frac{\phi^2(n+\theta+1)(j+\theta+1)}{n+\bar{n}+\bar{i}+\bar{j}+2\theta+1} - \frac{\phi\langle(n+\theta)(j+\theta+1) + (n+\theta+1)(j+\theta)\rangle}{n+\bar{n}+\bar{i}+\bar{j}+2\theta}\right\}\right]$$

$$\Pi_{22} = -\frac{h^2(1-\nu)\pi}{24\ L_x\ L_s^{\ 3}}\ m\left[\sum_{n=0}^{k}\ \sum_{\bar{n}=0}^{k}\ \sum_{\bar{i}=0}^{\bar{k}}\ b_{mn}\ d_{m\bar{n}}\ A_{\bar{i}}\left\{\frac{(n+\theta)(\bar{n}+\alpha)}{n+\bar{n}+\bar{i}+\alpha+\theta+1}\right.\right.$$

$$- \frac{\beta(n+\theta)(\bar{n}+\alpha+1)}{n+\bar{n}+\bar{i}+\alpha+\theta+2} + \frac{\gamma(n+\theta)(\bar{n}+\alpha+\beta)}{n+\bar{n}+\bar{i}+\alpha+\theta+\beta+1} - \frac{\Phi(n+\theta+1)(\bar{n}+\alpha)}{n+\bar{n}+\bar{i}+\alpha+\theta}$$

$$\left.\left. + \frac{\beta\Phi(n+\theta+1)(\bar{n}+\alpha+1)}{n+\bar{n}+\bar{i}+\alpha+\theta+1} - \frac{\Phi\gamma(n+\theta+1)(\bar{n}+\alpha+\beta)}{n+\bar{n}+\bar{i}+\alpha+\theta+\beta}\right\}\right]$$

$$\Pi_{23} = \frac{h^2(1-\nu)\pi^2}{16\ L_x^{\ 2}\ L_s^{\ 2}}\ m^2\left[\sum_{n=0}^{k}\ \sum_{\bar{n}=0}^{k}\ \sum_{\bar{i}=0}^{\bar{k}}\ \sum_{\bar{j}=0}^{\bar{k}}\ c_{mn}\ c_{m\bar{n}}\left\{\frac{1}{n+\bar{n}+\bar{i}+j+2\Omega+1}\right.\right.$$

$$\left.\left. + \frac{\delta^2}{n+\bar{n}+\bar{i}+\bar{j}+2\Omega+3} - \frac{2\delta}{n+\bar{n}+\bar{i}+\bar{j}+2\Omega+2}\right\}\right]$$

$$\Pi_{24} = \frac{h^2(1-\nu)\pi^2}{8\ L_x^{\ 2}\ L_s^{\ 2}}\ m^2\left[\sum_{n=0}^{k}\ \sum_{\bar{n}=0}^{k}\ \sum_{\bar{i}=0}^{\bar{k}}\ c_{mn}\ d_{m\bar{n}}\ A_{\bar{i}}\left\{\frac{(\bar{n}+\alpha)}{n+\bar{n}+\bar{i}+\alpha+\Omega}\right.\right.$$

$$- \frac{\beta(\bar{n}+\alpha+1)}{n+\bar{n}+\bar{i}+\alpha+\Omega+1} + \frac{\gamma(\bar{n}+\alpha+\beta)}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta} - \frac{\delta(\bar{n}+\alpha)}{n+\bar{n}+\bar{i}+\alpha+\Omega+1}$$

$$\left.\left. + \frac{\delta\beta(\bar{n}+\alpha+1)}{n+\bar{n}+\bar{i}+\alpha+\Omega+2} - \frac{\delta\gamma(\bar{n}+\alpha+\beta)}{n+\bar{n}+\bar{i}+\alpha+\Omega+\beta+1}\right\}\right]$$

$$\Pi_{25} = \frac{h^2 \pi^3}{12 L_s L_x^3} m^3 \left[\sum_{i=0}^{k} \sum_{j=0}^{k} \sum_{\bar{i}=0}^{\bar{k}} b_{mn} d_{m\bar{n}} A_{\bar{i}} \left\{ \frac{1}{n+\bar{n}+\bar{i}+\alpha+\theta+1} \right. \right.$$

$$- \frac{\beta}{n+\bar{n}+\bar{i}+\alpha+\theta+2} + \frac{\gamma}{n+\bar{n}+\bar{i}+\alpha+\theta+\beta+1} - \frac{\Phi}{n+\bar{n}+\bar{i}+\mu+\theta+2}$$

$$\left. \left. + \frac{\beta\Phi}{n+\bar{n}+\bar{i}+\alpha+\theta+3} - \frac{\Phi\gamma}{n+\bar{n}+\bar{i}+\alpha+\theta+\beta+1} \right\} \right]$$

$$\Pi_u = X_L \left[\sum_{n=0}^{k} b_{mn} \left\{ (\xi_2^{n+\theta+1} - \xi_1^{n+\theta+1}) - \Phi(\xi_2^{n+\theta+2} - \xi_1^{n+\theta+2}) \right\} \left\{ \frac{\sin m\pi\eta_2 - \sin m\pi\eta_1}{m\pi} \right\} \right]$$

$$\Pi_v = Y_L \left[\sum_{n=0}^{k} c_{mn} \left\{ (\xi_2^{n+\Omega+1} - \xi_1^{n+\Omega+1}) - \delta(\xi_2^{n+\Omega+2} - \xi_1^{n+\Omega+2}) \right\} \left\{ \frac{\cos m\pi\eta_2 - \cos m\pi\eta_2}{m\pi} \right\} \right]$$

$$\Pi_w = Z_L \left[\sum_{n=0}^{k} d_{mn} \left\{ (\xi_2^{n+\alpha+1} - \xi_1^{n+\alpha+1}) - \beta(\xi_2^{n+\alpha+2} - \xi_1^{n+\alpha+2}) + \gamma(\xi_2^{n+\alpha+\beta+1} - \xi_1^{n+\alpha+\beta+1}) \right\} \left\{ \frac{\cos m\pi\eta_1 - \cos m\pi\eta_2}{m\pi} \right\} \right] \quad (2.17)$$

2.6 Minimum of the Total Potential Energy

In the above expressions the constants $b_{m1}, \ldots, b_{mk}, c_{m1}, \ldots, c_{mk}, d_{m1}, \ldots, d_{mk}$, are 3k linearly independent parameters for each m yet to be determined. Since the components of displacement are now defined in terms of only 3k independent quantities, the parameters $b_{m1}, \ldots, d_{mk}$ behave as generalized coordinates and, in effect, the system has only 3k degrees of freedom. For the system to be in equilibrium, the variation in the total potential energy must be zero.

$$\delta\widehat{\Pi} = \sum_{n=0}^{k} (\widehat{\Pi},_{b_{mn}} \delta_{b_{mn}} + \widehat{\Pi},_{c_{mn}} \delta_{c_{mn}} + \widehat{\Pi},_{d_{mn}} \delta_{d_{mn}}) = 0 \tag{2.18}$$

for arbitrary values of $\delta_{b_{mn}}$, $\delta_{c_{mn}}$, $\delta_{d_{mn}}$. Equation 2.18 is satisfied if and only if

$$\begin{aligned}
&\widehat{\Pi},_{b_{m1}} = 0 \ , \quad \widehat{\Pi},_{b_{m2}} = 0 \ , \quad \ldots \ , \quad \widehat{\Pi},_{b_{mk}} = 0 \\
&\widehat{\Pi},_{c_{m1}} = 0 \ , \quad \widehat{\Pi},_{c_{m2}} = 0 \ , \quad \ldots \ , \quad \widehat{\Pi},_{c_{mk}} = 0 \\
&\widehat{\Pi},_{d_{m1}} = 0 \ , \quad \widehat{\Pi},_{d_{m2}} = 0 \ , \quad \ldots \ , \quad \widehat{\Pi},_{d_{mk}} = 0
\end{aligned} \tag{2.19}$$

Equations 2.19 represent for each value of m, a system of 3k linearly independent simultaneous equations in the unknown parameters b_{m1}, ..., d_{mk}. The solution of these parameters and the subsequent evaluation of the displacements at any point on the shell are discussed in the next chapter.

CHAPTER III

COMPUTER SOLUTION

3.1 General

The numerical calculations were made on a Model 360/50 IBM computer. The program is sufficiently general to handle a cylindrical shell with arbitrary curvatures, dimensions, boundary conditions on the straight edges and loading. The varying parameters are input from data cards as needed. A listing of the program is given in Appendix B, and a general flow chart appears in Figure 4.

3.2 Discussion of the Programming Technique

On applying equation 2.19 to equation 2.16, 3k simultaneous equations are obtained for each m (see equations 2.19). These can be reorganized and written in a matrix form as follows:

$$\left[\begin{array}{c|c|c} [BB] & [BC] & [BD] \\ \hline [CB] & [CC] & [CD] \\ \hline [DB] & [DC] & [DD] \end{array}\right] \cdot \left[\begin{array}{c} b_{mn} \\ \hline c_{mn} \\ \hline d_{mn} \end{array}\right] = \left[\begin{array}{c} [YUX] \\ \hline [YUY] \\ \hline [YUZ] \end{array}\right] \qquad (3.1)$$

symbolically,

$$[X] \cdot [DISP] = [Y] \qquad (3.2)$$

where the submatrices [BB], [CC], [DD], [BC], [BD], and [CD] are each square and of dimension k. The elements of each submatrix are obtained by partially differentiating Π with respect to the associated first alphabet. Submatrices [YUX], [YUY] and [YUZ] associated with the loads are obtained by partially differentiating Π with respect to b, c, and d, respectively. The [DISP] column matrix which contains 3k unknowns, $b_{m1}, \ldots, d_{mk}$ is obtained by using the STQN[1] subroutine for solving a set of simultaneous linear equations.

A close examination of [X] reveals that it is symmetrical and can be written as:

$$[X] = \left[\begin{array}{c|c|c} [BB] & [BC] & [BD] \\ \hline [BC] & [CC] & [CD] \\ \hline [BD] & [CD] & [DD] \end{array}\right] \qquad (3.3)$$

SYMMETRICAL

Consequently it is necessary to evaluate only the upper half of the [X] matrix and then using symmetry obtain the remaining portion.

From the [DISP] matrix, which gives the coefficients b_{mn}, c_{mn}, and d_{mn} of the terms of the assumed power series

---

[1]STQN subroutine uses the method of elimination and back substitution for inverting the matrix.

(equation 2.13), the displacements u, v, and w can be easily obtained at any point on the shell.

The program given in Appendix B has been written to give the displacements directly at every tenth point at a section half way between the curved edges for v and w displacements and at the curved edges for the u displacement. If necessary, this can easily be modified to obtain displacements at any other points on the shell. The flow chart shown in Figure 4 provides a general idea of the programming steps involved.

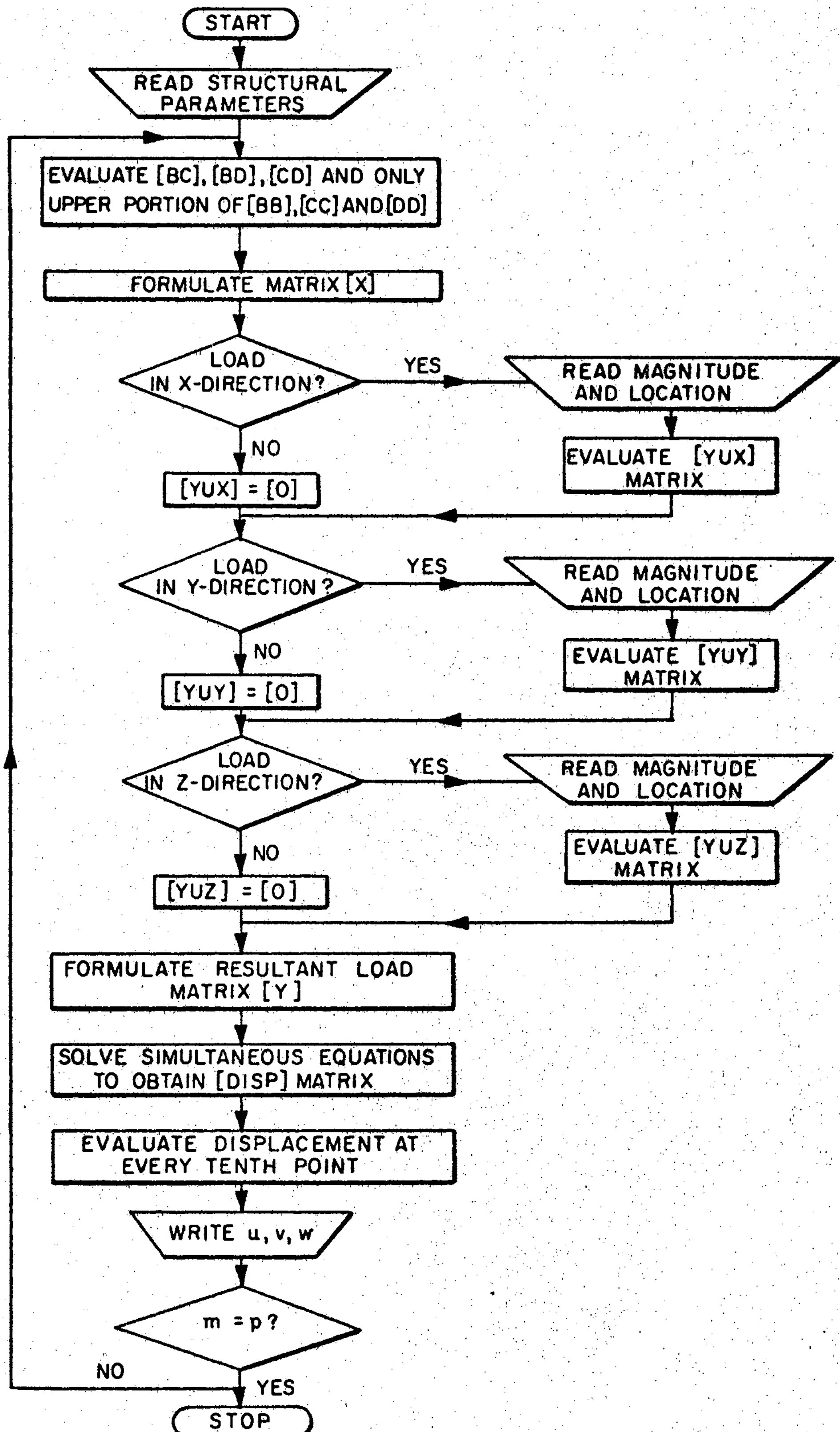


Figure 4. General Flow Chart for Computer Program

CHAPTER IV

NUMERICAL RESULTS

4.1 Convergence of the Solution

Theoretically, the accuracy of the solution increases as more terms of the assumed displacement functions are considered. An exact solution can be obtained if the number of these terms (values of k and p) are infinite but from a practical consideration only a limited number can be considered. The question then arises: what degree of accuracy is desired and what values of k and p should be used? The answer is a complex one, depending on the computer time and storage location available, the geometrical characteristics of the problem, loading conditions, and for what purpose the results are to be used.

A few trials were made with different values of k and p for a flat plate simply supported on all edges and uniformly loaded. The results are shown in Table II. It is seen that increasing only p or only k does not substantially increase the accuracy but it is necessary to select a suitable combination of values in both directions, which depend on the accuracy desired. For circular shells this finding is substantiated by Figure 5 which shows the

TABLE II

ACCURACY OF SOLUTION FOR DIFFERENT VALUES OF k AND p

Type of structure: Flat plate simply supported on all edges

Load: Uniformly distributed load = ρ_o

Properties: $L_s/L_x = 0.25$, $h/L_s = 0.005$, $\nu = 0.3$

Exact solution (2): $w_{MAX} = (\bar{w}\ \rho_o\ L_s^4)/D$, where $\bar{w} = 0.01282$

p	k	$\bar{w}$	Percent Error
1	1	0.011992006	- 6.4586
1	2	0.014683465	+ 14.5356
1	3	0.014683465	+ 14.5022
1	4	0.014679187	+ 14.5022
1	6	0.014679191	+ 14.5022
1	7	0.014679191	+ 14.5022
1	9	0.014679191	+ 14.5022
1	14	0.014679191	+ 14.5022
3	4	0.012425754	- 3.0752
3	7	0.012425728	- 3.0754
3	9	0.012425703	- 3.0756
3	14	0.012425703	- 3.0756
5	4	0.012923400	+ 0.8066
5	7	0.012923428	+ 0.8068
5	9	0.012923456	+ 0.8069
5	14	0.012923456	+ 0.8069
7	4	0.012784404	- 0.2777
7	9	0.012784339	- 0.2782
7	14	0.012784339	- 0.2782
9	4	0.012831831	+ 0.0922

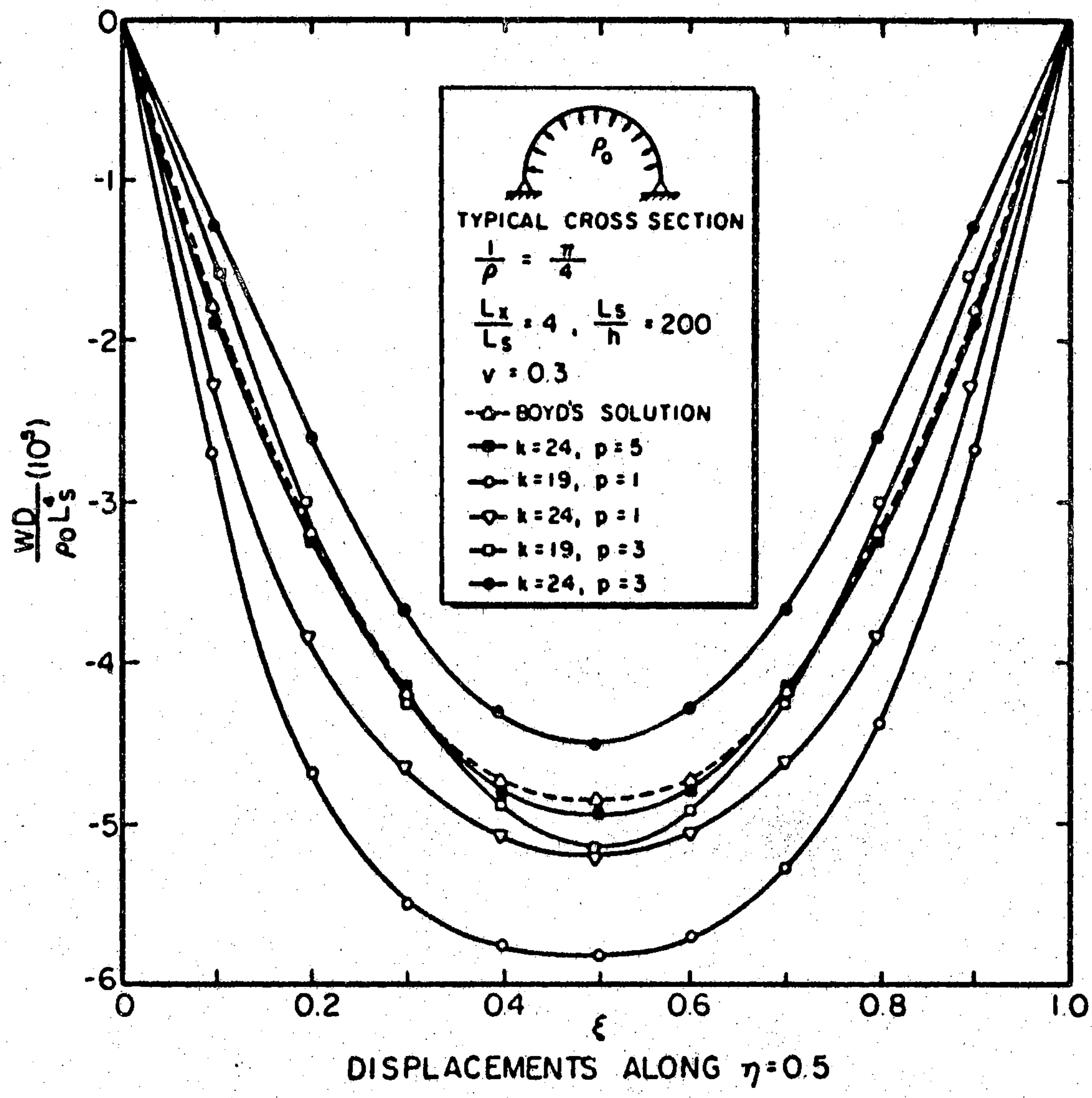


Figure 5. Convergence of Solution for Different Values of k and p for a Circular Shell

radial displacements for various combinations of k and p and also the solution obtained by Boyd. As an example, a look at the curves represented by $k = 19$, $p = 3$, and $k = 24$, $p = 3$ shows that the former gives better results.

Similar investigations for noncircular shells show that the number of terms required for convergence increase with an increase in the number of terms in the curvature expression. Also, more terms are required for the fixed edge or the free edge boundary conditions and line or point loads.

4.2 Comparisons with Known Results for Flat Plates

The program developed was tested for several cases of flat plates. The values of the deflections at the geometrical center of the plate are shown in Table III, together with the exact solutions given by Timoshenko (2). It was found that the simply supported plate with a uniform load compared most accurately with the exact solution. Errors associated with the case of a concentrated load at the center of a simply supported plate increased as the $\frac{L_s}{L_x}$ ratio was increased. For free boundary conditions along the straight edge, the error was relatively large. For the cases of the fixed straight edges, good results were obtained.

4.3 Circular Shells

Three cases for the circular cylindrical shell were studied.

TABLE III

MAXIMUM DEFLECTIONS FOR A FLAT PLATE FOR DIFFERENT BOUNDARY CONDITIONS AND LOADS

$$w_{MAX} = \bar{w}\,\frac{\rho_o L_s^4}{D} \quad \text{(Uniformly Distributed Load)}$$

$$w_{MAX} = \bar{w}\,\frac{P L_s^2}{D} \quad \text{(Point Load)}$$

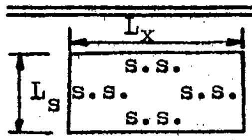
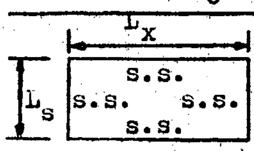
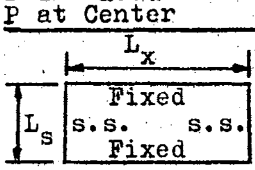
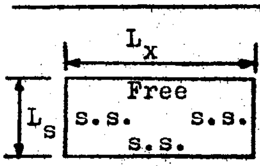
BOUNDARY CONDITION AND LOAD	$\frac{L_s}{L_x}$	$\frac{h}{L_s}$	$\bar{w}$ (Timoshenko)	$\bar{w}$ (Approx. Solution)	Percent Error
 Uniformly Distributed Load Intensity = ρ_o	0.25	0.005	0.01282	0.01278	- 0.3
	1.00	0.005	0.004060	0.004060	0.0
 Point Load P at Center	1.00	0.005	0.01160	0.01151	- 0.8
	3.00	0.005	0.01690	0.01831	+ 8.0
 Uniformly Distributed Load Intensity = ρ_o	0.50	0.005	0.00260	0.00261	+ 0.3
	1.00	0.005	0.00192	0.00192	0.0
	3.00	0.005	0.0001442	0.0001443	+ 0.1
 Uniformly Distributed Load Intensity = ρ_o	0.50	0.005	0.11360	0.12136	+ 6.8
	1.00	0.005	0.01286	0.01490	+15.8
	3.00	0.005	0.0001876	0.0002287	+21.9

Figure 6 shows the radial deflections for a simply supported circular shell under uniform pressure. For the same number of terms in the displacement functions the maximum error when compared with Boyd's values was less than one percent.

Figure 7 shows the radial and circumferential displacements for the same shell discussed above but under a point load applied at the geometric center of the shell.

In order to compare this method of analysis with experimental results under a variable loading intensity, the shell shown in Figure 8 was analyzed. This shell has been analyzed and tested experimentally by Lundgren.

The shell was 2 cm. thick, had a length of 3 m. and an arc length of 5 m. with a radius of curvature of 9 m. For practical purposes it was assumed to be simply supported on all edges. The vertical displacement curves for the three cases are shown in Figure 9. At the center of the shell the results agree with Lundgren's experimental as well as theoretical values. However, at the springings, the results obtained did not agree closely with either Lundgren's experimental or theoretical values. As pointed out by Lundgren this is due to the fact that the edge conditions at the springings were rather obscure in the experimental set-up, and an approximate theory was used by him in his theoretical work. It is possible that the method presented here provides better results at the springings than Lundgren's experimental or theoretical results.

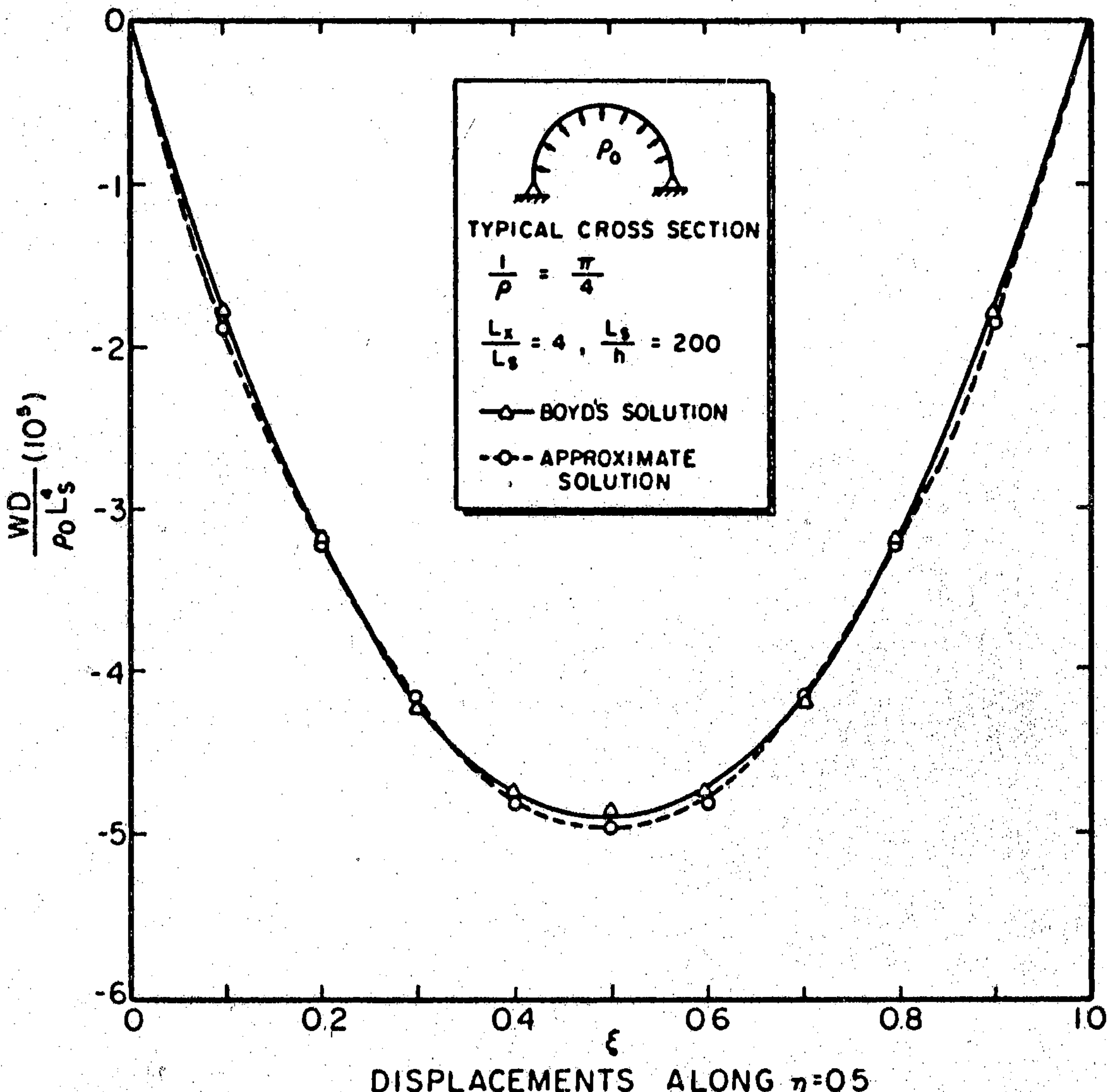


Figure 6. Radial Deflections for a Simply Supported Circular Shell Under Uniform Pressure Loading

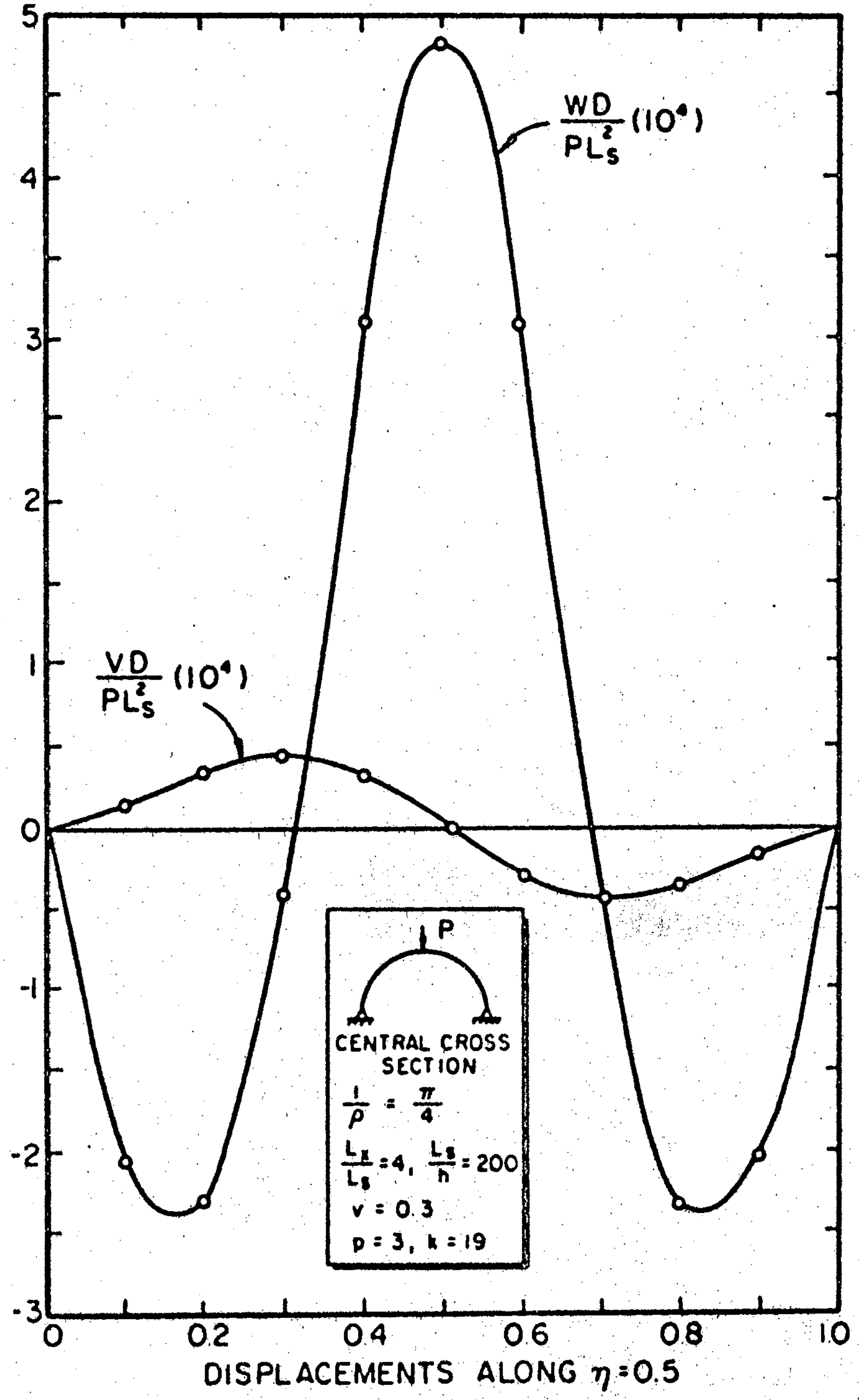


Figure 7. Radial and Circumferential Deflections for a Simply Supported Circular Shell Under a Point Load Applied at the Geometric Center

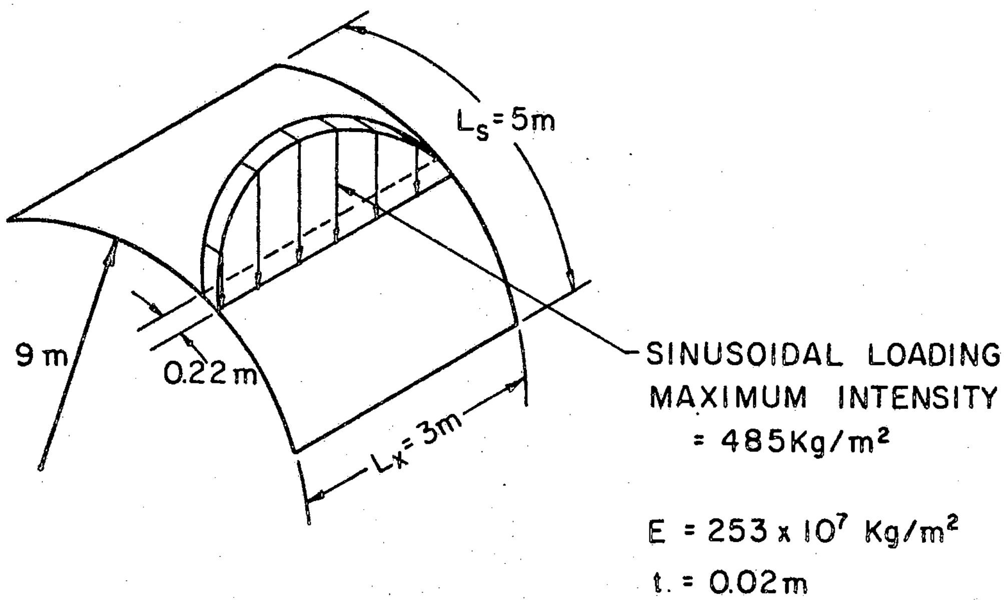


Figure 8. Simply Supported Circular Shell Under a Radial Sinusoidal Line Load

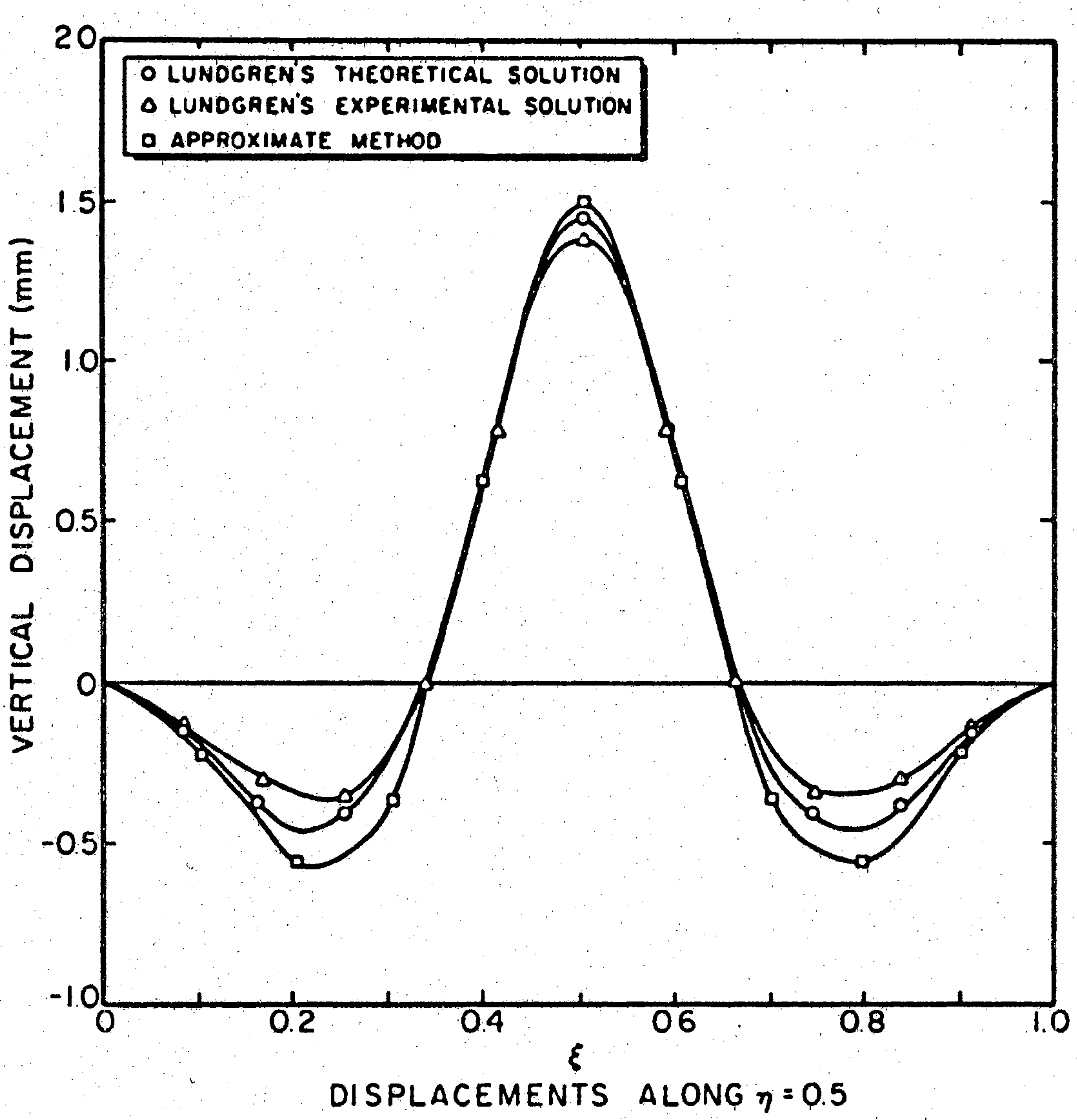


Figure 9. Vertical Displacements for a Simply Supported Circular Shell Under a Radial Sinusoidal Line Load

4.4 Noncircular Shells

A detailed study of the noncircular cylindrical shell studied by Boyd with $\frac{1}{\rho} = \frac{\pi}{4}\left(1 - \frac{\xi^2}{2}\right)$ was made. Figure 10 shows the radial and circumferential displacements when the shell is under a uniform pressure and the straight edges are either simply supported or fixed. As expected the simply supported shell deflects more than a shell with fixed edges. Figure 11 shows similar results under a point load placed at the shell center.

A study was also made to examine the shell deflections under different $\frac{L_x}{L_s}$ and $\frac{L_s}{h}$ ratios. Tables IV and V show radial and circumferential deflections for different thicknesses and constant length ratio $\left(\frac{L_x}{L_s} = 4\right)$ and Tables VI and VII show these deflections when the thickness remains constant $\left(\frac{L_s}{h} = 200\right)$ but the length ratio $\frac{L_x}{L_s}$ varies. Boyd's results for all cases are also shown and the percentage error based on his results evaluated. Some of the results shown in the tables together with others are plotted in Figures 12 to 15.

For the shell studied, the radius of curvature on the right hand side is greater indicating a flatter surface. A negative displacement on this side indicates that due to a uniform pressure this portion displaces in the same direction as that of the uniform pressure, while on the left hand side the displacement is in the opposite direction as that of the pressure.

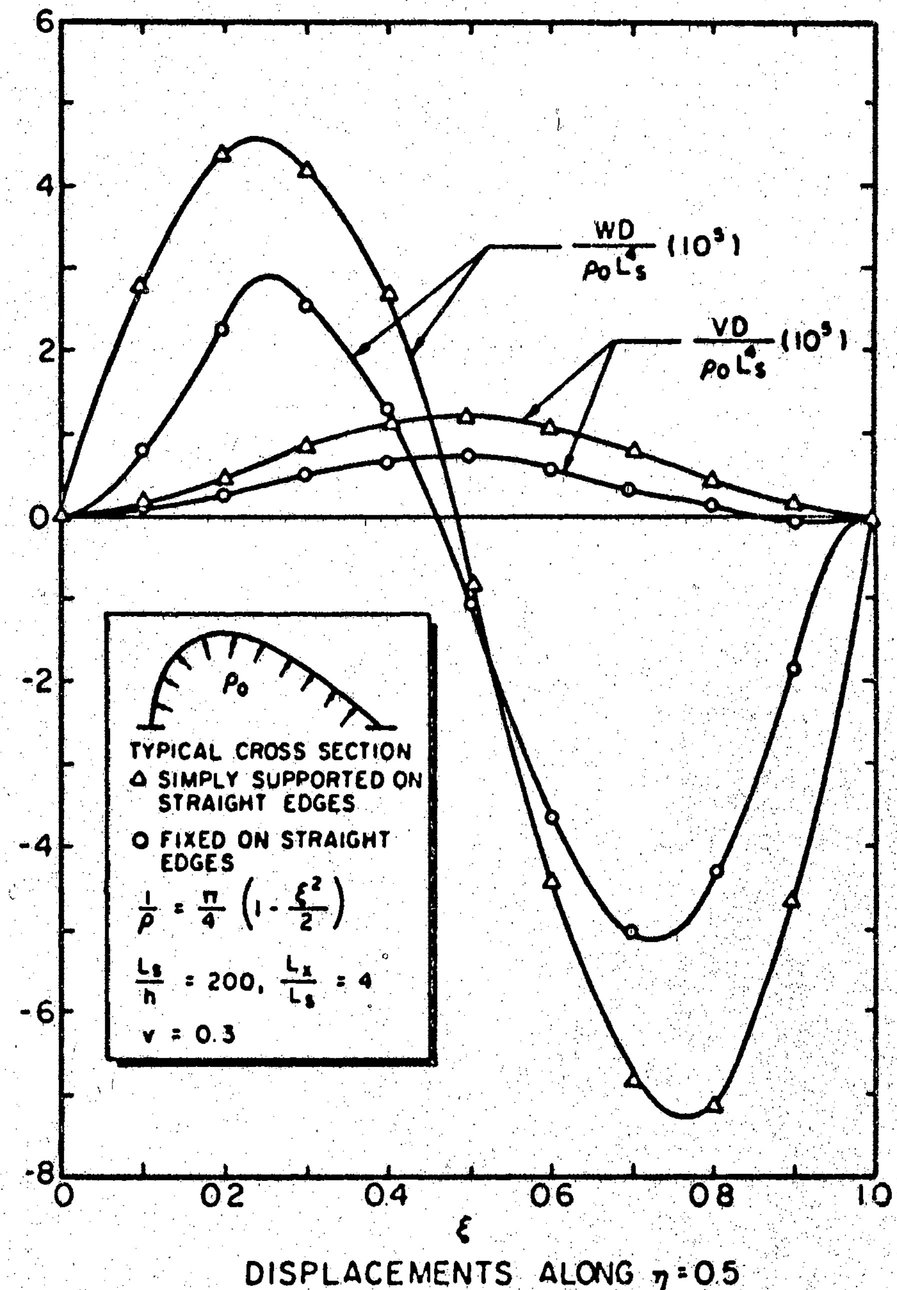


Figure 10. Comparison of Displacements for a Noncircular Cylindrical Shell Under Uniform Pressure and Different Boundary Conditions on the Straight Edges

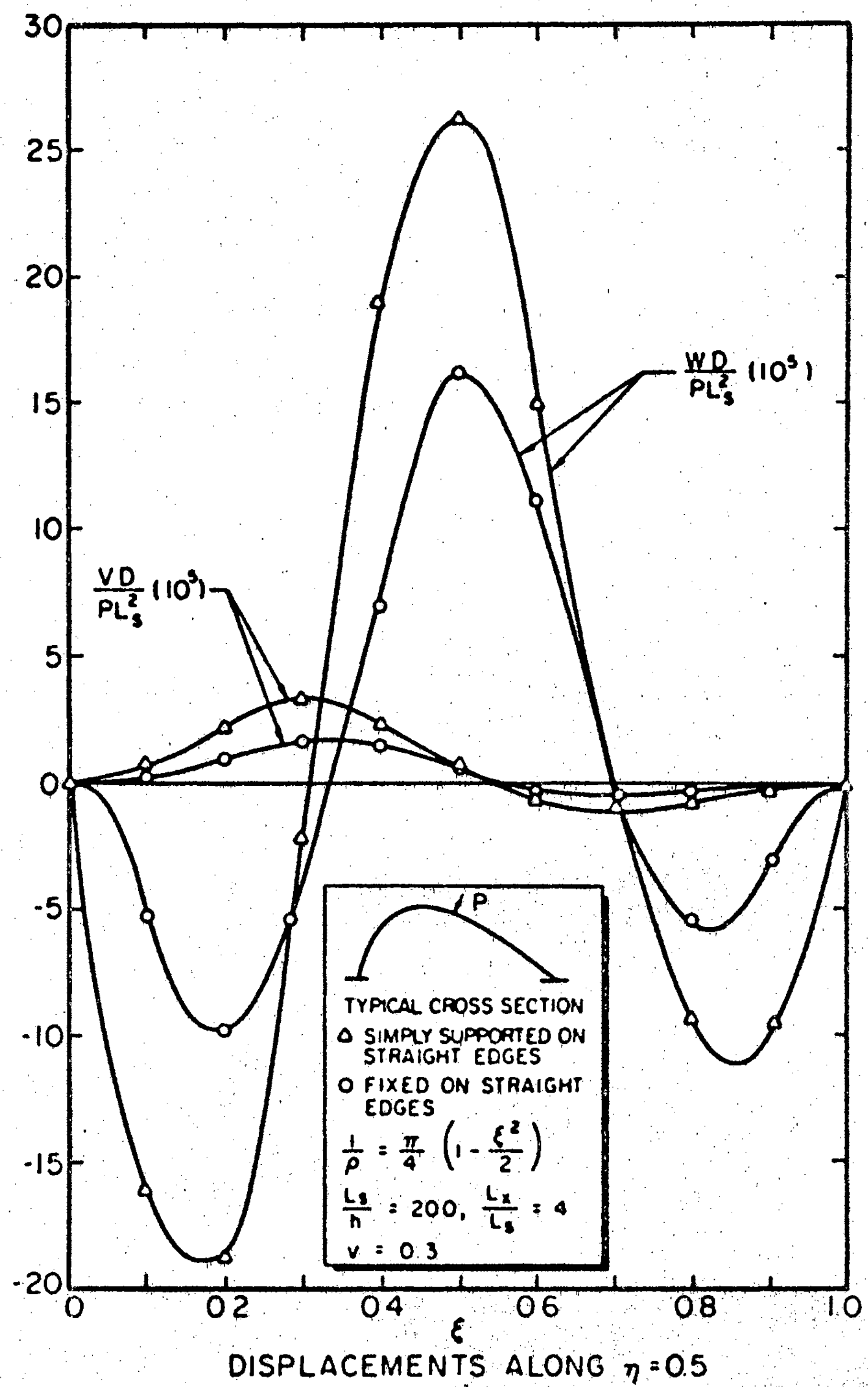


Figure 11. Comparison of Displacements for a Noncircular Cylindrical Shell Under a Radially Directed Point Load at the Shell Center and Different Boundary Conditions on the Straight Edges

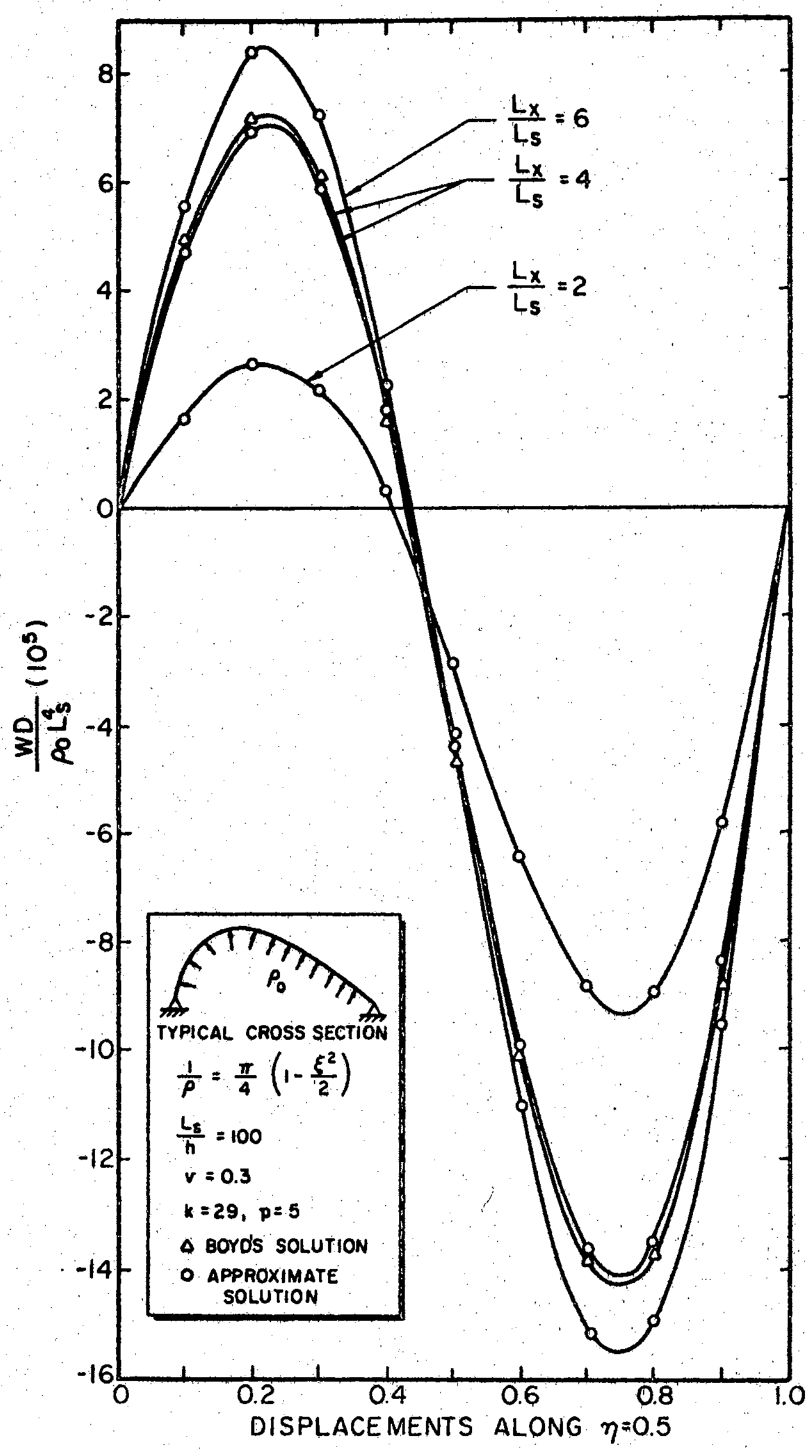


Figure 12. Radial Deflections for a Simply Supported Open Noncircular Cylindrical Shell Under Uniform Radial Pressure with $L_s/h = 100$ and Different L_x/L_s Ratios

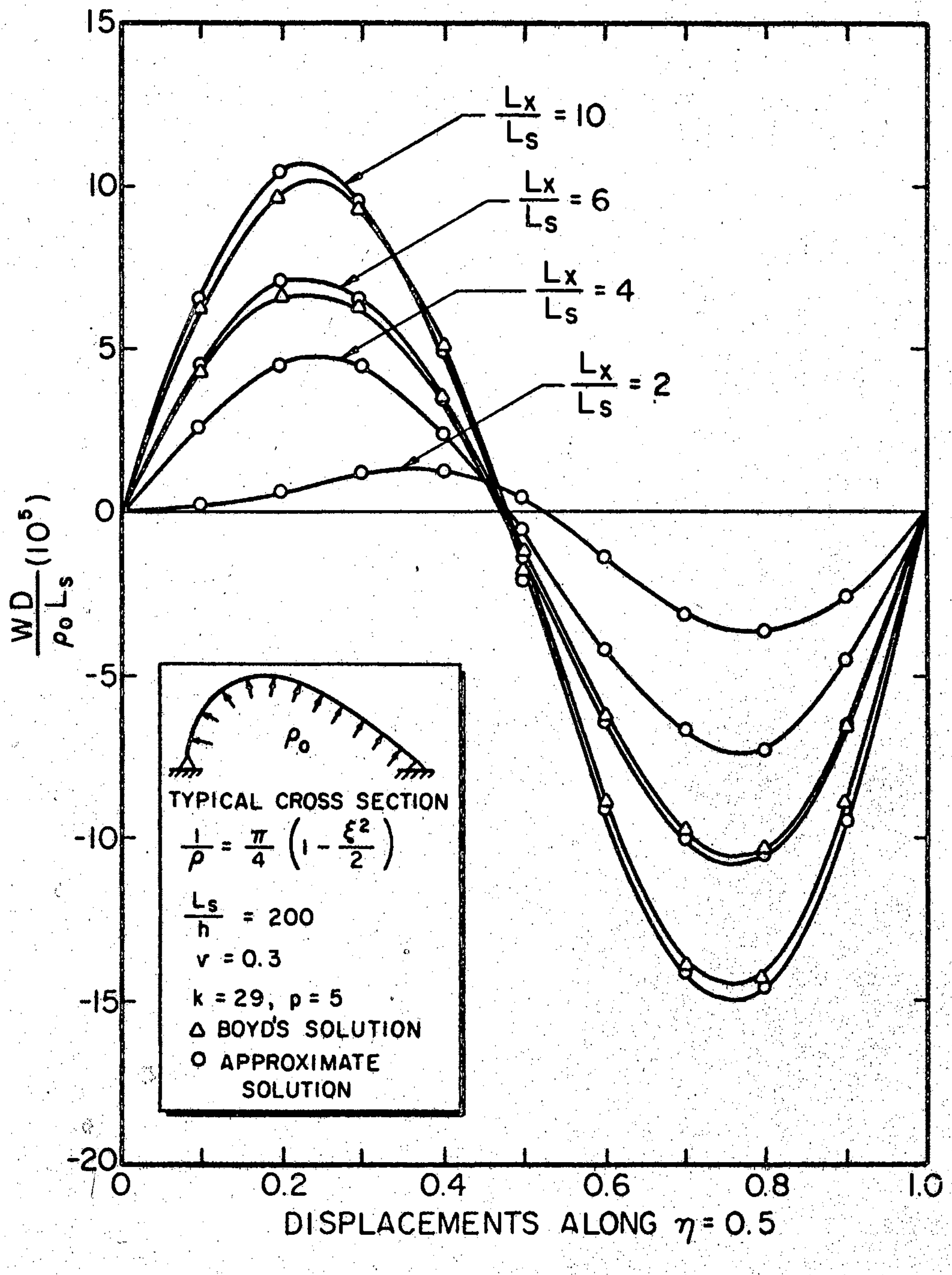


Figure 13. Radial Deflections for a Simply Supported Open Noncircular Cylindrical Shell Under Uniform Radial Pressure with $L_s/h = 200$ and Different L_x/L_s Ratios

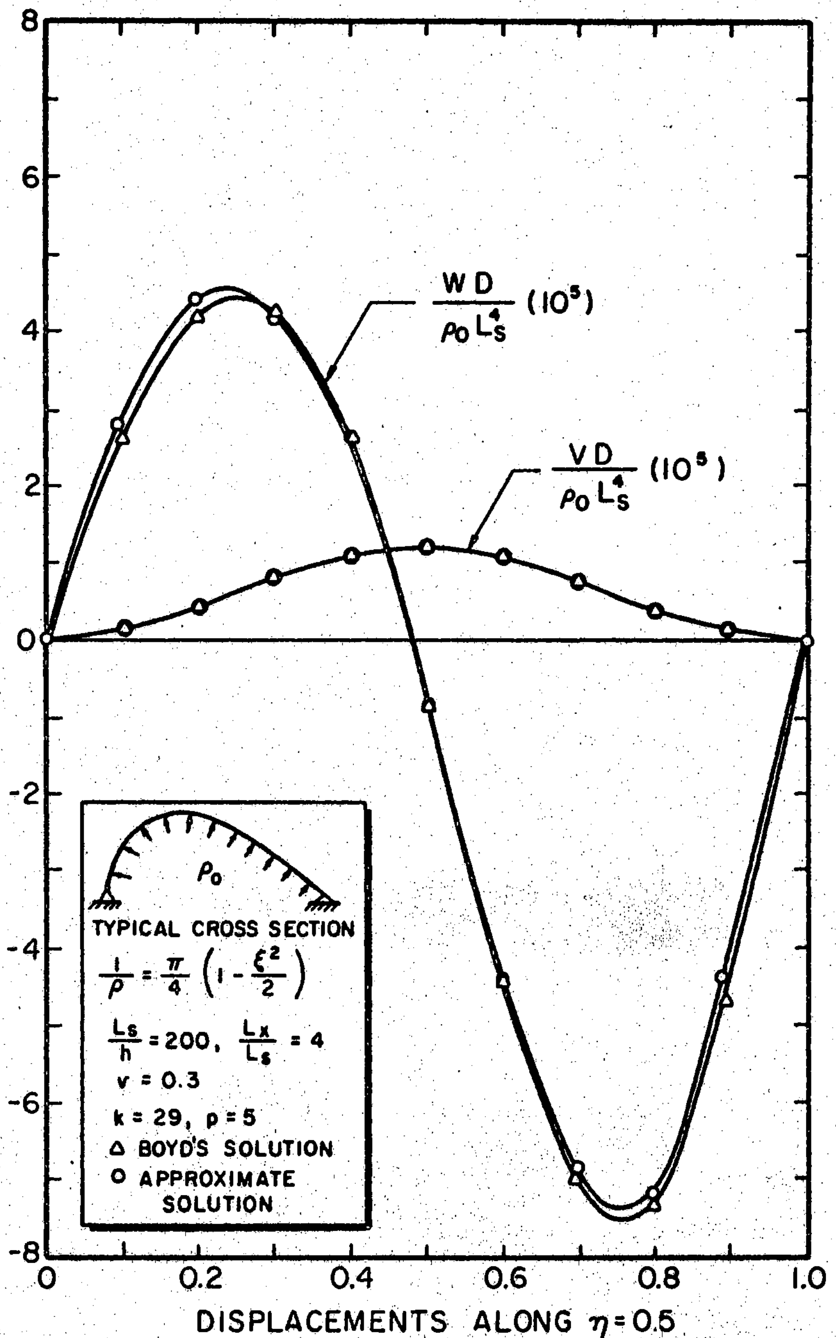


Figure 14. Radial and Circumferential Deflections for a Simply Supported Open Noncircular Cylindrical Shell Under Uniform Radial Pressure with $L_s/h = 200$ and $L_x/L_s = 4$

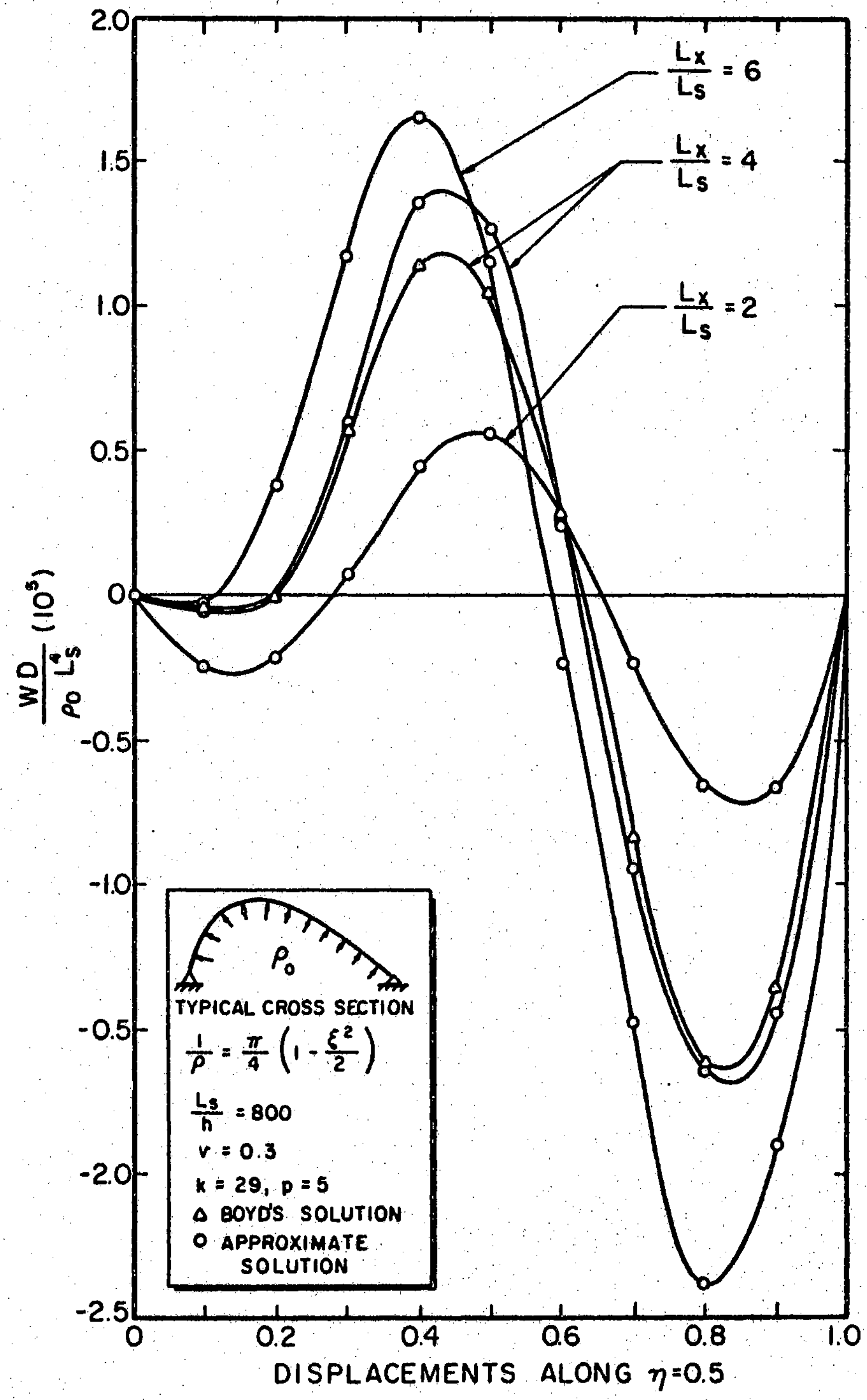


Figure 15. Radial Deflections for a Simply Supported Open Noncircular Cylindrical Shell Under Uniform Radial Pressure with $L_s/h = 800$ and Different L_x/L_s Ratios

TABLE IV

RADIAL DEFLECTION COMPARISON FOR A SIMPLY SUPPORTED OPEN NONCIRCULAR CYLINDRICAL SHELL UNDER UNIFORM RADIAL PRESSURE WITH $L_x/L_s = 4$ AND DIFFERENT L_s/h RATIOS

$$\frac{1}{\rho} = \frac{\pi}{4}\left(1 - \frac{\xi^2}{2}\right) \qquad w = \frac{\overline{W}\, \rho_o\, L_s^{\,4}}{D(10^5)} \text{ at } \eta = 0.5$$

	$\overline{W}$ for $\frac{L_s}{h} = 100$			$\overline{W}$ for $\frac{L_s}{h} = 200$			$\overline{W}$ for $\frac{L_s}{h} = 800$		
ξ	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error
0.0	0	0	0	0	0	0	0	0	0
0.1	4.853	4.576	-6.06	2.587	2.731	+5.93	-0.036	-0.043	+12.00
0.2	7.461	6.999	-6.60	4.231	4.449	+5.15	-0.020	-0.024	+12.00
0.3	6.211	5.951	-4.37	4.275	4.412	+3.20	0.585	0.600	+ 2.58
0.4	1.691	1.761	+4.00	2.509	2.526	+0.68	1.134	1.369	+20.72
0.5	- 4.698	- 4.412	-6.49	-0.664	-0.689	+3.77	1.079	1.281	+18.76
0.6	-10.056	- 9.836	-2.24	-4.285	-4.232	-1.24	0.289	0.253	-12.46
0.7	-13.836	-13.605	-1.70	-7.025	-6.825	-2.84	-0.817	-0.934	+14.37
0.8	-13.775	-13.533	-1.79	-7.546	-7.222	-4.29	-1.601	-1.646	+ 2.79
0.9	- 8.689	- 8.584	-1.22	-5.039	-4.752	-5.69	-1.369	-1.444	+ 5.50
1.0	0	0	0	0	0	0	0	0	0

TABLE V

CIRCUMFERENTIAL DEFLECTION COMPARISON FOR A SIMPLY SUPPORTED OPEN NONCIRCULAR CYLINDRICAL SHELL UNDER UNIFORM RADIAL PRESSURE WITH $L_x/L_s = 4$ AND DIFFERENT L_s/h RATIOS

$$\frac{1}{\rho} = \frac{\pi}{4}\left(1 - \frac{\xi^2}{2}\right) \qquad V = \frac{\bar{V}\,\rho_0\,L_s^4}{D(10^5)} \text{ at } \eta = 0.5$$

ξ	$\bar{V}$ for $\frac{L_s}{h} = 100$			$\bar{V}$ for $\frac{L_s}{h} = 200$			$\bar{V}$ for $\frac{L_s}{h} = 800$		
	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error
0.0	0	0	0	0	0	0	0	0	0
0.1	0.306	0.314	+2.42	0.134	0.133	-0.75	-0.015	-0.017	+12.50
0.2	0.881	0.917	+4.03	0.439	0.446	+1.59	-0.036	-0.047	+29.70
0.3	1.504	1.568	+4.23	0.807	0.821	+1.73	-0.021	-0.033	+59.47
0.4	1.938	2.006	+3.51	1.103	1.113	+0.91	0.052	0.045	-15.55
0.5	2.013	2.068	+2.73	1.209	1.210	+0.08	0.136	0.147	+ 7.91
0.6	1.699	1.738	+2.24	1.080	1.072	-0.74	0.190	0.205	+ 8.16
0.7	1.114	1.139	+2.26	0.758	0.748	-1.32	0.177	0.186	+ 4.86
0.8	0.479	0.491	+2.35	0.371	0.367	-1.08	0.108	0.113	+ 4.39
0.9	0.053	0.056	+4.60	0.079	0.081	+2.53	0.030	0.032	+ 6.25
1.0	0	0	0	0	0	0	0	0	0

TABLE VI

RADIAL DEFLECTION COMPARISON FOR A SIMPLY SUPPORTED OPEN NONCIRCULAR CYLINDRICAL SHELL UNDER UNIFORM RADIAL PRESSURE WITH $L_s/h = 200$ AND DIFFERENT L_x/L_s RATIOS

$\frac{1}{\rho} = \frac{\pi}{4}\left(1 - \frac{\xi^2}{2}\right)$ $\qquad W = \frac{\bar{W}\, p_o\, L_s^4}{D(10^5)}$ at $\eta = 0.5$

ξ	$\bar{W}$ for $\frac{L_x}{L_s} = 2$			$\bar{W}$ for $\frac{L_x}{L_s} = 4$			$\bar{W}$ for $\frac{L_x}{L_s} = 6$			$\bar{W}$ for $\frac{L_x}{L_s} = 10$		
	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error
0.0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	0.166	0.179	+ 7.83	2.587	2.731	+5.93	4.251	4.429	+ 4.18	6.214	6.586	+ 5.99
0.2	0.674	0.646	− 4.15	4.231	4.449	+5.15	6.741	7.039	+ 4.42	9.767	10.386	+ 6.34
0.3	1.223	1.226	+ 0.25	4.275	4.412	+3.20	6.495	6.665	+ 2.62	9.346	9.699	+ 3.78
0.4	1.118	1.296	+15.94	2.509	2.526	+0.68	3.525	3.472	− 1.50	5.059	4.924	− 2.67
0.5	0.336	0.324	− 3.58	−0.664	−0.689	+3.77	− 1.223	− 1.418	+15.91	− 1.760	− 2.138	+21.48
0.6	−1.441	−1.496	+ 3.83	−4.285	−4.232	−1.24	− 6.300	− 6.462	+ 2.57	− 8.902	− 9.220	+ 3.57
0.7	−3.084	−3.169	+ 2.74	−7.025	−6.825	−2.84	− 9.893	−10.057	+ 1.66	−13.828	−14.163	+ 2.42
0.8	−3.788	−3.794	+ 0.15	−7.546	−7.222	−4.29	−10.443	−10.503	+ 1.57	−14.296	−14.664	+ 2.57
0.9	−2.744	−2.779	+ 1.27	−5.039	−4.752	−5.69	− 6.758	− 6.829	+ 1.05	− 9.283	− 9.447	+ 1.77
1.0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE VII

CIRCUMFERENTIAL DEFLECTION COMPARISON FOR A SIMPLY SUPPORTED OPEN NONCIRCULAR CYLINDRICAL SHELL UNDER UNIFORM RADIAL PRESSURE WITH L_s/h = 200 AND DIFFERENT L_x/L_s RATIOS

$$\frac{1}{\rho} = \frac{\pi}{4}\left(1 - \frac{\xi^2}{2}\right) \qquad V = \frac{\bar{V}\, p_o\, L_s^4}{D(10^5)} \text{ at } \eta = 0.5$$

ξ	$\bar{V}$ for $\frac{L_x}{L_s} = 2$			$\bar{V}$ for $\frac{L_x}{L_s} = 4$			$\bar{V}$ for $\frac{L_x}{L_s} = 6$			$\bar{V}$ for $\frac{L_x}{L_s} = 10$		
	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error	Boyd's Solution	Approximate Solution	Percent Error
0.0	0	0	0	0	0	0	0	0	0	0	0	0
0.1	0.034	0.033	− 2.95	0.134	0.133	− 0.75	0.204	0.210	+ 2.80	0.291	0.304	+ 4.47
0.2	0.096	0.094	− 2.94	0.439	0.446	+ 1.59	0.680	0.705	+ 3.67	0.973	1.027	+ 5.55
0.3	0.213	0.199	− 6.57	0.807	0.821	+ 1.73	1.235	1.280	+ 3.63	1.766	1.859	+ 5.27
0.4	0.339	0.333	− 1.74	1.103	1.113	+ 0.91	1.655	1.703	+ 2.92	2.358	2.459	+ 4.28
0.5	0.427	0.434	+ 1.59	1.209	1.210	+ 0.08	1.778	1.817	+ 2.17	2.525	2.606	+ 4.03
0.6	0.425	0.435	+ 2.26	1.080	1.072	− 0.74	1.560	1.587	+ 1.74	2.207	2.274	+ 2.58
0.7	0.321	0.325	+ 1.19	0.758	0.748	− 1.32	1.083	1.101	+ 1.63	1.527	1.566	+ 2.55
0.8	0.159	0.161	+ 1.21	0.371	0.367	− 1.08	0.531	0.538	+ 1.40	0.750	0.767	+ 2.27
0.9	0.026	0.026	0.00	0.079	0.081	+ 2.53	0.121	0.122	+ 0.78	0.174	0.178	+ 2.30
1.0	0	0	0	0	0	0	0	0	0	0 v	0	0

Figure 15 reveals that a very thin shell $\left(\frac{L_s}{h} = 800\right)$ has a tendency to undergo two displacement reversals as compared to only one displacement reversal for a relatively thicker shell $\left(\frac{L_s}{h} = 200 \text{ or } 100\right)$ under uniform radial pressure. A possible explanation for this is that a flat surface deflects in the same direction as the uniform pressure and this effect is increased as the shell thickness decreases. The shell in question is 'flat' at the right hand side and deflects in the same direction as the pressure regardless of the thickness of the shell. On the left hand side the radius of curvature apparently is not large enough to let it deflect in the same direction as the load for $\frac{L_s}{h} = 100$ or 200, but for a thinner shell $\left(\frac{L_s}{h} = 800\right)$ the same radius of curvature is sufficient to deflect the shell in the same direction as the pressure, though only for a small portion near the support. Consequently, for a very thin shell there results a deflection curve that changes signs twice. A limiting case of this would be a flat plate where displacements will occur only in the direction of the pressure. This also suggests that predominantly the displacements are due to bending rather than due to stretching.

4.5 Accuracy of the Donnell Equations

Although it was not the specific object of this investigation to study the accuracy of Donnell's assumptions, a few remarks will be made here, since most of the results obtained from this investigation were compared to Boyd's results who used Donnell's assumptions regarding the contribution of the shearing force Q_s to the equilibrium of forces and the contribution of the circumferential displacement v to the expressions for the change of curvature and twist. For the cases studied here, there was no significant difference between Boyd's solution and the approximate method. Since both methods are approximate (although they use different approximations) very little can be said about the relative accuracy of either. For short thin shells under uniform load either method is expected to give reliable results. However, for very long and relatively thicker shells under some loads producing a large shear force, this approximate method would prove more versatile.

CHAPTER V

SUMMARY AND CONCLUSIONS

5.1 Summary

A method has been presented to determine the deformations of a general noncircular cylindrical shell using an energy method and an approximate technique where displacements are assumed as finite power series. Special cases of the flat plate, circular shell, and the noncircular shell used by Boyd were investigated and the following observations made:

1. Through comparison of deflections obtained by other methods for identical shells, this method of analysis was shown to give valid results.

2. The method is valid for open noncircular cylindrical shells having curvatures expressible as power series. The curved ends of the shell must be simply supported, but any arbitrary boundary conditions can be imposed on the straight edges.

3. In this method uniform, line, or point loads anywhere on the shell in any direction can be considered.

4. For the same degree of accuracy, convergence of the solution is most easily obtained for a flat plate. More

values of k and p are required as the number of terms in the curvature expression increase.

5. Convergence for the simply supported boundary condition along the straight edges is most easily obtained, and is most difficult for the free edges. The free edge condition along $\xi = 0$ cannot be obtained for all the cases since the power series here has to be approximated by a single term only.

6. Less values of p and k are required for convergence for a uniformly distributed load than for either a line or point load.

7. By treating the general solution associated with a point or a line load in any direction as an influence function it is possible to obtain a solution for any arbitrary loading condition. This fact can be used to obtain solutions to problems that have either a vertical, hydrostatic, parabolic, or any other shaped loading function.

8. A 'flatter' portion of the shell deflects in the same direction as the applied pressure and this effect is increased as the shell becomes thinner. The portion of the shell having a relatively smaller radius of curvature tends to deflect in the opposite direction so that the overall displacement of the shell is predominantly due to bending rather than stretching.

9. Although different approximations were used in this method and the method of Boyd, the results were close enough to be acceptable for Engineering purposes.

10. This method does not give very accurate results for short shells because an extremely large number of terms are required for convergence of solution in this case.

5.2 Conclusions

The method presented provides engineers with an approximate technique for calculating displacements for open non-circular cylindrical shells having curvatures expressible as power series. The curved ends of the shell must be simply supported but any arbitrary boundary conditions can be imposed on the straight edges. Classical, exact methods of analytical solution cannot be applied to take into account satisfactorily different types of loading functions which must be expanded into a series. The method used here, satisfactorily overcomes this limitation and provides an approach where any arbitrary load placed anywhere on the shell surface can be considered. It is necessary to select a suitable combination of the number of terms in both directions of the assumed displacement functions for proper convergence of the solution. A large number of terms in one direction only with an insufficient number in the other will not give satisfactory results. The number of these terms required for convergence increases as the shell geometry becomes more complicated. Also, more terms are required for the fixed edge than for the simply supported edge and it is easier to obtain convergence for uniformly distributed loads than either for point or line loads.

A shell where the radius of curvature is constant (example a flat plate or a circular shell) and under a uniform pressure will deflect in the direction of the load but for a variable radius of curvature, the 'flatter' portion of the shell deflects in the direction of the applied pressure and the remaining portion deflects in the opposite direction, so that the overall effect of bending is predominant rather than that of stretching. This effect becomes more predominant as the shell becomes thinner. The results obtained by this method give answers that are close to Boyd's for most cases, but it is expected that for relatively thick and long shells under shear loads this method would prove more versatile.

5.3 Suggestions for Further Work

During this study, many interesting topics were noted which should be studied.

It may be possible to introduce arbitrary boundary conditions even on the curved edges through the choice of polynomial functions for displacements in both the longitudinal and circumferential directions.

The possibility of taking into account discontinuous boundary conditions on any edge needs some attention.

Additional properties of the shell should be incorporated into this theory. For example, when applying this method to the analysis of shell structures for aircrafts, including helicopters; submarines, and space vehicles, it would be desirable to incorporate anisotropic material properties as well as variable thickness.

An investigation should be made into the possibility of extending this method for structures where the curvature is not constant at each section of the shell, but varies with length. An example of this would be an open noncircular tapering cylindrical panel.

Other approximate methods may be considered. In general, such methods can be classified into three basic groups as follows:

1. Methods which satisfy the governing differential equations but not the boundary conditions. Examples of this method are the Point matching method and the Trefftz-Morley method.

2. Methods which satisfy the boundary conditions but not the differential equations. The method discussed in this research belongs to this group. Also the interior collocation, Kantorovich, and the Galerkin methods are included in this group. For a brief and precise discussion of the methods of the first two groups the reader is referred to reference 15.

3. Methods that satisfy neither the differential equations nor the boundary conditions. The well-known finite difference method (16) and the finite element method for usual types of elements (17 and 18) belong to this group. Most of the methods discussed above require the use of a digital computer, mainly because of the necessity of

inverting a large matrix. An approximate solution where it would not be necessary to invert a matrix and probably eliminate the use of the digital computer would be most welcome. Such a method might use the results obtained from this investigation as a guide.

This method could also be extended to multiple bay shells where each bay has the same or different radius of curvature.

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APPENDIX A

DERIVATION OF THE STRAIN ENERGY EXPRESSION OF A NONCIRCULAR CYLINDRICAL SHELL

A.1 Assumptions

1. The shell is cylindrical, i.e., its cross section is characterized by the plane curve resulting from the intersection of the median surface and a plane normal to the axis of the cylinder.

2. The right-handed coordinate system shown in Figure 1 gives the coordinates of any point (x,s,z) in the shell.

3. The material of the shell is isotropic, homogeneous and elastic.

4. The thickness of the shell is very small compared to the other dimensions of the shell.

5. The deformations u, v, and w are small compared to the thickness of the shell and do not significantly change the geometry of the shell.

6. The Kirchoff-Love assumptions of thin-walled shell theory are applied; i.e., normals to the median surface of the undeformed shell remain straight, unextended, and normal to the median surface after deformation.

7. The loading is applied at the median surface.

8. The stresses at any point in the shell wall are related to the strains through Hooke's Law for plane stress.

A.2 Relation Between Stress Resultants

With reference to Figures 16, 17, and 18, the stress resultants N (membrane) and M (bending and twisting) are related to the axial, circumferential, and shear stresses σ_x, σ_s, and τ_{xs} (= τ_{sx}) at any distance z from the median surface, by the following relations:

$$N_x = \int_{-h/2}^{h/2} \sigma_x \left[1 - \left(\frac{z}{r}\right)\right] dz$$

$$N_{xs} = \int_{-h/2}^{h/2} \tau_{xs} \left[1 - \left(\frac{z}{r}\right)\right] dz$$

$$N_s = \int_{-h/2}^{h/2} \sigma_s \, dz$$

$$M_x = \int_{-h/2}^{h/2} \sigma_x \left[1 - \left(\frac{z}{r}\right)\right] z \, dz$$

$$M_{xs} = - \int_{-h/2}^{h/2} \tau_{xs} \left[1 - \left(\frac{z}{r}\right)\right] z \, dz$$

$$M_s = \int_{-h/2}^{h/2} \sigma_s \, z \, dz$$

$$M_{sx} = \int_{-h/2}^{h/2} \tau_{sx} \, z \, dz$$

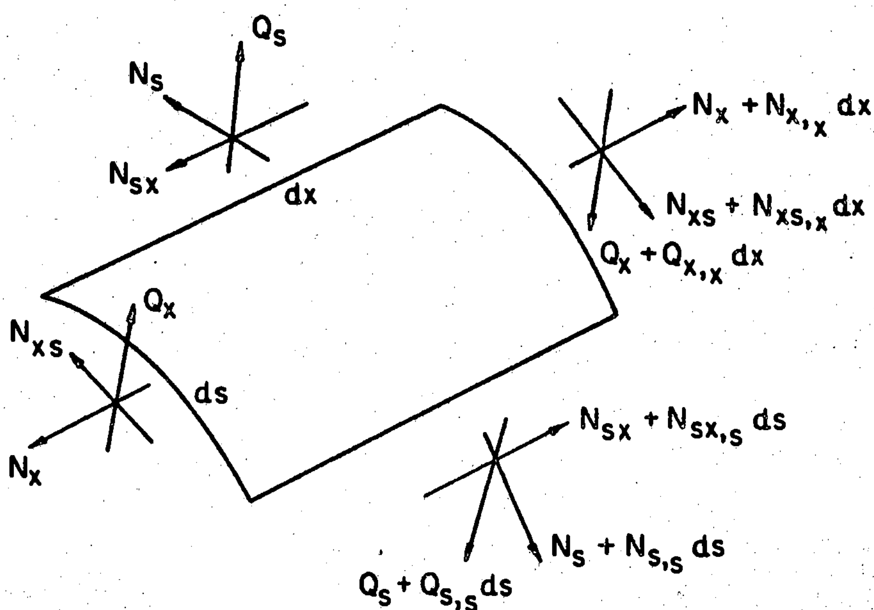

Figure 16. Sign Convention for Membrane and Transverse Shear Resultants

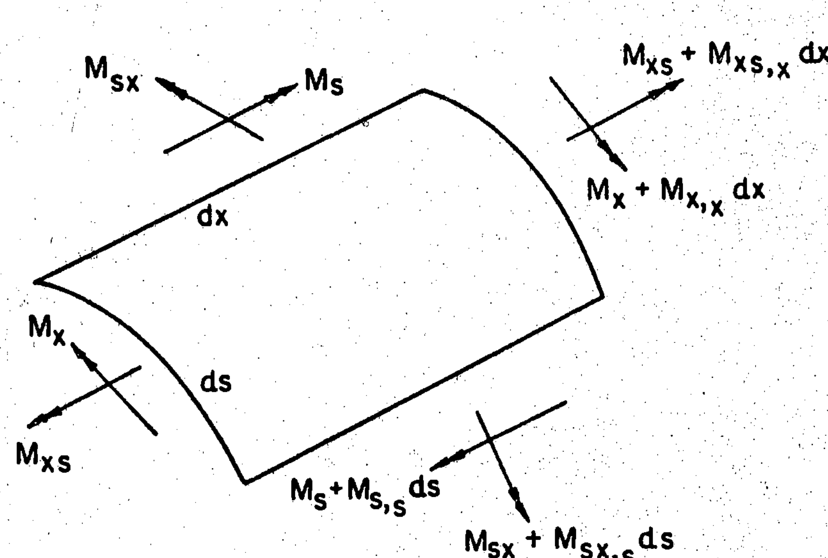

Figure 17. Sign Convention for Bending and Twisting Moment Resultants

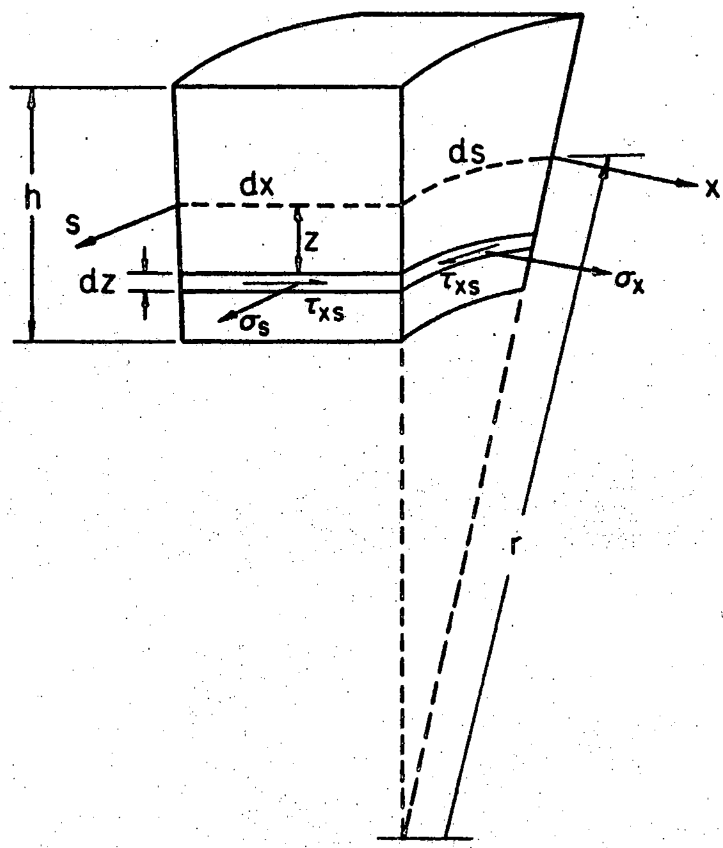

Figure 18. Sign Convention for Stresses on the Element

$$Q_x = M_{x,x} + M_{sx,s}$$

$$Q_s = M_{s,s} - M_{xs,x} \qquad (A.1)$$

A.3 Strain-Displacement Relations

With assumptions 5 and 6, the axial, circumferential, and radial displacements at any point in the shell wall, u_z, v_z, and w_z, respectively, can be expressed in terms of the corresponding median surface displacements u(x,s), v(x,s), and w(x,s) as well as the axial and circumferential components of rotation of the normal at the median surface ω_x and ω_s, respectively.

$$\begin{aligned} u_z &= u + z\ \omega_s \\ v_z &= v + z\ \omega_x \\ w_z &= \omega \end{aligned} \qquad (A.2)$$

where

$$\begin{aligned} \omega_x &= w,_s + \frac{v}{r} \\ \omega_s &= -\ w,_x \end{aligned} \qquad (A.3)$$

and

$$\omega_{x,x} = -\ \omega_{s,s} + \frac{1}{r}\ v,_x \qquad (A.4)$$

The strains at any point in the shell wall are related to the corresponding displacements by means of the well-known strain displacement relations expressed in cylindrical coordinates. Hence,

$$e_x = u_{z,x}$$

$$e_s = \frac{1}{1-z/r}\left[v_{z,s} - \frac{w_z}{r}\right] \qquad (A.5)$$

$$e_{xs} = e_{sx} = v_{z,x} + \frac{1}{1-z/r} u_{z,s} \qquad (A.5)$$

in which e_x, e_s, and e_{xs} are the axial, circumferential, and shearing strains, respectively, describing the state of strain in any plane tangential to the cylindrical surface [r(s), z]. In terms of the displacement and rotation components of the median surface, the strain displacement relations become:[1]

$$\begin{aligned} e_x &= u_{,x} + z\,\omega_{s,x} \\ e_s &= \frac{1}{1-z/r}\left[v_{,s} - \frac{w}{r} - z\,\omega_{x,s}\right] \\ e_{xs} &= \frac{1}{1-z/r}\left[u_{,s} + z\,\omega_{s,s}\right] + v_{,x} - z\,\omega_{x,x} \end{aligned} \qquad (A.6)$$

It is also a convenience to have these strains expressed in terms of the corresponding median surface strains. From equations A.3, A.4, and A.6

$$\begin{aligned} e_x &= e_{xo} + z\,\omega_{s,x} \\ e_s &= e_{so} + \frac{z}{1-z/r}\left[\omega_{x,s} - \frac{e_{so}}{r}\right] \\ e_{xs} &= e_{xso} + \frac{z}{1-z/r}\left[\omega_{s,s} - \left(1 - \frac{z}{r}\right)\omega_{x,x} + \left(\frac{1}{2r}\right)(e_{xso} - 2\omega_z)\right] \\ &= e_{xso} - \frac{z}{1-z/r}\left[\left(2 - \frac{z}{r}\right)\omega_{x,x} - \frac{e_{xso}}{r}\right] \end{aligned} \qquad (A.7)$$

in which

$$\begin{aligned} e_{xo} &= u_{,x} \\ e_{so} &= v_{,s} - \left(\frac{w}{r}\right) \\ e_{xso} &= u_{,s} + v_{,x} \end{aligned} \qquad (A.8)$$

---

[1]Physical interpretation of $\omega_{s,x}$, $\omega_{x,s}$, and $\omega_{x,x}$ is available in many references (see Reference 1).

Equation A.7 can also be presented in the form

$$e_x = e_{xo} - z\,\kappa_x$$

$$e_s = e_{so} - \left(\frac{z}{1-z/r}\right)\kappa_s$$

$$e_{xs} = \left(\frac{1}{1-z/r}\right) e_{xso} - \left(1 + \frac{z}{1-z/r}\right)\kappa_{xs} \qquad (A.9)$$

in which

$$\kappa_x = -\,\omega_{s,x} = \omega,_{xx}$$

$$\kappa_s = \omega_{x,s} - \left(\frac{e_{so}}{r}\right)$$

$$= \left[\omega,_s + \left(\frac{v}{r}\right)\right],_s - \left(\frac{1}{r}\right)\left[v,_s - \left(\frac{w}{r}\right)\right]$$

$$\kappa_{xs} = \omega_{x,x} = -\,\omega_{s,s} + \frac{1}{2r}\left(e_{xso} + 2\omega_z\right)$$

$$= \left[\omega,_s + \left(\frac{v}{r}\right)\right],_x \qquad (A.10)$$

A.4 Stress Resultants in Terms of Displacements

The stresses at any point in the shell wall are related to the strains through Hooke's Law for plane stress.

$$\sigma_x = \frac{E}{1-\nu^2}\left(e_x + \nu\, e_s\right)$$

$$\sigma_s = \frac{E}{1-\nu^2}\left(e_s + \nu\, e_x\right)$$

$$\tau_{xs} = \tau_{sx} = \frac{E}{2(1+\nu)}\, e_{xs} \qquad (A.11)$$

in which E is the Young's modulus and ν the Poisson's ratio.

Substitution of these stresses into the right hand side of Equation A.1, subsequent integration and use of Equations A.9 and A.10 yield

$$N_x = \frac{Eh}{(1-\nu^2)} (e_{xo} + \nu \, e_{so}) - \left(\frac{D}{r}\right) \omega_{s,x}$$

$$N_s = \frac{Eh}{(1-\nu^2)} (e_{so} + \nu \, e_{xo}) - \left(\frac{cD}{r}\right) \omega_{x,s} - \left(\frac{e_{so}}{r}\right)$$

$$N_{xs} = \frac{Eh}{2(1+\nu)} e_{xso} + \frac{1}{2(1-\nu)} \left(\frac{D}{r}\right) \omega_{x,x}$$

$$N_{sx} = \frac{Eh}{2(1+\nu)} e_{xso} - \frac{1}{2(1-\nu)} \left(\frac{cD}{r}\right)\left[\omega_{x,x} - \left(\frac{e_{xso}}{r}\right)\right]$$

$$M_x = - D \left[- \omega_{s,x} + \nu \, \omega_{x,s} + \left(\frac{e_{xo}}{r}\right)\right]$$

$$M_s = - D \left[c \, \omega_{x,s} - \nu \, \omega_{s,x} - c \left(\frac{e_{so}}{r}\right)\right]$$

$$M_{xs} = D(1-\nu) \, \omega_{x,x}$$

$$M_{sx} = - D(1-\nu) \left[\frac{1+c}{2} \omega_{x,x} - \frac{c}{2} \left(\frac{e_{xso}}{r}\right)\right]$$

$$Q_x = - D \left\{- \omega_{s,xx} + \frac{1-\nu}{2} [(1+c) \, \omega_{x,x}]_{,s} + \nu \, \omega_{x,xs} + \left(\frac{e_{xo}}{r}\right)_{,s} - \left[(1-\nu) \frac{c}{2} \left(\frac{e_{xso}}{r}\right)\right]_{,s}\right\}$$

$$Q_s = - D \left\{(c \, \omega_{x,s})_{,s} + (1-\nu) \, \omega_{x,xx} - \nu \, \omega_{s,xs} - \left[c \left(\frac{e_{so}}{r}\right)\right]_{,s}\right\} \qquad (A.12)$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$c = 12 \left(\frac{r}{h}\right)^2 \left\{\frac{r}{h} \, \mathrm{Log} \left[\frac{1 + \frac{h}{2r}}{1 - \frac{h}{2r}}\right] - 1\right\} \simeq 1 \qquad (A.13)$$

A.5 Strain Energy Expression

In accordance with the basic assumptions, the strain energy U stored in a volume of a cylindrical shell is

$$U = \frac{1}{2} \iiint_{vol} (\sigma_x \, e_x + \sigma_s \, e_s + \tau_{xs} \, e_{xs}) \left[1 - \frac{z}{r}\right] dx \, ds \, dz \tag{A.14}$$

Introduction of Equations A.7 into Equations A.14 and subsequent use of Equations A.1 facilitate the reduction of the volume integral to a surface integral involving quantities pertinent to the median surface only. Thus the strain energy function becomes

$$U = \frac{1}{2} \int_{L_s} \int_{L_x} [N_x \, e_{xo} + N_s \, e_{so} + N_{sx} \, e_{xso} + M_x \, \omega_{s,x} - M_s \, \omega_{x,s} + (M_{xs} - M_{sx}) \omega_{x,x}] \, dx \, ds \tag{A.15}$$

It is desirable to have the strain energy expressed only in terms of quantities characterizing the state of deformation of the median surface. Introduction of Equation A.12 into A.14, yields

$$\begin{aligned} U = \frac{1}{2} \int_{L_s} \int_{L_x} \frac{Eh}{(1-\nu^2)} \Bigg\{ & \left[e_{xo}{}^2 + e_{so}{}^2 + 2\nu \, e_{xo} \, e_{so} + \frac{1-\nu}{2} e_{xso}{}^2 \right] \\ & + D \Bigg[(\omega_{s,x})^2 + c(\omega_{x,s})^2 - 2\nu \, \omega_{x,s} + \omega_{x,s} \\ & + \frac{(1-\nu)}{2} (3+c)(\omega_{x,x})^2 - \left(\frac{2}{r}\right) (\omega_{s,x} \, e_{xso} + c \, \omega_{x,s} \, e_{so}) \\ & - (1-\nu)\left(\frac{c}{r}\right) \omega_{x,x} \, e_{xso} + c \left(\frac{e_{so}}{r}\right)^2 + (1-\nu) \left(\frac{c}{2}\right) \\ & \left(\frac{e_{xso}}{r}\right)^2 \Bigg] \Bigg\} dx \, ds \end{aligned} \tag{A.16}$$

Equation A.16, when combined with Equations A.8 and A.10, yields U in terms of the median surface displacements

$$
\begin{aligned}
U = \frac{Eh}{2(1-\nu^2)} \int_{L_s} \int_{L_x} \Bigg\langle & (u,_x)^2 + \left[v,_s - \left(\frac{w}{r}\right)\right]^2 \\
& + 2\nu\, u,_x \left[v,_s - \left(\frac{w}{r}\right)\right] + \frac{(1-\nu)}{2} (u,_s + v,_x)^2 \\
& + \frac{h^2}{12} \Bigg\{ (w,_{xx})^2 + c \left[w,_{ss} + \left(\frac{w}{r^2}\right) - \left(\frac{r,_s}{r^2}\right) v\right]^2 \\
& + 2\nu\, w,_{xx} \left[w,_{ss} + \left(\frac{1}{r}\right) v,_s - \left(\frac{r,_s}{r^2}\right) v\right] \\
& + \frac{c(1-\nu)}{2} \left[w,_{xs} - \left(\frac{1}{r}\right) u,_s\right]^2 + \frac{3}{2}(1-\nu)\left[w,_{xs} + \left(\frac{1}{r}\right) v,_x\right]^2 \\
& + \left(\frac{2}{r}\right) w,_{xx}\, u,_x \Bigg\} \Bigg\rangle dx\, ds
\end{aligned}
\qquad (A.17)
$$

APPENDIX B

COMPUTER PROGRAM

```
C     REFERENCE:
C
C     AN APPROXIMATE ANALYSIS OF OPEN NONCIRCULAR CYLINDRICAL SHELLS
C
C
C     PURPOSE:
C
C          TO EVALUATE , USING THE ENERGY METHOD , THE DISPLACEMENTS
C     IN AN OPEN NONCIRCULAR CYLINDRICAL SHELL SIMPLY SUPPORTED AT THE
C     CURVED ENDS AND WITH ANY BOUNDARY CONDITIONS ON THE STRAIGHT
C     EDGES , SUBJECT TO ANY ARBITRARY LOADING.
C
C
C     PROGRAMMER:
C
C
C     ASHOK NAIN
C     GRADUATE ASSISTANT
C     SCHOOL OF CIVIL ENGINEERING
C     OKLAHOMA STATE UNIVERSITY
C     STILLWATER , OKLAHOMA
C
C
C     GENERAL COMMENTS :
C     THE PROGRAM HAS BEEN WRITTEN TO GIVE THE DISPLACEMENTS DIRECTLY
C     AT EVERY TENTH POINT AT A SECTION HALF WAY BETWEEN THE CURVED
C     EDGES FOR V AND W DISPLACEMENTS AND AT THE CURVED EDGES FOR THE
C     U DISPLACEMENT
C
C
C     DESCRIPTION OF PARAMETERS:
C
C
C     (AAX) = MATRIX OF COEFFICIENTS OF POLYNOMIAL FOR SHELL CURVATURE
C     (X) = STRAIN ENERGY MATRIX
C     (BB),(CC),(DD),(BC),(BD),(CD) = SUBMATRICES OF MATRIX (X)
C     (A) = (X) STORED COLUMNWISE
C     (Y) = COLUMN MATRIX DUE TO LOAD
C     (YUX),(YUY),YUZ) = COLUMN MATRICES DUE TO LOAD IN X,S,Z DIRECTIONS
C     (DISP) = MATRIX CONTAINING UNKNOWN COEFFICIENTS  B(I),C(I),D(I)
C     (U),(V),(W) = DISPLACEMENT MATRICES IN X,S,Z DIRECTIONS
C     (PX),(PY),(PZ) = MATRICES OF LOADS IN X,S,Z DIRECTIONS
C     (SPX1),(SPX2),(SPY1),(SPY2),(SPZ1),(SPZ2) AND (EPX1),(EPX2),
C     (EPY1),(EPY2),(EPZ1),(EPZ2) = MATRICES DEFINING LOCATION AND
C     LIMITS OF LOAD MATRICES (PX),(PY),(PZ)
C     NN = HIGHEST POWER OF COEFFICIENT OF POLYNOMIAL FOR CURVATURE
C     K = NO. OF TERMS LESS ONE FOR POWER SERIES IN CURVED DIRECTION
C     MZ = NO. OF TERMS FOR SERIES ALONG STRAIGHT EDGES (P)
C     ZI=N
C     ZJ=N BAR
C     L1,L2,L3 = NO. OF LOADING FUNCTIONS IN X,S,Z DIRECTIONS
C     HOB = H/LS
C     HOA = H/LX
C     PR = POISSON'S RATIO
C     PIE = 3.141593
```

```
C      ALFA,BETA,GAMA,DELTA,OMEGA,THETA,PHI = PARAMETERS DEFINING
C      BOUNDARY CONDITIONS ON THE STRAIGHT EDGES OF THE SHELL
C
C
C      SUBROUTINES REQUIRED :
C
C
C      1. RRAY - CONVERTS (X) INTO (A) AND VICE VERSA
C      2. STQN - SOLVES A SET OF SIMULTANEOUS LINEAR EQUATIONS .
C
C
C      INPUT FORMAT SPECIFICATIONS :
C
C      1 ST CARD ------- NN,K,MZ (ALL INTEGERS)
C                        PUNCHED RIGHT JUSTIFIED WITH FORMAT (3I2)
C      2 ND CARD ------- AAX(I) (REAL)
C                        PUNCHED RIGHT JUSTIFIED WITH FORMAT (4D15.8)
C      3 RD CARD ------- PR,BOA,HOB,HOA,PIE (ALL REAL)
C                        PUNCHED RIGHT JUSTIFIED WITH FORMAT (5D14.8)
C      4 TH CARD ------ THETA,PHI,OMEGA,DELTA (ALL REAL)
C                        PUNCHED RIGHT JUSTIFIED WITH FORMAT (4D14.8)
C      5 TH CARD ------- ALFA,BETA,GAMA (ALL REAL)
C                        PUNCHED RIGHT JUSTIFIED WITH FORMAT (3D14.8)
C      6 TH CARD ------- L1,L2,L3 (ALL INTEGERS)
C                        PUNCHED RIGHT JUSTIFIED WITH FORMAT (3I2)
                         TYP. LOAD CARD -- THE NUMBER OF THESE DEPENDS
                         ON THE LOADS SPECIFIED BY L1,L2,L3. ALL LOADS
                         IN X DIRECTION ARE CONSIDERED FIRST, LOADS IN
                         S DIRECTION NEXT AND LASTLY THE LOADS IN THE
                         Z DIRECTION.
                         PX(I),SPX1(I),SPX2(I),EPX1(I),EPX2(I)
                         PY(I),SPY1(I),SPY2(I),EPY1(I),EPY2(I)
                         PZ(I),SPZ1(I),SPZ2(I),EPZ1(I),EPZ2(I)
C                        PUNCHED RIGHT JUSTIFIED WITH FORMAT (5D14.8)
C
C
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION AAX(10),X(90 ,90) ,A(8100) ,Y(90)  ,
      1DISP(90),U(10),V(10),W(10), YUX(30),YUY(30),YUZ(30)
       DIMENSION  PX(10) ,SPX1(10) ,SPX2(10) ,EPX1(10) ,EPX2(10)  ,
      1           PY(10) ,SPY1(10) ,SPY2(10) ,EPY1(10) ,EPY2(10)  ,
      2           PZ(10) ,SPZ1(10) ,SPZ2(10) ,EPZ1(10) ,EPZ2(10)
       EQUIVALENCE  (X(1,1) ,A(1))
  100  FORMAT (3I2)
  101  FORMAT (4D15.8)
  102  FORMAT (5D14.8)
  104  FORMAT (5D14.8)
  106  FORMAT (4D14.8)
  107  FORMAT (3D14.8)
  108  FORMAT (3I2)
  300  FORMAT (3X,5D15.8)
  311  FORMAT(1X ,  'THE ABOVE DISPLACEMENTS ARE TO BE MULTIPLIED BY A
      1FACTOR (H*H*B*A*PO/D)')
  312  FORMAT (1X,'U DISPLACEMENT')
  313  FORMAT (1X,'V DISPLACEMENT')
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314    FORMAT (1X,'W DISPLACEMENT')
323    FORMAT (1X, 'VALUES INDICATE DISPLACEMENTS AT ETA = 0. FOR
      1DISPLACEMENTS AT ETA = 1 ,MULTIPLY BY MINUS ONE')
2000   READ (5,100) NN,K,MZ
       NNN=NN+1
       KZ=K+1
       KN=2*KZ
       KK=3*KZ
       READ(5,101) (AAX(II),II=1,NNN)
       READ (5,102) PR,BOA,HOB,HOA,PIE
       READ (5,106)  THETA,PHI,OMEGA,DELTA
       READ (5,107) ALFA,BETA,GAMA
       READ (5,108) L1,L2,L3
       DO 1011 I=1,10
       U(I   ) = 0.0D0
       V(I   ) = 0.0D0
       W(I   ) = 0.0D0
1011   CONTINUE
       IF (L1.EQ.0) GO TO 1016
       DO 980  I=1,L1
980    READ (5,104) PX(I),SPX1(I), SPX2(I), EPX1(I), EPX2(I)
1016   IF (L2.EQ.0) GO TO 1017
       DO 971  I=1,L2
971    READ (5,104) PY(I),SPY1(I), SPY2(I), EPY1(I), EPY2(I)
1017   IF (L3.EQ.0) GO TO 1018
       DO 931  I=1,L3
931    READ (5,104) PZ(I),SPZ1(I), SPZ2(I), EPZ1(I), EPZ2(I)
1018   CONTINUE
       DO 1000 MM=1,MZ,2
       EM = MM
       DO 1012 I=1,KK
       DO 1013 J=1,KK
1013   X(I,J) = 0.0D0
       DISP(I) = 0.0D0
       A(I) = 0.0D0
1012   Y(I) = 0.0D0
       DO 1014 I=1,KZ
       YUX(I) = 0.0D0
       YUY(I) = 0.0D0
1014   YUZ(I) = 0.0D0
C      EVALUATION OF SIGMA M
       ONE = EM
       TWO = EM*EM
       THR = EM*EM*EM
       FOR = EM*EM*EM*EM
C      GENERATION OF BB MATRIX
       DO 200 IZ=1,KZ
       ZI=IZ-1
       DO 201JZ=IZ,KZ
       ZJ=JZ-1
       QB=ZI+ZJ+ 2.0D0*THETA
       P1 = 1.0D0/(QB+1.0D0)+PHI*PHI/(QB+3.0D0) - (2.0D0*PHI   )/(QB+2.
      10D0)
       PY1=(PIE*PIE*BOA*P1)/2.0D0
       PYM1 = PY1 * TWO
```

```
      QB=ZI+ZJ+ 2.0D0*THETA
      P7=((ZI+THETA)*(ZJ+THETA)/(QB-1.0D0+0.1D-50))+(PHI*PHI*(ZI+THETA+
     1 1.0D0 ) * (ZJ+THETA +1.0D0)/(QB + 1.0D0))   -  (((ZI+THETA) * (
     2 ZJ+ THETA +1.0D0 ) + (ZI+ THETA + 1.0D0) * (ZJ+ THETA))*PHI/
     3(QB + 0.1D-50 ))
      PY7 = ((1.0D0-PR) *P7 ) /(4.0D0*BOA)
      P21=0.0 D0
      DO  202 IIZ=1,NNN
      ZII=IIZ-1
      DO 203 JJZ=1,NNN
      ZJJ=JJZ-1
      QS = ZI+ZJ+ZII+ZJJ+2.0D0 *THETA
  203 P21 = P21 + (AAX(IIZ)*AAX(JJZ)) *(((ZI+THETA)*(ZJ+THETA) / (QS-1.0
     1D0+0.1D-50))+((ZI+THETA+1.0D0)*(ZJ+THETA+1.0D0)*PHI*PHI/(QS+1.0D0
     2))-(((((ZI+THETA)*(ZJ+THETA+1.0D0))+((ZI+THETA+1.0D0) *(ZJ+THETA
     3))) *PHI    /(QS+0.1D-50 )))
  202 CONTINUE
      PY 21 = ((1.0D0 -PR) *HOB*HOB *P21 )/(48.0D0 *BOA)
  201 X (IZ,JZ)=(PYM1+PY7+PY21) *2.0D0
  200 CONTINUE
C     GENERATION OF CC MATRIX
      DO 210 IZ=1,KZ
      ZI=IZ-1
      DO 211 JZ=IZ,KZ
      ZJ=JZ-1
      QH = ZI+ZJ+2.0D0*OMEGA
      P2=((ZI+OMEGA)*(ZJ+OMEGA)/(QH-1.0D0+0.1D-50))+(DELTA*DELTA*(ZI+OME
     1GA+1.0D0)*(ZJ+OMEGA+1.0D0)   /(QH + 1.0D0))    -  (((ZI+OMEGA) * (
     2 ZJ+ OMEGA +1.0D0 ) + (ZI+ OMEGA + 1.0D0) * (ZJ+ OMEGA))*DELTA/
     3(QH + 0.1D-50))
      PY2 = P2 /(2.0D0 *BOA)
      P8 = 1.0D0/(QH+1.0D0) + DELTA*DELTA/(QH+3.0D0)  -(2.0D0*DELTA)/
     1(QH+2.0D0)
      PY8 = ((1.0D0-PR) *PIE*PIE*BOA *P8)  /4.0D0
      PYM8 = PY8 * TWO
      P13=0.0 D0
      P23=0.0 D0
      DO 212 IIZ=1,NNN
      ZII=IIZ-1
      DO 213 JJZ=1,NNN
      ZJJ=JJZ-1
      QG = ZI+ZJ+2.0D0*OMEGA  +ZII+ ZJJ
      P13 = P13 + (AAX(IIZ) * AAX(JJZ) *ZII *ZJJ) * ((1.0D0/ (QG+1.0D0
     1))+(DELTA*DELTA/   (QG + 3.0D0)) - (2.0D0*DELTA / (QG+2.0D0)))
  213 P23 = P23 + (AAX(IIZ) * AAX(JJZ)             ) * ((1.0D0/ (QG+1.0D0
     1))+(DELTA*DELTA /  (QG + 3.0D0)) - (2.0D0*DELTA / (QG+2.0D0)))
  212 CONTINUE
      PY13= (HOB*HOB*P 13) /(24.0D0 *BOA )
      PY 23 = ((1.0D0-PR) *HOB*HOA*PIE*PIE*P23 )/16.0D0
      PYM23 = PY23 * TWO
  211 X(KZ+IZ ,KZ+JZ) = (PY2+PYM8+PY13+PYM23) *2.0D0
  210 CONTINUE
C     GENERATION OF DD MATRIX
      DO 220 IZ=1,KZ
      ZI=IZ-1
```

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      DO 221 JZ=IZ,KZ
      ZJ=JZ-1
      QD = ZI+ZJ+2.0D0*ALFA
      P10 = (1.0D0/(QD+1.0D0)) + (BETA * BETA/(QD+3.0D0)) +
     1(GAMA*GAMA/(QD+1.0D0 + 2.0D0 * BETA)) - (2.0D0*BETA / (QD+ 2.0D0
     2)) + (2.0D0*GAMA/(QD +1.0D0 +BETA))
     3 - (2.0D0 * BETA * GAMA /(QD + 2.0D0 +BETA))
      PY10= ((PIE*PIE*PIE*PIE*HOA*HOA*BOA*P10)/24.0D0 )
      PYM10 = PY10 * FOR
      RI = ZI+ ALFA +BETA
      RJ = ZJ+ ALFA +BETA
      P11 = ((ZI+ALFA) * (ZI+ALFA -1.0D0) * (ZJ+ALFA) * (ZJ+ALFA-1.0D0)
     1/(QD-3.0D0+0.1D-50))+ (BETA*BETA*(ZI+ALFA+1.0D0)* (ZI+ALFA) *
     2(ZJ+ALFA+1.0D0)*(ZJ+ALFA)/(QD-1.0D0+0.1D-50)) + (GAMA*GAMA*RI*(RI-
     11.0D0)*(RJ-1.0D0)*RJ/(QD+2.0D0*BETA-3.0D0+0.1D-50)) -(BETA*((ZI+
     4ALFA)* (ZI+ALFA-1.0D0)*(ZJ+ALFA+1.0D0) * (ZJ+ALFA) + (ZJ+ALFA) * (
     5 ZJ+ALFA-1.0D0)*(ZI+ALFA+1.0D0)*(ZI+ALFA))/(QD-2.0D0+0.1D-50))+(((
     6ZI + ALFA )* (ZI+ALFA-1.0D0)*RJ*(RJ-1.0D0) + (ZJ+ALFA) * (ZJ+ALFA
     7-1.0D0)*RI*(RI-1.0D0)) *GAMA/(QD+BETA-3.0D0+0.1D-50))-(((ZI+
     8ALFA+1.0D0)*(ZI+ALFA)*RJ*(RJ- 1.0D0)+(ZJ+ALFA+1.0D0)*(ZJ+ALFA)*RI*
     9(RI-1.0D0))*BETA*GAMA/(QD+ BETA-2.0D0+0.1D-50))
      PY11= (HOB*HOB*P11)/(24.0D0 *BOA )
      P17 = ((ZJ+ALFA)*(ZJ+ALFA-1.0D0)* (1.0D0/(QD-1.0D0+0.1D-50)-
     1BETA /(QD+0.1D-50    )+GAMA/(QD+BETA-1.0D0+0.1D-50)))-(BETA*(
     2ZJ+ALFA+1.0D0)*(ZJ+ALFA)*(1.0D0/(QD+0.1D-50    ) - BETA/(QD+1.0D0)
     3+GAMA/(QD+BETA+0.000001D0)))   +(GAMA  *
     4  (ZJ+ALFA+ BETA) *(ZJ+ALFA +BETA  -1.0D0 ) *
     5(1.0D0/(QD+BETA-1.0D0+0.1D-50)-BETA/ (QD+BETA+0.1D-50) +GAMA
     6/(QD+2.0D0 *BETA -1.0 D0+0.1D-50)))
      PY17= -((PR*PIE*PIE*HOA*HOB*P17)/12.0D0 )
      PYM17 = PY17 * TWO
      P20 = ((ZI+ALFA)*(ZJ+ALFA)/(QD-1.0D0+0.1D-50))+(BETA*BETA*(ZI+ALFA
     1+1.0D0)*(ZJ+ALFA+1.0D0)/(QD+1.0D0))  +  (GAMA*GAMA*RI*RJ
     2/(QD+(2.0D0*BETA)-1.0D0+0.1D-50))- (((ZI+ALFA)*(ZJ+ALFA+1.0D0) +
     3(ZI+ALFA+1.0D0)*(ZJ+ALFA))*BETA / (QD+0.1D-50    ))  +  (((ZI+ALFA)
     4*(ZJ+ALFA+BETA) + (ZI+ALFA+BETA)*(ZJ+ALFA))* GAMA / (QD+BETA-
     51.0D0+0.1D-50))- (((ZI+ALFA+1.0D0) * (ZJ+ALFA+BETA)+(ZI+ALFA+BETA)
     6 *(ZJ+ALFA+1.0D0))*BETA*GAMA/(QD+BETA-0.1D-50    ))
      PY20 = ((1.0D0-PR) *PIE*PIE*HOA*HOB*P20  )/ 12.0D0
      PYM20 = PY20 * TWO
      P 3=0.0D0
      P14=0.0D0
      P12=0.0D0
      DO 222 IIZ=1,NNN
      ZII=IIZ-1
      DO 223 JJZ=1,NNN
      ZJJ=JJZ-1
      QA = ZI+ZJ+ZII+ZJJ+(2.0D0*ALFA)
      P3 = P3+(AAX(IIZ)*AAX(JJZ)) * (1.0D0/(QA+1.0D0) + BETA *BETA /
     1(QA+3.0D0) + GAMA*GAMA /(QA+1.0D0+(2.0D0*BETA)) - 2.0D0*BETA /
     2(QA+2.0D0) + 2.0D0*GAMA / (QA+BETA+ 1.0D0) - 2.0D0*BETA *GAMA /
     3(QA+BETA+2.0D0))
      P14 = P14 + (AAX(IIZ)*AAX(JJZ)) *  ((ZJ+ALFA)*(ZJ+ALFA-1.0D0)*
     1(1.0D0/(QA-1.0D0+0.1D-50)-BETA/(QA+0.1D-50)     + GAMA/(QA+ BETA
     2-1.0D0+0.1D-50))-   BETA*(ZJ+ALFA+1.0D0) * (ZJ+ALFA)*(1.0D0 /
```

```
     3(QA+0.1D-50    ) - BETA/(QA+1.0D0) + GAMA/(QA+BETA+0.1D-50   ))
     4+  GAMA*(ZJ+ALFA+BETA)*(ZJ+ALFA+BETA-1.0D0) * (1.0D0/(QA+BETA -
     51.0D0+0.1D-50)-BETA/(QA+BETA+0.1D-50)      +GAMA/ (QA+(2.0D0*BETA)
     6-1.0D0+0.1D-50)))
      DO 224 IHZ=1,NNN
      ZIH=IHZ-1
      DO 225 JHZ=1,NNN
      ZJH=JHZ-1
      QF = ZI+ZJ+ZII+ZJJ+ZIH+ZJH+(2.0D0*ALFA)
 225  P12 = P12+(AAX(IIZ)*AAX(JJZ) *AAX(IHZ) * AAX(JHZ)) * (1.0D0/
     1(QF+1.0D0)+BETA*BETA/(QF+3.0D0) + GAMA*GAMA/(QF+1.0D0+(2.0D0*
     2BETA)) - 2.0D0*BETA / (QF+2.0D0) +2.0D0*GAMA /(QF+BETA+1.0D0) -
     32.0D0 *BETA*GAMA / (QF+BETA+2.0D0))
 224  CONTINUE
 223  CONTINUE
 222  CONTINUE
      PY3= (P3)/(2.0D0 *BOA)
      PY14=(HOB*HOB*P14) /(12.0D0 *BOA)
      PY12=(HOB*HOB*P12) /(24.0D0 *BOA)
 221  X(KN+IZ,KN+JZ) = (PYM10+PY11+PYM17+PYM20+PY3+PY14+PY12) *2.0D0
 220  CONTINUE
C     GENERATION OF BC MATRIX
      DO 230 IZ=1,KZ
      ZI=IZ-1
      DO 231 JZ=1,KZ
      ZJ=JZ-1
      QX = ZI+ZJ+OMEGA+THETA
      P5 = ((ZJ+OMEGA)/(QX+0.1D-50    )) +(DELTA*PHI  *(ZJ+OMEGA+1.0D0)
     1/(QX+2.0D0)) -((DELTA*(ZJ+OMEGA+1.0D0) +PHI*(ZJ+OMEGA)) /
     2(QX+1.0D0))
        PY5=-(PR*PIE*P5)
      PYM5 = PY5 * ONE
      P9 = ((ZI+THETA)/(QX+0.1D-50    )) +(DELTA*PHI  *(ZI+THETA+1.0D0)
     1/(QX+2.0D0)) -((PHI *(ZI+THETA+1.0D0)+ DELTA*(ZI+THETA)) /
     2(QX+1.0D0))
      PY9 = ((1.0D0-PR) *PIE *P9) /2.0D0
      PYM9 = PY9 * ONE
 231  X(IZ,KZ+JZ) = PYM5+PYM9
 230  CONTINUE
C     GENERATION OF BD MATRIX
      DO 240 IZ=1,KZ
      ZI=IZ-1
      DO 241 JZ=1,KZ
      ZJ=JZ-1
      P 6=0.0D0
      P22=0.0D0
      DO 242 IIZ= 1,NNN
      ZII=IIZ-1
      QM= ZI+ZJ+ZII+ALFA+THETA
      P6 = P6 +AAX(IIZ)* (1.0D0/(QM+1.0D0) - BETA/(QM+2.0D0) +GAMA/
     1(QM+BETA+1.0D0) - PHI  /(QM+2.0D0) + PHI  *BETA/(QM+3.0D0) -
     2PHI  *GAMA/(QM+BETA+2.0D0))
 242  P22 = P22+AAX(IIZ)*((ZI+THETA)*(ZJ+ALFA )/(QM+1.0D0) - BETA *
     1(ZI+THETA)*(ZJ+ALFA+1.0D0)/(QM+2.0D0)  +  GAMA*(ZI+THETA) *
     2(ZJ+ALFA+BETA)/(QM+BETA+0.1D-50+1.0D0)- PHI   * (ZI+THETA+1.0D0)
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```
     3*(ZJ+ALFA)/(QM+0.1D-50    )  +  BETA*PHI  *(ZI+THETA+1.0D0)*
     4(ZJ+ALFA+1.0D0) /(QM+1.0D0) - GAMA*PHI  *(ZI +THETA+1.0D0) *
     5(ZJ+ALFA+BETA) / (QM+BETA+0.1D-50   ))
      PY6 = (PR*PIE) * P6
      PYM6 = PY6 * ONE
      PY25 = (PIE*PIE*PIE*HOA*HOA/12.0D0) *P6
      PYM25 = PY25 * THR
      PY22 =-((PIE*(1.0D0-PR) *HOB*HOB *P22) /24.0D0)
      PYM22 = PY22 * ONE
241   X(IZ,KN+JZ) = PYM6+PYM22+PYM25
240   CONTINUE
C     GENERATION OF CD MATRIX
      DO 250 IZ=1,KZ
      ZI=IZ-1
      DO 251 JZ=1,KZ
      ZJ=JZ-1
      P 4=0.0 D0
      P16=0.0  D0
      P19=0.0D0
      P24=0.0D0
      P15=0.0 D0
      DO 252 IIZ=1,NNN
      ZII=IIZ-1
      QC= ZI+ZJ+ZII+ALFA+OMEGA
      P4=P4+AAX(IIZ)*((ZI+OMEGA)/(QC+0.1D-50    )- DELTA*(ZI+OMEGA+1.0D0)
     1 /( QC+1.0D0)-BETA*(ZI+OMEGA)/(QC+1.0D0) + BETA*DELTA*(ZI+OMEGA+
     21.0D0) / (QC+2.0D0) +(ZI    + OMEGA) * GAMA/(QC+BETA+0.1D-50     )-
     3DELTA*GAMA*(ZI+OMEGA+1.0D0)/( QC+BETA+1.0D0))
      P16 = P16+AAX(IIZ)*ZII*((ZJ+ALFA)*(ZJ+ALFA-1.0D0) *(1.0D0/
     1(QC-1.0D0+0.1D-50)-DELTA/ (QC+0.1D-50))-   BETA*(ZJ+ALFA+1.0D0)
     2*(ZJ+ALFA)*(1.0D0/(QC+0.1D-50    ) - DELTA/(QC+1.0D0))  +  GAMA *
     3(ZJ+ ALFA+BETA) *(ZJ+ALFA+BETA-1.0D0) *(1.0D0/(QC+BETA-1.0D0+
     4 0.1D-50) -DELTA /(QC+BETA +0.1D-50)))
      P19= P19 +AAX(IIZ) * ZII*(1.0D0/(QC+1.0D0) - BETA/(QC+2.0D0) +
     1GAMA/(QC+BETA+1.0D0) - DELTA/(QC+2.0D0)  + DELTA*BETA/(QC+3.0D0)
     2  - DELTA*GAMA/(QC+BETA+2.0D0))
      P24 =P24+AAX(IIZ) *((ZJ+ALFA)/(QC+0.1D-50    ) -BETA*(ZJ+ALFA+
     11.0D0)/(QC+1.0D0)+GAMA*(ZJ+ALFA+BETA)/(QC+BETA+0.1D-50    ) -
     2DELTA*(ZJ+ALFA)/(QC+1.0D0)  +  DELTA*BETA *(ZJ+ALFA+1.0D0) /
     3(QC+2.0D0) -DELTA*GAMA*(ZJ+ALFA+BETA) /(QC+BETA+1.0D0))
      DO 253 IHZ=1,NNN
      ZHI=IHZ-1
      DO 254 JHZ=1,NNN
      ZHJ=JHZ-1
      QL = ZI+ZJ+ZII+ZHI+ZHJ+OMEGA+ALFA
254   P15 = P15 +AAX(IIZ)*AAX(IHZ)*AAX(JHZ)*ZII*(1.0D0/(QL+1.0D0) -
     1BETA/(QL+2.0D0) +GAMA/(QL+BETA+1.0D0) - DELTA/(QL +2.0D0) +DELTA *
     2BETA/(QL+3.0D0) -DELTA* GAMA/(QL+BETA+2.0D0))
253   CONTINUE
252   CONTINUE
      PY4 = -(1.0D0/BOA )*P4
      PY18 = -((PR*PIE*PIE*HOA*HOB)/12.0D0) *P4
      PYM18 = PY18 * TWO
      PY16=-((HOB*HOB*P16)/(12.0D0   *BOA))
      PY19=-((PR*PIE*PIE*HOA*HOB*P19)/12.0D0 )
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```
       PYM19 = PY19 * TWO
       PY24 = (((1.0D0 - PR) *PIE*PIE *HOA*HOB *P24 ) /8.0D0 )
       PYM24 = PY24 * TWO
       PY15=(HOB*HOB*P15)/(12.0D0 *BOA )
 251   X(KZ+IZ,KN+JZ)= PY4+PY16+PYM19+PYM24+PY15+PYM18
 250   CONTINUE
C      GENERATION OF X MATRIX
       DO 401 I=1,KK
       DO 401 J=I,KK
       X(J,I) = X(I,J)
 401   CONTINUE
C      GENERATION OF Y MATRIX
C      LOAD IN X DIRECTION
       DO 968 I=1,KZ
 968   YUX(I) = 0.0D0
       IF (L1.EQ.0 ) GO TO 969
       DO 963   I=1,L1
       DO 960  IZ = 1,KZ
       ZI = IZ - 1
       IF (SPX1(I) .EQ. SPX2(I) )  GO TO 966
       PUX =   ((SPX2(I)**(ZI+THETA+1.0D0) - SPX1(I)**(ZI+THETA+1.0D0))
      1/(ZI+THETA+1.0D0))  - PHI  * ((SPX2(I)**(ZI+THETA+2.0D0) - SPX1
      2(I) **(ZI+THETA+2.0D0))/(ZI+THETA+2.0D0))
       GO TO 967
 966   PUX = (SPX2(I)**(ZI+ THETA)) - (PHI  *(SPX2(I)**(ZI+THETA+1.0D0)))
 967   IF (EPX1(I) .EQ. EPX2(I)) GO TO 950
       XX1 = DSIN(EM*PIE*EPX1(I))/EM
       XX2 = DSIN(EM*PIE*EPX2(I))/EM
       XXU = -((XX2 - XX1 )*PX(I) /(6.0D0*PIE))
       GO TO 961
 950   XX1 = DCOS(EM*PIE*EPX1(I))
       XXU = ((XX1) *PX(I) /(6.0D0      ))
 961   CONTINUE
 960   YUX (IZ) = (XXU *PUX) + YUX(IZ)
 963   CONTINUE
C      LOAD IN Y DIRECTION
 969   DO 978 I=1,KZ
 978   YUY(I) = 0.0D0
       IF (L2.EQ.0 ) GO TO 979
       DO 973   I=1,L2
       DO 970  IZ = 1,KZ
       ZI = IZ - 1
       IF (SPY1(I) .EQ. SPY2(I) )  GO TO 976
       PUY =   ((SPY2(I)**(ZI+OMEGA+1.0D0) - SPY1(I)**(ZI+OMEGA+1.0D0))
      1/(ZI+OMEGA+1.0D0))  - DELTA* ((SPY2(I)**(ZI+OMEGA+2.0D0) - SPY1
      2(I) **(ZI+OMEGA+2.0D0))/(ZI+OMEGA+2.0D0))
       GO TO 977
 976   PUY = (SPY2(I)**(ZI+ OMEGA)) - (DELTA*(SPY2(I)**(ZI+OMEGA+1.0D0)))
 977   IF (EPY1(I) .EQ. EPY2(I)) GO TO 940
       XY1 = DCOS(EM*PIE*EPY1(I))/EM
       XY2 = DCOS(EM*PIE*EPY2(I))/EM
       XYU = +((XY2 - XY1 )*PY(I) /(6.0D0*PIE))
       GO TO 941
 940   XY1 = DSIN(EM*PIE*EPY1(I))
       XYU = ((XY1) *PY(I) /(6.0D0      ))
```

```
941   CONTINUE
970   YUY (IZ) = (XYU *PUY) + YUY(IZ)
973   CONTINUE
C     LOAD IN Z DIRECTION
979   DO 988 I=1,KZ
988   YUZ(I) = 0.0D0
      IF (L3.EQ.0 ) GO TO 909
      DO 933   I=1,L3
      DO 930  IZ = 1,KZ
      ZI = IZ - 1
      IF (SPZ1(I) .EQ. SPZ2(I) )  GO TO 936
      PUZ =  ((SPZ2(I)**(ZI+ALFA +1.0D0) - SPZ1(I)**(ZI+ALFA +1.0D0))
     1/(ZI+ALFA +1.0D0))  - BETA * ((SPZ2(I)**(ZI+ALFA +2.0D0) - SPZ1
     2(I) **(ZI+ALFA +2.0D0))/(ZI+ALFA +2.0D0)) + ((SPZ2(I)**(ZI+ALFA +
     3BETA+1.0D0) - SPZ1(I)**(ZI+ALFA+BETA+1.0D0)) *GAMA  /
     4(ZI+ ALFA +BETA + 1.0D0 ))
      GO TO 937
936   PUZ = (SPZ2(I)**(ZI+ ALFA )) - (BETA *(SPZ2(I)**(ZI+ALFA+1.0D0)))
     1 + (GAMA * (SPZ2(I) ** (ZI+ALFA+ BETA)))
937   IF (EPZ1(I) .EQ. EPZ2(I)) GO TO 910
      XZ1 = DCOS(EM*PIE*EPZ1(I))/EM
      XZ2 = DCOS(EM*PIE*EPZ2(I))/EM
      XZU = +((XZ2 - XZ1 )*PZ(I) /(6.0D0*PIE))
      GO TO 911
910   XZ1 = DSIN(EM*PIE*EPZ1(I))
      XZU = ((XZ1) *PZ(I) /(6.0D0      ))
911   CONTINUE
930   YUZ (IZ) = (XZU *PUZ) + YUZ(IZ)
933   CONTINUE
C     RESULTANT LOAD MATRIX
909   DO 920 I=1,KZ
920   Y(I    ) =  (YUX(I ))
      DO 921 I=1,KZ
921   Y(I+KZ) =  (YUY(I ))
      DO 922 I=1,KZ
922   Y(I+KN) =  (YUZ(I ))
C     GENERATION OF DISP MATRIX
      CALL RRAY (2,KK,KK,90,90,A,X)
      CALL  STQN  (A,Y, KK,0)
      DO 403 I=1,KK
403   DISP(I)=Y(I)
C     GENERATION OF DISPLACEMENT CURVES
C     AT MIDDLE OF SHELL (V,W) AND AT ENDS (U)
      WRITE (6,312)
      WRITE (6,323)
      DO 500  IS=1,10
      ES =  IS
      SI = ES /10.0D0
      DO  501  I=1,KZ
      ZI=I-1
      U1        = DISP (    I) *(SI**(ZI+THETA) - PHI   *(SI**(ZI+THETA
     1+ 1.0D0 )))
      U(IS   ) = U(IS   ) + U1
501   CONTINUE
500   CONTINUE
```

```
      WRITE (6,300) (U(IS),IS=1,10)
      WRITE (6,313)
      DO 600  IS=1,10
      ES =  IS
      SI = ES /10.0D0
      DO  601  I=1,KZ
      ZI=I-1
      V1       =(DISP (KZ+I) *(SI**(ZI+OMEGA) - DELTA*(SI**(ZI+OMEGA
     1+ 1.0D0 ))) )*DSIN(EM*PIE/2.0D0)
      V(IS  ) = V(IS  ) + V1
  601 CONTINUE
  600 CONTINUE
      WRITE (6,300) (V(IS),IS=1,10)
      WRITE (6,314)
      DO 700  IS=1,10
      ES =  IS
      SI = ES /10.0D0
      DO  701  I=1,KZ
      ZI=I-1
      W1       =(DISP(KN+I) * (SI**(ZI+ALFA ) - BETA *(SI**(ZI+ALFA+
     11.0D0)) + GAMA* (SI**(ZI+ALFA+BETA))))*DSIN(EM*PIE/2.0D0)
      W(IS  ) = W(IS  ) + W1
  701 CONTINUE
  700 CONTINUE
      WRITE (6,300) (W(IS),IS=1,10)
      WRITE  (6,311)
 1000 CONTINUE
      GO TO 2000
      STOP
      END
C
C
C
C
C        SUBROUTINE  RRAY
C
C        PURPOSE:
C           CONVERT DATA ARRAY FROM SINGLE TO DOUBLE DIMENSION OR VICE
C           VERSA.  THIS SUBROUTINE IS USED TO LINK THE USER PROGRAM
C           WHICH HAS DOUBLE DIMENSION ARRAYS AND THE SSP SUBROUTINES
C           WHICH OPERATE ON ARRAYS OF DATA IN A VECTOR FASHION.
C
C        USAGE:
C           CALL  RRAY (MODE,I,J,N,M,S,D)
C
C        DESCRIPTION OF PARAMETERS:
C           MODE - CODE INDICATING TYPE OF CONVERSION
C                     1 - FROM SINGLE TO DOUBLE DIMENSION
C                     2 - FROM DOUBLE TO SINGLE DIMENSION
C           I    - NUMBER OF ROWS IN ACTUAL DATA MATRIX
C           J    - NUMBER OF COLUMNS IN ACTUAL DATA MATRIX
C           N    - NUMBER OF ROWS SPECIFIED FOR THE MATRIX D IN
C                  DIMENSION STATEMENT
C           M    - NUMBER OF COLUMNS SPECIFIED FOR THE MATRIX D IN
C                  DIMENSION STATEMENT
```

```
C              S    - IF MODE#1, THIS VECTOR CONTAINS, AS INPUT, A DATA
C                     MATRIX OF SIZE I BY J IN CONSECUTIVE LOCATIONS
C                     COLUMN-WISE.  IF MODE#2, IT CONTAINS A DATA MATRIX
C                     OF THE SAME SIZE AS OUTPUT.  THE LENGTH OF VECTOR S
C                     IS IJ, WHERE IJ#I*J.
C              D    - IF MODE#1, THIS MATRIX (N BY M) CONTAINS, AS OUTPUT,
C                     A DATA MATRIX OF SIZE I BY J IN FIRST I ROWS AND
C                     J COLUMNS.  IF MODE#2, IT CONTAINS A DATA MATRIX OF
C                     THE SAME SIZE AS INPUT.
C
C           REMARKS:
C              VECTOR S CAN BE IN THE SAME LOCATION AS MATRIX D.  VECTOR S
C              IS REFERRED AS A MATRIX IN OTHER SSP ROUTINES, SINCE IT
C              CONTAINS A DATA MATRIX.
C              THIS SUBROUTINE CONVERTS ONLY GENERAL DATA MATRICES (STORAGE
C              MODE OF 0).
C
C           SUBROUTINES AND FUNCTION SUBROUTINES REQUIRED:
C              NONE
C
      SUBROUTINE  RRAY (MODE,I,J,N,M,S,D)
      DOUBLE PRECISION S,D
      DIMENSION S(1),D(N,1)
      IF(MODE-1) 100, 100, 120
  100 DO 110 K=1,J
      DO 110 L=1,I
      KK = (K-1)*I+L
  110 D(L,K)=S(KK)
      RETURN
  120 DO 130 K=1,J
      DO 130 L=1,I
      KK = (K-1)*I+L
  130 S(KK) = D(L,K)
      RETURN
      END
C
C
C
C
C     SUBROUTINE STQN
C
C     PURPOSE :
C          OBTAIN SOLUTION OF A SET OF SIMULTANEOUS LINEAR EQUATIONS
C          AX=B
C     USAGE :
C          CALL STQN (A,B,N,KS)
C     DESCRIPTION OF PARAMETERS :
C          A - MATRIX OF COEFFICIENTS STORED COLUMNWISE . THESE ARE
C              DESTROYED IN THE COMPUTATION . THE SIZE OF MATRIX A
C              IS N BY N .
C          B - VECTOR OF ORIGINAL CONSTANTS (LENGTH N) . THESE ARE
C              REPLACED BY FINAL SOLUTION VALUES, VECTOR X .
C          N - NUMBER OF EQUATIONS AND VARIABLES . N MUST BE GREATER
C              THAN ONE .
C          KS- OUTPUT DIGIT
```

```
C                 0 FOR A NORMAL SOLUTION
C                 1 FOR A SINGULAR SET OF EQUATIONS
C
C     REMARKS :
C          MATRIX A MUST BE GENERAL.
C
C     SUBROUTINE AND FUNCTION SUBROUTINES REQUIRED:
C        NONE
C
C     METHOD:
C        STQN USES THE METHOD OF ELIMINATION AND BACK SUBSTITUTION
C        FOR INVERTING THE MATRIX
C
      SUBROUTINE STQN (A,B,N,KS)
      REAL*8 A,B,TOL,BIGA,SAVE,DABS
      DIMENSION A(1),B(1)
      TOL=0.0D0
      KS=0
      JJ=-N
      DO 65 J=1,N
      JY=J+1
      JJ=JJ+N+1
      BIGA=0.0D0
      IT=JJ-J
      DO 30 I=J,N
      IJ=IT+I
      IF(DABS(BIGA)-DABS(A(IJ))) 20,30,30
   20 BIGA=A(IJ)
      IMAX=I
   30 CONTINUE
      IF(DABS(BIGA)-TOL) 35,35,40
   35 KS=1
      RETURN
   40 I1=J+N*(J-2)
      IT=IMAX-J
      DO 50 K=J,N
      I1=I1+N
      I2=I1+IT
      SAVE=A(I1)
      A(I1)=A(I2)
      A(I2)=SAVE
   50 A(I1)=A(I1)/BIGA
      SAVE=B(IMAX)
      B(IMAX)=B(J)
      B(J)=SAVE/BIGA
      IF(J-N) 55,70,55
   55 IQS=N*(J-1)
      DO 65 IX=JY,N
      IXJ=IQS+IX
      IT=J-IX
      DO 60 JX=JY,N
      IXJX=N*(JX-1)+IX
      JJX=IXJX+IT
   60 A(IXJX)=A(IXJX)-(A(IXJ)*A(JJX))
   65 B(IX)=B(IX)-(B(J)*A(IXJ))
```

```
   70 NY=N-1
      IT=N*N
      DO 80 J=1,NY
      IA=IT-J
      IB=N-J
      IC=N
      DO 80 K=1,J
      B(IB)=B(IB)-A(IA)*B(IC)
      IA=IA-N
   80 IC=IC-1
      RETURN
      END
```

VITA 3

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