ELEMENTARY APPROACHES TO WARING'S PROBLEM

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in partial fulfillment of the requirements
for the degree of
Doctor of Education
May, 1970

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ACKNOWLEDGEMENTS

I would like to express my sincere appreciation to Dr. Jeanne Agnew for the guidance and assistance which she provided during the preparation of this thesis. I also wish to express my appreciation to the other members of my advisory committee, Dr. H. S. Mendenhall and Dr. E. K. McLachlan for serving as my committee chairmen, and Drs. Milton Berg, W. Ware Marsden, and Vernon Troxel, for their assistance.

I would like to express a special thanks to Dr. L. Wayne Johnson for the wise counsel and encouragement that he has given me.

To Dr. Emily Chandler Pixley, I express my gratitude for introducing me to number theory and Waring's Problem. I am indeed grateful for her inspired teaching and for the enthusiasm for number theory which she conveys.

Finally, I would like to express my gratitude to my wife, Ruth. Without her encouragement and assistance this task could not have been completed.

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CHAPTER I

HISTORY OF WARING'S PROBLEM

In 1770, Edward Waring [51] stated the famous conjecture which has become known as Waring's problem. In his book, Meditationes Algebraicae, under "Theorema XLVII", he states eleven propositions. The fifth and ninth of these propositions imply Waring's conjecture. They read as follows:

> 5. Omnis integer numerus est quadratus; vel e duobus, tribus vel quatuor quadratis compositus.
>
> . . . .
>
> 9. Omnis integer numerus vel est cubus, vel e duobus, tribus, 4,5,6,7,8, vel novem cubis compositus: est etiam quadrato-quadratus, vel e duobus, tribus, etc. usque ad novedecim compositus, et sic deinceps;

The translation of proposition 5 is: Every integer is a square; or the sum of two, three, or four squares. Proposition 9 states that every integer is either a cube or the sum of two, three, 4,5,6,7,8, or nine cubes; every integer is also a fourth-power or the sum of two, three, etc., up to nineteen fourth-powers, and so on.

Proposition 5 and 9 are special cases of what has come to be known as Waring's conjecture. This conjecture states that for every positive integer k there exists a smallest positive integer $g(k)$ such that any positive integer n can be expressed as the sum of at most $g(k)$ positive kth powers. Waring never actually made the general statement that today bears his name, and he never gave any proof or arguments for his assertions. It seems likely that Waring's original statement was made from the examination of a number of particular cases.

Research relating to Waring's problem has been extensive. Basically it can be classified into the attempt to prove the existence of $g(k)$ and the attempt to establish bounds and eventually an exact formula for $g(k)$. This thesis will give a brief discussion of the work dealing with existence for $g(k)$ and will concentrate on the determination of a value of $g(k)$ for $k = 2, 3, 4, 5$, and 6. This work primarily involves algebraic identities and inequalities and should give the undergraduate or advanced high school student some feeling for the elementary research methods which characterize the approach to this problem from 1772 to the present.

Although Waring made his conjecture in 1770, it was not until 1909, 139 years later, that Hilbert [22] was able to prove the existence of $g(k)$ for k in general. Hilbert's proof was based on considerations drawn from integral calculus and was quite complicated (using a 25 - fold integral in his first paper). This proof was reconsidered and simplified by Hausdorff [21], Stridsberg [48], and Remak [41]. Remak actually succeeded in eliminating all reference to the integral calculus, and his proof, although not easy, is purely algebraic. Hilbert's proof was a tremendous breakthrough, for at that time it was only known that $g(2) = 4$ and $g(3) = 9$, and there were no other proofs to indicate that Waring's conjecture was true. Unfortunately, the methods used by Hilbert were too specialized for general applications and did not contribute to any other notable results.

Ten years after Hilbert's famous proof, G. H. Hardy [19] and

J. E. Littlewood developed a new method for the solution of the Waring conjecture which has proven to be a standard technique in analytic number theory. This new method is completely independent of Hilbert's solution and is based on Cauchy's Theorem and the theory of analytic functions. It reduces the problem to a new question, namely; is a certain coefficient in the expansion of an infinite series positive and under what conditions? The Hardy and Littlewood method not only yields a proof of the existence of $g(k)$ but also gives asymptotic formulae for the number of representations of any integer n as the sum of $g(k)$ kth powers.

A third method of proof of Waring's conjecture was provided by I. M. Vinogradov [49]. His method resembles that of Hardy and Littlewood, but leads more rapidly to some of their results and provides a simpler solution of Waring's problem. Vinogradov also uses Cauchy's Theorem for the determination of the number of representations of n as the sum of $g(k)$ kth powers $(r_{k,s}(n))$, but he shows that it is simpler to work with finite exponential sums instead of with power series.

The methods of Hardy and Littlewood and Vinogradov are both analytic in nature, and in 1942, Y. V. Linnik [17] was the first to present a proof of Waring's conjecture without using such techniques. This new method is based on Schnirelmann's density and reduces Waring's problem to the proof that the sum of a sufficiently large number of sequences is a sequence of positive density. Linnik's method is strictly an existence proof and does not provide asymptotic formulae for $r_{k,s}(n)$ or an upper bound for $G(k)$ as the Hardy-Littlewood and Vinogradov methods do.

In order to illustrate the meaning of $g(k)$, Table I lists the values of $g(k)$ for $k = 2, 3, 4, 5, 6$, and Tables II, III, IV, V, VI list all the integers from 1 to 100 with a decomposition of these integers into the minimum number of squares, cubes, biquadrates, fifth-powers, and sixth-powers. The examination of the following tables will tend to give some insight into the possible size of $g(k)$.

TABLE I

VALUES OF $g(k)$ FOR SMALL POWERS

k	$g(k)$
2	4
3	9
4	19
5	37
6	73

TABLE II

DECOMPOSITION INTO SQUARES

Number	Squares Required	Number	Squares Required
$1 = 1^2$	1	$26 = 5^2 \quad 1^2$	2
$2 = 2 \cdot 1^2$	2	$27 = 5^2 + 2 \cdot 1^2$	3
$3 = 3 \cdot 1^2$	3	$28 = 5^2 + 3 \cdot 1^2$	4
$4 = 2^2$	1	$29 = 5^2 + 2^2$	2
$5 = 2^2 + 1^2$	2	$30 = 5^2 + 2^2 + 1^2$	3
$6 = 2^2 + 2 \cdot 1^2$	3	$31 = 5^2 + 2^2 + 2 \cdot 1^2$	4
$7 = 2^2 + 3 \cdot 1^2$	4	$32 = 2 \cdot 4^2$	2
$8 = 2 \cdot 2^2$	2	$33 = 2 \cdot 4^2 + 1^2$	3
$9 = 3^2$	1	$34 = 5^2 + 3^2$	2
$10 = 3^2 + 1^2$	2	$35 = 5^2 + 3^2 + 1^2$	3
$11 = 3^2 + 2 \cdot 1^2$	3	$36 = 6^2$	1
$12 = 3 \cdot 2^2$	3	$37 = 6^2 + 1^2$	2
$13 = 3^2 + 2^2$	2	$38 = 6^2 + 2 \cdot 1^2$	3
$14 = 3^2 + 2^2 + 1^2$	3	$39 = 6^2 + 3 \cdot 1^2$	4
$15 = 3^2 + 2^2 + 2 \cdot 1^2$	4	$40 = 6^2 + 2^2$	2
$16 = 4^2$	1	$41 = 5^2 + 4^2$	2
$17 = 4^2 + 1^2$	2	$42 = 5^2 + 4^2 + 1^2$	3
$18 = 2 \cdot 3^2$	2	$43 = 5^2 + 2 \cdot 3^2$	3
$19 = 2 \cdot 3^2 + 1^2$	3	$44 = 6^2 + 2 \cdot 2^2$	3
$20 = 4^2 + 2^2$	2	$45 = 6^2 + 3^2$	2
$21 = 4^2 + 2^2 + 1^2$	3	$46 = 6^2 + 3^2$	2

TABLE II (CONTINUED)

Number	Squares Required	Number	Squares Required
$22 = 4^2 + 2^2$	3	$47 = 6^2 + 3^2 + 2\cdot1^2$	4
$23 = 2\cdot3^2 + 2^2 + 1^2$	4	$48 = 3\cdot4^2$	3
$24 = 4^2 + 2\cdot2^2$	3	$49 = 7^2$	1
$25 = 5^2$	1	$50 = 7^2 + 1^2$	2
$51 = 7^2 + 2\cdot1^2$	3	$76 = 2\cdot6^2 + 2^2$	3
$52 = 6^2 + 4^2$	2	$77 = 8^2 + 3^2 + 2^2$	3
$53 = 7^2 + 2^2$	2	$78 = 7^2 + 5^2 + 2^2$	3
$54 = 7^2 + 2^2 + 1^2$	3	$79 = 7^2 + 5^2 + 2^2 + 1^2$	4
$55 = 7^2 + 2^2 + 2\cdot1^2$	4	$80 = 8^2 + 4^2$	2
$56 = 6^2 + 4^2 + 2^2$	3	$81 = 9^2$	1
$57 = 7^2 + 2\cdot2^2$	3	$82 = 9^2 + 1^2$	2
$58 = 7^2 + 3^2$	2	$83 = 9^2 + 2\cdot1^2$	3
$59 = 7^2 + 3^2 + 1^2$	3	$84 = 8^2 + 4^2 + 2^2$	3
$60 = 7^2 + 3^2 + 2\cdot1^2$	4	$85 = 7^2 + 6^2$	2
$61 = 6^2 + 5^2$	2	$86 = 7^2 + 6^2 + 1^2$	3
$62 = 6^2 + 5^2 + 1^2$	3	$87 = 7^2 + 6^2 + 2\cdot1^2$	4
$63 = 6^2 + 5^2 + 2\cdot1^2$	4	$88 = 2\cdot6^2 + 4^2$	3
$64 = 8^2$	1	$89 = 8^2 + 5^2$	2
$65 = 8^2 + 1^2$	2	$90 = 9^2 + 3^2$	2
$66 = 8^2 + 2\cdot1^2$	3	$91 = 9^2 + 3^2 + 1^2$	3
$67 = 7^2 + 2\cdot3^2$	3	$92 = 9^2 + 3^2 + 2\cdot1^2$	4
$68 = 8^2 + 2^2$	2	$93 = 8^2 + 5^2 + 2^2$	3
$69 = 8^2 + 2^2 + 1^2$	3	$94 = 9^2 + 3^2 + 2^2$	3
$70 = 6^2 + 5^2 + 3^2$	3	$95 = 9^2 + 3^2 + 2^2 + 1^2$	4
$71 = 6^2 + 5^2 + 3^2 + 1^2$	4	$96 = 8^2 + 4^2$	2
$72 = 2\cdot6^2$	2	$97 = 9^2 + 2\cdot4^2$	3
$73 = 8^2 + 3^2$	2	$98 = 2\cdot7^2$	2
$74 = 7^2 + 5^2$	2	$99 = 2\cdot7^2 + 1^2$	3
$75 = 7^2 + 5^2 + 1^2$	3	$100 = 10^2$	1

TABLE III

DECOMPOSITION INTO CUBES

Number	Cubes Required	Number	Cubes Required
$1 = 1^3$	1	$26 = 3\cdot 2^3 + 2\cdot 1^3$	5
$2 = 2\cdot 1^3$	2	$27 = 3^3$	1
$3 = 3\cdot 1$	3	$28 = 3^3 + 1^3$	2
$4 = 4\cdot 1^3$	4	$29 = 3^3 + 2\cdot 1^3$	3
$5 = 5\cdot 1^3$	5	$30 = 3^3 + 3\cdot 1^3$	4
$6 = 6\cdot 1^3$	6	$31 = 3^3 + 4\cdot 1^3$	5
$7 = 7\cdot 1^3$	7	$32 = 4\cdot 2^3$	4
$8 = 2^3$	1	$33 = 4\cdot 2^3 + 1^3$	5
$9 = 3^3 + 1^3$	2	$34 = 4\cdot 2^3 + 2\cdot 1^3$	6
$10 = 2^3 + 2\cdot 1^3$	3	$35 = 3^3 + 2^3$	2
$11 = 2^3 + 3\cdot 1^3$	4	$36 = 3^3 + 2^3 + 1^3$	3
$12 = 2^3 + 4\cdot 1^3$	5	$37 = 3^3 + 2^3 + 2\cdot 1^3$	4
$13 = 2^3 + 5\cdot 1^3$	6	$38 = 3^3 + 2^3 + 3\cdot 1^3$	5
$14 = 2^3 + 6\cdot 1^3$	7	$39 = 3^3 + 2^3 + 4\cdot 1^3$	6
$15 = 2^3 + 7\cdot 1^3$	8	$40 = 5\cdot 2^3$	5
$16 = 2\cdot 2^3$	2	$41 = 5\cdot 2^3 + 1^3$	6
$17 = 2\cdot 2^3 + 1^3$	3	$42 = 5\cdot 2^3 + 2\cdot 1^3$	7
$18 = 2\cdot 2^3 + 2\cdot 1^3$	4	$43 = 3^3 + 2\cdot 2^3$	3
$19 = 2\cdot 2^3 + 3\cdot 1^3$	5	$44 = 3^3 + 2\cdot 2^3 + 1^3$	4
$20 = 2\cdot 2^3 + 4\cdot 1^3$	6	$45 = 3^3 + 2\cdot 2^3 + 2\cdot 1^3$	5
$21 = 2\cdot 2^3 + 5\cdot 1^3$	7	$46 = 3^3 + 2\cdot 2^3 + 3\cdot 1^3$	6
$22 = 2\cdot 2^3 + 6\cdot 1^3$	8	$47 = 3^3 + 2\cdot 2^3 + 4\cdot 1^3$	7
$23 = 2\cdot 2^3 + 7\cdot 1^3$	9	$48 = 6\cdot 2^3$	6
$24 = 3\cdot 2^3$	3	$49 = 6\cdot 2^3 + 1^3$	7
$25 = 3\cdot 2^3 + 1^3$	4	$50 = 6\cdot 2^3 + 2\cdot 1^3$	8
$51 = 3^3 + 3\cdot 2^3$	4	$76 = 4^3 + 2^3 + 4\cdot 1^3$	6
$52 = 3^3 + 3\cdot 2^3 + 1^3$	5	$77 = 4^3 + 2^3 + 5\cdot 1^3$	7
$53 = 3^3 + 3\cdot 2^3 + 2\cdot 1^3$	6	$78 = 2\cdot 3^3 + 3\cdot 2^3$	5

TABLE III (CONTINUED)

Number	Cubes Required	Number	Cubes Required
$54 = 2\cdot3^3$	2	$79 = 2\cdot3^3+3\cdot2^3 + 1^3$	6
$55 = 2\cdot3^3 + 1^3$	3	$80 = 4^3 + 2\cdot3^3$	3
$56 = 2\cdot3^3 + 2\cdot1^3$	4	$81 = 3\cdot3^3$	3
$57 = 2\cdot3^3 + 3\cdot1^3$	5	$82 = 3\cdot3^3 + 1^3$	4
$58 = 2\cdot3^3 + 4\cdot1^3$	6	$83 = 3\cdot3^3 + 2\cdot1^3$	5
$59 = 3^3 + 4\cdot2^3$	5	$84 = 3\cdot3^3 + 3\cdot1^3$	6
$60 = 3^3 + 4\cdot2^3 + 1^3$	6	$85 = 3\cdot3^3 + 4\cdot1^3$	7
$61 = 3^3 + 4\cdot2^3 + 2\cdot1^3$	7	$86 = 2\cdot3^3 + 4\cdot2^3$	6
$62 = 2\cdot3^3 + 2^3$	3	$87 = 2\cdot3^3 + 4\cdot2^3 + 1^3$	7
$63 = 2\cdot3^3 + 2^3 + 1^3$	4	$88 = 4^3 + 3\cdot3^3$	4
$64 = 4^3$	1	$89 = 3\cdot3^3 + 2^3$	4
$65 = 4^3 + 1^3$	2	$90 = 3\cdot3^3 + 2^3 + 1^3$	5
$66 = 4^3 + 2\cdot1^3$	3	$91 = 4^3 + 3^3$	2
$67 = 4^3 + 3\cdot1^3$	4	$92 = 4^3 + 3^3 + 1^3$	3
$68 = 4^3 + 4\cdot1^3$	5	$93 = 4^3 + 3^3 + 2\cdot1^3$	4
$69 = 4^3 + 5\cdot1^3$	6	$94 = 4^3 + 3^3 + 3\cdot1^3$	5
$70 = 2\cdot3^3 + 2\cdot2^3$	4	$95 = 4^3 + 3^3 + 4\cdot1^3$	6
$71 = 2\cdot3^3 + 2\cdot2^3 + 1^3$	5	$96 = 4^3 + 4\cdot3^3$	5
$72 = 4^3 + 2^3$	2	$97 = 3\cdot3^2 + 2\cdot2^3$	5
$73 = 4^3 + 2^3 + 1^3$	3	$98 = 3\cdot3^3 + 2\cdot2^3 + 1^3$	6
$74 = 4^3 + 2^3 + 2\cdot1^3$	4	$99 = 4^3 + 3^3 + 2^3$	3
$75 = 4^3 + 2^3 + 3\cdot1^3$	5	$100 = 4^3 + 3^3 + 2^3 + 1^3$	4

TABLE IV

DECOMPOSITION INTO BIQUADRATES

Number	Biquadrates Required	Number	Biquadrates Requried
$1 = 1^4$	1	$26 = 2^4 + 10\cdot 1^4$	11
$2 = 2\cdot 1^4$	2	$27 = 2^4 + 11\cdot 1^4$	12
$3 = 3\cdot 1^4$	3	$28 = 2^4 + 12\cdot 1^4$	13
$4 = 4\cdot 1^4$	4	$29 = 2^4 + 13\cdot 1^4$	14
$5 = 5\cdot 1^4$	5	$30 = 2^4 + 14\cdot 1^4$	15
$6 = 6\cdot 1^4$	6	$31 = 2^4 + 15\cdot 1^4$	16
$7 = 7\cdot 1^4$	7	$32 = 2\cdot 2^4$	2
$8 = 8\cdot 1^4$	8	$33 = 2\cdot 2^4 + 1^4$	3
$9 = 9\cdot 1^4$	9	$34 = 2\cdot 2^4 + 2\cdot 1^4$	4
$10 = 10\cdot 1^4$	10	$35 = 2\cdot 2^4 + 3\cdot 1^4$	5
$11 = 11\cdot 1^4$	11	$36 = 2\cdot 2^4 + 4\cdot 1^4$	6
$12 = 12\cdot 1^4$	12	$37 = 2\cdot 2^4 + 5\cdot 1^4$	7
$13 = 13\cdot 1^4$	13	$38 = 2\cdot 2^4 + 6\cdot 1^4$	8
$14 = 14\cdot 1^4$	14	$39 = 2\cdot 2^4 + 7\cdot 1^4$	9
$15 = 15\cdot 1^4$	15	$40 = 2\cdot 2^4 + 8\cdot 1^4$	10
$16 = 2^4$	1	$41 = 2\cdot 2^4 + 9\cdot 1^4$	11
$17 = 2^4 + 1^4$	2	$42 = 2\cdot 2^4 + 10\cdot 1^4$	12
$18 = 2^4 + 2\cdot 1^4$	3	$43 = 2\cdot 2^4 + 11\cdot 1^4$	13
$19 = 2^4 + 3\cdot 1^4$	4	$44 = 2\cdot 2^4 + 12\cdot 1^4$	14
$20 = 2^4 + 4\cdot 1^4$	5	$45 = 2\cdot 2^4 + 13\cdot 1^4$	15
$21 = 2^4 + 5\cdot 1^4$	6	$46 = 2\cdot 2^4 + 14\cdot 1^4$	16
$22 = 2^4 + 6\cdot 1^4$	7	$47 = 2\cdot 2^4 + 15\cdot 1^4$	17
$23 = 2^4 + 7\cdot 1^4$	8	$48 = 3\cdot 2^4$	3
$24 = 2^4 + 8\cdot 1^4$	9	$49 = 3\cdot 2^4 + 1^4$	4
$25 = 2^4 + 9\cdot 1^4$	10	$50 = 3\cdot 2^4 + 2\cdot 1^4$	5
$51 = 3\cdot 2^4 + 3\cdot 1^4$	6	$76 = 4\cdot 2^4 + 12\cdot 1^4$	16

TABLE IV (CONTINUED)

Number	Biquadrates Required	Number	Biquadrates Required
$52 = 3\cdot 2^4 + 4\cdot 1^4$	7	$77 = 4\cdot 2^4 + 13\cdot 1^4$	17
$53 = 3\cdot 2^4 + 5\cdot 1^4$	8	$78 = 4\cdot 2^4 + 14\cdot 1^4$	18
$54 = 3\cdot 2^4 + 6\cdot 1^4$	9	$79 = 4\cdot 2^4 + 15\cdot 1^4$	19
$55 = 3\cdot 2^4 + 7\cdot 1^4$	10	$80 = 5\cdot 2^4$	5
$56 = 3\cdot 2^4 + 8\cdot 1^4$	11	$81 = 3^4$	1
$57 = 3\cdot 2^4 + 9\cdot 1^4$	12	$82 = 3^4 + 1^4$	2
$58 = 3\cdot 2^4 + 10\cdot 1^4$	13	$83 = 3^4 + 2\cdot 1^4$	3
$59 = 3\cdot 2^4 + 11\cdot 1^4$	14	$84 = 3^4 + 3\cdot 1^4$	4
$60 = 3\cdot 2^4 + 12\cdot 1^4$	15	$85 = 3^4 + 4\cdot 1^4$	5
$61 = 3\cdot 2^4 + 13\cdot 1^4$	16	$86 = 3^4 + 5\cdot 1^4$	6
$62 = 3\cdot 2^4 + 14\cdot 1^4$	17	$87 = 3^4 + 6\cdot 1^4$	7
$63 = 3\cdot 2^4 + 15\cdot 1^4$	18	$88 = 3^4 + 7\cdot 1^4$	8
$64 = 4\cdot 2^4$	4	$89 = 3^4 + 8\cdot 1^4$	9
$65 = 4\cdot 2^4 + 1\cdot 1^4$	5	$90 = 3^4 + 9\cdot 1^4$	10
$66 = 4\cdot 2^4 + 2\cdot 1^4$	6	$91 = 3^4 + 10\cdot 1^4$	11
$67 = 4\cdot 2^4 + 3\cdot 1^4$	7	$92 = 3^4 + 11\cdot 1^4$	12
$68 = 4\cdot 2^4 + 4\cdot 1^4$	8	$93 = 3^4 + 12\cdot 1^4$	13
$69 = 4\cdot 2^4 + 5\cdot 1^4$	9	$94 = 3^4 + 13\cdot 1^4$	14
$70 = 4\cdot 2^4 + 6\cdot 1^4$	10	$95 = 3^4 + 14\cdot 1^4$	15
$71 = 4\cdot 2^4 + 7\cdot 1^4$	11	$96 = 6\cdot 2^4$	6
$72 = 4\cdot 2^4 + 8\cdot 1^4$	12	$97 = 6\cdot 2^4 + 1^4$	7
$73 = 4\cdot 2^4 + 9\cdot 1^4$	13	$98 = 6\cdot 2^4 + 2\cdot 1^4$	8
$74 = 4\cdot 2^4 + 10\cdot 1^4$	14	$99 = 6\cdot 2^4 + 3\cdot 1^4$	9
$75 = 4\cdot 2^4 + 11\cdot 1^4$	15	$100 = 3^4 + 2^4 + 3\cdot 1^4$	5

TABLE V

DECOMPOSITION INTO FIFTH-POWERS

Number	Fifth-Powers Required	Number	Fifth-Powers Required
$1 = 1^5$	1	$26 = 26\cdot 1^5$	26
$2 = 2\cdot 1^5$	2	$27 = 27\cdot 1^5$	27
$3 = 3\cdot 1^5$	3	$28 = 28\cdot 1^5$	28
$4 = 4\cdot 1^5$	4	$29 = 29\cdot 1^5$	29
$5 = 5\cdot 1^5$	5	$30 = 30\cdot 1^5$	30
$6 = 6\cdot 1^5$	6	$31 = 31\cdot 1^5$	31
$7 = 7\cdot 1^5$	7	$32 = 2^5$	1
$8 = 8\cdot 1^5$	8	$33 = 2^5 + 1^5$	2
$9 = 9\cdot 1^5$	9	$34 = 2^5 + 2\cdot 1^5$	3
$10 = 10\cdot 1^5$	10	$35 = 2^5 + 3\cdot 1^5$	4
$11 = 11\cdot 1^5$	11	$36 = 2^5 + 4\cdot 1^5$	5
$12 = 12\cdot 1^5$	12	$37 = 2^5 + 5\cdot 1^5$	6
$13 = 13\cdot 1^5$	13	$38 = 2^5 + 6\cdot 1^5$	7
$14 = 14\cdot 1^5$	14	$39 = 2^5 + 7\cdot 1^5$	8
$15 = 15\cdot 1^5$	15	$40 = 2^5 + 8\cdot 1^5$	9
$16 = 16\cdot 1^5$	16	$41 = 2^5 + 9\cdot 1^5$	10
$17 = 17\cdot 1^5$	17	$42 = 2^5 + 10\cdot 1^5$	11
$18 = 18\cdot 1^5$	18	$43 = 2^5 + 11\cdot 1^5$	12
$19 = 19\cdot 1^5$	19	$44 = 2^5 + 12\cdot 1^5$	13
$20 = 20\cdot 1^5$	20	$45 = 2^5 + 13\cdot 1^5$	14
$21 = 21\cdot 1^5$	21	$46 = 2^5 + 14\cdot 1^5$	15
$22 = 22\cdot 1^5$	22	$47 = 2^5 + 15\cdot 1^5$	16
$23 = 23\cdot 1^5$	23	$48 = 2^5 + 16\cdot 1^5$	17
$24 = 24\cdot 1^5$	24	$49 = 2^5 + 17\cdot 1^5$	18
$25 = 25\cdot 1^5$	25	$50 = 2^5 + 17\cdot 1^5$	19
$51 = 2^5 + 19\cdot 1^5$	20	$76 = 2\cdot 2^5 + 12\cdot 1^5$	14
$52 = 2^5 + 20\cdot 1^5$	21	$77 = 2\cdot 2^5 + 13\cdot 1^5$	15

TABLE V (CONTINUED)

Number	Fifth-Powers Required	Number	Fifth Powers Required
$53 = 2^5 + 21\cdot1^5$	22	$78 = 2\cdot2^5 + 14\cdot1^5$	16
$54 = 2^5 + 22\cdot1^5$	23	$79 = 2\cdot2^5 + 15\cdot1^5$	17
$55 = 2^5 + 23\cdot1^5$	24	$80 = 2\cdot2^5 + 16\cdot1^5$	18
$56 = 2^5 + 24\cdot1^5$	25	$81 = 2\cdot2^5 + 17\cdot1^5$	19
$57 = 2^5 + 25\cdot1^5$	26	$82 = 2\cdot2^5 + 18\cdot1^5$	20
$58 = 2^5 + 26\cdot1^5$	27	$83 = 2.2^5 + 19\cdot1^5$	21
$59 = 2^5 + 27\cdot1^5$	28	$84 = 2\cdot2^5 + 20\cdot1^5$	22
$60 = 2^5 + 28\cdot1^5$	29	$85 = 2\cdot2^5 + 21\cdot1^5$	23
$61 = 2^5 + 29\cdot1^5$	30	$86 = 2\cdot2^5 + 22\cdot1^5$	24
$62 = 2^5 + 30\cdot1^5$	31	$87 = 2\cdot2^5 + 23\cdot1^5$	25
$63 = 2^5 + 31\cdot1^5$	32	$88 = 2\cdot2^5 + 24\cdot1^5$	26
$64 = 2\cdot2^5$	2	$89 = 2\cdot2^5 + 25\cdot1^5$	27
$65 = 2\cdot2^5 + 1\cdot1^5$	3	$90 = 2\cdot2^5 + 26\cdot1^5$	28
$66 = 2\cdot2^5 + 2\cdot1^5$	4	$91 = 2\cdot2^5 + 27\cdot1^5$	29
$67 = 2\cdot2^5 + 3\cdot1^5$	5	$92 = 2\cdot2^5 + 28\cdot1^5$	30
$68 = 2\cdot2^5 + 4\cdot1^5$	6	$93 = 2\cdot2^5 + 29\cdot1^5$	31
$69 = 2\cdot2^5 + 5\cdot1^5$	7	$94 = 2\cdot2^5 + 30\cdot1^5$	32
$70 = 2\cdot2^5 + 6\cdot1^5$	8	$95 = 2\cdot2^5 + 31\cdot1^5$	33
$71 = 2\cdot2^5 + 7\cdot1^5$	9	$96 = 3\cdot2^5$	3
$72 = 2\cdot2^5 + 8\cdot1^5$	10	$97 = 3\cdot2^5 + 1^5$	4
$73 = 2\cdot2^5 + 9\cdot1^5$	11	$98 = 3\cdot2^5 + 2\cdot1^5$	5
$74 = 2\cdot2^5 + 10\cdot1^5$	12	$99 = 3\cdot2^5 + 3\cdot1^5$	6
$75 = 2\cdot2^5 + 11\cdot1^5$	13	$100 = 3\cdot2^5 + 4\cdot1^5$	7

TABLE VI

DECOMPOSITION INTO SIXTH-POWERS

Number	Sixth-Powers Required	Number	Sixth-Powers Required
$1 = 1^6$	1	$26 = 26\cdot 1^6$	26
$2 = 2\cdot 1^6$	2	$27 = 27\cdot 1^6$	27
$3 = 3\cdot 1^6$	3	$28 = 28\cdot 1^6$	28
$4 = 4\cdot 1^6$	4	$29 = 29\cdot 1^6$	29
$5 = 5\cdot 1^6$	5	$30 = 30\cdot 1^6$	30
$6 = 6\cdot 1^6$	6	$31 = 31\cdot 1^6$	31
$7 = 7\cdot 1^6$	7	$32 = 32.1^6$	32
$8 = 8\cdot 1^6$	8	$33 = 33\cdot 1^6$	33
$9 = 9\cdot 1^6$	9	$34 = 34\cdot 1^6$	34
$10 = 10\cdot 1^6$	10	$35 = 35\cdot 1^6$	35
$11 = 11\cdot 1^6$	11	$36 = 36\cdot 1^6$	36
$12 = 12\cdot 1^6$	12	$37 = 37\cdot 1^6$	37
$13 = 13\cdot 1^6$	13	$38 = 38\cdot 1^6$	38
$14 = 14\cdot 1^6$	14	$39 = 39\cdot 1^6$	39
$15 = 15\cdot 1^6$	15	$40 = 40\cdot 1^6$	40
$16 = 16\cdot 1^6$	16	$41 = 41\cdot 1^6$	41
$17 = 17.1^6$	17	$42 = 42\cdot 1^6$	42
$18 = 18\cdot 1^6$	18	$43 = 43\cdot 1^6$	43
$19 = 19\cdot 1^6$	19	$44 = 44\cdot 1^6$	44
$20 = 20\cdot 1^6$	20	$45 = 45\cdot 1^6$	45
$21 = 21\cdot 1^6$	21	$46 = 46\cdot 1^6$	46
$22 = 22\cdot 1^6$	22	$47 = 47\cdot 1^6$	47
$23 = 23\cdot 1^6$	23	$48 = 48\cdot 1^6$	48
$24 = 24\cdot 1^6$	24	$49 = 49\cdot 1^6$	49
$25 = 25\cdot 1^6$	25	$50 = 50\cdot 1^6$	50

TABLE VI (CONTINUED)

Number	Sixth-Powers Required	Number	Sixth-Powers Required
$51 = 51 \cdot 1^6$	51	$76 = 2^6 + 12 \cdot 1^6$	13
$52 = 52 \cdot 1^6$	52	$77 = 2^6 + 13 \cdot 1^6$	14
$53 = 53 \cdot 1^6$	53	$78 = 2^6 + 14 \cdot 1^6$	15
$54 = 54 \cdot 1^6$	54	$79 = 2^6 + 15 \cdot 1^6$	16
$55 = 55 \cdot 1^6$	55	$80 = 2^6 + 16 \cdot 1^6$	17
$56 = 56 \cdot 1^6$	56	$81 = 2^6 + 17 \cdot 1^6$	18
$57 = 57 \cdot 1^6$	57	$82 = 2^6 + 18 \cdot 1^6$	19
$58 = 58 \cdot 1^6$	58	$83 = 2^6 + 19 \cdot 1^6$	20
$59 = 59 \cdot 1^6$	59	$84 = 2^6 + 20 \cdot 1^6$	21
$60 = 60 \cdot 1^6$	60	$85 = 2^6 + 21 \cdot 1^6$	22
$61 = 61 \cdot 1^6$	61	$86 = 2^6 + 22 \cdot 1^6$	23
$62 = 62 \cdot 2^6$	62	$87 = 2^6 + 23 \cdot 1^6$	24
$63 = 63 \cdot 2^6$	63	$88 = 2^6 + 24 \cdot 1^6$	25
$64 = 2^6$	1	$89 = 2^6 + 25 \cdot 1^6$	26
$65 = 2^6 + 1^6$	2	$90 = 2^6 + 26 \cdot 1^6$	27
$66 = 2^6 + 2 \cdot 1^6$	3	$91 = 2^6 + 27 \cdot 1^6$	28
$67 = 2^6 + 3 \cdot 1^6$	4	$92 = 2^6 + 28 \cdot 1^6$	29
$68 = 2^6 + 4 \cdot 1^6$	5	$93 = 2^6 + 29 \cdot 1^6$	30
$69 = 2^6 + 5 : 1^6$	6	$94 = 2^6 + 30 \cdot 1^6$	31
$70 = 2^6 + 6 \cdot 1^6$	7	$95 = 2^6 + 31 \cdot 1^6$	32
$71 = 2^6 + 7 \cdot 1^6$	8	$96 = 2^6 + 32 \cdot 1^6$	33
$72 = 2^6 + 8 \cdot 1^6$	9	$97 = 2^6 + 33 \cdot 1^6$	34
$73 = 2^6 + 9 \cdot 1^6$	10	$98 = 2^6 + 34 \cdot 1^6$	35
$74 = 2^6 + 10 \cdot 1^6$	11	$99 = 2^6 + 35 \cdot 1^6$	36
$75 = 2^6 + 11 \cdot 1^6$	12	$100 = 2^6 + 36 \cdot 1^6$	37

In Table II, it can be seen that many of the integers from 1 to 100 require at least four squares for their decomposition, and it should also be noted that no integer in the table requires more than four squares. The integer

$$7 = 2^2 + 1^2 + 1^2 + 1^2$$

for example, cannot be expressed as the sum of less than four squares. Thus Table II implies that $g(2) \geq 4$ and indicates that there is a strong possibility that $g(2) = 4$. A proof that $g(2) = 4$ will be contained in Chapter II.

The integers that require four squares can be proved to be exclusively those of the form $4^k(8n + 7)$. A proof of this theorem will also be included in Chapter II.

The integers from 1 to 100 can be expressed as the sum of at most nine cubes, as seen in Table III. The integer

$$23 = 2 \cdot 2^3 + 7 \cdot 1^3$$

cannot be expressed as the sum of less than nine cubes, thus Table III implies that $g(3) \geq 9$. It is important to note that 23 is the only integer in the table that requires nine cubes and that there are only three integers, (15, 22, 50), that require eight cubes. Dickson [13] proved that every integer except 23 and 239 can be expressed as the sum of eight or less cubes, and it seems that there are only 15 integers (15, 22, 50, 114, 167, 175, 186, 212, 231, 238, 303, 364, 420, 428, and 454), that require eight cubes.

Table IV illustrates that part of Waring's problem which is considered by many number theorists to be the most interesting as well as the most difficult. It can be seen that

$$79 = 4 \cdot 2^4 + 15 \cdot 1^4$$

cannot be expressed by less than nineteen fourth powers. Thus $g(4) \geq 19$ It should be noticed that 79 is the only value in the table that requires 19 fourth powers, and 63 and 78 are the only integers requiring 18 fourth powers.

In Table V one can see that no integer from 1 to 100 requires 37 fifth powers. However, if we examine the table closely, we can see a pattern that will give us a number requiring 37 fifth powers.

$31 = 2^5 - 1 =$	$31 \cdot 1^5$	requires	31	fifth	powers
$63 = 2 \cdot 2^5 - 1 =$	$2^5 + 31 \cdot 1^5$	"	32	"	"
$95 = 3 \cdot 2^5 - 1 =$	$2 \cdot 2^5 + 31 \cdot 1^5$	"	33	"	"
$127 = 4 \cdot 2^5 - 1 =$	$3 \cdot 2^5 + 31 \cdot 1^5$	"	34	"	"
$159 = 5 \cdot 2^5 - 1 =$	$4 \cdot 2^5 + 31 \cdot 1^5$	"	35	"	"
$191 = 6 \cdot 2^5 - 1 =$	$5 \cdot 2^5 + 31 \cdot 1^5$	"	36	"	"
$223 = 7 \cdot 2^5 - 1 =$	$6 \cdot 2^5 + 31 \cdot 1^5$	"	37	"	"
$255 = 8 \cdot 2^5 - 1 =$	$3^5 + 12 \cdot 1^5$	"	13	"	"

Now, $3^5 = 243$, and $7 \cdot 2^5 = 224$, so it is apparent that these integers cannot be used in the decomposition of 223 into a minimum number of fifth powers. Thus, 223 requires 37 fifth powers, and this implies that $g(5) \geq 37$. The next number in the pattern established above, is 255, and this integer can be represented by $3^5 + 12 \cdot 1^5$. Hence, 225 is the sum of 13 rather than 38 fifth powers, as one might suspect.

No integer in Table VI requires 73 sixth powers, but it can be seen that $63 = 2^6 - 1$ does require 63 sixth powers. Following a pattern similar to that used for fifth powers, we may locate an integer that requires 73 sixth powers.

$63 = 2^6 - 1 = 63 \cdot 1^6$	requires	63	sixth	powers
$127 = 2 \cdot 2^6 - 1 = 2^6 + 63 \cdot 1^6$	"	64	"	"
$191 = 3 \cdot 2^6 - 1 = 2 \cdot 2^6 + 63 \cdot 1^6$	"	65	"	"
$255 = 4 \cdot 2^6 - 1 = 3 \cdot 2^6 + 63 \cdot 1^6$	"	66	"	"
$319 = 5 \cdot 2^6 - 1 = 4 \cdot 2^6 + 63 \cdot 1^6$	"	67	"	"
$383 = 6 \cdot 2^6 - 1 = 5 \cdot 2^6 + 63 \cdot 1^6$	"	68	"	"
$447 = 7 \cdot 2^6 - 1 = 6 \cdot 2^6 + 63 \cdot 1^6$	"	69	"	"
$511 = 8 \cdot 2^6 - 1 = 7 \cdot 2^6 + 63 \cdot 1^6$	"	70	"	"
$575 = 9 \cdot 2^6 - 1 = 8 \cdot 2^6 + 63 \cdot 1^6$	"	71	"	"
$639 = 10 \cdot 2^6 - 1 = 9 \cdot 2^6 + 63 \cdot 1^6$	"	72	"	"
$703 = 11 \cdot 2^6 - 1 = 10 \cdot 2^6 + 63 \cdot 1^6$	"	73	"	"
$767 = 12 \cdot 2^6 - 1 = 3^6 + 38 \cdot 1^6$	"	39	"	"

Since $3^6 = 729$ and $11 \cdot 2^6 = 704$, these integers can not be used to represent 703 as a sum of sixth powers. Hence, $10 \cdot 2^6 + 63 \cdot 1^6$ is the decomposition of 703 into the minimum number of sixth powers, since $63 \cdot 1^6$ cannot be replaced by 3^6. Therefore an integer, which requires 73 sixth powers, has been found, and this implies that $g(6) \geq 73$. It might be expected that the next integer, 767, in the pattern established above, is the sum of 74 sixth powers, but $767 = 3^6 + 38 \cdot 1^6$, and is the sum of only 39 sixth powers.

From Tables II, III, IV, V, VI, we proved that $g(2) \geq 4$, $g(3) \geq 9$, $g(4) \geq 19$, $g(5) \geq 37$, $g(6) \geq 73$, but the method that we used to get these values becomes more difficult as k increases so that other methods must be used to obtain more information concerning $g(k)$.

A rather simple lower bound for the value of $g(k)$ was determined early in the study of this problem. This result and some of the work preceding it is presented below.

J. A. Euler, stated in 1772 that in order to express every integer as a sum of kth powers, at least $I(k) = 2^k + [(\frac{3}{2})^k] - 2$ terms are necessary. In other words

$$g(k) \geq 2^k + [(\frac{3}{2})^k] - 2 .$$

<u>Theorem 1.1.</u> $g(k) \geq I(k) = 2^k + [(\frac{3}{2})^k] - 2, \ k = 1, 2, \ldots$

Proof: Let

$$n = 2^k \ [(\frac{3}{2})^k] - 1 \tag{1.1}$$

n is a natural number, and since $[x] < x$, we have

$$n < 2^k \ (\frac{3}{2})^k - 1 = 3^k - 1 .$$

Thus

$$n < 3^k . \tag{1.2}$$

By definition of $g(k)$

$$n = x_1^k + x_2^k + \ldots + x_{g(k)}^k \tag{1.3}$$

where x_i $[i = 1, 2, \ldots, g(k)]$ are non-negative integers. From (1.2) and (1.3)

$$x_1^k + x_2^k + \ldots + x_{g(k)}^k < 3^k .$$

Therefore each $x_i [i = 1, 2, \ldots, g(k)]$ must be less than 3. Then each x_i can take only the values $0, 1,$ and 2. Suppose that there are a different integers among the x_i's equal to 2, b diffierent integers equal to 1, and c different integers equal to 0. Clearly, a, b, and c are non-negative integers, and

(1.4) $$g(k) = a + b + c \geq a + b .$$

Since

$$\underbrace{2^k + 2^k + \ldots + 2^k}_{a \text{ times}} = 2^k a \quad \text{and} \quad \underbrace{1^k + 1^k + \ldots + 1^k}_{b \text{ times}} = b .$$

from (1.3)

(1.5) $$n = 2^k a + b .$$

Then $$n \geq 2^k a$$

and from (1.1)

$$n < 2^k [(\tfrac{3}{2})^k] .$$

This gives

$$2^k a \leq n < 2^k [(\tfrac{3}{2})^k]$$

and

$$a < [(\tfrac{3}{2})^k] .$$

This imples

(1.6) $$a \leq [(\tfrac{3}{2})^k] - 1 .$$

From (1.5)

$$b = n - 2^k a$$

and

(1.7) $$a + b = a + n - 2^k a = n - (2^k - 1)a .$$

Since k is a natural number, $2^k - 1$ is also a natural number. Multiply (1.6) by $2^k - 1$ and obtain

$$(2^k - 1)a \leq (2^k - 1)([(\tfrac{3}{2})^k] - 1) .$$

Then by (1.4), (1.7), (1.6), and (1.1)

$$\begin{aligned} g(k) \geq a + b = n - (2^k - 1)a &\geq n - (2^k - 1)([(\tfrac{3}{2})^k]-1) \\ &= n - (2^k [(\tfrac{3}{2})^k] - 1) + 2^k + [(\tfrac{3}{2})^k] - 2 \\ &= 2^k + [(\tfrac{3}{2})^k] - 2 \end{aligned}$$

Therefore

$$g(k) \geq 2^k + [(\tfrac{3}{2})^k] - 2 .$$

In this formula,

$$I(2) = 2^2 + [\tfrac{9}{4}] - 2 = 4 + 2 - 2 = 4$$

$$I(3) = 2^3 + [\tfrac{27}{8}] - 2 = 8 + 3 - 2 = 9$$

$$I(4) = 2^4 + [\tfrac{81}{16}] - 2 = 16 + 5 - 2 = 19$$

$$I(5) = 2^5 + [\tfrac{243}{32}] - 2 = 32 + 7 - 2 = 37$$

$$I(6) = 2^6 + [\tfrac{729}{64}] - 2 = 64 + 11 - 2 = 73$$

Thus $g(2) \geq 4$, $g(3) \geq 9$, $g(4) \geq 19$, $g(5) \geq 37$, and $g(6) \geq 73$. These lowers bounds for $g(k)$ are exactly those we derived previously. However, a lower bound for $g(k)$ may now be determined for any value of k. This finding of a lower bound for $g(k)$ was relatively easy, but it is much more complicated to find the exact value of $g(k)$.

Waring was not the first to state that every integer is the sum of four squares. This was known as Bachet's theorem, called after C. J. Bachet, who remarked in 1621 that any number is either a square or the sum of 2, 3, or 4 squares. Girard and Fermat also stated this theorem remarking that Diaphantus seemed to have known it. As usual, Fermat stated that he possessed a proof of this theorem, but he never published it. Léonard Euler made many unsuccessful attempts to prove this result, and the first published proof was given in 1770 by Lagrange who acknowledged his great dependence on results obtained by Euler.

In 1770, Lagrange was able to prove that $g(2) = 4$, but it was a more formidable task to prove that $g(3) = 9$. The first attempts at this proof were in the form of tables of the smallest number of positive cubes into which whole numbers can be decomposed (similar to Table III). At the suggestion of Jacobi, Zornow constructed such a table in 1835 for each integer $\leq 3{,}000$. Dashe extended this table to 12,000 in 1851, and in 1903, Von Sterneck [50] continued this table to 40,000. These tables verified that all integers up to 40,000 can be represented by at most 8 cubes (except 23 and 239, which require 9 cubes). All numbers between 454 and 40,000 require at most 7 cubes, and all numbers between 8042 and 40,000 require at most 6 cubes. From these tables it was presumed that every integer greater than 8042 is the sum of at most 6 cubes.

Maillet [31] was the first to find an upper bound for $g(3)$, when he showed in 1895 that $g(3) \leq 21$. By a variation of Maillet's proof, Fleck [15] was able to prove that $g(3) \leq 13$ in 1906. The big breakthrough came in 1909 when Wieferich [54] finally proved $g(3) = 9$. However, due to an oversight, there was a gap in the proof that was finally filled by Kempner [24] in 1912. In referring to Wieferich's proof, Landau said that it was one of the most satisfying advances in number theory.

Proofs of $g(3) \leq 21$, $g(3) \leq 17$ and $g(3) \leq 13$ will be included in Chapter III, and a proof that $g(3) = 9$ will also be given.

It was exceedingly difficult to establish that $g(4) = 19$. The ablest mathematicians of the nineteenth and twentieth centuries have attempted to determine $g(4)$, but without success. When Liouville proved that $g(4) \le 53$ in 1859, this was the first actual proof associated with $g(k)$ except for Lagrange's famous proof. This upper bound was reduced to 47, 45, 41, 39, 38 and finally to 37 by Wieferich [55] in 1909. In his doctoral dissertation, Baer [3] gave a simpler proof that $g(4) \le 37$, and it was not until 1933 that this value could be improved upon. Emily Chandler [5] succeeded in proving $g(4) \le 35$ in her dissertation in 1933, and this result is still the best available today. There has never been a counterexample to disprove that $g(4) = 19$, and it is surprising that in the 36 years since Chandler's proof, her upper bound has not been improved upon.

Liouville's important result of $g(4) \le 53$ will be proved in Chapter IV, and by very interesting methods it will also be shown that $g(4)$ is less than 45, 41, 39, 38, and finally 37. These proofs are elegent in their simplicity and lead one to the illusion that still lower bounds may be easily found. It would be worthwhile for anyone who has taken, or is taking number theory, to go through these proofs.

The value of $g(5)$ has also not been determined. Maillett [32] was the first to find an upper bound for $g(5)$ when he proved that $g(5) \le 192$ in 1895. Fleck reduced this bound by about 36, Wieferich [53] proved $g(5) \le 59$, Baer [3] proved $g(5) \le 58$, and Dickson [9] finally proved $g(5) \le 54$ in 1933. This was the best upper bound for $g(5)$ for 26 years until Chen [6] supposedly proved $g(5) \le 40$ in 1959. Thus $37 \le g(5) \le 40$, and $g(5)$ is almost determined. There is no obvious reason why fourth and fifth powers should present so much more difficulty than all of the other powers, but they are still the only powers for which $g(k)$ is not determined.

Sixth powers seem to have a special significance in Waring's problem, for in 1936, Pillai [39] and Dickson [11] independently determined $g(n)$ for $n > 6$, subject to certain restrictions. However, the proof that $g(6) = 73$ was more difficult. In 1907 Fleck [16] established the first upper bound for $g(6)$ when he proved $g(6) \leq 184 \cdot g(3) + 59$. Since $g(3)$ was only known to be ≤ 13, this bound was 2451, which is a long way from the ideal of 73, but using Wieferich's proof that $g(3) = 9$ the bound would be 1715. By an interesting method, Kempner [25] was able to prove that $g(6) \leq 970$ in 1912, and this upper bound was lowered to 478, 183, 160, 115, 110, and 104 until in 1940, Pillai [40] finally proved that $g(6) = 73$.

The proofs of theorems relating to the determination of $g(5)$ and $g(6)$ tend to be quite lengthy and analytic in nature. For this reason the proof that $g(5) \leq 59$ is not given in detail. On the basis of this result, it is proven that $g(6) \leq 184 \cdot g(3) + 59$, and $g(6) \leq 970$, thus establishing a upper bounds for $g(5)$ and $g(6)$.

It has been proven in Theorem 1.1 that

$$g(k) \geq I(k) = 2^k + [(\tfrac{3}{2})^k] - 2 .$$

The conjecture that $g(k) = I(k)$ is referred to as the Ideal Waring Theorem. The history of the attempts to prove this theorem and their success is outlined in Chapter VI.

It is apparent that only the smaller integers of

$$n = X_1^k + X_2^k + \ldots + X_{g(k)}^k$$

require $g(k)$ terms. For example, for $k = 3$, $n = 23$ and $n = 239$ are the only integers which require 9 cubes [13]. This has lead to the definition of $G(k)$, where $G(k)$ is the number of kth powers required to represent every sufficiently large number. Theorems concerning $G(k)$ are more difficult and analytic than those dealt with in this thesis. The determination of $G(k)$ will not be discussed in this report.

CHAPTER II

SQUARES

Lagrange's proof in 1770, that every integer is the sum of four squares, was rather involved. Three years later, L. Euler gave a much simpler proof. A modification of Euler's proof has become a standard theorem in most textbooks on number theory, and this proof will now be given.

An integral part of the following proof is the identity

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2)$$
$$= (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)^2$$
$$+ (x_1y_3 - x_3y_1 + x_4y_2 - x_2y_4)^2 + (x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)^2 \quad .$$

This identity is known as Euler's identity and can be verified by multiplying out both sides of the equation. From this identity it is apparent that the product of two numbers that are the sum of four squares is also the sum of four squares. If it could be shown that every odd prime is the sum of at most four squares, then all primes would be the sum of at most four squares, since $2 = 1^2 + 1^2$. Then Lagrange's theorem would follow by the Fundamental Theorem of Arithmetic. It will now be proved that every odd prime is the sum of at most four squares.

Lemma 2.1. Let p be an odd prime. There exists an integer m where

$1 \le m < p$ such that $mp = x_1^2 + x_2^2 + x_3^2 + x_4^2$ for some integers x_1, x_2, x_3, x_4.

Proof: Let x_i, where $i = 0, 1, 2, \ldots, \frac{p-1}{2}$, represent the integers that lie in the interval $0 \le x_i \le \frac{p-1}{2}$. There are $\frac{p-1}{2} + 1 = \frac{p+1}{2}$ such integers. No two integers x_i^2 are congruent modulo p, for if

$$x_i^2 \equiv x_j^2 \pmod{p} \quad i \neq j,\ j = 0, 1, 2, \ldots, \frac{p-1}{2}$$

then

$$(x_i - x_j)(x_i + x_j) \equiv 0 \pmod{p} .$$

This would imply $x_i - x_j$ or $x_i + x_j$ is divisible by p, but this is impossible since each is less than p.

Similarly, let $\{y_i, i = 0, \ldots, \frac{p-1}{2}\}$ represent the $\frac{p+1}{2}$ integers in the same interval and form the numbers $-1 - y_i^2$. These integers are also incongruent modulo p, for if

$$-1 - y_i^2 \equiv -1 - y_j^2 \pmod{p} \quad i \neq j$$

then

$$y_i^2 \equiv y_j^2 \pmod{p} .$$

This congruence has been shown to be impossible.

Since there are $p+1$ integers in the sets x_i^2 and $-1 - y_i^2$ taken together, two of them must be congruent modulo p. Let x^2 and $-1 - y^2$ be these integers, then

$$x^2 \equiv -1 - y^2 \pmod{p}$$

and

$$x^2 + y^2 + 1^2 + 0^2 \equiv 0 \pmod{p}$$

hence

$$x^2 + y^2 + 1^2 + 0^2 = mp, \quad \text{where } m \text{ is a positive integer.}$$

Now,

$$x^2 < \frac{p^2}{4}, \quad y^2 < \frac{p^2}{4},$$

and since

$$p^2 > 2, \; 2p^2 = p^2 + p^2 > p^2 + 2$$

which implies

$$p^2 > \frac{p^2}{2} + 1 .$$

Therefore

$$0 < mp = x^2 + y^2 + 1^2 < \frac{p^2}{4} + \frac{p^2}{4} + 1 = \frac{p^2}{2} + 1 < p^2 .$$

This shows that $m < p$, and as a result $1 \leq m < p$.

<u>Lemma 2.2</u>. If p is an odd prime and m is the least positive integer such that $mp = x_1^2 + x_2^2 + x_3^2 + x_4^2$, then m is odd.

Proof: There is at least one such m. If x_i is even, so is x_i^2, and if x_i is odd, then x_i^2 is also odd. Suppose that m is even, then so is $mp = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Then the x_i are (1) all even, (2) all odd, or (3) two are even and two are odd. In any of these cases, the x_i can be numbered and grouped in pairs in so that $x_1+x_2 \equiv 0 \pmod 2$ $x_3 + x_4 \equiv 0 \pmod 2$. Then, $(x_1 + x_2)/2$ and $(x_3 + x_4)/2$ are integers, and so are $(x_1 - x_2)/2$ and $(x_3 - x_4)/2$. Therefore,

$$\left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{x_1 - x_2}{2}\right)^2 + \left(\frac{x_3 + x_4}{2}\right)^2 + \left(\frac{x_3 - x_4}{2}\right)^2 =$$

$$\frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2} + \frac{x_4^2}{2} = \frac{m}{2} p .$$

Thus there is an integer $\frac{m}{2}$ smaller than m such that $\frac{mp}{2}$ is the sum of four squares. This contradicts the minimality of m, therefore m must be odd.

Lemma 2.3. Let p be an odd prime and m the least positive integer less than p such that $mp = x_1^2 + x_2^2 + x_3^2 + x_4^2$. If m is not 1, there exists a positive integer n where $n < m$ and $nm = y_1^2 + y_2^2 + y_3^2 + y_4^2$.

Proof: By Lemma 2.2, m is odd, and if it is not 1, then $3 \leq m < p$. Let y_i be chosen, for $i = 1, 2, 3$, and 4, in such a way that

$$y_i \equiv x_i \pmod{m} \quad , \quad |y_i| < \frac{m}{2} \ .$$

This can be done, since $\frac{-m-1}{2} \leq y \leq \frac{m-1}{2}$ is a complete set of residues. Then since

$$y_i^2 \equiv x_i^2 \pmod{m} \quad ,$$

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 \pmod{m} \ .$$

Hence

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 \equiv mp \equiv 0 \pmod{m}$$

and

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = mn \ .$$

The integer n is not 0, for if it were, $y_i = 0$ and it follows that $x_i \equiv 0 \pmod{m}$. Then m divides x_i, m^2 divides Σx_i^2, m^2 divides mp and this implies that m divides p. But, this is a contradiction since $1 < m < p$.

Furthermore, since

$$|y_i| < \frac{m}{2}, \ y_i^2 < \frac{m^2}{4} \text{ and } \Sigma y_i^2 = mn < 4 . \frac{m^2}{4} = m^2 .$$

Therefore, $n < m$ and $1 \leq n < m$.

Theorem 2.4. Every odd prime is a sum of at most four positive squares.

Proof: From Lemma 2.3 and Euler's identity

$$\begin{aligned} m^2np &= (x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) \\ &= (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3)^2 \\ &+ (x_1y_3 - x_3y_1 + x_4y_2 - x_2y_4)^2 + (x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2)^2 \\ &= A_1^2 + A_2^2 + A_3^2 + A_4^2 . \end{aligned}$$

But since $y_i \equiv x_i \pmod{m}$

$$x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv mp \equiv 0 \pmod{m}$$

and

$$x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 \equiv x_1x_2 - x_1x_2 + x_3x_4 - x_3x_4 \equiv 0 \pmod{m} .$$

Similarly, $A_3 \equiv 0 \pmod{m}$ and $A_4 \equiv 0 \pmod{m}$, implying that m divides A_i and m^2 divides A_i^2 . Therefore,

$$np = \left(\frac{A_1}{m}\right)^2 + \left(\frac{A_2}{m}\right)^2 + \left(\frac{A_3}{m}\right)^2 + \left(\frac{A_4}{m}\right)^2$$

with $0 < n < m$. But this conclusion contradicts the fact that m was chosen as the least positive integer such that mp is the sum of four squares. Therefore $m = 1$ and every odd prime is a sum of at most four positive squares.

Theorem 2.5. Every integer is a sum of at most four positive squares.
Proof: By Theorem 2.4, every odd prime is the sum of at most four positive squares and since $2 = 1^2 + 1^2$, every prime is the sum of at most four positive squares. The theorem follows immediately from Euler's identity and the fact that any integer can be expressed as a Product of primes.

In Chapter IV, which deals with fourth powers, it is essential to know what integers require four squares and what integers can be expressed as a sum of three or fewer squares. An important theorem will now be proved that integers of the form $4^r(8n + 7)$ require four squares. Of this theorem, Dickson [14] has said; "the following result is used more often than any other in researches in the theory of numbers."

It should be noted that part of this theorem will be proved using Fermat's "method of infinite descent." Hollingshead [23] has suggested that this method should be one of the important topics treated in any high school number theory course, and there is an entire chapter dealing with Fermat's method in the SMSG publication, "Essays on Number Theory II", written for high school students. Thus, Theorem 2.6 would be an illustration of the usefulness of Fermat's "method of infinite descent."

Theorem 2.6. Positive integers of the form $4^r(8n + 7)$, with r and n integers greater than zero, are not the sum of three squares.
Proof: If an integer x is even $(x = 2m)$, then

$$x^2 = (2m)^2 \equiv 0 \pmod 4,$$

and if x is odd $(x = 2m + 1)$,

$$x^2 = (2m + 1)^2 \equiv 1 \pmod 4 \text{ and } x^2 = (2m + 1)^2 \equiv 1 \pmod 8 .$$

For any integer x, it follows that

$$x^2 \equiv 0, 1, \text{ or } 4 \pmod 8 ,$$

and from this

$$x^2 + y^2 + z^2 \equiv 0, 1, 2, 3, 4, 5, \text{ or } 6 \pmod 8,$$

where y and z are arbitrary integers. Therefore $x^2 + y^2 + z^2 \not\equiv 7 \pmod 8$, and integers of the form $8n + 7$ cannot be represented as the sum of three squares.

Suppose that $4^r(8n + 7) = x^2 + y^2 + z^2$, for $r \geq 1$. Then

$$x^2 + y^2 + z^2 \equiv 0 \pmod 4 .$$

But, from above, this is true only if x, y, and z are all even. Let $x = 2a_1$, $y = 2b_1$, $z = 2c_1$, hence

$$x^2 + y^2 + z^2 = (2a_1)^2 + (2b_1)^2 + (2c_1)^2 = 4(a_1^2 + b_1^2 + c_1^2) .$$

Therefore,

$$\frac{1}{4}(x^2 + y^2 + z^2) = a_1^2 + b_1^2 + c_1^2$$

or

$$4^{r-1}(8n + 7) = a_1^2 + b_1^2 + c_1^2 .$$

In a similar manner,

$$a_1^2 + b_1^2 + c_1^2 \equiv 0 \pmod 4 ,$$

and a_1, b_1, c_1 are even. Then

$$a_1^2 + b_1^2 + c_1^2 = (2a_2)^2 + (2b_2)^2 + (2c_2)^2 = 4(a_2^2 + b_2^2 + c_2^2)$$

and

$$4^{r-2}(8n + 7) = a_2^2 + b_2^2 + c_2^2 .$$

Repeating this argument will show that

$$4^{r-j}(8n + 7) = a_j^2 + b_j^2 + c_j^2 .$$

For $j = r$,

$$8n + 7 = a_r^2 + b_r^2 + c_r^2 .$$

But this is a contradiction. Thus no positive integer of the form $4^r(8n + 7)$ is a sum of three squares.

For several theorems in Chapter IV, it is essential to know that all integers not of the form $4^r(8n + 7)$ can be expressed as the sum of three squares. However, the proof of this theorem involves the use of the theory of ternary forms and is much more difficult than the proof of Theorem 2.6. For these reasons, Theorems 2.11 and 2.12 will be stated without proof. Proofs of these theorems may be found in Landau [27]. At first reading, it might be suggested that the proof of Theorem 2.13 could be omitted.

<u>Definition 2.7</u>. If x_1, x_2, x_3 are integral values, and if the numbers a_{kg} , for $1 \leq k \leq g \leq 3$, are integral coefficients, then

$$F = F(x_1, x_2, x_3) = a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2$$
$$+ 2a_{23}x_2x_3 + a_{33}x_3^2$$

is called a ternary form.

Definition 2.8. The determinant $d = |a_{kg}|$ is called the discriminant of the ternary form F, where

$$d = |a_{kg}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} .$$

Definition 2.9. If

$$F(x_1, x_2, x_3) = \sum_{k,g} a_{kg} x_k x_g, \quad G(x_1, x_2, x_3) = \sum_{k,g} b_{kg} x_k x_g$$

are ternary forms, then F is equivalent to G if there are 9 integers C_{kg} of determinant

$$|C_{kg}| = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = 1$$

for which the 3 equations

$$x_k = \Sigma\, C_{kg} y_g$$

formally transform $F(x_1, x_2, x_3)$ into $G(y_1, y_2, y_3)$.

Definition 2.10. F is called definite, if $F > 0$ for all integral values of x_1, x_2, x_3 that do not all vanish simultaneously.

Theorem 2.11. $F = \sum_{k,g=1}^{3} a_{kg}\, x_k\, x_g$ is definite if and only if all of the following hold:

$$a_{11} > 0 , \; b = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0 , \; d > 0 .$$

Theorem 2.12. Every definite ternary form with discriminant 1 is equivalent to the form $x_1^2 + x_2^2 + x_3^2$.

Theorem 2.13. If $n > 0$ is not of the form $4^r(8k + 7)$, $r \geq 0$, $k \geq 0$, then n can be written as a sum of three squares.

Proof: If $n \equiv 1, 2, 3, 5$, or $6 \pmod 8$, then n is not of the form $4^r(8k + 7)$. To prove the theorem, it will be sufficient to show that n is the sum of three squares for one of the above forms. If $n = x^2 + y^2 + z^2$, then

$$4n = (2x)^2 + (2y)^2 + (2z)^2 \text{ , and } 4^m n = (2^m x)^2 + (2^m y)^3 + (2^m z)^2 .$$

In a similar way, if $4^m n = x^2 + y^2 + z^2$, then

$4^{m-1} n = (\frac{x}{2})^2 + (\frac{y}{2})^2 + (\frac{z}{2})^2 = x_1^2 + y_1^2 + z_1^2$, and finally

$n = (\frac{x}{2^m})^2 + (\frac{y}{2^m})^2 + (\frac{z}{2^m})^2 = x_m^2 + y_m^2 + z_m^2$.

Therefore, let $n \equiv 1, 2, 3, 5$, or $6 \pmod 8$. By Theorem 2.12, the theorem will be proved if a definite ternary form of discriminant 1, which represents n, can be found. By Theorem 2.11, nine members, $a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, x_1, x_2, x_3$, must be found that satisfy the following conditions:

$$n = a_{11} x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2x + 2a_{23}x_2x_3 + a_{33}x_3^2 ,$$

$$a_{11} > 0 ,$$

$$a_{11} a_{22} - a_{12}^2 > 0 ,$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 1 .$$

Assume

$$a_{13} = 1,\ a_{23} = 0,\ a_{33} = n,\ x_1 = 0,\ x_2 = 0,\ x_3 = 1\ .$$

It will be shown that suitable values of a_{11}, a_{12} and a_{22} may be determined such that the above four conditions are still satisfied.

From the above assumptions, a_{11}, a_{12} and a_{22} must satisfy the following conditions:

$$a_{11} > 0\ ,$$

$$b = a_{11}\, a_{22} - a_{12}^{\ 2} > 0\ ,$$

$$a_{22} = bn - 1\ ,$$

$$\text{since}\quad \begin{vmatrix} a_{11} & a_{12} & 1 \\ a_{12} & a_{22} & 0 \\ 1 & 0 & n \end{vmatrix} = (a_{11}\, a_{22} - a_{12}^{\ 2})\, n - a_{22} = bn - a_{22} = 1\ .$$

The problem can then be reduced to showing $b > 0$, and since $a_{12}^2 = -b + a_{11}\,(bn - 1)$, $-b$ must be shown to be a quadratic residue mod $(bn - 1)$.

Let $n \equiv 2$ or $6 \pmod 8$. Then $(4n, n - 1) = 1$, and by Dirichlet's Theorem on Arithmetic Progressions, there is a prime

$$p \equiv n-1 \pmod{4n}\ .$$

Let $b = 4v + 1$, then $b > 0$ and $p = bn - 1$.

Since $p \equiv 1 \pmod 4$ and $(b, p) = 1$, it follows by properties of quadratic residues that

$$\left(\frac{-b}{p}\right) = \left(\frac{p}{b}\right) = \left(\frac{bn - 1}{b}\right) = \left(\frac{-1}{b}\right) = 1 .$$

Hence, $-b$ is a quadratic residue mod $(bn - 1)$, as was to be proved.

Let $n \equiv 1, 3,$ or $5 \pmod 8$. If $n \equiv 3 \pmod 8$, let $c = 1$, and if $n \equiv 1$ or $5 \pmod 8$, set $c = 3$. In both cases $\frac{cn - 1}{2}$ is odd, thus $(4n, \frac{cn - 1}{2}) = 1$. Again, by Dirichlet's Theorem on Arithmetic Progressions there is a prime

$$p \equiv \frac{cn - 1}{2} \pmod{4n} .$$

Let

$$p = \frac{cn - 1}{2} + 4nv = \frac{1}{2}\,[(8v + c)\,n-1] .$$

If $b = 8v + c$, then $b > 0$ and $2p = bn - 1$.

Then

$$b \equiv 3 \pmod 8 \text{ and } p \equiv 1 \pmod 4 \text{ for } n \equiv 1 \pmod 8 ,$$

$$b \equiv 1 \pmod 8 \text{ and } p \equiv 1 \pmod 4 \text{ for } n \equiv 3 \pmod 8 ,$$

$$b \equiv 3 \pmod 8 \text{ and } p \equiv 3 \pmod 4 \text{ for } n \equiv 5 \pmod 8 .$$

For $b \equiv 1$ or $3 \pmod 8$,

$$\left(\frac{-2}{b}\right) = 1$$

and by properties of quadratic residues,

$$\left(\frac{-b}{p}\right) = \left(\frac{p}{b}\right) = \left(\frac{p}{b}\right)\left(\frac{-2}{b}\right) = \left(\frac{-2p}{b}\right) = \left(\frac{1 - bn}{b}\right) = \left(\frac{1}{b}\right) = 1 .$$

Thus, $-b$ is a quadratic residue mod p. Since $-b \equiv 1^2 \pmod 2$, it follows that $-b$ is a quadratic residue mod $2p$.

For $n \equiv 1, 2, 3, 5,$ or $6 \pmod 8$, it has now been proven that $b > 0$ and $-b$ is a quadratic residue mod $(bn-1)$. The theorem then follows by Theorem 2.12.

CHAPTER III

CUBES

The technique used by Maillet in his search for an upper bound for $g(3)$ is quite different from the approaches used in any of the following chapters. Maillet's method is to determine an interval with the property that every integer contained in it can be represented as the sum of 21 or fewer cubes. The bounds of this interval are manipulated and it is determined that from a certain point onward, successive intervals always overlap. Then every integer can be represented by at most 21 cubes.

In his proofs that $g(3) \leq 21$ and $g(3) \leq 17$, Maillet makes use of several identities. The use of identities is a common element in many of the theorems of Chapters II, III, IV, and V, and in most cases it is the basis for the proof of these theorems.

Maillet [31] begins his proof of $g(3) \leq 21$ with the identity

$$(3.1) \qquad (\alpha + x)^3 + (\alpha - x)^3 = 2\alpha(\alpha^2 + 3x^2)$$

where α and x are integers. If

$$(3.2) \qquad 0 \leq x \leq \alpha$$

the two cubes on the left hand side of (3.1) will be positive. Let x_1, x_2, x_3 and x_4 be four values of x, each satisfying (3.2). If these values are substituted in (3.1) and the equations added together, the following equation is obtained

$$(3.3)\qquad \sum_{i=1}^{4} [(\alpha + x_i)^3 + (\alpha - x_i)^3] = 2\alpha[4\alpha^2 + 3(x_1^2+x_2^2+x_3^2+x_4^2)] .$$

It is apparent that the left hand side of (3.3) is the sum of eight positive cubes.

Let m be a number such that

$$(3.4)\qquad 0 \le m \le \alpha^2$$

By Theorem 2.5 $m = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and each x_i satisfies (3.2). Then by (3.3), the following lemma has been established.

<u>Lemma 3.1</u> Any integer of the form

$$(3.5)\qquad 2\alpha\,[4\alpha^2 + 3m]$$

is the sum of eight or fewer cubes provided $0 \le m \le \alpha^2$.

Consider an integer of the form

$$(3.6)\qquad 2A = 2\alpha(4\alpha^2 + 3m) + 2\alpha'(4\alpha'^2 + 3m')$$

where

$$(3.7)\qquad 0 \le m \le \alpha^2 \quad\text{and}\quad 0 \le m' \le \alpha'^2 .$$

From (3.6)

$$(3.8)\qquad 2A - 8(\alpha^3 + \alpha'^3) = 6(\alpha m + \alpha' m') = 6A'.$$

The following lemma will now be proved.

<u>Lemma 3.2.</u> Any integer of the form

$$(3.9)\qquad 2A = 8(\alpha^3 + \alpha'^3) + 6A'$$

is the sum of 16 or fewer cubes of positive integers provided

(3.10) $$\alpha\alpha' \le A' \le \alpha'^3$$

$$\alpha \le \alpha' \le \alpha^2 \quad \text{and} \quad (\alpha, \alpha') = 1 .$$

Proof: Consider the integer

(3.11) $$A' = \alpha m + \alpha' m' .$$

According to (3.7), $0 \le A' \le \alpha^3 + \alpha'^3$.

Suppose that A' is given, and that when α and α' are determined it will be demanded that $(\alpha, \alpha') = 1$ and that $\alpha < \alpha'$. Equation (3.11) will now be solved in such a way that m and m' are integers and (3.7) holds.

Consider

(3.12) $$\frac{A - \alpha m}{\alpha'} = m'$$

Let $0 \le A' \le \alpha'^3$ and let m in (3.12) take on the values

(3.13) $$0, 1, 2, \ldots, \alpha' - 1$$

where it is presumed

(3.14) $$\alpha' < \alpha^2 .$$

Consider the following numbers

(3.15) $$A', A' - \alpha, A' - 2\alpha, \ldots, A' - \alpha(\alpha' - 1) .$$

There are α' numbers in (3.15) and they are non congruent modulo α' . To see this, suppose that any two numbers $A' - L\alpha$ and $A' - L'\alpha$ of (3.15) are congruent modulo α' . Then, $A' - L\alpha \equiv A' - L'\alpha \pmod{\alpha'}$ and $\alpha(L' - L) \equiv 0 \pmod{\alpha'}$. Since $(\alpha, \alpha') = 1$, then $L' = L$.

Thus, one of the integers in (3.15) is congruent to zero modulo α'

and the corresponding value of m' in (3.12) is integral and could be written as

$$m' = \frac{A' - L'\alpha}{\alpha'}$$

Since, $A' \leq \alpha'^3$ and $0 \leq L' \leq \alpha'$, then $m' \leq \alpha'^2$, and m' will be positive if $A' \geq \alpha\alpha'$.

Therefore if

$$\alpha\alpha' \leq A' \leq \alpha'^3, \tag{3.16}$$

$$\alpha \leq \alpha' \leq \alpha^2 \text{ and } (\alpha, \alpha') = 1 .$$

Integral values of m and m' can be found that satisfies (3.7) and (3.11). Thus, by (3.8), (3.6), (3.5) and (3.3) the lemma is established.

Lemma 3.3. Any integer B such that

$$8(\alpha^3 + \alpha'^3) + 6\alpha\alpha' \leq B \leq 8(\alpha^3 + \alpha'^3) + 6\alpha'^3 \tag{3.17}$$

is the sum of 21 or fewer cubes provided $\alpha < \alpha' < \alpha^2$ and $(\alpha, \alpha') = 1$.

Proof: From equations (3.9) and (3.16)

$$8(\alpha^3 + \alpha'^3) + 6\alpha\alpha' \leq 2A \leq 8(\alpha^3 + \alpha'^3) + 6\alpha'^3$$

and $2A$ differs from $8(\alpha^3 + \alpha'^3)$ by a multiple of six. Unity is the cube of an integer and if at most five unities are added to each of these numbers, then the lemma is established.

At this point in his proof, Maillet has proved that every integer in a certain interval is the sum of at most 21 cubes. It now remains to be shown that by manipulating α and α', the intervals obtained will overlap, and this will imply that all integers from a certain point onward are the sums of 21 or fewer cubes.

<u>Theorem 3.4</u>. Any integer greater than 14,372 can be represented as the sum of 21 or fewer cubes.

Proof: In equation (3.17) let α and α' be $\gamma - 1$ and γ. The conditions of (3.17), $\gamma-1 < \gamma < (\gamma-1)^2$ and $(\gamma-1, \gamma) = 1$ will be satisfied if $\gamma \geq 3$. Equation (3.17) will then hold for any number B such that

$$(3.18) \qquad 8[(\gamma-1)^3 + \gamma^3] + 6\gamma(\gamma-1) \leq B \leq 8[(\gamma-1)^3 + \gamma^3] + 6\gamma^3 .$$

If γ and $\gamma+1$ are substituted into (3.17) for α and α', the conditions of (3.17) are satisfied for $\gamma \geq 2$. Equation (3.18) will then become

$$(3.19) \qquad 8[\gamma^3 + (\gamma+1)^3] + 6\gamma(\gamma+1) \leq B' \leq 8[\gamma^3 + (\gamma+1)^3] + 6(\gamma+1)^3$$

It is now important to find out if the intervals defined in (3.18) and (3.19) overlap. It is obvious that

$$8[\gamma^3 + (\gamma-1)^3] + 6\gamma(\gamma-1) \leq 8[\gamma^3 + (\gamma+1)^3] + 6\gamma(\gamma+1)$$

when γ is a positive integer. The intervals would overlap if

$$8[(\gamma-1)^3 + \gamma^3] + 6\gamma^3 \geq 8[\gamma^3 + (\gamma+1)^3] + 6\gamma(\gamma+1)$$

It is not readily apparent that this inequality holds, so reducing the inequality

$$3\gamma^3 - 27\gamma^2 - 3\gamma - 8 \geq 0$$

is obtained. This inequality holds for $\gamma \geq 10$.

For $\gamma = 10$, the greatest lower bound of the intervals defined in (3.18) and (3.19) is

$$8[(\gamma-1)^3 + \gamma^3] + 6\gamma(\gamma-1) = 8[7^3+8^3] + 6\cdot 10.9 = 14,372 .$$

It has thus been determined that from 14,372 onward successive intervals as defined in (3.18) and (3.19), will always overlap. Therefore, every integer greater than 14,372 is in some interval and thus can be represented as the sum of 21 or fewer cubes.

To complete Maillet's proof it must be shown that all integers up to 14,372 are the sum of 21 or fewer cubes.

Theorem 3.5. Any positive integer is the sum of 21 or fewer cubes.

Proof: By Dickson's tables [13] all integers less than 560,000 are the sum of at most eight cubes except for 23 and 239 which are represented as the sum of nine cubes. Thus, by these tables and Theorems 3.4, any positive integer is the sum of 21 or fewer cubes.

Maillet was able to lower the upper bound of $g(3)$ to 17 from 21 using the same general pattern but with arguments that are a little more complicated.

In equation (3.12) assume that α and α' are odd and relatively prime and that

(3.20) $$\alpha < \alpha' < \frac{\alpha^2}{8} , \quad 8\alpha\alpha' \le A' \le \alpha'^3 .$$

Let m in equation (3.12) take on the values

(3.21) $$0, 1, 2, \ldots, 8\alpha' - 1 .$$

Among the corresponding numbers

(3.22) $\qquad A', \ A'-\alpha, \ A'-2\alpha, \ \ldots, \ A'-(8\alpha'-1)\alpha$

exactly 8 of them are divisible by α'. The eight corresponding values of m and m' given by equation (3.12) will be integral. These values will be positive because of (3.20). It will be useful to write m and m' in the form

(3.23) $$m = m_1 + j\alpha' \quad , \quad m' = m_1' - j\alpha$$
$$(j = 0, 1, 2, \ldots, 7) \ .$$

Since α' is odd, the numbers $0, \alpha', 2\alpha', \ldots, 7\alpha'$ are non-congruent to one another modulo 8. This is also true for the numbers $0, \alpha$, $2\alpha, \ldots, 7\alpha$. Among the integers $m_1 + j\alpha'$ there are only three numbers at most of the form $4^h(8n + 7)$ for h and n non-negative integers. This is also true for $m_1' - j\alpha$. Then from the 8 systems of values of m and m', there will be at least two for which neither m nor m' will be of the form $4^h(8n+7)$. Choose one of these systems. Then by Theorem 2.13, m and m' will each be the sum of three squares.

Let x_1, x_2, x_3 be three values of x satisfying (3.2.). If these values are substituted in (3.1) and if the resulting equations are added together, the following equation is obtained

(3.24) $$\sum_{i=1}^{3} [(\alpha + x_i)^3 + (\alpha = x_i)^3] = 6\alpha[\alpha^2 + (x_1^2+x_2^2+x_3^2)].$$

If $m = x_1^2 + x_2^2 + x_3^2$ with the conditions

(3.25) $$0 \leq m \leq \alpha^2 \quad \text{and} \quad m \neq 4^h(8n + 7) \ ,$$

then the following lemma is established.

<u>Lemma 3.6</u> Every integer of the form $6\alpha(\alpha^2 + m)$ is the sum of 6 or fewer cubes if $0 \leq m \leq \alpha^2$ and $m \neq 4^h(8n+7)$.

Following a pattern similar to that used for Lemma 3.2, consider

$$6A = 6\alpha(\alpha^2 + m) + 6\alpha'(\alpha'^2 + m') \tag{3.26}$$

where m satisfies (3.25) and m' its analog. Then

$$6A - 6(\alpha^3 + \alpha'^3) = 6(\alpha m + \alpha' m') = 6A' , \tag{3.27}$$

where

$$A' = \alpha m + \alpha' m' . \tag{3.28}$$

By (3.25) $$0 \leq A' \leq \alpha^3 + \alpha'^3 .$$

If α and α' are odd and relatively prime, by the previous method it follows that two positive integers m and m' can be found that will satisfy (3.25) and (3.28) and they will each be the sum of these squares.

Then since (3.25) is the sum of 12 cubes and since

$$8\,\alpha\,\alpha' \leq A' \leq \alpha'^3$$

the following lemma is established.

<u>Lemma 3.7</u> Every integer $6A$ such that

$$6(\alpha^3 + \alpha'^3) + 48\,\alpha\,\alpha' \leq 6A \leq 6(\alpha^3 + \alpha'^3) + 6\alpha'^3 \tag{3.29}$$

where $\alpha < \alpha' < \frac{\alpha^2}{8}$ and α, α' are odd and $(\alpha, \alpha') = 1$, is the sum of 12 or fewer cubes of positive integers.

To obtain a general theorem from Lemma 3.7, the same method that was used in Theorem 3.4 will be successful. Let γ be odd, $\alpha=\gamma-2$,

$\alpha' = \gamma$ and $\gamma \geq 13$. By the same reasoning as Lemma 3.7, if γ is above a finite limit, then the intervals (3.29) obtained by letting γ vary will overlap and the theorem below follows.

<u>Theorem 3.8</u>. Every integral multiple of six above a certain finite limit is the sum of 12 or fewer cubes of positive integers.

For $\gamma = 13$, the lower limit of the interval defined by (3.29) has a value of 28,032. It is this apparent that Dickson's tables [13], showing that all integers less than 560,000 are the sum of 9 or fewer cubes, are sufficient to fill the gap left in Theorem 3.8 and prove the following theorem.

<u>Theorem 3.9</u>. Every integral multiple of six is the sum of 12 or fewer cubes of positive integers.

Fleck [15] was able to improve on Maillet's upper bound for $g(3)$ in a simple way. It is surprising that Maillet did not see this himself.

It is well known that $\alpha^3 \equiv \alpha \pmod 6$ for any integer. Consider the integers $6m + \alpha$ $(\alpha = 1, 2, 3, 4, 5)$.

Now

$$6m + \alpha = \alpha^3 + 6\mu .$$

By Theorem 3.9, 6μ is the sum of at most 12 cubes. Thus, every integer of the form $6m + \alpha$ is the sum of 13 cubes. This proves the following theorem.

<u>Theorem 3.10</u>. Every positive integer can be expressed as the sum of 13 or fewer cubes.

It will now be proved that $g(3) = 9$. In Chapter I, it was proven that $g(3) \geq 9$, and it was shown that 23 requires 9 cubes.

Dickson [13] has proven that 23 and 239 are the only integers which require nine cubes, and all other positive integers can be expressed as the sum of eight or fewer cubes. It will be shown below that every integer greater than 5^{33} can be expressed as a sum of at most eight cubes. Actually, the proof given below is due to Watson [52] and shows that $G(3) \leq 8$. The integers less than 5^{33} are known to be the sum of 9 or fewer cubes by Dickson's tables for cubes. Thus, it will follow that $g(3) = 9$.

For notational purposes, C_k will be used to denote the sum of k or fewer cubes of positive integers.

The proof begins with two theorems which are usually found in elementary number theory texts and are stated here without proof.

Theorem 3.11. The congruence

(3.30) $$X^3 \equiv n \pmod 5$$

is solvable (uniquely) for every n .

Theorem 3.12. The congruence

(3.31) $$X^3 \equiv n \pmod{5^r}$$

a) always has a unique solution if $n \not\equiv 0 \pmod 5$,

b) is solvable for $\nu(n) > 0$ provided that $\nu(n)$ either divides by three or is not less than r .

Lemma 3.13. If there exists an m satisfying the three conditions

(3.32) $$(m, 6) = 1$$

(3.33) $$\frac{3}{4} m^3 < N < \frac{3}{2} m^3$$

(3.34) $$N \equiv 3m \pmod{6m} ,$$

then N is C_6.

Proof: From (3.33) and (3.34)

$$8N = 6\,m^3 + 6mk\ , \quad 0 < k < m^2\ .$$

Hence

$$6mk \equiv 8N - 6m^3 \equiv 24m - 6m \equiv 18m \pmod{48m}\ .$$

Then, because $(m, 6) = 1$, $k \equiv 3 \pmod 8$.

By Lemma 4.13, k is the sum of three odd squares.

Let $k = x_1^2 + x_2^2 + x_3^2$. Then,

$$\begin{aligned} 8N &= 6m^3 + 6m(x_1^2 + x_2^2 + x_3^2) \\ &= \sum_{i=1}^{3} \{(m+x_i)^3 + (m-x_i)^3\} \end{aligned}$$

identically. Since each $X_i \le k^{1/2}$ and m and x_i are odd, $8N$ is the sum of six positive even cubes. Thus,

$$8N = \sum_{i=1}^{6} (2y_i)^3 = 8 \sum_{i=1}^{6} y_i^3$$

$$N = \sum_{i=1}^{6} y_i^3$$

and the lemma is proved.

Lemma 3.14. If there exists an m such that

(3.35) $$3/4\ m^3 < n - x^3 - y^3 < 3/2\ m^3$$

(3.36) $$x^3 + y^3 \equiv n \pmod m$$

and

(3.37) $$x + y \equiv n + 3 \pmod 6$$

then n is C_8 .

Proof: Assume there exists an m such that (3.35), (3.36), and (3.37), are satisfies. Let $N = n - x^3 - y^3$. If $(m, 6) = 1$, then (3.34) may be broken up into

(3.38) $$x^3 + y^3 \equiv n - 3m \pmod{m}$$

and

(3.39) $$x^3 + y^3 \equiv n - 3m \pmod{6} .$$

From (3.38) it obviously follows that

$$x^3 + y^3 \equiv n \pmod{m} .$$

Since $x^3 \equiv x \pmod{6}$ and $y^3 \equiv y \pmod{6}$

(3.40) $$x^3 + y^3 \equiv x + y \pmod{6} .$$

By (3.32) $$m \equiv 1 \pmod{6} \quad \text{or} \quad m \equiv 5 \pmod{6}$$

thus

(3.41) $$3m \equiv 3 \pmod{6} .$$

Therefore, by (3.40) and (3.41), (3.39) becomes

$$x + y \equiv n + 3 \pmod{6} .$$

Let N satisfy (3.33), then

$$3/4\ m^3 < n - x^3 - y^3 < 3/2\ m^3 .$$

Then, if there exists an m satisfying (3.35), (3.36) and (3.37), m also satisfies (3.32), (3.33) and (3.34), thus by Lemma 3.13, N is C_6 . But $n = N + x^3 + y^3$, therefore n is C_8 .

Let $m = 5^r$, with r defined by

$$5^{3r} \le n < 5^{3(r+1)} \tag{3.42}$$

then $r \geq 10$ if $n \geq 5^{30}$. If

$$0 \le x < 1/2\, m \tag{3.43}$$

and

$$n - 7/8\, m^3 > y^3 \geq \max\left(n - \frac{3}{2} m^2 , 0\right) , \tag{3.44}$$

then (3.35) will hold. The inequality (3.44) defines an interval whose length is greater than $m/120$. The interval will be smallest when

$$n = 5^{3r+3} - 1 = 125m^3 - 1$$

and even then, it is much larger than $m/120$.

Watson [52] now proceeds to prove that $G(3) \le 8$ by using Lemma 3.14.

<u>Theorem 3.15</u>. If n is an integer and $n \geq 5^{33}$, then n is C_8 .
Proof: Let $m = 5^r$, with r defined by (3.42). If it can be shown that this value of m satisfies (3.35), (3.36), and (3.37), then n is C_8 by Lemma 3.14. However, it is pointed out above, that (3.43) and (3.44) imply (3.35). Watson's method of proof is to show that if $m = 5^r$, then an x and y may be found that satisfies (3.36), (3.37), (3.43), and (3.44), thus implying that subject to certain restrictions, n is C_8 .

Consider any $n > 5^{33}$. If $n/125$ is integral and C_8 , then

$$\frac{n}{125} = \frac{n}{5^3} = \sum_{i=1}^{8} x_i^3$$

and

$$n = \sum_{i=1}^{8} (5x_i)^3 .$$

Thus, if $n/125$ is integral and C_8, it follows that n is C_8. Then for $\nu(n) = 3, 4,$ or 5, $\nu(n/5^3) = 0, 1$ or 2, and these cases will be dealt with below.

Watson shows that for $\nu(n) = 0, 1, 2,$ and ≥ 6, (3.36) (3.37), (3.43) and (3.44) will be satisfied.

For $\nu(n) \geq 6$. Let $x = 25X$ and $y = 25Y$. It will be shown that (3.36), (3.37), (3.43) and (3.44) are satisfied.

Since $6 \cdot 5^{-6} m < \frac{1}{2} 5^{-2} m$, (3.43) will be satisfied if a suitable X can be found such that

$$0 \leq X < 6 \cdot 5^{-6} m .$$

From (3.44) $y = 5^2 Y$ must belong to an interval of length greater than $\frac{m}{120}$. That is, Y must belong to an interval of length greater than $\frac{m}{5^2 \cdot 120} = \frac{m}{24 \cdot 5^3}$. Now

$$\frac{m}{5^6} < \frac{m}{24 \cdot 5^3} .$$

Therefore, if Y is in an interval of length $5^{-6} m$, there exists an equivalent y in the required interval. Let Y_0 be any value of Y in the required interval.

From (3.36)

$$x^3 + y^3 \equiv 5^6 X^3 + 5^6 Y^3 \equiv n \pmod{m} .$$

Then

$$X^3 + Y^3 \equiv 5^{\nu(n)-6} n_0 \pmod{5^{-6} m}$$

and

$$X^3 \equiv -Y_0 + 5^{\nu(n)-6} n_0 \pmod{5^{-6} m}$$

$$\equiv a \pmod{5^{-6} m}$$

for some a. This congruence is solvable by Theorem 3.12. Let X_0 be a solution.

From (3.37)

$$x + y \equiv 5^2 X + 5^2 Y \equiv n+3 \pmod 6 ,$$

that is

$$X + Y \equiv n + 3 \pmod 6$$

and

$$X = -Y + n + 3 \pmod 6 \cdot$$

Now solve

$$X \equiv X_0 \pmod{5^{r-6}}$$

and

$$X \equiv -Y_0 + n - 3 \pmod 6 \cdot$$

by the Chinese Remainder Theorem, and the result is $X \equiv X_1 \pmod{6 \cdot 5^{-6}}$. This will provide the desired result.

The case $\nu(n) = 0$ will be dealt with here. The cases for $\nu(n) = 1$ and 2 are handled in a similar manner.

Let $\nu(n) = 0$. This implies $n \not\equiv 0 \pmod 5$. Now for any λ

$$(1+\lambda^3) = (1+\lambda)(1-\lambda+\lambda^2)$$

and

$$\nu(1+\lambda^3) = \nu(1+\lambda) + \nu(1-\lambda+\lambda^2) .$$

But since $1-\lambda+\lambda^2 \equiv 0 \pmod 5$ has no solution,

$$\nu(1-\lambda+\lambda^2) = 0 .$$

Therefore

$$\nu(1+\lambda^3) = \nu(1+\lambda) . \tag{3.45}$$

Consider

$$(1+\lambda^3)\, X^3 \equiv n \pmod{5^{r-5}} \tag{3.46}$$

with $$X \not\equiv 0 \pmod 5) .$$

If $\lambda \not\equiv -1 \pmod 5$ then $\nu(1+\lambda) = \nu(1+\lambda^3) = 0$. Hence for some a ,

$$(1+\lambda^3)\, a \equiv 1 \pmod{5^{r-5}} .$$

The congruence (3.46) can be written

$$X^3 \equiv n' \pmod{5^{r-5}} .$$

This congruence is solvable, by Theorem 3.12.

Now, λ must be chosen so that $\nu(1+\lambda) = 0$, and it may also be chosen such that

$$\lambda^2 \equiv 1201 \pmod{5^5} . \tag{3.47}$$

Let $n \not\equiv 0 \pmod{5^3}$, then $\nu(n) \neq 3$.

From (3.46)

$$X^3 + (\lambda X)^3 \equiv n \pmod{5^{r-5}} . \tag{3.48}$$

If

$$Y \equiv \lambda X \pmod{5^{r-5}} \tag{3.49}$$

then

(3.50) $$X^3 + Y^3 \equiv n \pmod{5^{r-5}} .$$

Since $X \not\equiv 0 \pmod 5$ and $\lambda^2 \equiv 1201 \pmod{5^5}$, then

$$\lambda \not\equiv 0 \pmod 5 , \text{ thus } Y \not\equiv 0 \pmod 5 .$$

Therefore,

(3.51) $$XY \not\equiv 0 \pmod 5 .$$

Since the modulus in (3.49) is 5^{r-5}, X and Y can be chosen so that

(3.52) $$0 \leq X < 5^{r-5} , \text{ and } 0 \leq Y < 5^{r-5} .$$

Let

(3.53) $$x = X + 5^{r-5} u \text{ and } y = Y + 5^{r-5} v .$$

Then (3.36) becomes

$$(X + 5^{r-5}u)^3 + (Y + 5^{r-5}v)^3 \equiv n \pmod{5^r}$$

and

(3.54) $$\begin{aligned}(X + 5^{r-5}u)^3 + (Y+5^{r-5}v)^3 &\equiv X^3 + 3X^2 5^{r-5}u + Y^3 + 3Y^2 5^{r-5}v \pmod{5^r} \\ &\equiv X^3 + Y^3 + 3\cdot 5^{r-5} (X^2u + Y^2v) \pmod{5^r} \\ &\equiv n \pmod{5^r}\end{aligned}$$

if $2(r-5) \geq r$, that is, if $r \geq 10$.

But from (3.49)

(3.55) $$X^3 + Y^3 = n + k\, 5^{r-5} .$$

Substitute (3.55) in (3.54), then

$$3 \cdot 5^{r-5} (X^2 u + Y^2 v) \equiv k\ 5^{r-5} \pmod{5^r} .$$

Cancel 5^{r-5}, hence

$$3(X^2 u + Y^2 v) \equiv k \pmod{5^5} .$$

Substitute $Y = \lambda X$ in the above congruence.

Then,

$$3X^2 (u + \lambda^2 v) \equiv k \pmod{5^5} .$$

Since, $X \not\equiv 0 \pmod 5$,

(3.56) $$u + \lambda^2 v \equiv M_1 \pmod{5^5}$$

for some M_1. Now use the Chinese Remainder Theorem to find an M such that

$$M \equiv M_1 \pmod{5^5}$$

and

$$M \equiv n + 3 \pmod 6 .$$

Then by (3.47), (3.36) and (3.37) together become

(3.57) $$u + \mu v \equiv M \pmod{6 \cdot 5^5}$$

where $\mu = \lambda^2 = 1201$.

Now, (3.57) is to be solved. By (3.52) and (3.53), $0 \le u \le 1561$ and ensures (3.43) and (3.44) hold. By (3.52) and (3.53), the interval defined by (3.44) permits at least $[5^5/120] = 26$ consecutive integral values of v. By a transformation these values may be considered the values $0, 1, 2, \ldots, 25$. If $v = [M/\mu] \le 15$, then this provides a solution of (3.57).

Thus, for $r \geq 10$, $n \geq 5^{33}$, (3.36), (3.37), (3.43) and (3.44) hold. Hence n is C_8 by Lemma 3.14.

<u>Theorem 3.16</u>. $g(3) = 9$.

Proof: If n is an integer then n is C_8 if $n \geq 5^{33}$, by Theorem 3.15. By Dickson's tables [13] all integers less than 5^{33} are C_9. Thus, all positive integers n are C_9 or $g(3) = 9$.

CHAPTER IV

BIQUADRATES

In Chapter I, it was mentioned that the value of $g(4)$ has not been precisely determined. From Theorem 1.1, $g(4) \geq 19$, and if an upper bound could be found for $g(4)$, then one might suppose that it would not be too difficult to reduce this upper bound towards the goal of 19.

In this chapter, Liouville's classical proof that $g(4) \leq 53$ will be given, as well as succeeding proofs that $g(4) \leq 45, 41, 39, 38$, and 37, basically due to Lucas, Lucas, Fleck, Landau, and Baer respectively. The improvement of the upper bound of $g(4)$ by a few integers is not too important, especially when one realizes that the ideal is 19. However, these proofs are an excellent illustration of how progress is made in number theory in proving or attempting to prove a conjecture. A person studying these theorems should realize that the proof of $g(4) \leq 53$ is not a very significant result in itself, but the fact that Liouville's proof was the first to establish an upper bound for $g(4)$ is important. The method of proof used by Liouville is also important in that all succeeding improvements of his result used his basic method to obtain their proofs.

Many of the identitites and theorems proved in this chapter were once well known in number theory, but few of them are found in the standard number theory text books today. Most undergraduate students of number theory would probably not be familiar with them, and could broaden their knowledge of number theory by studying them. Since many of the proofs are simple, the undergraduate student might enjoy obtaining the proofs for himself or trying to improve on the results.

Theorem 4.4 is Liouville's proof that $g(4) \leq 53$. In this proof, a simpler identity established by Lucas will be used, instead of the original identity used by Liouville.

Lemma 4.1. Every integer of the form $6a^2$ is the sum of 12 biquadrates, if a is a positive integer.

Proof: Let a be a positive integer. Then by Theorem 2.2

$$a = x_1^2 + x_2^2 + x_3^2 + x_4^2 .$$

Now,

$$\begin{aligned} 6(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 = {} & (x_1 + x_2)^4 + (x_1 - x_2)^4 + (x_1 + x_3)^4 \\ & + (x_1 - x_3)^4 + (x_1 + x_4)^4 + (x_1 - x_4)^4 \\ & + (x_2 + x_3)^4 + (x_2 - x_3)^4 + (x_2 + x_4)^4 \\ & + (x_2 - x_4)^4 + (x_3 + x_4)^4 + (x_3 - x_4)^4 . \end{aligned}$$

This identity was first established by Lucas [29] in 1876. Since

$$6a^2 = 6(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 ,$$

it follows that $6a^2$ is the sum of 12 biquadrates.

Theorem 4.2. Every positive integer of the form $6m$ is the sum of not more than 48 fourth powers.

Proof: Every positive integer m is the sum of four squares, by Theorem 2.5. Thus,

$$m = a_1^2 + a_2^2 + a_3^2 + a_4^2$$

and

$$6m = 6(a_1^2 + a_2^2 + a_3^2 + a_4^2)$$
$$= 6a_1^2 + 6a_2^2 + 6a_3^2 + 6a_4^2 \ .$$

By Lemma 4.1, $6a_i^2$ is the sum of 12 biquadrates. Therefore, $6m$ is the sum of $12 + 12 + 12 + 12 = 48$ biquadrates.

For convenience, the notation $n = B_i$ will be used to mean that n can be expressed as the sum of at most i positive biquadrates. For example, Theorem 4.2 states that $6m = B_{48}$.

<u>Corollary 4.3.</u> If m is the sum of 3 squares, then $6m = B_{36}$.

<u>Theorem 4.4.</u> Every positive integer can be expressed as the sum of not more than 53 fourth powers.

Proof: Any positive integer n is of the form $6m + r$, where $m \geq 0$ and $r = 0, 1, 2, 3, 4,$ or 5. By Theorem 4.2, $6m$ is the sum of 48 biquadrates. Since r is expressible by at most $5 \cdot 1^4$, then n is the sum of $58 + 5 = 53$ biquadrates.

The proof of the above theorem is dependent on expressing a positive integer as the sum of four squares, but by Theorem 2.13, it is known that certain integers can be expressed as the sum of three squares. This information will now be used to establish the following theorem, due to Lucas [28].

<u>Theorem 4.5.</u> Every positive integer can be expressed as the sum of not more than 45 fourth powers.

Proof: By Theorem 2.13, if $m = 8h + j$ ($j = 1, 2, 3, 5$, or 6) and $m > 0$, then m is the sum of three squares. Let

$$m = a_1^2 + a_2^2 + a_3^2 ,$$

then

$$\begin{aligned} n &= 6m + r \\ &= 6a_1^2 + 6a_2^2 + 6a_3^2 + r \\ &= B_{12} + B_{12} + B_{12} + B_5 \\ &= B_{41} , \end{aligned}$$

by Lemma 4.1.

If $m = 8h$ or $8h + 4$, since $8h - 27 \equiv 5 \pmod 8$ and $8h + 4 - 27 \equiv 1 \pmod 8$, it follows that $m - 27 \equiv 5$ or $1 \pmod 8$. Now, if $m-27 \geq 0$ it is the sum of three squares, since $m - 27$ is not of the form $4^r(8n + 7)$. Hence,

$$\begin{aligned} n &= 6m + r \\ &= 6(m - 27) + 6 \cdot 27 + r \\ &= 6(a_1^2 + a_2^2 + a_3^2) + 2 \cdot 3^4 + r \\ &= 6a_1^2 + 6a_2^2 + 6a_3^2 + 2 \cdot 3^4 + r \\ &= B_{12} + B_{12} + B_{12} + B_2 + B_5 \\ &= B_{43}. \end{aligned}$$

If $m = 8h + 7$ and $m > 14$, then

$$m - 14 = 8h + 7 - 14 \equiv 1 \pmod 8$$

and $m-14$ is a sum of three squares. Therefore,

$$\begin{aligned} n &= 6m + r \\ &= 6(m - 14) + 6\cdot 14 + r \\ &= 6(a_1^2 + a_2^2 + a_3^2) + 3^4 + 3 + r \\ &= 6a_1^2 + 6a_2^2 + 6a_3^2 + 3^4 + 3 + r \\ &= B_{12} + B_{12} + B_{12} + B_1 + B_3 + B_5 \\ &= B_{45}. \end{aligned}$$

Since m was chosen > 27 and > 14, it remains to establish how many fourth powers are required to represent $n = 6m + r$ for $m \leq 27$. Now, $6\cdot 27 + 5 = 167$ and all positive integers less than 167 are the sum of not more than 19 fourth powers, as can be seen in Bretschneider's [4] tables which give the decomposition of numbers ≤ 4100 into a sum of biquadrates.

Therefore, any positive integer can be expressed as the sum of not more than 45 fourth powers.

In order to improve his upper bound for $g(4)$, Lucas divided the integers into classes modulo 48 instead of modulo 8. Modulo 48 was not an arbitrary choice, as may be seen in the following lemma, which is also essential in the remaining theorems in this chapter.

Lemma 4.6. Every positive integer of the form $48h + j$, for $j = 6, 12, 18, 30,$ or 36, is the sum of not more than 36 fourth powers.

Proof: If $m = 8h + j$ ($j = 1, 2, 3, 5,$ or 6) and $m > 0$, then m is the sum of three squares, by Theorem 2.13. Hence, $6m = B_{36}$ by Corollary 4.3. Therefore, every integer having one of the following forms is the sum of not more than 36 fourth powers:

$$6(8h + 1) = 48h + 6 \qquad 6(8h + 5) = 48h + 30$$
$$6(8h + 2) = 48h + 12 \qquad 6(8h + 6) = 48h + 36$$
$$6(8h + 3) = 48h + 18.$$

Thus, the lemma is proved.

So far, it has been shown that $g(4) \leq 45$, but by the above lemma, certain numbers require only 36 biquadrates. Thus, Lemma 4.6 would appear to be a significant result, for if every integer could be expressed in the form of $48h + j$ ($j = 6, 12, 18, 30,$ or 36) plus a certain number of biquadrates, it seems that the upper bound for $g(4)$ could be reduced. A table will now be given to see if every integer can be expressed in the form $48h + j$ ($j = 6, 12, 18, 30,$ or 36) plus at most five biquadrates. If this can be done, it will prove that $g(4) = 36 + 5 = 41$.

By Lemma 4.6, $48h + 36 = B_{36}$. Hence,

$$48h + 37 = (48h + 36) + 1^4 = B_{37}$$
$$48h + 38 = (48h + 36) + 2\cdot 1^4 = B_{38}$$
$$48h + 39 = (48h + 36) + 3\cdot 1^4 = B_{39}$$
$$48h + 40 = (48h + 36) + 4\cdot 1^4 = B_{40}$$
$$48h + 41 = (48h + 36) + 5\cdot 1^4 = B_{41}$$
$$48h + 42 = (48h + 36) + 6\cdot 1^4 = B_{42}.$$

However, $48h + 42 = B_{42}$ is unsatisfactory, since each number is to be expressed as a sum of not more than 41 biquadrates. But,

$$
\begin{aligned}
48h + 42 &= 48(h - 1) + 6 + 3^4 + 3\cdot 1^4 &&= B_{40} \\
48h + 43 &= 48(h - 1) + 6 + 3^4 + 4\cdot 1^4 &&= B_{41} \\
48h + 44 &= 48h + 12 + 2\cdot 2^4 &&= B_{38} \\
48h + 45 &= 48h + 12 + 2\cdot 2^4 + \quad 1^4 &&= B_{39} \\
48h + 46 &= 48h + 12 + 2\cdot 2^4 + 2\cdot 1^4 &&= B_{40} \\
48h + 47 &= 48h + 12 + 2\cdot 2^4 + 3\cdot 1^4 &&= B_{41} \; .
\end{aligned}
$$

It has now been shown that $48h + j$ $(j = 36, 37, \ldots, 47)$ is the sum of not more than 41 biquadrates. In a similar manner, it can be shown that $48h + j$ $(j = 0, 1, \ldots, 35)$ can be expressed in the form $48h + r$ $(r = 6, 12, 18, 30, \text{ or } 36)$ plus at most five biquadrates. This is shown in Appendix A.

Although

$$48h + 46 = (48h + 12) + 2\cdot 2^4 + 2\cdot 1^4 = B_{40} \; ,$$

this result can be improved upon by writing

$$48h + 4 = (48h + 30) + 2^4 = B_{37} \; .$$

Whenever more than one representation is available for an integer, the one requiring the smallest number of biquadrates is used.

<u>Theorem 4.7.</u> $\qquad g(4) \leq 41$

Proof: Let n be a positive integer, then $n = 48h + j$ for
0,
$j = 1, 2, \ldots, 47$. If $j = 6, 12, 18, 30, \text{ or } 36$, then by Lemma 4.2, $n = B_{36}$. If j is not $6, 12, 18, 30, \text{ or } 36$, then $48n + j$ can be expressed in the form $48k + r$ $(r = 6, 12, 18, 30, \text{ or } 36)$ plus

at most five biquadrates.[1]

For $48(h - 4) + 30$ to be positive, h must be four or greater. Thus, it has been proved that every integer $\geq 48.4 = 192$ is the sum of $36 + 5 = 41$ biquadrates. Bretschneider [4] proved that all integers $\leq 4{,}100$ are B_{19} . Therefore, all integers are the sum of not more than 41 biquadrates.

The method of proof used in Theorem 4.7 will be used in the following theorems to reduce the upper bound for $g(4)$. This method consists of refinements of the upper bound which can be established from some integer on. For the numbers up to that point, the result is established by direct calculation. Baer's [3] proof that all integers $\leq 93^4 - 456$ are B_{38} will be necessary to complete the proof of the remaining theorems of this chapter, and will be assumed.

In his proof of $g(4) \leq 41$, Lucas [29] states how many biquadrates each residue class modulo 48 requires, but he does not show this. Appendix A was made up by this writer to complete the proof of Lucas. It should be noted that Lucas states $48h + 45 = B_{39}$, when he could have proved that $48h + 45 = B_{37}$, as is shown in Appendix A.

In Theorem 4.7, integers of the form $48h + 11$, $48h + 27$, and $48h + 43$ require 41 biquadrates, and those of the form $48h + 10$, $48h + 26$, and $48h + 42$ require 40 biquadrates. For the upper bound of $g(4)$ to be reduced to 39, these integers would have to be expressed as a sum of fewer biquadrates. This will now be proved using the method due to Fleck [15].

[1]See Appendix A

Theorem 4.8. $g(4) \le 39$

Proof: From Theorem 4.7, it will be sufficient to prove that $48h + r$ (r = 10, 11, 26, 27, 43, and 42) $= B_{39}$.

Now,

$$48h + 10 = 48(h - 2) + 24 + 82$$
$$= 48m + 24 + 3^4 + 1^4$$
$$= 6 \cdot 4(2m + 1) + 3^4 + 1^4 ,$$

where $m = h - 2$. If $2m + 1$ is not of the form $8n + 7$, then $6 \cdot 4(2m + 1) = B_{36}$, by Corollary 4.3. Hence, $6 \cdot 4(2m + 1) + 3^4 + 1^4 = B_{36} + B_2 = B_{38}$. Suppose $2m + 1 = 8n + 7$. Since $1^4 = 5^4 - 13.48$,

$$48h + 10 = 48m + 24 + 3^4 + 1^4$$
$$= 48(m - 13) + 24 + 5^4 + 1^4$$
$$= 6 \cdot 4[8(n - 3) + 5] + 5^4 + 1^4 .$$

By Corollary 4.3., $6 \cdot 4[8(n - 3) + 5] = B_{36}$, and

$$6 \cdot 4[8(n - 3) + 5] + 5^4 + 1^4 = B_{36} + B_2 = B_{38} .$$

Therefore, $48h + 10 = B_{38}$.

Similarly,

$$48h + 26 = 48h + 24 + 2 \cdot 1^4$$
$$= 6 \cdot 4(2h + 1) + 2 \cdot 1^4 .$$

If $2h + 1 \neq 8n + 7$, then $6 \cdot 4(2h + 1) = B_{36}$. Hence,

$$6 \cdot 4(2h + 1) + 2 \cdot 1^4 = B_{38} .$$

Also,

$$48h + 42 = 6 \cdot 4(2h + 1) + 2^4 + 2 \cdot 1^4 = B_{39} ,$$

if $2h + 1 = 8n + 7$. Let $2h + 1 = 8n + 7$, then

$$\begin{aligned} 48h + 42 &= 6 \cdot 4(2h + 1) + 2^4 + 2 \cdot 1^4 \\ &= 6 \cdot 4[8(n - 3) + 5] + 5^4 + 2^4 + 1^4 \\ &= B_{39} . \end{aligned}$$

Then, $48h + 42 = B_{39}$.

Finally,

$$\begin{aligned} 48h + 43 &= 6 \cdot 4[2(h - 3) + 1] + 2 \cdot 3^4 + 1^4 \\ &= B_{39} , \end{aligned}$$

if $2(h - 3) + 1 \neq 8n + 7$. If $2(h - 3) + 1 = 8n + 7$,

$$\begin{aligned} 48h + 43 &= 6 \cdot 4[8(n - 3) + 5] + 5^4 + 2 \cdot 3^4 \\ &= B_{39} . \end{aligned}$$

Therefore, $48h + 43 = B_{39}$.

Now,

$$48h + 11 = 48h + 10 + 1^4 = B_{39} ,$$

and

$$48h + 27 = 48h + 26 + 1^4 = B_{39} .$$

Consequently,

$$g(4) \leq 39 .$$

Therefore, the theorem is proven.

In Theorem 4.8, the integers requiring 39 biquadrates, are of the form $48h + r$ ($r = 1, 11, 17, 25, 27, 33, 41, 42, 43$). In order to prove $g(4) \le 38$, the above integers would have to be shown to be B_{38}. Several lemmas, necessary for Landau's [4] proof of $g(4) \le 38$, will now be established.

Lemma 4.9. Any odd integer is of the form $x_1^2 + x_2^2 + 2x_3^2$.

Proof: Let n be an arbitrary integer. Then any number of the form $4n + 2$ is not of the form $4^r(8t + 7)$. Hence, $4n + 2$ is the sum of three squares, by Theorem 2.13. Let $4n + 2 = a^2 + b^2 + c^2$. Since $4n + 2$ is not divisible by 4, this implies that a, b, and c cannot all be even. However, $4n + 2$ is even, therefore the number of odd integers among a, b, and c must be even. Let a and b be odd, then c must be even. The integers $a + b$ and $a - b$ are even. Hence, $a + b = 2x_1$ and $a - b = 2x_2$. Now, $a = x_1+x_2$, $b = x_1-x_2$, and if $c = 2x_3$,

$$\begin{aligned} 4n + 2 &= a^2 + b^2 + c^2 \\ &= (x_1 + x_2)^2 + (x_1 - x_2)^2 + 4x_3^2 \\ &= 2x_1^2 + 2x_2^2 + 4x_3^2 . \end{aligned}$$

Therefore,

$$2n + 1 = x_1^2 + x_2^2 + 2x_3^2 ,$$

and the lemma is proven.

Lemma 4.10. $$6(x_1^2 + x_2^2 + 2x_3^2)^2 = B_{11}$$

Proof: In Lucas' identity in Lemma 4.1, let $x_3=x_4$. Then $(x_3-x_4)^4=0$, and one of the biquadrates in the identity is zero. Therefore,

$$6(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 = 6(x_1^2 + x_2^2 + 2x_3^2)^2 = B_{11} .$$

Corollary 4.11. If u is an odd integer, $6u^2 = B_{11}$.

Proof: Let u be an odd integer. By Lemma 4.9., $u = x_1^2 + x_2^2 + 2x_3^2$. Hence, by Lemma 4.10, $6u^2 = B_{11}$.

Lemma 4.12. If u is an odd integer, $24 \cdot 4u^2 = B_{11}$.

Proof: If u is an odd integer, $u = x_1^2 + x_2^2 + 2x_3^2$ by Lemma 4.9. Now,

$$\begin{aligned} 24 \cdot 4u^2 &= 16 \cdot 6u^2 \\ &= 4^2 \cdot 6(x_1^2 + x_2^2 + 2x_3^2)^2 \\ &= 6[(2x_1)^2 + (2x_2)^2 + 2(2x_3)^2]^2 \\ &= 6(a^2 + b^2 + 2c^2)^2 \\ &= B_{11} , \end{aligned}$$

by Lemma 4.10.

Lemma 4.13. Let $8n + j = x_1^2 + x_2^2 + x_3^2$. Then, one of the x_i is odd if $j = 1$ or 5 , two of the x_i are odd if $j = 2$ or 6 , and all the x_i are odd if $j = 3$.

Proof: If $8n + j = x_1^2 + x_2^2 + x_3^2$, then $j = 1, 2, 3, 4, 5,$ or 6, by Theorem 2.13. If x_i is an odd integer, $x_i^2 \equiv 1 \pmod 8$, and if x_i is even, $x_i^2 \equiv 0$ or $4 \pmod 8$. If exactly one of the x_i is odd, then $x_1^2 + x_2^2 + x_3^2 \equiv 1$ or $5 \pmod 8$. If exactly two of the x_i are odd, $x_1^2 + x_2^2 + x_3^2 \equiv 3 \pmod 8$.

Lemma 4.14. $48h + 18 = B_{33}$, $48h + r$ $(r = 12, 36) = B_{34}$, and $48h + r$ $(r = 6, 30) = B_{35}$.

Proof: By Lemma 4.13, $8h + 1$ is the sum of three squares of which one is odd. Hence,

$$\begin{aligned} 48h + 6 &= 6(8h + 1) \\ &= 6(x^2 + y^2 + z^2) \\ &= 6x^2 + 6y^2 + 6z^2 \\ &= B_{11} + B_{12} + B_{12} \\ &= B_{35} , \end{aligned}$$

by Lemma 4.1 and Corollary 4.11. Every integer of the form $8h + 2$ is the sum of three squares, two of which are odd.
Therefore,

$$\begin{aligned} 48h + 12 &= 6(8h + 2) \\ &= 6(x^2 + y^2 + z^2) \\ &= B_{11} + B_{11} + B_{12} \\ &= B_{34} . \end{aligned}$$

Since $8h + 3$ is the sum of three odd squares,

$$48h + 18 = 6(8h + 3) = B_{11} + B_{11} + B_{11} = B_{33} .$$

Similarly,

$$48h + 30 = 6(8h + 5) = B_{11} + B_{12} + B_{12} = B_{35} ,$$

and

$$48h + 36 = 6(8h + 6) = B_{11} + B_{11} + B_{12} = B_{34} .$$

<u>Corollary 4.15</u>. $48h + 1 = B_{37}$, and $48h + r$ $(r = 11, 17, 33) = B_{38}$.

Proof: By Lemma 4.14,

$$48h + 1 = 48(h - 4) + 30 + 2\cdot 3^4 = B_{37} ,$$

$$48h + 11 = 48(h - 2) + 25 + 3^4 + 1^4 = B_{38} ,$$

$$48h + 17 = 48(h - 2) + 30 + 3^4 + 2\cdot 1^4 = B_{38} ,$$

$$48h + 33 = 48h + 30 + 3\cdot 1^4 = B_{38} .$$

Lemma 4.16. Every integer of the form $8h + 5$ or $8h + 6$ is the sum of three squares, of which one is twice an odd integer.

Proof: If $n \equiv 5 \pmod 8$, then $n = x_1^2 + x_2^2 + x_3^2$ where x_i is odd, $x_2^2 \equiv 0 \pmod 8$, and $x_3^2 \equiv 4 \pmod 8$, by Lemma 4.14. Now, $x_3 = 2t$. Suppose t is even $(t = 2r)$, then $x_3^2 = 4t^2 = 16r^2 \equiv 0 \pmod 8$, which is a contradiction. Thus t is odd. By the same lemma, if $n \equiv 6 \pmod 8$, then $n = y_1^2 + y_2^2 + y_3^2$ where y_1, y_2 are odd, and $y_3^2 \equiv 4 \pmod 8$. Since $y_3 = 2k$, suppose k is even $(k = 2s)$. Then $y_3^2 = 4k^2 = 16s^2 \equiv 0 \pmod 8$, which is a contradiction. Thus the lemma follows.

Lemma 4.17. $48h + 25 = B_{36}$.

Proof: Let $u = 48h + 25$ where $u > 13^4$. Then

$$u - 1^4 = 48h + 24 = 24(2h + 1) ,$$

$$u - 5^4 = 48h - 600 = 24(2h - 25) ,$$

$$u - 7^4 = 48h - 2376 = 24(2h - 99) ,$$

$$u - 13^4 = 48h - 28536 = 24(2h - 1189) .$$

The integers $2h + 1$, $2h - 25$, $2h - 99$, $2h - 1189$ are all positive, and one of them is congruent to five modulo 8. Suppose $h = 4n$, then $2h - 99 = 8h - 99 \equiv 5 \pmod 8$. If $h = 4n + 1$; $2h - 1189 = 8n - 1187 \equiv 5 \pmod 8$. If $h = 4n + 2$, $2h + 1 \equiv 8n + 5 \pmod 8$. Finally, for $h = 4n + 3$, $2h - 25 = 8n - 19 \equiv 5 \pmod 8$. By Lemma 4.16, one of the integers $2h + 1$, $2h - 25$, $2h - 99$, or $2h - 1189$ can be expressed as the sum of three squares of which one is twice an odd number. Therefore,

$$\begin{aligned} 24(8t + 5) &= 24(x^2 + y^2 + z^2) \\ &= 24((2u)^2 + y^2 + z^2) \\ &= 24\cdot 4u^2 + 24y^2 + 24z^2 \\ &= 24\cdot 4u^2 + 6(2y)^2 + 6(2z)^2 \\ &= B_{11} + B_{12} + B_{12} \\ &= B_{35} , \end{aligned}$$

by Lemmas 4.1 and 4.12. Hence, $u = 48h + 25 = B_{36}$.

Corollary 4.18. $48h + 27 = B_{38}$, $48h + 41 = B_{37}$, and $48h + r$ $(r = 42, 43) = B_{38}$.

Proof: By Lemma 4.17,

$$\begin{aligned} 48h + 27 &= 48h + 25 + 2\cdot 1^4 = B_{38} , \\ 48h + 41 &= 48h + 25 + 2^4 = B_{37} , \\ 48h + 42 &= 48h + 25 + 2^4 + 1^4 = B_{38} , \\ 48h + 43 &= 48(h - 3) + 25 + 2\cdot 3^4 = B_{38} . \end{aligned}$$

Theorem 4.19. $$g(4) \leq 38$$

Proof: The integers which require 39 biquadrates in Theorem 4.8, are of the form $48h + 1$, $48h + 11$, $48h + 17$, $48h + 25$, $48h + 27$, $48h + 33$, $48h + 41$, $48h + 42$, and $48h + 43$. By Lemmas 4.14 and 4.17, and Corollaries 4.15 and 4.18, all of the above integers are the sum of not more than 38 biquadrates.

Lemma 4.20. $$48h + 1 = B_{36}$$

Proof: Let $u = 48h + 1$ where $u > 13^4$. Then

$$u - 1^4 = 48h \qquad\qquad = 24\cdot 2h$$
$$u - 5^4 = 48h - 624 = 24(2h - 26)$$
$$u - 7^4 = 48h - 2400 = 24(2h - 100)$$
$$u - 13^4 = 48h - 28560 = 24(2h - 1190) \ .$$

The integers $2h$, $2h - 26$, $2h - 100$, and $2h - 1190$ are all positive. Since

$$2h - 26 = 8n - 26 \equiv 6 (\text{mod } 8) \text{ if } h = 4n \ ,$$
$$2h - 100 = 8n - 98 \equiv 6 (\text{mod } 8) \text{ if } h = 4n + 1 \ ,$$
$$2h - 1900 = 8n - 1186 \equiv 6 (\text{mod } 8) \text{ if } h = 4n + 2 \ ,$$
$$2h \qquad\quad = 8n + 6 \qquad \equiv 6 (\text{mod } 8) \text{ if } h = 4n + 3 \ ,$$

one of the integers $2h$, $2h - 26$, $2h - 100$, or $2h - 1190$ is congruent to $6 (\text{mod } 8)$. Hence, one of these integers is equal to $x^2 + y^2 + z^2$ where x is twice an odd integer $(x = 2k)$, by Lemma 4.8. Therefore,

$$\begin{aligned} 24(x^2 + y^2 + z^2) &= 24\cdot 4k^2 + 6(2y)^2 + 6(2z)^2 \\ &= B_{11} + B_{12} + B_{12} \\ &= B_{35} \ , \end{aligned}$$

by Lemmas 4.1 and 4.12. Thus $u = 48h + 1 = B_{36}$.

<u>Lemma 4.21.</u> $\qquad 24m + 9 = B_{36}$

Proof: Let $u = 24m + 9$ where $u > 21^4$. Then

$$u - 3^4 = 24m - 72 = 24(m - 3) \ ,$$
$$u - 9^4 = 24m - 6552 = 24(m - 273) \ ,$$
$$u - 15^4 = 24m - 50616 = 24(m - 2109) \ ,$$
$$u - 21^4 = 24m - 194472 = 24(m - 8103) \ .$$

Since

$$
\begin{aligned}
m - 3 &= 8k - 3 \equiv 5 \pmod 8) \text{ if } m = 8k \\
m - 3 &= 8k - 2 \equiv 6 \pmod 8) \text{ if } m = 8k + 1 \\
m - 2109 &- 8k - 2107 \equiv 5 \pmod 8) \text{ if } m = 8k + 2 \\
m - 2109 &= 8k - 2106 \equiv 6 \pmod 8) \text{ if } m = 8k + 3 \\
m - 8103 &= 8k - 8099 \equiv 5 \pmod 8) \text{ if } m = 8k + 4 \\
m - 8103 &= 8k - 8098 \equiv 6 \pmod 8) \text{ if } m = 8k + 5 \\
m - 273 &= 8k - 267 \equiv 5 \pmod 8) \text{ if } m = 8k + 6 \\
m - 273 &= 8k - 266 \equiv 6 \pmod 8) \text{ if } m = 8k + 7 ,
\end{aligned}
$$

one of the integers $m - 3$, $m - 273$, $m - 2109$, or $m - 8103$ is congruent to 5 modulo 8 if m is even, or if m is odd, one of these integers is congruent to six modulo 8. Hence, one of the numbers $m - 3$, $m - 273$, $m - 2109$, or $m - 8103$ is the sum of three squares $(x^2 + y^2 + z^2)$, where x is twice an odd integer $(x = 2t)$, by Lemma 4.16. Thus,

$$
\begin{aligned}
24(x^2 + y^2 + z^2) &= 24 \cdot 4t^2 + 6(2y)^2 + 6(2z)^2 \\
&= B_{11} + B_{12} + B_{12} \\
&= B_{35}
\end{aligned}
$$

by Lemma 4.1 and 4.12. Therefore, $u = 24m + 9 = B_{36}$.

<u>Lemma 4.22</u>. All integers of the form $48h + 1$ or $48h + 33$ can be represented by at most 34 biquadrates.

Proof: For every integer of the form $48k + 1$ or $48k + 33$, one can find 16 integers $a_i (i = 1, 2, \ldots, 16)$ such that $1 - a_i^4$ and $33 - a_i^4$ is of the form $48\alpha_i$, where α_i takes on every value in the least positive residue class modulo 16. This is shown in the following table.

a_i	1	5	7	11	13	17	19	23	25	29	31	37	43	49	55	61	
$\frac{1-a_i^4}{48}=\alpha_i\equiv$	0	3	14	15	13	4	5	10	6	1	8	11	7	12	2	9	(mod 16)

a_i	3	9	15	21	27	33	39	45	51	57	63	69	75	81	87	93	
$\frac{33-a_i^4}{48}=\alpha_i\equiv$	15	8	2	13	1	14	12	11	3	4	6	9	5	10	0	7	(mod 16)

For convenience, the symbol $\{{1 \atop 33}\}$ will be used to mean 1 or 33 . For each α_i in the table, there exists an a_i , where $0 \le a_i \le 93$, and

$$\{{1 \atop 33}\} - a_i^4 \equiv 48\alpha_i \pmod{48\cdot16} .$$

Let

$$\alpha_i \equiv 6 - h \pmod{16}$$

then

$$\{{1 \atop 33}\} - a_i^4 = 48\cdot16k + 48(6 - h) ,$$

where k is some integer. If s is of the form $48h + 1$ or $48h + 33$, then

$$\begin{aligned} s = a_i^4 &= 48h + \{{1 \atop 33}\} - a_i^4 \\ &= 48\cdot16k + 48\cdot6 \\ &= 2^4\cdot6(8k + 3) \\ &= 2^4\cdot6(u_1^2 + u_2^2 + u_3^2) \\ &= 2^4(B_{11} + B_{11} + B_{11}) \\ &= 2^4\cdot B_{33} \end{aligned}$$

by Lemma 4.13 and Corollary 4.11. Since $2^4\cdot x^4 = (2x)^4$, 2^4B_{33} is also the sum of 33 biquadrates. Hence,

$$s = a_i^4 = B_{33} + B_{34} .$$

It was assumed above that $h > -1$. Then, $s - a_i^4 > -480$ and $s > 93^4 - 480$, since $0 < a_i \leq 93$. It has been proven that all integers of the form $48h + 1$ or $48h + 33$, which are $> 93^4 - 480$, are B_{34} .

<u>Corollary 4.23.</u> $48h + 17 = B_{35}$

Proof:

$$48h + 17 = 48h + 1 + 2^4 = B_{35} .$$

<u>Lemma 4.24.</u> $24(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 = B_{12}$

Proof:

$$\begin{aligned} 24(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 &= (x_1 + x_2 + x_3 + x_4)^4 + (x_1 + x_2 + x_3 - x_4)^4 \\ &+ (x_1 + x_2 + x_3 - x_4)^4 + (x_1 - x_2 + x_3 + x_4)^4 \\ &+ (x_1 + x_2 - x_3 + x_4)^4 + (x_1 - x_2 - x_3 - x_4)^4 \\ &+ (x_1 - x_2 - x_3 + x_4)^4 + (x_1 - x_2 - x_3 - x_4)^4 \\ &+ (2x_1)^4 + (2x_2)^4 + (2x_3)^4 + (2x_4)^4 \\ &= B_{12} . \end{aligned}$$

<u>Corollary 4.25.</u> $24(x_1^2 + x_2^2 + x_3^2)^2 = B_{11}$.

<u>Lemma 4.26.</u> $48h + \{{9 \atop 25}\} = B_{35}$.

Proof: [3] Let s be an integer of the form $48h + 9$ or $48h + 25$. Then, $s - 24$ is of the form $48h + 33$ or $48h + 1$. By Lemma 4.22, there exists an integer $a_i > 0$ such that

$$s - 24 - a_i^4 = 2^4 \cdot 6(8k + 3)$$

where k is a positive integer. Now

$$s - a_i^4 = 2^4 \cdot 6(8k + 3) + 24 = 24(32k + 12 + 1) = 24(16L + 13)$$

where $L = 2k$ is an integer ≥ 0. Since $16L + 13 \equiv 5 \pmod 8$, $16L + 13 = v_1^2 + v_2^2 + v_3^2$. If v is an odd integer, $v \equiv 1$ or $9 \pmod{16}$, and if v is even, then $v \equiv 0$ or $4 \pmod{16}$. Therefore, $v_1^2 + v_2^2 + v_3^2 \equiv 13 \pmod{16}$ implies that one of the squares, say v_1^2, is $\equiv 9 \pmod{16}$, a second square, say v_2^2, is $\equiv 4 \pmod{16}$, and $v_3^2 \equiv 0 \pmod{16}$. Hence, $v_1 \equiv 3$ or $5 \pmod 8$, $v_2 \equiv 2$ or $6 \pmod 8$, and $v_3 \equiv 0$ or $4 \pmod 8$. By Theorem 2.13, v_1 and v_2 are the sum of three squares, and by Theorem 2.5, v_3 is the sum of four squares. Therefore,

$$\begin{aligned} s = a_i^4 + 24(16L + 13) &= a_i^4 + 24(v_1^2 + v_2^2 + v_3^2) \\ &= B_1 + B_{11} + B_{11} + B_{12} \\ &= B_{35} \end{aligned}$$

by Lemma 4.24 and Corollary 4.25.

Since $k > -1$, $s > 93^4 - 456$. It has now been proven that every integer of the form $48h + 9$ or $48h + 25$, which is $> 93^4 - 456$, is B_{35}. Thus, the lemma is proven

<u>Theorem 4.27.</u> $g(4) \leq 37$.

Proof: In Theorem 4.19, the integers which require 38 biquadrates are of the form $48h + r$ ($r = 1, 9, 11, 27, 33, 42, 43$). By Lemma 4.22, $48h + 1 = B_{34}$ and $48h + 33 = B_{34}$. Since $48h + 9$ and $48h + 25$

are B_{35} , by Lemma 4.25, then

$$48h + 11 = 48(h - 2) + 25 + 3^4 + 1^4 = B_{37}$$

$$48h + 27 = 48h + 25 + 2 \cdot 1^4 = B_{37}$$

$$48h + 42 = 48(h - 1) + 9 + 3^4 = B_{36}$$

$$48h + 43 = 48h(h - 1) + 9 + 3^4 + 1^4 = B_{37} .$$

Therefore, every integer can be expressed as the sum of 37 or less biquadrates.

The upper bound of 37 for $g(4)$ is still distant from Waring's conjectured value of 19. Chandler [5] proved that $g(4) \leq 35$ by extensive analytic work which involved proving that all integers $< 10^{26}$ are B_{19} . This result has not been improved on in 36 years, indicating why $g(4) = 19$ is regarded as the most difficult portion of Waring's conjecture.

From the theorems and lemmas of Chapter IV, it appears that 35 is the best result that can be obtained using present information. There appear to be several possible approaches to take in reducing 35 towards 19. One method would be to find new identities which would yield better results than those used in Lemmas 4.1 and 4.24. Many of the theorems of Chapter IV are dependent on the representation of an integer as the sum of a minimum number of squares. If more was known about the representations of integers as sums of squares, it appears quite likely that the upper bound of $g(4)$ could be reduced.

CHAPTER V

FIFTH AND SIXTH POWERS

The techniques involved in estimating $g(k)$ for $k = 5$ and 6 are similar to and more complicated than those used for $k = 3$. In both cases an algebraic identity plays an important role. Some of these results and arguments are outlined here. A more detailed discussion of these cases can be found in [53], [16], [25] and [3].

In his article "Zur Darstellung der Zahlen als Summen von 5^{ten} und 7^{ten} Potenzen positiver ganzer Zahlen", Wieferich [53] proves that $g(5) \leq 59$. This proof is not algebraic, as were the proofs for squares and fourth powers, and is thus more difficult to follow. Wieferich's proof consists of showing that if $z > 2^{545}$, then subject to certain restrictions, integers ε, A, B_1, and z^1 can be found such that

$$z = \varepsilon + A^5 + B_1^5 + z^1 . \tag{5.1}$$

It will be pointed out later that the number ε is 0 or 1, and that z^1 is the sum of at most 56 fifth powers. It will then follow from (5.1) that $g(5) \leq 59$. The integers $< 2^{545}$ are shown to be the sum of at most 59 fifth powers by constructing suitable tables to cover this range. Wieferich does not give the reasoning behind his proof, he merely states that he can always find over forty numbers, subject to certain conditions, by which he can define A, B_1, and z^1 in (5.1).

If z is a positive integer, Wieferich ingeneously develops various equations with restrictions on their variables until he obtains

$$z = \varepsilon + A^5 + B_1^5 + z^1$$

where ε is chosen as 0 if z is odd, but ε is 1 if z is even. The number B_1 has been arrived at by a complex process

$$B_1^5 = 15^5 \cdot 2^{v + 8 + n} \cdot B^5$$

where v is determined by

$$P + Q < z - \varepsilon < 2^5 (P + Q)$$

for

$$P = 15\cdot 2^{v + 7} (2^{2v - 6} \cdot 1377 + 1)^2 - 1579\cdot 2^{5v + 6}$$

$$Q = 15^5 \cdot 2^{5v + 50}$$

and n is determined by

$$n = 2, 3, 4, 5, 6$$

$$v + 3 + n \equiv 0 \pmod 5) .$$

The term B is chosen in such a way that

$$0 < M_1 < 2^{4v - 18}$$

where

$$M_1 = M - 15^4 \cdot 2^n \cdot B^5 .$$

The number M is of the form $4N + 3$ and is dependent on the choice of A^5.

$$A^5 = z - \varepsilon - P - 15\cdot 2^{v + 8} \cdot M$$

where A is determined such that

$$0 \leq A < 15 \cdot 2^{v + 8} .$$

The restrictions on these integers imply that v is greater than 95 . If $z > 2^{545}$, v is greater than 95 , thus the theorem will be proven for $z > 2^{545}$.

The only remaining letter in (5.1) to be explained is z^1 , which is probably the most important term in the expression. Wieferich defines z^1 to be

$$15 \cdot 2^{v + 3} \cdot \Sigma[43 \cdot 2^{2v} + y_{1i}^2 + y_{2i}^2 + y_{3i}^2]^2 - 1579 \cdot 2^{5v + 6}$$

where y_{xi} are numbers such that $y_{xi} < 2^{v - 2}$.

If α, u_i, v_i, w_i are any numbers, the following is an identity.

$$\begin{aligned} A_i = {} & (8\alpha + u_i)^5 + (8\alpha - u_i)^5 + (8\alpha + v_i)^5 + (8\alpha - v_i)^5 \\ & + (8\alpha + w_i)^5 + (8\alpha - w_i)^5 + (\alpha + u_i + v_i + w_i)^5 \\ & + (\alpha - u_i - v_i - w_i)^5 + (\alpha - u_i + v_i + w_i)^5 \\ & + (\alpha + u_i - v_i - w_i)^5 + (\alpha + u_i - v_i + w_i)^5 \\ & + (\alpha - u_i + v_i - w_i)^5 + (\alpha + u_i + v_i - w_i)^5 \\ & + (\alpha - {}_u{}^i - v_i + w_i)^5 \\ & = 2\alpha\{60\ [43^2 + u_i^2 + v_i^2 + w_i^2]^2 - 8 \cdot 1579\alpha^4\} \end{aligned}$$

Then

$$\sum_{i=1}^{4} A_i = 15 \cdot 2^3 \cdot \alpha \sum_{i=1}^{4} [43 \cdot \alpha^2 + u_i^2 + v_i^2 + w_i^2]^2 - 2^6 \cdot 1579 \cdot \alpha^5 .$$

If

$$\alpha = 2^v,\ u_i = y_{1i},\ v_i = y_{2i},\ w_i = y_{3i} .$$

Then

$$\sum_{i=1}^{4} A_i = 15 \cdot 2^{v+3} \cdot \sum_{i=1}^{4} [43 \cdot 2^{2v} + y_{1i}^2 + y_{2i}^2 + y_{3i}^2]^2 - 2^6 \cdot 1579 \cdot 2^{5v} .$$

Thus,

$$\sum_{i=1}^{4} A_i = z^1 .$$

A_i is the sum of 14 fifth powers, and $z^1 = \sum_{i=1}^{4} A_i$ is the sum of at most $4 \cdot 14 = 56$ fifth powers. Since

$$z = \varepsilon + A^5 + B_1^5 + z^1 ,$$

Wieferich thus concludes that any $z > 2^{545}$ can be represented by the sum of at most 59 fifth powers.

Wieferich then constructed a table that showed that the numbers from 1 to 500 can be represented as the sum of at most 37 fifth powers and the numbers from 500 to 3,000 can be represented by the sum of at most 28 fifth powers. To establish that $g(5) \leq 59$, it remains to show that all integers between 3,000 and 2^{545} can be expressed as the sum of 59 or less fifth powers.

Wieferich established that all integers $z < 2^{545}$ can be written as

$$z = A_1^5 + A_2^5 + \ldots + A_{21}^5 + A_{22}^5 + z_{22} \tag{5.2}$$

where $z_{22} < 50{,}000$. This equation plays an important role in Wieferich's proof. If z can be represented by equation (5.2), then z is the sum of 22 fifth powers and z_{22}. If z_{22} could be shown to be the sum of fewer than 37 fifth powers, then it would follow that z would be the sum of at most 59 fifth powers.

Therefore, to complete Wieferich's proof that $g(5) \leq 59$, it must be proved that all integers $z < 2^{545}$ can be represented by (5.2), and that z_{22} be proven to be the sum of fewer than 37 fifth powers.

Lemma 5.1. All integers $z < 2^{545}$ can be represented by

$$z = A_1^5 + A_2^5 + \dots + A_{21}^5 + A_{22}^5 + Z_{22}$$

where $Z_{22} < 50{,}000$.

Proof: Let $$z_1 = z - A_1^5$$

By a suitable choice of A_1 it can be arranged so that

$$z_1 < 5 \cdot z^{4/5}$$

If $$z_2 = z_1 - A_2^5$$

by choice of A_2, it can be arranged that

$$z_2 < 5 \cdot z_1^{\frac{4}{5}} < 5 \cdot 5^{\frac{4}{5}} \cdot z^{(\frac{4}{5})^2} = 5^{1 + \frac{4}{5}} \cdot z^{(\frac{4}{5})^2} .$$

For $z_3 = z_2 - A_3^5$ Wieferich found that he could make

$$z_3 < 5 \cdot z_2^{\frac{4}{5}} < 5^{1 + \frac{4}{5} + (\frac{4}{5})^2} \cdot z^{(\frac{4}{5})^2} .$$

By continuing this process he found that

$$z_{v-1} < 5^{1 + \frac{4}{5} + \dots + (\frac{4}{5})^{v-2}} \cdot z^{(\frac{4}{5})^{v-1}} .$$

Let $z_v = z_{v-1} - A_v^5$. By a suitable choice of A_v, it can be arranged that

$$z_v < 5 \cdot z_{v-1}^{\frac{4}{5}} < 5^{1 + \frac{4}{5} + \dots + (\frac{4}{5})^{v-1}} \cdot z^{(\frac{4}{5})^v}$$

or

$$z_v < 5^{5 - 5 \cdot (\frac{4}{5})^v} \cdot z^{(\frac{4}{5})^v} = 5^5 \cdot \left(\frac{z}{5^5}\right)^{(\frac{4}{5})^v} .$$

If the upper bound of 2^{545} is used for z and if $z = 22$, then

$$z_{22} < 5^5 \cdot \left(\frac{2^{545}}{5^5}\right)^{(\frac{4}{5})^{22}} < 50{,}000$$

Now that it has been shown that $z_{22} < 50{,}000$, it remains to be proven that z_{22} is the sum of at most 37 fifth powers. This was established by Wieferich [53] in the following way.

Lemma 5.2. If $z_{22} < 50{,}000$, then z_{22} is the sum of 37 or fewer fifth powers.

Proof: Let z_{22} be an integer such that $z_{22} < 50{,}000$. If $z_{22} \leq 3{,}000$, then from the previously mentioned tables, z_{22} is the sum of at most 37 fifth powers.

Let $3{,}000 < z_{22} < 50{,}000$.

If $33{,}500 \leq z_{22} < 50{,}000$,

let $z_{22} = 8^5 + z_{23} = 32{,}768 + z_{23}$.

If $17{,}500 \leq z_{22} < 33{,}500$,

let $z_{22} = 7^5 + z_{23} = 16{,}807 + z_{23}$.

Then $500 < z_{23} < 17{,}500$.

If $8{,}500 \leq z_{23} < 17{,}500$,

let $z_{23} = 6^5 + z_{24} = 7{,}776 + z_{24}$.

This implies that $550 < z_{24} < 10{,}000$

If $4{,}000 < z_{24} < 10{,}000$,

set

$$z_{24} = \varepsilon \cdot 5^5 + z_{26}$$
$$= \varepsilon \cdot 3125 + z_{26}$$

where $\varepsilon = 1$ or 2 .

$$500 < z_{26} < 4{,}000 .$$

If $$3{,}000 < z_{26} < 4{,}000 ,$$

set $$z_{26} = 4^5 + z_{27}$$
$$= 1{,}024 + z_{27} ,$$

then $$500 < z_{27} < 3{,}000 .$$

Thus z_{27} is the sum of 28 or fewer fifth powers. Therefore, z_{22} is the sum of at most 37 fifth powers.

By using Lemmas 5.1 and 5.2 a final result dealing with fifth powers can be obtained.

Theorem 5.3. All integers $z < 2^{545}$ can be represented as the sum of 59 or fewer fifth powers.

Proof: Since $z < 2^{545}$, by Lemma 5.1

$$z = A_1^5 + A_2^5 + \ldots + A_{21}^5 + A_{22}^5 + Z_{22}$$

where $z_{22} < 50{,}000$. By Lemma 5.2, z_{22} is the sum of 37 or fewer fifth powers. Therefore, z is the sum of at most $22 + 37 = 59$ fifth powers.

Sixth pwers seem to have a special significance in Waring's conjecture. The first general determination of $g(n)$ was accomplished independently by Dickson [11] and Pillai [39]. Dickson determined a formula for $g(n)$, subject to certain restrictions, for $n > 6$. The following chapter outlines the results in this case. Thus, $g(6)$ is the one remaining value of $g(n)$ to be discussed.

Fleck [16] made the first important contribution concerning sixth powers when he proved that $g(6) \leq 184 \cdot g(3) + 59$. This was a notable step because it proved that $g(6)$ was finite, and it also gave a method by which the problem might be attacked. An integral part of this method was the construction of the following identity:

$$60\,(a^2 + b^2 + c^2 + d^2)^3 =$$

$$\begin{aligned}
&(a + b + c)^6 + (-a + b + c)^6 + (a - b + c)^6 + (a + b - c)^6 \\
&+ (a + b + d)^6 + (-a + b + d)^6 + (a - b + d)^6 + (a + b - d)^6 \\
&+ (a + c + d)^6 + (-a + c + d)^6 + (a - c + d)^6 + (a + c - d)^6 \\
&+ (b + c + d)^6 + (-b + c + d)^6 + (b - c + d)^6 + (b + c - d)^6 \\
&+ 2(a + b)^6 + 2(a - b)^6 + 2(a + c)^6 + 2(a - c)^6 \\
&+ 2(a + d)^6 + 2(a - d)^6 + 2(b + c)^6 + 2(b - c)^6 \\
&+ 2(b + d)^6 + 2(b - d)^6 + 2(c + d)^6 + 2(c - d)^6 \\
&+ 36a^6 + 36b^6 + 36c^6 + 36d^6 \,.
\end{aligned}$$

where the right hand side is the sum of $1\cdot 16 + 2\cdot 12 + 36\cdot 4 = 184$ sixth powers. By Theorem 2.2, any positive integer may be expressed as the sum of $a^2 + b^2 + c^2 + d^2$. Thus the following lemma has been proven.

<u>Lemma 5.4</u>. If n is any positive integer, then $60n^3$ can be expressed as the sum of 184 sixth powers.

This lemma was not discovered by accident. Fleck specifically attempted to express $(a^2 + b^2 + c^2 + d^2)^3$ or a multiple of it as the sum of sixth powers of the form $(\alpha a + \beta b + \gamma c + \delta d)^6$. The coefficients α, β, γ, δ must be so determined that in the expanded sum of $(a^2 + b^2 + c^2 + d^2)^3$, all the terms which contain odd powers of a, b, c, d will disappear. With these ideas in mind, Fleck derived a method which determined the α, β, γ, and δ, as illustrated in Lemma 5.4. With this lemma, an upper bound for $g(6)$

can be found, as is proven in the following theorem.

Theorem 5.5. $g(6) \le 184 \cdot g(3) + 59$

Proof: Let m be any positive integer. Then

$$\begin{aligned} 60m &= 60(n_1^3 + n_2^3 + \ldots + n_{g(3)}^3) \\ &= 60n_1^3 + 60n_2^3 + \ldots + 60n_{g(3)}^3 \ . \end{aligned}$$

By Lemma 5.4, each $60n_i^3$ is the sum of 184 sixth powers. Hence, $60m$ is the sum of $184 \cdot g(3)$ sixth powers. Since any integer is of the form $60m + r$, for $r = 0, 1, \ldots, 59$, it follows that $g(6) \le 184 \cdot g(3) + 59$.

Corollary 5.6. $g(6) \le 1715$

Proof: Since $g(3) = 9$, $g(6) \le 184 \cdot 9 + 59 = 1715$.

The proof that $g(6) \le 1715$ is far from a proof of $g(6) = 73$, but it did inspire Kempner to approach the problem in a manner similar to that used by Fleck.

Kempner [25] derived the following identity:

$$\begin{aligned} (5.3)\ 120(a^2+b^2+c^2+d^2)^3 =& (a+b+c+d)^6+(a-b+c+d)^6+(a+b-c+d)^6+(a+b+c-d)^6 \\ & +(a-b-c+d)^6+(a-b+c-d)^6+(a+b-c-d)^6(a-b-c-d)^6 \\ & +8(a+b)^6+8(a-b)^6+8(a+c)^6+8(a-c)^6+8(a+d)^6+8(a-d)^6 \\ & +8(b+c)^6+8(b-c)^6+8(b+d)^6+8(b-d)^6+8(c+d)^6+8(c-d)^6 \\ & +(2a)^6 + (2b)^6 + (2c)^6 + (2d)^6 \ . \end{aligned}$$

There are $8 + 8 \cdot 12 + 4 = 108$ sixth powers on the right hand side of this identity. This proves the following lemma.

Lemma 5.7. For any positive integer n, $120n^3$ is the sum of 108 sixth powers.

With this lemma, it was easy for Kempner [25] to improve on Fleck's results, as is shown in the following theorem.

Theorem 5.8. $g(6) \le 108\, g(3) + 119$

Proof: If m is any positive integer,

$$\begin{aligned} 120m &= 120\,(n_1^3 + n_2^3 + \dots + n_{g(3)}^3) \\ &= 120n_1^3 + 120n_2^3 + \dots + 120n_{g(3)}^3 . \end{aligned}$$

Each $120n_i^3$ is the sum of 108 sixth powers by the previous lemma. Thus, $120m$ is the sum of $108\cdot g(3)$ sixth powers. Any integer is of the form $120m + r$, where $r = 0, 1, \dots, 119$.

Therefore, $g(6) \le 108\cdot g(3) + 119$.

Corollary 5.9. $g(6) \le 1091$

Proof: Since $g(3) = 9$, $g(6) \le 108\cdot 9 + 119 = 1091$.

An integral part of Theorem 5.8 is the representation of n_i as the sum of four squares. If n_i could be expressed as the sum of fewer than four squares, it would follow that $120n_i^3$ could be represented by fewer than 108 sixth powers, and the upper bound for $g(6)$ could be reduced. As in Chapter IV, Theorem 2.13 and Lemma 4.9 appear to play a significant role in the reduction of the upper bound for $g(k)$. Once again, the importance of knowledge concerning the representation of a number as the sum of squares is emphasized. Kempner [25] used Theorem 2.6 and Lemma 4.9 to prove the following theorem.

Theorem 5.10. $g(6) \le 107\cdot g(3) + \alpha \quad (0 \le \alpha \le 119)$

Proof: By Theorem 2.13 and Lemma 4.3, every integer is of the form $a^2 + b^2 + c^2$ or of the form $a^2 + b^2 + 2c^2$. If $n = a^2 + b^2 + c^2$ is used in the identity of Lemma 5.7, this implies that $d = 0$ and $120n^3$ will be the sum of 107 sixth powers. If $n = a^2 + b^2 + 2c^2$, then $c = d$ in (5.3) and since $8(c - d)^6 = 0$, the right hand side of the identity will be composed of 100 sixth powers. Thus all integers can be represented by at most $107 \cdot g(3) + \alpha$ $(0 \leq \alpha \leq 119)$ sixth powers.

Corollary 5.11. $g(6) \leq 1026$

α can be expressed in the form $\beta \cdot 2^6 + \gamma$, where β is 0 or 1. Then the maximum number of sixth powers occurs when $\beta = 0$ and $\gamma = 63 \cdot 1^6$. Hence, $107 \cdot g(3) + \alpha = 107 \cdot 9 + \beta \cdot 2^6 + \gamma \leq 963 + 63 = 1026$.

In most elementary text books on number theory congruences play a very important role, and the Chinese Remainder Theorem is also regarded as an essential topic. The proof of the following lemma is an excellent illustration of the application of several of the properties of congruences and also includes a practical application of the Chinese Remainder Theorem. This lemma could thus serve as a useful supplement to any first course in number theory.

If A is any positive integer, the congruence

$$(5.4) \qquad A \equiv z_1^6 + z_2^6 + \ldots + z_7^6 \pmod{120}$$

is solvable, since the congruence $A = y_1^6 + y_2^6 + \ldots + y_7^6$ is solvable modulo 3, 5, 8 by $y_1, \ldots, y_7, y_1', \ldots, y_7', y_1'', \ldots, y_7''$ and the solution of the congruences

$$z_i \equiv y_i \pmod{3}$$

$$z_i \equiv y_i' \pmod 5$$
$$z_i \equiv y_i'' \pmod 8)$$

satisfy (5.4). For example, let $A = 1607$. Then

$$1607 \equiv 2 \pmod 3$$
$$1607 \equiv 2 \pmod 5$$
$$1607 \equiv 7 \pmod 8 .$$

Thus, $y_1 = y_2 = 1$, $y_3 = \ldots = y_7 = 0$, $y_1' = y_2' = 1$, $y_3' = \ldots = y_7' = 0$, and $y_1'' = y_2'' = \ldots = y_7'' = 1$. By the Chinese Remainder Theorem, the values of z_i may be found by solving the simultaneous congruences

$$\begin{array}{ll} z_i \equiv 1 \pmod 3 & z_j \equiv 0 \pmod 3 \\ z_i \equiv 1 \pmod 5 & z_j \equiv 0 \pmod 5 \\ z_i \equiv 1 \pmod 8 & z_j \equiv 1 \pmod 8 \end{array}$$

where $i = 1, 2$, and $j = 3, 4, 5, 6, 7$. Hence, $z_1 = z_2 = 1$, $z_3 = \ldots = z_7 = 105$, and $1607 \equiv 1^6 + 1^6 + (105)^6 + \ldots + (105)^6 \pmod{120}$. The next lemma follows easily from above.

<u>Lemma 5.12.</u> The congruence

$$A \equiv z_1^6 + z_2^6 + \ldots + z_7^6 \pmod{120}$$

is solvable if A is any positive integer.

This lemma is the core of Kempner's proof [25] of $g(6) \leq 970$.

<u>Theorem 5.13.</u> $g(6) \leq 970$

Proof: In Lemma 5.12 , it may be assumed that $z_i \leq 119$ for $i = 1, \ldots, 7$. Let $A \geq 7.119^6$. Then, by the above lemma , there exist seven sixth powers which may be subtracted from A, and the resulting positive integer will be divisible by 120. However, by Lemma 5.7 and Theorem 5.10, it is known that every positive multiple of 120 can be expressed as the sum of $107 \cdot 9 = 963$ sixth powers. Therefore, every integer $\geq 7 \cdot 119$ can be represented by means of $963 + 7 = 970$ sixth powers. The integers $\leq 7 \cdot 119^6$ are the sum of not more than 186 sixth powers as calculated by Baer [3]. Thus the theorem is proven.

CHAPTER VI

THE IDEAL WARING THEOREM

In Theorem 1.1 it was proven that

$$g(k) \geq I(k)$$

where

$$I(k) = 2^k + [(\tfrac{3}{2})^k] - 2 .$$

If the values of $I(k)$ are examined for $k = 2, 3, 4, 5,$ and 6, it is found that $I(2) = 4$, $I(3) = 9$, $I(4) = 19$, $I(5) = 37$, and $I(7) = 73$. These values are not only lower bounds to $g(k)$, but it appears that $g(k) = I(k)$ in these cases. It has been conjectured that $g(k) = I(k)$ for every positive integer k, and this prediction has been called the Ideal Waring Theorem.

Let

$$3^k = 2^k\,[(\tfrac{3}{2})^k] + r , \quad 0 < r < 2^k ,$$

$$q = [(\tfrac{3}{2})^k] \quad \text{and} \quad f = [(\tfrac{4}{3})^k] .$$

Dickson [10, 11] and Pillai [39] independently proved that for $k > 6$ and $k > 7$ respectively,

(6.1) $$g(k) = I(k) \quad \text{if} \quad r \leq 2^k - q - 3 .$$

Dickson [11] was also able to prove that for $k > 6$, if

$$r > 2^k - q$$

then

$$(6.2) \qquad g(k) = \begin{cases} I(k) + f & \text{if } 2^k = fq + f + q \\ I(k) + f - 1 & \text{if } 2^k \leq fq + f + q \end{cases}$$

It can be shown that for $r > 2^k - q$

then

$$2^k < fq + f + q .$$

Since Dickson's first proof was for $r \leq 2^k - q - 3$ and his second one was for $r > 2^k - q$, it can be seen that there is a gap to be filled.

Niven [35] was able to prove

$$g(k) = I(k) \quad \text{if} \quad r = 2^k - q - 2 .$$

Dickson [11] was able to show that with

$$3^k = 2^k + q + r \quad , \qquad 0 < r < 2^k$$

it is impossible for r to be equal to $2^k - q - 1$. Rubugunday [43] showed that $r = 2^k - q$ is also impossible.

Thus, $g(k) = I(k)$ for $r \leq 2^k - q - 3$ and for $r = 2^k - q - 2$ and since $r = 2^k - q - 1$ and $r = 2^k - q$ are impossible, then it has been proven that

$$g(k) = I(k) \qquad \text{for} \qquad r < 2^k - q .$$

Since Dickson proved (6.2) if $r > 2^k - q$, the conjectured result would need to be modified in this case. However, the Ideal Waring Theorem would be proved (except for $k=4$ and $k=5$), if it could be shown that there are no r such that $r > 2^k - q$. This appears very likely. Dickson [12] has shown that $r \leq 2^k - q - 3$ for

$4 < k \leq 400$, and Mahler [30] has proved that $r > 2^k - q$ is possible for only a finite number of positive integers k if at all.

By using an IBM 7090 computer, Stemmler [47] was able to extend Dickson's results. She was able to prove that up to $k = 200{,}000$, it is true that $2^k \geq q + r$ and the Ideal Waring Theorem thus holds for these values.

Mahler's proof is based on a theorem by Ridout [42] on rational approximations of algebraic numbers. This theorem [30: 123] states:

> Let ζ be any algebraic number other than 0 ; let $P_1, \ldots, P_s, Q_1, \ldots, Q_t$ be finite sets of distinct primes; and let α, β, γ, c be real numbers satisfying
>
> $$0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1,\ \gamma > \alpha + \beta,\ c > 0$$
>
> Let p, q be restricted to be integers of the form
>
> $$p = p^* P_1^{h_1} \ldots P_s^{h_s},\ q = q^* Q_1^{k_1} \ldots Q_t^{k_t},$$
>
> where $h_1, \ldots, h_s, k_1, \ldots, k_t$ are non-negative integers and p^* , q^* are integers satisfying
>
> $$0 < |p^*| \leq cp^{\alpha},\quad 0 < q^* \leq cq^{\beta} .$$
>
> There exists a positive number C depending on $\zeta, \alpha, \beta, \gamma, c$ and the primes $P_1, \ldots, Q_1, \ldots$, such that, for all p and q of the above form, we have.
>
> $$\left|\zeta - \frac{p}{q}\right| > \frac{C}{q^{\gamma}} \text{ provided } \zeta - \frac{p}{q} \neq 0 .$$

The constant C used by Ridout can not be determined by his method. If C could be evaluated, it would be known whether Stemmler has completed the proof of the Ideal Waring Theorem, or at least the value of k to which her work would have to be extended in order to complete the proof.

Thus, the determination of $g(k)$ is now complete except for $k=4$ and 5, and the uncertainty whether or not $r > 2^k - q$ for any r .

A SELECTED BIBLIOGRAPHY

(1) Ayoub, Raymond. An Introduction to the Analytic Theory of Numbers. Providence, R. I.: American Mathematical Society, 1963.

(2) Bachman, Paul. Niedere Zahlentheorie, Zweiter Teil: Additive Zahlentheorie. Leipzig: 1910.

(3) Baer, W. S. Beitrage zum Waringschen Problem, Dissertation, Gottingen, 1913.

(4) Bretschneider, G. A. "Tafeln für die Zerlegung der Zahlen bis 4100 in Biguadrate", Journal für die Reine und Angewandte Mathematik, 46 (1853), pp. 1-28.

(5) Chandler, Emily. Waring's Theorem for Fourth Powers, Dissertation, University of Chicago, 1933.

(6) Chen, Jing-Jung. "Waring's Problem for $g(5)$", Science Record, 3 (1959), pp. 327-330.

(7) Chowla, S. "Pillai's Exact Formulae for the Number $g(n)$ in Waring's Problem", Proceedings of the Indian Academy of Science, (A), 4 (1936), P. 261.

(8) Dickson, Leonard E. History of the Theory of Numbers, Washington: Carnegie Institute, Vol. II, 1919.

(9) ______. "Recent Progress on Waring's Theorem and its' Generalizations", Bulletin of the American Mathematical Society, 39 (1933), pp. 701-727.

(10) ______. "Proof of the Ideal Waring Theorem for Exponents 7-180", American Journal of Mathematics, 58 (1936), pp. 521-529.

(11) ______. "Solution of Waring's Problem", American Journal of Mathematics, 58(1936), pp. 530-535.

(12) ______. "The Waring Problem and it's Generalizations", Bulletin of the American Mathematical Society, 42 (1936), pp. 832 - 842.

(13) ______. "All Integers Except 23 and 239 are Sums of Eight Cubes", Bulletin of the American Mathematical Society, XLV (1939), pp. 588 - 591.

BIBLIOGRAPHY (CONT'D)

(14) ____________. Modern Elementary Theory of Numbers, Chicago: University of Chicago Press, 1939.

(15) Fleck, Albert. "Uber die Darstellung ganzer Zahlen als Summen von Biquadraten ganzer Zahlen", Sitzungsberichte der Berliner Mathematischen Gesellschaft, 5 (1906), pp. 2-9.

(16) ____________. "Uber die Darstellung ganzer Zahlen als Summen van sechsten Potenzen ganzer Zahlen", Mathematische Annalen, LXIV (1907), pp. 561-565.

(17) Gelfond, A. O. and Linnik, Y. V. Elementary Methods in Analytic Number Theory, translated by Amie Feinstein, revised and edited by L. J. Morell, Chicago: Rand McNally and Co., 1965.

(18) Hardy, G. H. Some Famous Problems of the Theory of Numbers, Oxford: Clarendon Press, 1920.

(19) ____________. Collected Papers of G. H. Hardy, Oxford: Clarendon Press, Vol. 1, 1966.

(20) Hardy, G. H. and Wright, E. M. An Introduction to the Theory of Numbers, 4th edition, Oxford: Clarendon Press, 1960.

(21) Havsdorff, F. "Zur Hilbertschen Lösung des Waringschen Problems", Mathematische Annalen, LXVII (1909), pp. 301-305.

(22) Hilbert, David. "Beweis fur die Darstellbarkeit der ganzen Zahlendurcheine feste Anzahl n-ter Potenzen (Waringsches Problem)", Nachrichten von der Koniglichen Gesellschaft der Wissenschaften zu Gottingen, (1909), pp. 17-36, Mathematische Annalen, LXVII (1909), pp. 281-300.

(23) Hollingshead, Irving. "Number Theory--a Short Course for High School Seniors", Mathematics Teacher, LX (1967), pp. 222-227.

(24) Kempner, Aubrey. "Uber das Waringsche Problem und einige Verallgemeinerungen", Dissertation, Gottingen, 1912.

(25) ____________. "Bemerkungen zum Waringschen Problem", Mathematische Annalen, LXXII (1912), pp. 387-399.

(26) Landau, Edmund. "Uber die Darstellung einer ganzen Zahl als Summe von Biquadraten", Rendiconti del Circolo Mathematico di Palermo, XXIII (1907), pp. 91-96.

BIBLIOGRAPHY (CONT'D)

(27) ____________. Elementary Number Theory, New York: Chelsea, 1958.

(28) Lucas, Edouard. "Sur la decomposition des nombres en bicarres", Nouvelle Correspondence Mathematiques, 4(1878), pp. 323-325.

(29) ____________. "Sur un theoreme de M. Liouville concernant la decomposition des nombres en bicarres", Nouvelles Annales de Mathematiques, (2), 17 (1878), pp. 536-537.

(30) Mahler, K. "On the Fractional Parts of the Powers of a Rational Number (II)", Mathematika, 4 (1957), pp. 122-124.

(31) Maillet, Edouard. "Sur la decomposition d'un nombre entier en une somme de cubes d'entiers positifs", Conferences Association fraincais pour l'avancement des Sciences, (2) 24 (1895), pp. 242-247.

(32) ____________. "Quelques extensions du theoreme de Fermat sur les nombres polygones", Journal de Mathematiques pures et appliquees, (5), II (1896), pp. 363-380.

(33) Mathematical Association of America. Course Guide for the Training of Junior and Senior High Mathematics, Teachers, 1961

(34) ____________. Course Guide for the Training of Teachers of Elementary School Mathematics, 1964.

(35) Niven, Ivan. "An Unsolved Case of the Waring Problem", American Journal of Mathematics, 66 (1944), pp. 137-143.

(36) Niven, Ivan and Zuckerman, Herbert S. An Introduction to the Theory of Numbers, 2nd edition, New York: John Wiley and Sons, 1966.

(37) Ostmann, Hans-Heinrich. Additive Zahlentheorie, zweiter Teil, Berlin: Springer-Verlag, 1956.

(38) Palama, Guiseppe. "Il Problema di Waring", Bollettino della Unione Mathematica Italiana, (3), 12 (1957), pp. 83-100.

(39) Pillai, S. S. "On Waring's Problem", Journal of the Indian Mathematical Society, (2), 2 (1936), pp. 16-44.

(40) ____________. "On Waring's Problem g(3) = 73", Proceedings of the Indian Academy of Science, (A), 12 (1940), pp. 30-40.

BIBLIOGRAPHY (CONT'D)

(41) Remak, Robert. "Bemerkung zu Herrn Stridsbergs Beweis des Waringschen Theorems", Mathematische Annalen, LXXII (1912), pp. 153-156.

(42) Ridout, D. "Rational Approximations to Algebraic Numbers", Mathematika, 4 (1957), pp. 125-131.

(43) Rubugunday, R. K. "On g(k) in Waring's Problem", Journal of the Indian Mathematical Society, (2), 6(1942), pp. 192-198.

(44) School Mathematics Study Group. Essays on Number Theory I, II, 1960.

(45) Shanks, Daniel. Solved and Unsolved Problems in Number Theory, Washington: Spartan Books, Vol. 1, 1962.

(46) Sierpinski, W. Elementary Theory of Numbers, Warszawa, Poland: Panstwowe Wydawnict wo Naukowe, 1965.

(47) Stemmler, R. M. "The Ideal Waring Theorem for Exponents 401-200,000", Mathematics of Computation, 18 (1964), pp. 144 146.

(48) Stridsberg, Erik. "Sur la demonstration de M. Hilbert du theoreme de Waring", Mathematische Annalen, LXXII (1912), pp. 145-152.

(49) Vinogradov, I. M. The Method of Trigonometrical Sums in the Theory of Numbers, translated and annoted by K. F. Roth and Anne Davenport, New York: Interscience Publishers, 1954.

(50) Von Sterneck, C. A. "Tafeln für die Zerlegung der Zahlen bis 4100 in Biquadrate", Journal für die Reine und Angewandte Mathematik, 46 (1853), pp. 1-28.

(51) Waring, Edward. Meditationes Algebraicae, Cambridge: J. Archdeacon, 1782.

(52) Watson, G. L. "A Simple Proof that All Large Integers are Sums of at Most Eight Cubes", Mathematical Gazette, 37 (1953), pp. 209-211.

(53) Wieferich, Arthur. "Zur Darstellung der Zahlen als Summen von 5-ten und 7-ten Potenzen positiver ganzer Zahlen", Mathematische Annalen, LXVII (1909), pp. 61-75.

BIBLIOGRAPHY (CONT'D)

(54) ____________. "Beweis des Satszes dass sicheine jede ganze Zahl als Summe von höchstens neun positiven Kuben darstellen lasst", Mathematische Annalen, LXVI (1909), pp. 95-101.

(55) ____________. "Uber die Darstellung der Zahlen als Summen von Biquadraten", Mathematische Annalen, 66 (1909), pp. 106-108.

APPENDIX

$g(4) \le 41$

$48h = 48(h-4)+30+2\cdot 3^4 = B_{38}$
$48h+1 = 48(h-4)+30+2\cdot 3^4+1^4 = B_{39}$
$48h+2 = 48(h-1)+18+2\cdot 2^4 = B_{38}$
$48h+3 = 48(h-2)+18+3^4 = B_{37}$
$48h+4 = 48(h-1)+36+2^4 = B_{37}$
$48h+5 = 48(h-1)+36+2^4+1^4 = B_{38}$
$48h+6 = B_{36}$
$48h+7 = 48h+6+1^4 = B_{37}$
$48h+8 = 48h+6+2\cdot 1^4 = B_{38}$
$48h+9 = 48h+6+3\cdot 1^4 = B_{39}$
$48h+10 = 48h+6+4\cdot 1^4 = B_{40}$
$48h+11 = 48h+6+5\cdot 1^4 = B_{41}$
$48h+12 = B_{36}$
$48h+13 = 48h+12+1^4 = B_{37}$
$48h+14 = 48h+12+2\cdot 1^4 = B_{38}$
$48h+15 = 48(h-2)+30+3^4 = B_{37}$
$48h+16 = 48(h-2)+30+3^4+1^4 = B_{38}$
$48h+17 = 48(h-2)+30+3^4+2\cdot 1^4 = B_{39}$
$48h+18 = B_{36}$
$48h+19 = 48h+18+1^4 = B_{37}$
$48h+20 = 48h+18+2\cdot 1^4 = B_{38}$
$48h+21 = 48h(h-2)+36+3^4 = B_{37}$
$48h+22 = 48h+6+2^4 = B_{37}$
$48h+23 = 48h+6+2^4+1^4 = B_{38}$
$48h+24 = 48(h-3)+6+2\cdot 3^4 = B_{38}$
$48h+25 = 48(h-3)+6+2\cdot 3^4+1^4 = B_{39}$
$48h+26 = 48(h-3)+6+2\cdot 3^4+2\cdot 1^4 = B_{40}$
$48h+27 = 48(h-3)+6+2\cdot 3^4+3\cdot 1^4 = B_{41}$
$48h+28 = 48h+12+2^4 = B_{37}$
$48h+29 = 48h+12+2^4+1^4 = B_{38}$
$48h+30 = B_{36}$
$48h+31 = 48h+30+1^4 = B_{37}$
$48h+32 = 49h+30+2\cdot 1^4 = B_{38}$
$48h+33 = 48h+30+3\cdot 1^4 = B_{39}$
$48h+34 = 48h+18+2^4 = B_{37}$
$48h+35 = 48h+18+2^4+1^4 = B_{38}$
$48h+36 = B_{36}$
$48h+37 = 48h+36+1^4 = B_{37}$
$48h+38 = 48h+36+2\cdot 1^4 = B_{38}$
$48h+39 = 48\ (h-1)+6+3^4 = B_{37}$
$48h+40 = 48\ (h-1)+6+3^4+1^4 = B_{38}$
$48h+41 = 48\ (h-1)+6+3^4+2\cdot 1^4 = B_{39}$
$48h+42 = 48\ (h-1)+6+3^4+3\cdot 1^4 = B_{40}$
$48h+43 = 48\ (h-1)+6+3^4+4\cdot 1^4 = B_{41}$
$48h+44 = 48h+12+2\cdot 2^4 = B_{38}$
$48h+45 = 48h+12+3^4 = B_{37}$
$48h+46 = 48h+30+2^4 = B_{37}$
$48h+47 = 48h+30+2^4+1^4 = B_{38}$

VITA

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