ANALYSIS OF PLATES WITH FREE EDGES ON ELASTIC WINKLER FOUNDATIONS BY THE GRID FRAMEWORK METHOD

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PREFACE

The application of the grid framework analogy to the analysis of plates on elastic foundations investigated in this thesis is the culmination of the author's studies at Oklahoma State University. With the rapid increase in the availability of digital computers in the consulting field, more structural engineers are entering the profession with a knowledge of computer techniques. In addition, the use of a high speed computer makes the solution of many difficult problems much easier for the consulting office. This investigation provides a numerical technique for the solution of an elastic plate supported by an elastic foundation. The grid framework method is readily adapted to many frame and grid analysis programs now available.

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CHAPTER I

INTRODUCTION

Discussion of the Problem

The recent increase in the availability of high speed digital computers has greatly affected the field of structural engineering. The accompanying increase in the availability of computer software has made use of the computer relatively easy for routine arithmetic operations. Because of these facts, the analysis of highly indeterminant framework structures poses few problems for the structural engineer. In addition, a highly accurate structural analysis of a continuous elastic medium is now possible if the physical properties of the medium can be simulated by an equivalent framework structure. The significance of this possibility is that problems which have been concerned with continuous elastic structures and which have usually been beyond the ability of the consulting engineer can now be solved using available techniques. Such a problem is the analysis of an elastic plate resting upon an elastic foundation.

The solution of this problem can be approximated in either of two ways: by the use of a finite element approach; or by the use of a line element approach. These methods produce comparable results. The finite element method allows deformations to occur at the joints of the approximating system, and the mathematical relationships are best defined in terms of displacement functions. On the other hand, the line element method uses deformable members with continuous joints and lends itself to more commonly used techniques of analysis such as the stiffness method. This property of the line element approach makes it more readily adaptable by the average consulting engineer. It is, therefore, the purpose of this thesis to develop in terms of the stiffness or displacement method of structural analysis an equivalent grid framework model for an elastic plate resting upon an elastic foundation. Such a method of solution offers many advantages: first, the introduction of different boundary conditions will not change the procedure nor will it alter the governing equation; secondly, the analysis will closely follow methods used to analyze other framework structures and will either be familiar to the consulting engineer or available through commercial computer facilities. Finally, plates of irregular boundaries or internal configurations can be analyzed with the same ease as plates with more regular conformation.

History of the Problem

The basic differential equation governing the deflection of thin plates was developed by Lagrange and Sophie

Germain (1) in 1811 and was the first major contribution in the theory of thin plates to be presented.

Poisson (2), in a paper on elasticity published in 1829, set forth the first investigation of the problem of an elastic plate using the general equations of elasticity. Poisson's derivation contains a set of boundary conditions; thus, he was able to obtain the solution for circular plates under symmetrical loading conditions.

Kirchhoff (3), in a paper published in 1850, derived the governing equation and the corresponding boundary conditions by using the energy principles or the principle of least work. In this paper, Kirchhoff was able to reduce by one the number of boundary conditions necessary to describe a free edge as proposed by Poisson.

The first study of elastic foundations was conducted in 1867 by E. Winkler (4) and was concerned with beams on elastic foundations. This theory was extended in 1888 by H. Zimmerman (5) in his treatise which was directed toward foundation problems under railroad tracks.

In the particular area of plates resting upon elastic foundations the first contribution was made in 1881 by H. Hertz (6), who investigated the problem of an infinite floating plate subjected to a concentrated load. Hertz used the assumption that the intensity of the foundation reaction was proportional to the deflection of the elastic plate. This assumption was identical to that proposed by Winkler in the previously discussed paper on beams resting on elastic foundations.

In 1923 Westergaard (7) extended the theory of plates on elastic foundations to the analysis of infinite pavements. This work was then expanded to include practical applications of pavement designs. In the design of pavements, Westergaard's solution is generally used today, thereby assuming that the slab under consideration has infinite dimensions.

More recent discussions of the problem of plates on elastic foundations have been presented by Timoshenko and Woinowsky-Krieger (8), who have developed a solution for a circular plate with a center load. In 1953 R. K. Livesley (9) presented a formal solution for the case of a semi-infinite plate and an infinite quadrant, simply supported along their edges, in terms of double Fourier transforms. In addition, the "method of images" was investigated by Arnold D. Kerr (10) and presented in 1963.

The analysis of elastic mediums by the use of a grid framework model was first presented in 1941 by A. Hrennikoff (11) in a paper which proposed a square grid model and utilized a constant Poisson's ratio of 1/3. Other writers such as Newmark (12), Ang and Newmark (13), and Yettram and Husain (14) have refined the technique by developing more general plane framework models. In addition, the special case of Poisson's ratio equal to zero was investigated by Christensen (15), Lightfoot (16), and Yettram and Husain (14).

An interesting application of numerical models to the problem of plates on elastic foundations was presented in 1967 by Hudson and Matlock (17) in a paper concerning cracked pavement slabs with non-uniform support. This approach used a finite element grid model with infinitely stiff edge members, torsional resistant cross members, and flexible, deformable joints. The mathematical formulation of the problem then assumed a finite difference approach by relating the equilibrium expressions formed from a study of a free body of a general joint. The solution of the resulting set of simultaneous differential equations was accomplished by using a cross iteration technique to obtain the deformation of each joint. Whereas this procedure provided a numerical method which was quite flexible to obtain answers for a previously difficult problem, the formulation of the method did not relate to techniques readily available to the average structural engineer.

The stiffness method of structural analysis is a matrix algebra representation of the slope deflection equations familiar to structural engineers and has been presented by N. Willems and the writer (18) and others (19), (20), (21). This method is particularly suited to high speed computation as the repetetive manipulation of many terms is necessary. The suitability of the stiffness method to problems involving a large number of members was demonstrated by Eiseman, Namyet, and Woo (22) in 1962. This property indicates that the method is suited to solve the stiffness equations necessary to define a grid framework used to represent an elastic medium.

Definition of Terms and Basic Assumptions

In the analysis of plates on elastic foundations the term "thin plate" refers to a plate for which the thickness is small in comparison to its other dimensions. For the purpose of this paper, a plate will be considered thin when its thickness is less than one-twentieth (1/20) of its next smallest dimension.

In addition to the relative thickness of the subject plate, a distinction is usually made between thin plates with small deflections and thin plates with large deflections. In the classical derivation of the governing equation for a plate on an elastic foundation, the assumption is usually made that the deflections are small in comparison to the thickness of the plate. To allow for a comparison of results, thin plates with small deflections are considered in this thesis.

The equivalent grid framework model used to approximate the actions of an elastic plate is composed of members that are similar to those used in any regular grid structure. For the development of this model the assumptions are made that the material is homogeneous, isotropic, and continuous. In addition, it is assumed that the modulus of elasticity is a known constant and is the same in both tension and compression and that the material deformations follow Hooke's law.

In the analysis of plates resting on an elastic foundation, it is commonly assumed that the support offered by the foundation is proportional to the deflection of the plate. This assumption was first introduced by E. Winkler (4) and the corresponding foundation is usually referred to as a "Winkler foundation."

Many subsoils display deformations localized mainly in the loaded region, and for such soils close agreement between computations based upon Winkler's hypothesis and test results is usually observed. For instance, Westergaard (7) developed his theory for the design of infinite slabs on the above assumption, and the resulting calculated values have been shown to agree closely with experimental results.

Discussion of the Procedure

The procedure followed in the development of a grid framework approach to the solution of plates resting on elastic foundations consists first, in Chapter II, of a discussion of the stiffness approach to the solution of general grids. The general stiffness equations and matrices are developed for grid structures, and the solution of these equations is explained.

Chapter III applies the principles of the stiffness equations for grid structures to the development of a grid framework model for an elastic plate element. The model matrix equation is then solved in terms of the actual unit displacements of the plate element, and the properties of the model grid members are thereby established.

An extension of the grid framework model stiffness equation is presented in Chapter IV where the effects of an elastic Winkler foundation are introduced. These foundation effects are assembled into a new matrix, K, combined with the joint stiffness matrix, and methods of solution of the governing equation are presented.

The application of the method to plates of various shapes and subjected to various loading conditions is the subject of Chapter V. The results of the grid framework method are compared to deformations obtained by other methods for four different types of plate problems.

In Chapter VI, the versatility of the method is demonstrated through the analysis of plates with variable rigidities. Solutions are presented for tapered concrete pavement slabs with stiffened and unstiffened edges, and for slabs tapered in two directions.

The results of the investigation and the analysis procedure are summarized in Chapter VII and conclusions as to the suitability of the method are presented.

CHAPTER II

ANALYSIS OF GRIDS BY THE STIFFNESS METHOD

Introduction

The high speed digital computer has made the stiffness method an efficient tool for the structural engineer. Using matrix algebra, the method can be organized into a highly systematic procedure which is readily programmed for computer application. The stiffness method is particularly suited to the analysis of framed structures and is, therefore, applicable to the problem of solving grid framework systems.

In this chapter, the member stiffness method will be presented and applied to grid structures. Stiffness influence coefficients are presented for a general grid member, and these are assembled into a member-oriented stiffness matrix. This matrix is then rotated by means of angular transformation matrices to form the structure oriented member stiffness matrix for a general grid member. The general grid member stiffness matrices are then assembled into the structure joint stiffness matrix, and methods of solution for the general matrix equation are discussed.

Member Stiffness Matrices for Grid Elements

The fundamental matrix equation of the stiffness method is

$$\{A\} = [S]\ \{D\} \qquad (2\text{-}1)$$

which states that the actions of a system can be expressed in terms of the displacements of the system by the formulation of a stiffness matrix representing actions due to unit values of the displacements. In the stiffness method this equation is used to ensure the equilibrium of forces at the various joints.

The stiffness matrix is a square, symmetrical matrix composed of stiffness influence coefficients. A stiffness influence coefficient is the force produced by a unit deformation of a given member in a particular direction.

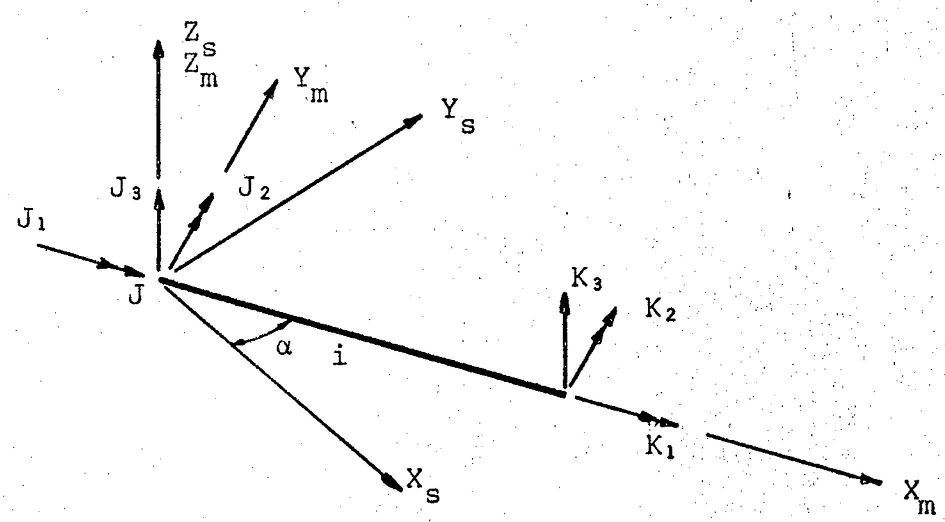
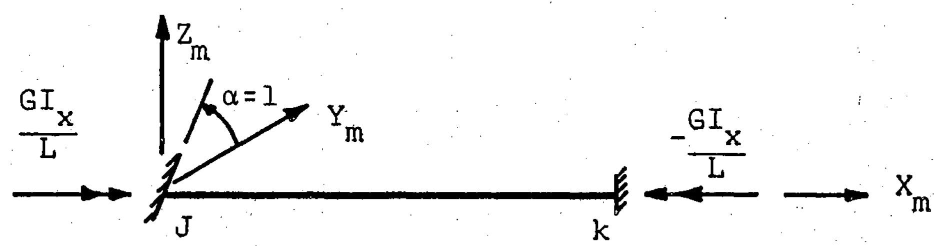


Figure 1. General Member of a Structural Grid

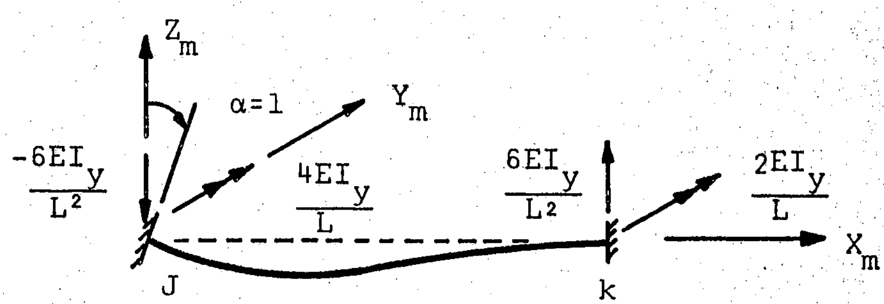
Consider for example, a general member i of a structural grid shown in Figure 1. The centroid of the member lies along the X_m axis and the Y_m and Z_m axes are also assumed to occur about these principal axes. Therefore, the shear center of the member is considered to coincide with the centroidal axis. The X_s, Y_s and Z_s axes are assumed to be the structure oriented coordinate system and are arbitrarily chosen for the convenience of future calculations.

The ends of the member i are denoted J and K, and at each end there are three possible deformations: a joint translation in the Z_m direction and member rotations about the X_m and Y_m axes. If the member is allowed to deform one deformation at a time, and the resulting forces are recorded in matrix form, the resulting matrix is the stiffness matrix for the member. The three deformations for the J end and their associated reactions are given in Figure 2 and the similar values for the K end of the member

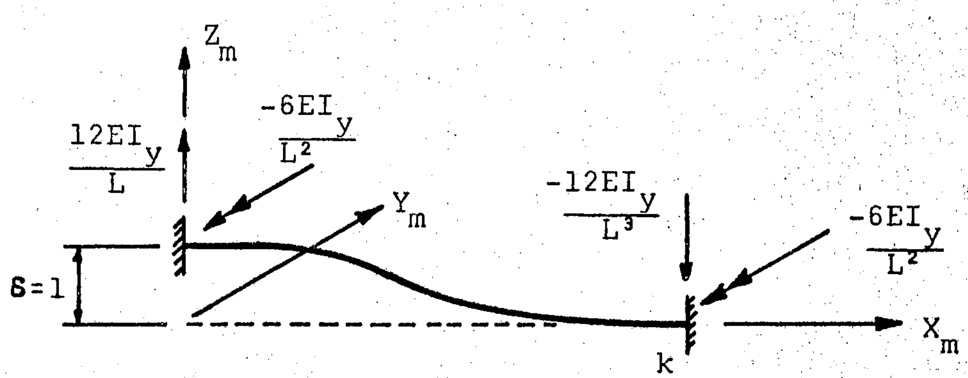
$$[S_m]_i = \left[\begin{array}{ccc|ccc} \frac{GI_x}{L} & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{4EI_y}{L} & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} \\ 0 & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} \\ \hline -\frac{GI_x}{L} & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{2EI_y}{L} & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} \\ 0 & \frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} \end{array}\right] \quad (2-2)$$



(1) Unit rotation about X_m axis

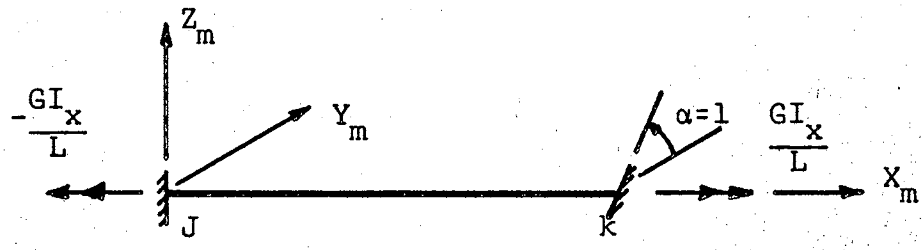


(2) Unit rotation about Y_m axis

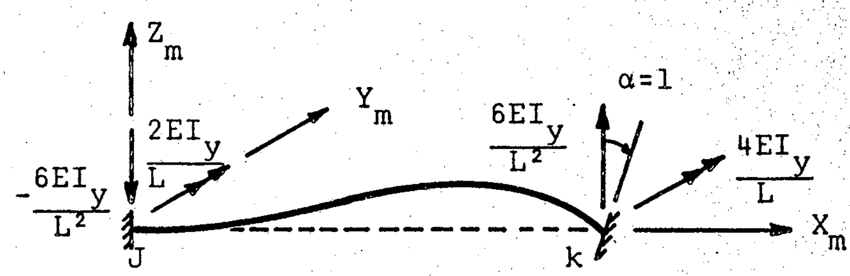


(3) Unit Deformations at J End of Member

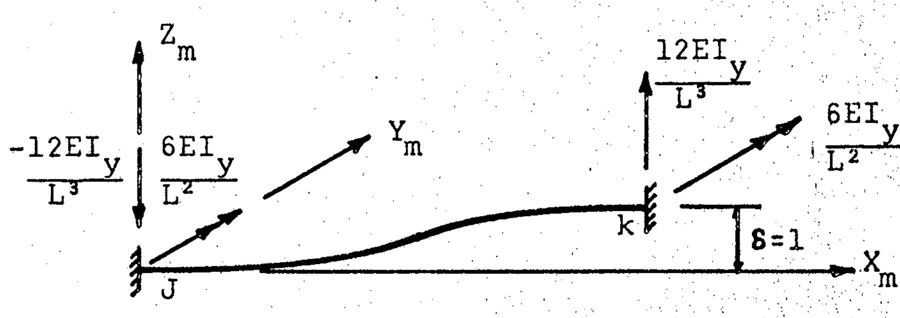
Figure 2. Unit Deformations at J End of Member



(1) Unit rotation about X_m axis



(2) Unit rotation about Y_m axis



(3) Unit translation in Z_m direction

Figure 3. Unit Deformations at K End of Member

are given in Figure 3. The resulting stiffness matrix for a general grid member in terms of the member oriented axes is then given in Equation (2-2) where G is the shear modulus of the member, E is the modulus of elasticity, I_x and I_y are the moments of inertia about the X_m and Y_m axes respectively and L is the length of the member.

Rotation of Stiffness Matrices

While Equation (2-2) represents the stiffness matrix for a general member of a structural grid, it is in terms of the member oriented axes. However, in order to combine the stiffness matrices for all members of a structural grid into the structure stiffness matrix, it is necessary for the stiffness influence coefficients to be in terms of a single reference coordinate system. This reference coordinate system is chosen in relation to the complete structural system and is known as the structure oriented system. Although the member oriented reference system for some members of a particular grid will coincide with the structure oriented system, this is not the case for all members. In the situation where the two coordinate systems are not identical, the member stiffness matrices must be rotated by means of angular transformation matrices to the structure oriented coordinate system.

Referring again to Figure 1, it can be seen that the axes of the member oriented system, denoted by the subscripts m, are rotated an angle α from the structure oriented system, which is indicated by the subscripts s. The transformation matrix relating the deformations of the structure oriented and member oriented systems is of the form

$$\begin{Bmatrix} JI_m \\ J2_m \\ J3_m \end{Bmatrix} = \begin{bmatrix} \text{Cos } \alpha & \text{Sin } \alpha & 0 \\ -\text{Sin } \alpha & \text{Cos } \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} JI_s \\ J2_s \\ J3_s \end{Bmatrix} \qquad (2\text{-}3)$$

or, in general terms

$$\{D_m^J\} = [R_o]\{D_s^J\} \qquad (2\text{-}4)$$

where D_m^J are the deformations at the J end of the member in terms of the member oriented system and D_s^J represents the deformations at the J end in terms of the structure oriented system. R_o is the angular transformation matrix composed of the direction cosines of the member. When deformations at both ends of the member are considered simultaneously, the relationship may be expressed as

$$\begin{Bmatrix} D_m^J \\ \hline D_m^K \end{Bmatrix} = \left[\begin{array}{c|c} R_o & 0 \\ \hline 0 & R_o \end{array}\right] \begin{Bmatrix} D_s^J \\ \hline D_s^K \end{Bmatrix} \qquad (2\text{-}5)$$

or simply

$$\{D_m\} = [Q]\{D_s\} \qquad (2\text{-}6)$$

where Q is the complete angular transformation matrix for a grid structure and is given by

$$Q = \left[\begin{array}{c|c} R_o & 0 \\ \hline 0 & R_o \end{array}\right] \tag{2-7}$$

A similar expression may be written for the actions at the ends of the member in the form:

$$\left\{\begin{array}{c} A_m^J \\ \hline A_m^K \end{array}\right\} = \left[\begin{array}{c|c} R_o & 0 \\ \hline 0 & R_o \end{array}\right] \left\{\begin{array}{c} A_s^J \\ \hline A_s^K \end{array}\right\} \tag{2-8}$$

or, in matrix form:

$$\{A_m\} = [Q]\{A_s\} \tag{2-9}$$

which is similar to the expressions given for angular rotation of the deformations presented in Equation (2-6).

The general stiffness equation for structural systems is given as

$$\{A_s\} = [S_s]\{D_s\} \tag{2-10}$$

in terms of the structure oriented coordinate system and

$$\{A_m\} = [S_m]\{D_m\} \tag{2-11}$$

in terms of the member oriented system. As a general rule, however, it is more convenient to express the actions and deformations in terms of the structure oriented system whereas the physical properties of the member which are used to formulate the member stiffness matrix are given in the member oriented system. Substituting the expressions from Equations (2-6) and (2-9) into Equation (2-11) yields:

$$[Q]\{A_s\} = [S_m][Q]\{D_s\} \qquad (2\text{-}12)$$

When both sides of this equation are premultiplied by the inverse of the Q matrix, Equation (2-12) becomes

$$\{A_s\} = [Q^{-1}][S_m][Q]\{D_s\} \qquad (2\text{-}13)$$

Comparing Equation (2-13) with Equation (2-10), it can be seen that

$$[S_s] = [Q^{-1}][S_m][Q] \qquad (2\text{-}14)$$

which gives the relationship between the member stiffness matrix expressed in terms of the structure oriented coordinate system and the same matrix in terms of the member-oriented coordinate system. It should be noted that because of the particular properties of the angular transformation matrix Q, the inverse Q^{-1} is also equal to the transpose Q^T. This property allows the general stiffness matrix of Equation (2-14) to be obtained without the necessity of inverting a large Q matrix.

Using the relationship derived above and inserting the particular member stiffness matrix for a structural grid, the rotational process of the general grid member stiffness matrix is given in Equation (2-15), and the rotated matrix in expanded form is expressed in Equation (2-16) where C represents cos α and S represents sin α. Therefore, it can be seen that if the member stiffness matrix is written in terms of the member oriented coordinate system, it is

$$
S_s = \left[\begin{array}{ccc|ccc}
\text{Cos}\,\alpha & -\text{Sin}\,\alpha & 0 & 0 & 0 & 0 \\
\text{Sin}\,\alpha & \text{Cos}\,\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \text{Cos}\,\alpha & -\text{Sin}\,\alpha & 0 \\
0 & 0 & 0 & \text{Sin}\,\alpha & \text{Cos}\,\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{ccc|ccc}
\frac{GI_x}{L} & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\
0 & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} \\
0 & -\frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} \\
\hline
-\frac{GI_x}{L} & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\
0 & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} \\
0 & \frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3}
\end{array}\right]
\left[\begin{array}{ccc|ccc}
\text{Cos}\,\alpha & \text{Sin}\,\alpha & 0 & 0 & 0 & 0 \\
-\text{Sin}\,\alpha & \text{Cos}\,\alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \text{Cos}\,\alpha & \text{Sin}\,\alpha & 0 \\
0 & 0 & 0 & -\text{Sin}\,\alpha & \text{Cos}\,\alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \qquad (2\text{-}15)
$$

$$
\left[\begin{array}{ccc|ccc}
\frac{GI_x}{L}C^2+\frac{4EI_y}{L}S^2 & \frac{GI_x}{L}CS-\frac{4EI_y}{L}CS & \frac{6EI_y}{L^2}S & -\frac{GI_x}{L}C^2+\frac{2EI_y}{L}S^2 & -\frac{GI_x}{L}CS-\frac{2EI_y}{L}CS & -\frac{6EI_y}{L^2}S \\
\frac{GI_x}{L}CS-\frac{4EI_y}{L}CS & \frac{GI_x}{L}S^2+\frac{4EI_y}{L}C^2 & \frac{6EI_y}{L^2}C & -\frac{GI_x}{L}CS-\frac{2EI_y}{L}CS & -\frac{GI_x}{L}S^2+\frac{2EI_y}{L}C^2 & \frac{6EI_y}{L^2}C \\
\frac{6EI_y}{L^2}S & \frac{6EI_y}{L^2}C & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}S & -\frac{6EI_y}{L^2}C & -\frac{12EI_y}{L^3} \\
\hline
-\frac{GI_x}{L}C^2+\frac{2EI_y}{L}S^2 & -\frac{GI_x}{L}CS-\frac{2EI_y}{L}CS & \frac{6EI_y}{L^2}S & \frac{GI_x}{L}C^2+\frac{4EI_y}{L}S^2 & \frac{GI_x}{L}CS-\frac{4EI_y}{L}CS & \frac{6EI_y}{L^2}S \\
-\frac{GI_x}{L}CS-\frac{2EI_y}{L}CS & -\frac{GI_x}{L}S^2+\frac{2EI_x}{L}C^2 & -\frac{6EI_y}{L^2}C & \frac{GI_x}{L}CS-\frac{4EI_y}{L}CS & \frac{GI_x}{L}S^2+\frac{4EI_y}{L}C^2 & \frac{6EI_y}{L^2}C \\
-\frac{6EI_y}{L^2}S & \frac{6EI_y}{L^2}C & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}S & \frac{6EI_y}{L^2}C & \frac{12EI_y}{L^3}
\end{array}\right] = S_s \qquad (2\text{-}16)
$$

an easy operation to transform these results to a structurally oriented system. Indeed, the member stiffness matrix can be determined directly in the structure-oriented coordinate system by the use of the expressions given in Equation (2-16).

The necessary direction cosines used in Equation (2-16) are directly obtainable from the coordinates of each end of the grid member. If the coordinates of the J end of the member are denoted by X_j and Y_j and the K end coordinates by X_k and Y_k, the direction cosines for any angle α are given by

$$\cos \alpha = \frac{X_k - X_j}{L} \text{ and } \cos \beta = \sin \alpha = \frac{Y_k - Y_j}{L} \quad (2\text{-}17)$$

where

$$L = \sqrt{(X_k - X_j)^2 + (Y_k - Y_j)^2} \quad (2\text{-}18)$$

and then all necessary quantities used to formulate the member stiffness matrix for a general grid member in terms of the structure oriented coordinate system are known.

Solution of the General Equation

Once the individual member stiffness matrices for each member of the structural grid have been obtained in terms of the structure oriented axes, the joint stiffness matrix may be formed. This formulation is accomplished by the superposition of the individual member stiffness matrices as dictated by the grid joint conditions. In other words, at each joint there are three possible deformations in terms of the structure oriented coordinate system. However, because of the total structure compatability conditions, all members entering any given joint are subjected to the same three possible deformations at that joint. Therefore, the stiffness influence coefficients for all of these members, associated with the same three possible deformations, must be superimposed to reflect the total stiffness of the system. The results of the superposition of the individual member stiffness matrices is known as the joint stiffness matrix.

Upon the completion of the structure joint stiffness matrix the governing stiffness equation

$$\{A_s\} = [S_s]\{D_s\} \tag{2-19}$$

can be solved for the unknown deformations and these deformations will then form the basis of calculations for any other unknown value the designer wishes to compute. The method of solution of Equation (2-19) can take one of two forms. First, the matrix expression may be treated as a set of linear simultaneous equations and may be solved using an elimination or iteration technique such as the Gauss-Seidel or the Crout method. Or, the system may be solved by matrix inversion. In the latter method both sides of Equation (2-19) are premultiplied by the inverse of the joint stiffness matrix S_s as

$$[S_s^{-1}]\{A_s\} = [S_s^{-1}][S_s]\{D_s\} \qquad (2\text{-}20)$$

which reduces to the form

$$[S_s^{-1}]\{A_s\} = \{D_s\} \qquad (2\text{-}21)$$

and provides the solution for the deformation matrix D_s.

CHAPTER III

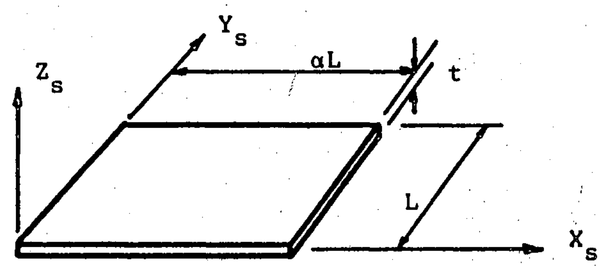
DEVELOPMENT OF A GRID FRAMEWORK MODEL

Introduction

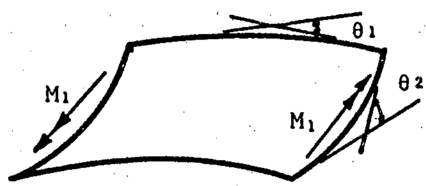
The governing equation for the deflection of a general elastic plate is given as

$$\nabla^4 w = \frac{q}{D} \qquad (3\text{-}1)$$

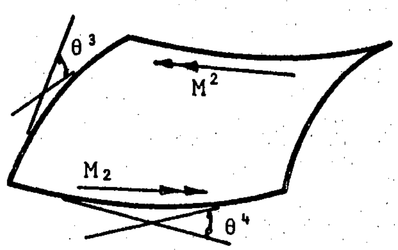
in which w is the vertical deflection, q is the load function, D is the flexural rigidity of the plate, and ∇^4 is the bi-Laplacian operator. In this chapter a rectangular grid framework model will be presented which will allow the numerical solution of the partial differential governing equation, Equation (3-1). Because of the assumptions used in developing the basic elastic plate theory, it is necessary to consider only bending displacements in the development of the grid framework model. Therefore, a grid framework model can be developed by equating the displacements of the grid model to the actual displacements of a plate element subjected to bending and twisting moments. This grid framework model consists of six members--four perimeter beams, each capable of resisting out of plane bending as well as torsion, and two diagonal beams capable of withstanding only out of plane bending. In this manner



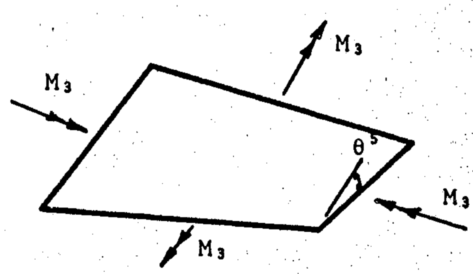
(a) Dimension of elastic plate element



(b) Application of Bending Moment M_1



(c) Application of Bending Moment M_2



(d) Application of Twisting Moment M_3

Figure 4. Element of a General Elastic Plate

a rectangular grid model with five cross sectional properties will define uniquely a rectangular element of a plate.

Properties of a Plate Element

Consider the rectangular elastic plate element shown in Figure 4-a which has exterior dimensions L and αL for the side lengths and t for the thickness. When this plate element is subjected to bending moments M_1, as shown in Figure 4-b, the angular rotation in the direction of bending is given as

$$\theta_1 = \frac{\alpha L M_1}{E(\frac{t^3}{12})} \qquad (3\text{-}2)$$

where E is the modulus of elasticity of the material. Taking Poisson's ratio as μ, the angular rotation in the orthogonal direction is

$$\theta_2 = \frac{\mu L M_1}{E(\frac{t^3}{12})} \qquad (3\text{-}3)$$

Similarly, when the element is subjected to bending moments M_2 along the other two edges, as shown in Figure 4-c, the angular rotation about the X and Y axes are given by

$$\theta_3 = \frac{L M_2}{E(\frac{t^3}{12})} \qquad (3\text{-}4)$$

and

$$\theta_4 = \frac{\mu \alpha L M_2}{E(\frac{t^3}{12})} \qquad (3\text{-}5)$$

Finally, if twisting moments of intensity M_3 are applied to all edges of the element as in Figure 4-d, the resulting angle of twist will be

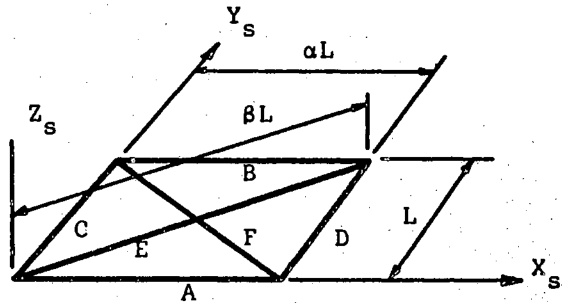
$$\theta_5 = \frac{\alpha L M_3 (1+\mu)}{E(\frac{t^3}{12})} \qquad (3\text{-}6)$$

Therefore, the deformations of a general plate element subjected to bending and twisting moments are known.

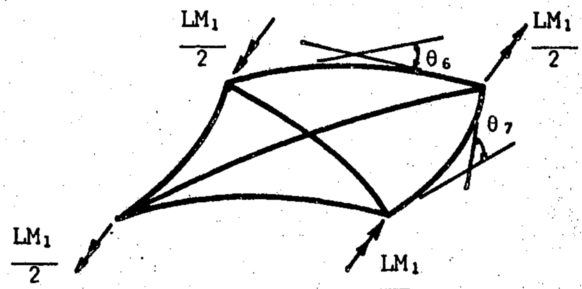
Development of the Stiffness Equation for the Equivalent Grid

An equivalent grid model of the plate element can be constructed of six members. The physical properties of the grid members can then be determined by equating the rotation of the grid nodes with those of the same size plate element. It is important that both the plate element and the equivalent grid structure be subjected to statically equivalent loads.

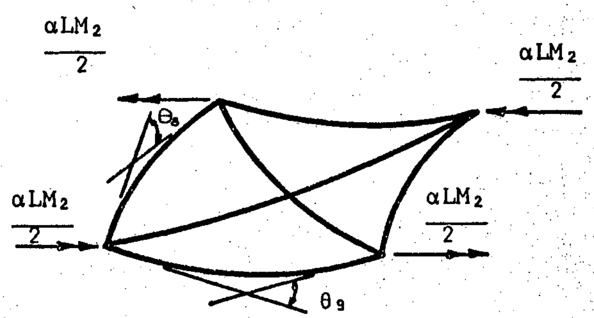
Consider, for example, a structural grid as shown in Figure 5-a composed of six members. The physical dimensions of the grid are L and αL as the lengths of the edge members and βL as the length of the diagonals. The two end members of length L have moments of inertia about



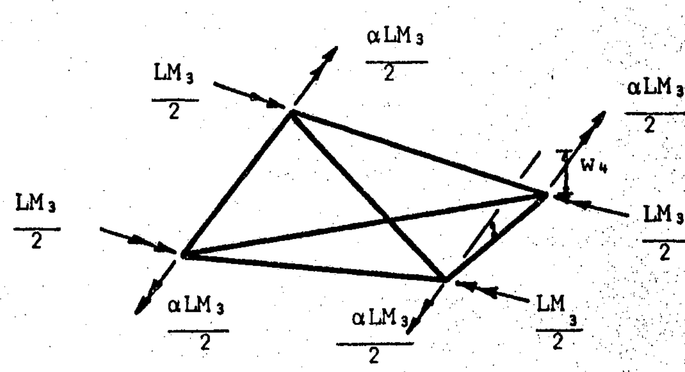
(a) Dimension of equivalent grid structure



(b) Applications of equivalent bending moments M_1



(c) Applications of equivalent bending moments M_2



(d) Applications of equivalent twisting moments M_3

Figure 5. Equivalent Grid Structure

their Y axis designated as I_e and torsional constants designated as GJ_e/E. Similarly, the two side members have moments of inertia and torsional constants equal to I_s and GJ_s/E respectively. The diagonal members are assumed to resist no torsion but have a moment of inertia equal to I_d.

The joint stiffness matrix for the complete system can now be formulated by substituting the above member properties into the general grid member stiffness matrix of Equation (2-16) and then assembling all of the resulting stiffness influence coefficients into the joint stiffness matrix. The location of the stiffness influence coefficients in the joint stiffness matrix is determined by the number of the deformation causing the action to occur. At each joint of the model structure, three deformations and three corresponding actions can occur. As shown in Figure 1, the J end actions are designated J1, J2, and J3 and the K end actions are designated K1, K2, and K3. If a numbering system is chosen for these actions that corresponds to the number of the joint, automation of computation is enhanced. For example, consider the system where

$$
\begin{aligned}
J1 &= 3n-2 \\
J2 &= 3n-1 \qquad (3\text{-}7) \\
J3 &= 3n
\end{aligned}
$$

where n is the number of the joint. As can be seen, a logical numbering relationship then exists between the number of actions and the joint designations. In addition, a similar set of expressions can be written for the

K end of the member. To formulate the joint stiffness matrix for the model structure, consider the member stiffness matrices for each member. First, for the side members A and B, the member stiffness matrices are as given in Equation (3-8). Secondly, for the end members where $\sin \alpha = 1.0$ and $\cos \alpha = 0.0$; the member stiffness matrices are of the form of Equation (3-9). For the diagonal member E, $\sin \alpha = 1/\beta$ and $\cos \alpha = \alpha/\beta$ and for diagonal member F, $\sin \alpha = 1/\beta$ and $\cos \alpha = \alpha/\beta$. The member stiffness matrices for these members are given in Equations (3-10) and (3-11) respectively.

Once the member stiffness matrices are complete, the joint stiffness matrix, and then the governing stiffness equation, can be assembled. This final matrix equation for the structural grid model is of the form of Equation (3-12) where A_i^J denotes action number i occurring at joint j and D_i^j is the deformation in the i direction of joint j.

Solution of the Equivalent Grid Stiffness Equation

In order to obtain the physical constants of the equivalent grid framework members such that the model represents the actions of a plate element, the deformations of the grid framework model are equated to the corresponding deformations of the plate element. The deformations of the grid model may be found by solving Equation (3-12) when the model is alternately subjected to loads

$$[S_m]_A = [S_m]_B = \left[\begin{array}{ccc|ccc}
\frac{GJ_s}{\alpha L} & 0 & 0 & -\frac{GJ_s}{\alpha L} & 0 & 0 \\
0 & \frac{4EI_s}{\alpha L} & -\frac{6EI_s}{\alpha^2 L^2} & 0 & \frac{2EI_s}{\alpha L} & \frac{6EI_s}{\alpha^2 L^2} \\
0 & -\frac{6EI_s}{\alpha^2 L^2} & \frac{12EI_s}{\alpha^3 L^3} & 0 & -\frac{6EI_s}{\alpha^2 L^2} & -\frac{12EI_s}{\alpha^3 L^3} \\
\hline
-\frac{GJ_s}{\alpha L} & 0 & 0 & \frac{GJ_s}{\alpha L} & 0 & 0 \\
0 & \frac{2EI_s}{\alpha L} & -\frac{6EI_s}{\alpha^2 L^2} & 0 & \frac{4EI_s}{\alpha L} & \frac{6EI_s}{\alpha^2 L^2} \\
0 & \frac{6EI_s}{\alpha^2 L^2} & -\frac{12EI_s}{\alpha^3 L^3} & 0 & \frac{6EI_s}{\alpha^2 L^2} & \frac{12EI_s}{\alpha^3 L^3}
\end{array}\right] \qquad (3\text{-}8)$$

$$[S_m]_C=[S_m] = \left[\begin{array}{ccc|ccc} \frac{4EI_e}{L} & 0 & \frac{6EI_e}{L^2} & \frac{2EI_e}{L} & 0 & -\frac{6EI_e}{L^2} \\ 0 & \frac{GJ_e}{L} & 0 & 0 & -\frac{GJ_e}{L} & 0 \\ \frac{6EI_e}{L^2} & 0 & \frac{12EI_e}{L^3} & \frac{6EI_e}{L^2} & 0 & -\frac{12EI_e}{L^3} \\ \hline \frac{2EI_e}{L} & 0 & \frac{6EI_e}{L^2} & \frac{4EI_e}{L} & 0 & -\frac{6EI_e}{L^2} \\ 0 & -\frac{GJ_e}{L} & 0 & 0 & \frac{GJ_e}{L} & 0 \\ -\frac{6EI_e}{L^2} & 0 & -\frac{12EI_e}{L^3} & -\frac{6EI_e}{L^2} & 0 & \frac{12EI_e}{L^3} \end{array}\right] \quad (3\text{-}9)$$

$$
[S_m]_E = \left[\begin{array}{ccc|ccc}
\frac{4EI_d}{\beta L}\left(\frac{1}{\beta^2}\right) & -\frac{4EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & +\frac{2EI_d}{\beta L}\left(\frac{1}{\beta^2}\right) & -\frac{2EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) \\
-\frac{4EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{4EI_d}{\beta L}\left(\frac{\alpha^2}{\beta^2}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & -\frac{2EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{2EI_d}{\beta L}\left(\frac{\alpha^2}{\beta^2}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) \\
\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & \frac{12EI_d}{\beta^3 L^3} & \frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & -\frac{12EI_d}{\beta^3 L^3} \\
\hline
\frac{2EI_d}{\beta L}\left(\frac{1}{\beta^2}\right) & -\frac{2EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & \frac{4EI_d}{\beta L}\left(\frac{1}{\beta^2}\right) & -\frac{4EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) \\
-\frac{2EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{2EI_d}{\beta L}\left(\frac{\alpha^2}{\beta^2}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & -\frac{4EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{4EI_d}{\beta L}\left(\frac{\alpha^2}{\beta^2}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) \\
-\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & -\frac{12EI_d}{\beta^3 L^3} & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & \frac{12EI_d}{\beta^3 L^3}
\end{array}\right] \qquad (3\text{-}10)
$$

$$
[S_m]_F = \left[\begin{array}{ccc|ccc}
\frac{4EI_d}{\beta L}\left(\frac{1}{\beta^2}\right) & \frac{4EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & \frac{2EI_d}{\beta L}\left(\frac{1}{\beta^2}\right) & \frac{2EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) \\
\frac{4EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{4EI_d}{\beta L}\left(\frac{\alpha^2}{\beta^2}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & \frac{2EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{2EI_d}{\beta L}\left(\frac{\alpha^2}{\beta^2}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) \\
-\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & \frac{12EI_d}{\beta^3 L^3} & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & -\frac{12EI_d}{\beta^3 L^3} \\
\hline
\frac{2EI_d}{\beta L}\left(\frac{1}{\beta^2}\right) & \frac{2EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & \frac{4EI_d}{\beta L}\left(\frac{1}{\beta^2}\right) & +\frac{4EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) \\
\frac{2EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{2EI_d}{\beta L}\left(\frac{\alpha^2}{\beta^2}\right) & -\frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & \frac{4EI_d}{\beta L}\left(\frac{\alpha}{\beta^2}\right) & \frac{4EI_d}{\beta L}\left(\frac{\alpha^2}{\beta^2}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) \\
\frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & -\frac{12EI_d}{\beta^3 L^3} & \frac{6EI_d}{\beta^2 L^2}\left(\frac{1}{\beta}\right) & \frac{6EI_d}{\beta^2 L^2}\left(\frac{\alpha}{\beta}\right) & \frac{12EI_d}{\beta^3 L^3}
\end{array}\right] \qquad (3\text{-}11)
$$

$$
\begin{bmatrix} A_1^1 \\ A_2^1 \\ A_3^1 \\ A_1^2 \\ A_2^2 \\ A_3^2 \\ \hline A_1^3 \\ A_2^3 \\ A_3^3 \\ A_1^4 \\ A_2^4 \\ A_3^4 \end{bmatrix}
=
\left[\begin{array}{cccccc:cccccc}
\frac{GJ_s}{\alpha L}+\frac{4EI_e}{L}+\frac{4EI_d}{\beta^3 L} & -\frac{4\alpha EI_d}{\beta^3 L} & -\frac{6EI_e}{L^2}-\frac{6EI_d}{\beta^3 L} & -\frac{GJ_s}{\alpha L} & 0 & 0 & \frac{2EI_e}{L} & 0 & \frac{6EI_e}{L^2} & \frac{2EI_d}{\beta^3 L} & -\frac{2\alpha EI_d}{\beta^3 L} & \frac{6EI_d}{\beta^3 L^2} \\
-\frac{4\alpha EI_d}{\beta^3 L} & \frac{4EI_s}{\alpha L}+\frac{GJ_e}{L}+\frac{4\alpha^2 EI_d}{\beta^3 L} & \frac{6EI_s}{\alpha^2 L^2}+\frac{6\alpha EI_d}{\beta^3 L^2} & 0 & \frac{2EI_s}{\alpha L} & -\frac{6EI_s}{\alpha^2 L^2} & 0 & -\frac{GJ_e}{L} & 0 & -\frac{2\alpha EI_d}{\beta^3 L} & \frac{2\alpha^2 EI_d}{\beta^3 L} & \frac{6\alpha EI_d}{\beta^3 L^2} \\
-\frac{6EI_e}{L^2}-\frac{6EI_d}{\beta^3 L^2} & \frac{6EI_s}{\alpha^2 L^2}+\frac{6\alpha EI_d}{\beta^3 L^2} & \frac{12EI_s}{\alpha^3 L^3}+\frac{12EI_e}{L^3}+\frac{12EI_d}{\beta^3 L^3} & 0 & \frac{6EI_s}{\alpha^2 L^2} & -\frac{12EI_s}{\alpha^3 L^3} & -\frac{6EI_e}{L^2} & 0 & -\frac{12EI_e}{L^3} & -\frac{6EI_d}{\beta^3 L^2} & \frac{6\alpha EI_d}{\beta^3 L^2} & -\frac{12EI_d}{\beta^3 L^3} \\
-\frac{GJ_s}{\alpha L} & 0 & 0 & \frac{GJ_s}{\alpha L}+\frac{4EI_e}{L}+\frac{4EI_d}{\beta^3 L} & \frac{4\alpha EI_d}{\beta^3 L} & \frac{6EI_e}{L^2}-\frac{6EI_d}{\beta^3 L^2} & \frac{2EI_d}{\beta^3 L} & \frac{2\alpha EI_d}{\beta^3 L} & \frac{6EI_d}{\beta^3 L^2} & \frac{2EI_e}{L} & 0 & \frac{6EI_e}{L^2} \\
0 & \frac{2EI_s}{\alpha L} & \frac{6EI_s}{\alpha^2 L^2} & \frac{4\alpha EI_d}{\beta^3 L} & \frac{4EI_s}{\alpha L}+\frac{GJ_e}{L}+\frac{4\alpha^2 EI_d}{\beta^3 L} & \frac{6EI_s}{\alpha^2 L^2}-\frac{6\alpha EI_d}{\beta^3 L^2} & \frac{2\alpha EI_d}{\beta^3 L} & \frac{2\alpha^2 EI_d}{\beta^3 L} & \frac{6\alpha EI_d}{\beta^3 L^2} & 0 & -\frac{GJ_e}{L} & 0 \\
0 & -\frac{6EI_s}{\alpha^2 L^2} & -\frac{12EI_s}{\alpha^3 L^3} & -\frac{6EI_e}{L^2}-\frac{6EI_d}{\beta^3 L^2} & -\frac{6EI_s}{\alpha^2 L^2}-\frac{6\alpha EI_d}{\beta^3 L^2} & \frac{12EI_s}{\alpha^3 L^3}+\frac{12EI_e}{L^3}+\frac{12EI_d}{\beta^3 L^3} & -\frac{6EI_d}{\beta^3 L^2} & -\frac{6\alpha EI_d}{\beta^3 L^2} & -\frac{12EI_d}{\beta^3 L^3} & -\frac{6EI_e}{L^2} & 0 & -\frac{12EI_e}{L^3} \\
\hdashline
\frac{2EI_e}{L} & 0 & -\frac{6EI_e}{L^2} & \frac{2EI_d}{\beta^3 L} & \frac{2\alpha EI_d}{\beta^3 L} & -\frac{6EI_d}{\beta^3 L^2} & \frac{GJ_s}{\alpha L}+\frac{4EI_e}{L}+\frac{4EI_d}{\beta^3 L} & \frac{4\alpha EI_d}{L^2} & \frac{6EI_e}{L^2}+\frac{6EI_d}{\beta^3 L^2} & -\frac{GJ_s}{\alpha L} & 0 & 0 \\
0 & -\frac{GJ_e}{L} & 0 & \frac{2\alpha EI_d}{\beta^3 L} & \frac{2\alpha^2 EI_d}{\beta^3 L} & -\frac{6\alpha EI_d}{\beta^3 L^2} & \frac{4\alpha EI_d}{\beta^3 L} & \frac{4EI_s}{\alpha L}+\frac{GJ_e}{L}+\frac{4\alpha^2 EI_d}{\beta^3 L} & \frac{6EI_s}{\alpha^2 L^2}+\frac{6\alpha EI_d}{\beta^3 L^2} & 0 & \frac{2EI_s}{\alpha L} & -\frac{6EI_s}{\alpha^2 L^2} \\
\frac{6EI_e}{L^2} & 0 & -\frac{12EI_e}{L^3} & \frac{6EI_d}{\beta^3 L^2} & \frac{6\alpha EI_d}{\beta^3 L^2} & -\frac{12EI_d}{\beta^3 L^3} & \frac{6EI_e}{L^2}+\frac{6EI_d}{\beta^3 L^2} & \frac{6EI_s}{\alpha^2 L^2}+\frac{6\alpha EI_d}{\beta^3 L^2} & \frac{12EI_s}{\alpha^3 L^3}+\frac{12EI_e}{L^3}+\frac{12EI_d}{\beta^3 L^3} & 0 & \frac{6EI_s}{\alpha^2 L^2} & -\frac{12EI_s}{\alpha^3 L^3} \\
\frac{2EI_d}{\beta^3 L} & -\frac{2\alpha EI_d}{\beta^3 L} & -\frac{6EI_d}{\beta^3 L^2} & \frac{2EI_e}{L} & 0 & -\frac{6EI_e}{L^2} & -\frac{GJ_s}{\alpha L} & 0 & 0 & \frac{GJ_s}{\alpha L}+\frac{4EI_e}{L}+\frac{4EI_d}{\beta^3 L} & \frac{4\alpha EI_d}{\beta^3 L} & \frac{6EI_e}{L^2}+\frac{6EI_d}{\beta^3 L^2} \\
-\frac{2\alpha EI_d}{\beta^3 L} & \frac{2\alpha^2 EI_d}{\beta^3 L} & \frac{6\alpha EI_d}{\beta^3 L^2} & 0 & -\frac{GJ_e}{L} & 0 & 0 & \frac{2EI_s}{\alpha L} & \frac{6EI_s}{\alpha^2 L^2} & \frac{4\alpha EI_d}{\beta^3 L} & \frac{4EI_s}{\alpha L}+\frac{GJ_e}{L}+\frac{4\alpha^2 EI_d}{\beta^3 L} & -\frac{6EI_s}{\alpha^2 L^2}-\frac{6\alpha EI_d}{\beta^3 L^2} \\
\frac{6EI_d}{\beta^3 L^2} & \frac{6\alpha EI_d}{\beta^3 L^2} & -\frac{12EI_d}{\beta^3 L^3} & \frac{6EI_e}{L^2} & 0 & -\frac{12EI_e}{L^3} & 0 & -\frac{6EI_s}{\alpha^2 L^2} & \frac{12EI_s}{\alpha^3 L^3} & \frac{6EI_e}{L}+\frac{6EI_d}{\beta^3 L^2} & \frac{6EI_s}{\alpha^2 L^2}-\frac{6\alpha EI_d}{\beta^3 L^2} & \frac{12EI_s}{\alpha^3 L^3}+\frac{12EI_e}{L^3}+\frac{12EI_d}{\beta^3 L^3}
\end{array}\right]
\begin{bmatrix} \theta_1^1 \\ \theta_2^1 \\ w^1 \\ \theta_1^2 \\ \theta_2^2 \\ w^2 \\ \hline \theta_1^3 \\ \theta_2^3 \\ w^3 \\ \theta_1^4 \\ \theta_2^4 \\ w^4 \end{bmatrix}
\tag{3-12}
$$

$LM_1/2$, $\alpha LM_2/2$ and the twisting moments as shown in Figures 5-b, 5-c, and 5-d. For each loading condition, two independent simultaneous equations are formed from the matrix relationships of Equation (3-12). By equating these independent equations, expressions are obtained for the deformations of the grid framework model in terms of the unknown member properties as

$$\theta_6 = \frac{L^2 M_1 \alpha}{2E} \times \frac{\beta^3 I_e + I_d}{\beta^3 I_e I_s + I_d I_s + \alpha^3 I_d I_e} \quad (3\text{-}13)$$

$$\theta_7 = \frac{L^2 M_1 \alpha^2}{2E} \times \frac{I_d}{\beta^3 I_e I_s + I_d I_s + \alpha^3 I_d I_e} \quad (3\text{-}14)$$

$$\theta_8 = \frac{L^2 M_2 \alpha}{2E} \times \frac{\beta^3 I_s + \alpha^3 I_d}{\beta^3 I_e I_s + I_d I_s + \alpha^3 I_d I_e} \quad (3\text{-}15)$$

$$\theta_9 = \frac{L^2 M_2 \alpha^3}{2E} \times \frac{I_d}{\beta^3 I_e I_s + I_d I_s + \alpha^3 I_d I_e} \quad (3\text{-}16)$$

$$\theta_{10} = \frac{L^2 M_3 \beta^3 \alpha}{2E[\beta^3 (\frac{GJ_s}{E}) + 2\alpha I_d]} \quad (3\text{-}17)$$

and

$$w_4 = -L\theta_{10} \quad (3\text{-}18)$$

Equating corresponding deformations, it can be seen that

$$\theta_1 = \theta_6 \quad (3\text{-}19)$$

$$\theta_2 = \theta_7 \quad (3\text{-}20)$$

$$\theta_3 = \theta_8 \tag{3-21}$$

$$\theta_4 = \theta_9 \tag{3-22}$$

and

$$\theta_5 = \theta_{10} \tag{3-23}$$

Expanding and solving Equations (3-19) through (3-23), the final beam properties are computed as

$$I_e = \frac{(\alpha^2 - \mu)L}{2\alpha(1 - \mu^2)} \cdot \frac{t^3}{12} \tag{3-24}$$

$$I_s = \frac{(1 - \alpha^2\mu)L}{2(1 - \mu^2)} \cdot \frac{t^3}{12} \tag{3-25}$$

$$I_d = \frac{\mu\beta^3 L}{2\alpha(1 - \mu^2)} \cdot \frac{t^3}{12} \tag{3-26}$$

$$\frac{GJ_s}{E} = \frac{(1 - 3\mu)L}{2(1 - \mu^2)} \cdot \frac{t}{12} \qquad (3\text{-}27$$

and

$$\frac{GJ_e}{E} = \frac{\alpha(1 - 3\mu)L}{2(1 - \mu^2)} \cdot \frac{t^3}{12} \tag{3-28}$$

Using Equations (3-24) through (3-28), a plate may be idealized into an equivalent grid framework model which, when analyzed by any standard frame or grid analysis computer program, will represent the actions of the original plate structure.

CHAPTER IV

DEVELOPMENT OF MODEL FOR PLATES ON ELASTIC FOUNDATIONS

Introduction

In the previous chapters the stiffness method of structural analysis was discussed and used to develop a grid framework model for the analysis of an elastic plate. The grid framework model, therefore, may be considered to be a means of solving the fourth order partial differential equation

$$\nabla^4 w = \frac{q}{D} \qquad (4\text{-}1)$$

which describes the deflection surface of an elastic plate, by the matrix expression

$$\{A\} = [S]\{D\} \qquad (4\text{-}2)$$

However, the development of the grid framework model in Chapter II did not include a discussion of support conditions for the elastic plate. While simple supports, fixed supports or free edges, as found in general elastic plate problems pose little difficulty for the grid framework method, the inclusion of elastic support conditions necessitates a re-formulation of the governing equation,

Equation (4-2). The consideration of these foundation support reactions and the accompanying modifications of the grid framework stiffness equations are discussed in the following sections.

Effects of Elastic Foundation Forces

The basic assumptions usually made in the analysis of plates resting on elastic foundations is that the intensity of the subgrade reaction is proportional to the deflection of the plate. This foundation reaction is expressed by the function kw where w is the deflection of the plate and k, expressed in pounds per square inch per inch of deflection, is known as the "modulus of the foundation." As discussed previously, this assumption was first made by E. Winkler, and the resulting foundation system is usually referred to as a Winkler foundation.

In considering the effects of the elastic foundation, the foundation reaction, kw, must be incorporated into the governing equation. Including these effects as part of the load expression in Equation (4-1) the governing equation becomes

$$\nabla^4 w = \frac{q - kw}{D} \qquad (4\text{-}3)$$

The solution of this equation by a grid framework model will now be investigated.

Derivation of Model Matrix Equation

Consider an equivalent grid framework model, similar to that discussed in Chapter III, resting on a system of spring supports. Such a model is shown in Figure 6. If

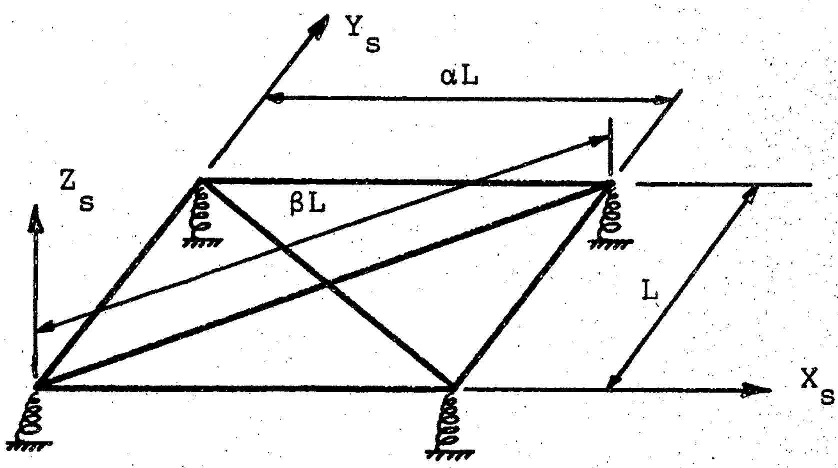


Figure 6. Equivalent Grid Model for a Plate With an Elastic Foundation

the spring constants are given as c, then a general corner reaction can be expressed as cw_i where w_i is defined as the vertical deflection of corner i. When the vertical deformation at each joint is considered, and these deformations are expressed in matrix notation, the vertical force matrix, F_v, is of the form

$$\{F_v\} = c\{W\} \qquad (4\text{-}4)$$

where c is a scalar multiplier representing the foundation spring constant and W is a column matrix of the vertical deflections. Including this expression of vertical foundation forces into the governing matrix equation, Equation (4-2) becomes

$$\{A\} + \{F_V\} = [S]\{D\} \quad \text{or} \quad \{A\} + c\{W\} = [S]\{D\} \qquad (4\text{-}5)$$

or, rearranging terms

$$\{A\} = [S]\{D\} - c\{W\} \qquad (4\text{-}6)$$

Equation (4-6) may now be written as

$$\{A\} = [S]\{D\} - c[I]\{W\} \qquad (4\text{-}7)$$

without changing the value of the original equation since I represents the identity or unit matrix.

Consider now only the term c[I]{W} of Equation (4-7). Written in its expanded form the term may be expressed as

$$c[I]\{W\} = c \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & 0 & & & & & \cdot \\ 0 & 0 & 1 & 0 & & & & & \cdot \\ 0 & 0 & 0 & 1 & & & & & \cdot \\ \cdot & & & & \cdot & & & & \cdot \\ \cdot & & & & & \cdot & & & \cdot \\ \cdot & & & & & & \cdot & & \cdot \\ \cdot & & & & & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ w_1 \\ 0 \\ 0 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ cw_1 \\ 0 \\ 0 \\ cw_2 \\ \cdot \\ \cdot \\ cw_n \end{bmatrix} \qquad (4\text{-}8)$$

However, because of the particular laws of matrix multiplication, the results of Equation (4-8) may be obtained in more than one manner. For example, a new matrix K may be defined in such a way that

$$[K]\{D\} = c[I]\{W\} \tag{4-9}$$

or

$$[K]\begin{bmatrix} 0_{1x} \\ 0_{1y} \\ w_1 \\ 0_{2x} \\ 0_{2y} \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ cw_1 \\ 0 \\ 0 \\ cw_2 \\ \cdot \\ \cdot \\ \cdot \\ cw_n \end{bmatrix} \tag{4-10}$$

For this relationship to be valid, the matrix K must be of the form

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & & \cdot \\ 0 & 0 & c & 0 & 0 & 0 & & & & & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & & \cdot \\ 0 & 0 & 0 & 0 & 0 & c & & & & & \cdot \\ \cdot & & & & & & \cdot & & & & \cdot \\ \cdot & & & & & & & \cdot & & & \cdot \\ \cdot & & & & & & & & \cdot & & \cdot \\ \cdot & & & & & & & & & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c \end{bmatrix} \tag{4-11}$$

where K is a diagonal matrix, and every third term of the diagonal represents the foundation spring constant c, and all other terms are equal to zero.

Substituting Equation (4-9) into Equation (4-7), the governing matrix equation may now be written as

$$\{A\} = [S]\{D\} - [K]\{D\} \qquad (4\text{-}12)$$

or, collecting terms,

$$\{A\} = [S - K]\{D\} \qquad (4\text{-}13)$$

Comparing Equation (4-13) with the general stiffness matrix equation, Equation (4-2), it can be seen that for an equivalent grid framework model for a plate resting upon an elastic foundation the stiffness matrix is of the form

$$[S_k] = [S - K] \qquad (4\text{-}14)$$

where S represents the grid model stiffness matrix, and K represents the foundation spring constant matrix. Therefore, the stiffness matrix for a plate resting upon an elastic foundation, written in its expanded form, is given as

$$S_k = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & \cdot & \cdot & \cdot & \cdot & S_{1n} \\ S_{21} & S_{22} & S_{23} & S_{24} & & & & & \cdot \\ S_{31} & S_{32} & (S_{33}-c) & S_{34} & & & & & \cdot \\ S_{41} & S_{42} & S_{43} & S_{44} & & & & & \cdot \\ \cdot & & & & \cdot & & & & \cdot \\ \cdot & & & & & \cdot & & & \cdot \\ \cdot & & & & & & \cdot & & \cdot \\ \cdot & & & & & & & \cdot & \cdot \\ S_{n1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (S_{nn}-c) \end{bmatrix} \quad (4\text{-}15)$$

The significance of Equation (4-15) is that plates resting on elastic foundations can be analyzed by the grid framework method, using the same equations that are necessary for the analysis of general grids. The difference in the analysis of the two structural systems are the inclusion of the foundation spring constants in every third term on the diagonal of the overall joint stiffness matrix, and the difficulty in using the resulting internal member forces. These resulting member moments and shears represent the localized effects of the individual member stiffness and not the total joint forces which are of interest in the design.

In the analysis of elastic plates, the problem is usually considered to be solved when the deflections of the middle surface are known. The grid framework method provides these results. Once the deflections are known, the internal plate forces can be obtained by using a number of different approaches. An example of one numerical technique, the finite difference method, is presented in Appendix C.

CHAPTER V

APPLICATION OF THE METHOD

Introduction

While the development of theory and the derivation of equations are important in the field of structural engineering, the application of the methods to actual structural problems is of equal importance. In this section the analysis of four selected plates is presented. Each plate was completely analyzed, and vertical deflections were computed for all points on the middle surface. These analyses were accomplished on a General Electric 650 digital computer using a program written to analyze structural grids. After geometric properties of the members and the complete structure were designated, the following procedure of analysis was used:

1. Physical properties of each member were computed by evaluating Equations (3-24) through (3-28).
2. The thirty-six stiffness influence coefficients of Equation (2-16) were determined for each member, and the member stiffness matrix was formed.

3. Member stiffness matrices were next assembled into the structure joint stiffness matrix.
4. The effects of the elastic foundation forces were included as indicated by Equation (4-15).
5. The structure joint stiffness matrix was inverted.
6. External structure loads were read, and the resulting load vectors were formed.
7. Equation (4-13) was solved for the unknown joint displacements.

The basic logic of the computer program is given above, and a complete listing of the program is presented in Appendix A. Input data provided geometric properties of members and structure configuration, the modulus of the elastic foundation, and the external loads. Output was presented as the grid joint displacements, and all other processes were internal to the computer.

Solution of a Centrally Loaded Square Plate

A 12" x 12" square plate, centrally loaded, is analyzed in the first example. The thickness of the plate is 1/4 inch, Poisson's ratio equals 0.3 and the modulus of elasticity is taken as 30,000,000 psi. The modulus of the foundation is assumed as 200 lb./in^2/in., and a 1000 lb. load was applied at the center of the plate. The equivalent grid framework used to analyze this plate is as shown in Figure 7. The grid increment was chosen as 2 in., and the member and joint numbers are as shown. Due to the symmetrical properties of the plate system, it was necessary only to consider one quadrant of the equivalent grid model. The computer input necessary to analyze this plate problem is presented in Appendix B, and the final deformations are tabulated in Table I. A similar problem was solved by N. Willems (23) using the Ritz method of analysis, and the deflection at the center of the plate,

TABLE I

DEFLECTIONS OF A CENTRALLY LOADED SQUARE PLATE

Joint	Deformation	Joint	Deformation
1	-0.011838 in.	9	-0.026240 in.
2	-0.019723 in.	10	-0.035820 in.
3	-0.026240 in.	11	-0.045531 in.
4	-0.028841 in.	12	-0.050283 in.
5	-0.019723 in.	13	-0.028841 in.
6	-0.028218 in.	14	-0.039139 in.
7	-0.035820 in.	15	-0.050283 in.
8	-0.039139 in.	16	-0.056380 in.

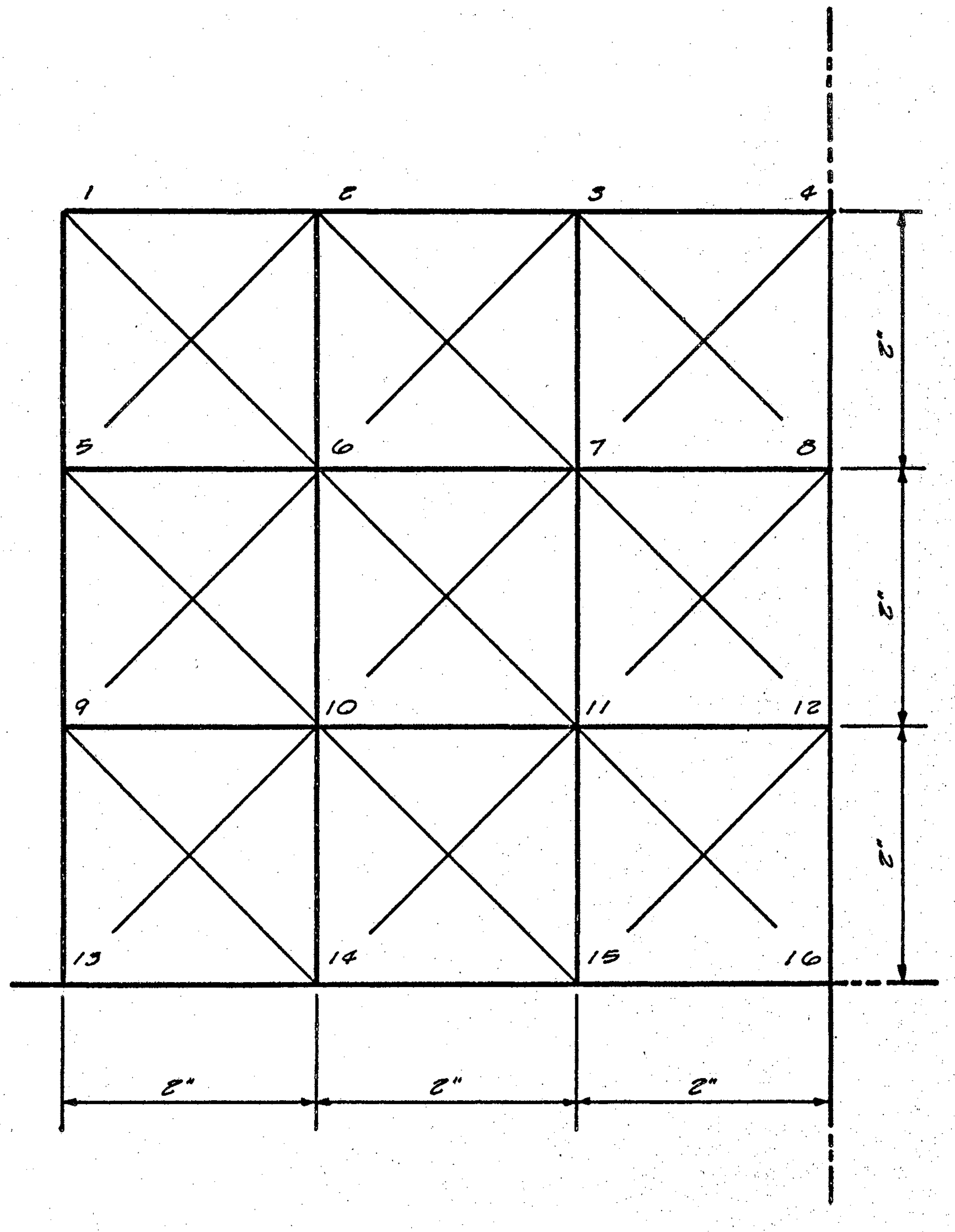


Figure 7. Equivalent Grid Framework Model for a Square Plate Centrally Loaded

under the load, was determined to be -0.0528 in. The results of the grid framework solution indicate that the maximum deflection at joint 16, the point of loading, is -0.056380 in. This is a difference of 0.00370 in. or approximately 6%. However, the Ritz method itself is an approximate solution, involving the truncation of a double trigonometric series, and the analysis was based upon the Westergaard assumption of a plate with infinite dimensions in one direction.

Solution of a Centrally Loaded Circular Plate

A second problem which demonstrates the application of the grid framework method in cases of nonrectilinear boundaries and also provides an additional check on the accuracy of the method is the solution of a centrally loaded circular plate. A ten inch diameter circular plate of 1/3 inch thickness is chosen. Poisson's ratio is set equal to 0.3, the modulus of elasticity is given as 30,000,000 psi, and the foundation constant is again set at 200 lb./in.^{2}/in. The plate is centrally loaded with a 640 pound load. The equivalent grid framework model is shown in Figure 8, and the final joint deformations are given in Table II.

A similar problem was analyzed by Timoshenko and Woinowsky-Kreiger (8), and a comparison of results is again possible. The maximum deflection under the load point was given in Table II as -0.040365 in., and the

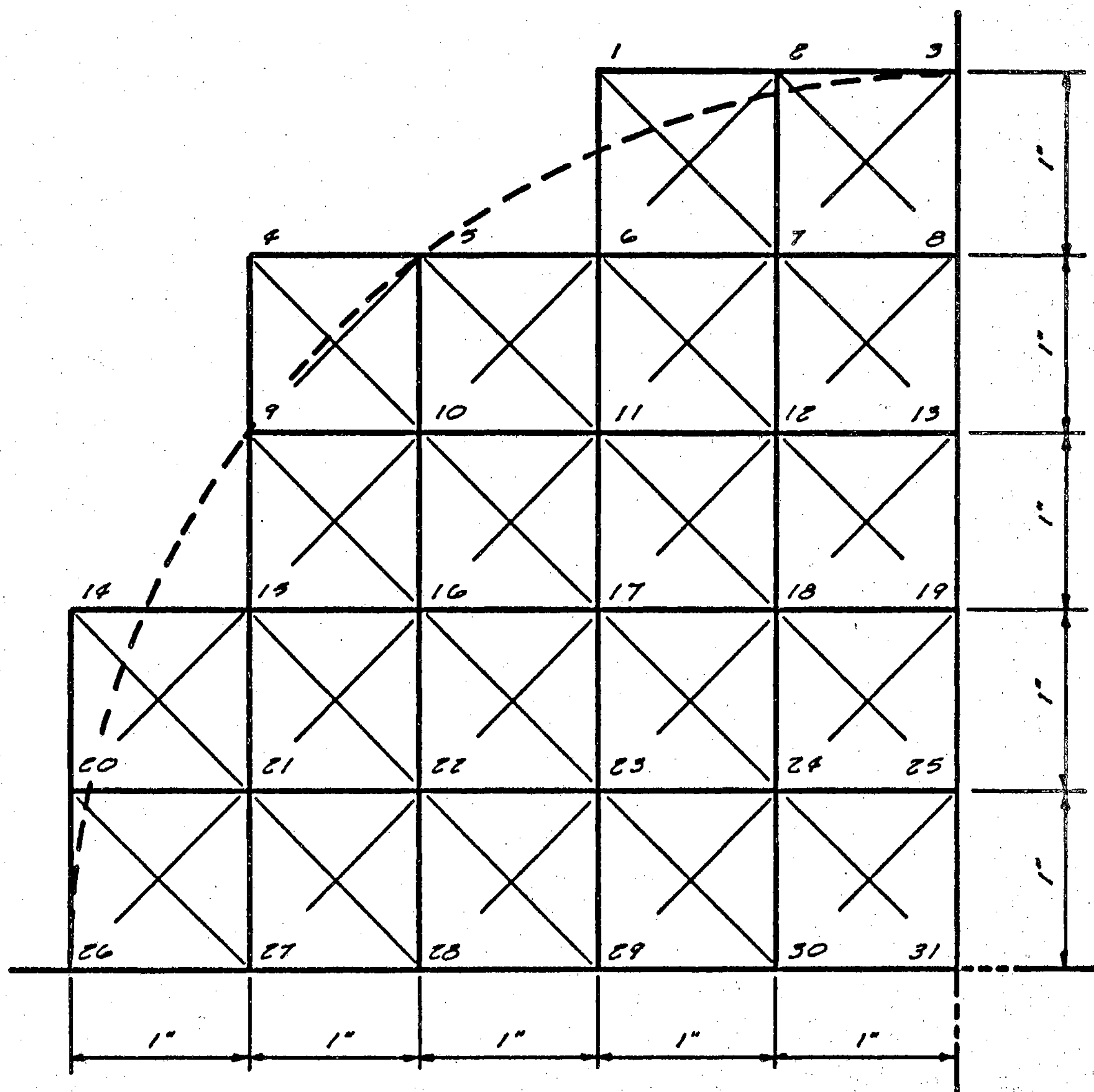


Figure 8. Equivalent Grid Framework Model for a Circular Plate Centrally Loaded

TABLE II

DEFLECTIONS OF A CENTRALLY LOADED CIRCULAR PLATE

Joint	Deformation	Joint	Deformation
1	-0.039668 in.	17	-0.040028 in.
2	-0.039714 in.	18	-0.040123 in.
3	-0.039732 in.	19	-0.040160 in.
4	-0.039598 in.	20	-0.039714 in.
5	-0.039699 in.	21	-0.039843 in.
6	-0.039785 in.	22	-0.039982 in.
7	-0.039843 in.	23	-0.040123 in.
8	-0.039864 in.	24	-0.040242 in.
9	-0.039699 in.	25	-0.040294 in.
10	-0.039808 in.	26	-0.039732 in.
11	-0.039908 in.	27	-0.039864 in.
12	-0.039982 in.	28	-0.040010 in.
13	-0.040010 in.	29	-0.040160 in.
14	-0.039668 in.	30	-0.040294 in.
15	-0.039785 in.	31	-0.040365 in.
16	-0.039908 in.		

deflection at the boundary of the plate was determined to be -0.03970 inches. Timoshenko computed deflections of -0.04300 in. at the center and -0.03910 in. at the edges of a similar plate. These results are within 6% at the center and approximately 2% at the boundary points. The same problem was again solved by Timoshenko using an approximate finite difference approach, and a deflection of -0.04180 in. was determined for the center point. This solution is within 3% of that obtained by the equivalent grid framework method.

Solution of an Edge Loaded Pavement Slab

A more realistic problem involving plates on elastic foundations is the analysis of concrete pavement slabs. In this area, checking the solution of a particular problem becomes more difficult because few problems have been accurately solved. The analysis presented in this section is of a 24' x 24' concrete pavement slab with a ten inch thickness. The modulus of elasticity of concrete is given as 3,000,000 psi and Poisson's ratio is taken as 0.20. The foundation modulus was assumed to be 200 lb./in.^{2}/in., and a concentrated load of 10 kip was applied at the center of one edge. The grid increment was chosen as 3 ft. and is shown in Figure 9.

This slab was analyzed by Hudson and Matlock (17) in their previously discussed paper, and their analysis indicated a deflection under the load of -0.018 inches. The results of the grid framework analysis are tabulated in Table III and give a deflection of -0.018024 inches at the point of loading. In addition, the computed deformations

TABLE III

DEFLECTIONS OF AN EDGE LOADED PAVEMENT SLAB

Joint	Deformation	Joint	Deformation
1	-0.000017 in.	24	0.000298 in.
2	-0.000011 in.	25	0.000329 in.
3	-0.000011 in.	26	0.000471 in.
4	-0.000014 in.	27	0.000449 in.
5	-0.000017 in.	28	0.000430 in.
6	-0.000035 in.	29	0.000331 in.
7	-0.000023 in.	30	0.000276 in.
8	-0.000017 in.	31	0.000802 in.
9	-0.000013 in.	32	0.000547 in.
10	-0.000013 in.	33	0.000125 in.
11	-0.000044 in.	34	-0.000696 in.
12	-0.000020 in.	35	-0.001202 in.
13	-0.000003 in.	36	0.000863 in.
14	0.000013 in.	37	0.000049 in.
15	0.000022 in.	38	-0.001534 in.
16	-0.000002 in.	39	-0.004502 in.
17	0.000035 in.	40	-0.006555 in.
18	0.000073 in.	41	0.000562 in.
19	0.000113 in.	42	-0.001055 in.
20	0.000136 in.	43	-0.004599 in.
21	0.000160 in.	44	-0.011561 in.
22	0.000198 in.	45	-0.018024 in.
23	0.000250 in.		

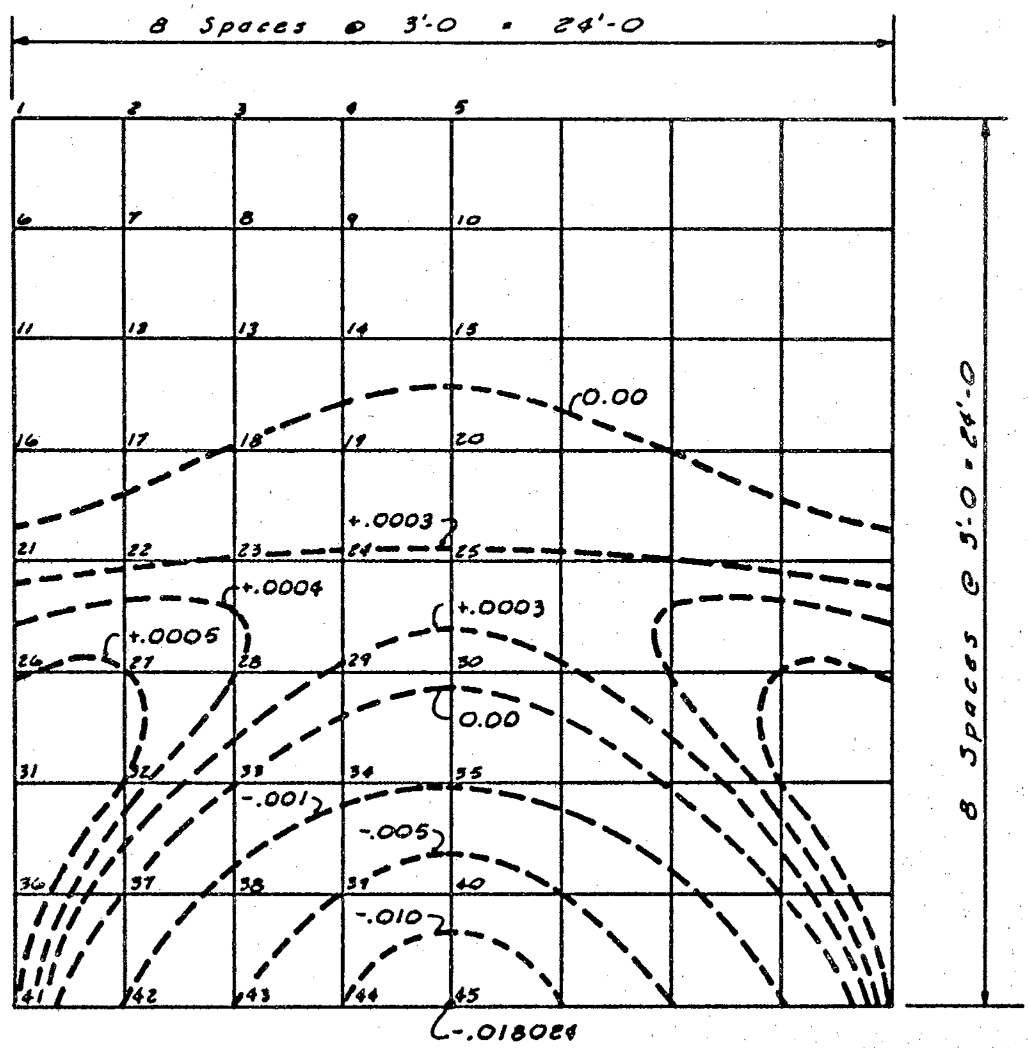


Figure 9. Equivalent Grid Framework Model and Deflection Contours for a Square Pavement Slab

at all other points on the plate closely match those obtained by Hudson and Matlock. The deflection contour lines are plotted over the grid framework of Figure 9, and it can be seen that there are areas where upward deflection of the plate is indicated. This condition of uplift is present in most loaded slabs on elastic foundations; however, the sensitivity in the analysis necessary to show this effect has not always been present in previously proposed methods.

Solution of an Edge Loaded Cracked Pavement Slab

The final example of the application of the grid framework method is the analysis of a cracked pavement slab. To illustrate this condition, the slab analyzed in the previous example is used again. For this case a crack was assumed through the midsection of the slab, and the 10 kip load was placed on one edge of the plate and centered over the crack. The results of this analysis are given in Table IV, and a deflection of -0.024225 in. is indicated under the load. This again compares favorably with a deflection of -0.025 in. computed by Hudson and Matlock for a similar slab.

TABLE IV

DEFLECTIONS OF AN EDGE LOADED
CRACKED PAVEMENT SLAB

Joint	Deformation	Joint	Deformation
1	-0.000022 in.	24	0.000353 in.
2	-0.000012 in.	25	0.000267 in.
3	-0.000011 in.	26	0.000347 in.
4	-0.000016 in.	27	0.000514 in.
5	-0.000025 in.	28	0.000623 in.
6	-0.000046 in.	29	0.000374 in.
7	-0.000024 in.	30	-0.000184 in.
8	-0.000012 in.	31	0.000667 in.
9	-0.000009 in.	32	0.000734 in.
10	-0.000013 in.	33	0.000536 in.
11	-0.000066 in.	34	-0.000675 in.
12	-0.000020 in.	35	-0.002819 in.
13	0.000012 in.	36	0.000842 in.
14	0.000030 in.	37	0.000529 in.
15	0.000034 in.	38	-0.000657 in.
16	-0.000045 in.	39	-0.004301 in.
17	0.000041 in.	40	-0.010434 in.
18	0.000112 in.	41	0.000821 in.
19	0.000150 in.	42	-0.000048 in.
20	0.000154 in.	43	-0.002834 in.
21	0.000080 in.	44	-0.010463 in.
22	0.000220 in.	45	-0.010463 in.
23	0.000341 in.		

CHAPTER VI

ANALYSIS OF PLATES OF VARIABLE RIGIDITY

Introduction

In the preceding chapter, the grid framework method of analysis for plates on elastic foundations was applied to four different plate problems. Each plate was completely analyzed, and the results were compared to deformations obtained by other methods of solution. In this manner, the accuracy and versatility of the method were established. However, all four of these example problems consisted of elastic plates of constant thickness.

The solution of the basic plate equation has been mainly concerned with plates of constant rigidity. However, plates of variable thickness are now being used more and more in engineering structures. The classical theories of plates on elastic foundations do apply to plates of variable rigidity, but, unfortunately, very few solutions have been developed for these cases. This lack of closed form solution for plates of variable thickness may be attributed to the increased mathematical complexity of the problem when exact solutions are desired.

The grid framework method lends itself ideally to the analysis of plates of variable rigidity. The basic stiffness equations were developed for an individual grid model, and each plate element makes its own contribution to the torsional and flexural rigidity of each corresponding grid member. Because of this relationship the physical properties of each equivalent grid element may vary from element to element, and the basic mathematical matrix relationships are not altered. In this chapter, the application of the grid framework method of analysis to plates of variable rigidity on elastic foundations will be demonstrated.

Analysis of a Tapered Concrete Pavement Slab

Consider first the analysis of a concrete pavement slab as shown in Figure 10. The portion of pavement to be analyzed is 24 ft. wide and 12 ft. long. The thickness of the slab is 12 inches at the crown and tapers to a thickness of 4 inches at the outside edges. The material constants for the concrete are a modulus of elasticity of 3,000,000 psi and a Poisson's ratio equal to 0.20. The foundation modulus is taken as 200 lb./in.^{2}/in. In order to extend the analysis of this slab to demonstrate still another application of the grid framework method, the slab is assumed to resist two moving wheel loads of 10,000 pounds each. This condition may be simulated by analyzing the slab with the two loads placed in a static condition at the edge of the pavement and then moving the loads one grid space inward in each succeeding analysis. If the

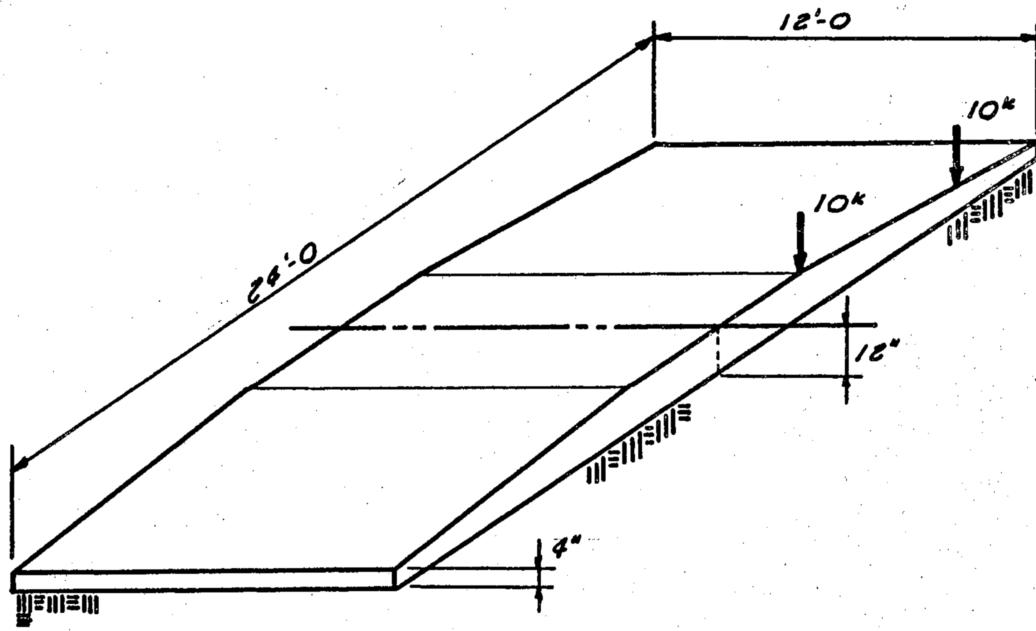


Figure 10. Tapered Concrete Pavement Slab

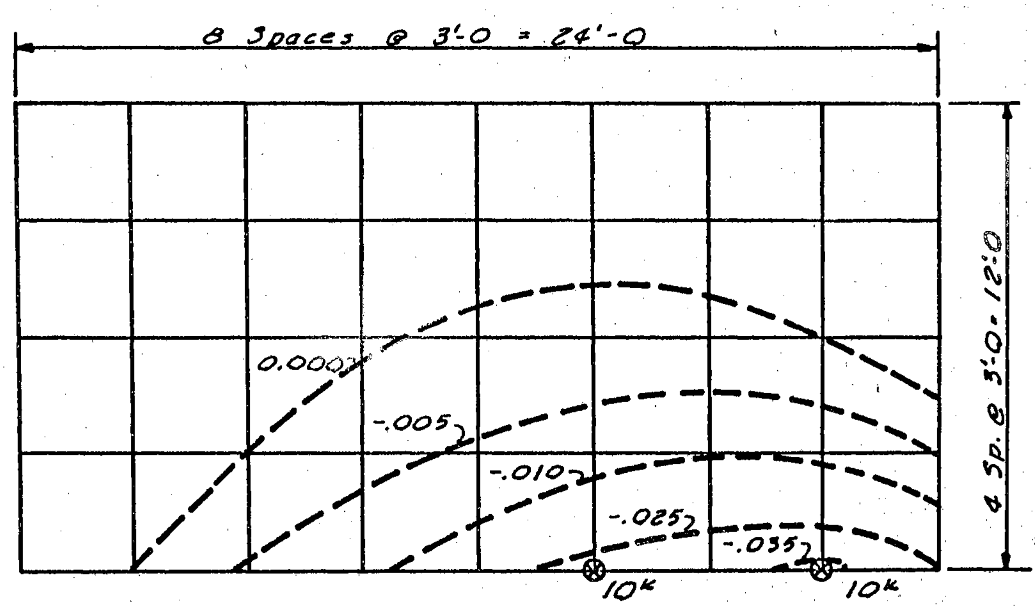


Figure 11. Tapered Concrete Pavement Slab
Loading Condition Number 1

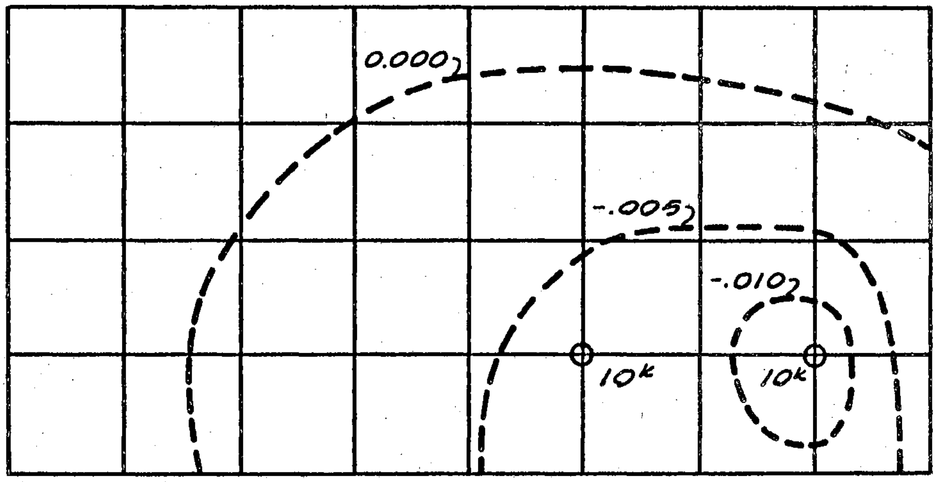


Figure 12. Tapered Concrete Pavement Slab
Loading Condition Number 2

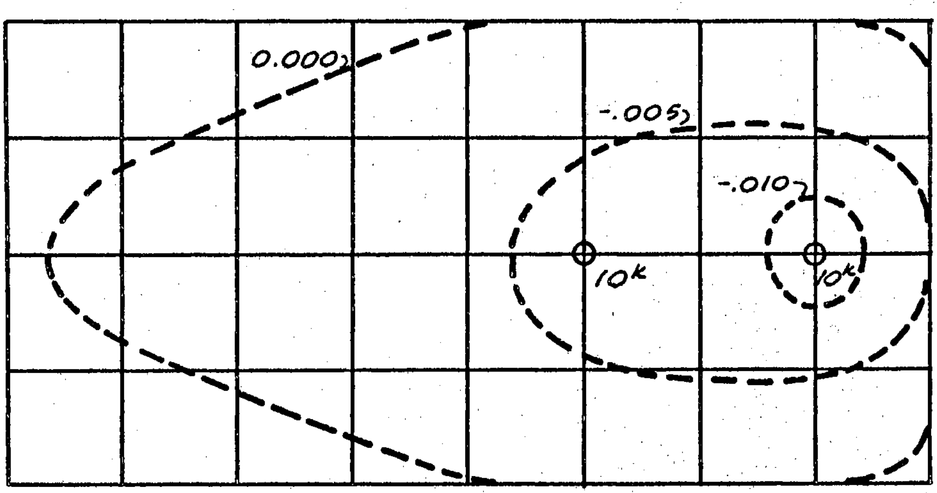


Figure 13. Tapered Concrete Pavement Slab
Loading Condition Number 3

grid increment is chosen at three feet, then one computer run is necessary for the two loads placed at 0 ft., 3 ft., and 6 ft. to analyze the slab completely. The resulting deformation contour lines are plotted for each load condition as given in Figures 11, 12 and 13, and the engineer can then visualize the effects of moving wheel loads on a pavement to be designed.

Analysis of Slabs with Stiffened Edges

The grid framework method is also suitable for approximate analysis of slabs with abrupt changes in rigidity such as pavements with curbs or otherwise stiffened edges. In order to apply the method to slabs with these conditions, the effects of the unusually stiff slab areas must be approximated in the equivalent grid members representing these rigid elements. As an example, consider again the tapered pavement slab analyzed in the previous section. For this illustration, assume the properties and configuration of the slab are as before with the addition of a 6 in. by 12 in. curb along the two exterior edges of the pavement. If loading condition Number 3 is repeated for this slab, a comparison may be obtained between deformations in a slab with stiffened edges and one with unstiffened edges. To effect this comparison, the deflection contours for this slab with stiffened edges are given in Figure 14. The deformation contours for the same slab with unstiffened edges were previously presented in

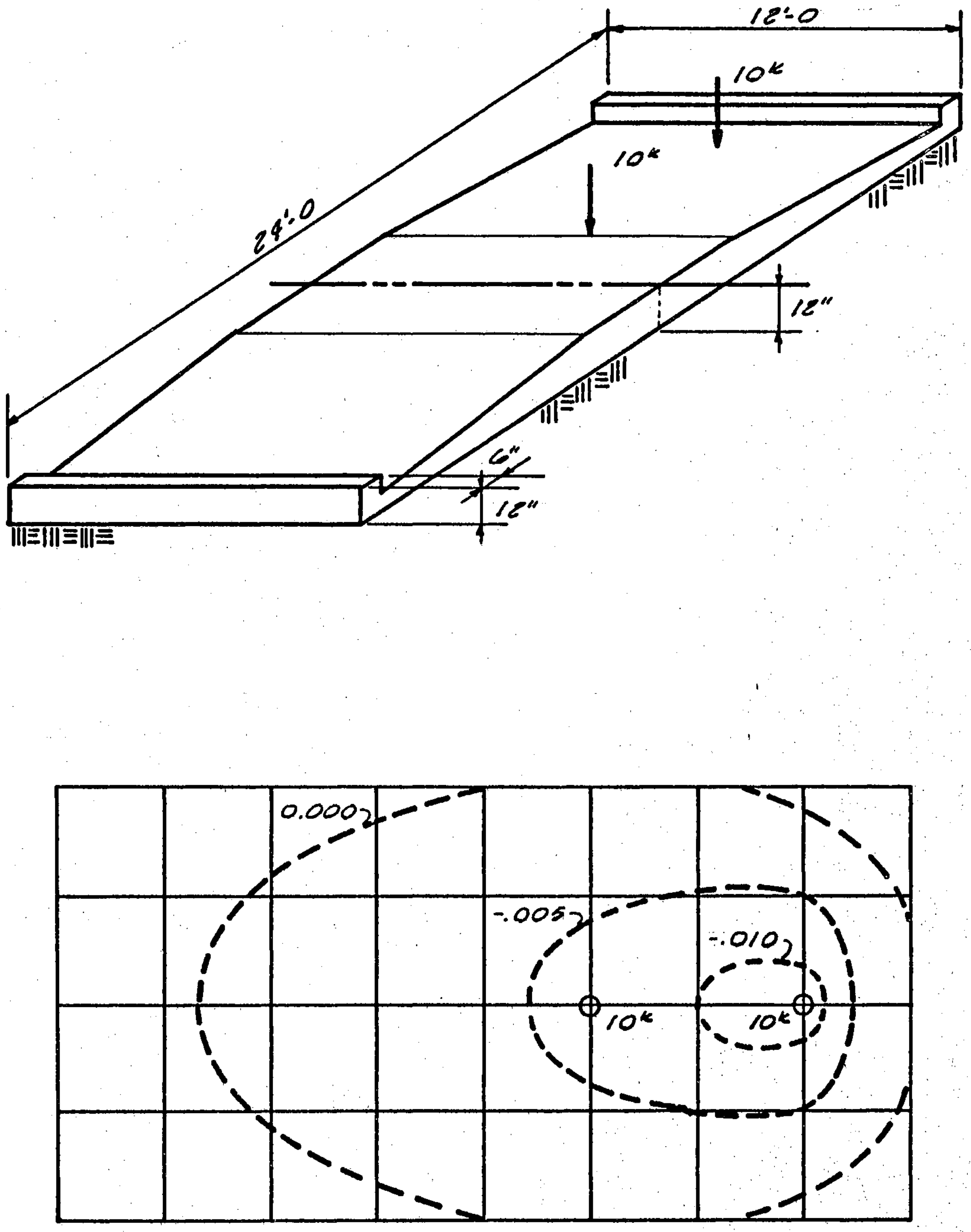


Figure 14. Slab Configuration and Deformation Contour Lines for an Edge Stiffened Concrete Pavement Slab

Figure 13.

Analysis of Doubly Tapered Slabs

As a final example of the application of the grid framework method to plates with variable rigidities, a plate tapered in two directions will be investigated. Such a plate is shown in Figure 15. The plate itself is 12 in. by 12 in. and varies in thickness from 0.30 in. at the thickest point to 0.18 in. at the thin corner. The modulus of elasticity is assumed to be 30,000,000 psi with Poisson's ratio taken as 0.30. The modulus of the foundation is assumed to be 200 lb./in.^{2}/in., and a 1000 lb. load was applied at the center of the plate. The grid increment was chosen to be two inches in each direction, and the resulting deformation contours are shown in Figure 16.

An examination of the deflection contours of Figure 16 shows that they are not symmetrical about the point of loading. This result is, of course, to be expected, as the nonsymmetrical contours are the result of the different rigidities of adjacent plate elements. However, in the past, such a complete deflection analysis of every part of a tapered plate would have been extremely difficult to obtain.

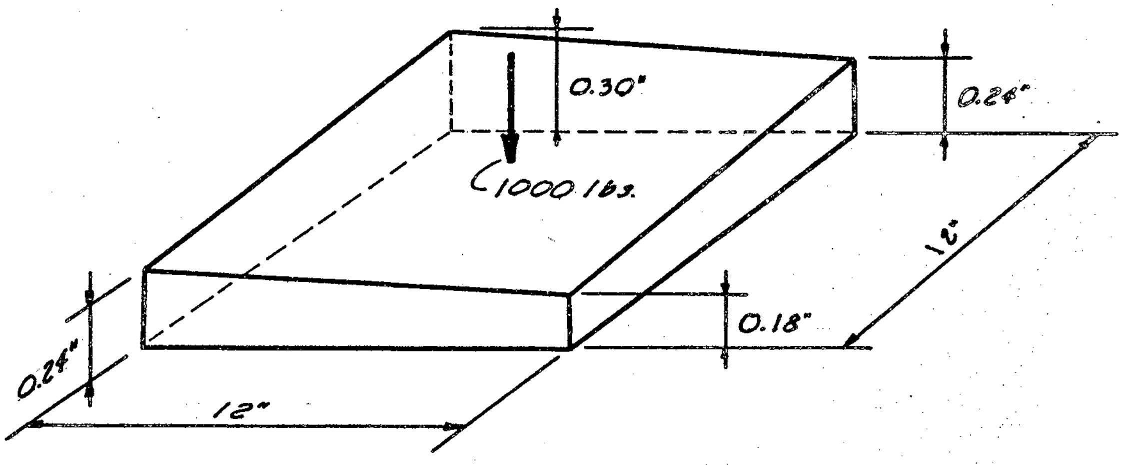


Figure 15. Doubly Tapered Elastic Plate

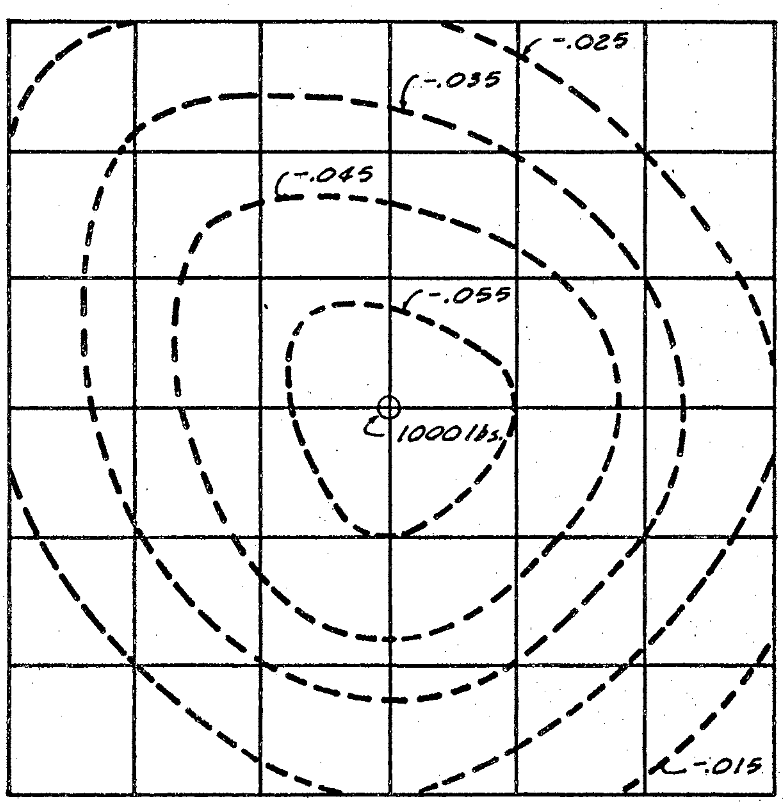


Figure 16. Deflection Contours for a Doubly Tapered Elastic Plate

CHAPTER VII

SUMMARY AND CONCLUSIONS

Summary

The development of an equivalent grid framework system to analyze plates on elastic foundations is presented.

An equivalent grid framework model is formulated to represent an element of an elastic medium, and the physical relationships of the model are presented in terms of stiffness influence coefficients. The resulting member stiffness matrices are assembled into the joint stiffness matrix, and the model deformations are equated to the known deformations of the original plate element. The resulting stiffness equation is then solved to provide physical constants for the grid framework model such that the actions of the elastic medium are reproduced. Next, the effects of elastic support conditions are introduced, and the elastic spring constant matrix, K, is defined and included in the general expression for the system stiffness matrix. The resulting matrix equation, when solved, provides the deformations of an equivalent grid framework system which represents the corresponding deformations of an elastic plate supported upon an elastic foundation.

Application of the method is demonstrated, and numerical results are compared with solutions obtained by other methods. It is then found that the grid framework method provides deformations which are in close agreement with other known results.

The versatility of the grid framework method is indicated by the variety of problems solved. In particular, it is shown that this method provides an easily applied, rapid solution for plates of variable rigidities on elastic foundations. In the past, the mathematical complexity of plates of this type has been a major problem to engineers concerned with pavement design.

In addition, the formulation of the equivalent grid is such that each plate element makes its own contribution to the flexural and torsional properties of the corresponding grid members. Therefore, plate discontinuities, such as cut-outs or irregular boundaries may be easily approximated by the proper choice of grid pattern.

Conclusions

Based upon the results of this investigation, the following conclusions are drawn:

1. The equivalent grid framework method provides an efficient and easily applied method of analysis for plates on elastic foundations.

2. The inclusion of elastic foundation effects in the matrix formulation of this problem is unique and has not been expressed before.

3. The results of the grid framework method correlate closely with results obtained by other methods.

4. The grid framework method is applicable to plates of irregular configuration or variable rigidities.

5. The grid framework method provides a systematic method of analysis which can be applied to existing computer programs readily available to consulting engineers.

6. Complex problems involving multiple load systems and a combination of boundary conditions can be solved with the same ease as problems of simple configuration.

Recommendation for Further Study

The method of analysis discussed in this work forms the basis for possible extension and further study. For example, while skew plates can be approximated by the orthogonal grid system presented, the results would undoubtedly be more accurate if a skew grid system were used. To facilitate this system, the stiffness influence coefficients should be derived in terms of a skew coordinate system. Once this has been accomplished, the equivalent grid system could be formulated and compared to the deformations of a skew element of a skew elastic plate. The remainder of the solution procedure would then follow the derivation presented in this work.

The analysis of ribbed plates is an area in which an extension of the method presented here would prove beneficial. A three dimensional equivalent grid system would be necessary to accomplish this type of analysis. Many existing computer programs are readily available to solve space frames, and application of the method would again follow the procedure presented in this thesis.

In addition, the grid framework method provides a means of approximating any continuous elastic medium. Solution of a framework system by the stiffness method of analysis is currently known by most structural engineers. The combination of these two facts indicates that this method can provide an easily understood and quickly applicable method of solution for many problems which currently are avoided by many engineers because of their mathematical complexity.

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APPENDIX A

COMPUTER PROGRAM FOR ANALYSIS OF PLATES ON ELASTIC FOUNDATIONS

```
C                  ANALYSIS OF PLATES ON ELASTIC FOUNDATIONS
C
C                  BY THE USE OF THE GRID FRAMEWORK METHOD
C
C     -----------------------------------------------------------------
C
C     ANALYSIS OF PLATES ON ELASTIC FOUNDATIONS
C
C     -----------------------------------------------------------------
C
C     1969 , . . . . . . . WILLIAM M. LUCAS, JR.
C
C     -----------------------------------------------------------------
C
C     PART 1 ..... INPUT AND PRINT STRUCTURE DATA
C
C     -----------------------------------------------------------------
      DIMENSION X(160),Y(160),JE(160),KE(160),XI(160),YI(160),XL(160),
     1CX(160),CY(160),JT(160),CONS(160),JED(160)
      DIMENSION SMD(6,6),A(160),AML(160,6),AE(160),AC(160),
     1D(160),AR(160),S(160,160)
      INTEGER B1,B2,B3,B4,RL(160),CRL(160)
    1 WRITE(6,1100)
 1100 FORMAT(42H1ANALYSIS OF PLATES ON ELASTIC FOUNDATIONS)
C
C.... A, STRUCTURE PARAMETERS AND ELASTIC MODULUS
C
      READ(5,101)NM,NJ,NR,NRJ,E,G,THK,POIS
  101 FORMAT (4I5,2F10.0,2F10.4)
      IF(NM.EQ.0)GO TO 1000
      N=3*NJ-NR
      WRITE(6,102)N,NM,NJ,NR,NRJ,E,G
  102 FORMAT(17H0STRUCTURE DATA   ,7H     N=I5,8H     NM=I5,8H     NJ=I5,
     18H     NR=I5,9H     NRJ=I5,7H     E=F10.0,7H     G=F10.0)
      READ(5,601)XINC,YINC,SCON
  601 FORMAT(3F20.8)
      CONST=XINC*YINC*SCON
      WRITE(6,602)SCON,CONST
  602 FORMAT(/15H SOIL CONSTANT=,F10.5,5X,16H JOINT CONSTANT=,F10.2)
C
C.... B, JOINT COORDINATES
C
      DO 1110 J=1,NJ
      READ(5,103)JT(J),X(J),Y(J),JED(J)
  103 FORMAT(I5,2F10.0,I5)
      WRITE(6,104)JT(J),X(J),Y(J)
  104 FORMAT(25H0COORDINATES OF JOINTS   ,3H J=I5,6H X(J)=F10.4,6H Y(J)=
     1F10.4)
 1110 CONTINUE
      READ(5,9300)LJL,LKL,LJV,LKV,LJD,LKD
```

```
 9300 FORMAT(6I5)
      XLV=X(LKV)-X(LJV)
      YLV=Y(LKV)-Y(LJV)
      XLVV=SQRT(XLV**2+YLV**2)
      XLL=X(LKL)-X(LJL)
      YLL=Y(LKL)-Y(LJL)
      XLLL=SQRT(XLL**2+YLL**2)
      CON=XLLL/XLVV
C
C.... C. MEMBER DESIGNATIONS AND PROPERTIES
C
      DO 120 I=1,NM
      READ(5,111)JE(I),KE(I),MDES,THK
  111 FORMAT(3I5,F10.4)
      KI=KE(I)
      JI=JE(I)
      XCL=X(KI)-X(JI)
      YCL=Y(KI)-Y(JI)
      XL(I)=SQRT(XCL**2+YCL**2)
      CX(I)=XCL/XL(I)
      CY(I)=YCL/XL(I)
      GO TO (241,242,243),MDES
  241 UP=((XL(I))/CON)-POIS*CON*XL(I)
      DN=2.0*(1.0-(POIS**2))
      RAT=(THK**3)/12.0
      YI(I)=RAT*UP/DN
      UPN=(XL(I))*(1.0-(3.0*POIS))
      DNN=2.*(1.0-(POIS**2))
      XI(I)=(UPN*RAT*E)/(DNN*G)
      GO TO 244
  242 UP=(XL(I))*((CON**2)-POIS)
      DN=2.0*CON*(1.0-(POIS**2))
      RAT=(THK**3)/12.0
      YI(I)=RAT*UP/DN
      UPN=CON*(XL(I))*(1.0-(3.0*POIS))
      DNN=2.*(1.0-(POIS**2))
      XI(I)=(UPN*RAT*E)/(DNN*G)
      GO TO 244
  243 RON=XL(I)/XLVV
      UP=POIS*(RON**2)*(XL(I))
      DN=2.*CON*(1.0-(POIS**2))
      RAT=(THK**3)/12.0
      YI(I)=RAT*UP/DN
      XI(I)=0.0
  244 CONTINUE
      WRITE(6,112)I,JI,KI,XI(I),YI(I),XL(I),CX(I),CY(I)
  112 FORMAT(35H0MEMBER DESIGNATIONS AND PROPERTIES/,3H I=I6,6H J(I)=I6,
     16H K(I)=I5,7H XI(I)=F10.6,7H YI(I)=F10.6,7H XL(I)=F10.2,7H CX(I)=F
     110.2,7H CY(I)=F10.2)
  120 CONTINUE
```

```
C
C.... D. JCINT RESTRAINT LIST ) ) CUMULATIVE RESTRAINT LIST
C
      NJ3=3*NJ
      DO 930 JK=1,NJ3
      RL(JK)=0
  930 CONTINUE
      IF(NRJ.EQ.0) GO TO 1141
      DO 1130 JKL=1,NRJ
      READ(5,121)K,RL(3*K-2),RL(3*K-1),RL(3*K)
  121 FORMAT(4I5)
      WRITE(6,122)K,RL(3*K-2),RL(3*K-1),RL(3*K)
  122 FORMAT(18H0JOINT RESTRAINTS ,3H K=I5,11H RL(3*KR2)=I5,11H RL(3*K-1
     1)=I5,9H RL(3*K)=I5)
 1130 CONTINUE
 1141 CRL(1)=RL(1)
      NJ3=3*NJ
      DO 131 K=2,NJ3
      K1=K-1
      CRL(K)=CRL(K1)+RL(K)
  131 CONTINUE
C
C     ---------------------------------------------------------------
C
C     PART 2 ..... STRUCTURE STIFFNESS MATRIX
C
C     ---------------------------------------------------------------
C
C.... A. GENERATION OF STIFFNESS MATRIX
C
      DO 2200 I=1,NM
      J1=3*JE(I)-2
      J2=3*JE(I)-1
      J3=3*JE(I)
      K1=3*KE(I)-2
      K2=3*KE(I)-1
      K3=3*KE(I)
      SCM1=(G*XI(I))/XL(I)
      SCM2=(4.0*E*YI(I))/XL(I)
      SCM3=(1.5*SCM2)/XL(I)
      SCM4=(2.0*SCM3)/XL(I)
      IF(RL(J1))133,132,133
  132 J1=J1-CRL(J1)
      GO TC 134
  133 J1=N+CRL(J1)
  134 IF(RL(J2))136,135,136
  135 J2=J2-CRL(J2)
      GO TC 137
  136 J2=N+CRL(J2)
  137 IF(RL(J3))139,138,139
```

```
 138 J3=J3-CRL(J3)
     GO TC 1140
 139 J3=N+CRL(J3)
1140 IF(RL(K1))142,141,142
 141 K1=K1-CRL(K1)
     GO TC 143
 142 K1=N+CRL(K1)
 143 IF(RL(K2))145,144,145
 144 K2=K2-CRL(K2)
     GO TC 146
 145 K2=N+CRL(K2)
 146 IF(RL(K3))148,147,148
 147 K3=K3-CRL(K3)
     GO TC 149
 148 K3=N+CRL(K3)
 149 SMD(4,4)=SCM1*CX(I)**2+SCM2*CY(I)**2
     SMD(1,1)=SMD(4,4)
     SMD(2,1)=(SCM1-SCM2)*CX(I)*CY(I)
     SMD(1,2)=SMD(2,1)
     SMD(5,4)=SMD(1,2)
     SMD(4,5)=SMD(5,4)
     SMD(4,3)=SCM3*CY(I)
     SMD(3,4)=SMD(4,3)
     SMD(3,1)=SMD(3,4)
     SMD(1,3)=SMD(3,1)
     SMD(6,4)=-SMD(1,3)
     SMD(4,6)=SMD(6,4)
     SMD(6,1)=SMD(4,6)
     SMD(1,6)=SMD(6,1)
     SMD(4,1)=-SCM1*CX(I)**2+SCM2*0.5*CY(I)**2
     SMD(1,4)=SMD(4,1)
     SMD(5,1)=-(SCM1+SCM2*0.5)*CX(I)*CY(I)
     SMD(1,5)=SMD(5,1)
     SMD(4,2)=SMD(1,5)
     SMD(2,4)=SMD(4,2)
     SMD(5,5)=SCM1*CY(I)**2+SCM2*CX(I)**2
     SMD(2,2)=SMD(5,5)
     SMD(5,3)=-SCM3*CX(I)
     SMD(3,5)=SMD(5,3)
     SMD(3,2)=SMD(3,5)
     SMD(2,3)=SMD(3,2)
     SMD(6,5)=-SMD(2,3)
     SMD(5,6)=SMD(6,5)
     SMD(6,2)=SMD(5,6)
     SMD(2,6)=SMD(6,2)
     SMD(5,2)=-SCM1*CY(I)**2+SCM2*0.5*CX(I)**2
     SMD(2,5)=SMD(5,2)
     SMD(6,6)=SCM4
     SMD(3,3)=SMD(6,6)
     SMD(6,3)=-SCM4
```

```
      SMD(3,6)=SMD(6,3)
      JI3=3*JE(I)
      JI2=JI3-1
      JI1=JI3-2
      IF(RL(JI1).NE.0) GO TO 151
 1150 S(J1,J1)=S(J1,J1)+SMD(1,1)
      S(J2,J1)=S(J2,J1)+SMD(2,1)
      S(J3,J1)=S(J3,J1)+SMD(3,1)
      S(K1,J1)=SMD(4,1)
      S(K2,J1)=SMD(5,1)
      S(K3,J1)=SMD(6,1)
  151 IF(RL(JI2).NE.0) GO TO 153
  152 S(J1,J2)=S(J1,J2)+SMD(1,2)
      S(J2,J2)=S(J2,J2)+SMD(2,2)
      S(J3,J2)=S(J3,J2)+SMD(3,2)
      S(K1,J2)=SMD(4,2)
      S(K2,J2)=SMD(5,2)
      S(K3,J2)=SMD(6,2)
  153 IF(RL(JI3).NE.0) GO TO 155
  154 S(J1,J3)=S(J1,J3)+SMD(1,3)
      S(J2,J3)=S(J2,J3)+SMD(2,3)
      S(J3,J3)=S(J3,J3)+SMD(3,3)
      S(K1,J3)=SMD(4,3)
      S(K2,J3)=SMD(5,3)
      S(K3,J3)=SMD(6,3)
  155 KI3=3*KE(I)
      KI2=KI3-1
      KI1=KI3-2
      IF(RL(KI1).NE.0) GO TO 157
  156 S(J1,K1)=SMD(1,4)
      S(J2,K1)=SMD(2,4)
      S(J3,K1)=SMD(3,4)
      S(K1,K1)=S(K1,K1)+SMD(4,4)
      S(K2,K1)=S(K2,K1)+SMD(5,4)
      S(K3,K1)=S(K3,K1)+SMD(6,4)
  157 IF(RL(KI2).NE.0) GO TO 159
  158 S(J1,K2)=SMD(1,5)
      S(J2,K2)=SMD(2,5)
      S(J3,K2)=SMD(3,5)
      S(K1,K2)=S(K1,K2)+SMD(4,5)
      S(K2,K2)=S(K2,K2)+SMD(5,5)
      S(K3,K2)=S(K3,K2)+SMD(6,5)
  159 IF(RL(KI3).NE.0) GO TO 2200
 1160 S(J1,K3)=SMD(1,6)
      S(J2,K3)=SMD(2,6)
      S(J3,K3)=SMD(3,6)
      S(K1,K3)=S(K1,K3)+SMD(4,6)
      S(K2,K3)=S(K2,K3)+SMD(5,6)
      S(K3,K3)=S(K3,K3)+SMD(6,6)
 2200 CONTINUE
```

```
C
C,... B, GENERATION OF SOIL CONSTANT MATRIX
C
      DO 7000 J=1,NJ
      KN=JED(J)
      KJ=3*J
      GO TC (701,702,703,704),KN
  701 CONS(KJ)=CONST
      GO TC 7000
  702 CONS(KJ)=CONST*0.5
      GO TC 7000
  703 CONS(KJ)=CONST*0.25
      GO TC 7000
  704 CONS(KJ)=CONST*0.75
 7000 CONTINUE
C
C,... C, GENERATION OF (S-K) MATRIX
C
      J1=0.0
      J2=N
      DO 900 I=1,NJ3
      IF(RL(I).EQ.1) GO TO 910
      J1=J1+1
      S(J1,J1)=S(J1,J1)+CONS(I)
      GO TC 900
  910 J2=J2+1
      S(J2,J2)=S(J2,J2)+CONS(I)
  900 CONTINUE
C
C,... D, INVERSION OF STIFFNESS MATRIX
C
      NODE=1
      CALL INVERT (S,N,S,0,DET,160,160,NODE)
C
C     ------------------------------------------------------------
C
C     PART 3 ..... INPUT AND PRINT LOAD DATA
C
C     ------------------------------------------------------------
C
C,... A, NUMBERS OF LOADED JOINTS AND MEMBERS
C
      READ(5,201)NLJ,NLM
  201 FORMAT(2I5)
      WRITE(6,202)NLJ,NLM
  202 FORMAT(11H0LOAD DATA ,5H NLJ=I5,5H NLM=I5)
C
C,... B, ACTIONS APPLIED AT JOINTS
C
      IF(NLJ)203,2210,203
```

```
  203 DO 209 J=1,NLJ
      READ(5,204)K,A(3*K-2),A(3*K-1),A(3*K)
  204 FORMAT(I5,5X,3F10.0)
      WRITE(6,205)K,A(3*K-2),A(3*K-1),A(3*K)
  205 FORMAT(27H0ACTIONS APPLIED AT JOINTS ,7H JOINT I5,10H X ACTION F10
     1.0,10H Y ACTION F10.0,10H Z ACTION F10.0)
  209 CONTINUE
C
C.... C, ACTIONS AT ENDS OF RESTRAINED MEMBERS DUE TO LOADS
C
 2210 IF(NLM)211,2220,211
  211 DO 219 J=1,NLM
      READ(5,212)I,AML(I,1),AML(I,2),AML(I,3),AM
  212 FORMAT(I5,5X,6F10.2)
      WRITE(6,213)I,AML(I,1),AML(I,2),AML(I,3),AML(I,4),AML(I,5),AML(I,6
     1)
  213 FORMAT(3H I=I5,10H AML(I,1)=F10.2,10H AML(I,2)=F10.2,10H AML(I,3,=
     1F10.2,10H AML(I,4)=F10.2,10H AML(I,5)=F10.2,10H AML(I,6)=F10.2)
  219 CONTINUE
C
C     ------------------------------------------------------------------
C
C     PART 4 ..... CONSTRUCTION OF VECTORS ASSOCIATED WITH LOADS
C
C     ------------------------------------------------------------------
C
C.... A, EQUIVALENT JOINT LOADS
C
 2220 IF(NLM)221,223,221
  221 DO 222 I=1,NM
      JI3=3*JE(I)
      AE(JI3-2)=AE(JI3-2)-AML(I,1)*CX(I)+AML(I,2)*CY(I)
      AE(JI3-1)=AE(JI3-1)-AML(I,1)*CY(I)-AML(I,2)*CX(I)
      AE(JI3)=AE(JI3)-AML(I,3)
      KI3=3*KE(I)
      AE(KI3-2)=AE(KI3-2)-AML(I,4)*CX(I)+AML(I,5)*CY(I)
      AE(KI3-1)=AE(KI3-1)-AML(I,4)*CY(I)-AML(I,5)*CX(I)
      AE(KI3)=AE(KI3)-AML(I,6)
  222 CONTINUE
C
C.... B, COMBINED JOINT LOADS
C
  223 NNR=N+NR
      DO 227 J=1,NNR
      IF(RL(J))224,225,224
  224 K=N+CRL(J)
      GO TO 226
  225 K=J-CRL(J)
  226 AC(K)=A(J)+AE(J)
  227 CONTINUE
```

```
C
C      ---------------------------------------------------------------
C
C      PART 5 ..... CALCULATION AND OUTPUT OF RESULTS 9
C
C      ---------------------------------------------------------------
C
C.... A. JOINT DISPLACEMENTS AND SUPPORT REACTIONS
C
      DO 228 J=1,N
      DO 228 K=1,N
      D(J)=D(J)+S(J,K)*AC(K)
  228 CONTINUE
C
C      COMPUTE SUPPORT REACTIONS                        9
C
      N1=N+1
      NNR=N+NR
      DO 229 K=N1,NNR
      AR(K)=-AC(K)
      DO 229 J=1,N
      AR(K)=AR(K)+S(K,J)*D(J)
  229 CONTINUE
      NNR=N+NR
      J=N+1
      DO 232 ME1=1,NNR
      ME=NNR-ME1+1
      IF(RL(ME))231,2230,231
 2230 J=J-1
      D(ME)=D(J)
      GO TO 232
  231 D(ME)=0.0
  232 CONTINUE
      NJ3=3*NJ
      DO 237 ME=3,NJ3,3
      B3=ME-1
      B2=ME-2
      B4=ME
      B1=ME/3
      WRITE(6,236)B1,D(B2),D(B3),D(B4)
  236 FORMAT(7HUJOINT=I5,10H X DISPL.=F10.6,10H Y DISPL.=F10.6,10H Z DIS
     1PL.=F10.6)
  237 CONTINUE
      GO TO 1
 1000 STOP
      END

4576 WORDS OF MEMORY USED BY THIS COMPILATION
```

```
03-05-70   12,237

 CINVERT
       SUBRCUTINE INVERT( A, N, B, M, DETERM, MA, NA, NODE)
 C
 C     THIS SUBROUTINE IS A SUBPROGRAM FOR INVERTING A MATRIX OR SOLVING
 C     A SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS BY THE GAUSS-JORDAN
 C     METHCD.  AT EACH STAGE THE LARGEST ELEMENT OF THE SUBMATRIX UNDER
 C     CONSIDERATION IS FOUND, ROWS AND COLUMNS ARE INTERCHANGED TO MAKE
 C     THIS THE PIVOT ELEMENT, AND ALL ELEMENTS ARE DIVIDED BY IT.
 C
 C     THIS SUBROUTINE COMPUTES THE INVERSE AND THE DETERMINANT FOR THE
 C     MATRIX A OF ORDER N BY N,  IT ALSO SOLVES THE MATRIX EQUATION
 C     A*X = B WHERE B IS A RECTANGULAR MATRIX OF ORDER N BY M.
 C
 C     UPON EXITING FROM THIS SUBROUTINE, THE INVERSE OF A WILL HAVE
 C     REPLACED A AND X = A(INVERSE)*B WILL HAVE REPLACED B.
 C
 C     IF M = 0, ONLY THE INVERSE AND THE DETERMINANT OF A ARE COMPUTED.
 C     IN THIS CASE, B IS NOT TOUCHED AND IN ITS PLACE WE CAN ALSO HAVE A
 C     (IF THIS IS DONE, SET NA = MA).
 C
 C     THE VALUE OF NODE TRANSMITTED FROM THE CALLING PROGRAM TO THIS
 C     SUBRCUTINE CONTROLS THE WRITTING OF MESSAGES BY THIS SUBROUTINE
 C              IF THE INPUT VALUE OF NODE = 1, THE LOWER BOUND AND THE
 C              UPPER BOUND OF THE CONDITION NUMBER FOR THE MATRIX ARE
 C              WRITTEN OUT.  ALSO, A MESSAGE STATING THAT THE MATRIX IS
 C              NEAR-SINGULAR IS WRITTEN OUT IF THIS IS FOUND TO BE TRUE.
 C              IF THE INPUT VALUE OF NODE = 0, NONE OF THESE MESSAGES ARE
 C              WRITTEN OUT AND NEITHER THE LOWER BOUND NOR THE UPPER
 C              BOUND OF THE CONDITION NUMBER IS CALCULATED.
 C              IRREGARDLESS OF THE INPUT VALUE OF NODE, A MESSAGE STATING
 C              THAT NO SOLUTION EXISTS IS WRITTEN OUT IF A SINGULAR
 C              MATRIX IS ENCOUNTERED.
 C
 C     THE VALUE OF NODE RETURNED FROM THIS SUBROUTINE TO THE CALLING
 C     PROGRAM IS USED TO INDICATE CONDITIONS FOUND BY THIS SUBROUTINE
 C              NODE = 0 IF THE MATRIX WAS FOUND TO BE SINGULAR.
 C              NODE = 2 IF THE MATRIX WAS FOUND TO BE NEAR-SINGULAR.
 C              NODE = 1 IF NEITHER OF THE ABOVE WAS FOUND TO BE TRUE.
 C
 C     IT SHOULD BE NOTED THAT ACTUALLY FOR A SINGULAR SYSTEM, EITHER NO
 C     SOLUTION EXISTS AT ALL, OR THERE WILL BE AN INFINITE NUMBER OF
 C     SOLUTIONS.
 C
 C     DETERM = THE DETERMINANT OF MATRIX A.
 C     CNLB = THE LOWER BOUND OF THE CONDITION NUMBER FOR MATRIX A.
 C     CNUB = THE UPPER BOUND OF THE CONDITION NUMBER FOR MATRIX A.
 C
 C     THE CONDITION NUMBER OF A MATRIX IS THE RATIO OF THE MAXIMUM
 C     EIGENVALUE TO THE MINIMUM EIGENVALUE.  HOWEVER,9COMPUTING THIS
 C     QUANTITY IS A COMPLEX9AND LENGTHY PROCESS.  THERE ARE MORE EASILY
 C     EVALLATED QUANTITIES WHICH BOUND THE CONDITION NUMBER AND THESE
 C     ARE THE ONES CALCULATED.
```

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```
C
      DIMENSION A( MA, MA), B( MA, NA), INDEX(160, 2), IPIVOT(160)
      EQUIVALENCE ( IROW, JROW), ( ICOLUM, JCOLUM), ( AMAX, T, SWAP)
C.... INITIALIZATION,
   10 DETERM = 1.0
      NODEIN = NODE
      NODE = 1
   15 DO 20 J=1,N
   20 IPIVCT(J) = 0
   30 DO 550 I=1,N
C.... SEARCH FOR PIVOT ELEMENT.
   40 AMAX = 0.0
   45 DO 105 J=1,N
      IF(IPIVOT(J) - 1) 60, 105, 60
   60 DO 100 K=1,N
      IF(IPIVOT(K) - 1) 80, 100, 715
   80 IF(ABS (AMAX) - ABS (A(J,K))) 85, 100, 100
   85 IROW = J
   90 ICOLUM = K
      AMAX = A(J,K)
  100 CONTINUE
  105 CONTINUE
C.... SET ZERO EQUAL TO 10**-5 TIMES THE MAGNITUDE OF THE LARGEST
C.... ORIGINAL ELEMENT,
      IF(I - 1) 108, 106, 108
  106 A1BIG = AMAX
      ZERO = AMAX * 1.E-05
      IF(ZERO) 110, 715, 110
C.... CHECK FOR A SINGULAR MATRIX OR A NEAR-SINGULAR MATRIX.
  108 IF(ABS (AMAX) - ABS (ZERO)) 109, 110, 110
  109 IF (AMAX) 107, 715, 107
  107 NODE = 2
  110 IPIVCT(ICOLUM) = IPIVOT(ICOLUM) + 1
  260 INDEX(I,1) = IROW
  270 INDEX(I,2) = ICOLUM
C.... INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL.
  130 IF(IROW - ICOLUM) 140, 310, 140
  140 DETERM = -DETERM
  150 DO 200 L=1,N
  160 SWAP = A(IROW,L)
  170 A(IROW,L) = A(ICOLUM,L)
  200 A(ICOLUM,L) = SWAP
      IF(M) 310, 310, 210
  210 DO 250 L=1,M
  220 SWAP = B(IROW,L)
  230 B(IROW,L) = B(ICOLUM,L)
  250 B(ICOLUM,L) = SWAP
C.... DIVIDE PIVOT ROW BY PIVOT ELEMENT.                9
  310 PIVOT=(A(ICOLUM,ICOLUM))**(-1)
  330 A(ICOLUM,ICOLUM) = 1.0
  340 DO 350 L=1,N
  350 A(ICOLUM,L) = A(ICOLUM,L) * PIVOT
```

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```
  355 IF(M) 380, 380, 360
  360 DO 370  L=1,M
  370 B(ICOLUM,L) = B(ICOLUM,L) * PIVOT
C.... REDUCE NON-PIVOT ROWS.
  380 DO 550 L1=1,N
  390 IF(L1 - ICOLUM) 400, 550, 400
  400 T = A(L1,ICOLUM)
  420 A(L1,ICOLUM) = 0.0
  430 DO 450 L=1,N
  450 A(L1,L) = A(L1,L) - A(ICOLUM,L) * T
  455 IF(M) 550, 550, 460
  460 DO 500 L=1,M
  500 B(L1,L) = B(L1,L) - B(ICOLUM,L) * T
  550 CONTINUE
C.... INTERCHANGE COLUMNS.
  600 DO 710 I=1,N
  610 L = N + 1 - I
  620 IF(INDEX(L,1) - INDEX(L,2)) 630, 710, 630
  630 JROW = INDEX(L,1)
  640 JCOLUM = INDEX(L,2)
  650 DO 705 K=1,N
  660 SWAP = A(K,JROW)
  670 A(K,JROW) = A(K,JCOLUM)
  700 A(K,JCOLUM) = SWAP
  705 CONTINUE
  710 CONTINUE
      DO 730 K=1,N
      IF(IPIVOT(K) - 1) 715, 730, 715
  730 CONTINUE
      IF(NCDEIN .EQ. 0) RETURN
      IF(NCDE .EQ. 1) GO TO 802
      WRITE (6,801)
  801 FORMAT ( //// 10X, 68HTHE MATRIX IS NEAR-SINGULAR SINCE ONE OF THE
     1 PIVOT ELEMENTS IS SMALL , / 10X, 58HCOMPARED TO THE MAGNITUDE OF
     2THE LARGEST ORIGINAL ELEMENT,  )
C.... SEARCH FOR THE LARGEST ELEMENT IN THE INVERTED MATRIX.
  802 A2BIG = 0.0
      DO 803 J=1,N
      DO 803 K=1,N
      IF(ABS (A2BIG) .LT. ABS (A(J,K))) A2BIG = A(J,K)
  803 CONTINUE
C.... DETERMINE THE LOWER BOUND AND THE UPPER BOUND OF THE CONDITION
C.... NUMBER FOR THE MATRIX.
      CNLB = ABS (A1BIG * A2BIG)
      CNUB = CNLB * FLOAT(N * N)
      WRITE (6,804) CNLB, CNUB
  804 FORMAT ( //// 10X, 42HTHE LOWER BOUND OF THE CONDITION NUMBER = ,
     1 E15.8,     / 10X, 42HTHE UPPER BOUND OF THE CONDITION NUMBER = ,
     2 E15.8, / 10X, 70HIF THE CONDITION NUMBER IS LARGER THAN 10**8 OR
     310**9, THEN THE MATRIX , / 10X, 67HIS ILL-CONDITIONED AND IT IS FA
     4IRLY SAFE TO CONCLUDE THAT SOLUTIONS , / 10X, 64HOF LINEAR EQUATIO
     5NS ASSOCIATED WITH THIS MATRIX ARE MEANINGLESS.    )
```

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          RETURN
      715 NODE = 0
          DETERM = 0.0
          WRITE (6,800)
      800 FORMAT ( //// 10X, 47HNO SOLUTION EXISTS SINCE THE MATRIX IS SINGU
         1LAR     )
          RETURN
          END

23697 WORDS OF MEMORY USED BY THIS COMPILATION
```

APPENDIX B

SAMPLE DATA FOR SQUARE PLATE PROBLEM

DATA FOR SQUARE PLATE ON ELASTIC FOUNDATION

```
 42   16    8    7  30000000  12000000        0.25         0.3
               2.0                   2.0                200.0
  1        0.        0.    3
  2        0.        2.    2
  3        0.        4.    2
  4        0.        6.    3
  5        2.        0.    2
  6        2.        2.    1
  7        2.        4.    1
  8        2.        6.    2
  9        4.        0.    2
 10        4.        2.    1
 11        4.        4.    1
 12        4.        6.    2
 13        6.        0.    3
 14        6.        2.    2
 15        6.        4.    2
 16        6.        6.    3
  1    2    1    5   1    6
  1    2    1  0.25
  2    3    1  0.25
  3    4    1  0.25
  5    6    1  0.25
  6    7    1  0.25
  7    8    1  0.25
  9   10    1  0.25
 10   11    1  0.25
 11   12    1  0.25
 13   14    1  0.25
 14   15    1  0.25
 15   16    1  0.25
  1    5    2  0.25
  5    9    2  0.25
  9   13    2  0.25
  2    6    2  0.25
  6   10    2  0.25
 10   14    2  0.25
  3    7    2  0.25
  7   11    2  0.25
 11   15    2  0.25
  4    8    2  0.25
```

8	12	2	0.25		
12	16	2	0.25		
1	6	3	0.25		
5	2	3	0.25		
2	7	3	0.25		
6	3	3	0.25		
3	8	3	0.25		
7	4	3	0.25		
5	10	3	0.25		
9	6	3	0.25		
6	11	3	0.25		
10	7	3	0.25		
7	12	3	0.25		
11	8	3	0.25		
9	14	3	0.25		
13	10	3	0.25		
10	15	3	0.25		
14	11	3	0.25		
11	16	3	0.25		
15	12	3	0.25		
4	1	0	0		
8	1	0	0		
12	1	0	0		
13	0	1	0		
14	0	1	0		
15	0	1	0		
16	1	1	0		
1	0				
16			0.	0.	-250.

APPENDIX C

THE SOLUTION OF PLATE MOMENTS AND SHEARS BY THE FINITE DIFFERENCE METHOD

The Solution of Plate Moments and Shears by the Finite Difference Method

The grid framework method calculates the joint displacements for all joints of the equivalent grid model. In addition, most grid programs will also compute the member forces for each member. However, when using this procedure for the analysis of an elastic plate, the internal member forces as computed cannot be used. As explained in Chapter IV, the grid member forces represent the localized effects of the individual member stiffness and not the total joint forces. The plate moments M_x, M_y, and M_{xy}, and the plate shears Q_x and Q_y can be obtained from the internal grid forces, but that procedure may entail somewhat lengthy computations. However, these forces are readily computed using the well known plate formulas applied either to the average curvatures of each element or to the vertical deflections. If these plate forces are to be computed from the deflections of the middle surface, the simplest approach is by finite differences.

Consider the general expressions for plate moments and shears as given by Timoshenko and Woinowsky-Krieger (8) as

$$M_x = -D(w_{xx} + \nu w_{yy}) \qquad \text{(C-1)}$$

$$M_y = -D(w_{yy} + \nu w_{xx}) \qquad (C-2)$$

$$M_{xy} = D(1 - \nu)w_{xy} \qquad (C-3)$$

$$Q_x = -D \frac{\partial}{\partial x}(w_{xx} + w_{yy}) \qquad (C-4)$$

and

$$Q_y = D \frac{\partial}{\partial y}(w_{xx} + w_{yy}) \qquad (C-5)$$

From Figure 17, the finite difference expressions for w_{xx}, w_{yy}, and w_{xy} are expressed as

$$w_{xx} = (w_{i+1,j} - 2w_{ij} + w_{i-1,j})/(\Delta x^2) \qquad (C-6)$$

$$w_{yy} = (w_{i,j+1} - 2w_{ij} + w_{i,j-1})/(\Delta y^2) \qquad (C-7)$$

$$w_{xy} = (-w_{i+1,j-1} + w_{i+1,j+1} + w_{i-1,j-1} - w_{i-1,j+1})/4\Delta x\Delta y \qquad (C-8)$$

Having the deflections for a given grid, the difference expressions of Equations (C-6) through (C-8) are readily formed. Substituting these values into the general equations for the plate moments and shears, the resulting internal plate forces M_x, M_y, M_{xy}, Q_x, and Q_y are obtained.

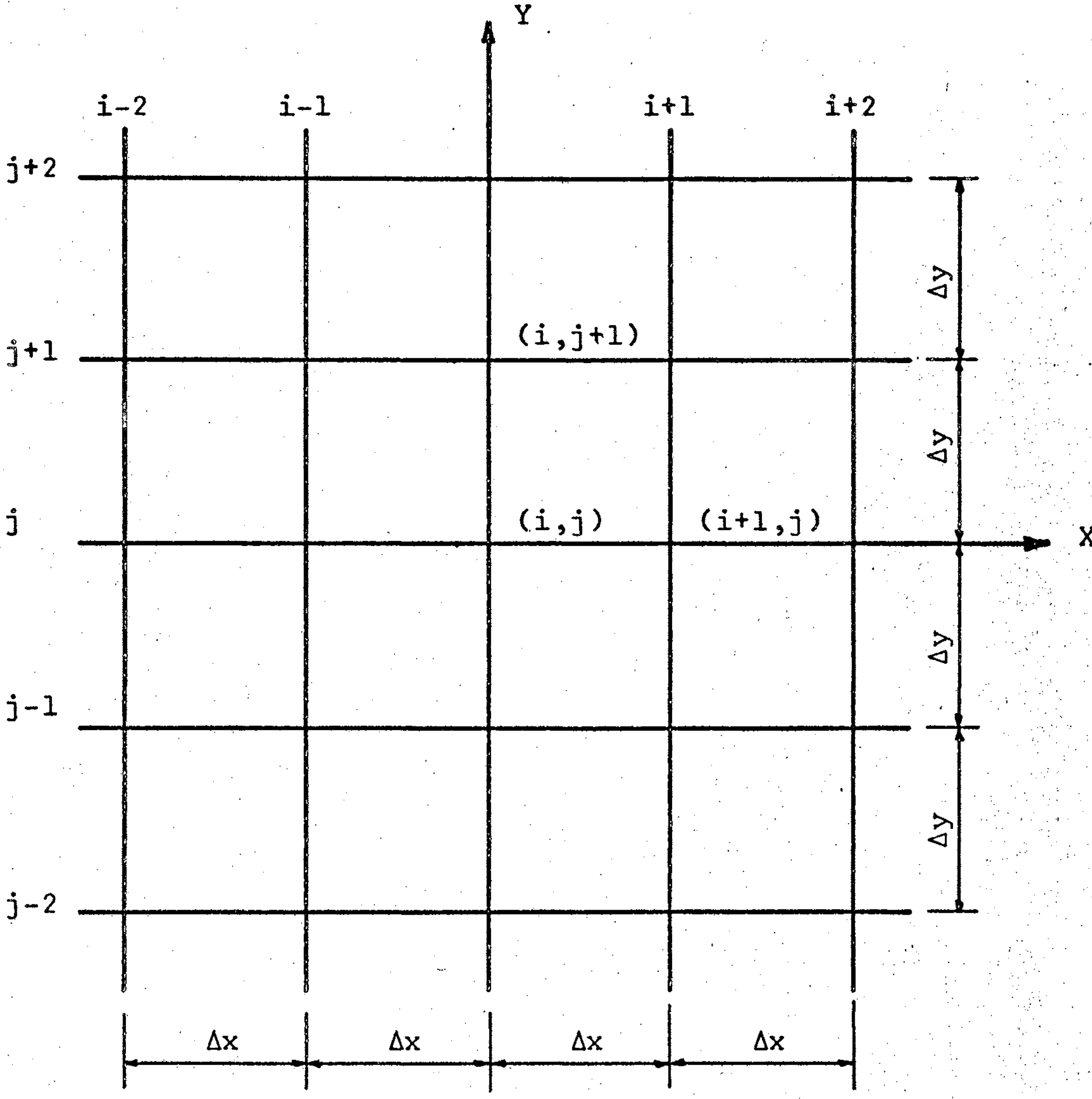

Figure 17. General Difference Grid

VITA

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