SUMMABILITY METHODS, SEQUENCE SPACES AND APPLICATIONS

By

Thomas Emmett Ikard

Bachelor of Arts
Panhandle State College
Goodwell, Oklahoma
1961

Master of Science
Oklahoma State University
Stillwater, Oklahoma
1963

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
May 1970

SUMMABILITY METHODS, SEQUENCE SPACES

AND APPLICATIONS

Thesis Approved:

Jeanne Agnew
Thesis Adviser

John Jewett

William Marsden

D. Durham
Dean of the Graduate College

PREFACE

Several recent texts which would be appropriate for the three semester-hour course in real analysis recommended for the General Curriculum in Mathematics for Colleges by the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America contain an introduction to the topics of divergent sequences and summability methods.

This collection of results on summability methods, sequence spaces, and applications is intended for those students who show an interest in investigating methods which are more general than convergence by which a number can be assigned to a sequence.

The writer acknowledges his indebtedness to Professors L. Wayne Johnson and John Jewett, and to each member of the mathematics faculty for the assistance and encouragement they have given. Professor Jeanne L. Agnew deserves whatever credit this work is due. Her patience and tolerance do not seem to have an upper bound. My family has lent me the moral support necessary to persevere in this effort, and my wife, Phyllis, has been my greatest help.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The topics of infinite sequences, convergence, and infinite series are introduced in elementary calculus. In most instances divergent sequences and series are not given much attention. Once a sequence or series is shown to be divergent, it is not usually regarded as an object of compelling interest. One of the simplest examples of a divergent series is the alternating series $\sum_{n=0}^{\infty} (-1)^n$ which seems at first glance to sum nicely to zero if it is written as

$$\sum_{n=0}^{\infty} (-1)^n = [1+(-1)] + [1+(-1)] + \ldots,$$

but the sequence of partial sums is an oscillating sequence of zeros and ones and does not converge. This is an excellent example of the fact that parentheses cannot be inserted or removed with impunity in the case of a divergent series.

Since the definitions of convergence of a sequence and of the sum of an infinite series are a way of assigning a number to a sequence, some students must wonder if there could be a number between zero and one which might be assigned to $\sum_{n=0}^{\infty} (-1)^n$ in some other natural way. Some students may discover that the sequence of arithmetic means of the sequence of partial sums of $\sum_{n=0}^{\infty} (-1)^n$ converges to 1/2, which is the arithmetic mean of 0 and 1.

In any case, the concept of the limit of a sequence can be extended for students who wonder about divergent series by an introduction to methods of summability, to the structure of the set of sequences, and to some applications of methods of summability.

Some of the more prominent mathematicians who have contributed to the theory of divergent sequences and series are Niels Henrik Abel (1802-1829), Emile Borel (1871-1956), Augustin Louis Cauchy (1789-1857), Ernesto Césaro (1859-1906), Peter Dirichlet (1805-1859), Leonhard Euler (1707-1783), Leopold Fejér (1880-1959), Jean Baptiste Joseph Baron de Fourier (1768-1830), David Hilbert (1862-1943), Otto Hölder (1859-1937), Gottfried Wilhelm von Leibniz (1646-1716), Gosta Mittag-Leffler (1846-1927), and Simeon Denis Poisson (1781-1840). Comprehensive collections of the theory and applications of divergent series were written by K. Knopp in 1928 and G. H. Hardy in 1949. Leibniz and Euler used divergent series in some of their works in analysis although Abel is reported to have written, "Divergent series are an invention of the devil, and it is shameful to base any demonstration on them whatsoever." Perhaps his remark stimulated mathematicians into efforts to make divergent series respectable.

The usual definitions and some theorems which follow readily from them are listed below for reference or for comparison with similar theorems concerning summability methods.

Definition 1.1. A sequence $x = \{x_n\}$, $n \in I^+$ of complex numbers is a function from I^+ into E.

It is customary to write $\{x_n\}$ for the sequence rather than $\{(n, x_n)\}$ where $x_n = f(n)$, since the domain of a sequence is always the positive or the nonnegative integers.

Definition 1.2. Let $\{a_n\}$ and $\{b_n\}$ be two sequences. These sequences are the same if and only if $a_n = b_n$ for every $n \in I^+$.

Definition 1.3. A sequence $\{a_n\}$ in E, the set of complex numbers, converges to a complex number a if and only if, given any real number $\mathcal{E} > 0$, there exists an integer N such that $n > N$ implies $|a - a_n| < \mathcal{E}$. a is called the limit of the sequence $\{a_n\}$, written $\lim a_n = a$.

Theorem 1.4. $\lim a_n = a$ and $\lim a_n = b$ implies $a = b$ and $a_n = a$ for every n implies $\lim a_n = a$.

Theorem 1.5. Let $\lim a_n = a$ and let $\lim b_n = b$. Then

i) $\lim (a_n + b_n) = a + b$,

ii) $\lim a_n b_n = ab$,

iii) if $c \in E$ then $\lim ca_n = ca$.

From the first theorem one can observe that the concept of convergence corresponds to the idea of a function defined on the set of convergent sequences. The second theorem states that the limit function is additive, multiplicative, and homogeneous.

The notion of a subsequence is frequently a useful tool.

Definition 1.6. A subsequence y of the sequence $\{n\}$, $n \in I^+$ is a function from I^+ into I^+ such that $y(i) < y(j)$ if $i < j$ for i, j in I^+.

Definition 1.7. If $x = \{x_n\}$ is a sequence of complex numbers and $y = \{n_i\}$ is a subsequence of I^+, then $x(y) = \{x_{n_i}\}$ is called a subsequence of x.

Boundedness and monotonicity are also properties which will be useful in what follows.

Definition 1.8. The sequence $\{a_n\}$ in E is bounded if and only if there exists a nonnegative number M such that $|a_n| \leq M$ for every $n \in I^+$.

The following theorem is a direct result of the last two definitions.

Theorem 1.9. Every subsequence of a bounded sequence in E is bounded.

Since a sequence is a function it can be characterized as monotone increasing or monotone decreasing if it is a sequence of real numbers.

Definition 1.10. A sequence $\{a_n\}$ in R is monotone nondecreasing if and only if $a_n \leq a_{n+1}$ for all $n \in I^+$. A sequence is monotone nonincreasing if and only if $a_{n+1} \leq a_n$ for all $n \in I^+$. A sequence is monotone if and only if it is monotone nondecreasing or monotone nonincreasing. A sequence $\{a_n\}$ is monotone increasing if and only if $a_n < a_{n+1}$ for all $n \in I^+$. A sequence $\{a_n\}$ is monotone decreasing if and only if $a_n > a_{n+1}$ for all $n \in I^+$. A sequence is strictly monotone if and only if it is monotone increasing or monotone decreasing.

The relationships of subsequences and monotonicity is clear from the following theorem.

Theorem 1.11. Every subsequence of a monotone nonincreasing (nondecreasing) sequence is monotone nonincreasing (nondecreasing).

Every subsequence of a monotone increasing (decreasing) sequence is monotone increasing (decreasing).

Sequences in R have the following important property.

Theorem 1.12. Every sequence of real numbers has a monotone subsequence.

The supremum or least upper bound and the infimum or greatest lower bound of a set of real numbers are defined next.

Definition 1.13. If $A \subset R$ then b is the supremum of A, written $b = \sup A$, if and only if for all $a \in A$, $a \leq b$ and for any x such that, for all $a \in A$, $a \leq x$ then $b \leq x$.

Definition 1.14. If $A \subset R$ then c is the infimum of A, written $c = \inf A$, if and only if for all $a \in A$, $c \leq a$ and for any y such that for all $a \in A$, $y \leq a$ then $y \leq c$.

The completeness of R and E can be stated in terms of Cauchy sequences.

Definition 1.15. A sequence $\{a_n\}$ in E is called a Cauchy sequence (or a fundamental sequence) if and only if for every real number $\varepsilon > 0$ there exists $N \in I^+$ such that $|a_m - a_n| < \varepsilon$ whenever $m, n > N$.

Theorem 1.16. If $\{a_n\}$ is Cauchy in E or in R then $\lim a_n$ exists and is an element of E or R respectively.

Theorem 1.17. The sequence $\{a_n\}$ in E is convergent if and only if $\{a_n\}$ is a Cauchy sequence in E.

Theorem 1.18. Every convergent sequence in E is bounded.

Theorem 1.19. A monotone sequence in R converges if and only if it is bounded.

There are, of course, sequences in R which are bounded which are not convergent. Consider the sequence $\{a_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ where $a_{2n} = 0$ and $a_{2n-1} = 1$, $n \in I^+$. It can be seen that the subsequences $\{a_{2n}\}$ and $\{a_{2n-1}\}$ are convergent sequences. This observation is included in the Bolzano-Weierstrass Theorem for sequences.

Theorem 1.20. Every bounded sequence in R has a convergent subsequence.

The concepts of limit superior and limit inferior for sequences in R are defined next.

Definition 1.21. Let $\{a_n\}$ be a sequence in R and let $u \in R$ such that

i) for every $\mathcal{E} > 0$ there exists $N \in I^+$ such that $n > N$ implies $a_n < u + \mathcal{E}$,

ii) for every $\mathcal{E} > 0$ and for every $m > 0$ there exists $n \in I^+$, $n > m$ such that $a_n > u - \mathcal{E}$.

Then u is the limit superior of $\{a_n\}$, written $u = \overline{\lim}\, a_n$. The limit inferior of $\{a_n\}$, written $\underline{\lim}\, a_n$, is defined to be $-\overline{\lim}\, b_n$ where $b_n = -a_n$ for all $n \in I^+$.

Thus the sequence $\{a_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ has $\overline{\lim}\, a_n = 1$, $\underline{\lim}\, a_n = 0$. The following theorem lists some of the more important properties of $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ and their connection with convergence.

Theorem 1.22. Let $\{a_n\}$ be a sequence in R. Then:

i) $\underline{\lim}\, a_n \leq \overline{\lim}\, a_n$,

ii) $\{a_n\}$ converges if and only if $\overline{\lim}\, a_n$, $\underline{\lim}\, a_n \in R$ and $\overline{\lim}\, a_n = \underline{\lim}\, a_n$. In this case $\lim a_n = \overline{\lim}\, a_n = \underline{\lim}\, a_n$.

Infinite series are defined in terms of their sequences of partial sums and the results for sequences are directly applicable.

Definition 1.23. Let $\{a_n\}$ be a sequence in E, and define

$$s_n = a_1 + \ldots + a_n = \sum_{k=1}^{n} a_k, \; n \in I^+.$$

The sequence $\{s_n\}$ is called an infinite series. The number a_n is called the nth term of the series. The series converges if and only if $\{s_n\}$ converges. Write $\sum_{k=1}^{\infty} a_k$ for $\{s_n\}$, and if $\lim s_n = s$, write

$$\sum_{k=1}^{\infty} a_k = s.$$

A series may sometimes be written more conveniently as $\sum_{n=0}^{\infty} a_n$ and

$$s_n = a_0 + a_1 + \ldots + a_n = \sum_{k=0}^{n} a_k.$$

Theorem 1.24. Let

$$a = \sum_{n=1}^{\infty} a_n \quad \text{and} \quad b = \sum_{n=1}^{\infty} b_n$$

in E. Then for any $\alpha, \beta \in E$,

$$\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n.$$

Theorem 1.25. Let $a_n \geq 0$ for all $n \in I^+$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\{s_n\}$ is bounded above.

Theorem 1.26. Let $\{a_n\}$ and $\{b_n\}$ be sequences in E such that $a_n = b_{n+1} - b_n$ for all $n \in I^+$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\lim b_n = b \in E$. In this case,

$$\sum_{n=1}^{\infty} a_n = \lim b_n - b_1.$$

Theorem 1.27. $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\mathcal{E} > 0$ there exists $N \in I^+$ such that $n > N$ implies $|a_{n+1} + \ldots + a_{n+p}| < \mathcal{E}$ for each $p \in I^+$.

Definition 1.28. If $a_n > 0$ for all $n \in I^+$, the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

is called an alternating series.

Theorem 1.29. If $\{a_n\}$ is a decreasing sequence converging to 0, the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Notice that in the case of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \, 1/n$$

the sequence of partial sums is the sequence $\{s_n\} = \{1, -1/2, 1/3, \ldots\}$ and

$$\sum_{n=1}^{\infty} (-1)^{n+1}\, 1/n$$

converges by Theorem 1.29. However

$$\sum_{n=1}^{\infty} |(-1)^{n+1}\, 1/n|$$

is the harmonic series which diverges. This concept is formalized in the next definition.

Definition 1.30. A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges. It is called conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem 1.31. Absolute convergence of $\sum_{n=1}^{\infty} a_n$ implies convergence.

The following topics will be used in Chapter IV.

Definition 1.32. Let $z_0 \in E$ and let $a_n \in E$ for $n \in I^+ \cup \{0\}$. Then the infinite series

$$a_0 + \sum_{n=1}^{\infty} a_n(z - z_0)^n,$$

or more briefly

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

is called a power series in $z - z_0$.

Theorem 1.33. Let

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

be a power series and let $\lambda = \overline{\lim} \sqrt[n]{|a_n|}$, $r = 1/\lambda$, (where $r = 0$ if $\lambda = +\infty$ and $r = +\infty$ if $\lambda = 0$). Then the series converges absolutely if $|z - z_0| < r$ and diverges if $|z - z_0| > r$.

The material included in this section is not exhaustive of the topics to be considered concerning convergence of sequences and series, but should suffice as a background for what is to follow. Additional material and proofs of theorems included here may be found in many books, for example [2], [7], and [10].

CHAPTER II

SUMMABILITY METHODS

The objective of this chapter is to consider methods of assigning a number to sequences which are divergent. One method, which would certainly make this investigation a short one, would be to assign each sequence in E the number 0. This would not produce many interesting results. One consideration to be kept in mind is that a worthwhile method of assigning a number to a sequence should not cause a convergent sequence to diverge. In other words, a desirable method should preserve the property of convergence when applied to convergent sequences. If it has become customary to assign the sequence $\{1+2^{1-n}\}$ the number 1, it might be desirable to continue the custom. Thus a desirable summability method might be required to assign convergent sequences their usual limits. Some definitions and theorems to formalize these concepts and some examples of summability methods follow.

Again let E represent the set of complex numbers, R the set of real numbers. Let s represent the set of sequences in E, c the set of convergent sequences in E, and $F[(0, \infty)]$ the set of complex functions defined on $(0, \infty) \subset R$.

Definition 2.1. Let $\{f_n(x)\}$ be a sequence of functions in $F[(0, \infty)]$ and let $\{z_n\}$ be a sequence of complex numbers. If

$$g(x) = \sum_{n=1}^{\infty} f_n(x) z_n$$

belongs to $F[(0,\infty)]$ and if $\lim_{x\to\infty} g(x) = t \neq \infty$, then $\{f_n(x)\}$ will be called a sequence-to-function summability method (or transformation), and $\{z_n\}$ will be said to be in the domain of $\{f_n(x)\}$.

The sequence-to-function transformation $\{f_n(x)\}$ operates on the sequence $\{z_n\}$ in a way suggested by the inner product of vectors. One would define an inner product of vectors in an infinite dimensional vector space to be the infinite series used to define g(x). For example, let $\{z_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ and let

$$\{f_n(x)\} = \{(\frac{x}{3x+1})^n\}.$$

Example 2.2.

$$\left\{\frac{x}{3x+1}, \left(\frac{x}{3x+1}\right)^2, \left(\frac{x}{3x+1}\right)^3, \ldots, \left(\frac{x}{3x+1}\right)^n, \ldots\right\} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{Bmatrix} = g(x).$$

$$g(x) = \frac{x}{3x+1} + \left(\frac{x}{3x+1}\right)^3 + \ldots + \left(\frac{x}{3x+1}\right)^{2n-1} + \ldots = \sum_{n=1}^{\infty} \left(\frac{x}{3x+1}\right)^{2n-1}.$$

The formula for the sum of a geometric series can be used to write this in closed form as

$$g(x) = \frac{x}{3x+1}\left(1 - \frac{x}{3x+1}\right)^{-2}.$$

Hence

$$g(x) = \frac{3x^2+x}{8x^2+6x+1},$$

and $\{z_n\}$ is assigned the number $\lim_{x\to\infty} g(x) = 3/8$. If $\{f_n(x)\} = \{1/2^n\}$, a sequence of constant functions in $F[(0, \infty)]$, then

Example 2.3.

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\right\} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = g(x) = \sum_{n=1}^{\infty} 2^{1-2n}.$$

In this instance

$$g(x) = 1/2\left(\frac{1}{1-1/4}\right) = 1/2(4/3) = 2/3,$$

and $\{z_n\}$ is assigned the number $\lim_{x \to \infty} g(x) = 2/3$. Notice that the sequence

$$\left\{\left(\frac{x}{3x+1}\right)^n\right\}$$

and the sequence $\left\{\frac{1}{2^n}\right\}$ assign the divergent sequence $\{1, 0, 1, 0, 1, 0, \ldots\}$ two different limits, 3/8 and 2/3 respectively. This is an indication of the variety which exists when different summability methods are applied.

It is quite possible that a particular sequence $\{f_n(x)\}$ will not transform a sequence to a function $g(x) \in F[(0, \infty)]$. Consider the sequence

$$\{f_n(x)\} = \left\{\left(\frac{2x}{x+1}\right)^n\right\}.$$

Here

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{2x}{x+1}\right)^{2n-1}$$

and for $x \geq 1$, $\frac{2x}{x+1} \geq 1$ so that

$$\sum_{n=1}^{\infty} \left(\frac{2x}{x+1}\right)^{2n-1}$$

diverges and g(x) is not defined. Thus in Definition 2. 1 the statement that g(x) belongs to $F[(0,\infty)]$ is not satisfied. In general it is necessary to assume, whenever the expression

$$g(x) = \sum_{n=1}^{\infty} f_n(x) z_n$$

appears, that $\{z_n\}$ is in the domain of $\{f_n(x)\}$ and that g(x) is a function in $F[(0,\infty)]$. Care must therefore be exercised in the application of the definitions and theorems in each particular case.

A case of particular importance arises when the functions $\{f_k(x)\}$ in Definition 2. 1 are step functions. That is, when $f_k(x)$ is constant on each interval $(n-1, n]$. Consider the function values at the right-hand endpoints n. Since the sequence $\{n\} = \{1, 2, 3, \ldots, n, \ldots\}$ is an element of s, the symbol $\{f_k(n)\}$ represents a sequence of sequences. That is, the continuous variable x is replaced by the discrete variable n and the set of function values

$$\{a_{nk} : a_{nk} = f_k(n),\ a_{nk} \in E\}$$

can be arranged in the rectangular array of an infinite matrix. As usual, n denotes the row subscript and k denotes the column subscript.

Definition 2. 4. Let (a_{nk}) be an infinite matrix of complex numbers and let $\{z_n\}$ be a sequence of complex numbers. If

$$\{z'_n\} = \{\sum_{k=1}^{\infty} a_{nk} z_k\}$$

belongs to s, then (a_{nk}) will be called a sequence-to-sequence summability method (or transformation), and $\{z_n\}$ will be said to be in the domain of (a_{nk}).

A summability matrix transforms sequences in the same way matrices transform vectors in a linear space. For example, let $\{z_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ and let $(a_{nk}) = (1/(n+1)^k)$ so that

Example 2.5.

$$(a_{nk})\{z_n\} = \begin{pmatrix} 1/2 & 1/4 & 1/8 & \ldots & 1/2^k & \ldots \\ 1/3 & 1/9 & 1/27 & \ldots & 1/3^k & \ldots \\ 1/4 & 1/16 & 1/64 & \ldots & 1/4^k & \ldots \\ \vdots & \vdots & \vdots & & \vdots & \\ 1/n+1 & 1/(n+1)^2 & 1/(n+1)^3 & \ldots & 1/(n+1)^k & \ldots \\ \vdots & \vdots & \vdots & & \vdots & \end{pmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{Bmatrix}$$

$$= \left\{ \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}}, \sum_{k=1}^{\infty} \frac{1}{3^{2k-1}}, \sum_{k=1}^{\infty} \frac{1}{4^{2k-1}}, \ldots, \sum_{k=1}^{\infty} \frac{1}{(n+1)^{2k-1}} \ldots \right\}$$

$$= \left\{ \frac{2}{3}, \frac{3}{8}, \frac{4}{15}, \ldots, \frac{n+1}{n(n+2)}, \ldots \right\}$$

$$= \{z'_n\} \text{ and } \lim z'_n = 0.$$

If (a_{nk}) is the infinite matrix where $a_{n,2n-1} = 1$ and $a_{ij} = 0$ otherwise, and if $\{z_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$, then

<u>Example 2.6.</u>

$$(a_{nk})(z_n) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{Bmatrix}$$

so that $\{z'_n\} = \{1, 1, 1, \ldots\}$ and $\lim z'_n = 1$. Notice that these matrices respectively assign the limits 0 and 1 to $\{1, 0, 1, 0, 1, 0, \ldots\}$, so that variety is still possible when infinite matrices instead of sequences of functions are used to transform divergent sequences.

There are situations where a particular infinite matrix cannot transform a particular sequence since $\sum_{k=1}^{\infty} a_{nk} z_n$ may be a divergent series. Consider the matrix (a_{nk}) where $a_{nk} = 1$ for $n, k \in I^+$ and the sequence $\{z_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$. In this case each series $\sum_{k=1}^{\infty} a_{nk} z_n$ is divergent. Again it will be necessary to assume whenever $\{z'_n\}$ is written as $\{\sum_{k=1}^{\infty} a_{nk} z_n\}$ that the matrix (a_{nk}) and the sequence $\{z_n\}$ are such that each series $\sum_{k=1}^{\infty} a_{nk} z_n$ is convergent, and to use caution in applying definitions and theorems to particular sequences and matrices.

The lemmas which follow are required for the proof of a fundamental theorem concerning summability methods. It will become evident that the proof is not short or easy. Shorter proofs using the methods of functional analysis are given in [11] and [13].

Lemma 2.7. Let $\{f_n(x)\}$ be a sequence of functions in $F[(0,\infty)]$ such that $\lim_{x\to\infty} f_n(x) = a_n \neq \infty$ for every $n \in I^+$. Then if there exists an $x_0 \in (0,\infty)$ and there exists $M \in (0,\infty)$ such that

$$\sum_{n=1}^{\infty} |f_n(x)| \leq M$$

for all $x > x_0 > 0$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Proof: $\lim_{x\to\infty} f_n(x) = a_n \neq \infty$ for all $n \in I^+$ implies that for every positive number $\mathcal{E}$ and each integer $p \in I^+$ there exists x, depending on n, written $x(n) > 0$ such that if $x > x(n)$ then $|a_n - f_n(x)| < \mathcal{E}/p$. Since $|a_n| - |f_n(x)| \leq |a_n - f_n(x)| < \mathcal{E}/p$, it follows that $|a_n| < |f_n(x)| + \mathcal{E}/p$ whenever $x > x(n)$.

Now

$$\sum_{n=1}^{\infty} |f_n(x)| \leq M$$

for all $x > x_0 > 0$ implies that

$$\sum_{n=1}^{p} |f_n(x)| \leq M$$

for every integer $p \in I^+$ whenever $x > x_0$. Given any $p \in I^+$,

$$\sum_{n=1}^{p} |a_n| < \sum_{n=1}^{p} (|f_n(x)| + \mathcal{E}/p)$$

for $x > \max\{x(1), \ldots, x(p)\}$, or

$$\sum_{n=1}^{p} |a_n| \leq M + \mathcal{E}$$

for $x > \max\{x_0, x(1), \ldots, x(p)\}$. Thus for each $p \in I^+$,

$$\sum_{n=1}^{p} |a_n| \leq M + \mathcal{E},$$

and the sequence of partial sums of $\sum_{n=1}^{\infty} |a_n|$ is bounded. This proves the lemma.

The next lemma concerns the behavior of a divergent sequence.

Lemma 2.8. If $\sum_{n=1}^{\infty} |u_n|$ is divergent, then there exists a sequence $\{z_n\}$ in E where $\lim z_n = 0$ and

$$\left\{ \left| \sum_{k=1}^{n} u_k z_k \right| \right\}$$

is not bounded.

Proof: Let $u_n = |u_n| e^{i\phi_n}$ and choose a real number $r > 1$. Then since $\sum_{n=1}^{\infty} |u_n|$ is divergent there exists $p_0 \in I^+$ such that

$$\sum_{n=1}^{p_0} |u_n| > r.$$

Now $r > 1$ implies $r^2 > r > 1$ and there exists $p_1 \in I^+$ such that

$$\sum_{n=p_0+1}^{p_1} |u_n| > r^2.$$

Similarly we have $r^3 > r^2 > r > 1$ and there exists $p_2 \in I^+$ such that

$$\sum_{n=p_1+1}^{p_2} |u_n| > r^3, \ \ldots \ .$$

Let

$$z_n = e^{-i\phi_n} \quad \text{for } 1 \leq n \leq p_0,$$

$$= \frac{e^{-i\phi_n}}{r} \quad \text{for } p_0 < n \leq p_1,$$

$$= \frac{e^{-i\phi_n}}{r^2} \quad \text{for } p_1 < n \leq p_2, \ldots$$

Note that $\lim z_n = 0$ and that

$$\sum_{n=1}^{\infty} u_n z_n = \sum_{n=1}^{p_0} |u_n| + 1/r \sum_{n=p_0+1}^{p_1} |u_n| + 1/r^2 \sum_{n=p_1+1}^{p_2} |u_n| + \ldots$$

$$> r + r + r + \ldots,$$

and the lemma is proved.

The following lemma discusses the properties of an infinite matrix whose rows and columns obey certain conditions.

<u>Lemma 2.9</u>. Let (s_{nk}), $n, k \in I^+$ be an infinite matrix of real numbers and let $\lim_{n \to \infty} s_{nk} = b_k$ for all $k \in I^+$. If

$$\sum_{k=1}^{\infty} |s_{nk}| = s_n$$

for every $n \in I^+$, then the sequence

$$\left\{ \sum_{k=1}^{p} |s_{nk}| \right\}$$

is bounded for each fixed $p \in I^+$.

Proof: The matrix is exhibited in Figure 2.1.

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} & \cdots & s_{1p} & \cdots \\ s_{21} & s_{22} & s_{23} & \cdots & s_{2p} & \cdots \\ s_{31} & s_{32} & s_{33} & \cdots & s_{3p} & \cdots \\ s_{n1} & s_{n2} & s_{n3} & \cdots & s_{np} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ \downarrow & \downarrow & \downarrow & & \downarrow & \end{pmatrix} \quad \begin{matrix} \Sigma |s_{1k}| = s_1 \\ \Sigma |s_{2k}| = s_2 \\ \Sigma |s_{3k}| = s_3 \\ \Sigma |s_{nk}| = s_n \\ \vdots \\ \end{matrix}$$

$$\begin{matrix} b_1 & b_2 & b_3 & \cdots & b_p & \cdots \end{matrix}$$

Figure 2.1.

The hypothesis that $\lim_{n \to \infty} s_{nk} = b_k$ justifies the appearance of column limits in the diagram. Further, $\lim_{n \to \infty} s_{nk} = b_k$ implies that for every positive number $\mathcal{E}$ and for each fixed integer $p \in I^+$ there exists $N(p) \in I^+$ such that if $n > N(p)$ then $|s_{nk} - b_k| < \mathcal{E}/p$. Hence $|s_{nk}| < |b_k| + \mathcal{E}/p$ and

$$\sum_{k=1}^{p} |s_{nk}| < \sum_{k=1}^{p} |b_k| + \mathcal{E}.$$

In Figure 2.1 this states that partial sums along the nth row are bounded by partial sums of the column limits for rows sufficiently far down in the array.

Now let us consider the rows above the (N+1)st row and the columns through the pth column. In this rectangular array in the upper left-hand corner of the diagram,

$$|s_{nk}| \leq \sum_{n=1}^{N} |s_{nk}|$$

so that

$$\sum_{k=1}^{p} |s_{nk}| < \left(\sum_{k=1}^{p} \sum_{n=1}^{N} |s_{nk}| \right) + 1,$$

where 1 is added to insure strict inequality. Therefore the partial row sums above the (N+1)st row are still bounded by the sum of the elements in the $n \times p$ rectangular array in the upper left-hand corner of the diagram plus one.

Suppose that

$$\left\{ \sum_{k=1}^{p} |s_{nk}| \right\}$$

is not bounded for some fixed $p \in I^+$, then for each $M > 0$ there exists $j \in I^+$ such that

$$\sum_{k=1}^{p} |s_{jk}| > M.$$

Let

$$M = \max \left\{ \sum_{k=1}^{p} |b_k| + \varepsilon, \left(\sum_{k=1}^{p} \sum_{n=1}^{N} |s_{nk}| \right) + 1 \right\}.$$

Now if $j > N$, then

$$\sum_{k=1}^{p} |s_{jk}| < \sum_{k=1}^{p} |b_k| + \varepsilon \leq M$$

and if $1 \leq j \leq N$, then

$$\sum_{k=1}^{p} |s_{jk}| < \left(\sum_{k=1}^{p} \sum_{n=1}^{N} |s_{nk}| \right) + 1 \leq M$$

which is a contradiction. Hence for each fixed $p \in I^+$ there exists $M_p > 0$ such that

$$\sum_{k=1}^{p} |s_{nk}| \leq M_p$$

for all $n \in I^+$.

In the case that

$$\left\{ \sum_{k=1}^{\infty} |s_{nk}| \right\}$$

is not a bounded sequence, the next lemma exhibits a convergent sequence $\{x_k\}$ such that the transform of $\{x_k\}$ by (s_{nk}) diverges.

Lemma 2.10. If

$$\sum_{k=1}^{\infty} |s_{nk}| = s_n$$

for all $n \in I^+$ and $\{s_n\}$ is not bounded, then there exists a sequence $\{x_k\}$ such that $|x_k| \leq 1$ for $k \in I^+$, $\lim x_k = 0$, and

$$\{x'_n\} = \left\{ \sum_{k=1}^{\infty} s_{nk} x_k \right\}$$

has a subsequence $\{x'_{n_i}\}$ which diverges.

Proof: If $\{s_n\}$ is not bounded then for every $r > 1$ there exists n_1 and q_1 in I^+ such that

$$\sum_{k=1}^{q_1} |s_{n_1 k}| > r^2.$$

Otherwise,

$$\sum_{k=1}^{q_1} |s_{n_1 k}| \leq r^2$$

for every q_1 and $n_1 \epsilon\ I^+$ would imply that $s_{n_1} \leq r^2$ for all $n_1 \epsilon\ I^+$, so that $\{s_n\}$ is bounded, contrary to hypothesis.

Let $\mathcal{E} > 0$, then there exists $N \epsilon\ I^+$ such that

$$\left| s_n - \sum_{k=1}^{m} |s_{nk}| \right| < \mathcal{E}$$

whenever $m > N$. Hence

$$s_n - \sum_{k=1}^{m} |s_{nk}| < \mathcal{E}$$

whenever $m > N$. Now

$$\sum_{k=1}^{\infty} |s_{nk}| = s_n$$

implies that

$$\sum_{k=m+1}^{\infty} |s_{nk}| < \mathcal{E}$$

whenever $m > N$. Thus there exists an integer $p_1 \geq q_1$ such that

$$\sum_{k=p_1+1}^{\infty} |s_{n_1k}| < \varepsilon .$$

Now the first p_1 terms of $\{x_k\}$ may be defined as follows,

$$x_1 = \frac{|s_{n_1 1}|}{r s_{n_1 1}} \quad \text{if } s_{n_1 1} \neq 0, \quad x_1 = 0 \text{ if } s_{n_1 1} = 0$$

$$x_2 = \frac{|s_{n_1 2}|}{r s_{n_1 2}} \quad \text{if } s_{n_1 2} \neq 0, \quad x_2 = 0 \text{ if } s_{n_1 2} = 0$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$x_{p_1} = \frac{|s_{n_1 p_1}|}{r s_{n_1 p_1}} \quad \text{if } s_{n_1 p_1} \neq 0, \quad x_{p_1} = 0 \text{ if } s_{n_1 p_1} = 0.$$

Notice that

$$\frac{|s_{n_1k}|}{s_{n_1k}} = \begin{cases} 1 & \text{if } s_{n_1k} > 0 \\ -1 & \text{if } s_{n_1k} < 0 \end{cases},$$

and that

$$s_{n_1k} x_k = \frac{|s_{n_1k}|}{r} \quad \text{for } 1 \leq k \leq p_1 .$$

Now consider the transform of the first p_1 terms of $\{x_k\}$ by the n_1st row of (s_{nk}),

$$s_{n_1 1}x_1 + s_{n_1 2}x_2 + \ldots + s_{n_1 p_1}x_{p_1} = \frac{1}{r} \sum_{k=1}^{p_1} |s_{n_1k}| .$$

Thus, $p_1 > q_1$ and

$$\frac{1}{r} \sum_{k=1}^{p_1} |s_{n_1k}| > r .$$

Also

$$|x_k| = \left| \frac{|s_{n_1k}|}{rs_{n_1k}} \right| = \frac{1}{r} < 1$$

if $s_{n_1k} \neq 0$ and $|x_k| = 0$ if $s_{n_1k} = 0$ so that $|x_k| < 1$ for $1 \leq k \leq p_1$. With the assumption that it will be possible to define $\{x_k\}$ so that $|x_k| < 1$ for every $k \in I^+$, let

$$x'_{n_1} = \frac{1}{r} \sum_{k=1}^{p_1} |s_{n_1k}| + \sum_{k=p_1+1}^{\infty} s_{n_1k} x_k .$$

Now

$$\sum_{k=p_1+1}^{\infty} |s_{n_1k}| < \varepsilon$$

and

$$\sum_{k=p_1+1}^{\infty} |s_{n_1k} x_k| < \sum_{k=p_1+1}^{\infty} |s_{n_1k}| < \varepsilon$$

so that

$$|x'_{n_1}| > \frac{1}{r} \sum_{k=1}^{p_1} |s_{n_1k}| - \sum_{k=p_1+1}^{\infty} |s_{n_1k}| > r - \varepsilon .$$

Consider the following diagram.

$$\begin{pmatrix}
s_{11} & s_{12} & s_{13} & \cdots & s_{1q_1} & \cdots & s_{1p_1} & \cdots & s_{1q_2} & \cdots & s_{1p_2} & \cdots \\
s_{21} & s_{22} & s_{23} & \cdots & s_{2q_1} & \cdots & s_{2p_1} & \cdots & s_{2q_2} & \cdots & s_{2p_2} & \cdots \\
s_{31} & s_{32} & s_{33} & \cdots & s_{3q_1} & \cdots & s_{3p_1} & \cdots & s_{3q_2} & \cdots & s_{3p_2} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
s_{n_11} & s_{n_12} & s_{n_13} & \cdots & s_{n_1q_1} & \cdots & s_{n_1p_1} & \cdots & s_{n_1q_1} & \cdots & s_{n_1p_2} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
s_{n_21} & s_{n_22} & s_{n_23} & \cdots & s_{n_2q_1} & \cdots & s_{n_2p_1} & \cdots & s_{n_2q_2} & \cdots & s_{n_2p_2} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{pmatrix}$$

The subscripts n_1 and p_1 have been used to define the first p_1 terms of $\{x_k\}$ and the first term x'_{n_1} of a subsequence of

$$\left\{ \sum_{k=1}^{\infty} s_{nk} x_k \right\}.$$

Now choose the distinguished subscripts $n_2 > n_1$ and $q_2 > q_1$ such that

$$\frac{1}{r^2} \sum_{k=p_1+1}^{q_2} |s_{n_2 k}| > M_{p_1} + r^2,$$

where M_{p_1} is the upper bound for

$$\sum_{k=1}^{p_1} |s_{n_k}|$$

which is provided for in Lemma 2.9. Next choose the subscript $p_2 \geq q_2$ such that

$$\sum_{k=p_2+1}^{\infty} |s_{n_2 k}| < \varepsilon .$$

Now define x_k for $p_1 < k \leq p_2$ as follows:

$$x_{p_1+1} = \frac{|s_{n_2 p_1+1}|}{r^2 s_{n_2 p_1+1}} \quad \text{if } s_{n_2 p_1+1} \neq 0, \quad x_{p_1+1} = 0 \text{ if } s_{n_2 p_1+1} = 0$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$x_{p_2} = \frac{|s_{n_2 p_2}|}{r^2 s_{n_2 p_2}} \quad \text{if } s_{n_2 p_2} \neq 0, \quad x_{p_2} = 0 \text{ if } s_{n_2 p_2} = 0 .$$

Again notice that

$$|x_k| = \left|\frac{|s_{n_2k}|}{r^2 s_{n_2k}}\right| = \frac{1}{r^2} < 1$$

for $p_1 < k \leq p_2$, and that

$$s_{n_2k}x_k = \frac{|s_{n_2k}|}{r^2}$$

for $p_1 < k \leq p_2$. The transform of the first p_2 terms of $\{x_k\}$ by the n_2th row of (s_{nk}) is

$$\frac{1}{r}\sum_{k=1}^{p_1} s_{n_2k}x_k + \frac{1}{r^2}\sum_{k=p_1+1}^{p_2} |s_{n_2k}|.$$

Again assume that $|x_k| < 1$ for all $k \in I^+$ and define

$$x'_{n_2} = \frac{1}{r}\sum_{k=1}^{p_1} s_{n_2k}x_k + \frac{1}{r^2}\sum_{k=p_1+1}^{p_2} |s_{n_2k}| + \sum_{k=p_2+1}^{\infty} s_{n_2k}x_k.$$

Now

$$|x'_{n_2}| > \left|\frac{1}{r}\sum_{k=1}^{p_1} s_{n_2k}x_k + \frac{1}{r^2}\sum_{k=p_1+1}^{p_2} |s_{n_2k}|\right| - \varepsilon,$$

$$> \left|\frac{1}{r^2}\sum_{k=p_1+1}^{p_2} |s_{n_2k}|\right| - \left|\frac{1}{r}\sum_{k=1}^{p_1} s_{n_2k}x_k\right| - \varepsilon.$$

Thus since $|x_k| < 1$ for $1 \leq k \leq p_1$,

$$\left|\frac{1}{r}\sum_{k=1}^{p_1} s_{n_2k}x_k\right| < M_{p_1}$$

and

$$|x'_{n_2}| > M_{p_1} + r^2 - M_{p_1} - \varepsilon$$

or $|x'_{n_2}| > r^2 - \varepsilon$.

Continuing in this fashion, for each $k \in I^+$ there exists $n \in I^+$ such that $|x_k| < r^{-n}$ and $\lim x_k = 0$. In addition for each i and each k in I^+ there exists n_i in I^+ such that $|x'_{n_i}| \geq r^k$. Thus $\{x'_{n_i}\}$ diverges and the lemma is proved.

To illustrate Lemma 2, 10, consider the infinite matrix (s_{nk}) where $s_{1k} = 1/2^k$ and

$$s_{nk} = \left(\frac{n-1}{n}\right)^{k-1}$$

for $n > 1$, as in the diagram below.

$$\begin{array}{ccccccc}
\left(\begin{array}{cccccc}
1/2 & 1/4 & 1/8 & \cdots & 1/2^k & \cdots \\
1 & 1/2 & 1/4 & \cdots & 1/2^{k-1} & \cdots \\
1 & 2/3 & 4/9 & \cdots & (2/3)^{k-1} & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \\
1 & \frac{n-1}{n} & \left(\frac{n-1}{n}\right)^2 & \cdots & \left(\frac{n-1}{n}\right)^{k-1} & \cdots \\
\vdots & \vdots & \vdots & & \vdots &
\end{array}\right)
&
\begin{array}{l}
\Sigma\,|s_{1k}| = 1 \\
\Sigma\,|s_{2k}| = 2 \\
\Sigma\,|s_{3k}| = 3 \\
\vdots \\
\Sigma\,|s_{nk}| = n \\
\vdots
\end{array}
\end{array}$$

$$\begin{array}{cccccc}
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\
1 & 1 & 1 & & 1 & \infty
\end{array}$$

Let $r = 2$ so that integers n_1 and q_1 must be found so that

$$\sum_{k=1}^{q_1} |s_{n_1 k}| > 4.$$

If $n_1 = 8$ and $q_1 = 6$ then

$$\sum_{k=1}^{6} (7/8)^{k-1} = \frac{1-(7/8)^6}{1-7/8} = 8\left(\frac{144,495}{262,144}\right) > 4.4,$$

using the formula for the sum of n terms of a geometric progression. Following the pattern of Lemma 2.10, let $\mathcal{E} = 1$ so there must be an integer $p_1 \geq q_1 = 6$ so that

$$\sum_{k=p_1+1}^{\infty} (7/8)^{k-1} \leq 1.$$

If $p_1 = 20$ then

$$\sum_{k=1}^{20} (7/8)^{k-1} = \left(\frac{8^{20} - 7^{20}}{8^{19}}\right) > 7.4$$

and

$$\sum_{k=21}^{\infty} (7/8)^{k-1} < .6 < 1.$$

This means that $x_k = 1/2$ for $1 \leq k \leq 20$, and that

$$x'_{n_1} = 1/2\ (7.4) + .6 = 4.3 > r - \mathcal{E} = 2 - 1 = 1.$$

In the next part of the lemma, n_2 and q_2 must be chosen so that $n_2 > 8$ and $q_2 > 20$ and at the same time

$$\frac{1}{4} \sum_{k=21}^{q_2} \left(\frac{n_2 - 1}{n_2}\right)^{k-1} > M_{20} + 4$$

or

$$\sum_{k=21}^{q_2} \left(\frac{n_2 - 1}{n_2}\right)^{k-1} > 4M_{20} + 16.$$

Note that M_{20}, which is the maximum term of

$$\left\{\sum_{k=1}^{20} \left(\frac{n-1}{n}\right)^{k-1}\right\} = \left\{n - (n-1)\left(\frac{n-1}{n}\right)^{19}\right\},$$

is less than or equal to 20. Hence n_2 and q_2 must be such that

$$\sum_{k=21}^{q_2} \left(\frac{n_2 - 1}{n_2}\right)^{k-1} > 96.$$

Now

$$\sum_{k=1}^{20} \left(\frac{124}{125}\right)^{k-1} < 20,$$

so let $n_2 = 125$ and let $q_2 = 340$. Thus

$$\sum_{k=21}^{340} \left(\frac{124}{125}\right)^{k-1} \doteq 100$$

and $p_2 \geq 340$ can be chosen so that

$$\sum_{k=p_2+1}^{\infty} \left(\frac{124}{125}\right)^{k-1} < 1.$$

This gives $x_k = 1/4$, $20 < k \leq p_2$ and

$$x'_{n_2} = \frac{1}{2} \sum_{k=1}^{20} s_{n_2k}\left(\frac{1}{2}\right) + \frac{1}{4} \sum_{k=21}^{p_2} |s_{n_2k}| + \sum_{k=p_2+1}^{\infty} s_{n_2k}x_k,$$

$$|x'_{n_2}| > 20 + 4 - 20 - 1 = 2^2 - \mathcal{E} = 3.$$

Continuing in this fashion, $\lim x_k = 0$ and $\{|x'_{n_i}|\}$ is not bounded.

Now the conditions under which a summability transformation will transform convergent sequences into convergent sequences can be examined.

Definition 2.11. Let $f_n(x)$ be a sequence-to-function summability method. If

$$g(x) = \sum_{n=1}^{\infty} f_n(x)z_n$$

belongs to $F[(0, \infty)]$ and if $\lim_{x \to \infty} g(x) = t \neq \infty$ for every convergent sequence $\{z_n\}$ in E, then $\{f_n(x)\}$ is a conservative sequence-to-function summability method. In this case $\{f_n(x)\}$ will also be called a Kojima sequence or a K-sequence.

The following theorem, which gives necessary and sufficient conditions that a sequence $\{f_n(x)\}$ should be conservative, was proved by Kojima in 1917 and extended by Schur in 1918. Its present form is the result of further extension and refinement by Agnew, Cooke, Hardy, and others.

Theorem 2.12. (Kojima-Schur) Let $\{f_n(x)\}$ be a sequence-to-function summability method. Then $\{f_n(x)\}$ is a K-sequence if and only if:

i) there exists $x_0 \in (0, \infty)$ and there exists $M \in (0, \infty)$ such that $\sum_{n=1}^{\infty} |f_n(x)| \leq M$, for all $x > x_0 > 0$,

ii) $\lim_{x \to \infty} f_n(x) = a_n \neq \infty$ for all $n \in I^+$,

iii) $\sum_{n=1}^{\infty} f_n(x) = f(x)$ and $\lim_{x \to \infty} f(x) = a \neq \infty$.

In this case

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \sum_{n=1}^{\infty} f_n(x) z_n = az + \sum_{n=1}^{\infty} a_n(z_n - z)$$

where $\lim z_n = z$.

Proof: a) The three conditions are sufficient.

Let $\{z_n\}$ be a convergent sequence, that is, let $\lim z_n = z$. This means that for all positive real numbers M and ε, there exists $N_1 \in I^+$ such that $|z_n - z| < \varepsilon/3M$ whenever $n > N_1$. Let

$$k = \max\{|z_n - z| : 1 \le n \le N_1\},$$

so that $|z_n - z| < k + 1$ for $1 \le n \le N_1$.

From conditions i and ii, and Lemma 2.7, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Hence, there exists an integer N_2 such that

$$\sum_{n=N_2+1}^{\infty} |a_n| \le M,$$

for every positive number M.

Choose $N = \max\{N_1, N_2\}$ and consider $|f_n(x) - a_n|$, where $1 \le n \le N$. From condition ii, there exists $x' \in R^+$ such that

$$|f_n(x) - a_n| < \frac{\varepsilon}{3N(k+1)}$$

whenever $x > x'$. Hence

$$\sum_{n=1}^{N} |f_n(x) - a_n| < \frac{\varepsilon}{3(k+1)}$$

for $x > x'$.

These bounds for $|z_n - z|$, $1 \le n \le N_1$; for

$$\sum_{n=N_2+1}^{\infty} |a_n|;$$

and for

$$\sum_{n=1}^{N} |f_n(x) - a_n|,$$

$x > x'$ will be used to show that

$$\lim_{x \to \infty} \sum_{n=1}^{\infty} f_n(x)(z_n - z) = \sum_{n=1}^{\infty} a_n(z_n - z).$$

Condition i implies that

$$\sum_{n=N+1}^{\infty} |f_n(x)| \le M$$

whenever $x > x_0 > 0$. Now let $\overline{x} = \max\{x_0, x'\}$ and write

$$\left| \sum_{n=1}^{\infty} (f_n(x)-a_n)(z_n-z) \right| = \left| \sum_{n=1}^{N} (f_n(x)-a_n)(z_n-z) + \sum_{n=N+1}^{\infty} (f_n(x)-a_n)(z_n-z) \right|$$

$$\le \sum_{n=1}^{N} |f_n(x)-a_n|\,|z_n-z| + \sum_{n=N+1}^{\infty} |f_n(x)|\,|z_n-z|$$

$$+ \sum_{n=N+1}^{\infty} |a_n|\,|z_n-z|.$$

With N chosen, and with $x > \overline{x}$,

$$\left| \sum_{n=1}^{\infty} (f_n(x)-a_n)(z_n-z) \right| < \frac{\varepsilon}{3(k+1)} \quad (k+1) + M \cdot \frac{\varepsilon}{3M} + M \cdot \frac{\varepsilon}{3M} = \varepsilon.$$

Hence

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} f_n(x)(z_n-z) = \sum_{n=1}^{\infty} a_n(z_n-z),$$

and

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} f_n(x)z_n - z \lim_{x\to\infty} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} a_n(z_n-z).$$

Condition iii implies that

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} f_n(x)z_n - za = \sum_{n=1}^{\infty} a_n(z_n-z),$$

since

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} f_n(x) = a.$$

Now if

$$\sum_{n=1}^{\infty} a_n(z_n - z)$$

converges, then

$$\lim_{x \to \infty} \sum_{n=1}^{\infty} f_n(x) z_n$$

will exist as a finite number in E. Now

$$\sum_{n=1}^{\infty} |a_n| = t \neq \infty$$

since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and $|z_n| < r \neq \infty$ since $\lim z_n = z$. Also $|z_n - z| \leq |z_n| + |z| < r + |z|$, and therefore,

$$\sum_{n=1}^{\infty} |a_n| |z_n - z| < \sum_{n=1}^{\infty} |a_n| (r + |z|) = (r + |z|)t,$$

so that

$$\sum_{n=1}^{\infty} a_n(z_n - z)$$

is absolutely convergent. Thus

$$\sum_{n=1}^{\infty} a_n(z_n - z)$$

is convergent and

$$\lim_{x \to \infty} \sum_{n=1}^{\infty} f_n(x) z_n = \lim_{x \to \infty} g(x) = az + \sum_{n=1}^{\infty} a_n(z_n - z),$$

a finite number in E.

This proves that i, ii, and iii are sufficient.

b) The three conditions are necessary.

Lemma 2.10 will help to show that condition i is necessary. Suppose condition i is not satisfied. That is, suppose that given $x \in (0, \infty)$ and given $M \in (0, \infty)$ there exists $x_0 > x$ such that

$$\sum_{n=1}^{\infty} |f_n(x_0)| > M.$$

Let $f_n(x) = u_n(x) + iv_n(x)$, where $u_n(x)$ and $v_n(x)$ are real. Then there exists a sequence $\{y_n\}$ such that $\lim y_n = x_0$ and there exist sequences

$$\{s_n\} = \left\{\sum_{k=1}^{\infty} |u_k(y_n)|\right\} \quad \text{and} \quad \{t_n\} = \left\{\sum_{k=1}^{\infty} |v_k(y_n)|\right\}$$

such that for every $M > 0$, either $\overline{\lim}\ s_n > M$ or $\overline{\lim}\ t_n > M$. Suppose that for every $M > 0$, $\overline{\lim}\ s_n > M$ and write $s_{nk} = u_k(y_n)$, so that

$$s_n = \sum_{k=1}^{\infty} |s_{nk}|.$$

Lemma 2.10 shows that there exists a sequence $\{x_k\}$ which converges to zero, but the sequence

$$\{x'_n\} = \left\{\sum_{k=1}^{\infty} s_{nk}x_k\right\}$$

has a subsequence $\{x'_{n_i}\}$ which diverges. This means that if condition i is not satisfied then there is at least one convergent sequence which is transformed by $\{f_n(x)\}$ into a divergent sequence. Thus condition i is necessary.

For the second condition, let $z_n = 0$, $n \neq p$ and let $z_n = 1$ if $n = p$. Then $\lim z_n = 0$ and $g(x) = f_p(x)$ so that $\lim_{x \to \infty} f_n(x) = a_n \neq \infty$ for all $n \in I^+$ is necessary for $\lim_{x \to \infty} g(x) = z_0 \neq \infty$.

In the case of condition iii, let $z_n = 1$, $n \in I^+$ so that $\lim z_n = 1$. Then

$$g(x) = \sum_{n=1}^{\infty} f_n(x),$$

and therefore,

$$\sum_{n=1}^{\infty} f_n(x) = f(x),$$

where $\lim_{x \to \infty} f(x) = a \neq \infty$, is necessary for $\lim_{x \to \infty} g(x) = z_0 \neq \infty$.

This proves the theorem.

Example 2.13: An example of a K-sequence is the sequence $\{f_n(x)\} = \{[(e^{-x}+1)^2 + n^2]^{-1}\}$. Here

$$\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{\infty} [(e^{-x}+1)^2 + n^2]^{-1},$$

and it can be shown that

$$\sum_{n=1}^{\infty} |f_n(x)| = \frac{\pi}{2(e^{-x}+1)} \coth \pi(e^{-x}+1) - \frac{1}{2(e^{-x}+1)^2} .$$

Thus,

$$\sum_{n=1}^{\infty} |f_n(x)| \leq \frac{\pi}{2} \coth 2\pi - \frac{1}{8},$$

and

$$\sum_{n=1}^{\infty} |f_n(x)| \leq M$$

for $x \in (0, \infty)$. Also

$$\lim_{x \to \infty} f_n(x) = \lim_{x \to \infty} [(e^{-x}+1)^2 + n^2]^{-1} = (1 + n^2)^{-1}$$

for all $n \in I^+$.

Last,

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} f_n(x) = \lim_{x\to\infty} \frac{\pi \coth[\pi(e^{-x}+1)]}{2(e^{-x}+1)} - \frac{1}{2(e^{-x}+1)^2}$$

$$= \frac{\pi}{2} \coth \pi - \frac{1}{2}$$

This means that if $\lim z_n = z$ then

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left[\frac{\pi}{2} \coth \pi - \frac{1}{2}\right] z \quad + \sum_{n=1}^{\infty} \frac{z_n - z}{1+n^2}.$$

Let

$$\{z_n\} = \{\frac{1+n^2}{2^n}\}$$

so that $\lim z_n = 0$ and

$$\lim_{x\to\infty} g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Example 2.14. A K-sequence which transforms every convergent sequence into a sequence which has the limit zero is the sequence

$$\{f_n(x)\} = \left\{\frac{1}{x^2+4n^2\pi^2}\right\}.$$

Here

$$\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{\infty} \frac{1}{x^2+4n^2\pi^2} = \frac{1}{2x}\left(\frac{1}{2} - \frac{1}{x} + \frac{1}{e^x-1}\right),$$

from results in the theory of functions of a complex variable, and

$$\sum_{n=1}^{\infty} |f_n(x)| \leq \frac{1}{20}\left(\frac{4}{10} + \frac{1}{e^{10}-1}\right)$$

whenever $x > 10 > 0$. In this example, $\lim_{x\to\infty} f_n(x) = 0$ for all $n \in I^+$ and

$$\lim_{x \to \infty} \sum_{n=1}^{\infty} f_n(x) = 0 .$$

Hence

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \left[0z + \sum_{n=1}^{\infty} 0(z_n - z)\right] = 0$$

for every convergent sequence $\{z_n\}$.

Now that the notion of K-sequence has been characterized, the same scrutiny can be applied to infinite matrices. The next definition and theorem will do this.

<u>Definition 2. 15</u>. Let (a_{nk}) be a sequence-to-sequence summability method. If every convergent sequence $\{z_n\}$ in E is in the domain of (a_{nk}), and if

$$\lim z'_n = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} z_k = t \neq \infty,$$

then (a_{nk}) is a <u>conservative</u> sequence-to-sequence summability method. In this case (a_{nk}) will also be called a <u>Kojima matrix</u> or a <u>K-matrix</u>.

<u>Theorem 2. 16</u>. Let (a_{nk}) be a sequence-to-sequence summability method. Then (a_{nk}) is a K-matrix if and only if:

i) there exists $n_0 \in I^+$ and there exists $M \in (0, \infty)$ such that $\sum_{n=1}^{\infty} |a_{nk}| \leq M$ for every $n > n_0 > 0$, $n \in I^+$,

ii) $\lim_{n \to \infty} a_{nk} = b_k \neq \infty$ for every fixed $k \in I^+$,

iii) $\sum_{k=1}^{\infty} a_{nk} = r_n$ and $\lim r_n = a \neq \infty$.

In this case

$$\lim z'_n = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} z_k = az + \sum_{k=1}^{\infty} b_k(z_k - z)$$

where $\lim z_n = z$.

Proof: Let $n, k \in I^+$ and for $n - 1 < x \leq n$ define $f_k(x) = a_{nk}$. This means that the rows of (a_{nk}) correspond to a sequence of step functions in $F[(0, \infty)]$. Thus condition i holds if and only if

$$\sum_{k=1}^{\infty} |f_k(x)| = \sum_{k=1}^{\infty} |a_{nk}| \leq M$$

for every $x > n_0$. Next, condition ii holds if and only if

$$\lim_{x \to \infty} f_k(x) = \lim_{n \to \infty} a_{nk} = b_k \neq \infty$$

for every $k \in I^+$. Lastly, condition iii holds if and only if

$$\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{\infty} a_{nk} = r_n$$

and $\lim r_n = a \neq \infty$. Thus Theorem 2.16 is a special case of the Kojima-Schur theorem. This means that

$$\lim_{x \to \infty} \sum_{k=1}^{\infty} f_k(x) z_k = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} z_k$$

$$= az + \sum_{k=1}^{\infty} b_k(z_k - z)$$

$$= \lim z'_n$$

whenever $\lim z_n = z$, and the theorem is proved.

<u>Example 2.17.</u> The matrix

$$(a_{nk}) = \left(\frac{3n-1}{n2^k}\right)$$

is an example of a K-matrix, since

$$\sum_{k=1}^{\infty} \frac{3n-1}{n2^k} = \frac{3n-1}{n} \leq 4$$

for all $n \in I^+$ and since $\lim (3n-1)/n = 3$. Also,

$$\lim_{n \to \infty} \frac{3n-1}{n2^k} = \frac{3}{2^k}$$

for all $n \in I^+$. Here the characteristic numbers are $a_k = 3/2^k$ and $a = 3$. In this case the transform of a constant sequence will converge to three times the constant. The sequence $\{z_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ is transformed by (a_{nk}) into the sequence $\{\frac{2(3n-1)}{3n}\}$ which converges to 2.

Example 2.18. The matrix $(a_{nk}) = 2^{-k}$ is a K-matrix because

$$\sum_{k=1}^{\infty} 2^{-k} = 1$$

for all $n \in I^+$, and $\lim 1 = 1$. Here $\lim_{n \to \infty} 2^{-k} = 2^{-k}$, and the characteristic numbers of (a_{nk}) are $a_k = 2^{-k}$ and $a = 1$. The sequence $\{z_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ is transformed into the sequence $\{2/3, 2/3, 2/3, \ldots\}$, a constant sequence with limit 2/3.

Notice that Theorem 2.16 and Theorem 2.12 not only characterize K-matrices and K-sequences, but they also state the relationship between $\lim z_n$ if $\{z_n\} \in c$ and $\lim z'_n$ where $\{z'_n\}$ is the transform of $\{z_n\}$. The numbers a and a_n in Theorem 2.12 and the numbers a and b_k in Theorem 2.16 are called respectively the characteristic numbers of the K-sequence $\{f_n(x)\}$ or the characteristic numbers of the K-matrix (a_{nk}) because of their role in determining the value of the transformed sequence.

The next definitions and theorems are concerned with sequences $\{f_n(x)\}$ and matrices (a_{nk}) which assign convergent sequences their usual limits.

Definition 2.19. Let $\{f_n(x)\}$ be a sequence-to-function summability method. If every convergent sequence $\{z_n\}$ in E is the domain of $\{f_n(x)\}$ and if

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \sum_{n=1}^{\infty} f_n(x) z_n = z$$

whenever $\lim z_n = z$, then $\{f_n(x)\}$ is a regular sequence-to-function summability method. In this case $\{f_n(x)\}$ will also be called a Toeplitz sequence or a T-sequence.

The following theorem concerning T-sequences was first proved by Toeplitz in 1911, extended by Silverman in 1913 and by Schur in 1920.

Theorem 2.20. (Toeplitz-Silverman) Let $\{f_n(x)\}$ be a sequence-to-function summability method. Then $\{f_n(x)\}$ is a T-sequence if and only if:

i) there exists $x_0 \in (0, \infty)$ and there exists $M \in (0, \infty)$ such that $\sum_{n=1}^{\infty} |f_n(x)| \leq M$, for all $x > x_0 > 0$,

ii) $\lim_{x \to \infty} f_n(x) = 0$ for all $n \in I^+$,

iii) $\sum_{n=1}^{\infty} f_n(x) = f(x)$ and $\lim_{x \to \infty} f(x) = 1$.

In this case

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \sum_{n=1}^{\infty} f_n(x) z_n = z$$

whenever $\lim z_n = z \neq \infty$.

Proof: a) i, ii, and iii are sufficient. Let $a = 1$ and $a_n = 0$ in Theorem 2.12 so that

$$\lim_{x \to \infty} g(x) = 1z + \sum_{n=1}^{\infty} 0(z_n - z) = z.$$

b) i, ii, and iii are necessary. Since every T-sequence is a K-sequence, and condition i is necessary for $\{f_n(x)\}$ to be a K-sequence, then condition i is necessary for $\{f_n(x)\}$ to be a T-sequence.

Let $z_n = 0$ if $n \neq p$ and let $z_p = 2$. In this case $\lim z_n = 0$ and

$$\sum_{n=1}^{\infty} f_n(x) z_n = 2 f_p(x).$$

This will be zero only if $\lim_{x \to \infty} f_p(x) = 0$. Hence condition ii is necessary.

Let $z_n = 1$ for all $n \in I^+$. In this case $\lim z_n = 1$ and

$$\sum_{n=1}^{\infty} f_n(x) z_n = \sum_{n=1}^{\infty} f_n(x).$$

This will be one only if condition iii is satisfied. This proves the theorem.

<u>Example 2.21.</u> The Mittag-Leffler sequences are a collection of T-sequences where

$$\{f_n(x)\} = \left\{ \frac{g(n)\, x^{n-1}}{E(x)} \right\}$$

such that $g(n) \geq 0$, $g(n) > 0$ for infinitely many integers $n \in I^+$, and

$$E(z) = \sum_{n=1}^{\infty} g(n) z^{n-1}$$

is an entire function. That is, E(z) is analytic in the finite complex plane. Now

$$\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{\infty} \frac{g(n)x^{n-1}}{E(x)}$$

$$= \frac{1}{E(x)} \cdot \sum_{n=1}^{\infty} g(n)x^{n-1}$$

$$= \frac{1}{E(x)} \cdot E(x) = 1,$$

so that

$$\lim_{x \to \infty} \sum_{n=1}^{\infty} f_n(x) = 1.$$

Also

$$\lim_{x \to \infty} f_n(x) = g(n) \lim_{x \to \infty} \frac{x^{n-1}}{E(x)} .$$

Now

$$\frac{x^{n-1}}{E(x)} = \left(\sum_{k=1}^{n} g(k)x^{k-n} + \frac{x^n}{x^{n-1}} \sum_{k=1}^{\infty} g(n+k)x^{k-1} \right)^{-1}$$

so

$$\lim_{x \to \infty} \frac{x^{n-1}}{E(x)} = \left(M + \lim_{x \to \infty} x \sum_{k=1}^{\infty} g(n+k)x^{k-1} \right)^{-1} = 0$$

since $g(n) > 0$ for infinitely many $n \in I^+$.

Example 2.22. A particular Mittag-Leffler sequence is the Borel sequence,

$$\{f_n(x)\} = \left\{ \frac{x^{n-1}}{e^x(n-1)!} \right\}.$$

For the Borel sequence,

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{e^x(n-1)!} = e^{-x}(e^x) = 1.$$

Now

$$\lim_{x\to\infty} f_n(x) = \frac{1}{(n-1)!} \lim_{x\to\infty} \frac{1}{\sum_{k=1}^{\infty} \frac{x^{k-n+1}}{(k-1)!}} = 0.$$

Hence

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} f_n(x) = 1$$

and $\lim_{x\to\infty} f_n(x) = 0$ for all $n \in I^+$. Consider the sequence $\{z_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$. Then

$$\sum_{n=1}^{\infty} f_n(x)z_n = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{e^x(2n-1)!} = e^{-x} \sinh x$$

and

$$\lim_{x\to\infty} e^{-x} \sinh x = 1/2.$$

so that $\{1, 0, 1, 0, 1, 0, \ldots\}$ is assigned the limit 1/2 by the Borel sequence.

Example 2.23. If $g(n) = (2n-2)!$ when $n = 2k-1$, and $g(n) = 0$ when $n = 2k$ then

$$E(z) = \sum_{n=1}^{\infty} \frac{z^{2n-2}}{(2n-2)!} = \cosh z.$$

Thus,

$$\{f_n(x)\} = \{ \frac{x^{n-1}}{\cosh x\,(2n-2)!}\}$$

and if $\{u_n\} = \{1, -(2), 4/2, -6/3, \ldots\}$ then

$$\sum_{n=1}^{\infty} f_n(x)u_n = \frac{1}{\cosh x} \sum_{n=1}^{\infty} \frac{(-x)^{n-1}}{(n-1)!} = \frac{1}{\cosh x} \cdot \frac{1}{e^x}$$

so that $\lim_{x \to \infty} g(x) = 0$. Note that, for this choice of g(n), the sequence $\{f_n(x)\}$ is quite powerful!

The next definition and theorem characterize T-matrices.

Definition 2.24. Let (a_{nk}) be a sequence-to-sequence summability method. If every convergent sequence $\{z_n\}$ in E is in the domain of (a_{nk}), and if

$$\lim z'_n = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} z_k = z$$

whenever $\lim z_n = z \neq \infty$, then (a_{nk}) is a regular sequence-to-sequence summability method. In this case (a_{nk}) will be called a Toeplitz matrix or a T-matrix.

Theorem 2.25. Let (a_{nk}) be a sequence-to-sequence summability method. Then (a_{nk}) is a T-matrix if and only if:

i) there exists $n_0 \in I^+$ and there exists $M \in (0, \infty)$ such that $\sum_{n=1}^{\infty} |a_{nk}| \leq M$ for every $n > n_0 > 0$, $n \in I^+$,

ii) $\lim_{n \to \infty} a_{nk} = 0$ for every fixed $k \in I^+$,

iii) $\sum_{k=1}^{\infty} a_{nk} = r_n$ and $\lim r_n = 1$.

In this case

$$\lim z'_n = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} z_k = 1z + \sum_{k=1}^{\infty} 0(z_k - z) = z$$

where $\lim z_n = z$.

Proof: a) i, ii, and iii are sufficient. Let a = 1 and $b_k = 0$ in Theorem 2.16 so that

$$\lim z'_n = 1z + \sum_{k=1}^{\infty} 0(z_k - z) = z.$$

b) i, ii, and iii are necessary. Since every T-matrix is a K-matrix, and condition i is necessary for (a_{nk}) to be a K-matrix, then condition i is necessary for (a_{nk}) to be a T-matrix.

Let $z_k = 0$ if $k \neq p$ and let $z_p = \pi$. Then

$$\sum_{k=1}^{\infty} a_{nk} z_k = \pi a_{np}.$$

Here $\lim z_n = 0$ and

$$\sum_{k=1}^{\infty} a_{nk} z_k = \pi a_{np}.$$

Now $\lim_{n \to \infty} \pi a_{np} = 0$ only if $\lim_{n \to \infty} a_{np} = 0$. This means that condition ii is necessary.

Let $z_n = 1$ for all $n \in I^+$ so that $\lim z_n = 1$ and

$$\sum_{k=1}^{\infty} a_{nk} z_k = \sum_{k=1}^{\infty} a_{nk}.$$

Now

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk}$$

will be one only if condition iii is satisfied, and the theorem is proved.

Example 2.26. An example of a T-matrix is the matrix (a_{nk}) where $a_{nk} = 1/n$ if $k \leq n$ and $a_{nk} = 0$ if $n < k$. This matrix is called the matrix of arithmetic means. For this matrix,

$$\sum_{k=1}^{\infty} |a_{nk}| = \sum_{k=1}^{n} \frac{1}{n} = 1$$

for all $n \in I^+$ and $\lim_{n\to\infty} a_{nk} = \lim_{n\to\infty} \frac{1}{n} = 0$ for all $k \in I^+$. Hence its characteristic numbers are $a_k = 0$ and $a = 1$. This matrix transforms $\{z_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ into the sequence

$$\{1, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \ldots, \frac{2}{2k-1}, \frac{1}{2}, \ldots\}$$

which has the limit 1/2.

Example 2.27. The matrices of Cesaro means of order $r > 0$ are a family of T-matrices where

$$a_{nk} = \frac{r\,\Gamma(n+1)\,\Gamma(r+n-k)}{\Gamma(n-k+1)\Gamma(r+n+1)}$$

if $k \le n$ and $a_{nk} = 0$ if $n < k$. Here

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

and $\Gamma(n) = (n-1)!$ for $n \in I^+$. Note that $r\Gamma(r) = \Gamma(r+1)$. Thus

$$\sum_{k=0}^{\infty} |a_{nk}| = \sum_{k=0}^{n} a_{nk} = \frac{r\,\Gamma(n+1)}{\Gamma(r+n+1)} \sum_{k=0}^{n} \frac{\Gamma(r+n-k)}{\Gamma(n-k+1)} .$$

Consider

$$\frac{1}{(1-z)^{j+1}} = \sum_{n=0}^{\infty} \frac{\Gamma(j+n+1)}{\Gamma(j+1)\,\Gamma(n+1)} z^n,$$

and

$$\frac{1}{(1-z)^{m+1}} = \sum_{n=0}^{\infty} \frac{\Gamma(m+n+1)}{\Gamma(m+1)\,\Gamma(n+1)} z^n$$

whenever $|z| < 1$. Then

$$\frac{1}{(1-z)^{j+1}} \cdot \frac{1}{(1-z)^{m+1}} = \sum_{n=0}^{\infty} \frac{\Gamma(m+j+n+2)}{\Gamma(m+j+2)\Gamma(n+1)} z^n$$

so that if the coefficients of $z^{n-k} \cdot z^k = z^n$ are equated,

$$\sum_{k=0}^{n} \left(\frac{\Gamma(j+n-k+1)}{\Gamma(j+1)\Gamma(n-k+1)} \cdot \frac{\Gamma(m+k+1)}{\Gamma(m+1)\Gamma(k+1)} \right) = \frac{\Gamma(m+j+n+2)}{\Gamma(m+j+2)\Gamma(n+1)} .$$

Now let $m = 0$ and let $j = r-1$, then for $r > 0$,

$$\sum_{k=0}^{n} \frac{\Gamma(r+n-k)}{\Gamma(r)\Gamma(n-k+1)} = \frac{\Gamma(r+n+1)}{\Gamma(r+1)\Gamma(n+1)} .$$

Hence,

$$\sum_{k=0}^{n} \frac{r\,\Gamma(n+1)\Gamma(r+n-k)}{\Gamma(r+n+1)\Gamma(n-k+1)} = 1,$$

and $a_k = 1$ for all $k \in I^+$. It can be shown that

$$\Gamma(n+1) = n^{n+\frac{1}{2}} e^{-n} 2\pi^{\frac{1}{2}} e^{q(n)}$$

where $\lim_{n \to \infty} nq(n) = 0$. This means that

$$\lim_{n \to \infty} \frac{\Gamma(r+1)\Gamma(n+1)\Gamma(r+n-k)}{\Gamma(r)\Gamma(n-k+1)\Gamma(r+n+1)}$$

$$= r \lim_{n \to \infty} \frac{\Gamma(n+1)\Gamma(r+n-k)}{\Gamma(n-k+1)\Gamma(r+n+1)}$$

$$= r \lim_{n \to \infty} \frac{n^{n+\frac{1}{2}} e^{-n} 2\pi^{\frac{1}{2}} e^{q(n)} (r+n-k-1)^{r+n-k-\frac{1}{2}} e^{-r-n+k+1} 2\pi^{\frac{1}{2}} e^{q(r+n-k-1)}}{(n-k)^{n-k+\frac{1}{2}} e^{-n+k} 2\pi^{\frac{1}{2}} e^{q(n-k)} (r+n)^{r+n+\frac{1}{2}} e^{-r-n} 2\pi^{\frac{1}{2}} e^{q(r+n)}}$$

$$= re \lim_{n \to \infty} \frac{n^{n+\frac{1}{2}} (r+n-k-1)^{r+n-k-\frac{1}{2}}}{(n-k)^{n-k+\frac{1}{2}} (r+n)^{r+n+\frac{1}{2}}}$$

$$= re \lim_{n\to\infty} \left(\frac{n}{r+n}\right)^{n+\frac{1}{2}} \left(\frac{r+n-k-1}{n-k}\right)^{n-k+\frac{1}{2}} \left(\frac{r+n-k-1}{r+n}\right)^{r} \left(\frac{1}{r+n-k-1}\right)$$

$$= re \lim_{n\to\infty} \left(\frac{1}{\frac{r}{n}+1}\right)^{n+\frac{1}{2}} \left(1+\frac{r-1}{n-k}\right)^{n-k+\frac{1}{2}} \left(1-\frac{k+1}{r+n}\right)^{r} \left(\frac{1}{r+n-k-1}\right)$$

$$= 0$$

for all $k \leq n$ and for all $r > 0$, so that $a = 0$ and the Cesaro matrices of all orders $r > 0$ are T-matrices. If $r = 1$, then $a_{nk} = 1/(n+1)$ if $k \leq n$ and $a_{nk} = 0$ if $n < k$ so that the sequence $\{z_n\} = \{1, 0, 1, 0, 1, 0, \ldots\}$ is transformed into the sequence $\{1/2, 1/3, 1/2, \ldots, 1/2, k/(2k-1), \ldots\}$ whose limit is $1/2$.

For every sequence $\{x_n\}$ in E there is an associated infinite series $\sum_{k=1}^{\infty} c_k$ where $c_1 = x_1$ and $c_k = x_k - x_{k-1}$ if $k > 1$. Thus $\sum_{k=1}^{n} c_k = x_n$, and $\{x_n\}$ is the sequence of partial sums of $\sum_{k=1}^{\infty} c_k$. This means that the results on sequence-to-function transformations can be extended to theorems on series-to-function transformations.

Definition 2.28. Let $h_k(x) \in F[(0, \infty)]$ for all $k \in I^+$, and let $\sum_{k=1}^{\infty} c_k$ be an infinite series of complex numbers such that

$$g(x) = \sum_{k=1}^{\infty} h_k(x) c_k$$

belongs to $F[(0, \infty)]$. If $\lim_{x\to\infty} g(x) = t \neq \infty$, then $\{h_k(x)\}$ will be called a series-to-function summability method, and $\sum_{k=1}^{\infty} c_k$ will be said to be in the domain of $\{h_k(x)\}$.

Just as before, care must be exercised in the application of this definition to particular sequences $\{h_k(x)\}$ and to particular series

$\sum_{k=1}^{\infty} c_k$ since $\sum_{k=1}^{\infty} h_k(x)c_k$ may not converge to a function in $F[(0, \infty)]$.

Conservative and regular series-to-function transformations are defined in a manner analogous to that for conservative and regular sequence-to-function transformations.

Definition 2.29. Let $\{h_k(x)\}$ be a series-to-function summability method. If

$$g(x) = \sum_{k=1}^{\infty} h_k(x)c_k$$

belongs to $F[(0, \infty)]$ and if $\lim_{x \to \infty} g(x) = t \neq \infty$ for every convergent series $\sum_{k=1}^{\infty} c_k$ in E, then $\{h_k(x)\}$ is a conservative series-to-function summability method. In this case $\{h_k(x)\}$ will also be called a β-sequence.

Definition 2.30. Let $\{h_k(x)\}$ be a series-to-function summability method. If

$$g(x) = \sum_{k=1}^{\infty} h_k(x)c_k$$

belongs to $F[(0, \infty)]$ for every convergent series $\sum_{k=1}^{\infty} c_k = t$ in E_1, and if

$$\lim_{x \to \infty} g(x) = \sum_{k=1}^{\infty} c_k = t,$$

then $\{h_k(x)\}$ is a regular series-to-function summability method. In this case $\{h_k(x)\}$ will also be called a γ-sequence.

The following lemma, first proved by R. Henstock, will be used in the proof of a theorem for series-to-function transformations which is the analogue of Theorem 2.12.

Lemma 2.31. Let $\{g_k(x)\}$ be a sequence of functions in $F[(0,\infty)]$ such that for every convergent series $\sum_{k=1}^{\infty} c_k$ in E, there exists $x_0 \epsilon R^+$ such that $\sum_{k=1}^{\infty} g_k(x)c_k$ converges for every fixed $x > x_0 > 0$. Then there exists a real number $M(x)$ for each fixed $x > x_0 > 0$ such that $|g_k(x)| \leq M(x)$ whenever $k \epsilon I^+$.

Proof: Suppose the lemma is false, that is, suppose that for every $r \epsilon I^+$ there is an $x > x_0$ such that $\{g_k(x)\}$ has a subsequence $\{g_{k_r}(x)\}$ where $|g_{k_r}(x)| > r^2$. Let $c_k = 0$ if $k \neq k_r$; let

$$c_{k_r} = \frac{|g_{k_r}(x)|}{r^2 g_{k_r}(x)}$$

if $g_{k_r}(x) \neq 0$; and let $c_{k_r} = 0$ if $g_{k_r}(x) = 0$.

Now

$$\sum_{k=1}^{\infty} |c_k| \leq \sum_{k=1}^{\infty} r^{-2},$$

which is a convergent series. Thus, $\sum_{k=1}^{\infty} c_k$ is absolutely convergent. Hence $\sum_{k=1}^{\infty} c_k$ is convergent, and

$$\sum_{k=1}^{\infty} g_k(x)c_k = \sum_{k=1}^{\infty} |g_k(x)| \geq \sum_{r=1}^{\infty} |g_{k_r}(x)| > \sum_{r=1}^{\infty} r^2 .$$

Thus, $\sum_{k=1}^{\infty} g_k(x)c_k$ is a divergent series. This contradicts the hypothesis, so the lemma is proved.

The next theorem was proved by Bosanquet in 1931. Cooke's modification of the proof is based upon the lemma of R. Henstock.

Theorem 2.32. Let $\{h_k(x)\}$ be a series-to-function summability method. Then $\{h_k(x)\}$ is a β-sequence if and only if:

i) there exists $x_0 \epsilon (0, \infty)$ and there exists $M \epsilon (0, \infty)$ such that

$$\sum_{k=1}^{\infty} |h_k(x) - h_{k+1}(x)| \leq M$$

for all $x > x_0 > 0$,

ii) $\lim\limits_{x \to \infty} h_k(x) = a_k \neq \infty$ for every $k \epsilon I^+$.

In this case

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \sum_{k=1}^{\infty} h_k(x) c_k$$

$$= a_1 t + \sum_{k=1}^{\infty} (a_k - a_{k+1})(t_k - t)$$

where $t_k = \sum\limits_{n=1}^{k} c_n$ and $\lim t_k = t \neq \infty$.

Proof: a) Conditions i and ii are sufficient. Let $\sum\limits_{k=1}^{\infty} c_k = t$ be a convergent series in E and write

$$\sum_{k=1}^{\infty} h_k(x) c_k = h_1(x) t + \sum_{k=1}^{\infty} (h_k(x) - h_{k+1}(x))(t_k - t).$$

The task at hand is to prove that $\sum\limits_{k=1}^{\infty} h_k(x) c_k$ converges if $\{h_k(x)\}$ satisfies i and ii above. If it can be shown that i and ii imply that

$$\sum_{k=1}^{\infty} (h_k(x) - h_{k+1}(x)) (t_k - t)$$

converges, the task will be accomplished.

Define $f_k(x) = h_k(x) - h_{k+1}(x)$ so that

$$\sum_{k=1}^{\infty} |f_k(x)| \leq M$$

for all $x > x_0 > 0$ if and only if

$$\sum_{k=1}^{\infty} |h_k(x) - h_{k+1}(x)| \leq M$$

for all $x > x_0 > 0$. Next, $\lim_{x \to \infty} f_k(x) = a'_k$ if and only if

$$\lim_{x \to \infty} h_k(x) - h_{k+1}(x) = a'_k = a_k - a_{k+1}.$$

Lastly,

$$\sum_{k=1}^{\infty} f_k(x) = h_1(x)$$

and

$$\lim_{x \to \infty} \sum_{k=1}^{\infty} f_k(x) = a_1 \neq \infty$$

if and only if $\lim_{x \to \infty} h_1(x) = a_1$. Now $\lim (t_k - t) = 0$, so that Theorem 2.12 applies, and

$$\sum_{k=1}^{\infty} (h_k(x) - h_{k+1}(x))\,(t_k - t)$$

converges for all $x > x_0 > 0$.

Now let M be the constant from condition i. Since $\lim t_k = t$, it is possible to choose N so that $|t_k - t| < \mathcal{E}/M$ whenever $k > N$. Hence

$$\sum_{k=1}^{\infty} h_k(x)c_k = h_1(x)t + \sum_{k=1}^{N} (h_k(x)-h_{k+1}(x))(t_k-t) + \sum_{k=N+1}^{\infty} (h_k(x)-h_{k+1}(x))(t_k-t)$$

$$= h_1(x)t + H_N(x) + H(x) .$$

Condition ii implies that

$$\lim_{x \to \infty} H_N(x) = \sum_{k=1}^{N} (a_k - a_{k+1})\,(t_k - t),$$

and condition i implies that

$$|H(x)| = \left| \sum_{k=N+1}^{\infty} (h_k(x) - h_{k+1}(x)\ (t_k - t) \right|$$

$$\leq \sum_{k=N+1}^{\infty} |h_k(x) - h_{k+1}(x)|\ |t_k - t|$$

$$< M \cdot \mathcal{E}/M = \mathcal{E}$$

for each fixed $x > x_0 > 0$. From condition ii, $\lim_{x \to \infty} h_1(x) = a_1$ so that

$$\lim_{x \to \infty} g(x) = a_1 t + \sum_{k=1}^{\infty} (a_k - a_{k+1})\ (t_k - t).$$

This proves that the conditions are sufficient.

b) The conditions are necessary. Now suppose that

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \sum_{k=1}^{\infty} h_k(x) c_k$$

exists whenever $\sum_{k=1}^{\infty} c_k$ converges, let $c_k = 0$, $k \neq N$, and let $c_N = 1$. This means that $g(x) = h_N(x)$. Thus condition ii, $\lim_{x \to \infty} h_k(x) = a_k$ for all $k \in I^+$ is necessary. Next write

$$\sum_{k=1}^{n} h_k(x) c_k = \sum_{k=1}^{n} h_k(x)\ (t_n - t_{k-1})$$

$$= \sum_{k=1}^{n} h_k(x) [\ (t_k - t) - (t_{k-1} - t)]$$

$$= \sum_{k=1}^{n-1} (h_k(x) - h_{k+1}(x)\ (t_k - t)) + h_1(x)t + (t_n - t) h_n(x).$$

From Lemma 2.31, $|h_n(x)| \leq M(x)$ for each fixed $x > x_0 > 0$ and for all $n \in I^+$. Hence

$$\lim_{n \to \infty} (t_n - t)\, h_n(x) = 0,$$

and since

$$\lim_{n \to \infty} \sum_{k=1}^{n} h_k(x) c_k = g(x),$$

then

$$\lim_{n \to \infty} \sum_{k=1}^{n-1} (h_k(x) - h_{k+1}(x))\,(t_k - t) + h_1(x)t = g(x)$$

for all fixed $x > x_0 > 0$. Since $\lim_{x \to \infty} g(x)$ exists by hypothesis, we can apply Theorem 2.12 to the sequence of functions

$$\{f_n(x)\} = \{h_n(x) - h_{n+1}(x)\}$$

and to the sequence $\{t_n\}$. Thus, the conditions are necessary and the theorem is proved. Note that the Kojima-Schur Theorem is the keystone of the results which characterize summability methods for sequences and series.

Example 2.33. A β-sequence can be constructed from the K-sequence

$$\{f_n(x)\} = \{(x^2 + 4n^2\pi^2)^{-1}\}$$

by letting

$$h_n(x) = \sum_{k=n}^{\infty} f_k(x).$$

Here $a_1 = 0$ and $a_k - a_{k+1} = 0$ for all $k \in I^+$ so $\lim_{x \to \infty} g(x) = 0$ and $\{h_n(x)\}$ transforms every convergent series into a function which has limit zero.

The next theorem is the analogue of Theorem 2.16 for infinite series.

Theorem 2.34. Let $\{h_k(x)\}$ be a series-to-function summability method. Then $\{h_k(x)\}$ is a γ-sequence if and only if:

i) There exists $x_0 \in (0, \infty)$, and there exists $M \in (0, \infty)$ such that

$$\sum_{k=1}^{\infty} |h_k(x) - h_{k+1}(x)| \leq M$$

for all $x > x_0 > 0$,

ii) $\lim_{x \to \infty} h_k(x) = 1$ for all $k \in I^+$.

In this case

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \sum_{k=1}^{\infty} h_k(x) c_k = t \neq \infty$$

whenever $\sum_{k=1}^{\infty} c_k = t$ is a convergent series in E.

Proof: a) Conditions i and ii are sufficient. Let $a_k = 1$ in Theorem 2.32 so that

$$\lim_{x \to \infty} g(x) = 1t + \sum_{k=1}^{\infty} 0(t_k - t) = t.$$

b) Conditions i and ii are necessary. Since a γ-sequence must be a β-sequence, and condition i is necessary for $\{h_k(x)\}$ to be a β-sequence, then condition i is necessary for $\{h_k(x)\}$ to be a γ-sequence. Let $c_k = 0$, $k \neq N$, and $c_N = 1$. Then $g(x) = h_N(x)$ and $t = 1$ so that condition ii is necessary. This proves the theorem.

Example 2.35. The Borel exponential sequence

$$\{h_k(x)\} = \left\{\frac{1}{k!} \int_0^x e^{-t} t^k dt\right\},$$

is a γ-sequence. Here integration by parts can be used to give

$$h_k(x) = \frac{e^{-x} x^{k+1}}{(k+1)!} + h_{k+1}(x).$$

Thus

$$h_k(x) - h_{k+1}(x) = \frac{e^{-x} x^{k+1}}{(k+1)!}$$

and

$$\sum_{k=1}^{\infty} |h_k(x) - h_{k+1}(x)| = e^{-x} \sum_{k=1}^{\infty} \frac{x^{k+1}}{(k+1)!} = e^{-x}(e^x - 1 - x) \leq 1$$

for all $x > 1 > 0$. Also

$$\lim_{x \to \infty} \int_0^x e^{-t} t^k dt = k!,$$

so that the Borel exponential sequence is indeed a γ-sequence. Let

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} (-1)^{k-1}$$

so that

$$\sum_{k=1}^{\infty} h_k c_k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \int_0^x e^{-t} t^k dt$$

$$= e^{-x}(e^x - 1) - e^{-x} \sum_{k=1}^{\infty} \frac{x^{2k-1}}{(2k-1)!}$$

$$= e^{-x}\left(e^x - 1 - \frac{e^x - e^{-x}}{2}\right)$$

$$= 1 - \frac{1}{2} - \frac{1}{e^x} + \frac{1}{e^{2x}}.$$

Hence $\lim_{x \to \infty} g(x) = 1/2$, and $\sum_{k=1}^{\infty} (-1)^{k-1}$ is assigned 1/2 for its limit.

It should now be apparent that the ways in which a divergent sequence can be assigned a number are many and varied. The concept of convergence can be viewed as a special case of more general methods which assign numbers to sequences. The divergence of a sequence or a series is no longer a cause for alarm, or for discarding it as totally useless. There may well be a summability method, which is applicable in a particular model of a physical situation, that can assign a number to the sequence. Aside from applications, there is ample opportunity to experiment with devising new summability methods.

The matrix of arithmetic means and the Cesaro matrices have special applications in dealing with divergent Fourier series, and the Borel and Mittag-Leffler sequences have applications concerning Taylor series of functions outside their circle of convergence. These applications will be examined in Chapter IV.

Many areas of interest concerning summability methods are now within view. The structure of the sets of K-sequences and T-sequences, of K-matrices and T-matrices have been examined by Agnew, Cooke, and others. K-sequences and K-matrices form an algebra, but the "nice" sequences and matrices, the T-sequences and matrices, are not so fortunate. The element-wise sum of two T-matrices may not yield a T-matrix. Hill, Cooke, Dienes and others have considered whether one value assigned to a divergent sequence is better, or more natural, than other values.

There is a considerable amount written on whether one summability method is stronger than another. Wilansky has given necessary and sufficient conditions for a summability method to be stronger than convergence. Zeller has given a criterion for testing the relative strengths of summability methods which belong to certain families. These and other topics are referred to in the books and articles listed in the bibliography.

Mazur, Ohrlich, Wlodarski, and most recently, Wilansky have examined the structure of the set of sequences itself. The use of the theory of linear spaces can lead to answers to questions about the size of the domain of a summability method and the relation of the domains of two summability methods to each other. This and other related topics will be the subject of the next chapter.

CHAPTER III

SEQUENCE SPACES

The notion of a linear space, or a vector space, in which there is defined a distance-like function has led to a branch of analysis called functional analysis. Examples of linear spaces, which should be familiar, are the spaces of n-tuples of real numbers or of complex numbers. In particular, the spaces of ordered pairs or ordered triples of real numbers from analytic geometry are indispensable to analysis of functions of several variables. Since sequences are a natural generalization of an ordered n-tuple, it should be expected that certain sets of sequences form linear spaces and that the domains of infinite matrices and distinguished subsets of their domains can be examined by the methods of functional analysis.

Some of the sets of sequences under consideration will be the set s of all sequences of complex numbers, the set m of all bounded sequences of complex numbers, the set c of all convergent sequences of complex numbers, and the set c_0 of all sequences of complex numbers which converge to zero. Sequences which converge to zero will be called null sequences, thus c_0 is the set of null sequences.

The study of the properties of sets of functions using the concept of a linear space was pioneered by Banach. Studies in the application of the methods of functional analysis to sequence spaces and infinite matrices are an area of recent research concerning summability methods. An introduction to sequence spaces is given in this chapter to demonstrate some of the types of problems concerning summability methods which are being examined using methods of functional analysis. The next definition is that of a linear space.

Definition 3.1. A *linear space X*, over a field $\mathfrak{F}$ of scalars, is a set for which an additional operation is defined making X a commutative group, and a multiplication by scalars is defined satisfying the following conditions:

i) $t(a+b) = ta + tb$,

ii) $(r+t)a = ra + ta$,

iii) $(rt)a = r(ta)$,

iv) $1a = a$,

where $a, b \in X$ and $r, t, 1 \in \mathfrak{F}$.

One example of a linear space is given by letting $X = \mathfrak{F} = R$, the set of real numbers. Another example of the same sort is given by letting $X = \mathfrak{F} = E$, the set of complex numbers. Since conditions i through iv do not require anything not already present in a field, it can be seen that any field can be considered to be a linear space over itself. An example of a linear space, which is basic to the study of sequence spaces, is a set M of functions whose range is a subset of a field $\mathfrak{F}$. Define $(f+g)(x) = f(x) + g(x)$ and define $(tf)(x) = t(f(x))$. Then M is a linear space over $\mathfrak{F}$. In the next theorem this fact is demonstrated for the particular case in which M is s, the set of all functions from I^+ to E.

Theorem 3.2. The set s is a linear space over the field E where $x+y = \{z_n + w_n\}$ and $tx = \{tz_n\}$ for $x = \{z_n\}$ and $y = \{w_n\}$ in s and t in E.

Proof: Let $x = \{z_n\}$ and let $y = \{w_n\}$ belong to s, and let r, t belong to E. Then s is a commutative group under the addition defined above since E is a commutative group under addition.

Now

$$t(x+y) = \{t(z_n + w_n)\} = \{tz_n + tw_n\} = \{tz_n\} + \{tw_n\} = tx + ty,$$

from the field properties in E and the definitions of addition and scalar multiplication. Thus i is satisfied. Also

$$(r+t)x = \{(r+t)z_n\} = \{rz_n + tz_n\} = \{rz_n\} + \{tz_n\} = rx + tx,$$

for the same reasons, and ii is satisfied.

From the definition of scalar multiplication, $(rt)x = \{(rt)z_n\}$. From associativity of multiplication in E, $\{(rt)z_n\} = \{r(tz_n)\}$. Using the definition of scalar multiplication again, $\{r(tz_n)\} = r(tx)$ so that iii is satisfied.

Lastly, $1x = \{1z_n\} = \{z_n\} = x$ from the definition of scalar multiplication, and the identity for multiplication in E. This means that iv is satisfied, and the theorem is proved.

Theorem 3.3. The sets c, c_0, and m are linear spaces over the field E.

Proof: Let addition and scalar multiplication be defined as in Theorem 3.2. Then c, c_0, and m are commutative subgroups of s under addition, as is shown below.

If $x = \{z_n\}$ and $y = \{w_n\}$ belong to m, then there exist M_x and M_y in $(0, \infty)$ such that $|z_n| < M_x$ and $|w_n| < M_y$ for all $n \in I^+$. Hence

$|z_n + w_n| \le |z_n| + |w_n| < M_x + M_y$ for all $n \in I^+$, and $x + y$ belongs to m. Since $|z_n| = |-z_n|$, each element of m has an additive inverse in m. Thus $x + (-y)$ belongs to m for all x and y in m. Addition is commutative in m since addition is commutative in E. This means that m is a commutative subgroup of s.

If $x = \{z_n\}$ and $y = \{w_n\}$ belong to c or c_0, then Theorem 1.5 implies that $x + y$ belongs to c or c_0 respectively. Again, Theorem 1.5 implies that if x belongs to c or c_0, then $-x$ belongs to c or c_0. Thus $x + (-y)$ belongs to c or c_0 whenever x and y belong to c or c_0. Addition is commutative in c and c_0 since addition is commutative in E. This means that c and c_0 are commutative subgroups of s.

Properties i, ii, iii, and iv are inherited by m, c, and c_0 from s. This proves the theorem.

Now that the set s and its subsets m, c, and c_0 have been shown to be linear spaces, the notion of distance in m, c, and c_0 will be explored. To appreciate the usefulness of a distance function, consider the linear space of R over itself. Here the distance from any point x of the real line to the origin has the handy representation, $|x|$. Note that $|tx| = |t|\,|x|$ in R, and that $|x-y|$ is the distance from x to y or from y to x. The set $\{x : |x - y| < r,\ r \in (0, \infty)\}$ is an open interval with midpoint y and length 2r. Recall that convergence of a sequence $\{x_n\}$ in R to the limit b requires that, for an arbitrarily small interval with b as midpoint, there exists an index N such that x_n is in the interval about b for all $n > N$.

In the linear space formed by taking E over itself, things look very similar to the situation in R. The distance from any point z of the complex plane to the origin is still written $|z|$, but recall that

$|z| = \sqrt{x^2 + y^2}$ where $z = x + iy$. Again $|tz| = |t|\,|z|$ and $|z - w|$ is the distance from z to w or from w to z. However, the set $\{z : |z - w| < r,\ r \in (0, \infty)\}$ is an open disc with center w. Here the convergence of a sequence $\{z_n\}$ in E to the limit u requires that, for an arbitrarily small disc with u as center, there exists an index N such that z_n is in the disc about u for all $n > N$.

The pertinent properties of $|x|$ in R or $|z|$ in E are collected to define the notion of a norm for a linear space X over E in the following definition.

Definition 3.4. A norm for a linear space X over E is a function $\emptyset$ from X to $[0, \infty)$, $\emptyset : X \to [0, \infty)$, which satisfies the following requirements:

i) $\emptyset(x) = 0$ if and only if x is the additive identity in X,

ii) $\emptyset(x) \geq 0$, for all $x \in X$,

iii) $\emptyset(-x) = \emptyset(x)$, for all $x \in X$,

iv) $\emptyset(x+y) \leq \emptyset(x) + \emptyset(y)$ for all $x, y \in X$,

v) $\emptyset(tx) = |t|\,\emptyset(x)$ for all $t \in E$ and all $x \in X$.

If X is a linear space over E with a norm defined on it, then X is called a normed linear space. It is customary to write $\|x\|$ for $\emptyset(x)$.

Since it is relatively easy to define a norm for m, c, and c_0, it will be desirable to concentrate on these spaces for much of what is to follow. It is possible to define a distance-like function in s. Wilansky discusses this topic and many others concerning sequence spaces in [11].

The following theorem yields the norm for m, c, and c_0 which was promised.

Theorem 3.5. $\|x\| = \sup\{|z_n| : x = \{z_n\}\}$ is a norm for m.

Proof: Note that $\|x\| = \sup\{|z_n| : x = \{z_n\}\}$ is well defined for all x in m since every bounded set of real numbers has a unique supremum.

Now $\sup\{|z_n| : x = \{z_n\}\} = 0$ if and only if $|z_n| = 0$ for all $n \in I^+$. Further, $|z_n| = 0$ if and only if $z_n = 0$. Hence $\|x\| = 0$ if and only if $x = \{0, 0, 0, \ldots\}$, and i of Definition 3.4 is satisfied.

Next, since $|z| \geq 0$ for all z in E, then $\|x\| = \sup\{|z_n| : x = \{z_n\}\} \geq 0$ in m so that ii is satisfied.

In E, $|-z| = |z|$, so that $\|-x\| = \|x\|$, and iii is satisfied.

Now let $A = \sup\{|z_n + w_n| : x = \{z_n\},\ y = \{w_n\}\}$, let $B = \sup\{|z_n| : x = \{z_n\}\}$, and let $C = \sup\{|w_n| : y = \{w_n\}\}$. Then for all $n \in I^+$, $B + C \geq |z_n| + |w_n| \geq |z_n + w_n|$. Further, for every $\mathcal{E} > 0$ there exists an integer N such that $A - \mathcal{E} < |z_N + w_N|$. This means that for every $\mathcal{E} > 0$ there exists an integer N such that

$$B + C \geq |z_N| + |w_N| \geq |z_N + w_N| > A - \mathcal{E}$$

and $B + C + \mathcal{E} > A$. Hence $B + C \geq A$ or $\|x\| + \|y\| \geq \|x+y\|$ for all x, y in m so that iv is satisfied.

Since $|t|\,|z_n| = |tz_n|$, $|t|\,\|x\| = \|tx\|$ so that v is satisfied. Thus $\|x\|$ is a norm for m and the theorem is proved.

Corollary 3.6. $\|x\| = \sup\{|z_n| : x = \{z_n\}\}$ is a norm for c and c_0.

Proof: c and c_0 are linear subspaces of m, so that each statement of Theorem 3.5 applies to them as well as to m, and the corollary is proved.

Since the generalization of the properties of $|x|$ in R and $|z|$ in E gave the concept of a norm for a linear space over E, it would seem reasonable that a generalization of $|x-y|$ in R or $|z-w|$ in E would lead to the concept of a distance measuring function which could be applied to m, c, and c_0.

<u>Definition 3.7</u>. A <u>metric</u> for a nonempty set X is a real function d of two variables satisfying for all x, y, z in X,

i) $d(x, y) \geq 0$,

ii) $d(x, y) = 0$ if and only if $x = y$,

iii) $d(x, y) = d(y, x)$,

iv) $d(x, y) \leq d(x, z) + d(z, y)$.

The next theorem shows that $\|x-y\|$ is a metric for m, c, and c_0.

<u>Theorem 3.8</u>. If $\emptyset$ is a norm for a linear space X then $d(x, y) = \emptyset(x-y)$ is a metric for X.

Proof: Property ii of Definition 3.4 insures that $d(x, y) \geq 0$, and i of Definition 3.4 implies that $d(x, y) = 0$ if and only if x-y is the additive identity in X, which is true if and only if $x = y$. $d(x, y) = d(y, x)$ since $\emptyset(x-y) = \emptyset(y-x)$ from iii of Definition 3.4. Now $x-y = (x-z) + (z-y)$ so that $d(x, y) \leq d(x, z) + d(z, y)$ from iv of Definition 3.4. This proves the theorem.

<u>Corollary 3.9</u>. $\|x-y\|$ is a metric for m, c, and c_0.

<u>Example 3.10</u>. The metric just given will now be illustrated

as it applies to some sequences in m. The sequences

$$x = \{1, 0, 1, 0, 1, 0, \ldots\},$$

$$y = \{1, 1/2, 1/4, \ldots, 2^{1-n}, \ldots\},$$

and

$$v = \{0, 3/2, 2/3, \ldots, (n+(-1)^n)/n, \ldots\}$$

belong to m, and so do the constant sequences $\overline{0} = \{0, 0, 0, \ldots\}$ and $\overline{1} = \{1, 1, 1, \ldots\}$. Now

$$\|x\| = \|y\| = \|\overline{1}\| = 1,$$

$\|v\| = 3/2$, and $\|\overline{0}\| = 0$. Also,

$$\|x-y\| = \|x-\overline{1}\| = \|y-v\| = \|x-\overline{0}\| = \|y-\overline{0}\|$$

$$= \|y-\overline{1}\| = \|\overline{1}-\overline{0}\| = \|v-\overline{1}\| = 1.$$

Finally,

$$\|x-v\| = \|v-\overline{0}\| = 3/2.$$

Note that x is not an element of c, $\lim y_n = 0$, $\lim v_n = 1$, $\lim \overline{0}_n = 0$, and $\lim \overline{1}_n = 1$. This should point out that the metric just defined for m, c, and c_0 does not have much relation to limit points and the distances between them in E with the metric $d(z, w) = |z-w|$.

Definition 3.7 defines a metric on a general nonempty set X, and thus in particular for a linear space. It is clear that the concept of a metric requires no linear space structure on the set for which the metric is defined. Indeed, some of the properties of sequence spaces to be examined in this chapter depend only on metric concepts, while some depend only on linear space concepts. The combination of the two yields even more information as will be seen.

Now that a metric is at hand for the spaces m, c, and c_0, sequences in these spaces and convergence of sequences in these spaces can be explored. Consider a sequence $\{x_n\}$ in m. This is a sequence $\{x_1, x_2, x_3, \ldots, x_n, \ldots\}$ where x_n is an element of m. That is,

$$x_n = \{z_{n1}, z_{n2}, z_{n3}, \ldots, z_{nk}, \ldots\}$$

where z_{nk}, the kth term of the sequence x_n, is an element of E. This situation corresponds to an infinite matrix (z_{nk}), whose rows are the elements x_n of a sequence $\{x_n\}$, (see Figure 3. 1, p. 73). With the diagram in mind, consider the following definition of convergence of a sequence in a metric space.

Definition 3. 11. Let $\{x_n\}$ be a sequence in a metric space X with metric $d(x, y)$. Then $\lim x_n = x \in X$ if and only if for every $\mathcal{E} > 0$ there exists $N \in I^+$ such that $d(x_n, x) < \mathcal{E}$ whenever $n > N$.

In the metric spaces m, c, and c_0, the definition would read $\lim x_n = x \in$ m, c, or c_0, if and only if for every $\mathcal{E} > 0$ there exists $N \in I^+$ such that $\|x_n - x\| < \mathcal{E}$ whenever $n > N$. Note that

$$\|x_n - x\| = \sup_k \{|z_{nk} - z_k| : x_n = \{z_{nk}\}, \ x = \{z_k\}\}.$$

From the diagram, this means that given $\mathcal{E} > 0$, one must be able to find a row in the array such that for all rows further down in the array, $|z_{nk} - z_k| < \mathcal{E}$ for all $k \in I^+$. In other words, each column $\{z_{nk}\}$, $n \in I^+$, must converge uniformly with respect to k, in the usual sense in E, to the corresponding z_k.

Convergence of a sequence $\{x_n\}$ to a limit x in a metric space is equivalent to $\lim d(x_n, x) = 0$ in R. From Theorem 1.4, limits of sequences in R and E are unique. The next theorem shows that this is also true in a metric space.

Theorem 3.12. Let $\{x_n\}$ be a sequence in a metric space X with metric d(x, y). Then $\lim x_n = x \in X$ and $\lim x_n = y \in X$ implies that $x = y$.

Proof: From Definition 3.7,

$$d(x, y) \leq d(x, x_n) + d(x_n, y)$$

so that

$$\lim d(x, y) \leq \lim d(x, x_n) + \lim d(x_n, y).$$

Hence, $d(x, y) \leq 0$, but $d(x, y) \geq 0$ by Definition 3.7, so that $d(x, y) = 0$ and $x = y$ by Definition 3.7. This proves the theorem.

Now consider a metric space X with metric $d_1(x, u)$ and another metric space Y with perhaps a different metric $d_2(y, v)$. Continuity of a function f from X to Y, $f: X \to Y$ is the subject of the next definition.

Definition 3.13. Let X and Y be metric spaces with metrics $d_1(x, u)$ and $d_2(y, v)$, respectively. Then a function $f: X \to Y$ is continuous if and only if for every sequence $\{x_n\}$ in X such that $\lim_{d_1} x_n = x$, $\lim_{d_2} f(x_n) = f(x)$ in Y.

Some functions connected with the linear structure of a space are addition, scalar multiplication, and the projection or coordinate functions. Addition as a function on a linear space is a function of two variables whose range is the linear space. That is, if L is a linear space, then addition is a function $\oplus$ from L X L to L; $\oplus : L \times L \rightarrow L$. Scalar multiplication is again a function of two variables whose range is the linear space. Here $\otimes$ is a function from E X L to L, $\otimes : E \times L \rightarrow L$, where L is a linear space over E. In a sequence space, the coordinate functions $P_n(x)$ are functions such that if $x = \{z_n\}$, then $P_n(x) = z_n$. Thus P_n is a function from m to E, $P_n : m \rightarrow E$. An important characteristic of m, c, and c_0 is that addition, scalar multiplication, and the coordinate functions are continuous. This is the topic of the next theorem.

Theorem 3.14. Addition, scalar multiplication, and the coordinate functions are continuous on m, c, and c_0.

Proof: Let $\lim (x_n, y_n) = (x, y)$ in m X m. That is, let $\lim x_n = x$ and let $\lim y_n = y$. Then since $\oplus(\{x_n\}, \{y_n\}) = \{x_n + y_n\}$ and $\oplus(x, y) = x + y$, it must be shown that $\lim (x_n + y_n) = x + y$. Now $\lim \|x_n - x\| = 0$ and $\lim \|y_n - y\| = 0$ implies that for every $\mathcal{E} > 0$ there exists an integer N_1 such that $\|x_n - x\| < \mathcal{E}/2$ whenever $n > N_1$, and there exists an integer N_2 such that $\|y_n - y\| < \mathcal{E}/2$ whenever $n > N_2$. Let $N = \max\{N_1, N_2\}$ so that $\|x_n - x\| + \|y_n - y\| < \mathcal{E}$ whenever $n > N$. Then

$$\|x_n - x + y_n - y\| = \|x_n + y_n - (x + y)\| < \mathcal{E}$$

whenever $n > N$, and addition is continuous on m.

Next let $\lim (t, x_n) = (t, x)$ where t is a scalar in E and $\{x_n\}$ is a sequence in m such that $\lim x_n = x$ in m. Now $\otimes(t, \{x_n\}) = \{tx_n\}$ and $\otimes(t, x) = tx$ so it must be shown that $\lim tx_n = tx$. Suppose $t \neq 0$. Now

$\lim \|x_n - x\| = 0$ implies that for every $\mathcal{E} > 0$ there exists an integer N such that $\|x_n - x\| < \mathcal{E}/|t|$ whenever $n > N$. Hence $|t| \|x_n - x\| < \mathcal{E}$, and $\|tx_n - tx\| < \mathcal{E}$ by Corollary 3.6 whenever $n > N$. If $t = 0$, then $tx_n = \{tz_{nk}\} = \{0, 0, 0, \ldots\}$ for every $n \in I^+$ and $tx = \{0, 0, 0, \ldots\}$ so that $\lim tx_n = tx$. This proves that scalar multiplication is continuous in m.

Now let $\{x_n\}$ be a sequence in m such that $\lim x_n = x$ in m. Then $P_k(x_n) = z_{nk}$ and $P_k(x) = z_k$. Hence it must be shown that $\lim z_{nk} = z_k$. Now for every $\mathcal{E} > 0$ there exists an integer N such that $\|x_n - x\| < \mathcal{E}$ whenever $n > N$, and since

$$\|x_n - x\| = \sup_k \{|z_{nk} - z_k| : x_n = \{z_{nk}\},\ x = \{z_k\}\},$$

it must also be true that $|z_{nk} - z_k| < \mathcal{E}$ for all $k \in I^+$. This shows that $\lim z_{nk} = z_k$ for all $k \in I^+$ and that the coordinate functions are continuous.

Since c and c_0 are linear subspaces of m, the arguments presented above also apply to c and c_0, so that the theorem is proved.

Now the interaction between linear and metric concepts can be seen quite clearly. The metric has been used in m, c, and c_0 to refine the linear structure by showing addition, scalar multiplication, and the coordinate functions to be continuous functions. Spaces in which the linear structure and the metric interact in this fashion are characterized in the next definition.

Definition 3.15. A space X is a linear metric space if and only if it is a linear space whose metric is such that addition and scalar multiplication are continuous.

Corollary 3.16. m, c, and c_0 are linear metric spaces.

Proof: Definition 3.15 and Theorem 3.14.

From Theorem 1.17, it is seen that a convergent sequence and a Cauchy sequence are equivalent in R and in E. Cauchy sequences can be defined in a general metric space, and a convergent sequence is a Cauchy sequence. It is not true for a general metric space that a Cauchy sequence is always convergent.

Definition 3.17. Let $\{x_n\}$ be a sequence in a metric space X with metric d(x, y). Then $\{x_n\}$ is a Cauchy sequence in X if and only if for every $\mathcal{E} > 0$ there exists an integer N such that $d(x_n, x_m) < \mathcal{E}$ whenever $m, n > N$.

Theorem 3.18. Let $\{x_n\}$ be a convergent sequence in a metric space X with metric d(x, y). If $\lim x_n = x \in X$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof: For every $\mathcal{E} > 0$ there exists an integer N such that $d(x_n, x) < \mathcal{E}/2$ and $d(x, x_m) < \mathcal{E}/2$ whenever $n, m > N$. Hence

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \mathcal{E}/2 + \mathcal{E}/2 = \mathcal{E}$$

whenever $m, n > N$ and the theorem is proved.

The completeness property of R states that every Cauchy sequence in R converges to an element of R. E is also complete so that every Cauchy sequence of complex numbers converges to a complex number. The next definition states this concept for a general metric space.

Definition 3.19. A metric space X is a complete metric space if and only if every Cauchy sequence in X converges to an element of X.

The set of rational numbers is the usual example of a metric space which is not complete.

Since linear metric spaces are the subject at hand, it is natural to consider complete linear metric spaces.

Definition 3.20. A complete linear metric space is called a Fréchet space.

If m, c, and c_0 are complete, then they are Fréchet spaces. The next theorem states that such is the case.

Theorem 3.21. m, c, and c_0 are Fréchet spaces.

Proof: Let $\{x_n\}$ be a Cauchy sequence in m. Then for every $\varepsilon > 0$ there exists an integer N such that $\|x_p - x_q\| < \varepsilon$ whenever $p, q > N$. Now $|z_{pk} - z_{qk}| \leq \|x_p - x_q\|$ for all $k \in I^+$ so that each column of $\{x_n\} = (z_{nk})$ is a Cauchy sequence in E. This means that each column of (z_{nk}) must converge to an element z_k in E, as in Figure 3.1, and that the convergence is uniform with respect to k. It remains to be shown that

$$\|x\| = \sup_k \{|z_k| : x = \{z_k\}\} = M < \infty$$

so that x belongs to m. Let $n_0 > N$ so that

$$|z_k| - |z_{n_0k}| \leq |z_{n_0k} - z_k| < \varepsilon$$

and

$$|z_k| < \varepsilon + |z_{n_0k}|$$

for all $k \in I^+$. Hence

$$\sup_k \{|z_k| : x = \{z_k\}\} \leq \sup_k \{|z_{n_0 k}| : x_{n_0} = \{z_{n_0 k}\}\} + \mathcal{E} = M_{n_0} + \mathcal{E},$$

Therefore $\|x\| < \infty$, and x is an element of m.

$$\begin{array}{ccccccccccc}
x_1 & = & z_{11} & z_{12} & z_{13} & \cdots & z_{1p} & \cdots & z_{1q} & \cdots \\
x_2 & = & z_{21} & z_{22} & z_{23} & \cdots & z_{2p} & \cdots & z_{2q} & \cdots \\
x_3 & = & z_{31} & z_{32} & z_{33} & \cdots & z_{3p} & \cdots & z_{3q} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\
x_n & = & z_{n1} & z_{n2} & z_{n3} & \cdots & z_{np} & \cdots & z_{nq} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\
x_{n_0} & = & z_{n_0 1} & z_{n_0 2} & z_{n_0 3} & \cdots & z_{n_0 p} & \cdots & z_{n_0 q} & \cdots \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \\
\downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow & \\
x & = & z_1 & z_2 & z_3 & \cdots & z_p & \cdots & z_q & \cdots
\end{array}$$

Figure 3. 1

Let $\{x_n\}$ be a Cauchy sequence in c. Then each row of (z_{nk}), $k \in I^+$, in Figure 3. 1 is a convergence sequence with limit w_k in E. Again, $\|x_m - x_n\| < \mathcal{E}$ whenever $m, n > N$ implies that $|z_{mk} - z_{nk}| < \mathcal{E}$ whenever $m, n > N$ so that each column of (z_{nk}) is a Cauchy sequence in E. Hence the column limits are justified in Figure 3. 1. Now this means that for every $\mathcal{E} > 0$ there exists an integer N_1 such that

$|z_p - z_{np}| < \mathcal{E}/3$ whenever $n > N_1$ and $|z_q - z_{nq}| < \mathcal{E}/3$ whenever $n > N_1$. Let n_0 be a fixed integer greater than N_1. Now since each row of (z_{nk}) is a convergent sequence, for every $\mathcal{E} > 0$ there exists an integer K which depends on n_0 such that $|z_{n_0p} - z_{n_0q}| < \mathcal{E}/3$ whenever $p, q > K$. Hence

$$|z_p - z_q| \leq |z_p - z_{n_0p}| + |z_{n_0p} - z_{n_0q}| + |z_{n_0q} - z_q|$$

$$< \mathcal{E}/3 + \mathcal{E}/3 + \mathcal{E}/3 = \mathcal{E}$$

whenever $p, q > K$. This proves that $\{z_k\}$ is a Cauchy sequence in E, and therefore $\{z_k\}$ is an element of c, so that c is a Fréchet space.

Since $\lim_{n\to\infty} z_{nk} = z_k$, $\lim_{k\to\infty} z_k = z$, and $\lim_{k\to\infty} z_{nk} = w_n$, one would suspect that $\lim_{n\to\infty} w_n$ exists and is equal to $\lim_{k\to\infty} z_k$. Such is the case, and the uniform convergence of the columns of (z_{nk}) is the key to proving this statement. First it will be shown that

$$\lim_{\substack{n\to\infty \\ k\to\infty}} z_{nk} = z.$$

For every $\mathcal{E} > 0$, there exists an integer N_1 such that $|z_{nk} - z_k| < \mathcal{E}/2$ whenever $n > N_1$ for every $k \in I^+$. Given $\mathcal{E} > 0$, choose an integer N_2 such that if $k > N_2$, $|z_k - z| < \mathcal{E}/2$. This can always be done since $\lim_{k\to\infty} z_k = z$. Let $N = \max\{N_1, N_2\}$ so that

$$|z_{nk} - z| \leq |z_{nk} - z_k| + |z_k - z| < \mathcal{E}/2 + \mathcal{E}/2 = \mathcal{E}$$

whenever $n > N$ and $k > N$. Next it will be shown that $\lim_{n\to\infty} w_n = z$. From what has just been done, for every $\mathcal{E} > 0$ there exists an integer N_3 so that $|z_{nk} - z| < \mathcal{E}/2$ whenever $n > N_3$ and $k > N_3$. Given $\mathcal{E} > 0$, for each $n \in I^+$, there exists an integer $N_4(n)$, which depends on n, such that $|w_n - z_{nk}| < \mathcal{E}/2$ whenever $k > N_4(n)$. Now for each $n > N_3$ choose $N_4(n)$ and choose a fixed integer $k_0 > N = \max\{N_3, N_4(n)\}$. This means that

$$|w_n - z| \leq |w_n - z_{nk_0}| + |z_{nk_0} - z| < \mathcal{E}/2 + \mathcal{E}/2 = \mathcal{E}$$

whenever $n > N$. Thus, $\lim_{n\to\infty} w_n = z$. To repeat, this is possible only because of the uniform convergence of the columns of (z_{nk}) which was used to show that

$$\lim_{\substack{n\to\infty \\ k\to\infty}} z_{nk} = z.$$

Now let $\{x_n\}$ be a sequence in c_0. As before, each column of $(z_{nk}) = \{x_n\}$ is a Cauchy sequence so that $\lim x_n = x = \{z_k\}$ and each row of $(z_{nk}) = \{x_n\}$ converges to zero so $\{w_n\} = \{0, 0, 0, \ldots\}$. Hence $\lim_{k\to\infty} z_k = 0$ and c_0 is complete.

Thus, m, c, and c_0 are Fréchet spaces, and the theorem is proved.

Fréchet sequence spaces with continuous coordinate functions are a distinguished collection of sequence spaces. The next definition gives the name of these spaces.

Definition 3.22. A Fréchet sequence space with continuous coordinate functions is an FK-space.

Theorem 3.23. m, c, and c_0 are FK-spaces.

Proof: Corollary 3.16, Theorem 3.21, and Definition 3.22 prove the theorem.

Another result from the application of functional analysis to summability methods is that the set of sequences which is the domain of an infinite matrix is an FK-space. That is, if A is an infinite matrix then $d_A = \{x : Ax \in s\}$ is an FK-space. Note that this statement does not require the matrix to be a summability matrix, nor does it require the transformed sequence to be convergent. The proof that d_A is an FK-space is not included here. Some of the concepts used in the proof require more background than the reader for whom this paper is intended may have. A proof that d_A is an FK-space for an arbitrary infinite matrix A can be found in [11].

Another set of sequences associated with an infinite matrix are those sequences which are transformed by the matrix into convergent sequences.

Definition 3.24. Let A be an infinite matrix. Then the set $c_A = \{x : Ax \in c\}$ is called the convergence domain of A.

The next theorem shows that if A is a K-matrix or a T-matrix, then $c_A \cap m$ is an FK-space.

Theorem 3.25. If A is a K-matrix or a T-matrix, then $c_A \cap m$ is an FK-space.

Proof: Let $x, y \in c_A \cap m$. Since $A(tx) = tAx$, and since $A(x+y) = Ax+Ay$, it is clear that $rx+ty$ belongs to $c_A \cap m$ for all x, y in $c_A \cap m$ and all r, t in E. Thus, $c_A \cap m$ is a linear space.

Since $c_A \cap m \subset m$, the metric $\|x - y\|$ on m applies to $c_A \cap m$ and $c_A \cap m$ is a metric space. Similarly, the properties of m assure that $c_A \cap m$ is a linear metric space.

Now let $\{x_n\}$ be a Cauchy sequence in $c_A \cap m$. It must be shown that $\lim x_n$ exists and is an element of $c_A \cap m$. From Theorem 3.22, $\lim x_n = x \in m$, so it must be shown that Ax exists and is convergent. Since $\{x_n\}$ is a Cauchy sequence in $c_A \cap m$, Ax_n exists and is convergent for all $n \in I^+$. That is,

$$Ax_m = \left\{ \sum_{k=1}^{\infty} a_{nk} z_{mk} \right\} = \{w_{mn}\},$$

and

$$\lim_{n \to \infty} w_{mn} = t_m \neq \infty$$

for all $m \in I^+$. In order for Ax to exist it must be shown that $\sum_{k=1}^{\infty} a_{nk} z_k$ exists and is finite for each $n \in I^+$. Since A is a K-matrix, Theorem 2.16 implies that

$$\sup \left\{ \sum_{k=1}^{\infty} |a_{nk}| : n \in I^+ \right\} = M < \infty,$$

and since x belongs to m, $\sup \{ |z_k| : x = \{z_k\}\} = J < \infty$. Thus,

$$\left| \sum_{k=1}^{\infty} a_{nk} z_k \right| \leq \sum_{k=1}^{\infty} |a_{nk} z_k| = \sum_{k=1}^{\infty} |a_{nk}| \, |z_k| \leq M \cdot J$$

so that $\sum_{k=1}^{\infty} a_{nk} z_k$ is absolutely convergent for all $n \in I^+$ and Ax exists. Now it must be shown that Ax belongs to c. If A is a K-matrix, then $\sum_{k=1}^{\infty} a_{nk} = r_n$ for each integer n and $\lim r_n = a \neq \infty$, by Theorem 2.16. Thus, $\{ \sum_{k=1}^{\infty} a_{nk} \}$ is a Cauchy sequence in E, and for every $\mathcal{E} > 0$ there exists an integer N such that

$$\left| \sum_{k=1}^{\infty} a_{nk} - \sum_{k=1}^{\infty} a_{mk} \right| < \mathcal{E}/J$$

whenever m, n > N. Recall that

$$\sup \{ |z_k| : x = \{z_k\}\} = J < \infty$$

so that

$$\left| \sum_{k=1}^{\infty} a_{nk} z_k - \sum_{k=1}^{\infty} a_{mk} z_k \right| \leq \left| \sum_{k=1}^{\infty} a_{nk} - \sum_{k=1}^{\infty} a_{mk} \right| \cdot J < \mathcal{E}.$$

Hence Ax is a Cauchy sequence in E, and this means that Ax belongs to c. This proves the theorem.

In the above theorem it was necessary to consider only those sequences in $c_A \cap m$ so that the metric for m could be used to full advantage. This result can be extended to c_A in the case that A is a K-matrix for which $a_{nk} = 0$ whenever $k > n$. In this situation, another metric exists in terms of which c_A itself is an FK-space.

Definition 3.26. If A is an infinite matrix such that $a_{nn} \neq 0$ for all $n \in I^+$ and $a_{nk} = 0$ whenever $k > n$ then A is called a triangle.

A property of triangles which will be useful later is the fact that a triangle A maps c_A one-to-one and onto c.

Theorem 3.27. If A is a triangle and x, y belong to c_A, then $Ax = Ay$ implies $x = y$ and $w = Ax$ has a unique solution for all $w \in c$.

Proof: Suppose $Ax = Ay$ in c. Then $Ax - Ay = A(x-y) = \{0, 0, 0, \ldots\}$ and this implies that

$$\sum_{k=1}^{n} a_{nk}(x_k - y_k) = 0$$

for all n. Now $a_{11}(x_1 - y_1) = 0$ implies $x_1 = y_1$ since $a_{11} \neq 0$.

Continuing in this fashion, $x_k = y_k$ for all $k \in I^+$ so that $x = y$.

Next consider $w = Ax$ where w is a fixed sequence in c. Then $w_1 = a_{11}x_1$, and $x_1 = w_1/a_{11}$ since $a_{11} \neq 0$. Next,

$$x_2 = \frac{w_2}{a_{22}} - \frac{a_{21}w_1}{a_{11}a_{22}} \quad \text{or} \quad x_2 = \sum_{k=1}^{2} b_{2k}w_k$$

where

$$b_{21} = \frac{-a_{21}}{a_{11}a_{22}} \quad \text{and} \quad b_{22} = \frac{1}{a_{22}} .$$

Similarly

$$x_n = \sum_{k=1}^{n} b_{nk}w_k$$

for each fixed $n \in I^+$. Let $b_{nk} = 0$ for $k > n$ so that $B = (b_{nk})$ is a triangle. Note that if $Ax = w$ then $Bw = x$ and B is a left inverse for A. Thus $w = Ax$ has a unique solution for all w in c, and the theorem is proved.

The metric for c_A when A is a conservative triangle is the subject of the next theorem.

Theorem 3.28. If A is a conservative triangle then

$$|x| = \sup \left\{ \left| \sum_{k=1}^{\infty} a_{nk}z_k \right| : n \in I^+ \right\}$$

is a norm for c_A, and $|x - y|$ is a metric for c_A. Note that $|x| = \|Ax\|$ and $|x - y| = \|Ax - Ay\|$.

Proof: Since A is conservative, $\{ |\sum_{k=1}^{\infty} a_{nk}z_k| \}$ is a convergent sequence of nonnegative real numbers so that $|x|$ exists, is finite, and is nonnegative. The fact that the supremum of a bounded set of real numbers is unique insures that $|x|$ is well defined for x in c_A.

Each sum is actually a finite sum because $a_{nk} = 0$ for $k > n$,

$$\left|\sum_{k=1}^{\infty} a_{nk} z_k\right| = \left|\sum_{k=1}^{n} a_{nk} z_k\right|$$

for all $n \in I^+$ so that $|x| = 0$ if and only if

$$\left|\sum_{k=1}^{n} a_{nk} z_k\right| = 0$$

for all $n \in I^+$. Now $a_{11}z_1 = 0$ implies $a_{22}z_2 = 0, \ldots,$ implies $a_{nn}z_n = 0$ for every $n \in I^+$. Since $a_{nn} \neq 0$ for all n, this implies that $z_n = 0$ for all n. Hence $|x| = 0$ if and only if $x = \{0, 0, 0, \ldots\}$.

$$\left|\sum_{k=1}^{n} a_{nk} z_k\right| = \left|\sum_{k=1}^{n} a_{nk}(-z_k)\right|$$

for all n implies that $|x| = |-x|$. Also

$$\left|\sum_{k=1}^{n} a_{nk}(z_k + w_k)\right| \leq \left|\sum_{k=1}^{n} a_{nk} z_k\right| + \left|\sum_{k=1}^{n} a_{nk} w_k\right|$$

for all n implies that $|x + y| \leq |x| + |y|$. Last

$$\left|\sum_{k=1}^{n} a_{nk}(tz_k)\right| = |t| \left|\sum_{k=1}^{n} a_{nk} z_k\right|$$

for all n implies that $|tx| = |t|\,|x|$, and this proves that $|x|$ is a norm for c_A. From Theorem 3.8, $|x - y|$ is a metric for c_A, and the theorem is proved.

<u>Theorem 3.29</u>. If A is a conservative triangle, then c_A is an FK-space.

Proof: The same arguments used in Theorem 3.25 can be used to show that c_A is a linear space.

Let $\{x_n\}$ and $\{y_n\}$ be sequences in c_A such that $\lim x_n = x$ in c_A and $\lim y_n = y$ in c_A. Also let t be an element of E. It must be shown that $\lim (x_n + y_n) = x + y$ and that $\lim (tx_n) = tx$. Now $\lim x_n = x$ in c_A implies that for every $\mathcal{E} > 0$ there exists an integer N_1 such that $|x_n - x| < \mathcal{E}/2$ whenever $n > N_1$ and $\lim y_n = y$ in c_A implies that there exists an integer N_2 such that $|y_n - y| < \mathcal{E}/2$ whenever $n > N_2$. Let $N = \max\{N_1, N_2\}$ so that

$$|x_n - x + y_n - y| = |x_n + y_n - (x+y)| \leq |x_n - x| + |y_n - y| < \mathcal{E}/2 + \mathcal{E}/2 = \mathcal{E}$$

whenever $n > N$. This means that addition is continuous in c_A. Since

$$|tx_n - tx| = |t|\ |x_n - x|,$$

if $t \neq 0$, then for every $\mathcal{E} > 0$ there exists an integer N such that $|x_n - x| < \mathcal{E}/|t|$, and $|tx_n - tx| < \mathcal{E}$ whenever $n > N$. Now let $t = 0$ so that $tx_n = tx = \{0, 0, 0, \ldots\}$ and $|tx_n - tx| < \mathcal{E}$ for all $n \in I^+$. Thus scalar multiplication is continuous in c_A. Now let $\{x_m\}$ be a Cauchy sequence in c_A. Since x_m is an element of c_A for all $m \in I^+$, it is clear that $Ax_m = w_m = \{w_n^m\}$ is in c for all $m \in I^+$. Since $\{x_m\}$ is Cauchy in c_A, for every $\mathcal{E} > 0$ there exists $n \in I^+$ such that

$$|x_p - x_q| = \|Ax_p - Ax_q\| < \mathcal{E}$$

whenever $p, q > N$. Thus $\|w_p - w_q\| < \mathcal{E}$ whenever $p, q > N$ and $\{w_m\}$ is a Cauchy sequence in c. This means that there exists w in c such that $\lim_{m \to \infty} w_m = w$ in c. Let x be the unique pre-image of w guaranteed by

Theorem 3.27. It will be shown that $\lim_{m \to \infty} x_m = x$ in c_A. Now

$$|x_m - x| = \|Ax_m - Ax\| = \|w_m - w\|.$$

Hence for every $\varepsilon > 0$ there exists an integer N such that

$$\|w_m - w\| = |x_m - x| < \varepsilon$$

whenever $m > N$. This proves that c_A is complete.

Now it must be shown that the coordinate functions, $P_q(x)$, are continuous on c_A. Let $\{x_n\}$ be a sequence in c_A such that $\lim_{n \to \infty} x_n = x$ in c_A. Then if $w_m = Ax_m$ and $w = Ax$, $\lim_{m \to \infty} w_m = w$ in c, from what was shown earlier. By the definition of $P_q(x)$,

$$|P_q(x_m) - P_q(x)| = |z_{mq} - z_q|$$

$$= \left| \sum_{k=1}^{q} b_{qk}(Ax_m - Ax) \right|$$

where $B = (b_{nk})$ is the left inverse of A given by Theorem 3.27. Since $Ax_m - Ax$ is an element of c,

$$\sup \left\{ \left| \sum_{k=1}^{\infty} a_{nk} z_{mk} - \sum_{k=1}^{\infty} a_{nk} z_k \right| : n \in I^+ \right\} = \|Ax_m - Ax\|,$$

and thus

$$\left| \sum_{k=1}^{q} b_{qk}(Ax_m - Ax) \right| \leq \|Ax_m - Ax\| \left| \sum_{k=1}^{q} b_{qk} \right| .$$

Hence for every $\varepsilon > 0$ and for each fixed $q \in I^+$ there exists an integer $N(q)$ such that

$$\|Ax_m - Ax\| = \|w_m - w\| < \frac{\varepsilon}{\left|\sum_{k=1}^{q} b_{qk}\right|}$$

whenever $m > N(q)$. This means that $|P_q(x_m) - P_q(x)| < \varepsilon$ whenever $m > N(q)$, and the coordinate functions are continuous on c_A. Thus c_A is an FK-space, and the theorem is proved.

A natural question to consider when two or more summability matrices are at hand is the question of whether they have the same convergence domains, whether the convergence domain of one contains the convergence domain of another, or whether there are any sequences that are in the intersection of the convergence domains of the matrices. At least a partial answer is available for the first part of this question.

Definition 3.30. Let A and B be K-matrices. Then A is equivalent to B if and only if $c_A = c_B$.

In a paper published in 1963, Wilansky stated the opinion that no really satisfactory characterization of convergence domains among FK-spaces exists. Note in particular that Definition 3.30 does not require that $\lim Ax = \lim Bx$ when $x \in c_A = c_B$, or that $\lim Ax = \lim x$ if $x \in c$. All that is required is that Ax and Bx belong to c for all x in $c_A = c_B$. If the convergence domains of T-matrices are the object of interest, Cooke has defined a more restricted equivalence which is given in the next definition.

Definition 3.31. Let A and B be T-matrices. Then A is absolutely equivalent to B if and only if $\lim (Ax - Bx) = 0$ for all x in $d_A \cap d_B \cap m \neq \emptyset$.

Note that this definition requires that Ax and Bx have the same limit for all x in $c_A \cap c_B \cap m$. It should be pointed out that it is possible for lim (Ax - Bx) to be zero when neither of Ax or Bx are convergent sequences. However, one would certainly have $\lim (Ax - Bx) = 0$ for all x in $c_A \cap c_B \cap m$ if A and B were absolutely equivalent. The following theorem of Cooke's, published in 1936, gives a necessary and sufficient condition for two T-matrices to be absolutely equivalent.

Theorem 3.32. If $A = (a_{nk})$ and $B = (b_{nk})$ are T-matrices, then A is absolutely equivalent to B if and only if

$$\lim_{n \to \infty} \left(\sum_{k=1}^{\infty} |a_{nk} - b_{nk}| \right) = 0.$$

Proof: a) The condition is sufficient. Since $x \in m$,

$$\sup \{ |z_k| : x = \{z_k\} \} = M < \infty$$

so that

$$Ax - Bx = (A - B)x = \sum_{k=1}^{\infty} (a_{nk} - b_{nk}) z_k.$$

Hence for every $\varepsilon > 0$ there exists an integer N such that

$$\sum_{k=1}^{\infty} |a_{nk} - b_{nk}| < \varepsilon / M$$

whenever $n > N$. This means that

$$\left| \sum_{k=1}^{\infty} a_{nk} z_k - \sum_{k=1}^{\infty} b_{nk} z_k \right| \leq \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| \, |z_k| < \varepsilon$$

whenever $n > N$, and this proves that the condition is sufficient.

b) The condition is necessary. The proof of necessity is similar to the proof of Lemma 2.10. It will be assumed that

$$\lim_{n \to \infty} \left(\sum_{k=1}^{\infty} |a_{nk} - b_{nk}| \right) = \alpha \neq 0$$

or that

$$\left\{ \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| \right\}$$

has at least one finite limit point $\alpha \neq 0$. Then a sequence $x = \{z_k\}$ will be constructed with $|z_k| \leq 1$ for all $k \in I^+$, and it will be shown that for this x, $\lim (Ax - Bx) \neq 0$.

The assumption that

$$\left\{ \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| \right\}$$

has at least one finite limit point $\alpha \neq 0$ is valid since if A and B are T-matrices then Theorem 2.25 implies that

$$\sum_{k=1}^{\infty} |a_{nk}| + \sum_{k=1}^{\infty} |b_{nk}| \leq M_A + M_B$$

for all $n \in I^+$. Hence

$$\sum_{k=1}^{\infty} |a_{nk} - b_{nk}| \leq M_A + M_B$$

for all $n \in I^+$, and Theorem 1.12 implies that

$$\left\{ \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| \right\}$$

has at least one finite limit point α, which is different from zero.

Now let

$$s_n = \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| = \sum_{k=1}^{\infty} |c_{nk}|$$

and let $\{s_{n_r}\}$ be a subsequence such that $\lim_{r\to\infty} s_{n_r} = \alpha$. Note that α is a real number. There exists an integer n_{r_1} such that $s_{n_{r_1}} > 3\alpha/4$ whenever $n_r \geq n_{r_1}$, since $\lim_{r\to\infty} s_{n_r} = \alpha$. Now choose an integer m_1 so that

$$\sum_{k=m_1+1}^{\infty} |c_{n_{r_1}k}| < \alpha/24,$$

and define the first m_1 terms of $\{z_k\}$ as follows. Let

$$z_k = \frac{|c_{n_{r_1}k}|}{c_{n_{r_1}k}} \quad \text{if} \quad c_{n_{r_1}k} \neq 0$$

and let $z_k = 0$ if $c_{n_{r_1}k} = 0$ for $1 \leq k \leq m_1$. Then

$$z'_{n_{r_1}} = \left|\sum_{k=1}^{\infty} c_{n_{r_1}k} z_k\right| \geq \sum_{k=1}^{\infty} |c_{n_{r_1}k}| - 2 \sum_{k=m_1+1}^{\infty} |c_{n_{r_1}k}|$$

$$> 3\alpha/4 - \alpha/12 = 2\alpha/3.$$

Since A and B are T-matrices, $\lim_{n\to\infty} c_{nk} = 0$ for all $k \in I^+$. This means that there exists an integer $n_{r_2} > n_{r_1}$ so that

$$\sum_{k=1}^{m_1} |c_{n_{r_2}k}| < \alpha/6,$$

and there exists an integer $m_2 > m_1$ such that

$$\sum_{k=m_2+1}^{\infty} |c_{n_{r_2}k}| < \alpha/6.$$

Now define $z_k = 0$ for $m_1 < k \leq m_2$ so that

$$z'_{n_{r_2}} = \left|\sum_{k=1}^{\infty} c_{n_{r_2}k}\right| \leq \sum_{k=1}^{m_1} |c_{n_{r_2}k}| + \sum_{k=m_2+1}^{\infty} |c_{n_{r_2}k}|$$

$$< \alpha/6 + \alpha/6 = \alpha/3.$$

If this process is continued, then each z_k will be zero, one or minus one and each term of $\{z'_{n_r}\}$, a subsequence of $|Ax - Bx|$, is either greater than $2\alpha/3$ or less than $\alpha/3$. Hence $|Ax - Bx|$ cannot converge to zero. This proves that the condition is necessary, and the theorem is proved.

Example 3.33. As an application of this theorem, it will be shown that the matrix A of arithmetic means and the matrix C_1 of Cesaro means of order one are absolutely equivalent. Recall from Chapter II that for A, $a_{nk} = 1/n$ if $k \leq n$ and $a_{nk} = 0$ if $n < k$. Also for C_1,

$$c_{nk} = \frac{1(\Gamma(n+1)\Gamma(n+1-k))}{\Gamma(n-k+1)\Gamma(n+2)} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$$

if $k \leq n$ and $c_{nk} = 0$ if $n < k$. Here

$$|a_{nk} - c_{nk}| = |1/n - 1/(n+1)| = \frac{1}{n(n+1)}$$

if $k \leq n$ and $|a_{nk} - c_{nk}| = 0$ if $n < k$. Hence

$$\sum_{k=1}^{\infty} |a_{nk} - c_{nk}| = \sum_{k=1}^{n} \frac{1}{n(n+1)} = \frac{1}{n+1},$$

and $\lim_{n \to \infty} 1/(n+1) = 0$. Thus A and C_1 are absolutely equivalent.

When it is possible to show that two summability matrices have the same convergence domain, some mathematicians have considered certain distinguished subsets of the convergence domains. If a distinguished subset of the convergence domain of two equivalent matrices depends only on the set which is the convergence domain, and not on the matrices, then the subset is said to be invariant. There has been a considerable amount of investigation into the invariance of some of the distinguished subsets. The distinguished subset which will be considered is defined next.

Definition 3.34. Let A be a K-matrix with convergence domain c_A. Then

$$B_A = \{x \in c_A : \text{There exists } M(x) \text{ in } (0, \infty) \text{ such that } \left|\sum_{k=1}^{m} a_{nk}x_k\right| < M(x) \text{ for all } m, n \text{ in } I^+\}.$$

Note that B_A is the subset of c_A for which the sequence of partial row sums is uniformly bounded for every row of $A = (a_{nk})$. The notation M(x) is intended to indicate that the number M is not necessarily the same for all x in B_A.

This set is used to define what is called the "mean value property" of a K-matrix. A theorem for real triangles concerning a property similar to the property defined by B_A is next.

Theorem 3.35. Let $A = (a_{nk})$ be a triangle where a_{nk} is real for all n, k in I^+, and let $x = \{x_k\}$ belong to the domain of A. If

i) $a_{nk} \neq 0, \quad 0 \leq a_{mk}/a_{nk} \leq J \ (1 \leq k \leq n \leq m),$

and

$$\text{ii)} \qquad \frac{a_{mk-1}}{a_{nk-1}} \geq \frac{a_{mk}}{a_{nk}} \quad (1 \leq k \leq n \leq m)$$

then for each $n \leq m$

$$\left|\sum_{k=1}^{n} a_{mk}x_k\right| \leq J \max_{1\leq r\leq n} \left\{\left|\sum_{k=1}^{r} a_{rk}x_k\right|\right\}.$$

Proof: The sum $\sum_{k=1}^{n} a_{mk}x_k$ can be written as

$$\sum_{k=1}^{n} \frac{a_{mk}}{a_{nk}} a_{nk}x_k$$

since $a_{nk} \neq 0$ for all $n \leq m$ by i. Now ii implies that a_{m1}/a_{n1} is the largest of the nonnegative ratios a_{mk}/a_{nk} for $1 \leq k \leq n \leq m$. Hence

$$\left|\sum_{k=1}^{n} a_{mk}x_k\right| \leq \frac{a_{m1}}{a_{n1}} \left|\sum_{k=1}^{n} a_{nk}x_k\right|,$$

and i implies that

$$\left|\sum_{k=1}^{n} a_{mk}x_k\right| \leq J \max_{1\leq r\leq n} \left\{\left|\sum_{k=1}^{r} a_{rk}x_k\right|\right\}.$$

This proves the theorem.

Example 3.36. As an example of a matrix which has this property, consider the matrix A of arithmetic means. Here $a_{nk} = 1/n \neq 0$ for $1 \leq k \leq n$, and if $n \leq m$ then $a_{mk}/a_{nk} = n/m$ so that $0 \leq n/m \leq 1$ when $n \leq m$. Also

$$\frac{a_{mk-1}}{a_{nk-1}} = n/m = \frac{a_{mk}}{a_{nk}}$$

for $1 \leq k \leq n \leq m$. Since $J = 1$, it is clear that

$$\left|\sum_{k=1}^{n} a_{mk}x_k\right| \leq \max_{1\leq r\leq n} \left|\sum_{k=1}^{r} a_{rk}x_k\right|$$

for the matrix of arithmetic means. The matrix C_1 of Cesaro means of order one is an entirely analogous example of a matrix with the property of Theorem 3.35.

To return to the set B_A and the mean value property for summability matrices, note that if it is required that x belong to c_A in Theorem 3.35, then

$$\left|\sum_{k=1}^{n} a_{mk}x_k\right| \leq |x|$$

for all $m, n \in I^+$. This means that if A satisfies the hypothesis of Theorem 3.35, then $c_A = B_A$. The next definition gives a formal statement of the mean value property.

Definition 3.37. Let A be a conservative triangle. Then A has the mean value property if and only if $c_A = B_A$.

The following theorem shows that the set B_A depends only on the set c_A, and not on the matrix A.

Theorem 3.38. Let A_1, A_2 be conservative triangles such that $c_{A_1} = c_{A_2}$ and let A_1 have the mean value property. Then A_2 has the mean value property. That is, B_{A_i} is invariant.

Proof: Since $B_{A_2} \subset c_{A_2}$ by definition, it must be shown that $c_{A_2} \subset B_{A_2}$ so that $c_{A_2} = B_{A_2}$. Now let x belong to c_{A_2} so that $A_2x = w_2$ in c. Then there exists a unique y in $c_{A_1} = c_{A_2}$ such that $A_1y = w_2$. Thus $A_2x = A_1y$ so that

$$\left|\sum_{k=1}^{n} a_{mk}^{(2)} x_k\right| = \left|\sum_{k=1}^{n} a_{mk}^{(1)} y_k\right|$$

and since A_1 has the mean value property,

$$\left|\sum_{k=1}^{n} a_{mk}^{(2)} x_k\right| < M(y)$$

for all $m, n \in I^+$. Hence $x \in B_{A_2}$ and $c_{A_2} \subset B_{A_2}$. This shows that $c_{A_2} = B_{A_2}$, and this proves the theorem.

This theorem justifies writing B without the subscript identifying a particular matrix. It must be remembered however, that the set B is invariant in the sense that it depends only on the convergence domain of a set of equivalent matrices. The convergence domain must still be identified by one of the equivalent matrices.

Example 3.39. An example of a conservative triangle which does not have the mean value property is the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{pmatrix}$$

Consider the sequence $x = \{1, 2, 3, 4, \ldots\}$. In this case $Ax = \{1, 1, 1, \ldots\}$ so that x belongs to c_A. The sequence of partial row sums $\{1, -1, -2, -3, \ldots\}$ is not bounded, however, so that B_A is a proper subset of c_A. Thus A cannot have the mean value property.

Several subsets of the convergence domain of a conservative matrix are known to be invariant. Others are still under investigation.

Another topic involved in characterizing convergence domains is that of determining the sequences which are a basis for the convergence domain. [11] and [12] contain more information concerning this topic.

The introduction to sequence spaces, to FK-spaces in particular, which is given here is only a beginning for many of the results which have come from the application of functional analysis. The reader can verify this easily in Chapters XI and XII of [11].

The next chapter will contain some of the applications of summability methods which were mentioned in Chapter II.

CHAPTER IV

APPLICATIONS

Under certain conditions, functions from E to E can be represented by an infinite series. Some examples of power series which should be familiar are

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \qquad \cos z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!},$$

and

$$\sin z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} .$$

These series representations are particular cases of the theory of representing functions by Taylor series. There are also series representations for certain functions from $[-\pi, \pi]$ to R by infinite series of trigonometric functions called Fourier series.

Taylor series are fundamental to the study of the class of analytic functions in the theory of functions of a complex variable. In fact, a function is analytic in a domain in E if and only if at each point of the domain it has a Taylor series representation valid in a neighborhood of the point. On the other hand, a power series with radius of convergence $r > 0$ represents a function which is analytic at every point within the circle of convergence.

Fourier series representations of real functions have wide application in the solution of differential equations. The differential equations which represent the motion of a vibrating string or a vibrating membrane are two standard applications. This is not too surprising, when the periodicity of a vibrating string or membrane and the periodicity of the trigonometric functions are considered. Since the solution of a differential equation with boundary conditions or initial conditions involves integration and the choice of a particular integral, it is clear that Fourier series representations allow evaluation of difficult integrals. Thus a differential equation may describe the motion of a physical object and Fourier series can be used to find the function which gives the position of the object at a given time.

Two types of applications of summability methods will be considered. By division, $1/(1-z)$ has the formal representation $\sum_{k=0}^{\infty} z^k$. It can be shown that

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

for all z such that $|z| < 1$. Now $1/(1-z)$ is defined for all $z \neq 1$, but the series $\sum_{k=0}^{\infty} z^k$ is convergent only for $|z| < 1$. By analytic continuation, $1/(1-z)$ can be represented by Taylor series at other points. This process requires that $1/(1-z)$ be represented by a family of Taylor series.

It will be shown that there is a T-sequence which transforms the sequence of partial sums of $\sum_{k=0}^{\infty} z^k$ into a function g(x) such that $\lim_{x \to \infty} g(x) = 1/(1-z)$ for all z in $E \setminus [1, \infty)$. Thus the Taylor series for $1/(1-z)$ can be transformed into a function g(x) whose limit as $x \to \infty$ is $1/(1-z)$ in a larger subset of E than the set $U = \{z: |z| < 1\}$. When Fourier series representations of functions are used, they may not converge, or they may converge to a value different from the value of the function at a given real number in the domain of the function. There are restrictions on the function which will be shown to guarantee that the transform of the sequence of partial sums of the Fourier series by the Cesaro matrix of order one will converge to the value of the function. Since the Cesaro matrix of order one is a T-matrix, convergence and limits of convergent sequences are preserved.

To summarize, summability methods will be applied to the problems of analytic continuation of Taylor series and convergence of Fourier series. Taylor series will be considered first.

Definition 4. 1. Let

$$f^{(n)}(a) = \frac{d^n f(a)}{dx^n}$$

and let f(z) be a function from E to E such that $f^{(n)}(a)$ exists for all $n \in I^+$. Then

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z - a)^k$$

is called the Taylor series of f(z) at a.

It should be pointed out that the series may diverge for all z except z = a. However, if it converges at a point $z_0 \neq a$, then it converges in the interior of some circle with a as center. From Theorem 1.33, if

$$a_n = \frac{f^{(n)}(a)}{n!}, \quad n = 0, 1, 2, \ldots,$$

then the radius of convergence is

$$r = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}} .$$

A Taylor series will be said to be convergent only if $r > 0$.

The star domain of a function will now be defined.

Definition 4.2. Let P be the set of finite singular points of f, and let c be a regular point of f. Then

$$D^* = E \setminus \bigcup_{z \in P} \{u : u = z + t(z - c),\ t \geq 0\}$$

is the star domain of f with respect to the point c.

Note that $\{u : u = z + t(z - c)\}$ is the ray with endpoint z which has the direction of the segment from c toward z. In other words, the star domain of f(z) with respect to c is the complex plane with the rays determined by c and the singular points of f(z) deleted from z on outward. Figure 4.1 may be helpful.

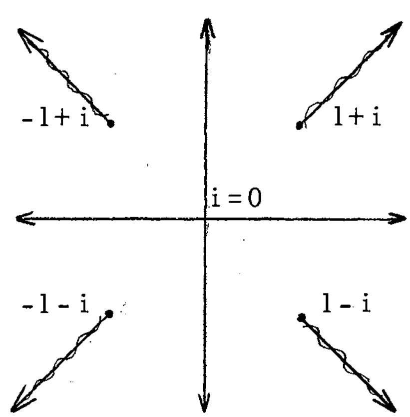

Figure 4.1

Thus D^* for $f(z) = 1/(z^4+4)$ with respect to $c = 0$ is E with the rays depicted ⇝ deleted. As another example, D^* for $f(z) = 1/(1-z)$ with respect to $z = 0$ is E with the ray $[1, \infty)$ deleted.

Dienes has proved the following theorem concerning the representation of a function f(z) by a power series in the star domain of f(z) with respect to the origin.

Theorem 4.3. Let

$$E(z) = \sum_{k=0}^{\infty} h(k)z^k$$

be an entire function with $h(k) \geq 0$ for every $k = 0, 1, 2, \ldots$. If for every $\mathcal{E} > 0$ E(z) converges uniformly to zero in $\mathcal{E} \leq \emptyset \leq 2\pi - \mathcal{E}$ as $|z|$ increases without bound, $z = re^{i\emptyset}$, then Mittag-Leffler's representation

$$f(z) = \lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{\sum_{k=0}^{\infty} s_k(z)h(k+1)x^{k+1}}{\sum_{k=0}^{\infty} h(k)x^k}$$

is valid in the star domain of f(z) with respect to $z = 0$. Here

$$s_k(z) = \sum_{n=0}^{k} \frac{f^{(n)}(0)z^n}{n!}$$

is the kth partial sum of the Maclaurin series of f(z).

The proof of this theorem can be found in [5], p. 309.

A theorem of LeRoy and Lindelöf shows that the functions

$$L_t(z) = \sum_{k=0}^{\infty} \left(\frac{z}{\log (k+t)}\right)^k, \quad t > 1,$$

satisfy the requirement that for every $\mathcal{E} > 0$, $L_t(z)$ converges uniformly to zero in $\mathcal{E} \leq \emptyset \leq 2\pi - \mathcal{E}$ as $|z|$ increases without bound, $z = re^{i\emptyset}$. This theorem can be found in [5], pp. 340-345. It is straight forward to show that $L_t(z)$ satisfies the other hypotheses of Theorem 4.3.

Theorem 4.4. If

$$h(k) = \frac{1}{[\log (k+t)]^k}, \quad t > 1,$$

then

$$L_t(z) = \sum_{k=0}^{\infty} h(k)z^k$$

is an entire function with $h(k) \geq 0$ for every $k = 0, 1, 2, \ldots$.

Proof: The radius of convergence of

$$\sum_{k=0}^{\infty} h(k)z^k$$

is infinite since

$$\lambda = \overline{\lim} \left(|h(k)|^{1/k} \right) = 0.$$

Thus $L_t(z)$ is an entire function. Since

$$\log (k+t) \geq \log t > \log 1 = 0,$$

$h(k)$ is positive for all $k = 0, 1, 2, \ldots$. This proves the theorem.

Now it is clear that the Lindelöf function, $L_t(z)$, can be used to define a subset of the set of T-sequences of Mittag-Leffler type which were mentioned in Chapter II. Thus if

$$f(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k,$$

a sequence

$$\{L_{tn}(x)\} = \frac{\dfrac{x^n}{\log (n+t)^n}}{\sum\limits_{k=0}^{\infty} \left(\dfrac{x}{\log (k+t)}\right)^k} \quad , \quad t > 1,$$

transforms the sequence

$$\{s_k\} = \left\{ \sum_{n=0}^{k} z^n \right\}$$

into a function g(x) such that $\lim_{x \to \infty} g(x) = f(z) = 1/(1-z)$ for all z in $E \setminus [1, \infty)$. This then is the application of T-sequences to the representation of a function by its Taylor series.

To summarize, a Lindelöf sequence will transform the sequence of partial sums of the Taylor series of f(z) into a function g(x) such that $\lim_{x \to \infty} g(x) = f(z)$ for all z in the star domain of f(z) with respect to a regular point of f(z).

This means that f(z) can be represented in its star domain with respect to the origin by its Taylor series at z = 0 and by the Lindelöf sequence. Thus the collection of Taylor series required by analytic continuation is reduced to two formulas.

Another method of representing a function f in a domain larger than the circle of convergence was developed by Borel.

Definition 4. 5. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for $|z| < r$, $0 < r < \infty$, then

$$F(zx) = \sum_{n=0}^{\infty} \frac{a_n z^n x^n}{n!}$$

where $x \in R$ is the <u>entire function associated with f(z).</u>

The next two theorems concern improper integrals of F(zx) and its derivatives. Proofs may be found in [5].

<u>Theorem 4.6.</u> Let

$$I_0(z) = \int_0^{\infty} e^{-x} F(zx)dx,$$

and let

$$I_m(z) = \int_0^{\infty} e^{-x} \frac{d^m F(zx)}{dx^m} dx, \ m \in I^+.$$

If these integrals are absolutely convergent for $z_0 = \varphi_0 e^{i\theta_0}$, they are absolutely convergent for all u such that $u = tz_0$, $0 \leq t \leq 1$, and $I_0(z)$ represents the analytic continuation of f(z) in the disc

$$|z - z_0/2| \leq |z_0/2|.$$

<u>Theorem 4.7.</u> If f(z) is analytic for $|z - z_0/2| \leq |z_0/2|$, then Borel's integrals are convergent for $u = tz_0$, $0 \leq t \leq 1$.

In the proof of Theorem 4.7 it is shown that Borel's integrals are convergent for all z such that $\mathrm{Re}(z/z_0) < 1$. Now $\{z : \mathrm{Re}(z/z_0) < 1\}$ is the half plane containing the origin whose edge is the line through z_0 perpendicular to the segment from the origin to z_0. Let $U = \{u : u$ is a singular point of $f\}$. Then both theorems apply to all z in the set

$$P = \bigcap_{u \in U} \{z : \mathrm{Re}(z/u) < 1\}.$$

If U is finite, P is the intersection of a finite collection of half-planes. In other words, P is a polygon. The diagram below illustrates the Borel polygon for $f(z) = 1/(z^4+4)$.

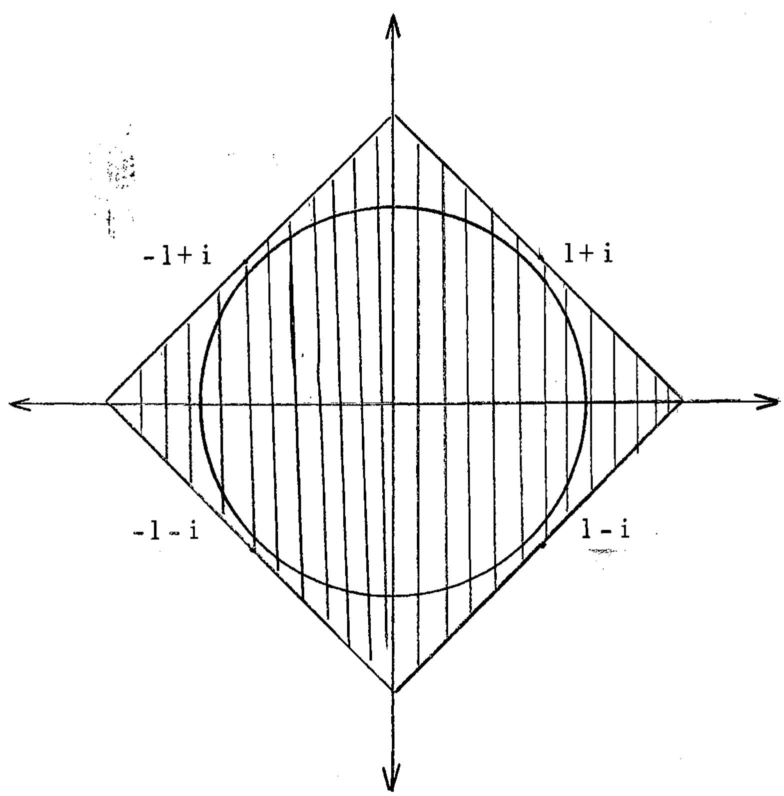

Figure 4.2

Here $U = \{1+i, -1+i, -1-i, 1-i\}$, and P is the shaded portion of the diagram, not including the boundary, which contains the circle of convergence of the Taylor series of $1/(z^4+4)$.

It can be shown that

$$\int_0^{\infty} e^{-x} F(zx)dx = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} \frac{s_n(z)x^n}{n!}$$

where

$$s_n(z) = \sum_{k=0}^{n} a_n z^n$$

is the nth partial sum of the Taylor series for f. From Chapter II, g(x) is the transform of $\{s_k(z)\}$ by the Borel T-sequence,

$$\{f_k(x)\} = \{\frac{e^{-x}x^k}{k!}\} .$$

Again note that f can be represented in P by two formulas,

$$\sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad \{\frac{e^{-x}x^k}{k!}\}.$$

The representation of real functions by Fourier series will now be considered. The results included here will contain the restriction that a function f(x) defined on $[-\pi, \pi]$ with function values in R be Lebesgue integrable on $[-\pi, \pi]$, written $f \in L[-\pi, \pi]$.

Definition 4.8. If $f \in L[-\pi, \pi]$, then the Fourier series for f is the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)(-\pi \leq x \leq \pi)$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx\, dx, \quad k = 0, 1, 2, \ldots$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx\, dx, \quad k = 1, 2, 3, \ldots$$

The numbers a_k and b_k are called the Fourier coefficients for f.

Example 4.9. As an example of a Fourier series representation of a function, let f(x) = 1 if $-\pi \leq x < 0$ and let f(x) = 0 if $0 \leq x \leq \pi$. Here

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} \text{cox } kx \, dx$$

so that $a_0 = (1/\pi)(\pi) = 1$, and

$$a_k = \frac{1}{\pi} \left(\frac{\sin kx}{k} \Big|_{-\pi}^{0} \right) = 0$$

for k = 1, 2, ... Next,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx = \frac{1}{\pi} \int_{-\pi}^{0} \sin kx \, dx,$$

and

$$b_k = \frac{1}{\pi} \left(\frac{-\cos kx}{k} \Big|_{-\pi}^{0} \right) = \frac{\cos k\pi - 1}{k\pi}$$

for k = 1, 2, ... This means that $b_k = 0$ if k = 2, 4, 6. . . and $b_k = -2/k\pi$ if k = 1, 3, 5, ... Thus the Fourier series representation for f(x) is

$$\frac{1}{2} - \frac{2}{\pi} \left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots \right].$$

To illustrate the fact that Fourier series may converge to a value other than the function value, note that f(0) = 0, but the series converges to 1/2 at x = 0.

Example 4.10. A further example is the Fourier series representation for f(x) = |x| on $[-\pi, \pi]$,

$$\frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1} + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \ldots\right].$$

and $x = 0$, $f(0) = |0| = 0$, and the Fourier series for $|x|$ at $x = 0$ is

$$\frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{9} + \frac{1}{25} + \ldots \frac{1}{(2k+1)^2} + \ldots\right].$$

It can be shown that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

so that the Fourier series for $|x|$ at $x = 0$ has the value

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\pi^2}{8}\right) = 0$$

which is the same as $f(0)$.

In the examples just given, the Fourier series representations both converge at $x = 0$, in one case to the function value, and in the other case to a value different from the function value. It is not so apparent whether or not they converge to $f(x)$ at $x \neq 0$ in $[-\pi, \pi]$.

The next two theorems give some information regarding the Fourier series representations of functions. Detailed proof of these can be found in [7].

Theorem 4.11. Let $f \in L[-\pi, \pi]$, let

$$s_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

for $-\pi \leq t \leq \pi$, then

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

if and only if

$$\lim_{n\to\infty} \frac{2}{\pi} \int_0^{\pi} \left[\frac{f(x+t) + f(x-t)}{2} - f(x)\right] D_n(t)dt = 0$$

where

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin[(2n+1)t/2]}{2\sin(t/2)},$$

$t \neq 0 \pm 2k\pi$, $D_n(0 \pm 2k\pi) = n + 1/2$.

Since C_1 summability of Fourier series is the topic of immediate concern, the next theorem gives necessary and sufficient conditions for the transform $\{C_1 s_n\}$ of the sequence $\{s_n\}$ to converge to the function f(x).

<u>Theorem 4. 12.</u> Let $f \in L[-\pi, \pi]$, let

$$s_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt),$$

and let

$$\sigma_n(t) = C_1 s_n(t) = \frac{1}{n}\left(\frac{a_0}{2} + \sum_{k=1}^{n} s_k(t)\right)$$

for $-\pi \leq t \leq \pi$. Then

$$f(x) = \lim_{n\to\infty} \sigma_n(x)$$

if and only if

$$\lim_{n\to\infty} \frac{2}{\pi} \int_0^{\pi} \left[\frac{f(x+t) + f(x-t)}{2} - f(x)\right] K_n(t)dt = 0$$

where

$$K_n(t) = \frac{1}{2n\sin(t/2)} \sum_{k=0}^{n-1} \sin(k + 1/2)t = \frac{\sin^2(nt/2)}{2n\sin^2(t/2)},$$

$t \neq 0 \pm 2k\pi$, $K_n(0 \pm 2k\pi) = n$.

Proof: Now $\lim_{n\to\infty} \sigma_n(x) = f(x)$ if and only if $\lim_{n\to\infty} (\sigma_n(x) - f(x)) = 0$, and the theorem will be proved if it can be shown that

$$\sigma_n(x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \left[\frac{f(x+t) + f(x-t)}{2} - f(x)\right] K_n(t)dt.$$

Since

$$s_n(x) = \frac{1}{\pi} \int_0^{\pi} [f(x+t) - f(x-t)]D_n(t)dt$$

from Theorem 4, 9,

$$\sigma_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{1}{\pi} \int_0^{\pi} [f(x+t) + f(x-t)]D_k(t)dt\right)$$

and

$$\sigma_n(x) = \frac{1}{\pi} \int_0^{\pi} [f(x+t) + f(x-t)] \left(\frac{1}{n} \sum_{k=0}^{n-1} D_k(t)\right) dt .$$

Now

$$\frac{1}{n} \sum_{k=0}^{n-1} D_k(t) = \frac{1}{2n \sin t/2} \sum_{k=0}^{n-1} \sin [(2k+1)t/2] = K_n(t) = \frac{\sin^2 (nt/2)}{2n \sin^2 (t/2)},$$

so

$$\sigma_n(x) = \frac{1}{\pi} \int_0^{\pi} [f(x+t) + f(x-t)] K_n(t)dt.$$

If $g(x) = 1$, then $s_0(x) = s_1(x) = \ldots = s_{n-1}(x) = 1$ so that $\sigma_n(x) = 1$ and $g(x+t) = g(x-t) = 1$. This means that

$$\frac{2}{\pi} \int_0^{\pi} K_n(t)\, dt = 1$$

and that

$$f(x) = \frac{2}{\pi} \int_0^{\pi} f(x)K_n(t)dt.$$

Thus

$$\sigma_n(x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \left[\frac{f(x+t) + f(x-t)}{2} - f(x)\right] K_n(t)dt$$

from which the theorem readily follows.

A sufficient condition which applies directly to the function f is given in the next theorem which shows that if f(x) is continuous and Lebesgue integrable on $[-\pi, \pi]$, then $\lim_{n \to \infty} \sigma_n(x) = f(x)$ for $-\pi \le x \le \pi$. Here continuity of f(x) on $[-\pi, \pi]$ means that the extension of f(x) by $f(u) = f(u+2\pi)$ is continuous on R. This requires that $f(-\pi) = f(-\pi+2\pi) = f(\pi)$.

<u>Theorem 4.13</u>. If $f \in L[-\pi, \pi]$ and f(x) is continuous on $[-\pi, \pi]$, then $\lim_{n \to \infty} \sigma_n(x) = f(x)$ for $-\pi \le x \le \pi$.

Proof: Let $\mathcal{E}$ be a positive real number. It must be shown that there exists an integer N such that $|\sigma_n(x) - f(x)| < \mathcal{E}$ whenever $n > N$.

The continuity of f implies that there exists a number δ where $0 < \delta < \pi$ such that $|f(y) - f(x)| < \mathcal{E}/2$ whenever $|y-x| < \delta$. Now if $0 \le t < \delta$, then $|x+t-x| = |x-t-x| < \delta$, and

$$\left|\frac{f(x+t) + f(x-t) - 2f(x)}{2}\right| \le \frac{1}{2} \ |f(x+t) - f(x)| + |f(x-t) - f(x)|$$

$$< \frac{1}{2} (\mathcal{E}/2 + \mathcal{E}/2) = \mathcal{E}/2.$$

This means that

$$\left|\frac{2}{\pi}\int_0^\delta\left[\frac{f(x+t)+f(x-t)}{2}-f(x)\right]K_n(t)dt\right| \leq \frac{\mathcal{E}}{2}\left(\frac{2}{\pi}\int_0^\delta K_n(t)dt\right).$$

Now $K_n(t) > 0$ so that if $0 < \delta < \pi$ then

$$\frac{2}{\pi}\int_0^\delta K_n(t)dt < \frac{2}{\pi}\int_0^\pi K_n(t)dt = 1.$$

Thus,

$$\left|\frac{2}{\pi}\int_0^\delta\left[\frac{f(x+t)+f(x-t)}{2}-f(x)\right]K_n(t)dt\right| < \frac{\mathcal{E}}{2}$$

for $0 < \delta < \pi$. For $t \geq \delta$,

$$K_n(t) \leq \frac{1}{2n(\sin^2 \delta/2)}$$

so that

$$\left|\frac{2}{\pi}\int_\delta^\pi\left[\frac{f(x+t)+f(x-t)}{2}-f(x)\right]K_n(t)dt\right|$$

$$\leq \frac{1}{2n\pi\sin^2\delta/2}\int_\delta^\pi (|f(x+t)-f(x)| + |f(x-t)-f(x)|)dt$$

$$\leq \frac{\mathcal{E}(\pi-\delta)}{4n\pi\sin^2\delta/2}.$$

Thus for a given $\mathcal{E}$ and corresponding δ, choose N so that

$$\frac{\pi-\delta}{4N\pi\sin^2(\delta/2)} < \frac{1}{2}.$$

That is, choose

$$N > \frac{\pi-\delta}{2\pi\sin^2(\delta/2)}.$$

Thus

$$|\sigma_n(x) - f(x)| = \left|\frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x+t) + f(x-t)}{2} - f(x)\right]K_n(t)dt\right|$$

$$\leq \left|\frac{2}{\pi}\int_0^{\delta}\left[\frac{f(x+t) + f(x-t)}{2} - f(x)\right]K_n(t)dt\right|$$

$$+ \left|\frac{2}{\pi}\int_{\delta}^{\pi}\left[\frac{f(x+t) + f(x-t)}{2} - f(x)\right]K_n(t)dt\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $n > N$. This proves the theorem.

Continuity is a rather strong condition to impose in order to guarantee that $\lim_{n \to \infty} \sigma_n(x) = f(x)$. Actually it can be shown that $\lim_{n \to \infty} \sigma_n(x) = f(x)$ for all x in the set

$$H = \{x \in [-\pi, \pi] : f(x) = \frac{d\,F(x)}{dx} \text{ where } F(x) = \int_0^x f(t)dt\}.$$

Now if f is Lebesgue integrable on $[-\pi, \pi]$ then the measure of $[-\pi, \pi]\setminus H$ is zero. Thus it can be said that for $f \in L[-\pi, \pi]$, $\lim_{n \to \infty} \sigma_n x = f(x)$ for "almost all" x in $[-\pi, \pi]$.

It must be pointed out that continuity is not sufficient to insure that $\lim_{n \to \infty} s_n(x) = f(x)$ on $[-\pi, \pi]$. There exists a function which is continuous on $[-\pi, \pi]$ but $\{s_n(x)\}$ diverges on a dense subset of $[-\pi, \pi]$. Worse yet, there exists a Lebesgue integrable function whose Fourier series diverges everywhere in $[-\pi, \pi]$. However, if a function is Lebesgue integrable on $[-\pi, \pi]$ its Fourier series must be summable to the function by the C_1 matrix almost everywhere in $[-\pi, \pi]$.

Further information concerning Fourier series can be found in [3], [6], [10], and [15].

A SELECTED BIBLIOGRAPHY

1. Agnew, R. P. "Properties of Generalized Definitions of Limit." Bulletin of the American Mathematical Society, 45 (1939), 689-730.

2. Apostol, Tom M. Mathematical Analysis. Reading, Mass.: Addison Wesley, 1957.

3. Bary, N. K. A Treatise on Trigonometric Series, Vol. I. New York: Macmillan, 1964.

4. Cooke, R. G. Infinite Matrices and Sequence Spaces. New York: Dover Publications, Inc., 1955.

5. Dienes, P. The Taylor Series. New York: Dover Publications, Inc., 1957.

6. Edwards, R. E. Fourier Series, A Modern Introduction, Vol. I. New York: Holt, Rinehart and Winston, Inc., 1967.

7. Goldberg, R. R. Methods of Real Analysis. New York: Blaisdell, 1964.

8. Hardy, G. H. Divergent Series. London: Oxford University Press, 1949.

9. Knopp, Konrad. Theory and Application of Infinite Series. London: Blackie and Son, 1928.

10. Widder, David V. Advanced Calculus. Englewood Cliffs, New Jersey: Prentice Hall, 1961.

11. Wilansky, A. Functional Analysis. New York: Blaisdell, 1964.

12. Wilansky, A. 'Distinguished Subsets and Summability Invariants." Journal Analyse Mathematique, 12 (1964), 327-350.

13. Wlodarski, L. "Sur les Methodes Continues de Limitation (I)," Studia Mathematica, XIV (1955), 161-187.

14. Zeller, K. Theorie der Limitierungsverfahren. Berlin: Springer-Verlag, 1958.

15. Zygmund, A. Trigonometric Series, Vol. I. Cambridge: University Press, 1959.

TABLE I

SELECTED T-MATRICES

$A = (a_{nk})$

Name	a_{nk}	Reference
Arithmetic means	$1/n,\ k < n$ $0,\ k > n$	[4, p. 68]
Cesaro matrix of order one, C_1	$1/(n+1),\ k \le n$ $0,\ k > n$	[4, p. 69]
Cesaro matrix of order $r > 0$, C_r	$\frac{r\Gamma(n+1)\Gamma(r+n-k)}{\Gamma(n-k+1)\Gamma(r+n+1)},\ k \le n$ $0,\ k > n$	[4, p. 69]
Abel	$\frac{n^k}{(n+1)^{k+1}}$	[4, p. 73]
Borel	$\frac{e^{-n}n^k}{k!}$	[4, p. 70]
Borel triangle, B_r, $r > 1$	$\frac{e^{-n/r}(n/r)^k}{k!},\ k \le n$ $0,\ k > n$	[4, p. 200]
Euler-Knopp, E_r, $0 < r < 1$	$\frac{\binom{n}{k} r^k(1-r)^{n-k}}{k!},\ k \le n$ $0,\ k > n$	[4, p. 200]
Nörlund	$p_{n-k}/P_n,\ k \le n\ (p_i > 0,\ i \in I^+)$ $0,\ k > n$ $[P_n = \sum_{i=1}^{n} p_i,\ \lim (p_n/P_n) = 0]$	[4, p. 73]
"Almost none"	$1/2,\ k = n$ or $k = n + 1$ $0,\ k \ne n$ and $k \ne n + 1$	[4, p. 220]
Raff	$1,\ n = k = 1$ or $n = k + 1$ 0, otherwise	[4, p. 178]

TABLE II

SELECTED T-SEQUENCES OF MITTAG-LEFFLER TYPE

Name	h(k)	Reference
Borel	$1/k!$	[4, p. 182]
Lindelöf	$[\log(k+t)]^{-k}$, $t > 1$	[4, p. 182]
Mittag-Leffler	$[\Gamma(1+\alpha k)]^{-1}$, $0 < \alpha < 2$	[4, p. 182]
Malmquist	$\left(\Gamma\left[1 + \frac{k}{(\log k)^{\alpha}}\right]\right)^{-1}$, $0 < \alpha < 1$ $(k \geq 2)$	[4, p. 182]

TABLE III

SELECTED T-SEQUENCES

Name	$f_k(x)$	Reference
Abel	$\frac{x^k}{(x+1)^{k+1}}$	[4, p. 218]
Bessel of order r	$2\,J^2_{k+r}(x)$	[4, p. 71]
Riesz of order r	$\left(1 - \frac{k}{x}\right)^r - \left(1 - \frac{k+1}{x}\right)^r$, $k+1 < x$ 0, $k+1 \geq x$.	[4, p. 72]
LeRoy	$\frac{\frac{kx}{x+1} + 1}{(k+1)} - \frac{\frac{(k+1)x}{x+1} + 1}{(k+2)}$	[4, p. 92]

VITA

THOMAS EMMETT IKARD

Candidate for the Degree of

Doctor of Education

Thesis: SUMMABILITY METHODS, SEQUENCE SPACES AND APPLICATIONS

Major Field: Higher Education

Biographical:

Personal Data: Born in Clayton, New Mexico, August 4, 1935, the son of A. Erma and Robert Burton Ikard.

Education: Graduated from Felt High School, Felt, Oklahoma, in 1953; Panhandle State College, Goodwell, Oklahoma, 1953-55, 1956-57, 1960-61; Southern Methodist University, Dallas, Texas, 1955-56; Oklahoma State University, 1960-63, 1964-69. Received the Bachelor of Arts degree in July 1961, Panhandle State College; received the Master of Science degree in August 1963, Oklahoma State University; completed the requirements for the Doctor of Education degree in May 1970, Oklahoma State University.

Professional Experience: Joined the mathematics department at Northwestern State College, Alva, Oklahoma, in 1963. Promoted to the rank of assistant professor in 1967.

Professional Organizations: Pi Mu Epsilon, Mathematical Association of America, American Association of University Professors, American Association for Higher Education, National Council of Teachers of Mathematics, Oklahoma Council of Teachers of Mathematics, Oklahoma Education Association