DEVELOPMENT AND ANALYSIS OF AN
ADVERTISING MODEL CONCEPT

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ADVERTISING MODEL CONCEPT

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PREFACE

The problem addressed in this dissertation is that of determining the optimal advertising expenditure in a competitive market. Specifically, a model concept is developed in which the assumption is made that the only competitive item of the marketing mix is effective advertising expenditure. As such, price, distribution, packaging, etc., are assumed to be essentially equal over all brands.

The model concept developed provides for a variable demand as a function of both time and total industry advertising, retention or habitual buying, and advertising carry-over. Two mathematical models are developed in this study. Model I provides for consideration of carry-over from advertising in previous periods, retention buying in future periods which may be attributed to present advertising, and variable demand as a function of time and total industry advertising. Model II is an extension of Model I in that the future carry-over effect of present advertising is considered.

Both models are analyzed in this study. A computer oriented "identical-competitor" equilibrium analysis is utilized to determine the influences of variable demand, retention buying, and advertising carry-over on optimal advertising. Where the equilibrium analysis fails to answer certain questions, a mathematical analysis is implemented. The results of these analyses are supported in a rather extensive example of profit maximization using each model.

The net result of this dissertation is the contribution of a new concept in mathematical advertising models. Also, significant contributions are made in answering the questions as to whether higher retention buying and carry-over motivate higher or lower optimal spending levels. Numerous lesser considerations and implications are provided for persons charged with determining advertising allocations.

My graduate work was made possible by the financial support of the National Science Foundation for one year and the National Aeronautics and Space Administration for two years. Both grants were through the Graduate Traineeship Program. Additional support was received from the Department of Industrial Engineering and Management. I am grateful to these organizations for their assistance.

I would like to express sincere appreciation to the Chairman of my Doctoral Committee, Professor James E. Shamblin, whose timely advice and encouragement has always been welcome. His guidance during this research has been very helpful and is reflected in many places throughout this dissertation. Dr. Shamblin has been a truly outstanding teacher and advisor. I would like to thank Professor Wilson J. Bentley for his inspiration and advice during my graduate study. Dr. Bentley provided me with the opportunity to engage in one year each of research, teaching, and applied studies. It has been an invaluable experience to associate with such an outstanding individual as Dr. Bentley. I would also like to express my gratitude to Professor G. T. Stevens, Jr., and Associate Dean V. S. Haneman, Jr. Dr. Stevens and Dr. Haneman have contributed advice, counsel, and encouragement throughout my graduate study. It has been a pleasure and privilege to have worked under such an outstanding Doctoral Committee.

I would like to thank Dr. Donald L. Stout who has had a great influence during my tenure in college. Dr. Stout's advice and example have been instrumental in a number of my activities and accomplishments.

Special credit is due Mrs. Margaret Estes for her excellent typing and advice in the final preparation of this study. Mr. Eldon Hardy also deserves recognition for his outstanding work in drawing the numerous figures and graphs herein.

I would like to thank my parents, Mr. and Mrs. R. T. Case, for the many sacrifices they have made and the inspiration they have given.

I would also like to express appreciation to my wife, Lynn, and daughter, Kristin. I hope my effort in graduate school has been worthy of my family's effort.

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CHAPTER I

INTRODUCTION

The objective of this research is to develop and analyze an improved mathematical tool to aid in the determination of an optimal advertising expenditure in a competitive market. The model concept developed by this research results in a significant advance in the published works in this area.

The model concept developed provides for consideration of variable demand as a function of both time and total industry advertising, advertising carry-over, and retention or habitual buying. Two profit oriented advertising models are developed. Model I considers retention sales in future periods which may be attributed to present period advertising, carry-over from previous advertising, and variable demand as a function of time and total industry advertising. Model II is an extension of Model I in that the future effect of carry-over from present period advertising is considered.

The formulation techniques developed are applicable in the analysis of areas other than advertising. The concept may be utilized where retention and carry-over effects are evident over time following initial or continuing capital expenditures. For example, evaluating research, the effects of quality change, etc., may be possible using a variation of this model concept.

In general, a brand's market sales (dollar sales during a given time interval) and market share (or brand share, a brand's fraction of total market sales during a given time interval) are functions of advertising expenditure, price, distribution, packaging, etc. These controllable variables are often referred to as elements of the marketing mix. A company usually tries to accomplish some objective -- maximization of profit, maintenance of constant brand share, etc. -- through manipulation of the elements of the marketing mix. This dissertation will address the case in which advertising is the primary force in attracting customers to buy the product class as well as the factor responsible for determining brand share within the industry. An advertising model will then be developed under the following assumption.

> Rivalry in the market is limited to promotional competition. It is assumed that all the other elements in the marketing mix - product quality, channels of distribution, price, etc. - are identical for all competitors.[1]

The objective of such an advertising model concept will be to maximize profit given estimates of competitor spending and parameters which describe each competitor as well as consumer response.

Management has had to rely upon judgment and experience in evaluating advertising budgets. Often decisions are reached by reference to rules of thumb relating advertising to sales projections and share of market statistics. For example, a well known, relatively safe rule of thumb is to budget advertising as a fixed percentage of projected sales equal to the industry average.[2] Rules of thumb, however, leave much room for improvement in the form of advertising models to be used as tools for management decision making.

In making decisions concerning an advertising budget there are several prominent characteristics which should be considered.

Retention Buying

Retention (or habitual) buying is characterized by various degrees of repeat buying by consumers from one purchase period to the next. For example, some frequently purchased grocery items tend to have fairly high habitual brand choice while infrequently purchased items with a low level of brand identification do not stimulate habitual buying.[3]

Advertising Carry-Over

Advertising carry-over is primarily a function of the media used in advertising. Depending upon the medium used, a particular advertisement may be more or less likely to be seen at a future date. Evidence indicates that this carry-over decays with the passage of time.

Variable Demand as a Function of Time

Many consumer products have seasonal or periodic demands. Alternatively, the projected demand for a product may be strictly increasing, decreasing, or constant. Another common possibility is an industry which experiences growth as well as seasonal demand.

Variable Demand as a Function of Industry Advertising

There are generally two possible (and probable) results from advertising activity which often occur concomitantly. The proporation of the market shared by competing firms could be changed or the total demand itself could be expanded. In general, for an established market, the total demand will increase at a decreasing rate as total industry promotion is increased.

The characteristics briefly discussed above all have main effects as well as interaction effects. In order to maximize profit or accomplish most other objectives it is difficult to find sufficient rules of thumb to allow for such influences. Therefore, this study concentrates on the development and analysis of a flexible and usable advertising model concept which logically incorporates the above characteristics.

The sole objective addressed in this dissertation is the maximization of profit when in competition with other manufacturers of the same product class. It is assumed that advertising is the only controllable variable of interest. An implicit assumption is made that each firm manufactures only the product under consideration; or, if diversified, that each product within a company is separate in terms of all operations. In the latter case, no "across-product" benefit or harm is assumed from consumer response to other company products. An advertising model concept, utilizing the above objective and assumptions, is developed. Included are the aforementioned characteristics which are known to influence optimal advertising. Two models are successfully completed and demonstrated. An analysis then determines the effects of the four characteristics on optimal advertising.

Literature Review

At this point the literature will be reviewed in order to acquaint the reader with the major advances in mathematical marketing models as related to the research in this dissertation. A complete review of the literature would not, however, be appropriate due to the volume of research available.

Within the past fifteen years there has been a considerable awakening in the marketing area to the insights available from mathematical models. Not only does a mathematical model play an explicative role and facilitate objective communications, but it is usually much less costly to manipulate a mathematical model to ascertain various input effects rather than to manipulate the actual environment.

Massy and Webster (1964) feel that studies in marketing utilizing applications of the scientific method can be categorized as either behavioral or optimization oriented in nature. The behavioral model

> ... attempts to summarize and hopefully quantify the behavior patterns of certain groups participating in the marketing system, in order to improve understanding and provide better forecasts of future behavior. Optimization models play the opposite role; they provide the value judgments that a manager needs in order to make decisions.[4]

In order to develop a valid optimization model, though, it is necessary to develop it in terms of human behavior.

Behavioral Research

In an excellent review of buyer behavior Sheth (1967) lists 371 references. His review does not encompass pure analytical optimization models. Sheth classifies the existing research according to the categories in Table I. This dissertation draws primarily upon the Operations Research models which Sheth classifies as adspective, omnispective, relational.[5] "An adspective concept is one definable in terms of observed entities, events, or relations."[6] "An omnispective concept is one whose observational content is conceptualized as public or overt."[7] "The relational concept is one in which the defining operations also introduce a relation or conjoint function involving two or more of the things or events."[8]

TABLE I

A CLASSIFICATORY SCHEME OF EXISTING RESEARCH ON BUYER BEHAVIOR

Adspective			
Omnispective		Propriospective	
Relational	Classificatory	Relational	Classificatory
a. Operations Research b. Experimentation c. Simulation	d. Market Segmentation e. Class Theories f. Reference Group Theories	g. Attitude and Preferences	h. Consumer Anticipations or Expectations

Ultraspective

Hypothetical		Fictional
Specified	Unspecified	
i. Perception j. Learning Theory	k. Cognitive Dissonance Theory l. Risk Taking m. Lewin's Field Theory	n. Motivation Research & Psychoanalytic Approaches

Source (Classificatory Scheme): Jagdish N. Sheth, "A Review of Buyer Behavior," Management Science, Vol. 13 (1967), p. B-721.

An Operations Research brand switching approach to buyer behavior can be broken down into two rather broad classifications: probabilistic and functional. Probabilistic brand switching models seek to predict a consumer's next purchase using first and higher order Markov chains (Kuehn, 1958) (Herniter and Magee, 1961) (Harary and Lipstein, 1962), semi-Markov chains (Howard, 1963), learning theory (Kuehn, 1962), nonstationary Bernoulli models (Howard, 1964), first order models with heterogeneity (Morrison, 1965), patterns of brand purchases after brand loyalty (Lawrence, 1969), etc. Such probabilistic models do not explicitly reflect the way in which merchandising factors influence the parameters of the models. Functional models do relate controllable merchandising variables as elements within the transition matrix of the switching model. It appears that probabilistic models are used primarily in an attempt to describe or predict individual consumer behavior on a purchase to purchase basis while functional models are used to describe aggregate consumer behavior on a period by period basis. The particular problem considered in this dissertation is the relationship of profit to advertising expenditure. As such, only the functional brand switching models will be of concern.

Probably the most often used functional models to describe consumer flow as a function of advertising have been those of Mills (1961) and Kuehn (1958, 1961). Mills' model assumes "a market, fixed in total unit volume, is shared among its brands in proportion to the brand promotional outlays."[9] Kuehn (1961) showed empirically in 1958 "that purchases of brands by a household prior to the most recent buying occasion have substantial effects upon its choice of a brand when the product is next purchased."[10] Thus it would appear that a first order Markov process would be inadequate to describe consumer buying behavior. However, Kuehn (1961) shows that a linear learning model which is dependent upon past purchasing history is mathematically equivalent to a first order Markov process. He then proceeds to develop a model that describes aggregate brand shifting from time period to time period. Kuehn's model allows for a retention buying or brand loyalty factor and the "non-loyal" consumers are then distributed among brands on the basis of relative promotional outlays much like Mills' model. A variation of Kuehn's approach to describe aggregate period to period consumer behavior will be used in the development of this dissertation. Others who have used an approach similar to that of Mills or Kuehn to describe aggregate consumer behavior are Friedman (1961), Herniter and Howard (1964), Reisman (1964), Shakun (1965), and Krishnan and Gupta (1967). Telser (1962) had excellent success in calculating the parameters of a different but similar model of the effect of advertising in the cigarette industry.

As can be seen, there has been much research in brand switching alone. Most of this research has been an attempt to describe a random consumer. Several of the methods mentioned earlier seem to accomplish this objective.[11] Yet there has been much difficulty in presenting such models so as to describe a heterogeneous body of consumers. On the other hand, there are functional models such as the one to be used in this research which describe aggregate consumer response on a period to period (monthly, etc.) basis as a function of advertising or other elements of the marketing mix. However, these do not effectively describe a single consumer on a purchase to purchase basis. Each type of model has its place in marketing research. The period to period functional model will be used in this study.

Characteristics Related to Advertising

A classic and often consulted piece of work is the empirical study by Vidale and Wolfe (1961) in which they identified three advertising parameters:

1. The sales decay constant.
2. The saturation level.
3. The response constant.[12]

The sales decay constant concept will be used directly and the saturation level and response constant indirectly in this dissertation. It should be noted that the sales decay constant is closely related to the retention buying behavior discussed in the "Statement of the Problem."

Jastram (1955) postulates one of the probable factors making for distributed lags in the impact of advertising in one period over sales in future periods to be "the type of advertising copy and the media used."[13] This is another type of advertising influence on future periods and it is referred to as advertising carry-over in the "Statement of the Problem." Jastram also makes reference to a germination period for a purchase decision. By a germination period he means the time elapsed between a consumer's first consideration of a product and his eventual decision to buy. The longer the germination period, the longer it will be before advertising shows its result in terms of sales.[14] Such a characteristic has not been considered directly in this research although it is easily incorporated and will again be discussed in Chapter III.

Shakun (1965) and Gupta and Krishnan (1967) used a differential equation approach to exhibit the decreasing rate of increasing industrial sales with increased advertising. Zentler and Ryde (1956) discuss a similar concept utilizing increasing and then diminishing returns to represent the individual's response to increased promotional activity. The Gompertz equation characterized by a small range of increasing returns followed by diminishing returns will be used in this research to describe the aggregate response to industry advertising.

Optimization Models

Now that the literature on behavior models and major characteristics related to this research has been reviewed it would be appropriate to consider optimization model developments. As Marschner says,

> The differences between practice and theory are so great as to suggest that there is opportunity for major reduction of confusion and increase in profits with better administration of advertising appropriations.[15]

Once again, probably the most frequently referenced mathematical advertising model is that of Kuehn (1961). Kuehn's model, although primarily concerned with advertising, incorporates provision for pricing and distribution effects. Although a very thorough advertising model, it does not explicitly include advertising carry-over or a variable market as a function of total industry advertising.

Mills (1961) developed an optimization model of promotional competition for n brands in an expanding market. His development also includes an algorithm for determination of an advertising equilibrium point for competitors with different logistics margins. Reisman (1964) uses a Lagrangian Multiplier approach to solve for equilibrium. Neither Mills nor Reisman consider the effects of habitual buying, advertising carry-over, or a variable market as a function of industry advertising.

Kotler (1965) develops a model for a new product with seasonal demand. His study involves examination of nine different merchandising strategies. Although Kotler addresses a problem different from that involved in this research, his work is a contribution to marketing models.

An intensive review was made of other articles in Journal of Industrial Engineering, Operations Research, Journal of Marketing Research, Journal of Marketing, Journal of Advertising Research, Management Science, and many other periodicals and books. Many of the findings were contributory to the researcher's understanding of marketing models but they either did not address any aspect of this dissertation or they were much too limited in scope, as compared to the literature reviewed above, to be mentioned here.

Summary of Research Approach

The thorough review of the literature pertaining to advertising models resulted in an observation that there has not been an advertising model developed which relates all of the major characteristics presented in the "Statement of the Problem." This dissertation research will involve a great deal of abstracting from previous analytical and empirical studies in order to incorporate such characteristics into a flexible profit-oriented advertising model. Each concept will be documented and discussed as needed in the basic development of the model.

Two mathematical models, one an extension of the other, will be developed. Computer search will be utilized with both models to determine the equilibrium advertising expenditure for two identical competitors. Computer search will also be used with both models to optimize advertising against a non-identical competitor. The results of the computer studies as well as mathematical analyses will be used to help determine the effects on optimal advertising of variable demand as a function of both time and advertising, retention buying, and advertising carry-over. Such effects on magnitude, amplitude, and phase of advertising will be readily apparent from a graphic display.

Assumptions in the application of the model concept and estimation of needed parameters are then discussed. Also, considerations of meaningful sensitivity analyses are presented. In conclusion, the results, findings and recommendations for future work are summarized.

Contributions of the Research

This research contributes a model which could conceivably be used, as presented, by a firm in an established industry which considers its only merchandising variable of interest to be advertising. Further, this research should be of significant value to many firms in a competitive industry where advertising is a controllable variable. The reason for such significance is the insight given the user in terms of the variable demand as a function of time and advertising, retention buying, and advertising carry-over characteristics which are often not considered or understood. Also, no other model considers all of these characteristics simultaneously.

Another contribution is the flexibility of the model in that known functions describing delayed initial effect of advertising, sales response to total industry advertising, and consumer behavior for a particular industry can be easily substituted for the corresponding functions used herein.

One of the most enlightening and unique aspects of this research is the realization and demonstration that reliable estimates of competitor's future expenditures are significant in the budgeting of advertising and the corresponding returns. No related research, to this author's knowledge, deals with such a fact. Also, considerable progress is made in answering the question as to whether higher carry-over and/or retention factors motivate higher or lower spending.

Another significant aspect of using such a model is the possibility of the user examining cost trade-offs. For example, changing product characteristics or advertising media may improve habitual buying or carry-over effects, respectively, and thus increase profit.

Such a model as described in this dissertation will certainly not make budgeting of advertising entirely mechanical. Management judgment is still of prime importance. However, such a model as this should improve management decision making either directly or indirectly.

FOOTNOTES

[1]Frank M. Bass et al., "Editorial Commentary to 'A Study in Promotional Competition'," _Mathematical Models and Methods in Marketing_ (Homewood, Illinois, 1961), p. 245.

[2]Alfred A. Kuehn, "A Model for Budgeting Advertising," _Mathematical Models and Methods in Marketing_, ed. Frank M. Bass et al. (Homewood, Illinois, 1961), p. 315.

[3]Alfred A. Kuehn, "How Advertising Performance Depends on Other Marketing Factors," _Managerial Marketing_, ed. Eugene J. Kelley and William Lazer (3rd ed., Homewood, Illinois, 1967), p. 565.

[4]William F. Massy and Frederick E. Webster, Jr., "Model Building in Marketing Research," _Journal of Marketing Research_, May, 1964, p. 10.

[5]Jagdish N. Sheth, "A Review of Buyer Behavior," _Management Science_, Vol. 13, No. 12 (1967), p. B-721.

[6]Ibid., p. B-720.

[7]Ibid., p. B-721.

[8]Ibid.

[9]Harlan D. Mills, "A Study in Promotional Competition," _Mathematical Models and Methods in Marketing_, ed. Frank M. Bass et al. (Homewood, Illinois, 1961), p. 281.

[10]Kuehn, "A Model for Budgeting Advertising," _Mathematical Models and Methods in Marketing_, p. 319.

[11]Donald G. Morrison, "Testing Brand-Switching Models," _Journal of Marketing Research_, III (1966), p. 408.

[12]M. L. Vidale and H. B. Wolfe, "An Operations-Research Study of Sales Response to Advertising," _Mathematical Models and Methods in Marketing_, ed. Frank M. Bass et al. (Homewood, Illinois, 1961), p. 365.

[13]Roy W. Jastram, "A Treatment of Distributed Lags in the Theory of Advertising Expenditure," _Journal of Marketing_, July, 1955, p. 36.

[14] Ibid., p. 37.

[15] Donald C. Marschner, "Theory Versus Practice in Allocating Advertising Money," The Journal of Business, Vol. 40 (1967), p. 286.

CHAPTER II

DEVELOPMENT OF BRAND-SWITCHING

In order to develop a profit maximizing advertising model, there must be a functional relationship between dollar sales and advertising. In this chapter a first order transition matrix will be developed to describe aggregate consumer flow in terms of brand share as a function of previous purchasing and advertising expenditure.

Brand-Shifting as a First Order Markov Process

The review of the literature in Chapter I provides evidence that much work has been done in the area of brand-switching on a purchase to purchase basis. Advertising and sales data are available on a period to period (monthly, quarterly, etc.) basis. It is a well-known fact that the buying habits of consumers, one to another, are not identical. Specifically, each consumer does not purchase an equal amount of a given class of products in a given time interval.[1,2]

It would appear that the use of a Markov process to describe brand-shifting would require the assumption that all purchasers of a given class of products buy one and only one unit per time period. Indeed, this is the case if the Markov process is to be used to describe consumer behavior on a purchase to purchase basis. Such a limiting and restrictive assumption is no longer applicable if the Markov process is used to describe an aggregate form of a period to period learning model. Further, such a model takes account of the fact that purchasers frequently buy two or more brands of a product based upon the preferences of individual members of their families. Whereas purchase to purchase analyses tend to overestimate a consumer's propensity to switch brands, the period to period analysis of the learning model focuses on changes in the mix of purchases within distinct time periods.

Kuehn (1961) presents the explanation and the development whereby he shows that the first order transition matrix presented in Figure 1 is an applicable model to describe "aggregate brand-shifting from time period to time period by consumers . . . ."[3] It consists of two basic parameters, a retention factor q_g for each brand g and a relative advertising attractiveness factor f_{gh} where $\sum_h f_{gh} = 1$. Due to the extensive development presented by Kuehn, this research will only take notice of the model and proceed with its use in further developments.

The interested reader will also want to review the article by Rohloff (1964). He describes an aggregate "gain-loss" brand-switching model as used by Lever Brothers. Gain-loss analysis is similar to the learning model and Markov process described earlier in that brand-switching is

		BRAND-NEXT PERIOD		
		1	2	3
BRAND-LAST PERIOD	1	$q_1+(1-q_1)f_{11}$	$(1-q_1)f_{12}$	$(1-q_1)f_{13}$
	2	$(1-q_2)f_{21}$	$q_2+(1-q_2)f_{22}$	$(1-q_2)f_{23}$
	3	$(1-q_3)f_{31}$	$(1-q_3)f_{32}$	$q_3+(1-q_3)f_{33}$

Figure 1. Transition Fractions From Period to Period

is studied from one time period to the next.[4]

A Descriptive Flow Model

For the remainder of this chapter a constant demand for a given product class will be assumed. Such an assumption will make the initial exposure to brand-switching easier to understand. Brand-switching under variable demand will be considered in Chapter III.

Brand-switching in a competitive market will now be conceptualized as it will be used in the profit model. As discussed previously, only the period to period characteristics are of interest in this model.

Consider the flow diagram of Figure 2. The idea for such a pictoral representation of brand share flow comes from Herniter and Howard (1964) on page 49. The diagram shown is for a two brand market. Such a diagram is easily extrapolated to 3, 4, or n brands. The parameters of the model are defined as follows:

$c_g(t)$ = the share of the market possessed by Brand g at time t. Note that $\sum_g c_g(t) = 1.0$.

q_g = the proportion of Brand g's market share which will be retained next period without advertising influence. This is the "brand-loyalty" or retention factor and it is assumed constant.

$f_{gh}(t)$ = a constant from time $t + \Delta$ (an instant after time t) through time $t + 1$ (one full time

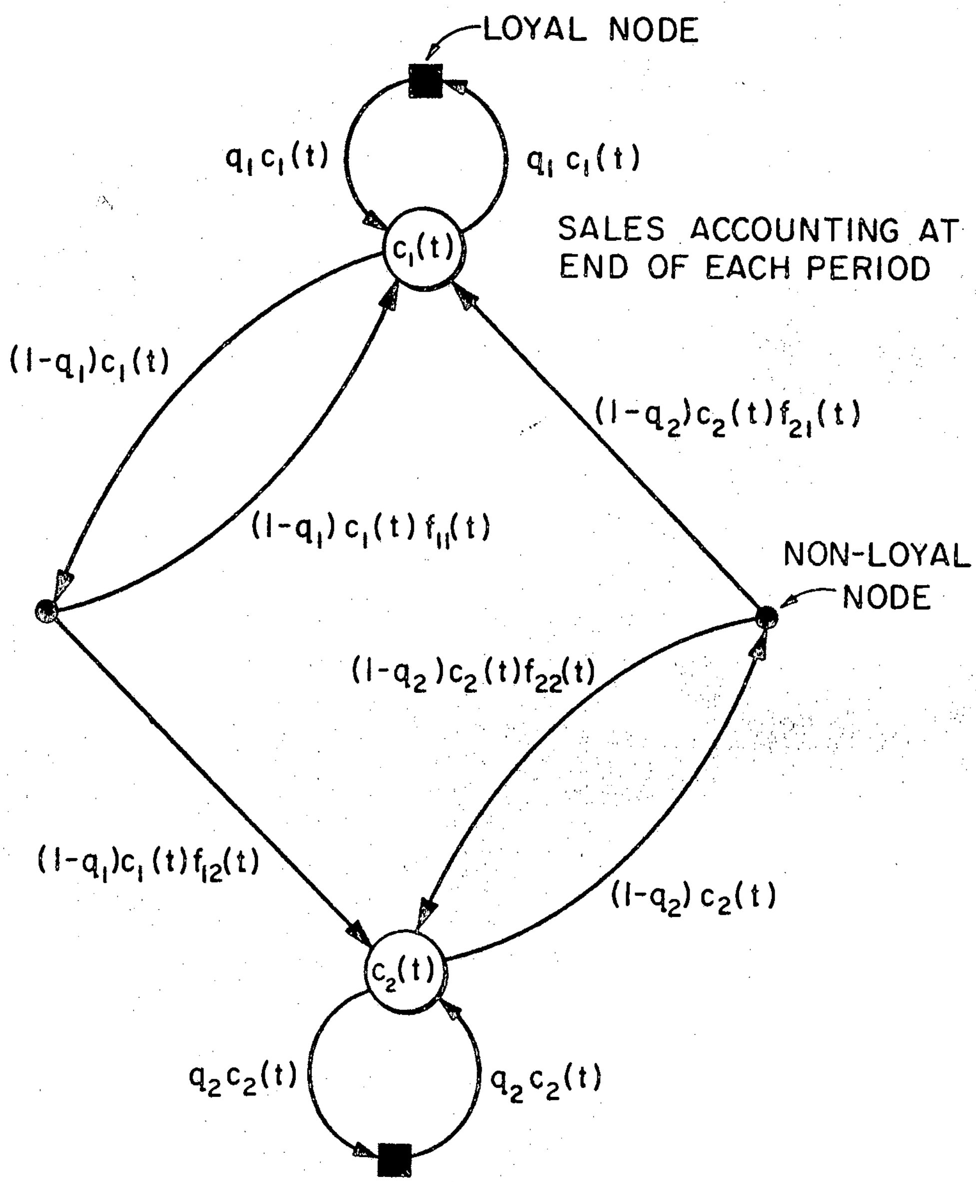

Figure 2. Conceptual Brand-Switching Flow Model

period after time t), its value set at the beginning of that period as a function of effective advertising at $t + \Delta$. As such, it is the fraction of Brand g's "potential brand-shifting fraction," $(1-q_g)c_g(t)$, which will shift to Brand h during the interval $[t + \Delta, t + 1]$ and be so reflected in sales data at $t + 1$. Figure 3 presents a view of successive time intervals of which $[t + \Delta, t + 1]$ is a part.

Conceptually, the loyal and non-loyal nodes of Figure 2 represent the consumer advertising influences and purchase transactions which take place in the period $[t + \Delta, t + 1]$, as influenced by effective advertising at the start of the period, $t + \Delta$. At the end of the period, $t + 1$, the various proportions of the market shown in Figure 2 become inputs to the periodic sales accounting procedure which in turn determines $c_g(t + 1)$. At time $t + 1 + \Delta$, a new cycle begins.

The brand share of firm 1, for example, at time $t + 1$ will be as follows:

$$c_1(t+1) = q_1c_1(t) + (1-q_1)c_1(t)f_{11}(t) + (1-q_2)c_2(t)f_{21}(t). \tag{2.1}$$

This is, of course, merely the first element of the product of last period's brand share vector times the transition matrix of Figure 1.

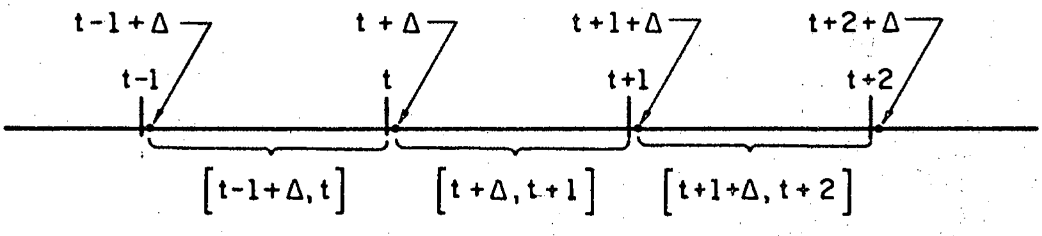

Figure 3. Successive Time Intervals: Terminology

$$[c_1(t+1), c_2(t+1)] = [c_1(t), c_2(t)] \begin{bmatrix} q_1+(1-q_1)f_{11}(t) & (1-q_1)f_{12}(t) \\ (1-q_2)f_{21}(t) & q_2+(1-q_2)f_{22}(t) \end{bmatrix}. \tag{2.2}$$

Of the n + 1 components (n competing brands) making up the right side of equation (2.1), the latter n are functions of effective advertising at $t + \Delta$. Their sum is equal to $c_1(t + 1) - q_1c_1(t)$. Thus, the fraction $c_1(t+1) - q_1c_1(t)$ represents that portion of Brand 1's share of market sales during $[t + \Delta, t + 1]$ which can be attributed to effective advertising at $t + \Delta$. Due to the retention factor q_1, the proportion $[c_1(t + 1) - q_1c_1(t)]q_1$ of market sales during $[t + 1 + \Delta, t + 2]$ can also be attributed to effective advertising at time $t + \Delta$. If

$S(t)$ = the demand for the product class (total demand for all brands) in terms of dollar sales during $[t + \Delta, t + 1]$

and if it is assumed that $S(t) = S(t + 1) = S(t + 2) = \ldots = S(\cdot)$, the undiscounted present and future dollar sales which can be attributed to effective advertising at $t + \Delta$ are:

$$\begin{aligned} \sum_{t''=1}^{\infty} DS_{1,t+\Delta}(t+t'') &= [c_1(t+1)-q_1c_1(t)][S(\cdot)+q_1S(\cdot)+q_1^2S(\cdot)+\ldots] \\ &= [c_1(t+1)-q_1c_1(t)][S(\cdot)][1+q_1+q_1^2+\ldots] \\ &= [c_1(t+1)-q_1c_1(t)][S(\cdot)]\left[\frac{1}{1-q_1}\right] \\ &= [(1-q_1)c_1(t)f_{11}(t)+(1-q_2)c_2(t)f_{21}(t)][S(\cdot)]\left[\frac{1}{1-q_1}\right] \\ &= \left[\sum_{g=1}^{2}(1-q_g)c_g(t)f_{g1}(t)\right][S(\cdot)]\left[\frac{1}{1-q_1}\right]. \end{aligned} \tag{2.3}$$

In Chapter III, equation (2.3) will be modified in order to treat the retention factor in conjunction with a variable demand as a function of time. The updated or generalized version of (2.3) will then be used in the profit models of Chapters IV and V.

FOOTNOTES

[1] Donald G. Morrison, "Interpurchase Time and Brand Loyalty," *Journal of Marketing Research*, III (1966), p. 289.

[2] Alfred A. Kuehn, "A Model for Budgeting Advertising," *Mathematical Models and Methods in Marketing*, ed. Frank M. Bass et al. (Homewood, Illinois, 1961), p. 322.

[3] Ibid., p. 319.

[4] Albert C. Rohloff, "New Ways to Analyze Brand-to-Brand Competition," *Toward Scientific Marketing*, ed. Stephen A. Greyser (Chicago, 1964), p. 255.

CHAPTER III

MATHEMATICAL TREATMENT OF CHARACTERISTICS AND FUNCTIONS TO BE USED IN THE ADVERTISING MODEL

In order to account for the effects of variable demand as a function of both time and total industry advertising, retention buying, and advertising carry-over, these characteristics must be expressed mathematically. Further, this mathematical treatment must be such that each characteristic accurately reflects its influence either solely (if simplifying assumptions negate the other characteristics) or in interaction with the other characteristics.

Variable Demand as a Function of Time

In Chapter II the assumption was made that demand for a class of products was constant over time. That assumption, although possible, was made only to facilitate an explanation of the brand-shifting model used. Now, a more general case of variable demand over time will be examined.

Consider the demand curve presented in Figure 4. The curve shown has the following mathematical expression:

$$f(t) = 1{,}000{,}000 + 240{,}000 \sin(\pi t/6) \; . \qquad (3.1)$$

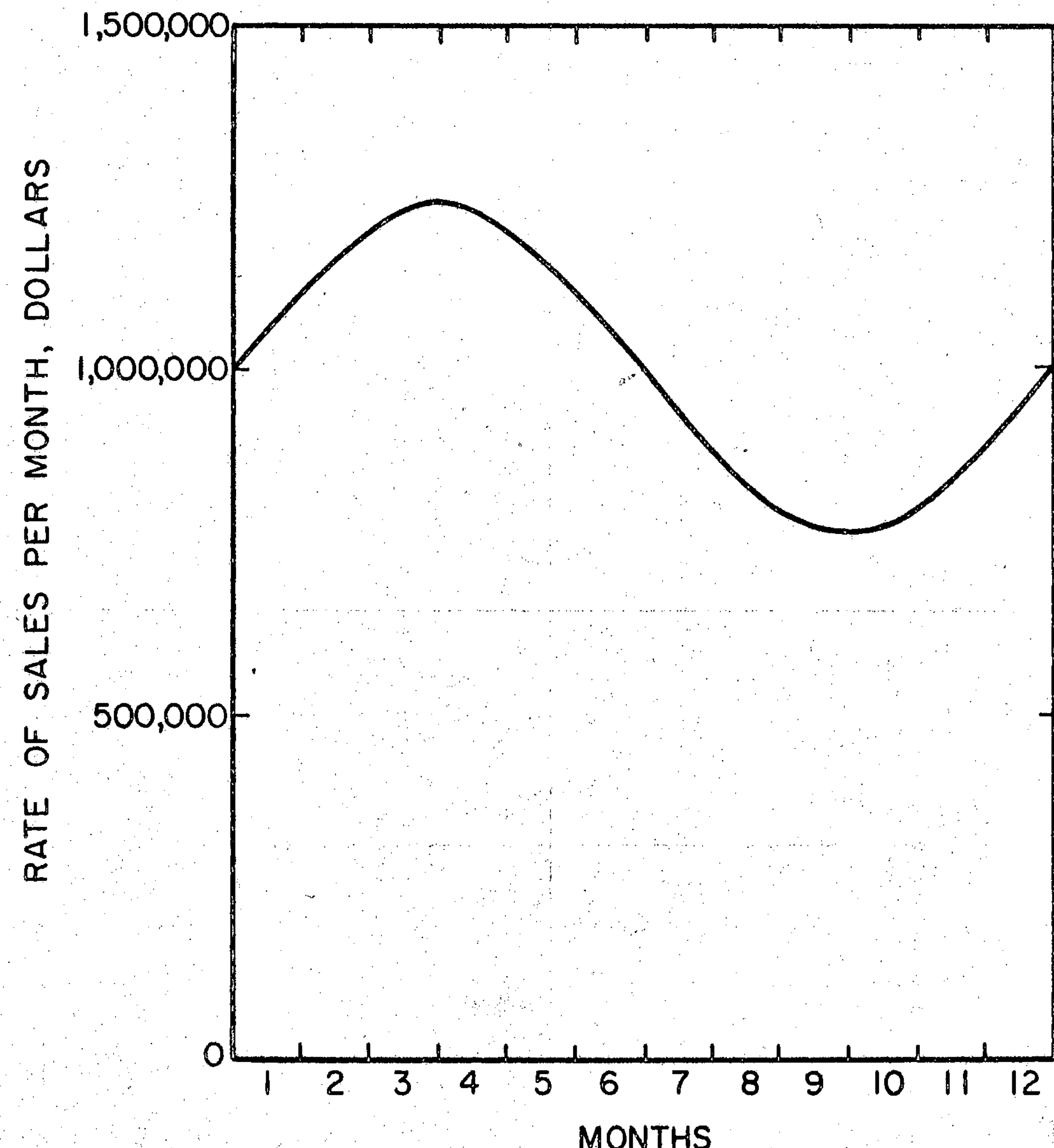

Figure 4. A Projected Periodic Demand Rate

Both the curve and the mathematical expression are continuous.

Due to the period by period nature of sales and advertising statistics and the profit oriented advertising model to be developed, it is desirable to have demand expressed in discrete periodic amounts. Examples throughout this dissertation will utilize the demand curve (later to be called potential demand) of Figure 4, and months will be considered periods. Each month, therefore, it is assumed that sales and advertising statistics of all brands are available and at the beginning of each month an optimal advertising expenditure is determined.

Under the assumptions detailed above, the continuous demand curve and its mathematical expression relate instantaneous rates of dollar sales in units of dollars per month. In order to determine demand in terms of dollar sales during Month 1, S(0), the expression (3.1) must be integrated over Month 1.

$$S(0) = \int_0^1 [1{,}000{,}000 + 240{,}000 \sin(\pi t/6)]dt \quad (3.2)$$

Similarly,

$$S(1) = \int_1^2 [1{,}000{,}000 + 240{,}000 \sin(\pi t/6)]dt \quad (3.3)$$

and generalizing,

$$S(t'-1) = \int_{t'-1}^{t'} [1{,}000{,}000 + 240{,}000 \sin(\pi t/6)]dt \quad (3.4)$$

Of course, projected demands in industry are often expressed directly on a discrete basis. The continuous sine-wave demand to be used throughout this dissertation is merely a construct to help in determining the effects of various characteristics including time variable demand. Virtually any real or hypothetical demand could be used.

Retention Buying

The reader has already been exposed to the retention factor for Brand g, q_g. The motivation for the mathematical use of this factor stems largely from the empirical verification of a Sales Decay Constant by Vidale and Wolfe (1961). "Under relatively constant market conditions, the rate of decrease is, in general, constant: that is, a constant percent of sales is lost each year."[1]

Momentarily assuming constant demand, equation (2.1) represents the brand share of Brand 1, based on sales during [t + Δ, t + 1], to be

$$c_1(t+1) = q_1 c_1(t) + (1-q_1)c_1(t)f_{11}(t) + (1-q_2)c_2(t)f_{21}(t). \tag{3.5}$$

Reconsidering the flow diagram of Figure 2 it can be seen that in the absence of further Brand 1 advertising ($f_{11}(t+1) = f_{21}(t+1) = 0$), Brand 1's market share at time t+2 will be

$$\begin{aligned} c_1(t+2) &= q_1 c_1(t+1) \\ &= q_1^2 c_1(t) + q_1[(1-q_1)c_1(t)f_{11}(t) + (1-q_2)c_2(t)f_{21}(t)]. \end{aligned} \tag{3.6}$$

If Brand 1 continues a "no advertising" policy such that the last effective advertising occurred at time $t + \Delta$, its brand share at any time $T \geq t + 1$ will be

$$\begin{aligned} c_1(t+t'') &= q_1^{(t''-1)} c_1(t+1) \\ &= q_1^{t''} c_1(t) + q_1^{(t''-1)} [(1-q_1)c_1(t)f_{11}(t) \\ &\qquad + (1-q_2)c_2(t)f_{21}(t)] . \end{aligned} \tag{3.7}$$

Expressed verbally, in the absence of future advertising, a firm's brand share diminishes by a constant fraction $1-q_g$ each time period. If, indeed, Brand 1 does continue to advertise, Brand 1's market share at time $t + 2$ will be

$$\begin{aligned} c_1(t+2) &= q_1 c_1(t+1) + (1-q_1)c_1(t+1)f_{11}(t+1) \\ &\qquad + (1-q_2)c_2(t+1)f_{21}(t+1) \\ &= q_1^2 c_1(t) + q_1[(1-q_1)c_1(t)f_{11}(t) + (1-q_2)c_2(t)f_{21}(t)] \\ &\qquad + (1-q_1)c_1(t+1)f_{11}(t+1) \\ &\qquad + (1-q_2)c_2(t+1)f_{21}(t+1) . \end{aligned} \tag{3.8}$$

It can be seen that regardless of future advertising (3.6 or 3.8), that portion of Brand 1's market share attributed to effective advertising at $t + \Delta$, $[(1-q_1)c_1(t)f_{11}(t) + (1-q_2)c_2(t)f_{21}(t)]$, diminishes by a fraction $1-q_1$ each period. That is, a fraction q_1 is retained each period.

Treatment of Time Variable Demand and Retention Buying Simultaneously

In making the transition from a constant demand to a time variable demand a distinction should be made concerning the use of the retention factor.

Method 1. It can be assumed that a fixed percentage, q_g, of Brand g's dollar sales are retained from period to period.

Method 2. It can be assumed that a fixed percentage, q_g, of Brand g's market share is retained from period to period.

Actually, there is no distinction to be made under an assumption of constant demand. While either method could have been used, the discussion in the previous section was written in terms of brand share retention, not dollar sales retention.

Example

Consider the following example. One brand, competing with one other brand, in a product class having a $1,000,000 demand during $[t + \Delta, t + 1]$, garners 30 per cent of that demand with effective advertising at $t + \Delta$. The firm in question has a retention factor of $q_h = .5$. Projected demand during $[t + 1 + \Delta, t + 2]$ is $1,200,000. What is the dollar value of sales due to retention buying during $[t + 1 + \Delta, t + 2]$ which may be attributed to effective advertising at $t + \Delta$?

Method 1. Sales due to effective advertising at $t + \Delta$ are .30 ($1,000,000) = $300,000 during $[t + \Delta, t + 1]$. The retention sales will be .5 ($300,000) = $150,000 during $[t + 1 + \Delta, t + 2]$.

Method 2. Sales due to effective advertising at $t + \Delta$ are .30 ($1,000,000) = $300,000 during $[t + \Delta, t + 1]$. The retained brand share will be .5 (.3) = .15. The retention sales will be .15 ($1,200,000) = $180,000 during $[t + 1 + \Delta, t + 2]$.

Note that the two concepts are different and yield different answers. Which concept should be used?

In their article, Vidale and Wolfe present three graphical sales histories of brands for which advertising has been discontinued. One product is a vivid example of a very seasonal product. It is seasonal in that, much like the sine-wave demand of Figure 4, the trend repeats itself each year. The graphical sales history of the product is operative over a five year span. The authors note that "the monthly sales averaged over a full year, 'decay' at a constant rate."[2] Such a statement, in terms of the model to be developed here, is misleading. It leads one to consider retention in terms of a fixed percentage of dollar sales. Method 1, however, would fail to sense seasonal fluctuations within a period of a year even though the period used in this model is the month. Vidale and Wolfe's presentation of brand sales data for an unpromoted brand shows quite clearly that seasonal fluctuations must be sensed in order to have a model which truly describes the empirical findings. The brief example at the beginning of this section shows how Method 2 "senses" a seasonal increase (or decrease) in demand provided the period length used in the model is small with respect to seasonal cycle length.

As another example of the difference between the two methods reconsider the previous example. Change only the projected demand for period $[t + 1 + \Delta, t + 2]$ to \$100,000.

Method 1. The retention sales predicted will still be \$150,000. Obviously, this prediction is out of line as it exceeds the entire projected demand for both competitors by 50 per cent. Sales due to retention would normally be much smaller than \$100,000.

Method 2. The retention sales will be .15(\$100,000) = \$15,000, a much more reasonable prediction.

As a final comparison of the two methods, let the time period of interest be one year. The projected yearly demand of the sine-wave curve is the sum of each month's demand.

$$\sum_{t=0}^{11} S(t) = \int_{0}^{12} [1{,}000{,}000 + 240{,}000 \sin(\pi t/6)]dt$$

$$= \$12{,}000{,}000 \quad . \tag{3.9}$$

Thus, on a yearly basis demand is constant at \$12,000,000. Under such constant demand, both methods give identical results as yearly sales and brand share decay at the same constant rate. With periods of one year, however, it is not possible to sense seasonality.

In summary of the above arguments, only Method 2 is seen to properly adjust to seasonal fluctuations. Therefore, Method 2, the method which considers retention in terms of brand share, will be used throughout this dissertation. As such, the discussion relating to equations (3.5) through (3.8) remains valid for variable as well as constant demand. This researcher knows of no other model which so senses empirically proven seasonal fluctuation in the use of its retention factor.

As discussed in Chapter II, the share of the total market during $[t + \Delta, t + 1]$ which can be attributed to effective advertising by Brand 1 at time $t + \Delta$ is

$$c_1(t+1) - q_1c_1(t) = (1-q_1)c_1(t)f_{11}(t) + (1-q_2)c_2(t)f_{21}(t) \quad . \qquad (3.10)$$

In terms of dollar sales we have

$$DS_{1,t+\Delta}(t+1) = [(1-q_1)c_1(t)f_{11}(t) + (1-q_2)c_2(t)f_{21}(t)][S(t)] \quad . \qquad (3.11)$$

The dollar sales during $[t + 1 + \Delta, t + 2]$ due to retention buying which may be attributed to advertising at $t + \Delta$ equals

$$DS_{1,t+\Delta}(t+2) = [(1-q_1)c_1(t)f_{11}(t) + (1-q_2)c_2(t)f_{21}(t)][q_1][S(t+1)] . \qquad (3.12)$$

In general,

$$DS_{1,t+\Delta}(t+t'') = [(1-q_1)c_1(t)f_{11}(t) + (1-q_2)c_2(t)f_{21}(t)] \cdot [q_1^{(t''-1)}][S(t+t''-1)] \qquad (3.13)$$

and

$$\sum_{t''=1}^{\infty} DS_{1,t+\Delta}(t+t'') = [(1-q_1)c_1(t)f_{11}(t)+(1-q_2)c_2(t)f_{21}(t)] \cdot [S(t)+q_1S(t+1)+q_1^2S(t+2) + \ldots]$$

$$= \sum_{g=1}^{2} (1-q_g)c_g(t)f_{g1}(t)] \cdot [S(t)+q_1S(t+1)+q_1^2S(t+2) + \ldots] . \qquad (3.14)$$

Note the resemblance between equations (3.14) and (2.3). The only difference is that in equation (2.3) a constant demand is assumed and thus $S(t) = S(t+1) = S(t+2) = \ldots = S(\cdot)$. In general, in equation (3.14), each period's demand differs, precluding the possibility of expressing the retention factor in a limiting form of geometric expansion. Each demand term in equation (3.14) must be calculated and then all terms summed. It is assumed that a reasonable demand curve will not continue to increase over time such that convergence is prohibited.

Obviously, for a product with a high retention factor (q_g = .9, for example) many terms must be summed before the remaining terms may be truncated. Appendix A treats a continuous approximation of such a sum. Included in Appendix A is a treatment of the time value of money. The time value of money will not be included in the remainder of the body of this research.

Advertising Carry-Over

Gundlach (1931) devised a method whereby the effect of advertisements in monthly journals could be measured. He keyed each month's advertising differently in order to determine the particular ad which motivated inquiry. His findings were that several months after the original advertising, inquiries due to that particular ad were still arriving.[3] Such an observed phenomenon will be referred to as advertising carry-over in this research.

In the model to be developed, carry-over will be expressed in terms of a coefficient of effective advertising carry-over for Brand g, b_g. A brand's "composite" advertising at time $t + \Delta$, for example, will be equal to $a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots$. Figure 5 shows graphically the carry-over effect as used herein. The carry-over is portrayed as decreasing by a constant amount, $1-b_g$, each period. Such an assumption is logical although carry-over does not have to be of this form. Any known rate or pattern of carry-over may be used. In fact, referring to the reference about a "germination period" by Jastram (1955) in which he notes that advertising for some products may not show immediate results, it may be desirable to first have an increasing carry-over effect followed by carry-over decay. Such a scheme is illustrated in Figure 6.

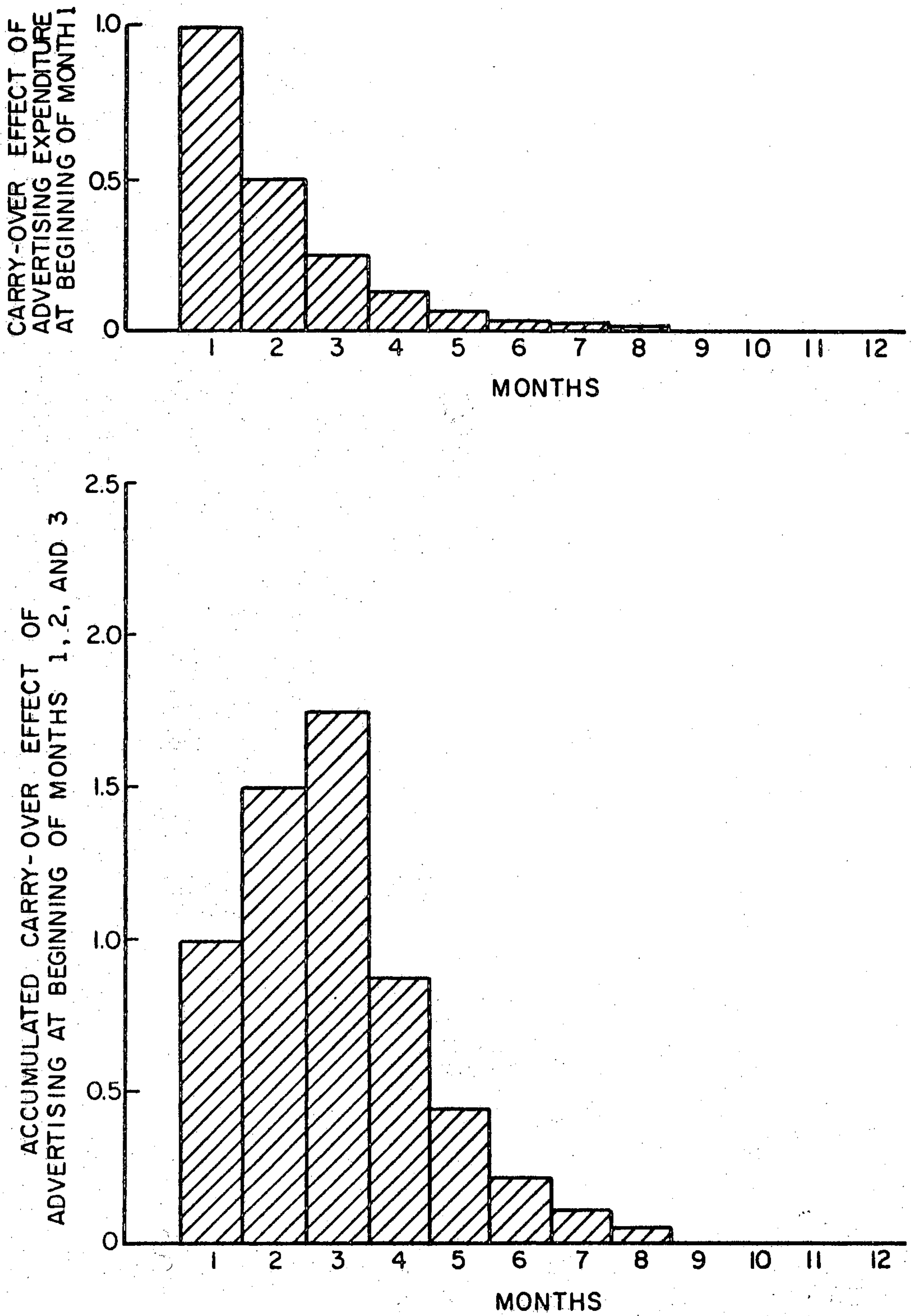

Figure 5. Carry-Over Effect With Decay

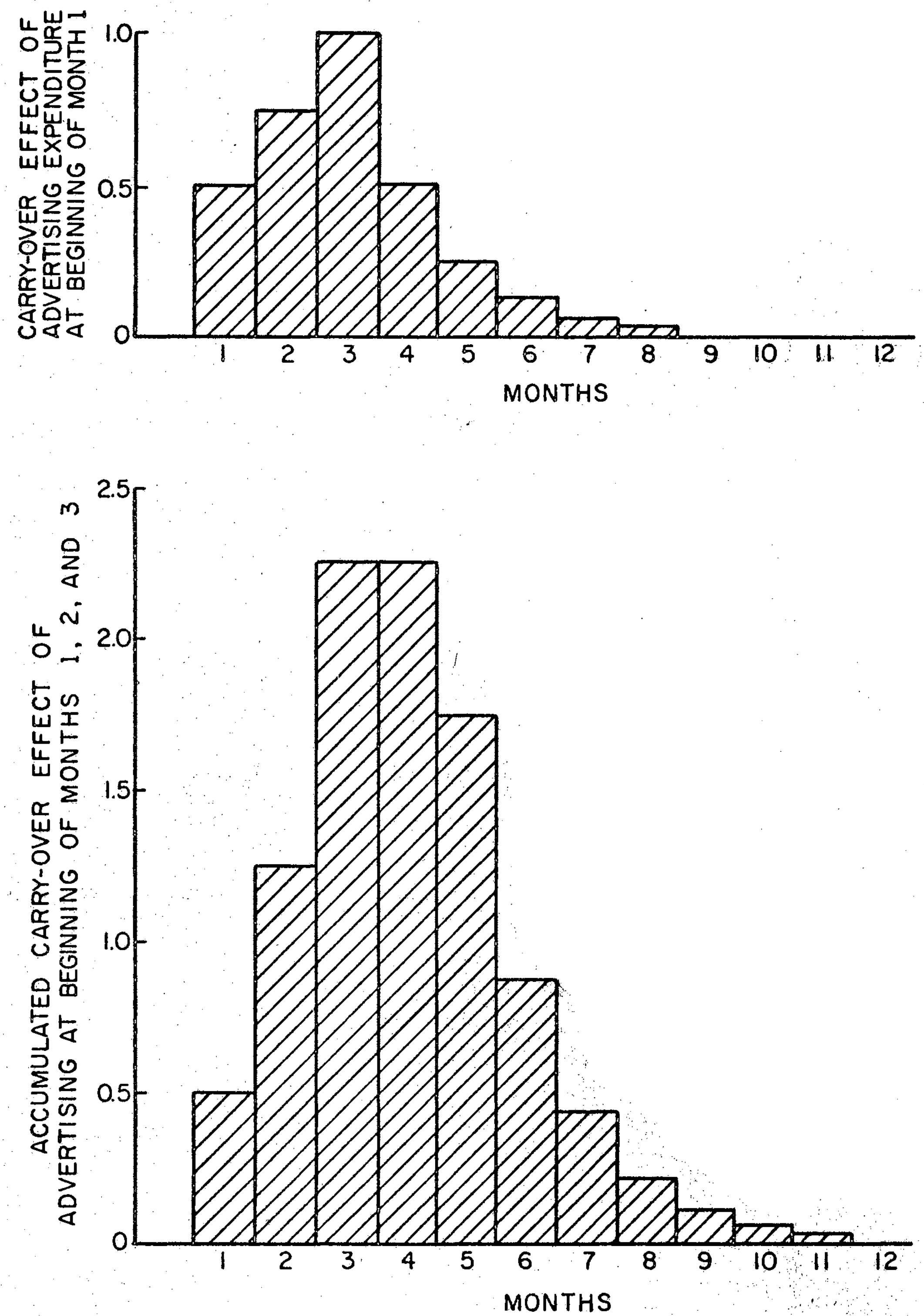

Figure 6. Carry-Over Effect With Initial Germination Period Followed by Decay

The Advertising Attractiveness Function

In Chapter II $f_{gh}(t)$ is described as follows:

$f_{gh}(t)$ = a constant from time $t + \Delta$ up to time $t + 1$, its value set at the beginning of that period by effective advertising at $t + \Delta$. As such, it is the fraction of Brand g's "potential brand-shifting fraction," $(1-q_g)c_g(t)$, which will shift to Brand h during the interval $[t + \Delta,\ t + 1]$ and be so reflected in sales data at $t + 1$.

In order to develop a profit model it is necessary to express $f_{gh}(t)$ as a function of advertising. It is necessary to assume that the advertising expenditure of each brand at time $t + \Delta$, $a_g(t)$, is "effectively" spent. That is, all brands spend advertising money with equal effectiveness. If desired, it would be possible to let

$$a_g(t) = \alpha_g x_g(t) \qquad (3.15)$$

where

α_g = the coefficient of effective spending by Brand g.

$x_g(t)$ = the actual advertising dollars spent at time $t + \Delta$ by Brand g.

However, this paper will assume $\alpha = 1$, or the money is effectively spent.

Since it is assumed that all brands spend money with equal effectiveness, it is reasonable that the relative effectiveness of Brand h in attracting Brand g's "potential brand-shifting fraction," $(1-q_g)c_g(t)$, is, not considering carry-over,

$$f_{gh}(t) = \frac{a_h(t)}{\sum_{g=1}^{n} a_g(t)} \quad . \tag{3.16}$$

That is, the portion of each brand's "potential brand-shifting fraction" which shifts to Brand h during $[t + \Delta, t + 1]$ is proportional to Brand h's relative spending at time $t + \Delta$ with respect to the industry as a whole. If the carry-over effect is present,

$$f_{gh}(t) = \frac{a_h(t) + b_h a_h(t-1) + b_h^2 a_h(t-2) + \ldots}{\sum_{g=1}^{n} [a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]} \quad . \tag{3.17}$$

That is, the portion of each brand's "potential brand-shifting fraction" which shifts to Brand h during $[t + \Delta, t + 1]$ is proportional to Brand h's relative composite advertising at time $t + \Delta$ with respect to the industry as a whole. As such,

$$f_{1h}(t) = f_{2h}(t) = \ldots = f_{hh}(t) = \ldots = f_{nh}(t) \quad . \tag{3.18}$$

The effect of increased advertising (or composite advertising) by Brand h on the "potential brand-shifting fraction," if the competion fails to react, is shown in Figure 7.

It is not necessary that $f_{gh}(t)$ be described as in equations (3.16) and (3.17). If some expression is known

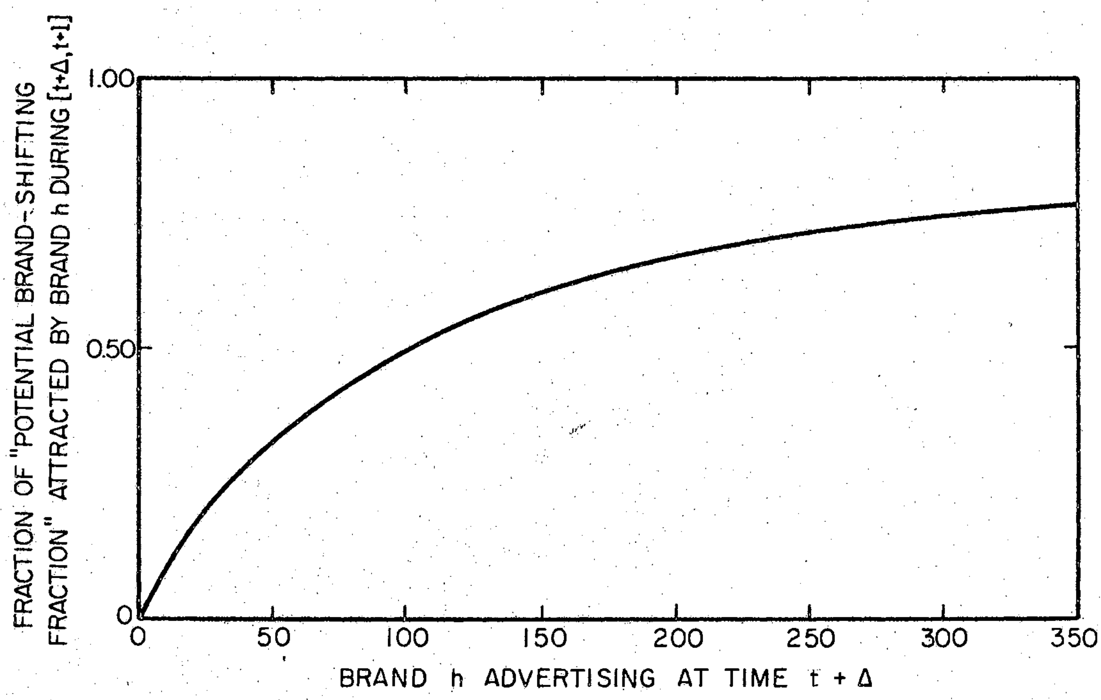

Figure 7. Effect of Increased Advertising by Brand h if Competition Composite Advertising is Fixed at 100 at Time t + Δ

to be different for a particular product class it can be used. Note that while equation (3.18) does not have to hold, equation (3.19) is essential as each row of the transition matrix of Figure 1 must sume to 1.

$$\sum_{h=1}^{n} f_{gh}(t) = 1.0 \qquad 0 \leq f_{gh}(t) \leq 1 \quad . \tag{3.19}$$

Variable Demand as a Function of Advertising

In writings on advertising models it is often suggested that as a firm increases its advertising expenditures its sales will increase at a decreasing rate. For example, Vidale and Wolfe (1961) hypothesize from their empirical studies that "sales generated per advertising dollar, when sales are at a level S, is given by r(M-S)/M where M is the Saturation Level" and r is a constant.[4] Zentler and Ryde (1956) refer to a similar concept when they rely on the opinions of publicity experts who confirm that the individual's response to steadily increasing promotional activity will begin with increasing returns followed by decreasing returns.[5] Previously they speak of total demand for a product class increasing with increased competitive advertising expenditures. Such a characteristic is not included in their model, however.

Kuehn (1961) points out that if the level of industry advertising influences total industry sales, one of the assumptions used in his model would be invalid.[6] Shakun (1965), through use of a differential equation approach, assumes that total market size increases at a decreasing rate with increasing advertising expenditures. Borrowing Vidale and Wolfe's terminology here, Shakun expresses the change of total industry sales with respect to the change of effective total industry advertising as $dS/dA = r(M-S)/M$.[7] Each variable retains its meaning as described earlier, but Shakun refers to total industry sales and advertising while Vidale and Wolfe refer to that of a firm or brand. Gupta and Krishnan (1967) also use a differential equation approach to consider "total market potential is a function of total effective promotional efforts of all competitors."[8]

Alderson and Green (1964) comment that

> Most characteristic of the response variables are advertising and selling. The presumption is that the market responds or may respond as advertising and selling expenditures increase.[9]

They then present two curves showing sales dollars versus advertising dollars. Both curves increase at a decreasing rate. One curve, however, shows that no sales are made until a certain minimum level of advertising has been reached. That level of advertising expenditure necessary may be called the threshold level, such a situation being possible if a certain advertising expenditure must be made before dealers will even stock the product.[10]

In summary of the discussion and findings thus far, it would be well to list some observations:

1. Most authors of optimization models acknowledge the effect of advertising on demand level.
2. It is generally agreed that the incremental effect of advertising decreases as expenditures increase.
3. There is some discussion of an initial range of increasing returns. Related to this is the threshold level.
4. All models seen have either completely disregarded or completely incorporated the effect of advertising on demand.

A model will now be described which may be used to completely disregard, completely incorporate, or partially incorporate demand as a function of total industry advertising. This model is very versatile and it represents a new concept in describing demand as a function of total industry advertising.

Consider Figure 8. Let "total potential demand" be defined as total industry demand for a product class, over time, as total effective industry advertising approaches infinity. As defined, total potential demand may be broken into four components as follows:

1. That fraction of total potential demand which buys the industry's product without advertising motivation. Their brand choice is not influenced by relative advertising.
2. That fraction of total potential demand which

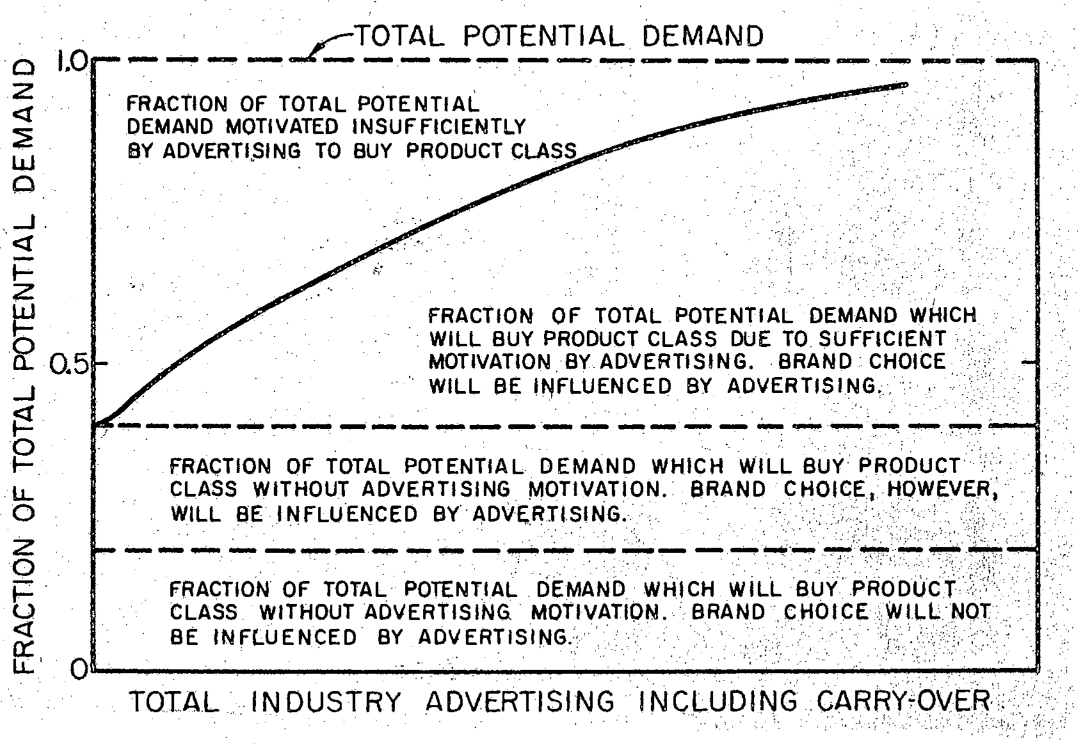

Figure 8. Influence of Total Industry Composite Advertising Expenditure on Demand

buys the industry's product without advertising motivation. Their brand choice is, however, influenced by relative advertising.

3. That fraction of total potential demand which buys the industry's product if sufficiently motivated by total industry advertising. Brand choice is, of course, influenced by relative advertising.

4. That fraction of total potential demand insufficiently motivated by total industry advertising to buy the industry's product.

It is assumed that categories 1 and 2 above remain constant. It is, however, recognized that while the sum of the fractions described in 1 and 2 will probably remain constant, their relative size may be a function of the level of industry advertising. Such a consideration will not be dealt with here.

In the development of an advertising optimization model the first component of total potential demand (described in 1 above) should be neglected. This is because advertising has no influence on this segment of total potential demand. Thus, only the fraction 1-D' as shown in Figure 8 will be considered. Therefore, let

Total Potential Demand (Rate) as a Function of Time $= 1{,}428{,}571 + 342{,}857 \sin(\pi t/6)$ (3.20)

$$D' = .3 \qquad (3.21)$$

Potential Demand (Rate as a Function of Time = (1−D') ∘ Total Potential Demand (Rate) as a Function of Time

$$= .7(1,428,571 + 342,857 \sin(\pi t/6))$$

$$= 1,000,000 + 240,000 \sin(\pi t/6) . \quad (3.22)$$

Verbally, "potential demand" is that fraction of "total potential demand" which is susceptible to advertising influence. Equations (3.20) and (3.21) are constructed such that (3.22), the potential demand of interest, is indeed the expression used originally in equation (3.1) to describe the rate of demand over time. S(t) will now be redefined as follows:

S(t) = the "potential demand" during the period $[t + \Delta, t + 1]$ in terms of dollar sales.

Also,

P(t) = the fraction of "potential demand" (not "total potential demand") which, due to total effective industry advertising at $t + \Delta$ (including carry-over), will purchase the product of the industry. As such, P(t) is a measure of response to total industry advertising. Potential demand is assumed to vary over time. The value of P(t) can be thought of as an advertising effectiveness coefficient. As such, P(t) represents the effectiveness of industry advertising at $t + \Delta$ in terms of the fraction of "potential demand" motivated or influenced in buying the industry's product. Therefore, the coefficient P(t) is assumed to carry over to all retention buying sales in future periods which are attributed to advertising at $t + \Delta$.

P(t) is described by a normalized Gompertz equation as follows:

$$P(t) = (D^{S^W} - D')/(1-D') \qquad \begin{array}{l} 0 < D' \leq D \leq 1 \\ 0 \leq S \leq 1 \\ 0 \leq W \end{array} \qquad (3.23)$$

where

$$W = U \sum_{g=1}^{n} [a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots] \qquad (3.24)$$

and U is a constant. Note that by subtracting D' and dividing by 1-D', the expression is normalized over the range of interest from Figure 8, 1-D'.

The Gompertz equation is used at this author's discretion for the following reasons:

1. P(t) increases at a decreasing rate after a small initial range of increasing returns (increasing at an increasing rate).
2. Depending upon the value of D, the expression P(t) can describe any situation along a continuum from fixed demand as a function of total industry advertising (D=1) to complete advertising model variable demand as a function of total industry advertising (D=D').

3. P(t) approaches 1.0 asymptotically. That is, "demand" approaches "potential demand" asymptotically as industry advertising increases.

Extending equation (3.14) to represent all present and future dollar sales of Brand 1 due to advertising at $t + \Delta$ and also considering demand to be a function of total industry advertising, one has

$$\sum_{t''=1}^{\infty} DS_{1,t+\Delta}(t+t'') = \left[\sum_{g=1}^{2} (1-q_g)c_g(t)f_{g1}(t)\right] \cdot [P(t)][S(t) + q_1 S(t+1) + \ldots] \qquad (3.25)$$

where

$$P(t) = \left[D^{S^{U\sum_{g=1}^{2}[a_g(t) + b_g a_g(t-1) + b_g^{2} a_g(t-2) + \ldots]}} - D'\right] \Big/ [1 - D'] . \qquad (3.26)$$

In the computer studies to be done, three cases will be considered. That is, demand will be assumed to be fixed as a function of total industry advertising (D=1.0), and demand will be assumed a complete function of advertising in the model (D=D'=.3) for the computer work of Chapters IV and V. Chapter VI, however, will address the case where demand is partially fixed and partially a function of total industry advertising (D=.65). The other parameters of the modified Gompertz equation will be as follows:

$$S = .6 \qquad (3.27)$$

$$U = .0000124 . \qquad (3.28)$$

A graphical representation of the function P(t) with $D=D'=.3$, $S=.6$, and $U=.0000124$ is shown in Figure 9.

Summary

In this chapter much of the mathematical background has been developed for the advertising models to be completed in Chapters IV and V. Specifically, demand has been presented as a function of both time and total industry advertising. Retention buying and advertising carry-over have been represented mathematically. Two methods of incorporating retention buying were considered. Of these, both were found to yield identical results under constant demand while only one method properly sensed fluctuations under variable demand.

The advertising attractiveness function was also presented. Throughout this research it is assumed to be the ratio of a given brand's composite advertising to the total industry composite advertising.

Finally, all of the characteristics and functions were combined into one term, equation (3.25). That term represents present and future dollar sales by Brand 1 which may be attributed to advertising at time $t + \Delta$. This term will be used in developing the models in the following two chapters.

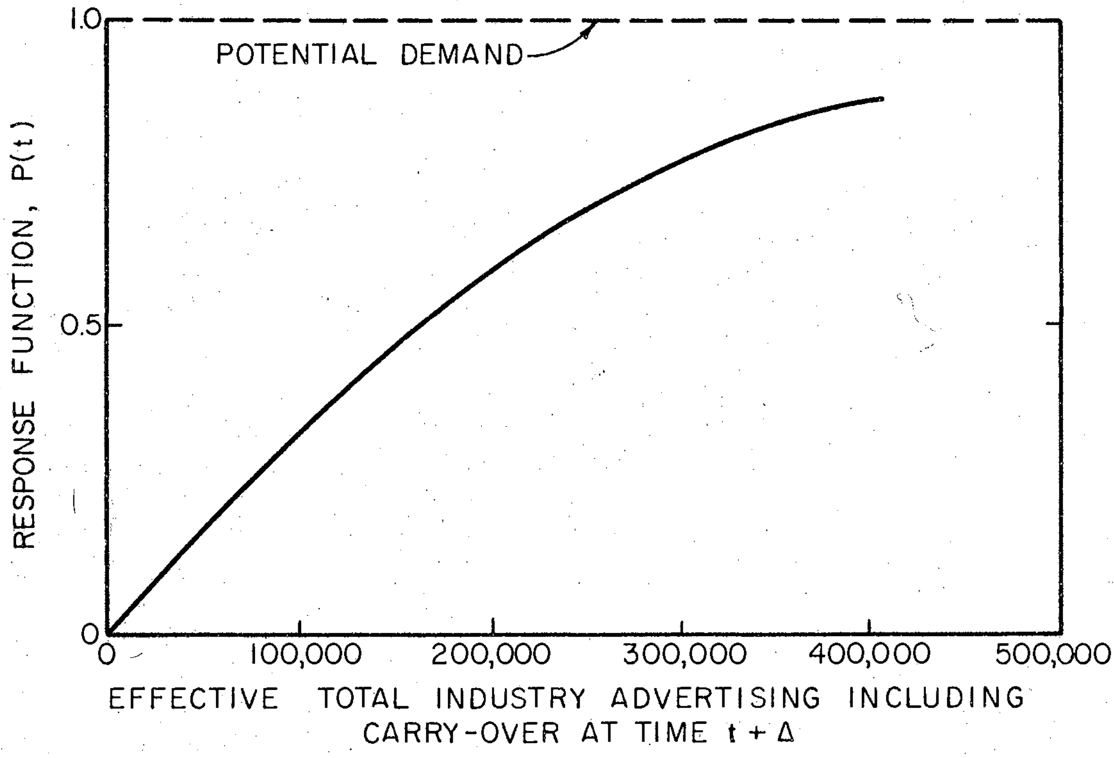

Figure 9. Response Curve P(t) Where Modified Gompertz Parameters are D=D'=.3, S=.6, U=.0000124

FOOTNOTES

[1]M. L. Vidale and H. B. Wolfe, "An Operations-Research Study of Sales Response to Advertising," Mathematical Models and Methods in Marketing, ed. Frank M. Bass et al. (Homewood, Illinois, 1961), p. 365.

[2]Ibid., p. 366.

[3]E. T. Gundlach, Facts and Fetishes in Advertising (Chicago, Illinois, 1931), p. 315.

[4]Vidale and Wolfe, "An Operations-Research Study of Sales Response to Advertising," Mathematical Models and Methods in Marketing, p. 369.

[5]A. P. Zentler and Dorothy Ryde, "An Optimum Geographical Description of Publicity Expenditure," Management Science, Vol. II (1956), p. 339.

[6]Alfred A. Kuehn, "How Advertising Performance Depends on Other Marketing Factors," Managerial Marketing, ed. Eugene J. Kelley and William Lazar (3rd ed., Homewood, Illinois, 1967), p. 565.

[7]Melvin F. Shakun, "Advertising Expenditures in Coupled Markets. A Game-Theory Approach," Management Science, Vol. II (1965), p. B-42.

[8]Shiv K. Gupta and K. S. Krishnan, "Mathematical Models in Marketing," Operations Research, Vol. 15 (1967), p. 1045.

[9]Wroe Alderson and Paul E. Green, Planning and Problem Solving in Marketing (Homewood, Illinois, 1964), p. 564.

[10]Ibid., p. 565.

CHAPTER IV

DEVELOPMENT AND ANALYSIS OF A PROFIT ORIENTED MODEL CONSIDERING PAST ADVERTISING EXPENDITURES

The objective of this chapter is to develop and analyze a profit oriented advertising model (or profit model) based on the characteristics of Chapter III. This model, to be referred to as Model I, will assume that the user has access to statistics showing the past advertising expenditures of competing firms. The user is also required to estimate the present period expenditures of competing firms. Further, it is assumed that the relative brand shares are known for the last period $[t - 1 + \Delta, t]$ in order to determine the optimal advertising expenditure at $t + \Delta$. Such information is available from market research firms.

Model I will be used to determine the equilibrium advertising expenditure and profit for each of two identical competitors under various parameter values. In this way some of the effects of the characteristics treated in Chapter III may be determined.

Development of the Model

At the outset it would be wise to establish exactly what is meant by "profit." In common usage, one usually thinks of profit during a time period $[t+\Delta, t+1]$ as income during that period less costs during that period. However, in the model to be developed, profit during $[t+\Delta, t+1]$ will have a different interpretation. Profit will be present and future (due to retention buying) income attributable to effective advertising, including carry-over, at $t+\Delta$, less present and future costs of related production and overhead, less the actual advertising expenditure at $t+\Delta$, $a_g(t)$. As such, $a_g(t)$ may be considered an investment with future returns. Income, less all costs except advertising, might then be considered the net present worth of the cash flows resulting from the investment $a_g(t)$.

In general terms, profit may be considered as follows:

$$\text{PROFIT} = \text{Dollar Sales} - \begin{array}{c}\text{All Costs Exclusive}\\ \text{of Advertising}\end{array} - \begin{array}{c}\text{Cost of}\\ \text{Advertising}\end{array} \tag{4.1}$$

If "all costs exclusive of advertising" may be considered a linear function of the number of units produced,

$$\text{PROFIT} = N(A_{t,t-1,\ldots})\, S - N(A_{t,t-1,\ldots})\, C - A_t \tag{4.2}$$

where

$N(A_{t,t-1,\ldots})$ = the number of units produced and sold. This number, in general, is a function of composite advertising (new advertising plus carry-over advertising) at $t + \Delta$.

S = the selling price per unit. This will be the price paid by an intermediary if the producer does not sell directly to the public. The price is assumed constant.

C = the cost per unit exclusive of advertising.

A_t = the actual advertising expenditure at $t + \Delta$.

Equation (4.2) can be rewritten as

$$PROFIT = N(A_{t,t-1,\ ...})\ S(1 - C/S) - A_t\ . \quad (4.3)$$

As shown in equation (4.3),

$N(A_{t,t-1,...})S$ = the total income from sales (a function of advertising).

$1 - C/S$ = the profit margin, before advertising expenditure, per dollar of sales.

Equation (4.3) is shown as a function of advertising expenditure in Figure 10. In Figure 10 no advertising carry-over is considered ($A_{t,t-1,\ ...} = A_t$).

In converting equation (4.3) to the terminology developed in Chapter III it should be noted that the term $N(A_{t,t-1,\ ...})S$ is represented by equation (3.25), the present and future dollar sales attributable to composite advertising at $t + \Delta$. Also, the term $1 - C/S$ is replaced

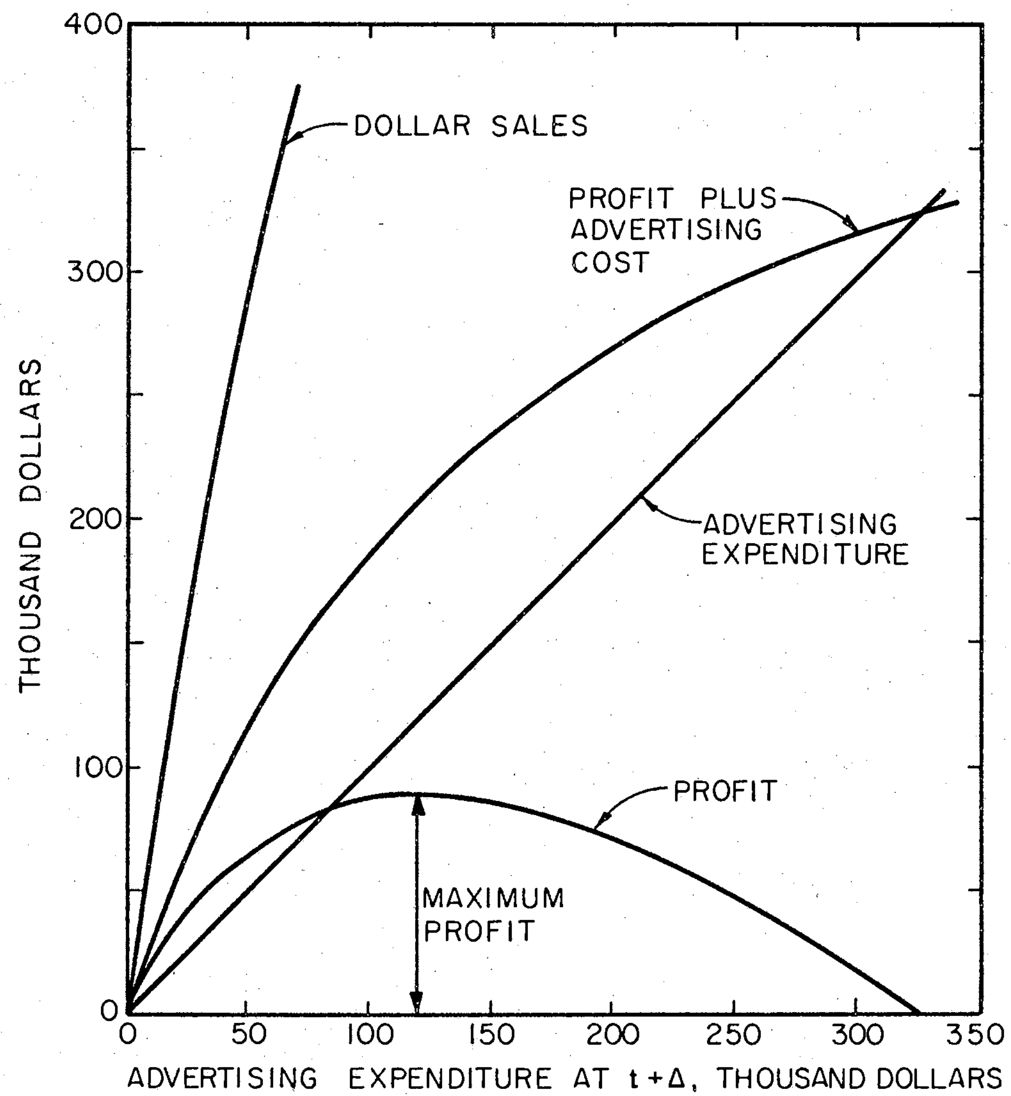

Figure 10. Dollar Sales, Profit Plus Advertising Cost, and Profit Versus Advertising Expenditure at t + Δ: Without Advertising Carry-Over

by $1 - r_g$ where

r_g = the fraction of each sales dollar of Brand g which is considered as cost before advertising. Such cost includes production, overhead, distribution and other associated costs. The factor r_g is assumed to be constant and a value of $r_g = .6$ is used throughout the remaining analysis.

The Cost Expression

Throughout the above discussion, costs before advertising were assumed to be a linear function of the number of items produced. Although a popular assumption, it is often more realistic to divide cost into two components, fixed cost and variable cost. Fixed costs are those which do not vary with the changes in the volume of activity; variable costs are those which vary directly with volume.[1] If the variable costs are then a linear function of production, equation (4.3) may be expressed as

$$\text{PROFIT} = N(A_{t,t-1,\ldots})S(1 - C/S) - F - A_t \tag{4.4}$$

where

$1 - C/S$ = the profit margin, before advertising and fixed costs, per dollar of sales.

F = the fixed cost.

The model to be developed in this chapter will consider all costs before advertising to be a linear function of production as in equation (4.3).

<u>Further Development of the Model</u>

Using the terminology discussed following equation (4.3) as well as the mathematical results from Chapter III, the profit equation can now be stated as a function of advertising. The equation will be written for a specific brand, Brand h, in competition with n-1 other firms.

$$\pi_{h,t+\Delta} = (1-r_h) \sum_{t''=1}^{\infty} DS_{h,t+\Delta}(t+t'') - a_h(t)$$

$$= (1-r_h)\left[\sum_{g=1}^{n} (1-q_g)c_g(t)f_{gh}(t)\right]\left[P(t)\right] \cdot$$

$$\left[S(t) + q_h S(t+1) + q_h{}^2 S(t+2) + \ldots\right] - a_h(t) \qquad (4.5)$$

where

$$P(t) = \left[D^{\frac{U}{S}\sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g{}^2 a_g(t-2) + \ldots]} - D'\right] \Big/ \left[1-D'\right] \qquad (4.6)$$

and

$$f_{gh}(t) = \frac{a_h(t) + b_h a_h(t-1) + b_h{}^2 a_h(t-2) + \ldots}{\sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g{}^2 a_g(t-2) + \ldots]}. \qquad (4.7)$$

The term $\pi_{h,t+\Delta}$ may be defined as

$\pi_{h,t+\Delta}$ = the total profit of Brand h which may be attributed to effective advertising, including carry-over, at $t + \Delta$. In other words, it is the total present and future income to Brand h from dollar sales attributed to composite advertising at $t + \Delta$, less all associated costs of production, overhead, and distribution, less the advertising expenditure of Brand h at time $t + \Delta$.

This profit equation does consider the past advertising levels of all firms. Also, it accounts for future retention sales given projected potential demand. Further, it considers the influence of total industry advertising upon industry demand. It does not consider the future influence of the carry-over advertising of $t + \Delta$. Such a model will be developed in Chapter V.

Analytical Solution for $P(t) = 1.0$

An often used assumption for advertising models is that industry demand is not a function of total industry advertising. Such an assumption is analogous to letting demand vary as a function of time only. In the model of equation (4.5) can let demand equal potential demand by letting $P(t) = 1.0$. If $P(t) = 1.0$ regardless of advertising level, demand is a function of time only. To examine this case let $D = 1$ (D' still equals .3) such that $P(t) = 1.0$. Equation (4.5) then reduces to

$$\pi_{h,t+\Delta} = (1-r_h)\left[\sum_{g=1}^{n}(1-q_g)c_g(t)f_{gh}(t)\right]\left[\text{DEMAND}\right] - a_h(t) \tag{4.8}$$

where

$$\text{DEMAND} = [S(t) + q_h S(t+1) + q_h^{\,2} S(t+2) + \ldots] \tag{4.9}$$

and

$$f_{gh}(t) = \frac{a_h(t) + b_h a_h(t-1) + b_h^2 a_h(t-2) + \ldots}{\sum_{g=1}^{n} [a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]} . \quad (4.10)$$

If equation (4.10) holds,

$$f_{1h}(t) = f_{2h}(t) = \ldots = f_{hh}(t) = \ldots = f_{nh}(t) . \quad (4.11)$$

That is, the attractiveness function for Brand h, $f_{gh}(t)$, has equal influence on each firm's "potential brand-shifting fraction," $(1-q_g)c_g(t)$. Therefore, equation (4.8) can be rewritten as

$$\pi_{h,t+\Delta} = (1-r_h)\left[\sum_{g=1}^{n} (1-q_g)c_g(t)\right]\left[\text{DEMAND}\right]\left[f_{gh}(t)\right] - a_h(t)$$

$$= (1-r_h)\left[\sum_{g=1}^{n} (1-q_g)c_g(t)\right]\left[\text{DEMAND}\right] \cdot$$

$$\left[\frac{a_h(t) + b_h a_h(t-1) + b_h^2 a_h(t-2) + \ldots}{\sum_{g=1}^{n} [a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]}\right] - a_h(t) . \quad (4.12)$$

In order to determine the optimal advertising expenditure at time $t+\Delta$, equation (4.13) must be solved for $a_h(t)$.

$$\frac{\partial \pi_{h,t+\Delta}}{\partial a_{h,(t)}} = 0 . \quad (4.13)$$

In order to proceed, let

$$X = (1-r_h)\left[\sum_{g=1}^{n} (1-q_g)c_g(t)\right]\left[\text{DEMAND}\right] . \quad (4.14)$$

Then

$$\pi_{h,t+\Delta} = X\left[\frac{a_h(t) + b_h a_h(t-1) + b_h^2 a_h(t-2) + \ldots}{\sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]}\right] - a_h(t) . \quad (4.15)$$

$$\frac{\partial \pi_{h,t+\Delta}}{\partial a_h(t)} = \frac{X}{\sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]}$$

$$- \frac{X[a_h(t) + b_h a_h(t-1) + b_h^2 a_h(t-2 + \ldots]}{\left[\sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]\right]^2} - 1 = 0 .$$

$$\frac{\partial \pi_{h,t+\Delta}}{\partial a_h(t)} = X \sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]$$

$$- X[a_h(t) + b_h a_h(t-1) + b_h^2 a_h(t-2) + \ldots]$$

$$- \left[\sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]\right]^2 = 0 . \quad (4.16)$$

Therefore, combining the first two of the three terms in equation (4.16),

$$X \sum_{\substack{g=1 \\ g\neq h}}^{n}[a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]$$

$$= \left[\sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]\right]^2 . \quad (4.17)$$

If all competitors are identical, i.e., with the same cost coefficients, retention factors, etc., then

$$a_1(t) = a_2(t) = \ldots = a_h(t) = \ldots = a_n(5) \quad (4.18)$$

at competitive equilibrium. Competitive equilibrium refers to that point at which a change in advertising expenditure (up or down) by any competitor will result in decreased profits. If all competitors are identical, it is also assumed that an expression similar to (4.18) holds for advertising in previous periods. In such a case, from equation (4.17)

$$X[n-1][a_h(t) + b_h a_h(t-1) + b_h{}^2 a_h(t-2) + \ldots]$$

$$= n^2[a_h(t) + b_h a_h(t-1) + b_h{}^2 a_h(t-2) + \ldots]^2 \quad (4.19)$$

and

$$X[n-1] = n^2[a_h(t) + b_h a_h(t-1) + b_h{}^2 a_h(t-2) + \ldots] . \quad (4.20)$$

Then the equilibrium advertising expenditure at $t + \Delta$ for each of n identical competitors is

$$a_h(t) = \left[\frac{n-1}{n^2}\right]\left[X\right] - b_h a_h(t-1) - b_h{}^2 a_h(t-2) - \ldots$$

$$= \left[\frac{n-1}{n^2}\right]\left[1-r_h\right]\left[\sum_{g=1}^{n} (1-q_g)c_g(t)\right]\left[\text{DEMAND}\right] - b_h a_h(t-1)$$

$$- b_h{}^2 a_h(t-2) - \ldots \quad . \quad (4.21)$$

If it cannot be assumed that all competitors are identical, and if it is assumed that no advertising carry-over exists,

$$b_1 = b_2 = \ldots = b_h = \ldots = b_n = 0 , \quad (4.22)$$

then the optimal level of advertising $a_h(t)$ can be determined analytically. Assume only two competitors, h = 1 and g = 2. From equation (4.17),

$$X[a_2(t)] = [a_1(t) + a_2(t)]^2$$

$$X[a_2(t)] = [a_1(t)^2 + 2a_1(t)a_2(t) + a_2(t)^2] \qquad (4.23)$$

$$a_1(t)^2 + 2a_1(t)a_2(t) + a_2(t)[a_2(t) - X] = 0 \quad . \qquad (4.24)$$

Solving for $a_1(t)$ we have

$$a_1(t) = \frac{-2a_2(t) \pm \sqrt{4a_2(t)^2 - 4a_2(t)[a_2(t) - X]}}{2}$$

$$a_1(t) = -a_2(t) + \sqrt{a_2(t)X}$$

$$= -a_2(t) + \sqrt{[a_2(t)][1-r_h]\left[\sum_{g=1}^{n} (1-q_g)c_g(t)\right][\text{DEMAND}]} \quad . \qquad (4.25)$$

Solving similarly for three brands and then solving inductively for n brands yields

$$a_h(t) = -\sum_{\substack{g=1 \\ g\neq h}}^{n} a_g(t) + \sqrt{\left[\sum_{\substack{g=1 \\ g\neq h}}^{n} a_g(t)\right]\left[1-r_h\right]\left[\sum_{g=1}^{n} (1-q_g)c_g(t)\right]\left[\text{DEMAND}\right]} \quad . \qquad (4.26)$$

If equation (4.22) does not hold, the analytical approach becomes infeasible. A computer approach is then necessary.

Explanation of Computer Analysis

In order to observe the model under the influences of a variable market as a function of time and industry advertising, retention buying, and advertising carry-over, it is desired to search for the optimal Brand h advertising expenditure when in competition with one other brand. This is done on a digital computer. In order to determine the optimal $a_h(t)$, the past advertising history of the competitor will have to be assumed. Since such an assumption represents only one of an infinite number of past advertising histories, an identical competitor will be assumed. Such a competitor will not only have identical parameter values as compared to Brand h, but he will also have access to this model. A situation as described above leads to the determination of an equilibrium advertising expenditure and profit for both competitors. As defined previously, competitive equilibrium refers to that point at which a change in advertising expenditure, up or down, will result in decreased profits.

The assumption of an identical competitor will provide a consistent example of competition for determining several effects of the various characteristics. In Chapter VI, a more typical example of this model will be used in which two non-identical competitors, one using this model, one using a rule of thumb, will be considered.

Search for Equilibrium

In general, the equilibrium point for the two identical competitors is found by first initializing each brand's advertising expenditure to some arbitrary value at time $t + \Delta$.

The Brand h expenditure is then incremented, holding its competitor constant, until roughly optimized in terms of maximum profit. The competitor's expenditure is then set equal to that of Brand h. Brand h is then incremented again. This procedure is continued until equilibrium advertising is determined to any preset accuracy.

The computer program to be used for the analysis in this chapter is shown, with explicative comment cards, in Appendix C. A sample output is also presented.

Successive Monthly Analysis

A method for determining the optimal advertising levels for successive months must be developed. Theoretically, the retention buying and advertising carry-over effects could continue indefinitely. In Chapter III and Appendix A, retention buying is considered over the entire future and therefore the DEMAND term (equation (4.9)) reflects retention buying over all time. Due to the possibility of rather large values of q_g, retention buying should be considered for many periods. Advertising carry-over is, however, a coefficient of advertising and cannot feasibly be considered for more than a few periods even on a high speed computer. Fortunately, most advertising carry-over lies in the range of

$$0 \leq b_g \leq .5 \qquad (4.27)$$

per month.[2] Therefore the influence of advertising carry-over diminishes rapidly.

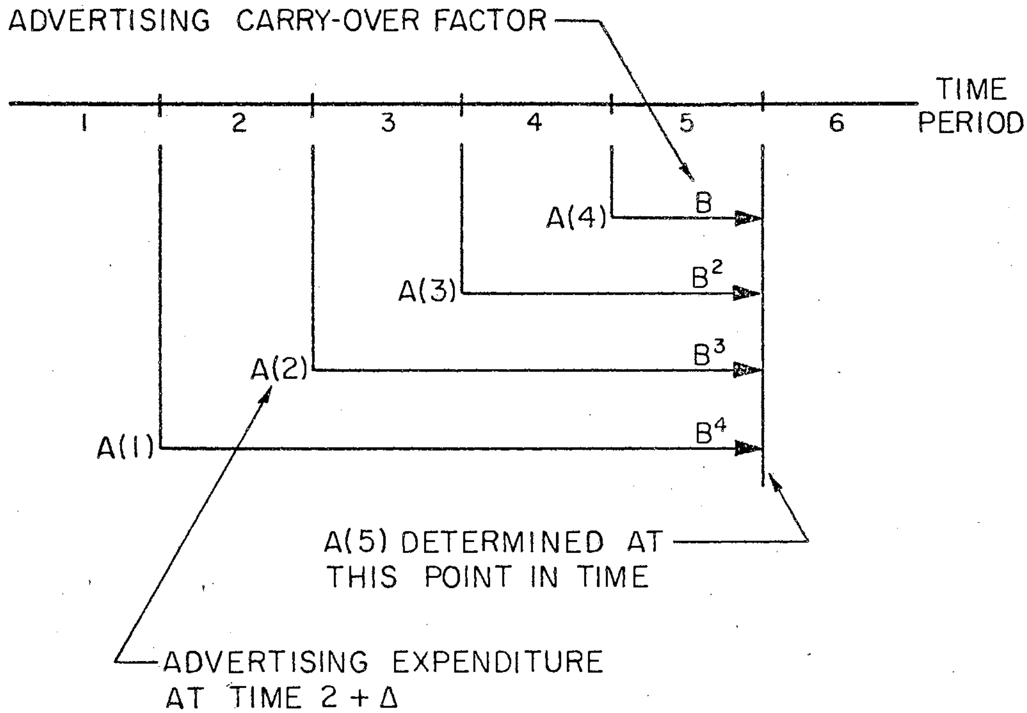

Figure 11. Timing Used in Computer Program of Chapter IV (Appendix C)

Figure 11 shows a pictorial view of the timing used in the computer program of Appendix C. Notice that advertising expenditures from the four previous periods, A(1), A(2), A(3), and A(4) ($a_g(t-4), a_g(t-3), a_g(t-2), a_g(t-1)$), are displaced through use of the carry-over coefficients B^4, B^3, B^2, and B (${b_g}^4$, ${b_g}^3$, ${b_g}^2$, b^g), to time 5 + Δ. At time 5 + Δ the optimal advertising expenditure, A(5) ($a_g(t)$), will be determined. Such advertising as determined, A(5), and the carry-over advertising from the four most recent expenditures, represents the composite advertising at time 5 + Δ.

If the original sine-wave demand is assumed to be stable from year to year, and if the equilibrium expenditures are to be determined each month, a starting place must be ascertained. If Month 1 is selected as the starting point, and the advertising has carry-over, the equilibrium expenditures in previous months must be known. It can be seen that regardless of where one starts, the advertising expenditure determined will not be truly "optimal" because the equilibrium expenditures are unknown.

If no advertising carry-over is present, past advertising, optimal or not, has no bearing on the present allocation. The smaller the carry-over factor, b_g, the smaller the influence of an initial estimate as to equilibrium advertising in previous periods. It is believed that an initial estimate of the equilibrium expenditures in recent periods will provide a way to begin the analysis. It is hypothesized that as successive advertising expenditures are found, there will be a tendency to approach the equilibrium pattern as if the initial estimates had indeed been equilibrium values themselves. Once the method has converged to the true equilibrium pattern of expenditures, continual sequencing will repeat the values obtained.

Consider the drawing of Figure 12. It shows a number of monthly decision stages which under the assumption of a stable pattern of potential demand from year to year, may be shown as a circular pattern. In order to determine the equilibrium advertising expenditure at each month, consider starting at Month 1. The expenditures at Months 9, 10, 11, and 12 must be estimated. At Month 2, only the expenditures at Months 10, 11, and 12 need be estimates. As successive stages are considered, it is anticipated that eventually the effect of initial advertising approximations at Months 9, 10, 11, and 12 will diminish and a repeating cycle of equilibrium advertising expenditures will result.

Results and Findings at Equilibrium; Mathematical Analysis

The results of the computer determination of equilibrium advertising expenditure will now be discussed. A mathematical analysis will be implemented when the findings from the "identical competitor" equilibrium studies do not also apply to a "non-identical competitor." Each of the four characteristics will be discussed in terms of observed and predicted influences on advertising with respect to the

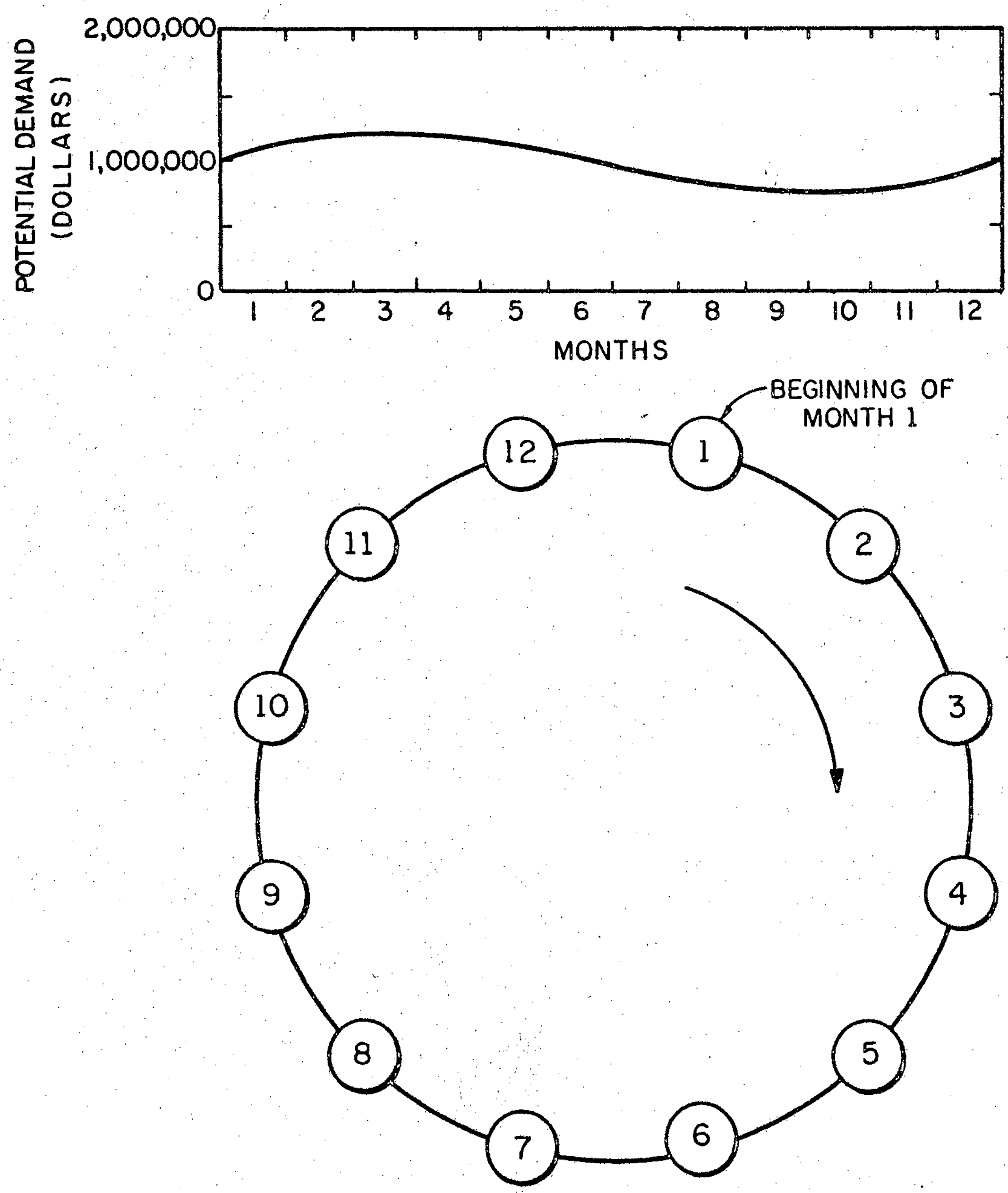

Figure 12. Successive Stages of Advertising Determination

model developed in this chapter. All combinations of the parameter values $b_g = 0.0, 0.5$ and $q_g = 0.0, 0.9$ were examined using the sine-wave potential demand of equation (3.1) and Figure 4 as discussed previously. The value of r_g (cost per dollar of sales before advertising) was set at .6. Further, both situations in which industry demand is and is not a function of total industry advertising were considered.

In regard to the theory, in the preceding section, that the influence of initial estimating would diminish, it was shown that in all cases the idea was correct. Of course, with $b_g = 0.0$ only one pass through each of the twelve months was necessary to determine the equilibrium pattern. As b_g increased, the influence of the initial estimates was found to last longer. For $b_g = 0.5$ it took about 2.5 passes through the twelve month circle before the yearly pattern had converged to cycling equilibrium advertising values.

Table II lists the equilibrium advertising values, the monthly spending as a percentage of the yearly budget, and the associated profits as determined for each of the two identical competitors. The advertising and profit results from Table II are shown graphically in continuous form in Figures 13 and 14. Figures 15 and 16 show plots of each month's relative advertising expenditure with respect to total yearly spending. Primarily from Figures 13, 14, 15, and 16 a number of conclusions will be made concerning the various characteristics shown empirically to affect

TABLE II

EQUILIBRIUM ADVERTISING, MONTHLY ADVERTISING AS A PERCENTAGE OF YEARLY BUDGET, AND CORRESPONDING PROFIT VALUES

	Demand not a Function of Total Industry Advertising				Demand as a Function of Total Industry Advertising			
Month	b_h=0.0 q_h=0.0	b_h=0.5 q_h=0.0	b_h=0.0 q_h=0.9	b_h=0.5 q_h=0.9	b_h=0.0 q_h=0.0	b_h=0.5 q_h=0.0	b_h=0.0 q_h=0.9	b_h=0.5 q_h=0.9
1	106140 8.85 106140	60410 9.75 151870	104640 8.72 104640	54410 8.79 154870	116390 9.88 22840	70820 11.65 68420	113730 9.04 21400	59410 9.14 75730
2	116780 9.73 116780	64880 10.47 168680	104490 8.71 104490	53750 8.68 155220	133800 11.36 33590	75870 12.48 91520	113460 9.01 21260	58230 8.96 76480
3	122920 10.24 122920	65790 10.62 180050	103130 8.59 103130	52520 8.48 153740	142960 12.14 40130	76720 12.62 106370	110990 8.82 19970	56000 8.62 74970
4	122920 10.24 122920	62900 10.16 182930	100930 8.41 100930	51040 8.24 150830	142960 12.14 40130	72950 12.00 110140	106890 8.49 17930	53200 8.19 71620
5	116780 9.73 116780	56990 9.20 176560	98490 8.21 98490	49720 8.03 147260	133800 11.36 33590	64240 10.57 103150	102120 8.11 15720	50530 7.78 67310
6	106140 8.85 106140	49640 8.01 162640	96450 8.04 96450	48900 7.90 143990	116390 9.88 22840	51700 8.51 87530	97980 7.78 13930	48770 7.51 63140
7	93860 7.82 93860	42820 6.91 144900	95360 7.95 95360	48810 7.88 141900	92440 7.85 11740	36620 6.03 67560	95690 7.60 13000	48520 7.47 60170
8	83220 6.94 83220	38350 6.19 128100	95510 7.96 95510	49480 7.99 141550	64670 5.49 3860	20840 3.43 47690	96020 7.63 13130	49920 7.68 59220
9	77080 6.42 77080	37440 6.04 116730	96870 8.07 96870	50710 8.19 143030	38500 3.27 580	8440 1.39 30640	98850 7.85 14300	52500 8.08 60650
10	77080 6.42 77080	40320 6.51 113840	99070 8.26 99070	52180 8.43 145950	38500 3.27 580	21260 3.50 17820	103270 8.20 16240	55420 8.53 64090
11	83220 6.94 83220	46230 7.46 120210	101510 8.46 101510	53510 8.64 149520	64670 5.49 3860	47040 7.74 21490	107980 8.58 18470	57870 8.91 68580
12	93860 7.82 93860	53590 8.65 134130	103550 8.63 103550	54320 8.77 152780	92440 7.85 11740	61250 10.08 42930	111770 8.88 20370	59290 9.13 72850

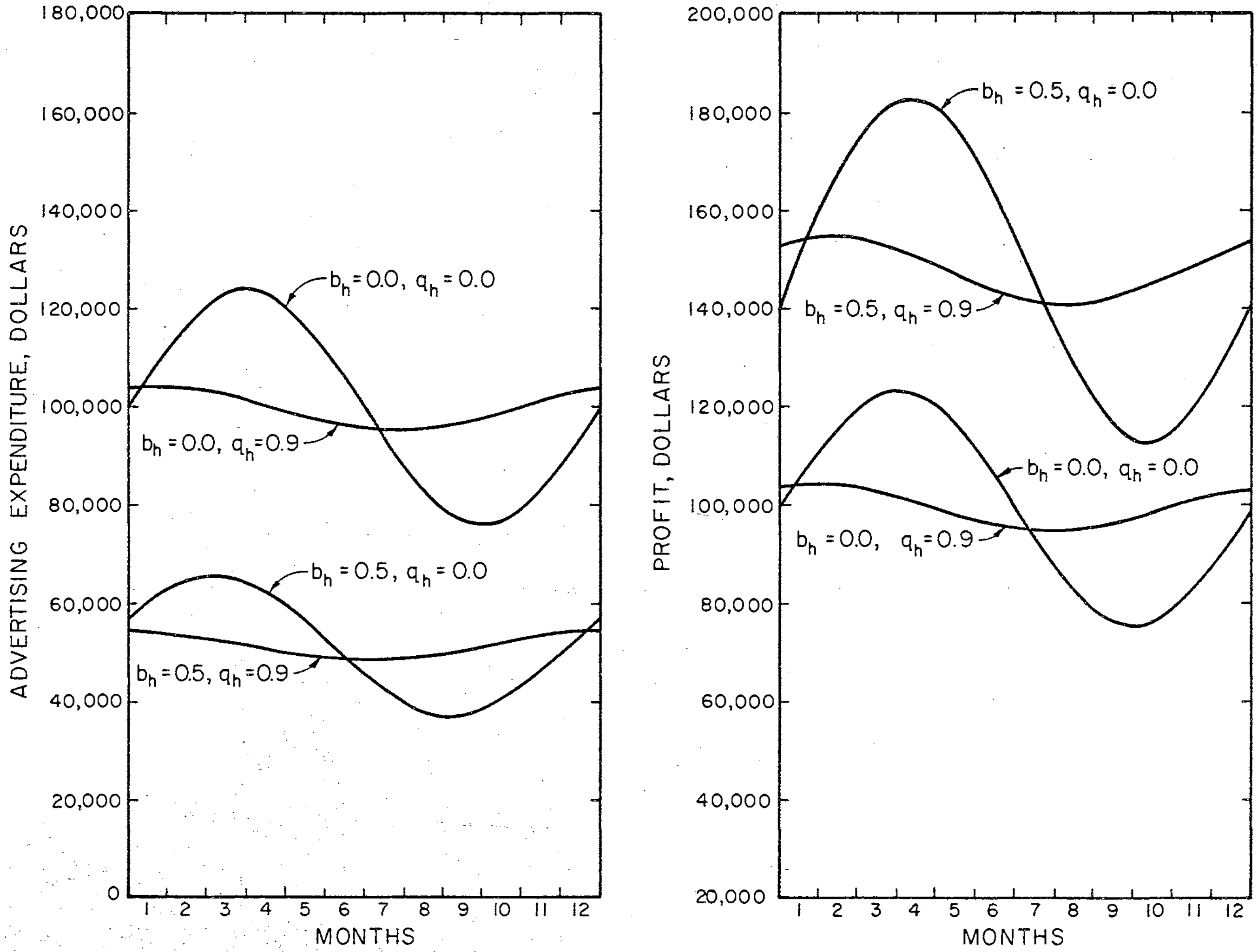

Figure 13. Model I Equilibrium Advertising and Profit: Demand not a Function of Total Industry Advertising

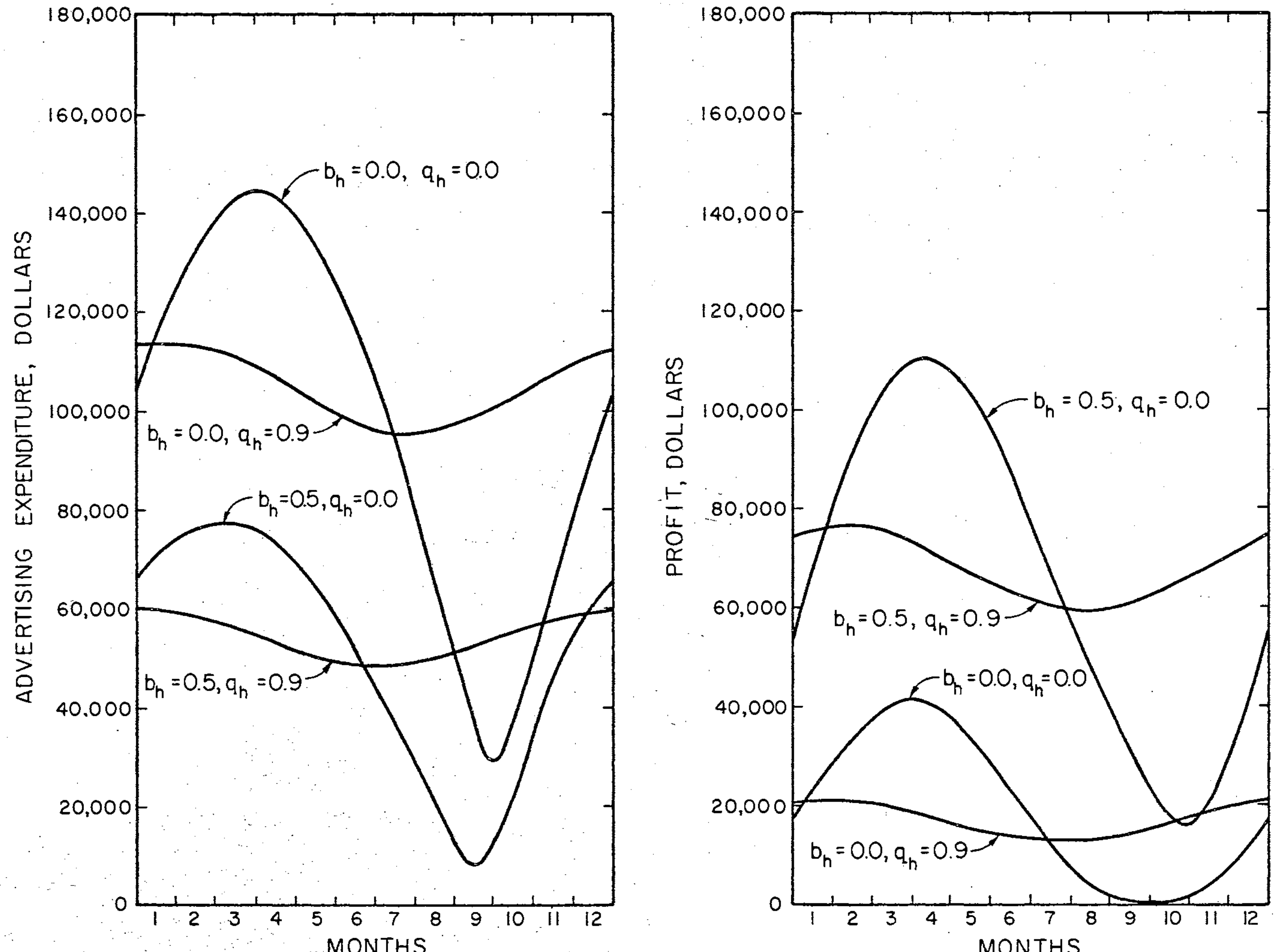

Figure 14. Model I Equilibrium Advertising and Profit: Demand a Function of Total Industry Advertising

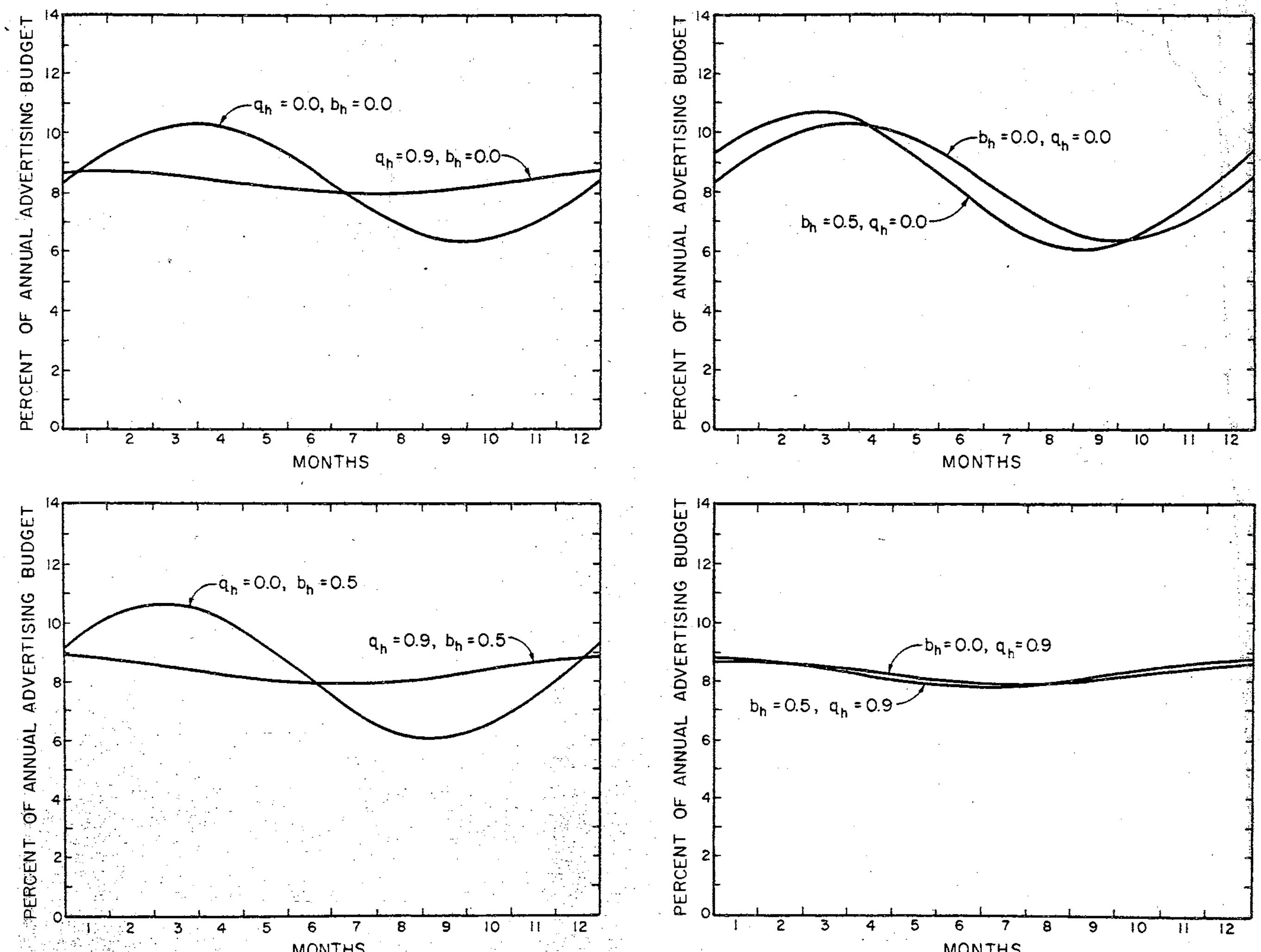

Figure 15. Model I Monthly Equilibrium Spending as Percentage of Yearly Budget: Demand not a Function of Industry Advertising

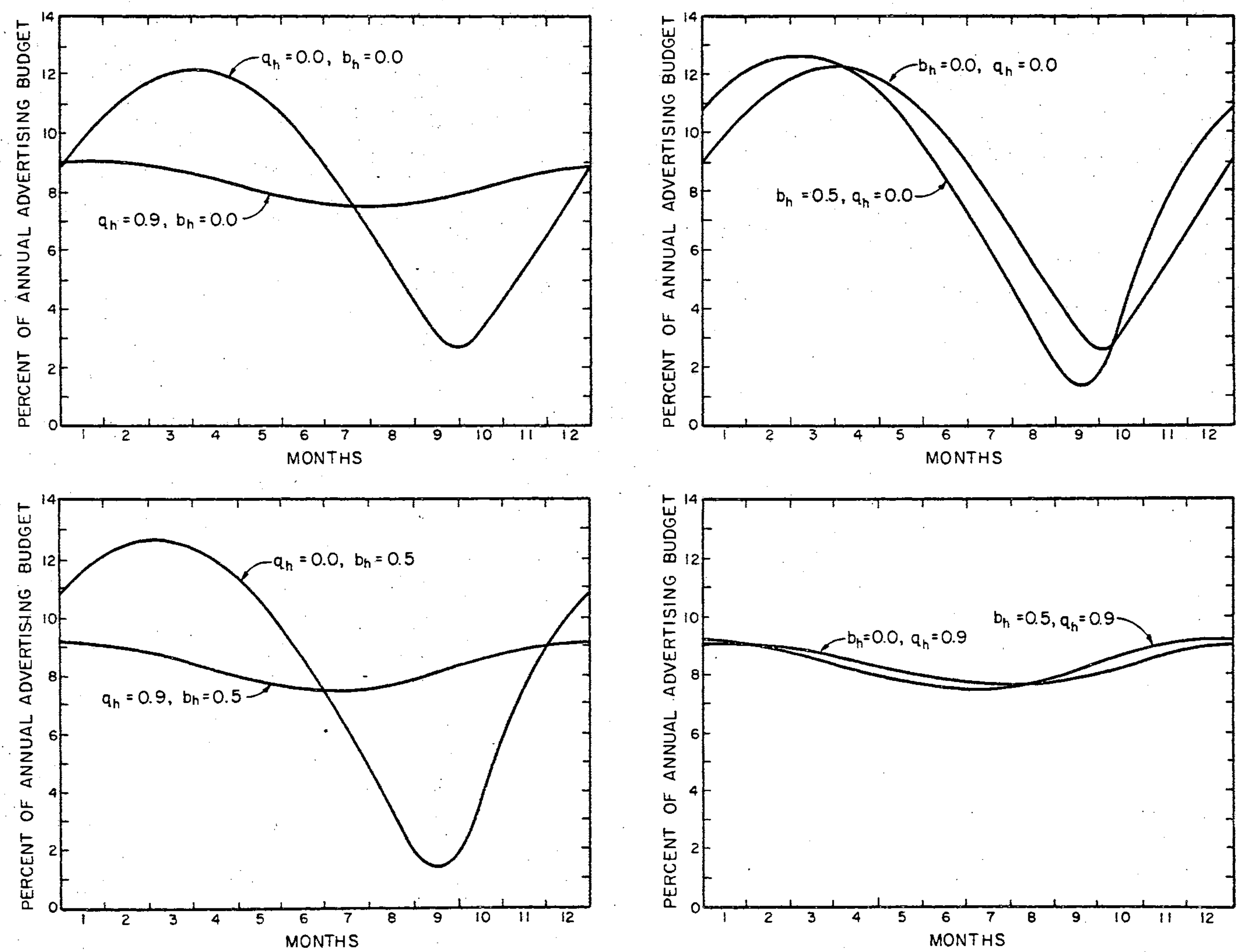

Figure 16. Model I Monthly Equilibrium Spending as Percentage of Yearly Budget: Demand a Function of Industry Advertising

advertising.

Demand as a Function of Time

The potential demand (also demand if P(t) = 1.0) rate is considered to vary as a function of time as

$$f(t) = 1{,}000{,}000 + 240{,}000 \sin(\pi t/6) \quad . \qquad (4.28)$$

Such a function has the familiar sine-wave shape about the value 1,000,000. Figures 13 and 14 show the influence of the sine-wave potential demand upon equilibrium advertising and profit. Such influence causes the general shape of the equilibrium advertising expenditure and profit pattern to assume a periodic, one cycle per period, form. It can be seen, however, that the relative amplitude, phase, and the proportional shape of the curves may differ from that of equation (4.28). In fact, the only curve which maintains the same relative amplitude, phase, and shape as that of equation (4.28) is one in which assumptions negate all of the other characteristics, i.e., $b_h = 0.0$, $q_h = 0.0$ and P(t) = 1.0. This result is an outgrowth of equation (4.21) in which the only variable on the right side is DEMAND. Note that the retention buying portion of demand equals zero since $q_h = 0.0$.

Other analyses were completed using constant, exponentially increasing, and exponentially decreasing potential demand curves. From the work with these functions, it was once again found that the potential demand lends its general shape to the pattern of equilibrium advertising expenditures. Such a shape is, much like the sine-wave results, subject to modification by the influence of other parameters.

Advertising Carry-Over

Advertising carry-over is another of the four major characteristics examined. Consider Figures 13 and 14. Probably the most noticeable of the influences of advertising carry-over is the greatly reduced equilibrium spending and the greatly increased equilibrium profits. It will be shown that even when in competition with a non-identical brand, advertising carry-over reduces advertising expenditures and increases profit in this model. To explain this phenomenon, consider total demand fixed with respect to total industry advertising. Referring again to equation (4.21), it is seen that the equilibrium advertising expenditure, $a_h(t)$, is some function of DEMAND less its effective advertising carry-over from past periods. In fact, the computer analysis shows that the equilibrium composite advertising for a given brand, at a given month, remains the same whether that composite advertising is made up of carry-over advertising, new advertising, or both. Such a statement applies for this model whether or not demand is a function of total industry advertising.

Consider the following example which relates to the above discussion:

Examine, at random, Month 5. Let total demand be fixed with respect to total industry advertising. If $b_h = 0.0$ and $q_h = 0.9$, the equilibrium advertising expenditure is \$98490 with a corresponding profit of \$98490. If, however, $b_h = 0.5$, that expenditure is reduced to \$49720, a savings of \$48770. Sales will certainly remain the same because only Month 5 is being considered, $P(t) = 1.0$, and each of the two identical competitors will split the demand evenly. Therefore, one would expect profit to increase by \$48770, the savings due to advertising carry-over at $b_h = 0.5$. Indeed, it does.

While the above example may have been obvious to the reader from equation (4.12), it is not so obvious from equation (4.5) that the same phenomenon occurs for a demand which is a function of total industry advertising.

Since the analysis of this chapter has been done in terms of an identical competitor, a broad statement that advertising carry-over reduces advertising expenditures and increases profit does not seem proper. The reason for this is that competition changes its expenditures exactly as does Brand h, and there is no constant pattern of competitive expenditures as a basis for comparison of different values of Brand h carry-over. However, reconsider the example given above in which the equilibrium advertising expenditure with $b_h = 0.5$ was \$49720 with a \$48770 carry-over for both competitors. Thus, the composite advertising for each competitor at the beginning of Month 5 was \$98490, the same as the actual advertising dollars spent during that month by each competitor when $b_h = 0.0$. Now, think of Brand h and its competitor as non-identical. Let Brand h have $b_h = 0.5$ and let the competition have $b_g = 0.0$. The equilibrium analysis at $b_h = 0.5$ can be thought of as optimizing the advertising expenditure of Brand h at the beginning of Month 5 as opposed to a competitor with $b_g = 0.0$ who spends \$98490. In other words, the equilibrium analyses at $b_h = 0.0$ and 0.5 may be considered as optimization analyses for Brand h versus a competitor who spends the same composite (new and/or carry-over) amount on advertising, regardless of his b_g, for a given potential demand and retention factor q_g. The outcome of the above discussion is that it can now be said that advertising carry-over reduces advertising expenditures and increases profit in this model.

The explanation for the observations and discussion in the preceding part of this section stems from Figures 10 and 17. Dollar sales, profit plus advertising cost, and profit in Figure 17 retain their general shape as in Figure 10. However, they have been displaced to the left by an amount equal to carry-over advertising from the four previous periods. The point of maximum profit will still occur at the point where the slope of the profit plus advertising

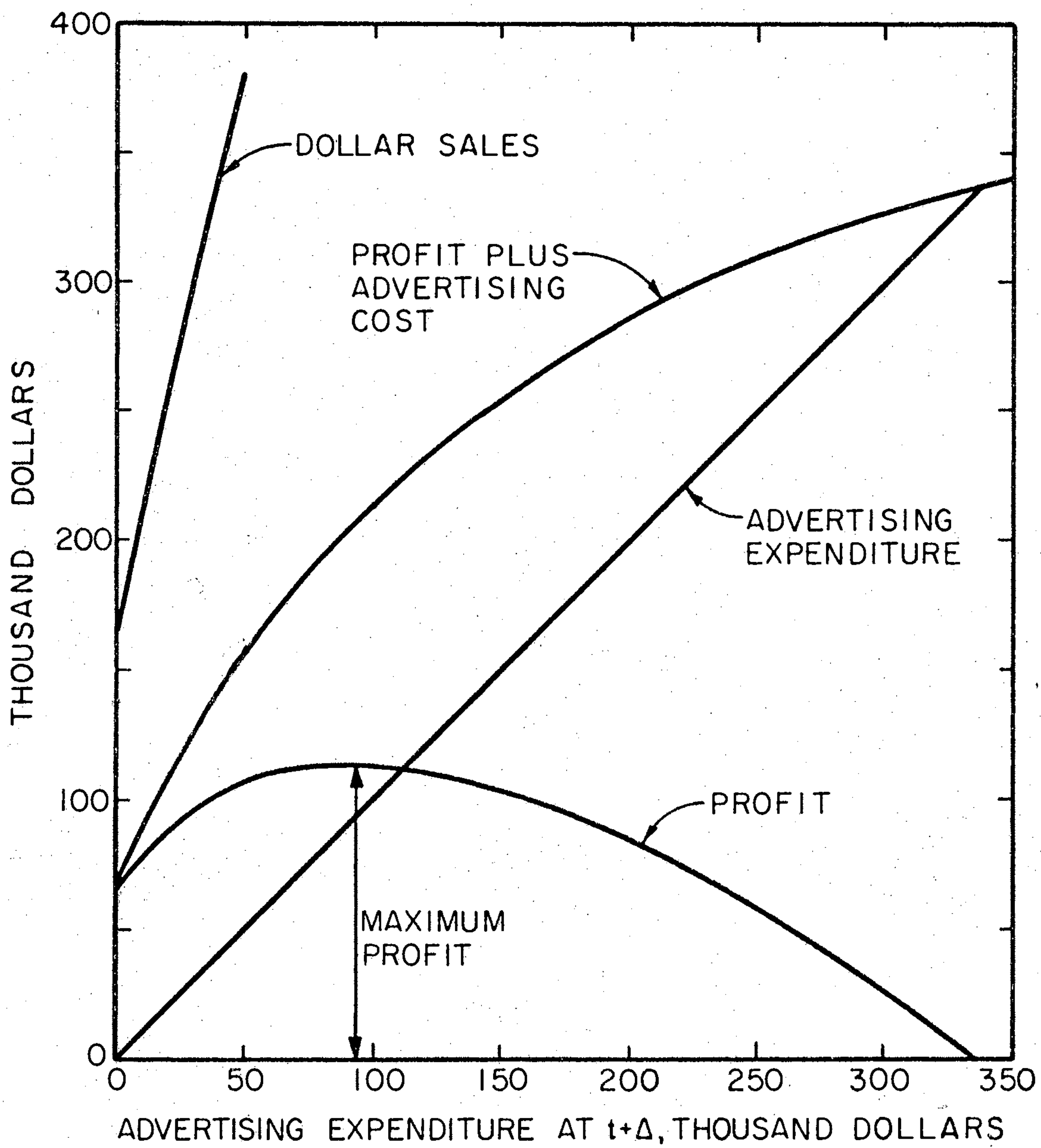

Figure 17. Dollar Sales, Profit Plus Advertising Cost, and Profit Versus Advertising Expenditure at t + Δ: With Advertising Carry-Over

cost line equals that of the advertising expenditure line. Since the profit plus advertising cost line has merely been displaced by an amount equal to the carry-over, its shape remaining intact, the optimal advertising will be less by an amount equal to advertising carry-over. The optimal profit will therefore be increased an equal amount. If the slope of the profit plus advertising cost curve is never equal to that of the advertising expenditure curve, it does not pay to increase advertising spending, the optimal level being zero.

From the above influence on spending and profit levels, advertising carry-over is seen to be a major factor in Model I. This characteristic, though, has two other significant effects as shown from the equilibrium figures. Consider Figures 15 and 16 where q_h is held fixed. Notice that the higher value of $b_h = 0.5$ causes relative advertising to slightly lead the projected potential demand (the equilibrium curve for $b_h = 0.0$, $q_h = 0.0$ is in-phase with the potential demand). Note also the slight increase of the relative amplitude for the higher value of carry-over. The slight change of phase and increase in relative amplitude with higher carry-over has also been noted by Kuehn (1967).

These effects may be logically explained by considering a transition from high to low potential demand. Due to carry-over, the firm significantly reduces spending (especially at low values of retention) during a demand low while expecting carry-over to supply a sizeable share of the composite advertising. After demand begins to increase, the firm must spend relatively more money due to the small amount of carry-over remaining from the previous few periods.

Retention Buying

Retention or habitual buying also has a great bearing on equilibrium advertising, but in a way different from that of advertising carry-over. Consider Figures 13 and 14. For a given level of b_h notice the degree of closeness in the average level of equilibrium advertising spending and profit over widely varying values of q_h. In fact, in Figure 13, the yearly totals of equilibrium advertising are the same for a given value of b_h, regardless of q_h. A similar statement can be made about yearly profit. The same cannot be said for the yearly advertising and profit totals in Figure 14. The reason relates to the variation of demand with industry advertising and it will be discussed in the next section.

From the above observations, it can be said that in the absence of demand as a function of industry advertising, the total yearly equilibrium advertising for identical competitors will be constant for a given value of carry-over regardless of the value of retention. A similar comment applies to equilibrium profit. Intuitively, one can see that in this case each firm could agree on a much lower level of advertising and thereby increase profit. However, such an act would be one of collusion. Further, one brand could then increase profit by higher advertising. Such a circumstance would violate the definition of competitive equilibrium.

Equilibrium advertising and profit for non-identical firms will not be constant for a given value of carry-over regardless of the value of retention. This can be seen by examining equation (4.14) which is repeated here for convenience:

$$X = \left[1-r_h\right]\left[\sum_{g=1}^{n}(1-q_g)c_g(t)\right]\left[\text{DEMAND}\right] . \qquad (4.29)$$

Actually, this term can be thought of by Brand h as "potential profit" before advertising and before subtracting the competitor's share. If q_h for Brand h remains constant, DEMAND (equation (4.9)) will remain constant. The term $1-r_h$ is fixed. The middle term will vary if $q_{g \neq h}$ varies. Assume only two competitors, h = 1 and g = 2. If $q_1 = q_2 = .5$, and the competitors are identical and at equilibrium, the center component would be

$$\sum_{g=1}^{2}(1-q_g)c_g(t) = [1-.5][.5]+[1-.5][.5] = .5 . \qquad (4.30)$$

If q_2 falls to .0, and if Brand 2 continues to advertise at all (so as to keep $c_1(t) < 1$),

$$\sum_{g=1}^{2}(1-q_g)c_g(t) = [1-.5]c_1(t)+[1-.0][1-c_1(t)]$$

$$= -.5c_1(t)+1 > .5 \qquad 0 \leq c_1(t) < 1 . \qquad (4.31)$$

Correspondingly, if q_2 increases to .9,

$$\sum_{g=1}^{2}(1-q_g)c_g(t) = [1-.5]c_1(t)+[1-.9][1-c_1(t)]$$
$$= .4c_1(t)+.1 < .5 \qquad 0 \leq c_1(t) < 1 . \tag{4.32}$$

In words, if the competitor's retention decreases, X of equations (4.14) and (4.29) will increase causing higher "potential profit" for Brand h and therefore one should expect higher Brand h advertising and profits when using Model I. The converse is true if the competitor's retention increases.

Again consider equations (4.14) and (4.29) for the case of a change of $q_{h=1}$ while $q_{g=2}$ remains constant. Now, DEMAND will also vary as does q_1 because, when h = 1,

$$\text{DEMAND} = S(t)+q_1S(t+1)+q_1^{2}S(t+2)+\ldots . \tag{4.33}$$

Initially, assume that

$$S(t) = S(t+1) = S(t+2) = \ldots = S(\cdot) . \tag{4.34}$$

Then,

$$\text{DEMAND} = \frac{1}{1-q_1}S(\cdot) \tag{4.35}$$

and

$$X = \left[1-r_1\right]\left[\sum_{g=1}^{2}(1-q_g)c_g(t)\right]\left[\frac{1}{1-q_1}\right]\left[S(\cdot)\right] . \tag{4.36}$$

Now, consider the components

$$\left[\sum_{g=1}^{2}(1-q_g)c_g(t)\right]\left[\frac{1}{1-q_1}\right]$$

where $q_1 = q_2 = .5$ and Brands 1 and 2 are identical competitors at equilibrium.

$$\left[\sum_{g=1}^{2}(1-q_g)c_g(t)\right]\left[\frac{1}{1-q_1}\right] = \left([1-.5][.5]+[1.5][.5]\right)\left(\frac{1}{1-.5}\right) = 1 \; . \tag{4.37}$$

If q_1 increases to .9 while q_2 remains constant at .5,

$$\left[\sum_{g=1}^{2}(1-q_g)c_g(t)\right]\left[\frac{1}{1-q_1}\right] = \left([1-.9]c_1(t)+[1-.5](10c_1(t)]\right)\left(10\right)$$

$$= [-.4c_1(t)+.5][10] > 1$$

$$0 \leq c_1(t) < 1 \; . \tag{4.38}$$

If q_1 falls to .0 while q_2 remains at .5,

$$\left[\sum_{g=1}^{2}(1-q_g)c_g(t)\right]\left[\frac{1}{1-q_1}\right] = \left([1-0]c_1(t)+[1-.5][10c_1(t)]\right)\left(1\right)$$

$$= .5c_1(t)+.5 < 1 \qquad 0 \leq c_1(t) < 1 \; . \tag{4.39}$$

In words, if potential demand is constant such that (4.34) holds and if Brand h has an increase in retention while the competition's retention factor remains constant, the term X of equations (4.14) and (4.29) will increase causing higher "potential profit" for Brand h. Therefore, in this case, one should expect higher Brand h advertising and profits when using Model I. The converse holds if Brand h retention decreases.

Note, however, in the general case of varying potential demand one cannot use the geometric expansion term in equation (4.35). As such, X is described as in equations (4.14) and (4.29) with DEMAND as described in equations (4.9) and (4.33). An increase in $q_{h=1}$ will have a definite tendency to increase X, the "potential profit" of Brand h. Yet, if the pattern of S(t), S(t+1), S(t+2), ... is decreasing steeply enough, X may decrease. A decrease in q_h may be accompanied by an increase in X if S(t), S(t+1), S(t+2), ... is increasing rapidly. These possibilities should not occur unless changes of $q_{h=1}$ are being examined in conjunction with relatively extreme potential demand fluctuations.

Considering Figures 15 and 16, it can be seen that the remaining effects of retention buying at equilibrium are twofold: a vast decrease in relative advertising amplitude and a significant phase shift in which advertising leads potential demand. These two effects were also noted by Kuehn (1967).

The reason for the relative stability and phase shift of advertising at high levels of q_h is the ability of this retention factor to help "sense" the future potential demand. The higher the value of the coefficient q_h, the less variation there is in the DEMAND term of equation (4.9) as the DEMAND term is calculated starting at different periods along the potential demand curve. One can see that for high q_h values, the DEMAND term fluctuates very little over time. It can also be seen that if q_h is high and if future values of S(t+1), S(t+2), etc., are at a high point, DEMAND will peak ahead of the potential demand curve, thus producing the phase shift. These conclusions were reflected in the DEMAND values computed using the program of Appendix B.

Thus, it is significant to conclude that for identical competitors at equilibrium the general influence of retention buying is not in the total budget expended over the period of a cycle, but rather in the timing and allocation of that budget on a period by period basis. For non-identical competitors one can expect retention buying to significantly affect the timing and amplitude of relative allocations, as well as the total budget.

Industry Demand as a Function of Total Industry Advertising

The final characteristic to be examined is that of demand as a function of total industry advertising. Consider the differences between Figures 13 and 14. Notice that the average equilibrium advertising expenditure (over a year) at a given value of b_h is roughly ($\pm$10%) the same whether or not demand is a function of total industry advertising. Indeed, this is not the general case nor is the preceding a general statement. However, such a circumstance will help one envision the influence of this characteristic. That is, observed differences cannot be attributed to significantly different average advertising levels.

The first dramatic effect is the significant reduction in equilibrium profit. The second main effect is the increase in amplitude variations of both equilibrium advertising and profit. A related observation is the sharp downward peak on the lower lobe of the advertising expenditure patterns at the low value of $q_h = 0.0$ in Figure 14. The above two effects of variable demand with respect to total industry advertising are easily seen by comparing Figure 14 with Figure 13 (and Figure 16 with Figure 15). The reasons for such effects are not as apparent.

It must be remembered that when demand is a function of total industry advertising, the composite advertising expenditure does not merely determine the allocation of the potential brand-switching fraction. It also serves to determine the absolute volume of demand in terms of dollar sales. At working or usual levels of advertising, both of these functions are in a state of decreasing returns. The maximum profit occurs at the point at which incremental profit is zero with an incremental increase in spending. Therefore, depending upon the shape of the response curve (P(t)), there is a tendency for demand to never reach potential demand, thus reducing sales and profit potential from the case where demand equals potential demand (P(t) = 1.0) as considered in Figures 13 and 15.

Figure 9 shows P(t) as a function of advertising. Such a function is assumed to still be applicable. Figure 10 shows a related curve. Dollar sales and profit before advertising in this figure may be considered a fraction of the product of P(t) and DEMAND as shown by equation (3.25).

As such, if at any time period the DEMAND term approaches a low enough value that profit before advertising falls below advertising expenditure, profit decreases to zero as does the optimal advertising expenditure. In such an instance it can be said that the level of demand and the response for the product class, in conjunction with cost and other parameters, will not sustain Brand h in the market. The reason for the relative stability of advertising and profit in Figure 14 for high values of retention is that the DEMAND term is rather stable over time and it is easily high enough to justify reasonably high and stable equilibrium advertising expenditures each period. On the other hand, for low values of retention, the DEMAND term is varying with much greater amplitudes, the lower of which are relatively close to the point at which advertising is not feasible (note the curves in Figure 14 for $b_h = 0.0$ and $q_h = 0.0$). The relatively higher equilibrium expenditures on the upper lobe of the advertising curves at $q_h = 0.0$ are due to the increased expenditures necessary as a result of the small carry-over advertising from "low expenditure" recent periods.

Summary

In this chapter, a model containing the mathematical development from Chapters II and III was completed as shown in equations (4.5), (4.6) and (4.7). Such a model can be used to maximize profits, as defined previously, through selection of the optimal advertising expenditure at each time period.

The model was used here to find the competitive equilibrium advertising expenditures under various assumptions and parameter values for the purpose of determining much of the influence of the four characteristics discussed in detail in Chapter III. The major findings concerning the four characteristics relating to Model I are as follows:

1. Demand as a Function of Time
    a. Causes equilibrium advertising and profit patterns to assume the same general shape as demand.
    b. If all other characteristics are negated, phase, relative amplitude, and shape are identical to that of demand at equilibrium.
2. Advertising Carry-Over
    a. Causes extreme magnitude differences in advertising and profit levels. With higher carry-over less new advertising is needed and profit is thereby increased.
    b. Causes slight increase in relative amplitude of advertising pattern at higher values of carry-over.
    c. Causes advertising to slightly lead potential demand.
3. Retention Buying
    a. Has a tendency to maintain the same total-cycle

equilibrium advertising and profit values for a given value of b_h when considering identical competitors, regardless of the value of q_h.

b. Can be expected to cause higher advertising and profits if the retention factor of competition drops. The converse is true if competitor's retention increases.

c. Can be expected to cause higher advertising and profits if the retention factor of Brand h increases, provided sharply decreasing potential demand does not negate this tendency. The opposite is true if q_h decreases.

d. Causes widely fluctuating intra-cycle advertising allocations and corresponding profit fluctuations at low values of q_h.

e. Causes advertising expenditures to considerably lead potential demand at high values of q_h.

4. Demand as a Function of Total Industry Advertising Expenditure

a. Causes higher relative advertising and profit fluctuations including distortion of the potential demand curve shape--especially in the steepest range of the response curve P(t).

b. Has influence such that it lends support and example to the belief that consumer response should be at a level to sustain profitable activity in order to merit investment of advertising money.

Chapter VI will consider the non-equilibrium case in which many of the above findings will be reconfirmed in terms of a general (not an identical) competitor. Also, Chapter VI will provide an opportunity to compare the above findings with those of the model to be developed in Chapter V, Model II.

FOOTNOTES

[1]Norman H. Barish, Economic Analysis (New York, 1962), p. 606.

[2]Alfred A. Kuehn, "How Advertising Performance Depends on Other Marketing Factors," Managerial Marketing, ed. Eugene J. Kelley and William Lazar (3rd ed., Homewood, Illinois, 1967), p. 564.

CHAPTER V

DEVELOPMENT AND ANALYSIS OF A PROFIT ORIENTED MODEL CONSIDERING PAST AND FUTURE ADVERTISING EXPENDITURES

The objective of this chapter is to develop and analyze a profit oriented advertising model (or profit model) which is an extension of Model I developed in Chapter IV. The major difference between the model developed here, to be referred to as Model II, and that in Chapter IV is in the usage, in this model, of estimates of future competition expenditures.

The mathematical model of equation (4.5) is written such that by finding the optimal advertising expenditure, profit, as defined, can be maximized. That model considers advertising carry-over from previous periods as well as retention buying into future periods. Provision is also made for demand to vary as a function of total industry advertising. However, it does not consider the future carry-over effect of advertising at time $t + \Delta$.

Theoretically, advertising carry-over and retention buying may continue into many future periods. In order to optimize advertising at $t + \Delta$ considering carry-over from previous advertising, retention buying, and future carry-over effects from advertising at $t + \Delta$, one must determine advertising so as to maximize profits over the present and all future periods. Such a model will be developed here.

Development of the Model

Profit was defined in Chapter IV to be

> ... present and future (due to retention buying) income attributable to effective advertising, including carry-over, at $t + \Delta$, less present and future costs of related production and overhead, less the actual advertising expenditure at $t + \Delta$, $a_g(t)$.

Profit will continue to assume the same basic meaning, only now the effect of carry-over advertising from $a_g(t)$ on future periods will be included. Profit will be defined to be present and future (due to retention buying) income attributable to effective advertising, including carry-over, at $t + \Delta$, $t + 1 + \Delta$, $t + 2 + \Delta$, ... , less present and future costs of related production and overhead, less the actual advertising expenditures at $t + \Delta$, $t + 1 + \Delta$, $t + 2 + \Delta$, ..., $a_g(t)$, $a_g(t+1)$, $a_g(t+2)$, ... The only variable to be manipulated will be advertising at $t + \Delta$, $a_g(t)$. Defining profit in such a way will indeed let one consider the influence of carry-over advertising from $t + \Delta$. However, it can also be seen that since $a_g(t)$ is the only variable to be manipulated, one must estimate not only the future advertising of competitors, but also his own. At first consideration, such a model does not appear practical in that competitor's moves are not easily predicted. This point will be considered in Chapter VII.

Also, it would appear that the decision maker has gone to great lengths to optimize $a_g(t)$ while committing himself to estimated, non-optimal, future expenditures. This handicap can be overcome as will be discussed in this and future chapters.

As described above profit can be written, using the same terminology as in Chapter IV, as follows:

$$\text{PROFIT} = \text{All Present and Future Dollar Sales} - \text{All Present and Future Related Costs Exclusive of Advertising} - \text{Cost of Present and Future Advertising} . \quad (5.1)$$

Equation (4.3) can then be expanded as:

$$\begin{aligned}\text{PROFIT} &= N(A_{t,t-1,\ \ldots})S(1-C/S) - A_t \\ &+ N(A_{t+1,t,\ \ldots})S(1-C/S) - A_{t+1} \\ &+ N(A_{t+2,t+1,\ \ldots})S(1-C/S) - A_{t+2} \\ &+ \ldots \\ &= \sum_{t'=t}^{\infty} N(A_{t',t'-1,\ \ldots})S(1-C/S) - A_{t'} \qquad (5.2)\end{aligned}$$

where all terminology remains as defined in Chapter IV. Expanding the terminology of equation (4.5) one can write the following equation for Brand h in competition with n-1 other firms.

$$\begin{aligned}\text{PROFIT} &= \sum_{t'=t}^{\infty} \pi_{h,t'+\Delta} \\ &= (1-r_h) \sum_{t'=t}^{\infty} \left[\sum_{t''=1}^{\infty} DS_{h,t'+\Delta}(t+t'') - a_h(t') \right]\end{aligned}$$

$$= (1-r_h)\left[\sum_{g=1}^{n}(1-q_g)c_g(t)f_{gh}(t)\right][P(t)]\cdot$$

$$[S(t)+q_hS(t+1)+q_h{}^2S(t+2)+\ \ldots] - a_h(t)$$

$$+ (1-r_h)\left[\sum_{g=1}^{n}(1-q_g)c_g(t+1)f_{gh}(t+1)\right][P(t+1)]\cdot$$

$$[S(t+1)+q_hS(t+2)+q_h{}^2S(t+3)+\ \ldots] - a_h(t+1)$$

$$+ \ldots \qquad - a_h(t+2)$$

$$+ \ldots \qquad - a_h(t+3)$$

$$\vdots \qquad (5.3)$$

where

$$P(t) = \left[{}_D{}^{U}_{S} \sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g{}^2 a_g(t-2) + \ldots]\ {}_{D'}\right] \Big/ [1 - D'] \qquad (5.4)$$

and

$$f_{gh}(t) = \frac{a_h(t) + b_h a_h(t-1) + b_h{}^2 a_h(t-2) + \ldots}{\sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g{}^2 a_g(t-2) + \ldots]}\ . \qquad (5.5)$$

In words,

$\sum_{t'=t}^{\infty} \pi_{h,t'+\Delta}$ = the total present and future income to Brand h from dollar sales attributed to composite advertising at $t + \Delta$, $t + 1 + \Delta$, $t + 2 + \Delta$, ... , less all associated costs of production, overhead, and distribution, less the advertising expenditures of Brand h at time $t + \Delta$, $t + 1 + \Delta$, $t + 2 + \Delta$, ...

The profit equation of (5.3) does consider the past advertising levels of all firms. Also, it accounts for future retention sales given projected potential demand. Further, it considers the influence of total industry advertising upon industry demand. Finally, the main difference between Model II and Model I is that the future influence of carry-over from advertising at $t + \Delta$ is considered in Model II.

Explanation of Computer Analysis

In order to observe the model under the influences of a variable market as a function of time and industry advertising, retention buying, and advertising carry-over, it is desired to search, utilizing the computer, for the optimal Brand h (a specific brand) advertising expenditure when in competition with one other brand. In order to determine the optimal $a_h(t)$, the past and future advertising history of the competitor will have to be assumed. For the same reasons presented in Chapter IV, an identical competitor, with identical parameter values as well as access to this model, will be used. Therefore, this model will first be considered in terms of equilibrium advertising expenditures and profits. Again, a more typical example will be presented in Chapter VI in which two non-identical competitors, one using this model, one using a rule of thumb, will be considered.

Search for Equilibrium

The same general method as described in Chapter IV -- initializing, incrementing, equating, etc., -- will be used to determine the equilibrium advertising expenditure and profit in this model.

The program to be used for the analysis in this chapter is shown, with explicative comment cards, in Appendix D. A sample output is also presented.

Successive Monthly Analysis

As in the previous chapter, a method is now needed to determine the optimal level of advertising expenditure over several successive months. The retention buying and carry-over effects are theoretically infinite in nature. Chapter III and Appendix A develop the DEMAND term (equation (4.9)) such that it reflects retention buying over all time. In Chapter IV, advertising carry-over from only the four previous periods was considered due to the typical range of $0 \leq b_g \leq .5$ per month. Now, by the same reasoning, it is desirable to include carry-over from advertising at $t + \Delta$ only four periods into the future. Not only does this make the use of the profit model in equation (5.3) feasible, but it is also practical and reasonable to truncate all remaining terms due to the rapidly diminishing effect of carry-over in the range $0 \leq b_g \leq .5$.

Figure 18 shows a pictorial view of the timing used in the computer program of Appendix D. Notice that each time to be considered, $5 + \Delta$, $6 + \Delta$, $7 + \Delta$, $8 + \Delta$, and $9 + \Delta$, has as inputs the four previous period's advertising reduced by B, B^2, B^3, and B^4 (b_g, b_g^2, b_g^3, and b_g^4). Also notice that the optimal amount to be determined at time $5 + \Delta$ is carried over four periods into the future, further period effects deemed negligible. There is no need to consider advertising at $10 + \Delta$ because carry-over advertising from time $5 + \Delta$ is truncated. Thus, to include expenditures at $10 + \Delta$ or further into the future has no affect on determining the optimal A(5).

As in the model of Chapter IV, if the original sine-wave demand is assumed to be stable from year to year, and if the equilibrium expenditures are to be determined each month, a starting place must be determined. If the advertising has a carry-over factor $b_g > 0$, then the equilibrium expenditures of the past four periods as well as the future four periods must be known. If these expenditures are unknown (or yet to be determined) and must be estimated, obviously the "equilibrium" advertising at $5 + \Delta$ will be subject to change as the true past and future equilibrium expenditures are determined. Reconsider the drawing in Figure 12. Using this model, the expenditures at the beginning of Months 9, 10, 11, 12, 2, 3, 4, and 5 must be estimated in order to determine the expenditure at the beginning of Month 1. At the next period, the expenditures at

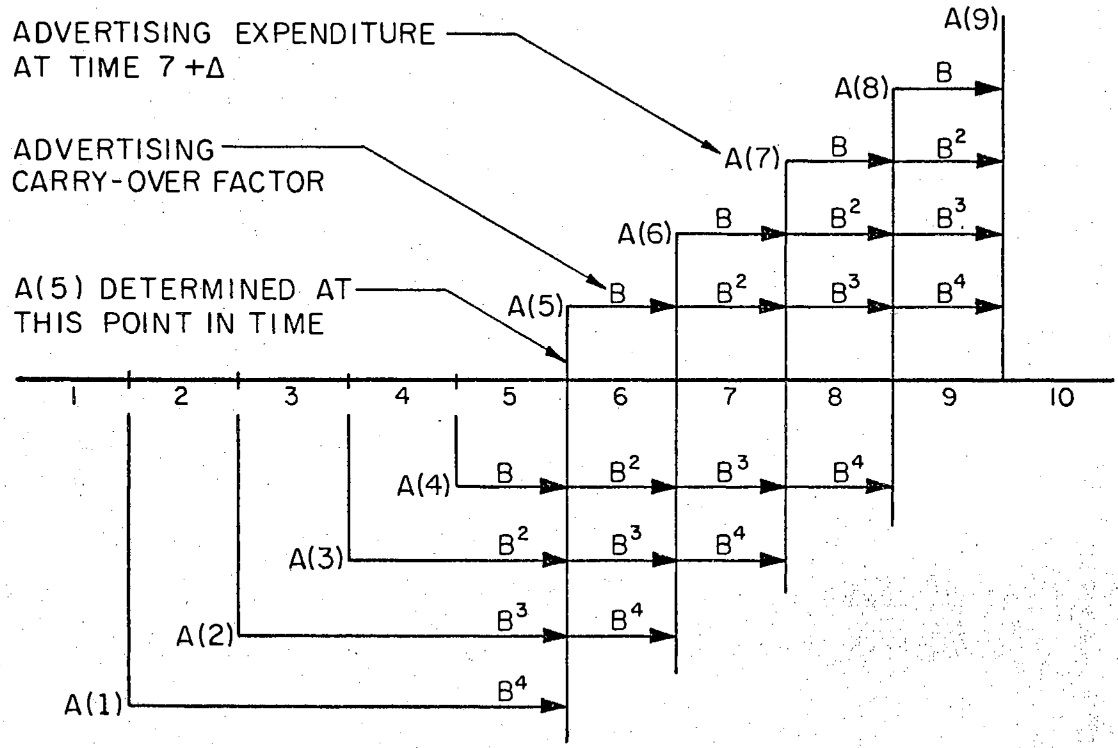

Figure 18. Timing Used in Computer Program of Chapter V (Appendix D)

the beginning of Months 10, 11, 12, 3, 4, 5, and 6 must be estimated. As the cycle progresses, period-by-period around the circle, the influence of the estimates of equilibrium spending at Months 9, 10, 11, and 12 diminish. However, during nearly the entire first year, the four future expenditures are only estimates and do have considerable influence. It is hypothesized that as successive advertising expenditures are found, and as the yearly cycle is repeatedly simulated, the effect of the initial estimates will diminish and the expenditures will approach a repeating equilibrium pattern.

It should be mentioned that if $b_1 = b_2 = \ldots = b_g = \ldots = b_n = 0$, Model II essentially reverts to Model I in that identical equilibrium values are obtained. Of course, if this is the case, only one pass through the twelve month cycle will determine the equilibrium expenditures. The above statements are not general in nature as they do not apply to non-identical competitors.

Results and Findings at Equilibrium; Mathematical Analysis

The results of the computer determination of equilibrium advertising expenditure will now be discussed. A mathematical analysis will be implemented when the findings from the "identical competitor" equilibrium studies do not also apply to a "non-identical competitor." As in the previous chapter, all combinations of $b_g = 0.0$, 0.5 and $q_g = 0.0$, 0.9

were examined. Potential demand was the sine pattern used throughout and cost before advertising per dollar of sales was set at $r_g = .6$. Demand was also considered variable as well as fixed with respect to total industry advertising.

Again the theory that the effect of the initial estimates would diminish was correct. With no carry-over, one pass through each of the twelve months determined the equilibrium pattern. On the other hand, as b_g was increased to .5, the lingering effect of the estimation of future expenditures was extensive. For $b_g = 0.5$ it took about six passes through the twelve month circle before the yearly pattern had converged to cycling equilibrium advertising values.

Table III lists the equilibrium advertising values, monthly spending as a fraction of the total yearly budget, and the associated profits as determined for each of the two identical competitors. The profits shown are those due only to composite advertising at $t + \Delta$ although the computer program was written in terms of profit as defined in this chapter. The reason for this is to present profit from both models on an equivalent basis for comparison purposes. The advertising and profit results from Table III are shown graphically in continuous form in Figures 19 and 20. Figures 21 and 22 show plots of each month's relative advertising expenditure with respect to yearly spending. While the influences of the various empirically shown characteristics are much the same as those discussed in Chapter IV,

TABLE III

EQUILIBRIUM ADVERTISING, MONTHLY ADVERTISING AS A PERCENTAGE OF YEARLY BUDGET, AND CORRESPONDING PROFIT VALUES

	Demand not a Function of Total Industry Advertising				Demand as a Function of Total Industry Advertising			
Month	b_h=0.0 q_h=0.0	b_h=0.5 q_h=0.0	b_h=0.0 q_h=0.9	b_h=0.5 q_h=0.9	b_h=0.0 q_h=0.0	b_h=0.5 q_h=0.0	b_h=0.0 q_h=0.9	b_h=0.5 q_h=0.9
1	106140 8.85 106140	117060 9.75 95220	104640 8.72 104640	105420 8.79 104370	116390 9.88 22840	133410 9.59 61310	113730 9.04 21400	122140 8.72 69400
2	116780 9.73 116780	125680 10.47 107880	104490 8.71 104490	104140 8.68 104610	133800 11.36 33590	140800 10.12 77580	113460 9.01 21260	120900 8.63 70290
3	122920 10.24 122920	127440 10.62 118400	103130 8.59 103130	101750 8.48 103610	142960 12.14 40130	142600 10.25 89350	110990 8.82 19970	118560 8.46 69580
4	122920 10.24 122920	121770 10.16 124070	100930 8.41 100930	98890 8.24 101640	142960 12.14 40130	138010 9.92 94070	106890 8.49 17930	115710 8.26 67520
5	116780 9.73 116780	110500 9.20 123060	98490 8.21 98490	96330 8.03 99240	133800 11.36 33590	127560 9.17 91080	102120 8.11 15720	113060 8.07 64590
6	106140 8.85 106140	96280 8.01 116000	96450 8.04 96450	94750 7.90 97040	116390 9.88 22840	114100 8.20 81010	97980 7.78 13930	111350 7.95 61650
7	93860 7.82 93860	83110 6.91 104610	95360 7.95 95360	94580 7.88 95630	92440 7.85 11740	99550 7.15 67710	95690 7.60 13000	111110 7.93 59380
8	83220 6.94 83220	74330 6.19 92120	95510 7.96 95510	95860 7.99 95390	64670 5.49 3860	88850 6.38 53650	96020 7.63 13130	112400 8.03 58430
9	77080 6.42 77080	72400 6.04 81760	96870 8.07 96870	98250 8.19 96390	38500 3.27 580	85420 6.14 42290	98850 7.85 14300	114930 8.21 59000
10	77080 6.42 77080	78210 6.51 75950	99070 8.26 99070	101110 8.43 98360	38500 3.27 580	92390 6.64 35300	103270 8.20 16240	118000 8.42 60960
11	83220 6.94 83220	89540 7.46 76910	101510 8.46 101510	103670 8.64 100760	64670 5.49 3860	107810 7.75 34990	107980 8.58 18470	120410 8.60 64070
12	93860 7.82 93860	103850 8.65 83870	103550 8.63 103550	105250 8.77 102960	92440 7.85 11740	121160 8.71 46030	111770 8.88 20370	122050 8.71 67050

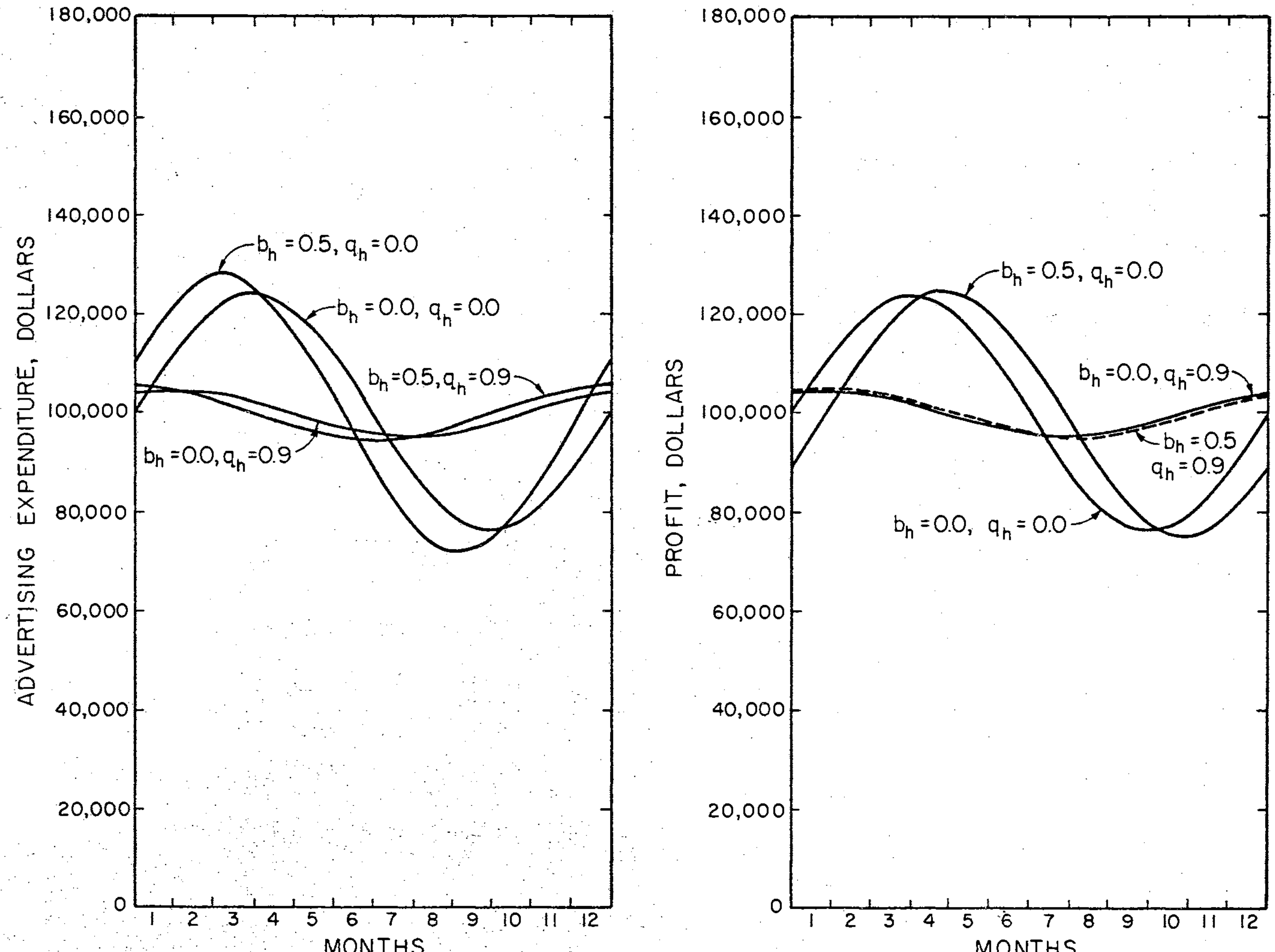

Figure 19. Model II Equilibrium Advertising and Profit: Demand not a Function of Total Industry Advertising

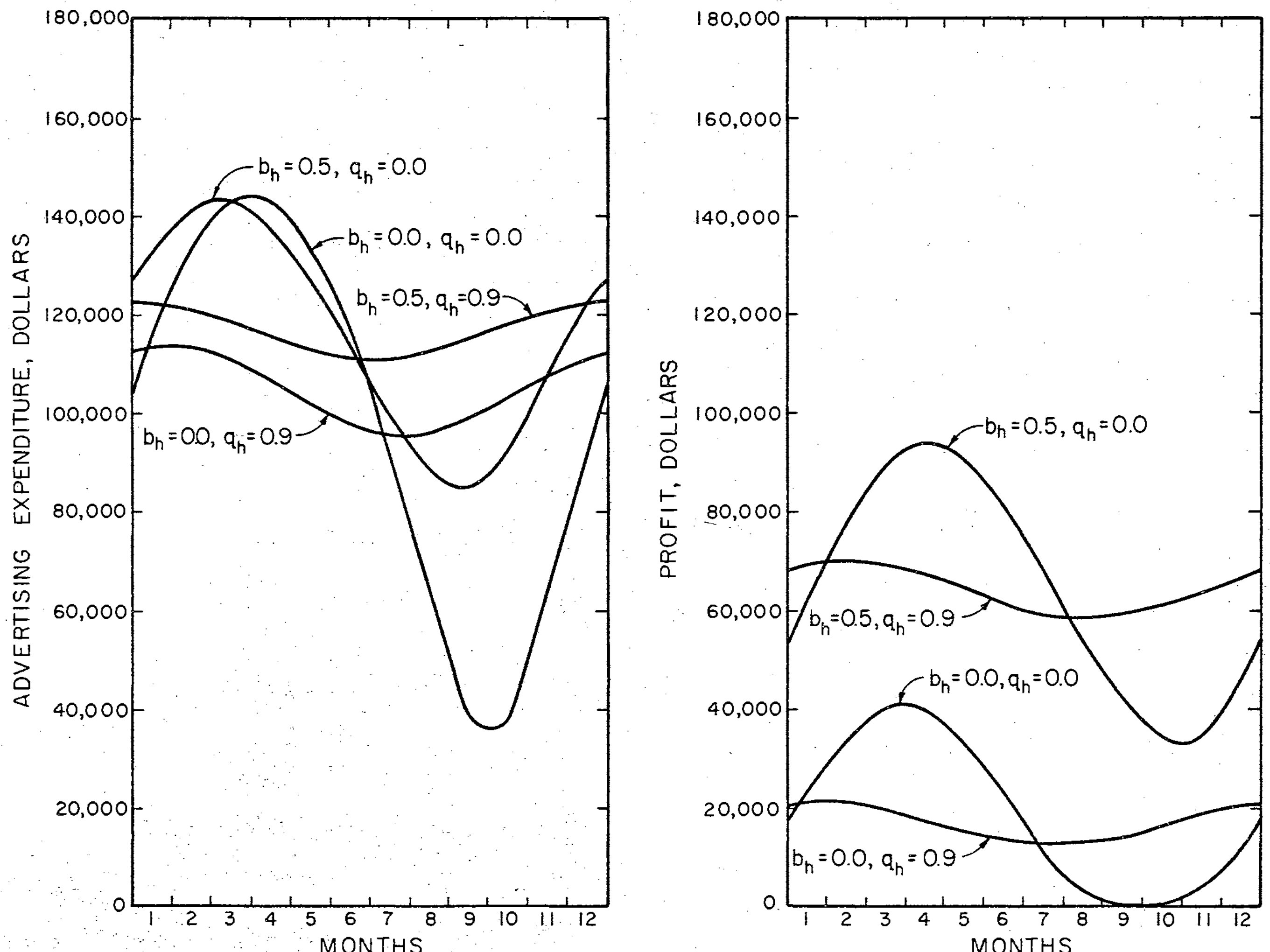

Figure 20. Model II Equilibrium Advertising and Profit: Demand a Function of Total Industry Advertising

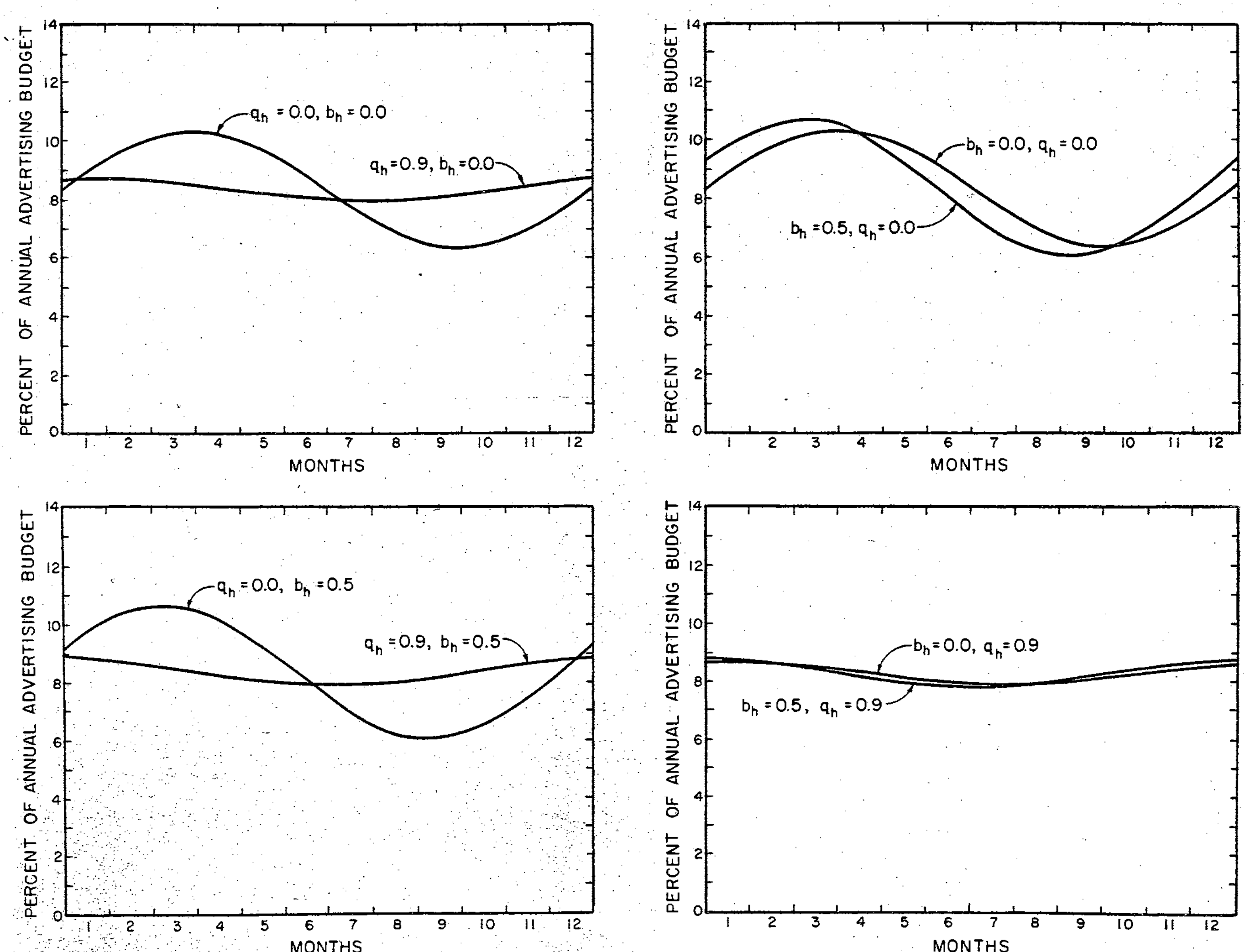

Figure 21. Model II Monthly Equilibrium Spending as Percentage of Yearly Budget: Demand not a Function of Industry Advertising

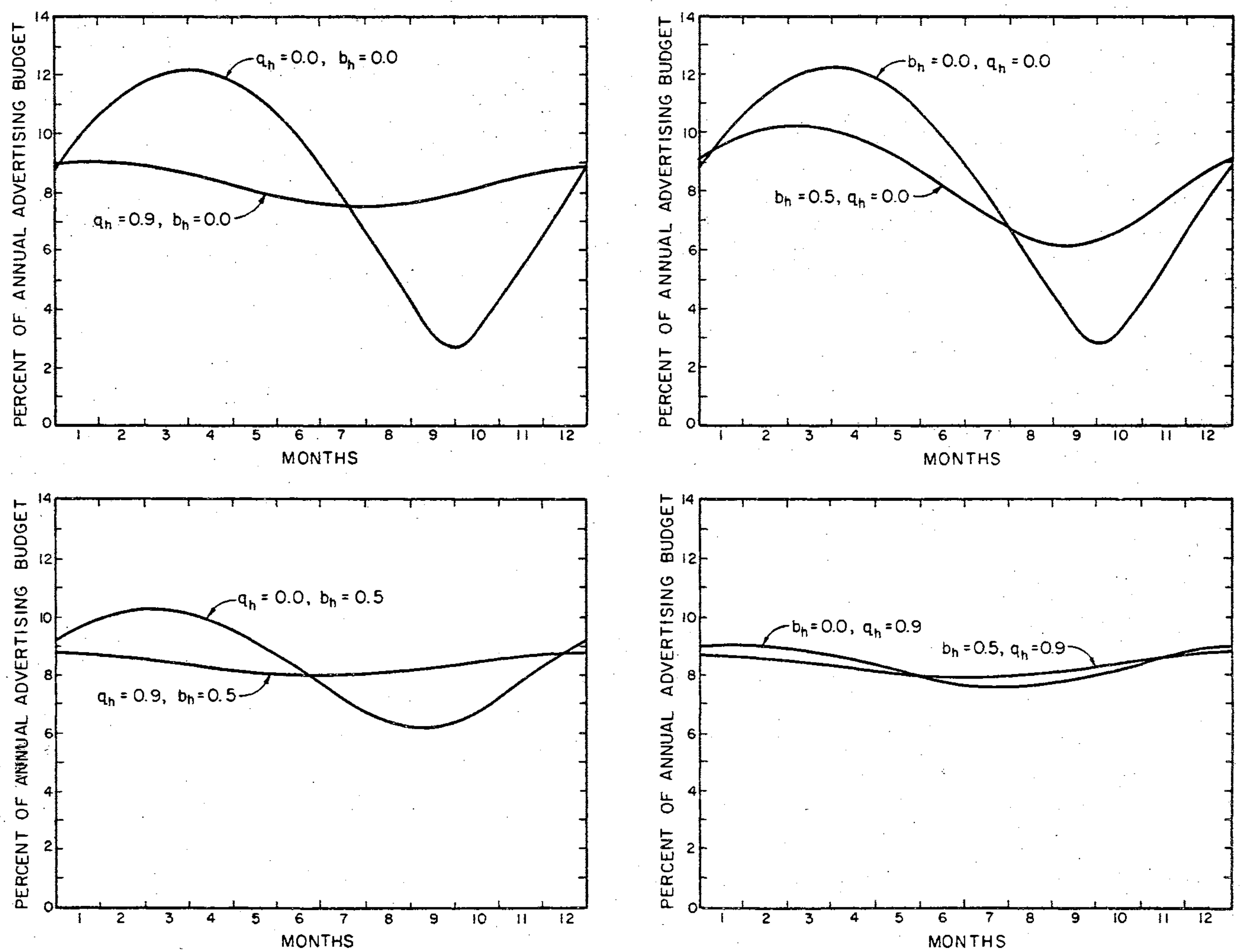

Figure 22. Model II Monthly Equilibrium Spending as Percentage of Yearly Budget: Demand a Function of Industry Advertising

there are some differences which merit discussion. Discussion of findings identical to those in Chapter IV will be held to a minimum to avoid repetition.

Demand as a Function of Time

The potential demand curve is again seen (in Figures 19 and 20) to lend its general sine-wave shape to equilibrium advertising expenditures and profits. The only advertising and profit pattern at equilibrium which is seen to maintain the exact relative amplitude, phase, and shape of potential demand in equation (4.28) is that in which $b_h = 0.0$, $q_h = 0.0$ and $P(t) = 1.0$. The effects of the carry-over and retention buying characteristics can be seen to alter the phase and amplitude of the advertising and profit in Figure 19. Demand as a function of total industry advertising is seen to distort the shape of the sine pattern in Figure 20.

Advertising Carry-Over

The equilibrium results using Model II can be seen to differ with the conclusions from the Summary of Chapter IV that:

2. Advertising Carry-Over
   a. Causes extreme magnitude differences in advertising and profit levels. With higher carry-over less new advertising is needed and thereby profit is increased (apparently contradicted

by Figures 19 and 20 where advertising levels are quite close regardless of the carry-over factor).

b. Causes slight increase in relative amplitude of advertising pattern at higher values of carry-over (apparently contradicted by Figure 22 where the lower value of carry-over has a larger relative amplitude).

The conclusion in part 2.a above was made in Chapter IV only after showing that Brand h could be considered in competition with a non-identical competitor with a fixed pattern of expenditures. Such a fixed pattern of advertising served as a stable basis for comparison of Brand h's expenditure patterns at two different levels of b_h. Consider the expenditure patterns of Figure 19 for Model II. Results of the analysis showed that for each of the four combinations of $b_h = 0.0$, 0.5 and $q_h = 0.0$, 0.9 the total yearly equilibrium expenditure was $1,200,000. In this case, however, if $b_h = 0.0$, the equilibrium composite advertising at each period was merely new advertising averaging $100,000 per month. If $b_h = 0.5$, the equilibrium composite advertising at each period was made up of carry-over from previous periods as well as an average of $100,000 per month of new advertising. Thus, it can be seen that the two equilibrium competitors cannot be considered as non-identical firms, one with a fixed pattern of advertising. As such, it cannot be said that finding 2.a above is contradicted by the results in Figures 19 and 20. However, result 2.a will be shown to be incorrect with respect to Model II.

An interesting phenomenon is expected to occur in the results of Chapter VI when a non-identical competitor is used. It is expected that the influence of advertising carry-over in Model II will differ from the result 2.a relating to Model I. The main question to be answered is whether higher values of advertising carry-over should motivate higher or lower spending. The following mathematical analysis will contribute toward answering that question.

Consider the drawings of Figures 10 and 17. These drawings are accurate in general shape as they were taken from computer print-out. Of major interest here is the shape of the profit plus advertising cost curve. Note that it increases at a decreasing rate. Such a curve asymptotically approaches the value

$$\left[1-r_h\right]\left[\sum_{g=1}^{n}(1-q_g)c_g(t)\right]\left[\text{DEMAND}\right]$$

which is seen to be the first term of the right hand side of equation (4.5) if $P(t) = 1.0$ and $f_{gh}(t) = 1.0$ (which will be the case as $a_h(t)$ approaches infinity). In other words, the general characteristics and shape of "profit plus advertising cost" are much like those of an increasing exponential curve. Due to the ease of working with an exponential function, one will be used as a construct to consider the influence of carry-over on the optimal level of advertising.

For the purpose of explanation, assume potential demand is constant. Further, let $P(t) = 1.0$ and let competitor's advertising be constant over time. Intuitively one can see that the optimal advertising level of Brand h will be constant each period. Also, it is easily seen that the profit plus advertising cost curve will remain the same each period. Let profit be defined as in equation (5.3) and be represented as

$$\sum_{t'=t}^{\infty} \pi_{h,t'+\Delta} = 1 - e^{-2z} - a_h(t) + 1 - e^{-2z} - a_h(t+1) \vdots \qquad (5.6)$$

where, since Brand h's optimal advertising is constant each month,

$$\ldots = a_h(t-1) = a_h(t) = a_h(t+1) = \ldots = a_h(\cdot) \qquad (5.7)$$

and

z = composite advertising of Brand h.

$$z = a_h(t) + b_h a_h(t-1) + b_h^2 a_h(t-2) + \ldots$$

$$= a_h(\cdot) \frac{1}{1-b_h} \quad . \qquad (5.8)$$

The term

$1 - e^{-2z}$ = the exponentially increasing form of "profit plus advertising cost."

Actually, since each of the terms in equation (5.6) is identical, one can maximize profit by finding the optimal value of "a" in

$$\pi = 1 - e^{-\frac{2a}{1-b}} - a \tag{5.9}$$

where the subscripts have been dropped for simplicity.

The object is now to find the optimal level of a.

$$\frac{\partial\pi}{\partial a} = \frac{2}{1-b}\, e^{-\frac{2a}{1-b}} - 1 = 0 \tag{5.10}$$

$$\frac{2}{1-b}\, e^{-\frac{2a}{1-b}} = 1$$

$$e^{-\frac{2a}{1-b}} = \frac{1-b}{2}$$

$$a = \frac{-(1-b)}{2} \ln \frac{1-b}{2} \quad . \tag{5.11}$$

As an example, consider solving for the optimal "a" when b = 0.0, 0.2, 0.4, 0.6, and 0.8. The values of z are composite advertising at each time period for each b value considered.

@ b = 0.0	a = .35	z = .35
@ b = 0.2	a = .37	z = .46
@ b = 0.4	a = .36	z = .6
@ b = 0.6	a = .32	z = .8
@ b = 0.8	a = .23	z = 1.15

Figure 23 presents the profit plus advertising cost curve as a function of composite Brand h advertising, z. The points at which the z axis is intersected by the various advertising expenditure lines represent the carry-over advertising,

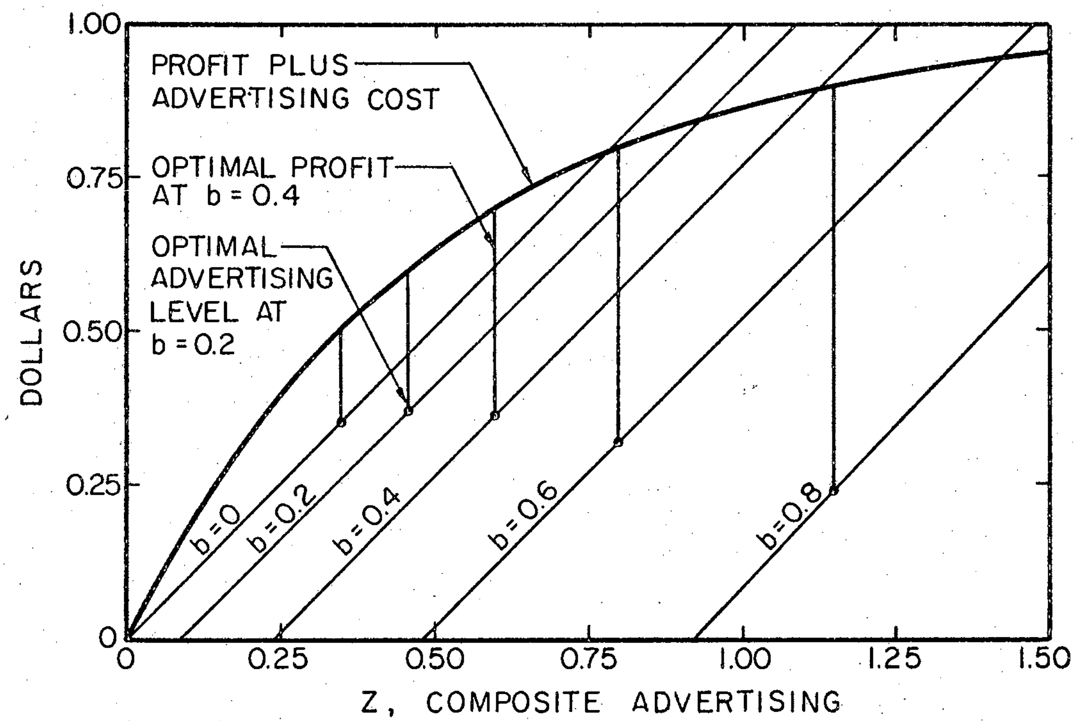

Figure 23. Profit Plus Advertising Cost, Optimal Advertising, and Optimal Profit at Selected Values of Advertising Carry-Over

respectively, at each of the levels of b shown above. For example, note that for b = .4, the optimal value of "a" determined yields a = .36. The carry-over advertising from previous periods amounts to .24 (i.e., $\frac{ab}{1-b} = \frac{(.36)(.4)}{(.6)} = .24$). The optimal composite amount is therefore .6 (i.e., $\frac{a}{1-b} = a + \frac{ab}{1-b} = .6$).

The important point is the slight increase in optimal advertising followed by a gradual decline in the optimal level as carry-over increases. Of course, throughout the range of carry-over the profit continues to increase. Also, unlike Model I, the optimal level of composite advertising does not recur at the point where the slope of "profit plus advertising cost" equals 1.0. Further, note that in the range $0 \leq b \leq .6$ the value of b makes little difference in the optimal advertising.

Although this discussion has been presented using hypothetical assumptions, the influence of the carry-over factor should have a similar bearing on optimal advertising even when the restrictive assumptions are removed. Thus, one might expect in the results of Chapter VI (using a non-identical competitor) to find relatively close spending levels over the range $0 \leq b_h \leq .5$ when using Model II. Profit, of course, should increase as carry-over increases.

Such a discussion as presented above will hopefully be a considerable contribution in helping to resolve the long unanswered question of whether to spend more money or less money as carry-over increases.

The apparent contradiction to part 2.b stems from Figure 22. Notice the two graphs in which q_h is held constant. It is quite obvious that the curves for high carry-over, $b_h = 0.5$, have considerably smaller relative amplitudes than the curves for $b_h = 0.0$. This is directly opposed to the conclusion of part 2.b above. Actually, the conclusion of part 2.b remains valid for this model as can be seen in Figure 21. The apparent contradiction relates to the following conclusion from the Chapter IV Summary:

4. Demand as a Function of Total Industry Advertising Expenditure

   b. Causes higher relative advertising and profit fluctuations including distortion of the potential demand curve shape -- especially in the steepest range of the response curve P(t).

At the higher level of advertising carry-over ($b_h = 0.5$) there is a higher level of composite advertising. The response curve value P(t) operates at the high and narrow (due to the flatness of P(t) in this region) range of values from .82 to .94 when $q_h = 0.0$. With no carry-over ($b_h = 0.0$), the response curve varies from .25 to .75 and causes large variations in dollar sales as can be seen from equation (3.25). The higher degree of stability at $b_h = 0.5$ as opposed to $b_h = 0.0$ is a result of the high level of composite advertising and the correspondingly high and stable values of P(t) which can be sustained with high carry-over. It can now be concluded that the apparent contradiction to 2.b is a result of the dominating effect of the variable demand as a function of industry advertising.

In all cases, the higher values of advertising carry-over cause advertising to lead potential demand slightly.

Retention Buying

Retention or habitual buying in this model retains many of the conclusions of Chapter IV regarding retention buying. There is still a tendency to maintain constant total-cycle equilibrium advertising and profit levels at any value of retention for identical competitors at a given value of carry-over. For example, from Figure 19, if $b_h = 0.0$, both competitors spend $1,200,000 yearly whether they are both at $q_h = 0.0$ or $q_h = 0.9$. A similar example can be shown if $b_h = 0.5$. Figure 20 shows a like tendency but demand as a function of industry advertising causes much lower spending during periods of low potential demand. High values of retention cause optimal advertising to lead potential demand by a large margin. Low values of retention cause widely fluctuating relative advertising and profit amplitudes.

At this point the exact likeness between the two models stops with respect to the retention factor. In Chapter IV it was shown that increasing a brand's retention would cause higher "profit potential" and thus higher advertising and profits. That explanation is certainly still applicable for the "present period" model of Chapter IV. Model II, however, takes into consideration the future influence of advertising. Therefore, the profit over five periods (the present and four future periods) is maximized here. The effect on advertising of varying q_h in this model will now be discussed. The following mathematical analysis contributes toward answering the question as to whether a higher retention factor motivates higher or lower optimal advertising. It will be seen that while a higher retention factor tends to cause higher "potential profit" (as explained in Chapter IV), there is another factor introduced which tends to result in lower spending. That discussion will be preceded by a short presentation of the mechanics of optimizing advertising using this model.

Consider equation (5.3). The first term is as follows:

$$\text{FIRST TERM} = \left[1-r_h\right]\left[\sum_{g=1}^{n}(1-q_g)c_g(t)f_{gh}(t)\right][P(t)] \cdot [\text{DEMAND}] - a_h(t) \qquad (5.12)$$

where DEMAND is as shown in equation (4.9) and

$$f_{gh}(t) = \frac{a_h(t) + b_h a_h(t-1) + b_h^2 a_h(t-2) + \ldots}{\sum_{g=1}^{n}[a_g(t) + b_g a_g(t-1) + b_g^2 a_g(t-2) + \ldots]} . \qquad (5.13)$$

FIRST TERM represents profit during the present period and is identical to Model I. Consider the increase in advertising $a_h(t)$ up to the point of maximum profit considering only this period, i.e., utilizing only FIRST TERM. Figure 17 is applicable and FIRST TERM is seen to be maximum where the slope of "profit plus advertising cost" equals 1.0. That is, of course, at the point where an increment of spending will yield an equal increment of profit before advertising -- the net gain being zero. Now, consider equation (5.3) in its entirety. If $a_h(t)$ is incremented upward by $1, the value of the first component of the first term will, say, increase by $.98 resulting in a net FIRST TERM loss of $.02. However, if $b_h = .25$, $f_{gh}(t+1)$ will be slightly increased causing perhaps an increase in SECOND TERM of $.10. The third, fourth, and fifth terms will also increase slightly. As $a_h(t)$ grows larger, however, the profit plus advertising cost curves approach a flatter region. A point is eventually reached at which the decrease in profit from FIRST TERM is not matched by at least a corresponding gain in future terms. It would therefore appear that the optimal level of advertising using this model is always above that of the one period model of Chapter IV.

Now, why may one expect that under certain conditions a higher retention buying value will result in decreased optimal spending? It was, of course, shown in Chapter IV that a higher retention factor causes higher "potential profit" which should result in higher spending and profits. The answer lies not in the present period term but rather in future period terms of Model II. Thus, the above "conditions" were not detected in Model I.

Consider two brands, $n = 2$, with $h = 1$ and $g = 2$. Let $q_1 = q_2 = .5$. From SECOND TERM,

$$\text{SECOND TERM} = \left[1-r_h\right]\left[\sum_{g=1}^{n}(1-q_g)c_g(t+1)f_{gh}(t+1)\right][P(t+1)].$$

$$[\text{DEMAND}_2] - a_h(t+1) \qquad (5.14)$$

where

$$\text{DEMAND}_2 = S(t+1) + q_h S(t+2) + q_h{}^2 S(t+3) + \ldots \qquad (5.15)$$

and

$$f_{gh}(t+1) = \frac{a_h(t+1) + b_h a_h(t) + b_h{}^2 a_h(t-1) + \ldots}{\sum_{g=1}^{n}[a_g(t+1) + b_g a_g(t) + b_g{}^2 a_g(t-1) + \ldots]} \qquad (5.16)$$

consider the element

$$\sum_{g=1}^{n}(1-q_g)c_g(t+1)f_{gh}(t+1).$$

$$\sum_{g=1}^{2}(1-q_g)c_g(t+1)f_{gh}(t+1) =$$

$$= (1-q_1)c_1(t+1)f_{11}(t+1) + (1-q_2)c_2(t+1)f_{21}(t+1)$$

$$= (1-.5)c_1(t+1)f_{11}(t+1) + (1-.5)c_2(t+1)f_{21}(t+1) \ . \qquad (5.17)$$

If $f_{gh}(t+1)$ is as described in equation (5.16),

$$f_{11}(t+1) = f_{21}(t+1) = f_{.1}(t+1) \ . \qquad (5.18)$$

Then

$$\sum_{g=1}^{2}(1-q_g)c_g(t+1)f_{g1}(t+1) =$$

$$= [.5c_1(t+1) + .5 - .5c_1(t+1)]f_{\circ 1}(t+1)$$

$$= [.5]f_{\circ 1}(t+1) \quad . \qquad (5.19)$$

It can be seen that the value $c_1(t+1)$ has no influence on the term of equation (5.17). As the $f_{\circ h}(t+1)$ term increases due to carry-over from advertising at $t + \Delta$, SECOND TERM profits will increase.

Now, let $q_1 = 0.0$ and $q_2 = 0.5$.

$$\sum_{g=1}^{2}(1-q_g)c_g(t+1)f_{g1}(t+1) =$$

$$= [(1-0)c_1(t+1) + (1-.5)(1-c_1(t+1))]f_{\circ 1}(t+1)$$

$$= [c_1(t+1) + .5 - .5c_1(t+1)]f_{\circ 1}(t+1)$$

$$= [.5c_1(t+1) + .5]f_{\circ 1}(t+1) \quad . \qquad (5.20)$$

Thus, with $q_1 < q_2$, higher spending at $t + \Delta$ causes a higher value of brand share $c_1(t+1)$ and therefore the left term of equation (5.17) is increased. Thus, a higher spending at $t + \Delta$ will not only cause an increase in $f_{gh}(t+1)$ due to carry-over but an increased brand share $c_1(t+1)$ will also have a tendency to increase SECOND TERM and future profits.

Now let $q_1 = 0.9$ and $q_2 = 0.5$.

$$\sum_{g=1}^{2}(1-q_g)c_g(t+1)f_{g1}(t+1) =$$

$$= [(1-.9)c_1(t+1) + (1-.5)(1-c_1(t+1))]f_{.1}(t+1)$$
$$= [.1c_1(t+1) + .5 - .5c_1(t+1)]f_{.1}(t+1)$$
$$= [-.4c_1(t+1) + .5]f_{.1}(t+1) \quad . \qquad (5.21)$$

Thus, if $q_1 > q_2$, higher spending at $t + \Delta$ causes a higher value of $c_1(t+1)$ and therefore the left term of equation (5.17) is decreased. The implication of the example in equation (5.21) is that when $q_1 > q_2$, increasing advertising $a_h(t)$ will cause an increase in $f_{gh}(t+1)$ due to carry-over, but increased brand share $c_1(t+1)$ will have a detrimental effect on SECOND TERM profit. That is, even though FIRST TERM of equation (5.12) may still be increasing as advertising is incremented upward, the future terms may be decreasing due to the phenomenon described by equation (5.21).

The net result of this discussion on retention buying is to show that for $q_h < q_g$, Brand h should advertise up to the point at which the decrease in present period profits by "overspending" is just equaled by the increase in future period gains from $a_h(t)$ carry-over. The optimal spending level will always be greater than or equal to that determined by Model I. If $q_h > q_g$, Brand h should again advertise up to the level at which present and future period profits fail to increase with an increased spending level. While carry-over tends to increase future period profits, the increasing level of present period brand share tends to decrease future period profits. As such, the optimal advertising pattern may be higher or lower than the advertising determined by Model I. It also may be lower than the optimal advertising pattern at a lower value of q_h. Note that with this model the rule that increasing q_h calls for higher advertising may not be at all true. Profit, however, should increase as q_h increases.

Although rather complicated, the above discussion contributes toward answering the question as to whether or not increased retention buying should call for more or less advertising. Much like the results determined for the carry-over factor, the optimal advertising level tends to increase and then decrease as the retention factor increases.

Demand as a Function of Total Industry Advertising

This characteristic's properties in the present model agree with those noted in Chapter IV. In the discussion on advertising carry-over it was shown that the effects of this characteristic actually dominated one of the influences of carry-over.

Summary

It can be seen that most of the findings outlined in the Summary of Chapter IV can still be considered valid. The results of the equilibrium analysis yielded the same conclusions as found in Chapter IV. The analytical analysis revealed differences between Model I and Model II in the influences of advertising carry-over and retention buying with respect to the level of optimal advertising.

These differences stem from the inclusion of future periods in Model II whereas Model I deals with only the present period. Such differences are discussed in some detail in the text of this chapter and will not be repeated here.

Another fact with implications for anyone desiring to use either model became apparent in this chapter. While the various characteristics presented mathematically in Chapter III may have the influences noted in the Summary of Chapter IV and the text of Chapter V, there may very well be combinations of parameter values and competitive expenditures which have dominating effects upon optimal advertising expenditures. Such dominating effects may appear to negate one or more of the influences noted in Chapters IV and V. Such was the case in this chapter when demand as a function of total industry advertising caused the relative amplitude of advertising at $b_h = 0.0$ to exceed that of $b_h = 0.5$.

CHAPTER VI

AN EXAMPLE OF PROFIT MAXIMIZATION

The objective of this chapter is to present an example of competition between two non-identical brands. The firm using the models developed previously will be designated as Brand h or Brand 1. The firm using a rule of thumb to budget advertising will be designated Brand g or Brand 2. The "present period" model developed in Chapter IV is called Model I; the "five period" model developed in Chapter V is called Model II.

The benefits derived from considering two non-identical competitors are several. Such an analysis should provide a direct comparison between Model I and Model II. Also, the validity of the findings and conclusions of the equilibrium studies as well as the predictions from the mathematical analyses of retention buying and advertising carry-over should be apparent. This competitive analysis should also give the reader a feel for the sensitivity of profit to advertising and the sensitivity of both profit and advertising to the retention and carry-over factors.

The Competitor, Brand g=2

Brand 2 of a given product class is produced by a firm which has heard that potential demand tends to lend its general shape to the optimal advertising pattern. Being in a stable but periodic industry, the company has estimated that the total potential demand (not potential demand) for the product follows equation (3.20). However, a fraction of .3 of that total potential demand will purchase the product class regardless of advertising and will remain oblivious to the advertising of Brands 1 and 2. Therefore, potential demand (that portion of total potential demand which can be influenced by advertising) is as defined in equation (3.22), the familiar sine pattern

$$f(t) = 1{,}000{,}000 + 240{,}000 \sin(\pi t/6) \qquad (6.1)$$

used throughout. Using a "rule of thumb" Brand 2 has decided to allocate advertising money at the beginning of each month. The expenditure is to be a fixed percentage, 7.5 per cent, of potential demand. Brand 2 feels that a fixed pattern of advertising such as described may not be the best competitive course of action but rather should be a good "middle of the road" decision. The monthly potential demand and Brand 2 advertising expenditure values are shown in Table IV.

TABLE IV

MONTHLY POTENTIAL DEMAND AND BRAND 2 ADVERTISING EXPENDITURE

Beginning of Month	Potential Demand Dollars	Brand 2 Advertising Expenditures, Dollars
1	1061410	79605
2	1167770	87583
3	1229180	92188
4	1229180	92188
5	1167770	87583
6	1061410	79605
7	938590	70394
8	832230	62416
9	770820	57811
10	770820	57811
11	832230	62416
12	938590	70394

Brand h=1

Brand 1 is produced by a firm which desires to maximize profits through the optimal adjustment of its advertising expenditure each month. Brand 1 has determined that about one half of the potential demand will purchase the product class even with no advertising motivation. However, this portion will be influenced in their buying by the relative advertising of each brand. Further, with advertising, the other half of potential demand may be motivated to buy, brand choice again influenced by relative advertising. Brand 1 has determined that the response curve P(t) of equations (3.23) and (3.24) has the following Gompertz curve parameters:

$$\begin{aligned} D &= .65 \\ D' &= .3 \\ S &= .6 \\ U &= .0000124\ . \end{aligned} \qquad (6.2)$$

Setting D = .65 when D' = .3 assures that one half of potential demand will certainly buy while the response of the other half is a function of advertising. The entire portion of potential demand which purchases the product class is influenced by relative composite advertising in determining brand choice. The response curve (P(t)) as described above is presented in Figure 24.

Brand 1 further estimates that both firms have an approximate advertising carry-over factor of $b_1 = b_2 = .25$ and a retention factor of $q_1 = q_2 = .5$. However, Brand 1

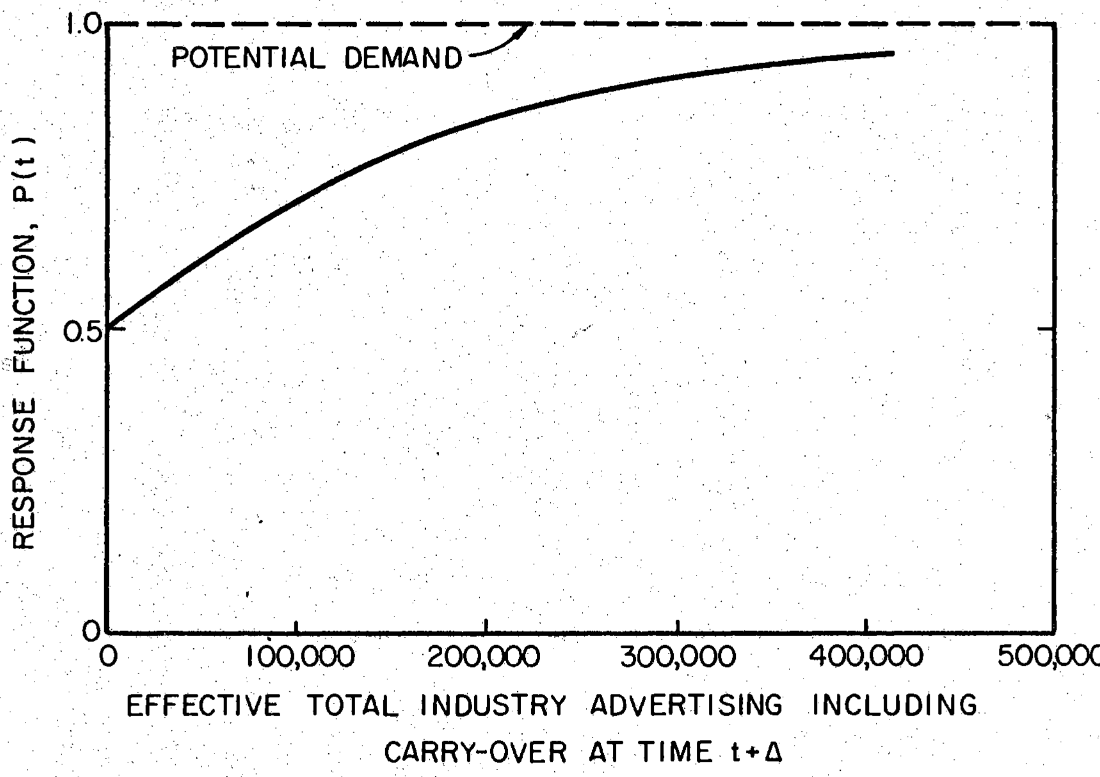

Figure 24. Response Curve P(t) Where Modified Gompertz Parameters are D=.65, D'=.3, S=.6, U=.0000124.

also wishes to determine the effect on advertising and profit levels if its parameters are allowed to vary considerably. Therefore, using both Models I and II, Brand 1 will determine optimal advertising and profits letting $b_1 = 0.0$, 0.25, and 0.5 while holding $q_1 = .5$. Also, retention can take on the values $q_1 = 0.0$, 0.5, and 0.9 holding $b_1 = .25$.

Brand 1 has further noticed that Brand 2 has been spending advertising money in a fixed pattern. This knowledge will allow Brand 1 to estimate future Brand 2 expenditures for use in Model II. Past advertising expenditures are available from company records or from information and data gathering research firms.

Explanation of Computer Analysis

The computer analysis of this model consists of determining and recording the optimal Brand 1 advertising expenditure and profit each month for each of the two models. As mentioned previously, Brand 1 will assume various values of carry-over and retention. Also, demand is assumed to be only a partial function of total industry advertising, a departure from the analyses of Chapters IV and V. A computer analysis is necessary due to the complexity of Models I and II shown in equations (4.5) and (5.3), respectively.

Search for Optimality

In order to arrive at a point of maximum profit, optimality is sought in a way quite similar to the search described in previous chapters. The advertising at time $t + \Delta$ is initialized at an obviously small value. The expenditure is then increased using wide increments until a near optimal expenditure is determined. The increment is then continually decreased until an optimum advertising level is determined to any precision desired. The major difference between the search for optimality in this chapter and the search for equilibrium in previous chapters in the stationarity of the competitor's expected advertising at $t + \Delta$. In this analysis, $a_1(t)$ is incremented to optimality while $a_2(t)$ remains fixed at a predetermined expected level of advertising.

Successive Monthly Analysis

In order to determine the optimal pattern of expenditures for Brand 1 it is assumed that Brand 2 will continue to advertise at the level shown in Table IV and potential demand will retain its pattern each year. In this analysis it is assumed that Brand 1 would like to know the optimal monthly expenditures as if all past advertising had been optimal.

Much like the successive monthly analyses of the previous two chapters, past "optimal" expenditures must be estimated for each of the models while Model II requires that future expenditures of both firms be estimated. Again for the purposes of computer analysis only the four most recent periods and the present period are used in each model. Model II, of course, also utilizes the future four periods. Since it is assumed that Brand 2 follows a consistent policy, it is not unreasonable to assume that Brand 1 can predict future Brand 2 spending. Once again, it is hypothesized that continual cycling, month by month, will result in the diminishing influence of initial estimating and the expenditures determined by both models will converge to their respective optimal patterns. Such a method was seen to work quite satisfactorily for the equilibrium studies of Chapters IV and V.

Results of Computer Analyses

The computer optimization was completed using the programs, shown with explicative comment cards, in Appendices E and F for Models I and II, respectively. Sample outputs for each program are also presented. Both programs had similar characteristics with respect to their predecessors, the equilibrium programs shown in Appendices C and D. As expected, both programs converged nicely to their optimal pattern after the proper number of yearly cycles of simulation. The program of Model II, due to the needed estimation of both past and future "optimal" expenditures, understandably required more time to converge.

The results of the optimization using Model I are shown in Table V. The optimal advertising expenditure and associated profit are presented for each month. Figures 25 and 26 represent these data graphically. Figure 25 shows $b_1 = 0.0$, 0.25, and 0.5, holding q_1 at 0.5; Figure 26 illustrates $q_1 = 0.0$, 0.5, and 0.9, holding b_1 at 0.25. The results using Model II are presented in Table VI and Figures 27 and 28. Figure 27 shows $b_1 = 0.0$, 0.25, and 0.5, holding q_1 at 0.5; Figure 28 illustrates $q_1 = 0.0$, 0.5, and 0.9, holding b_1 at .25.

Comparison of Findings With Those Expected From Previous Analyses

In this section it is hoped that the findings and predictions stemming from the equilibrium and mathematical analyses of Chapters IV and V will either be confirmed or denied. Several conclusions have been drawn from the equilibrium studies, but such items as absolute magnitudes of advertising and profits for non-identical competitors have only been predicted mathematically.

Before considering each characteristic separately as before, the reader should make note of one especially important result. In every single example shown in the tables or figures of this chapter, the profits using Model II are greater than or equal to those using Model I. The reasons for this phenomenon should be known and considered by anyone who attempts to utilize a quantitative model to

TABLE V

OPTIMAL MONTHLY BRAND 1 ADVERTISING EXPENDITURE AND CORRESPONDING PROFIT USING MODEL I

Month	$b_1 = .25$			$q_1 = .5$		
	q_1=0.0	q_1=.5	q_1=.9	b_1=.0	b_1=.25	b_1=.5
1	36260 22540	93100 114590	122640 196660	120360 87320	93100 114590	66930 140750
2	42380 28570	96880 120380	123530 190520	126920 90350	96880 120380	68040 149230
3	45230 32830	95710 118860	123350 185430	127370 87200	95710 118860	65400 149170
4	43970 33810	89900 110650	122340 182420	121670 78880	89900 110650	59710 140850
5	38990 31260	80890 98150	120890 182320	111230 67820	80890 98150	52330 126710
6	31780 26190	70980 84730	119300 185510	98700 57020	70980 84730	45180 110540
7	24480 20310	62880 73820	118560 194840	87460 49240	62880 73820	40240 96460
8	19140 15270	58880 68110	116970 200770	80650 46340	58880 68110	38960 88030
9	16930 12110	60010 69070	116240 206190	80090 48990	60010 69070	41620 87460
10	18050 11240	65970 76630	116770 208620	85910 56690	65970 76630	47500 95100
11	22220 12810	75220 89040	118530 207180	96630 67630	75220 89040	55090 109170
12	28790 16780	85200 103020	120780 202690	109290 78920	85200 103020	62240 125980

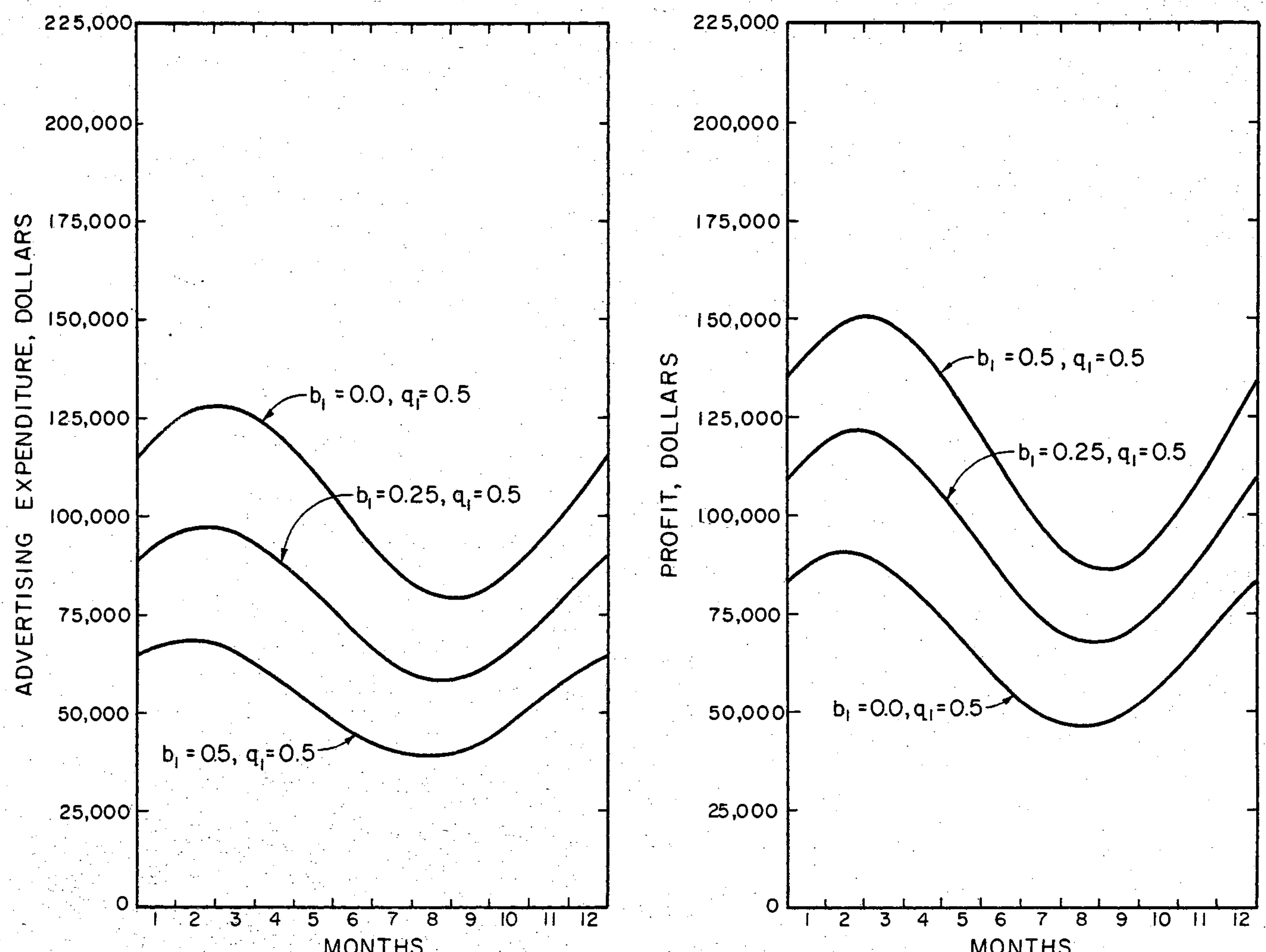

Figure 25. Optimal Brand 1 Advertising and Associated Profit at $b_1 = 0.0$, 0.25, 0.5 With $q_1 = 0.5$ Using Model I

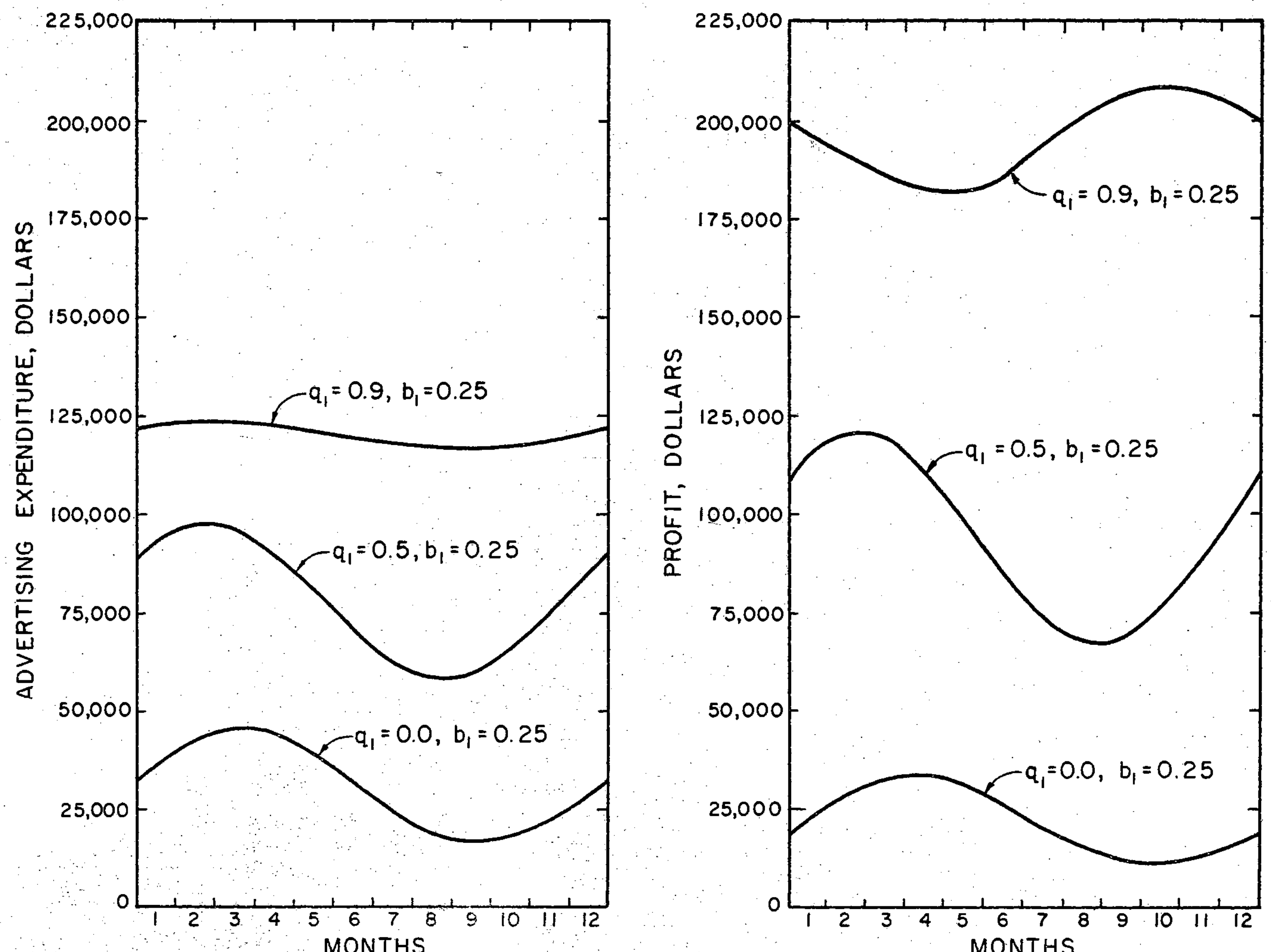

Figure 26. Optimal Brand 1 Advertising and Associated Profit at $q_1 = 0.0, 0.5, 0.9$ With $b_1 = 0.25$ Using Model I

TABLE VI

OPTIMAL MONTHLY BRAND 1 ADVERTISING EXPENDITURE AND CORRESPONDING PROFIT USING MODEL II

Month	b_1 = .25			q_1 = .5		
	q_1=0.0	q_1=.5	q_1=.9	b_1=.0	b_1=.25	b_1=.5
1	117400 40460	128000 119150	103600 204060	120400 87320	128000 119150	127700 161070
2	125700 50280	133100 125400	104700 197250	126900 90350	133100 125400	130400 171430
3	127000 56730	132300 124270	104100 191530	127400 87200	132300 124270	127800 173000
4	121200 57990	125800 116340	101700 187820	121700 78880	125800 116340	120300 165850
5	109600 54130	115100 103950	98200 187010	111200 67820	115100 103950	109700 152140
6	95100 46490	102800 90440	94600 189760	98700 57020	102800 90440	98300 135610
7	81500 37290	92200 79230	92100 196050	87500 49240	92200 79230	89300 120300
8	72600 28810	86400 73040	91900 204680	80600 46340	86400 73040	85800 109790
9	71100 22750	87100 73490	93600 212710	80100 48990	87100 73490	88900 107000
10	77200 20540	94200 80670	96100 216770	85900 56690	94200 80670	97700 113080
11	89400 23010	105600 93000	98700 215640	96600 67630	105600 93000	109600 126760
12	104200 30250	118000 107190	101500 210700	109300 78920	118000 107190	120100 144860

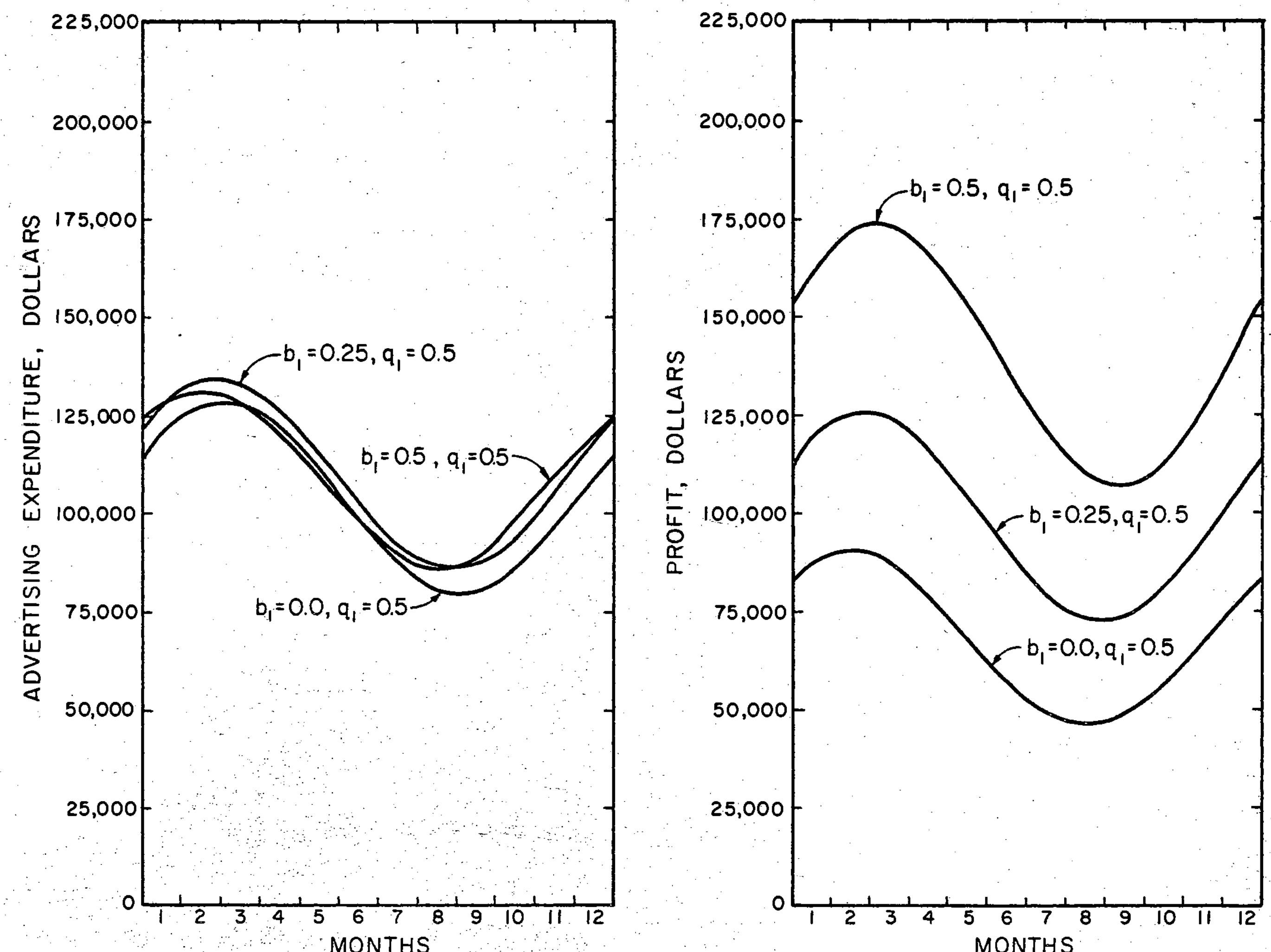

Figure 27. Optimal Brand 1 Advertising and Associated Profit at $b_1 = 0.0$, 0.25, 0.5 With $q_1 = 0.5$ Using Model II

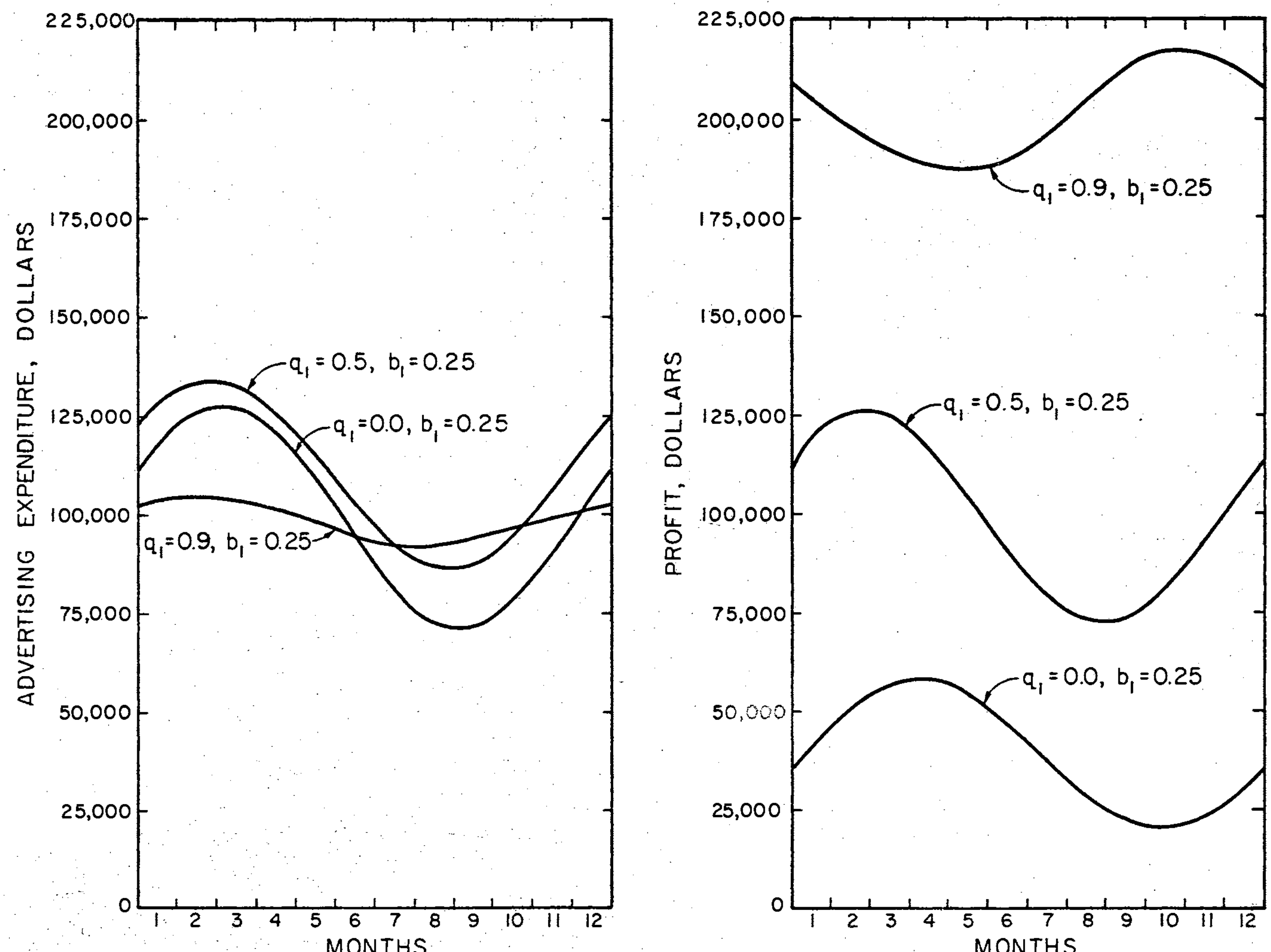

Figure 28. Optimal Brand 1 Advertising and Associated Profit at q_1 = 0.0, 0.5, 0.9 With b_1 = 0.25 Using Model II

budget advertising. The reasons and their implications will be discussed in some detail in Chapter VII.

Demand as a Function of Time

From each of the four figures it is easily seen that, as in the equilibrium studies, the general shape of the potential demand curve is lent to the optimal advertising and profit patterns. Expectedly, the phase and relative amplitude of advertising is quite varied with respect to the potential demand curve. In fact, not a single curve shown is a robust replica of potential demand. The reason is twofold: there is no analysis at $q_1 = 0.0$, $b_1 = 0.0$, and demand is a function of total industry advertising (actually half of potential demand is certain to buy, the other half requiring motivation from advertising).

Advertising Carry-Over

Figures 25 and 27 are of interest in the discussion to follow. They allow advertising carry-over to vary over the range of interest while holding retention buying at a constant moderate value of 0.5.

First examine Figure 25. Note that as b_1 increases from 0.0 to 0.5 the optimal advertising expenditure level decreases while profits correspondingly increase. This result was predicted from the equilibrium analysis of Chapter IV. The reason for this phenomenon is that Model I is a "present period only" model and the optimal composite advertising will always be at the point where the slope of "profit plus advertising cost" (Figures 10 and 17) equals 1.0. The higher carry-over factor causes a higher portion of composite advertising to be carry-over, thus the optimal level of new advertising is lower and profit is greater.

Figure 26 is not nearly as distinct in terms of significant magnitude effects on advertising levels caused by varying b_1. Reference to the discussion of carry-over in Chapter V will confirm that the very phenomenon obvious in Figure 26 was predicted mathematically. Figure 26 illustrates that the general level of advertising is lowest for $b_1 = 0.0$, highest for $b_1 = 0.25$, and in-between for $b_1 = 0.5$. However, the general level of advertising over the entire range is very close in terms of amplitude and phase. Also, just as shown in Figure 23, the profit levels do increase as does b_1. The results as discussed in this and the previous paragraph have a significant message for advertising personnel. These results and their meaning will again be considered in Chapter VII.

Another finding that has remained valid throughout this study is the slight phase shift resulting in optimal advertising leading potential demand at values of b_1 greater than zero. Both Figures 25 and 27 indicate the increased phase shift as b_1 climbs from 0.0 to 0.5. The phase shift apparent at $b_1 = 0.0$ is due to the fact that q_1 was held at 0.5 and therefore also caused phase shift.

Finally, it was shown in previous studies that advertising carry-over tended to cause slight, but definite, relative amplitude increases as opposed to relative amplitude at $b_1 = 0.0$. In Figure 25, the relative amplitude at $b_1 = 0.5$ is indeed greater than that at $b_1 = 0.0$. Of course, since the optimal composite advertising was identical in Model I regardless of the value of b_1, the response coefficient P(t) was also identical at each b_1 value during any given month. However, in Figure 27 it appears that such a conclusion concerning the influence of the factor b_1 is incorrect. Note that the relative amplitude of spending at both $b_1 = 0.0$ and $b_1 = 0.25$ is greater than that at $b_1 = 0.5$. The reason can be traced, much as in Chapter V, to the varying response coefficient P(t). While each month's spending is approximately the same, each month's composite advertising is considerably different in Model II due to the nature and values of the factor b_1. In fact, at $b_1 = 0.0$, P(t) varies from .79 to .88. At $b_1 = 0.5$, P(t) varies from .88 to .94. The reduction in variation of P(t) may be explained by noting that with higher carry-over, and therefore higher composite advertising, the response coefficient P(t) operates in a higher, flatter region, less sensitive to industry advertising fluctuations.

Retention Buying

Figures 26 and 28 are of interest in a discussion of the retention factor. The factor q_1 was allowed to assume values of 0.0, 0.5, and 0.9 while b_1 was held constant at 0.25.

In Figure 26, representing the results using Model I, it is easily seen that the prediction from the mathematical analysis of Chapter IV was correct with respect to Model I. Higher values of q_1 do increase "potential profit" and therefore motivate higher spending and profit levels.

In Figure 28, representing the results using Model II, it appears that the total yearly budgets required at each of the three levels of q_1 are quite close. A referral to the discussion on retention buying in Chapter V will reveal that the results of Figure 28 were relatively predictable. In the example presented here, at $q_1 = 0.5$, advertising $a_1(t)$ was increased so as to cause the present period profits to decrease while enlarging future period profits with carry-over advertising reflected in $f_{g1}(t+1)$, $f_{g1}(t+2)$, ..., and $P(t+1)$, $P(t+2)$, ... The "five period" profit was maximized at $q_1 = 0.0$, the same reasoning applied, only increased brand share obtained during the "present" period, $c_1(t+1)$, had a favorable effect on future period profit as shown by equation (5.20). Therefore, advertising at $q_1 = 0.0$ continued to climb, coming relatively close to that at $q_1 = 0.5$. However, at $q_1 = 0.9$, even though present period profits were still increasing by incrementing $a_1(t)$, the brand share gained during $[t + \Delta, t + 1]$ had a detrimental effect on future profit as shown by equation (5.21). Thus, optimal profit was at a lower level than might be expected. In line with this discussion, notice that at $q_1 = 0.0$ and 0.5 the advertising levels using Model I the "present-period-only" model were below those of Model II. Also, the level at $q_1 = 0.9$ in Model I was above that of Model II. The above discussion also has implications for advertisers which will be considered in Chapter VII.

An influence of retention buying which has continued to be quite predictable throughout this study in its propensity to cause a phase shift. That is, peaks in advertising spending occur ahead of peaks in potential demand. Optimal advertising is, as usual, seen to lead potential demand more as retention increases. Another influence is the increase in relative amplitude caused by decreasing the retention factor as evident from Figures 26 and 28.

Demand as a Function of Total Industry Advertising

In the example of this chapter demand was assumed to be a function of total industry advertising as shown in Figure 24. One half of potential demand was assured while the other half remained to be motivated by industry advertising. In Chapters IV and V the major influence of variable demand as a function of advertising was to cause distortion of the sine pattern of advertising. Specifically, the relative amplitude on the lower lobe of the spending curve was found to be quite low because the corresponding periodic low in potential demand could not profitably sustain higher advertising. This caused an increase in relative amplitude of advertising expenditures.

From Figures 25 through 28 there was no truly obvious effect of demand varying as a function of industry advertising. In general, this was due to the level of advertising which was certainly profitable in attracting the 50 per cent of potential demand assumed to purchase the product class without advertising motivation. Of course, the money spent by each brand tended to increase P(t), thereby supporting larger expenditures. As such, P(t) maintained a level which prevented the obvious elongations of the lower half of the advertising pattern evident in Chapters IV and V. The influence of advertising variable demand was still present in a non-obvious form, however, as the relative amplitude at $b_1 = 0.0$ was greater than that of $b_1 = 0.5$ using Model II. That aspect has previously been discussed.

Summary

In summary of this chapter, one can see that Model II is much more satisfactory in terms of profit maximization than Model I. Of course, it is also harder to use in terms of both mere mechanics as well as obtaining estimates of future spending.

The equilibrium and mathematical analyses presented in Chapters IV and V predicted trends and results which were supported by this computer analysis. The conclusions outlined in Chapter IV also held with the exceptions concerning the influence of the carry-over and retention factors on optimal spending (with Model II) as presented in some detail in Chapter V.

There were also seen to be several implications for persons budgeting advertising money. These implications as well as other considerations in using the models herein will be discussed in Chapter VII.

CHAPTER VII

CONSIDERATIONS IN THE USE OF MODELS I AND II

The objective of this chapter is to discuss the application of the models presented previously. Primarily, this discussion will consist of comments on various assumptions employed by the models as well as their objectives, the implications resulting from the findings of the previous three chapters, and thoughts concerning meaningful sensitivity analyses which might be performed in a "real world" situation.

Comments on the Applicability of Models I and II

In Chapters II and III a number of concepts and characteristics were developed mathematically for use in developing a model for budgeting advertising over time. In Chapters IV and V two models were developed, one an extension of the other to encompass the future effect, in terms of carry-over, of the advertising to be allocated at the beginning of the period under consideration. Chapters IV and V included mathematical and computer oriented analyses which provided conclusions and predictions concerning the influence of the various characteristics considered. Chapter VI then presented each model in a profit maximizing situation in which the results were compared with predictions.

In order to present the example in Chapter VI a number of assumptions were made concerning the necessary parameters and variables. Although different retention and carry-over factors were used, they were assumed to remain constant from each time period to the next. This appears to be a reasonable assumption as the retention buying factor is, price, product distribution and frequency of use considered equal over all brands, largely a function of the attributes of the brand. Included are quality, convenience of use, taste, and, in general, "likeability." Unless the characteristics of the product or general economic conditions change it is reasonable to assume q_g to remain constant, though perhaps different, for all brands. The carry-over factor is largely a function of the media, copy, and type of advertising engaged. Assuming a brand generally selects its various advertising media in roughly the same proportions each period, it can be assumed that the carry-over factor b_g remains constant each period.

The actual estimation of b_g and q_g is also a necessary consideration in using these models. In order to determine good statistical estimating procedures for finding b_g and q_g considerable research would probably be required. This author feels that a good estimate of q_g can be obtained using a regression analysis similar to that used by Telser (1964). Such a procedure requires past data on relative brand share and absolute or relative advertising expenditure between brands. This author does not have nearly the feel as to the best estimating procedure for b_g. It would appear that if a value of carry-over could be assigned to each type of frequently used media, a reasonable estimate of the overall carry-over factor could be easily determined. If a satisfactory method of determining the exact advertising which caused a purchase could be found, a step-wise regression analysis could be used to determine those factors about media, copy, message, etc., which set the carry-over level. Much of this type information is available from market research firms.

Another assumption was made that both potential demand and the response coefficient curve (P(t)) could be determined. There is a vast amount of literature on projecting future demands using exponential smoothing, time series, and other well refined techniques. The response curve is probably best estimated by a pooling of several pilot studies in typical communities. By increasing spending from zero to the saturation point in several steps, one can determine the approximate form and parameters of the response function. In the event that saturation is reached at a much lower level than optimal spending, P(t) may be considered equal to 1.0 and disregarded. The same may be said in the case of a necessity item in which demand is quite close to potential demand even without advertising. In the above two instances, advertising then goes solely to determine relative brand share.

In the above discussions on estimating parameters this author has not attempted to outline procedures for parameter determination. Rather, an attempt has been made to impress upon the reader that the parameters must be estimated and such estimation techniques must be given consideration. Although the accurate determination of parameters would take considerable time, they are not likely to change drastically unless brand characteristics change, the general mode of advertising changes, or a new brand or substitute good enters the market.

The assumption of knowing the past four period's expenditures for competing brands is not unreasonable. Such information is obtainable from past company records and/or market research firms. Both models also require an estimate of competition spending at the beginning of the period under consideration while Model II requires estimates of the four future period's spending as well. If a brand's competitors have generally advertised in a set pattern in the past without regard to their competitor's spending, future estimates can be quite accurate. If, however, the competition tends to vary its advertising pattern so as to be relatively unpredictable, the firm using Model II should be aware of any consequences resulting from inaccurate spending predictions.

The discussion throughout much of this dissertation has been with regard to only two competing firms. Actually, these models may be applied when n brands are in competition. Alternatively, all other competitors may be lumped and considered the "other" brand. The disadvantage to this is that carry-over and retention factors may now be variable unless equal for all competitor firms. If b_g and q_g are unequal for all competitor brands, b_g and q_g are weighted proportional to their spending level and brand share, respectively. Thus, as spending and brand share tend to change, the carry-over and retention factors change.

While it appears that many assumptions must be made, many parameters estimated, and competitor expenditures predicted, one fact must be noted. In order to use any of the models developed in the literature (and mentioned in Chapter I) many of the same or similar assumptions and estimates must be made. In a model which better describes actuality it is natural that it be more complex than many others. One strong advantage of both models presented in this dissertation is their ability to negate and eliminate certain expendable characteristics (for example, let $P(t) = 1.0$ or let potential demand be constant over time, etc.).

Implications for Advertisers

The analyses of Chapters IV, V, and VI resulted in many interesting findings from which inferences may be drawn. Of particular interest is a comparison between results of Model I and Model II and a discussion of the "philosophy" underlying each. Also of interest is the effect on optimal advertising and profit of incorrectly estimated values of b_g and q_g.

As noted in Chapter VI, in every instance the profit resulting from the allocation of advertising funds from Model II yielded profit greater than or equal to that of Model I. The profit used as a basis of comparison is the "present period" profit as defined in Chapter IV. Of course, optimal advertising in Model II was determined on the basis of profit over the present and four future periods. What causes this difference in profits?

Probably the best way to explain the difference is to examine the philosophy of each model in terms of what it is designed to accomplish. Model I has provision for considering carry-over advertising from past periods. It also accounts for future sales due to retention which may be attributed to advertising during the period under consideration. Its objective is to determine that expenditure which maximizes profit, as defined, for the period under consideration, the "present" period. Model II has all the provisions of Model I plus the ability to look at the future effects of carry-over from advertising spent at the beginning of the period under consideration. Its objective is still to determine the optimal present period advertising, however, in terms of maximizing present and future profits. The key explanation to describe the difference between the models is to say that Model I is a sub-optimization model while Model II optimizes with respect to an entire system of stages - that is, time periods. The net result is that Model I will have a higher profit during its first period of use while Model II will accept a lower profit in favor of setting advertising at a level which will yield low present profits but much higher future returns. The future result of spending "optimally" using Model I is to initiate a spending level which will be consistently too low or too high.

Another way of looking at the differences between the models is to recall that Model II is merely taking into account that which actually happens. In other words, today's advertising does affect future results. Model II attempts to allow for these effects while Model I disregards the future other than that of future buying due to retention. The important point is that present spending and budgeting do affect future results. As such, their effects should be considered by decision makers.

Another aspect of which decision makers should be aware is the influence of carry-over and retention characteristics on optimal advertising. It is seen, both from the mathematical analyses of Chapter V and the computer study of Chapter VI, that the carry-over and retention factors, b_1 and q_1, had little influence on the optimal level of spending in Model II. That implies that optimal advertising is rather insensitive to the estimation of b_1 and q_1. This is not to say that the optimal advertising level may not be more sensitive to estimation of competitor's factors, b_2 and q_2. On the other hand, profit is very sensitive to b_1 and q_1. A reasonable, but incorrect, estimation of b_1 and q_1 will therefore provide a fairly close estimate of optimal spending using Model II. Profit, however, may be quite different from that expected if b_1 and q_1 are incorrectly estimated. Thus, to optimize advertising should require only a rough estimate of b_1 and q_1. However, to predict the resulting profit requires a much closer estimate of the carry-over and retention factors. The use of a model such as Model I will always result in lower than truly optimal advertising expenditures when $q_1 \leq q_2$. At some point, Model I may result in higher than truly optimal spending when $q_1 > q_2$. Such was the case in Chapter VI. The reasons for this phenomenon are discussed in Chapter V. Also, when Model I is used and $b_1 \geq 0.0$, there is a tendency to underestimate the optimal spending level.

Meaningful Sensitivity Analyses

Many such studies as this dissertation include a sensitivity analysis of parameters. Such analyses may consider x levels each of n variables. Thus, nx computer or mathematical calculations need be made. In a competitive model such as this, if only two brands are considered, a correspondingly complete sensitivity analysis from which to draw definitive conclusions would require $(nx)^2$ such calculations. Due to the infeasibility of such a task using Models I and II, a verbal discussion of thoughts on practical sensitivity analyses will be presented. Many more simulations than are presented in this dissertation will influence the following discussion.

Perhaps the first question of concern to a potential user is that of the effects of incorrect estimates of competitor spending. Considerable work has been done in this area. The common approach is to consider a model such as Model I, without carry-over and often without retention buying. The competitor's expenditure is allowed to vary (usually from a pre-determined competitive equilibrium value) and a new optimum and the resulting profit found for a specific brand, Brand h. While such analyses are interesting and no doubt somewhat instructive, this author feels that the underlying assumptions stated above are much too limiting in view of empirical evidence to the contrary. A vast improvement in terms of a meaningful sensitivity study would be the use of Model I or a related version of such. At least past spending, retention buying, and variable demand should be considered. If the time value of money is considered important, DEMAND should be calculated as in Appendix A. Still such an approach will leave the user with incorrect and possibly very misleading results. The reason is that the future effects of such competitor spending deviations are not considered. Obviously Model II could be used for such a sensitivity analysis. However, its use at only one period will fail to consider that optimal future period spending will change due to a competitor's deviation at the "present" period. In other words, if the competitor were to increase his advertising at the "present" period by 15 per cent, say, Brand h could calculate its new optimal spending and profit this period also. However, in doing so, the Brand h future spending estimates used in Model II would no longer be optimal.

This author suggests the following approach to a meaningful sensitivity analysis of competitor deviations from expected spending. Consider a ± 10 per cent deviation for the entire pattern of competitor spending. Then calculate the resulting profits if Brand h fails to retaliate from its old budget. Finally, completely recalculate the optimal Brand h level or pattern of spending and the ensuing profits. The result will be such that Brand h will have a good idea of the optimal direction of change as well as the magnitude of increase or decrease in profits. This can be done at several percentage levels.

Brand h could also consider other non-optimal spending levels which might tend to force the competition to spend more or less. For example, from elementary sensitivity studies performed using Model I (with results similar to those of other authors) it was found that the optimal spending level for Brand h is slightly below its equilibrium level when a competitive brand is overspending considerably. This is at a considerable loss of profit to Brand h as opposed to both brands operating at equilibrium. The over-spending competitive brand is, however, receiving profits only slightly below those of both brands at equilibrium. If Brand h now deviates from his new optimum, his profits are very insensitive to change. But the competition's profits are very sensitive to a change in Brand h expenditure. A possible competitive strategy for Brand h is then to overspend, thus lowering the competitor's profits considerably. The possible result will be a mutual withdrawal to lower spending and higher profits for both. While the above discussion relates to Model I, the use of which was discouraged earlier in this section, similar such strategies could be explored using Model II.

Also of interest to the user of Models I and II is the effect of incorrectly estimating the carry-over and retention factors. The reader has already seen somewhat of a sensitivity analysis on the Brand h factors b_h and q_h (b_1 and q_1 in Chapter VI). No such analysis was performed while varying the competitor's factors which were assumed constant at $b_2 = 0.25$ and $q_2 = 0.5$ in Chapter VI.

This author would suggest the following ideas on determining the sensitivity of optimal advertising and profit to the carry-over and retention factors. First, the user has a reasonable idea of the accuracy of his estimates. If estimated statistically, a .95 confidence interval would be an excellent indicator of the accuracy of the estimates. Even if the user merely calls upon judgement and experience for his estimates, he probably has some conception of the accuracy of each. A recommended procedure would be to initially consider the expected value as well as the expected limits of accuracy of each of the two parameters. Enough combinations of the estimates and their expected boundary values should be considered both for Brand h and the competition, to determine which parameter ranges (if any) warrant more detailed research into estimation. For example, if one had estimated $b_1 = .25 \pm .10$ ($\pm$ 40 per cent) in the examples of Chapter VI, he would certainly not wish to estimate b_1 more closely if his only objective were to determine optimal spending.

Finally, the user may be concerned over his estimation of potential demand and the demand response curve (P(t)). Potential demand is relatively important as it sets the general level of spending and profits. The accuracy of its determination may depend, more or less, on the response curve. For example, if demand is highly variable with respect to total industry advertising, and perhaps retention buying is low, potential demand is quite important. As one recalls from Chapters IV and V it was noted that the above stated conditions result in rather abrupt reductions in spending and profits (see Figures 14 and 20) during potential demand lows. An incorrectly estimated potential demand and/or response curve could result in highly erroneous spending. If, however, the response curve ascends very quickly with respect to optimal advertising

($P(t) \cong 1.0$ at optimal advertising) or if demand is very near potential demand (for example, a necessary commodity), the response curve should require no further parameter estimation. In fact, it should possibly be eliminated completely in such a case.

As in the case of previous parameters discussed, if the optimal advertising level is sensitive to the response curve, more research on its shape and characteristics should be performed. An incorrectly estimated response curve ($P(t)$) could be disastrous if optimal advertising is quite sensitive as the parameters of the response curve are allowed to vary.

Summary

The presentation of this chapter is a compilation of many thoughts, recommendations, and aids in the use of these advertising models. Applicability, implications for advertisers and sensitivity analyses are considered in discussion form. Some of the discussion has also been presented earlier in this study. However, it was felt that a chapter such as this would best summarize a number of considerations in the use of Models I and II after the reader had become familiar with their development, analysis, and use.

CHAPTER VIII

CONCLUSION

The objective of this chapter is to present a brief statement of the problem and the approach used, the major findings, and recommendations for future work.

The Problem and the Approach

The problem addressed in this dissertation is one of budgeting advertising expenditure in a competitive market. The formulation techniques developed are applicable in the analysis of other areas involving retention and carry-over effects. Throughout this dissertation the assumption is made that the only competitive item of the marketing mix is advertising expenditure. Price, distribution, etc., are assumed equal over all brands. At the outset it was desired to include the characteristics:

1. Variable demand as a function of time.
2. Variable demand as a function of total industry advertising.
3. Retention or habitual buying.
4. Advertising carry-over.

Such characteristics are known to affect the magnitude and timing of optimal advertising expenditures.

The problem reduced to two major aspects. First, a mathematical model was needed in which the above characteristics were included. Secondly, an analysis of the model was needed in order to determine the effects of the above characteristics on the optimal advertising pattern over time.

The problem was initially approached by utilizing a considerable search of the relevant literature. Of particular interest were empirical and theoretical studies which provided a qualitative background for the quantitative treatment of the above characteristics in Chapter III. Also of interest were the various mathematical advertising models which have appeared in the literature. Such models have ranged from the extremely over-simplified to the mathematically complex. Unfortunately, the over-simplified models are generally too limited in the assumptions which must be made. The mathematically complex models generally place emphasis on the mathematics as opposed to the application of the models. This author sees the advertising model concept presented herein as one which is complex in view of its consideration of many facets related to optimal advertising. However, it is also seen as a rather simple, flexible concept from a mathematical viewpoint.

The basis for the development of two advertising models is presented in Chapters II and III. Chapter II presents a brand-switching concept which describes brand-switching as a function of relative advertising between brands on a period by period basis. Chapter III develops the four previously mentioned characteristics in a mathematical sense. In Chapters IV and V the actual models are developed. Model I in Chapter IV considers carry-over advertising from past advertising expenditures, future buying due to retention which may be attributed to the advertising expenditure under consideration, and variable demand as a function of both time and total industry advertising. Model II of Chapter V is identical to Model I with the exception that the future effect of carry-over from present advertising is also considered. The addition of this single element multiplies the mathematical size of the model by a factor of n, n being the total number of periods considered in Model II.

Chapters IV and V also present a computer simulation of the respective models under the assumption of two identical competitors at equilibrium. An analysis of information obtained from these studies as well as from mathematical studies results in a number of conclusions and predictions about optimal advertising expenditures. These conclusions and predictions are supported in a hypothetical profit maximizing example presented in Chapter VI. In Chapter VI a competitor with known parameters and a fixed pattern of advertising is considered. Brand 1, the brand using the advertising concept of this dissertation, utilizes Models I and II to optimize its advertising expenditures. Various levels of Brand 1 carry-over and retention are considered.

While Chapter VI concludes the development and resultant analyses, Chapter VII treats the applicability of Models I and II, their implications, and a discussion concerning meaningful sensitivity analyses.

The total approach to the problem can be described in five parts: 1) synthesis of available literature, 2) mathematical development of model components, 3) synthesis of components into two different models, 4) computer and mathematical studies of each model, and 5) summarization in terms of findings and conclusions.

Findings and Conclusions

It has been shown quite clearly that Model II is superior to Model I. Indeed, the reader may wonder why Model I was even developed and included in this study. The reason is simply that all of the models that this author has seen are essentially similar to Model I, often less complete. If a firm advertises through media in which there is considerable advertising carry-over, this author recommends strongly against the exclusive use of a model such as Model I. The reason, as stated in Chapter VII, is that a pattern or level of advertising expenditures will be initiated which is either too low or too high to fully take advantage of profit potential. Model II has its dysfunctions as well, however. Model II requires the estimation of future competitor expenditures. If the competitor does not have a predictable advertising strategy or expenditure pattern, his future spending may be indeterminable.

As a general rule it is suggested that Model I be used when both 1) the advertising carry-over factor of Brand h is at or very near zero and 2) the retention factors of all brands are nearly equal. Model II should definitely be used when carry-over is substantial and when competitor's expenditures are reasonably predictable. Judgement as to the extent of use of Model II should be used when carry-over is high and future competitor spending is unpredictable.

The numerous influences of the four previously listed characteristics were determined. Although the findings are essentially the same regardless of which model is used, there is a difference between Model I and Model II concerning the level of spending at different values of the carry-over and retention factors. In the findings to be presented below, Model II will prevail with respect to the above differences. This is because Model II is a more complete model and better represents reality. The following summary is a composite presentation of the influences of each of the four characteristics as determined from Chapters IV, V, and VI.

1. Variable Demand as a Function of Time
    a. Causes optimal advertising and profit patterns to assume the same general shape as potential demand over time.
    b. If all other characteristics are negated, phase, relative amplitude, and shape of

advertising and profit at equilibrium are identical to that of potential demand when an identical competitor is assumed.

2. Advertising Carry-Over

   a. As the carry-over factor increases there is a tendency to increase the level of optimal advertising followed by a tendency to decrease the optimal level. The slight degree of increase and then decrease indicates a tendency for optimal spending to be rather insensitive to a firm's own carry-over factor.

   b. Tends to increase profit as the carry-over factor increases. Profit appears to be quite sensitive to carry-over.

   c. Causes a slight increase in the relative amplitude of the optimal advertising pattern as the carry-over factor increases.

   d. Causes peaks in optimal advertising to slightly lead peaks in potential demand. This lead is increased by a higher value of the carry-over factor.

3. Retention or Habitual Buying

   a. Has a tendency to maintain the same total-cycle equilibrium advertising and profit values for a given value of the carry-over factor when considering identical competitors, regardless of the value of their common retention factor.

b. For the case in which $q_h \leq q_g$, advertising by Brand h should be increased to the point that the decrease in present period profits is not more than matched by the increase in future period profits. For the case considered ($q_h \leq q_g$), this policy will motivate a higher level of spending than that of Model I.

c. For the case in which $q_h > q_g$, the policy of 3.b above should again be followed up to a point. That point occurs when the advertising expenditure during the present period will capture an excessive brand share this period and cause future profits to diminish. In this case, advertising should be increased until present period profit gains no longer exceed future period profit losses. In such a case, the optimal advertising level at a high value of retention may be less than that at a lower retention value. Also, in this case, Model II will utilize a lower level of optimal advertising than will Model I.

d. Causes higher "profit potential" and therefore higher Brand h profits as the retention factor of Brand h increases, provided sharply decreasing potential demand does not negate this tendency. The converse is true if Brand h retention decreases.

e. Causes higher "potential profit" and therefore higher Brand h profits as the retention factor of Brand h's competition decreases. The converse is true if their retention factor increases.

f. Causes widely fluctuating intracycle Brand h optimal advertising allocations and corresponding profit fluctuations at low values of Brand h retention.

g. Causes the optimal advertising pattern of Brand h to considerably lead potential demand at high values of Brand h retention.

4. Demand as a Function of Total Industry Advertising Expenditure.

a. Causes higher relative advertising and profit amplitudes including distortion of the potential demand curve shape - especially when operating in the steep range of the response curve P(t).

b. Has influence such that it lends support and example to the belief that consumer response should be at a level to sustain profitable activity in order to merit investment of advertising money.

As noted in Chapter V, the above comments concerning the influence and tendencies of the four characteristics may appear incorrect at times. Actually, the influence, as stated, remains correct. A closer look will reveal that the cause of the apparent contradiction is either to be attributed to competitor spending and/or the dominance of one of the above characteristic's influences.

The findings as presented do not answer all of the questions one might have concerning the optimal budgeting of advertising in a competitive market. However, they do contribute significantly to the body of available knowledge, particularly in answering the questions of whether or not higher values of carry-over and retention motivate higher or lower optimal advertising. Also, these findings have contributed in terms of recognizing the complications introduced when demand is a function of total industry advertising.

Suggestions for Future Study

With respect solely to the work in this dissertation, the one area which could use considerably more study is that of sensitivity. Of particular interest is the sensitivity of Brand h optimal advertising and profit to the carry-over and retention parameter changes of competitors. Also, the sensitivity of optimal advertising to competitor underspending and overspending (with respect to equilibrium) is of interest at various ranges of parameter values. As discussed in Chapter VII, however, such a complete sensitivity analysis would be overwhelming. The results may well not merit the effort required in view of the fact that a firm can conduct a limited sensitivity analysis as described in Chapter VII. Such a limited analysis will not provide general rules of sensitivity but it will provide specific indicators for care in the use of an advertising model in a genuine application.

Perhaps the most logical area for future research is that of a marketing model which not only includes advertising as a controllable variable, but also includes other elements of the marketing mix such as price and distribution expense. Actually, considerable work has been done in this area although there is a lack of empirical evidence that the resultant models actually describe market activity.

Another area of some interest in terms of a pure advertising model is that of a multigrade-single product class model. Such an area is quite applicable to many of today's industries. Assume n firms, each of which makes m grades of the same general product class. Corresponding grades of the same general product across firms could be assumed to sell at the same price. A firm's advertising could then be for a specific grade of its product, for all products bearing the firm's name, or a mixture of the two. A similar problem could be addressed in which a firm is assumed to produce several diversified, non-related products.

The most sought after answer in marketing today appears to be that of how to describe purchase to purchase consumer behavior. While many techniques have been developed which seem to adequately describe consumer behavior, there has been very little published on describing individual consumer purchase to purchase behavior as a function of the marketing mix. A model which accurately describes individual consumer behavior as a function of the marketing mix would be a significant contribution to marketing studies. Such a model would provide for examination of much more individualized or directed advertising. For example, such a model may help determine to which homogeneous part of a heterogeneous population a certain type of advertising should be directed.

It is suggested that a model concept such as the one developed herein may be applicable to other areas involving retention and carry-over effects such as evaluating research, quality change, etc. The concept may also be utilized where the objective is perhaps maximizing the organization's public image as opposed to maximizing profit.

Of course, there are many related areas in which work remains to be initiated or extended. For one who endeavors to develop or use a model concept similar to that of this dissertation it is only imperative that he be aware of its limitations and assumptions. Such has been the background of this study throughout. While this dissertation is certainly only a small study with respect to the entire subject area, it is hoped that it represents some small contribution to the optimal budgeting of advertising.

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FOREWARD TO THE APPENDICES

Appendix A is a detailed development of a continuous approximation to a sum of many discrete terms. This is an original development which arose of necessity during the calculation of numerous DEMAND terms (equation (4.9)).

Appendices B, C, D, E, and F each include a brief summary of the calculations performed by their respective computer programs. The organization of input cards is presented in some detail. The programs contained therein are written in FORTRAN IV for use on an IBM 360 Model 50. A program listing and sample output is presented in each appendix. The listing is shown with the control cards necessary to adapt the program to the WATFOR terminal. The WATFOR terminal is an auxilliary "hands-on" input-output system which operates in conjunction with the 360/50.

The programs listed in Appendices B, C, D, E, and F are documented by the use of explicative "COMMENT" cards. Such comment cards are identified by a "C" as opposed to a number in their first column. The "COMMENT" cards are used to describe the calculations and/or the steps being performed in different sections of the program.

APPENDIX A

A CONTINUOUS APPROXIMATION TO A CONVERGING INFINITE SUM: THE 'DEMAND' TERM

This appendix refers particularly to the demand (or potential demand) component, including the use of the retention buying factor, of an equation such as (3.14). Included in this treatment will be the time value of money factor, ρ, corresponding to a "single-payment-present-worth" factor.

Consider the discrete portions of the graphs in Figure 29. An advertising expenditure is to be made at time $t + \Delta$ (i.e., at the beginning of Month 1). The first element of the top graph represents S(t), the time average of demand over Month 1. The first element of the middle graph represents q_g^0. The first element of the bottom graph is ρ^1, the factor necessary to find the worth of accounted (at t+1) sales at the time of the advertising expenditure, $t + \Delta$. The second element of each graph represents S(t+1), q_g^1, and ρ^2, respectively. Notice that the product, each month, of these three terms, summed from Month 1 to infinity equals

$$\sum_{t''=1}^{\infty} S(t+t''-1)q_g^{(t''-1)}\rho^{t''} = [\rho S(t) + \rho^2 q_g S(t+1) + \rho^3 q_g^2 S(t+2) + \ldots] . \qquad (A.1)$$

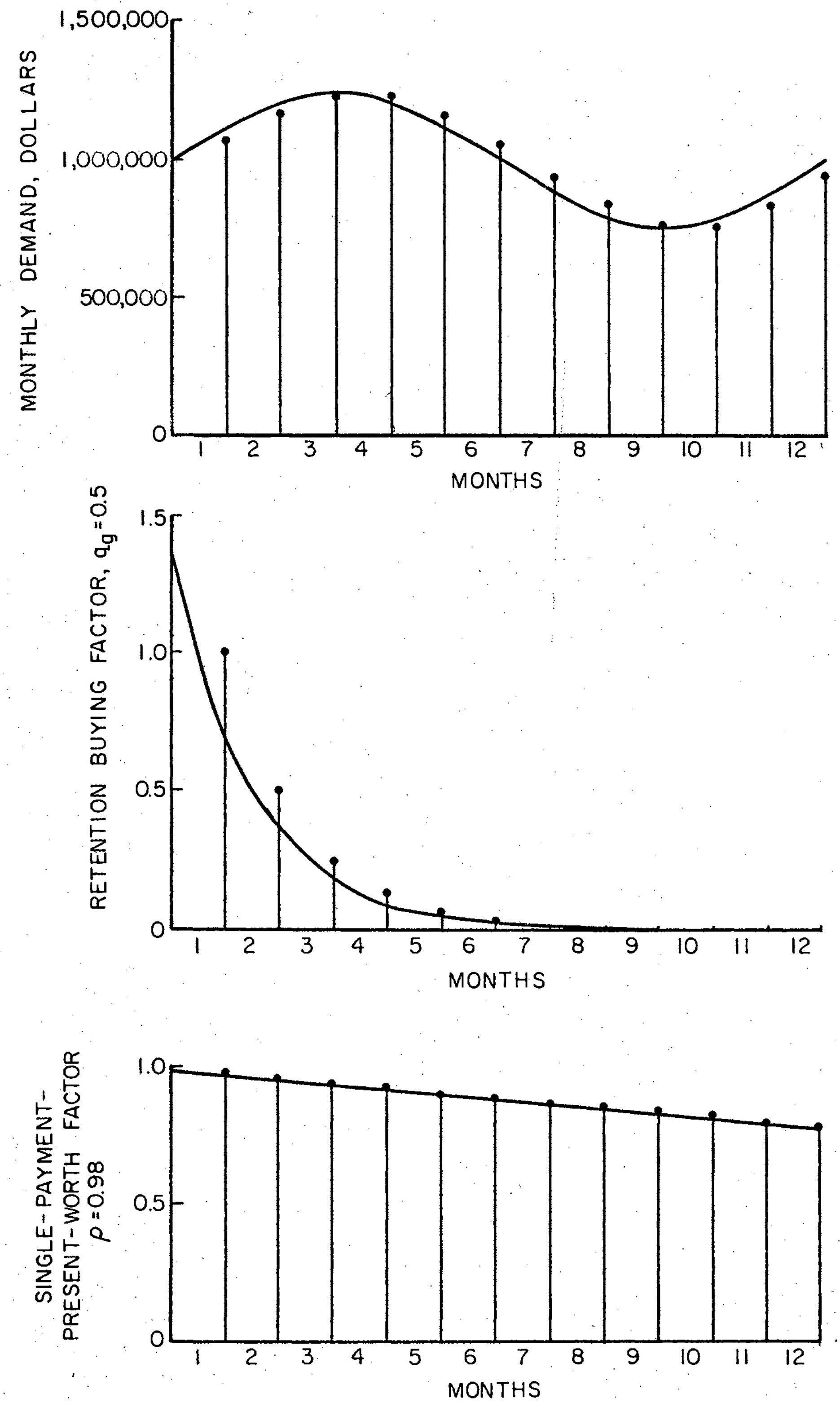

Figure 29. Continuous and Discrete Displays of Potential Demand, The Retention Buying Factor, and The Time Value of Money Factor

Note the similarity between (A.1) and the corresponding component of (3.14). The only difference is the inclusion of ρ, which if equal to 1.0, causes (A.1) to revert to the "demand" term in equation (3.14). Thus, the "demand" (later to be called DEMAND) term (the right hand side of equation (A.1)) is simply the sum of the product of the three elements at each time period from Month 1 into the future. This author knows that such a sum may require the inclusion of well over 75 time periods (at a high value of q_h) before truncation may take place without serious loss of accuracy. Thus, it was decided to try a continuous form of summation of products as follows:

$$\text{DEMAND} = \sum_{t''=1}^{\infty} S(t+t''-1)q_g^{(t''-1)}\rho^{t''} = \int_0^{\infty} f_1(t)f_2(t)f_3(t)dt \tag{A.2}$$

where

$f_1(t)$ = a continuous expression for demand (or potential demand) over time.

$f_2(t)$ = a continuous expression for q_g over time.

$f_3(t)$ = a continuous expression for ρ over time.

It has already been determined that the continuous expression for the sine-wave demand (or potential demand) used throughout this dissertation is

$$f_1(t) = 1{,}000{,}000 + 240{,}000 \sin(\pi t/6) \ . \tag{A.3}$$

It is now desired to determine $f_2(t)$ and $f_3(t)$. Due to their similarity in nature, the derivation for only $f_2(t)$ will be presented here.

The criterion for a continuous curve to represent the discrete values shown in the middle graph of Figure 28 is described by equation (A.4).

$$\int_0^1 f(t)\,dt = q_g^{\,0} = 1$$

$$\int_1^2 f(t)\,dt = q_g^{\,1}$$

$$\vdots$$

$$\int_{t^*}^{t^*+1} f(t)\,dt = q_g^{\,t^*} \quad . \tag{A.4}$$

It is desired to determine f(t).

$$\int_{t^*}^{t^*+1} f(t)\,dt = q_g^{\,t^*}$$

$$F(t^*+1) - F(t^*) = q_g^{\,t^*}$$

$$F'(t^*+1) - F'(t^*) = q_g^{\,t^*}\ln q_g$$

$$f(t^*+1) - f(t^*) = q_g^{\,t^*} \ln q_g$$

$$f(t^*+1) = f(t^*) + q_g^{\,t^*} \ln q_g$$

$$f(1) = f(0) + q_g^{\,0} \ln q_g \quad .$$

Let f(0) = b, a constant. Then

$$f(2) = b + q_g^{\,0} \ln q_g + q_g^{\,1} \ln q_g$$

$$f(3) = b + q_g^{\,0} \ln q_g + q_g^{\,1} \ln q_g + q_g^{\,2} \ln q_g$$

$$\vdots$$

$$f(t) = b + q_g^{\,0} \ln q_g + q_g^{\,1} \ln q_g + \ldots + q_g^{\,t-1} \ln q_g$$

$$= b + \ln q_g(1 + q_g + \ldots\ q_g^{\,t-1}) =$$

$$= b + \ln q_g \frac{1 - q_g^{\,t}}{1 - q_g} . \qquad (A.5)$$

The unknown intercept b can now be found as follows:

$$\int_{t*}^{t*+1} f(t)dt = \int_{t*}^{t*+1} b\ dt + \int_{t*}^{t*+1} \frac{\ln q_g}{1 - q_g}\ dt - \int_{t*}^{t*+1} \frac{\ln q_g}{1 - q_g} q_g^{\,t}\ dt$$

$$= q_g^{\,t*}$$

$$bt \Big|_{t*}^{t*+1} + \frac{\ln q_g}{1 - q_g} t \Big|_{t*}^{t*+1} - \frac{\ln q_g}{1 - q_g} \frac{q_g^{\,t}}{\ln q_g} \Big|_{t*}^{t*+1} = q_g^{\,t*}$$

$$b + \frac{\ln q_g}{1 - q_g} - \frac{(q_g^{\,t*+1} - q_g^{\,t*})}{1 - q_g} = q_g^{\,t*}$$

$$b + \frac{\ln q_g}{1 - q_g} - q_g^{\,t*} \frac{q_g - 1}{1 - q_g} = q_g^{\,t*}$$

$$b + \frac{\ln q_g}{1 - q_g} = q_g^{\,t*} - q_g^{\,t*} = 0$$

$$b = \frac{- \ln q_g}{1 - q_g} . \qquad (A.6)$$

Thus $f_2(t)$ reduces to $-\frac{q_g^{\,t} \ln q_g}{1 - q_g}$ and we have

$$f_2(t) = - \frac{q_g^{\,t} \ln q_g}{1 - q_g} . \qquad (A.7)$$

In a similar manner,

$$f_3(t) = - \frac{\rho^{t+1} \ln \rho}{1 - \rho} . \qquad (A.8)$$

If it is now desired to determine demand (or potential demand) starting at the beginning of month 1, discounted in terms of retention and time value of money, one can write

$$\int_0^\infty f_1(t)f_2(t)f_3(t)dt$$

$$= \int_0^\infty \left[(1{,}000{,}000 + 240{,}000 \sin(\pi t/6))\left(\frac{-(q_g{}^t)\ln q_g}{1-q_g}\right)\left(\frac{-(\rho^{t+1})\ln \rho}{1-\rho}\right)\right]dt$$

$$= 1{,}000{,}000 \frac{\ln q_g}{1-q_g}\frac{\ln \rho}{1-\rho}\, \rho\left[\int_0^\infty (\rho q_g)^t dt + .24 \int_0^\infty (\rho q_g)^t \sin(\pi t/6)dt\right]. \quad (A.9)$$

If

$$C = 1{,}000{,}000 \frac{\ln q_g}{1-q_g}\frac{\ln \rho}{1-\rho}\, p \quad (A.10)$$

then

$$\int_0^\infty f_1(t)f_2(t)f_3(t)dt = C\left[\frac{(\rho q_g)^t}{\ln \rho q_g}\Bigg|_0^\infty + .24 \int_0^\infty e^{t \ln \rho q_g} \sin(\pi t/6)dt\right]$$

$$= C\left[\frac{-1}{\ln \rho q_g} + .24 \frac{\pi/6}{(-\ln \rho q_g)^2 + (\pi/6)^2}\right]. \quad (A.11)$$

If, however, it is desired to determine demand (or potential demand), starting at the beginning of Month x + 1, discounted in terms of retention and the time value of money, one must displace the continuous form for q_g and ρ by x periods.

$$\int_x^\infty f_1(t)f_2(t)f_3(t)dt$$

$$= \int_x^\infty \left[(1{,}000{,}000 + 240{,}000 \sin(\pi t/6))\left(\frac{-(q_g{}^{(t-x)})\ln q_g}{1-q_g}\right) \cdot \left(\frac{-(\rho^{(t+1-x)})\ln \rho}{1-\rho}\right)\right]dt =$$

$$= 1{,}000{,}000 \, \frac{\ln q_g}{1 - q_g} \frac{\ln \rho}{1 - \rho} \, \rho \left[\int_x^\infty (\rho q_g)^{(t-x)} dt \right.$$

$$\left. + .24 \int_x^\infty (\rho q_g)^{(t-x)} \sin(\pi t/6) dt \right]$$

$$= C \left[\left. \frac{(\rho q_g)^{(t-x)}}{\ln \rho q_g} \right|_x^\infty + .24 \int_x^\infty e^{(t-x)\ln \rho q_g} \sin(\pi t/6) dt \right]$$

$$= C \left[\frac{-1}{\ln \rho q_g} + \left(.24 \, e^{(t-x)\ln \rho q_g} \right) \left(\left. \frac{(\ln \rho q_g) \, \sin(\pi t/6) - (\pi/6) \, \cos(\pi t/6)}{\ln \rho q_g)^2 + (\pi/6)^2} \right|_x^\infty \right) \right]$$

$$= C \left[\frac{-1}{\ln \rho q_g} - .24 \left(\frac{(\ln \rho q_g) \, \sin(\pi x/6) - (\pi/6) \, \cos(\pi x/6)}{(\ln \rho q_g)^2 + (\pi/6)^2} \right) \right]. \tag{A.12}$$

The calculations made using this technique were for $\rho = .98$ (corresponding roughly to i = 24% per year) and for $\rho = 1.0$ (i = 0%). Values of q_g ranged from .02 to .98 in small increments. This estimation method was found to be quite accurate (easily within 1%) for values of $q_g \geq .3$. At low values of q_g the continuous method was biased slightly low. At low values of q_g the discrete product summation was used due to its rapid convergence. At the high values of q_g the continuous method was certainly needed and it provided very accurate "sums."

In the actual body of the dissertation only the values $q_g = .0, .5, .9$ and $\rho = 1.0$ are used.

APPENDIX B

DETERMINATION OF THE 'DEMAND' TERM

The program listed in this appendix will determine the value of DEMAND:

$$\text{DEMAND} = S(t) + q_h S(t+1) + q_h^2 S(t+2) + \ldots \qquad (B.1)$$

The sample output following the program listing shows five values, E(5) through E(9). These are the respective DEMAND values needed for use in terms one through five of Model II. Only the value E(5) is needed in Model I.

As shown, the program will determine DEMAND where $\rho = 1.0$. If $\rho < 1.0$, the following change should be made.

Change 1: C(K) = (P*1000000.*DLOG(Q(K))*DLOG(P))/((1.-Q(K))*(1.-P))

This program is also only for use where the potential demand curve $f_1(t)$ is as follows:

$$f_1(t) = 1000000 + 240000 \sin(\pi t/6) \qquad (b.2)$$

As discussed in Appendix A, the number of summations necessary at low values of q_h is less than that number required at high values of q_h. In the program listed, each term $S(t)$, $q_h S(t+1)$, ... is determined and summed, sixty such terms being used when $q_h = 0.0$ or $q_h = 0.5$. At $q_h = 0.9$, many more than sixty terms are needed. Therefore, the continuous approach outlined in Appendix A is used when $q_h = 0.9$.

As presented, at Month 1,

$$E(5) = S(0) + q_h S(1) + q_h^2 S(2) + \ldots \qquad (B.3)$$

$$E(6) = S(1) + q_h S(2) + q_h^2 S(3) + \ldots \qquad (B.4)$$

and at Month 2,

$$E(5) = S(1) + q_h S(2) + q_h^2 S(3) + \ldots \qquad (B.5)$$

$$E(6) = S(2) + q_h S(3) + q_h^2 S(4) + \ldots \qquad (B.6)$$

Indeed, E(6) at Month 1 equals E(5) at Month 2. This, in general, is not the case when $\rho < 1.0$ as can be reasoned by considering equation (A.1).

Input data to this program should be as indicated:

Card 1: Value of π; col. 1-20 with up to 18 decimal places.

Card 2: Value of Monthly Present Worth Factor, ρ; cols. 1-5 with up to 4 decimal places.
Value of First Retention Factor, q_h; cols. 6-10 with up to 4 decimal places.
Value of Second Retention Factor, q_h; cols. 11-15 with up to 4 decimal places.
Value of Third Retention Factor, q_h; cols. 16-20 with up to 4 decimal places.

Card 3: Sequence of 15 Floating Point Integers - Begin With 0.0 if Starting at Time 0.0 on the Potential Demand Curve; each 5 cols. with decimal point but only zeroes in decimal places (i.e., 0.0 1.0 ... 14.0).

Card 4: Continue Card 3 - Two More Numbers:

(i.e., 15.0 16.0).

The listing of the statements and a sample output from this program are shown on the following pages.

```
$JOB 10322,444-42-7413,TIME=15          KEN CASE
C      SINE WAVE POTENTIAL DEMAND--DIRECT SUMMATION METHOD FOR Q=0.0 AND Q=0.5,
C      THEN CONTINUOUS METHOD OF APPENDIX A FOR Q=0.9
       DOUBLE PRECISION E(10),PI,DSIN,DLOG,P,Q(20),T(20),C(20),D(20),F(20
      1),DCOS
C
C      READ INPUT DATA AS EXPLAINED IN TEXT
       READ (5,5) PI, P, (Q(K),K=1,3), (T(M),M=1,17)
 5     FORMAT (F20.15/4F5.0/15F5.0/2F5.0)
       JJ=5
       KK=6
       LL=7
       MM=8
       NN=9
       K=3
       C(K)=(-1000000.*DLOG(Q(K)))/(1.-Q(K))
       D(K)=(((DLOG(P*Q(K)))**2)+((PI/6.0)**2))
 6     F(K)=DLOG(P*Q(K))
       I=1
C
C      BEGIN NEW PERIOD HERE
       DO 26 M=1,12
       Z=T(M)
C
C      WRITE MONTH UNDER CONSIDERATION
       WRITE (6,9) M
 9     FORMAT (6H1MONTH,I3)
C
C      THIS SECTION INVOLVES DIRECT SUMMATION OF TERMS TO DETERMINE E(5)
C      THROUGH E(9)
       DO 16 K=1,2
       T(M)=Z
       DO 8 J=5,9
       E(J)=0.0
       L=1
       E(J)=E(J)+((DCOS((PI*T(M))/6.)-DCOS((PI*(T(M)+1.))/6.))*(6./PI)*24
      10000.+1000000.)*(P**(J+L-5))
       T(M)=T(M)+1.
       DO 7 L=2,60
       E(J)=E(J)+((DCOS((PI*T(M))/6.)-DCOS((PI*(T(M)+1.))/6.))*(6./PI)*24
      10000.+1000000.)*(P**(J+L-5))*(Q(K)**(L-1))
 7     T(M)=T(M)+1.0
 8     T(M)=T(J+M-4)
C
C      WRITE VALUE OF RETENTION
       WRITE (6,11) Q(K)
 11    FORMAT (29H VALUE OF RETENTION FACTOR Q=,F5.3)
C
C      WRITE 'DEMAND' TERMS
       WRITE (6,17) JJ,E(5), KK,E(6),LL,E(7),MM,E(8),NN,E(9)
 17    FORMAT (5(3H E(,I1,2H)=,F11.2,2X)/)
 16    CONTINUE
C
C      THIS SECTION INVOLVES CONTINUOUS METHOD OF DETERMINING E(5) THROUGH E(9)
       K=3
       T(M)=Z
       DO 18 J=5,9
       E(J)=(P**(J-5))*C(K)*((-1./F(K))-((.24/D(K))*(F(K)*DSIN((PI*T(M))/
      16.)-PI/6.*DCOS((PI*T(M))/6.))))
 18    T(M)=T(M)+1.0
```

```
      C
      C      WRITE VALUE OF RETENTION
             WRITE (6,10) Q(K)
      10     FORMAT (29H VALUE OF RETENTION FACTOR Q=,F5.3)
      C
      C      WRITE 'DEMAND' TERMS
             WRITF (6,15) JJ,E(5), KK,E(6),LL,E(7),MM,E(8),NN,E(9)
      15     FORMAT (5(3H E(,I1,2H)=,F11.2,2X)/)
      C
      C      RETURN TO BEGIN A NEW PERIOD
      26     CONTINUE
      27     STOP
      19     END

      $ENTRY
```

```
MONTH  1
VALUE OF RETENTION FACTOR Q=0.000
E(5)= 1061409.43   E(6)= 1167773.69   E(7)= 1229183.12   E(8)= 1229183.12   E(9)= 1157773.59

VALUE OF RETENTION FACTOR Q=0.500
E(5)= 2239896.46   E(6)= 2356974.06   E(7)= 2378400.75   E(8)= 2298435.26   E(9)= 2138504.29

VALUE OF RETENTION FACTOR Q=0.900
E(5)=10464143.42   E(6)=10448658.33   E(7)=10312955.61   E(8)=10093396.68   E(9)= 9848812.19
```

APPENDIX C

CALCULATION OF EQUILIBRIUM ADVERTISING EXPENDITURE FOR EACH OF TWO IDENTICAL COMPETITORS USING MODEL I

The program listed in this appendix will calculate the equilibrium advertising expenditure for any number, Z, of identical competitors. The basis for this program is the advertising model of Chapter IV, Model I. The underlying equations of interest are (4.5), (4.6), and (4.7).

As described in Chapter IV, the past "equilibrium" advertising expenditures needed in this model when $b_h > 0.0$ are merely initial guesses or estimates. Since the competitors (Z = 2.0 is used in this research) are identical, one set of parameters will suffice as program inputs.

In order to approach a cycling equilibrium pattern of expenditures it may be necessary to go through the twleve month cycle several times. The number of time periods considered is controlled by the statement

```
IF(II.GE.12)GO TO 115
```

near the end of the program. If more than one cycle is desired, increase the decision level of II to, say, 36 for three complete cycles. The precision of calculation is controlled by card 150. A value of M = 4 will calculate the equilibrium advertising to the nearest ten dollars; M = 5 to the nearest dollar, etc.

The input data for this program involves a total of 15 cards plus the same number of DEMAND cards as periods to be considered. They are to be organized as follows:

Cards 1-4: Past equilibrium expenditure estimates in chronological order, $a_h(t-4)$ through $a_h(t-1)$; cols. 1-10 with decimal point.

Card 5: An initial value for the advertising expenditure to be determined, $a_h(t)$; cols. 1-10 with decimal point.

Card 6: Value of carry-over, b_h; cols. 1-5 with up to 4 decimal places.

Card 7: Value of Gompertz parameter, D; cols. 1-5 with up to 4 decimal places.

Card 8: Value of Gompertz parameter, D'; cols. 1-5 with up to 4 decimal places.

Card 9: Value determining precision of calculation of $a_h(t)$. This variable is overriden by card 150 in the program as shown, M; col. 1, with no decimal places.

Card 10: Value of each dollar of sales attributed to cost, r_h; cols. 1-5 with up to 4 decimal places.

Card 11: Value of Gompertz parameter, S; cols. 1-5 with up to 4 decimal places.

Card 12: Value of initial increment in determining

$a_h(t)$. This variable is overriden by STEP = 10000. in the program as shown, STEP; cols. 1-10 with decimal point.

Card 13: Value of Gompertz parameter, U; cols. 1-9, with up to 8 decimal places.

Card 14: Number of identical competitors, Z; cols. 1-5 with decimal point.

Card 15: Value of retention for all brands, q_h; cols. 1-5 with up to 4 decimal places.

Cards 16 - : (Use same number as periods to be considered) Month and value of DEMAND term, MONTH, E(5); cols. 1-5, right justify the month number; cols. 6-20, E(5), with decimal point. See Appendix B for determining E(5) value.

The program as described is listed starting on the next page. It is followed by a sample output.

```
$JOB 10322,444-42-7413,TIME=15          KEN CASE
      C      PROGRAM TO DETERMINE EQUILIBRIUM ADVERTISING EXPENDITURE FOR EACH
      C      OF Z IDENTICAL COMPETITORS USING MODEL 1
      1      DOUBLE PRECISION A(20),B,D,E(10),Q,R,S,STEP,J,Z,Y(10),GOMEXP(10),A
            1LAST,PRLAST,PROFIT,P(10),F(10),PN(10),AA,BB,YY,X,DABS,DP
      C
      C      READ INPUT DATA AS EXPLAINED IN TEXT
      2      READ (5,3) (A(J),J=1,5),B,D,DP,M,R,S,STEP,U,Z,Q
      3      FORMAT (F10.0/F10.0/F10.0/F10.0/F10.0/F5.0/F5.0/F5.0/I1/F5.0/F5.0/
            1F10.0/F9.0/F5.0/F5.0)
             II=0
             JJ=6
             KK=7
             LL=8
             MM=9
             NN=10
      C
      C      THE INPUT DATA TO BE READ NOW IS OUTPUT DATA FROM THE PROGRAM OF
      C      APPENDIX B
      170    READ (5,130) MONTH,E(5)
      130    FORMAT (I5,F15.0)
      150    M=6
             STEP=10000.
      C
      C      CALCULATE POTENTIAL PROFIT OF 'PRESENT' PERIOD
      44     Y(5)=(1.-R)*(1.-Q)*E(5)
      C
      C      THE FOLLOWING SEVEN STATEMENTS DETERMINE HOW MUCH EACH OF THE FOUR
      C      PREVIOUS ADVERTISING EXPENDITURES CONTRIBUTES TO THE RESPONSE CURVE
             GOMEXP(1)=Z*((B**4)*A(1))
             GOMEXP(2)=GOMEXP(1)+Z*((B**3)*A(2))
             GOMEXP(3)=GOMEXP(2)+Z*((B**2)*A(3))
             GOMEXP(4)=GOMEXP(3)+Z*(B*A(4))
             P(1)=((D**(S**(U*GOMEXP(1))))-DP)/(1.-DP)
             DO 63 J=2,4
      63     P(J)=((D**(S**(U*GOMEXP(J))))-(D**(S**(U*GOMEXP(J-1)))))/(1.-DP)
      C
      C      INITIALIZE LOGIC VARIABLES TO BE USED IN THE EQUILIBRIUM SEARCH PORTION
      C      OF THE PROGRAM
      52     N=0
             AA=0.0
             ALAST=0.0
      54     BB=AA
             AA=ALAST
             ALAST=A(5)
             YY=BB-ALAST
             IF(DABS(YY).LE..001)GO TO 100
      154    PRLAST=-10000.
      55     CONTINUE
      58     N=N+1
      C
      C      BEGIN CALCULATION OF PROFIT
             GOMEXP(5)=GOMEXP(4)+(Z-1.)*ALAST+A(5)
             PN(5)=((D**(S**(U*GOMEXP(5))))-DP)/(1.-DP)
             P(5)=PN(5)-(((D**(S**(U*GOMEXP(4))))-DP)/(1.-DP))
             F(5)=(A(5)+B*A(4)+(B**2)*A(3)+(B**3)*A(2)+(B**4)*A(1))/(A(5)+(Z-1.
            1)*ALAST+Z*(B*A(4)+(B**2)*A(3)+(B**3)*A(2)+(B**4)*A(1)))
             PROFIT=Y(5)*PN(5)*F(5)-A(5)
      C
      C      BEGIN EQUILIBRIUM SEARCH PORTION OF PROGRAM
```

```
86     IF(N.LE.2) GO TO 88
87     GO TO 92
88     IF(PROFIT.LE.PRLAST) GO TO 90
89     GO TO 97
90     A(5)=STEP/10.0
91     GO TO 100
92     IF(PROFIT.LE.PRLAST) GO TO 94
93     GO TO 97
94     A(5)=A(5)-(2.*STEP)
95     IF(A(5).LE.0.0) GO TO 120
       X=A(5)-ALAST
       IF(DABS(X).GE..001) GO TO 54
96     GO TO 100
97     PRLAST=PROFIT
98     A(5)=A(5)+STEP
99     GO TO 55
120    A(5)=STEP/10.0
       ALAST=A(5)
100    M=M-1
101    IF(M.LE.0) GO TO 103
       STEP=STEP/10.0
102    GO TO 154
103    A(5)=A(5)+STEP
C
C      AFTER COMPETITIVE EQUILIBRIUM IS DETERMINED, WRITE PARAMETERS, INPUTS,
C      ADVERTISING EXPENDITURES, RESPONSE COEFFICIENTS, AND PROFIT
135    WRITE (6,104)
104    FORMAT (1H1)
105    WRITE (6,106)
106    FORMAT ((20H SINE WAVE POTENTIAL)/)
107    WRITE (6,108) MONTH
108    FORMAT ((6H MONTH,I3)/)
       WRITE (6,30) D, DP, S, U
30     FORMAT ((27H GOMPERTZ PARAMETERS ARE D=,F5.2,3X,3HDP=,F5.2,3X,2HS=
      1,F5.2,3X,2HU=,F10.7)/)
       WRITE (6,160) B
160    FORMAT ((33H ADVERTISING CARRY-OVER FACTOR B=,F5.2)/)
113    WRITE (6,114) Q
114    FORMAT ((20H RETENTION FACTOR Q=,F5.2)/)
       WRITE (6,16) E(5)
16     FORMAT ((46H DEMAND DISCOUNTED IN TERMS OF RETENTION E(5)=,F12.2)/
      1)
       WRITE (6,42) Y(5)
42     FORMAT ((66H POTENTIAL PROFIT VALUE AS SEEN BY EACH IDENTICAL COMP
      1ETITOR Y(5)=,F16.8)/)
       DO 164 J=1,4
       WRITE (6,163) J,A(J)
163    FORMAT ((32H PAST ADVERTISING EXPENDITURE A(,I1,2H)=,F12.2)/)
164    CONTINUE
109    WRITE (6,110) A(5)
110    FORMAT ((42H EQUILIBRIUM ADVERTISING EXPENDITURE A(5)=,F12.2)/)
       DO 167 J=1,5
       WRITE (6,166) J,P(J)
166    FORMAT ((24H RESPONSE COEFFICIENT P(,I1,2H)=,F9.6)/)
167    CONTINUE
       WRITE (6,168) PN(5)
168    FORMAT ((36H OVERALL RESPONSE COEFFICIENT PN(5)=,F9.6)/)
117    WRITE (6,112) Z,PRLAST
112    FORMAT (31H EQUILIBRIUM PROFIT FOR EACH OF,F3.0,25H IDENTICAL COMP
      1ETITORS IS,F12.2)
```

```
C
C     PROCEED TO NEXT PERIOD
      II=II+1
      IF(II.GE.12)GO TO 115
      DO 180 J=1,4
180   A(J)=A(J+1)
      A(5)=10000.
      GO TO 170
115   STOP
116   END

$ENTRY
```

SINE WAVE POTENTIAL

MONTH 1

GOMPERTZ PARAMETERS ARE D= 0.30 DP= 0.30 S= 0.60 U= 0.0000124

ADVERTISING CARRY-OVER FACTOR B= 0.50

RETENTION FACTOR Q= 0.00

DEMAND DISCOUNTED IN TERMS OF RETENTION E(5)= 1061409.43

POTENTIAL PROFIT VALUE AS SEEN BY EACH IDENTICAL COMPETITOR Y(5)= 424563.77200000

PAST ADVERTISING EXPENDITURE A(1)= 8440.60

PAST ADVERTISING EXPENDITURE A(2)= 21255.10

PAST ADVERTISING EXPENDITURE A(3)= 47036.80

PAST ADVERTISING EXPENDITURE A(4)= 61254.50

EQUILIBRIUM ADVERTISING EXPENDITURE A(5)= 70820.10

RESPONSE COEFFICIENT P(1)= 0.003451

RESPONSE COEFFICIENT P(2)= 0.017444

RESPONSE COEFFICIENT P(3)= 0.078013

RESPONSE COEFFICIENT P(4)= 0.199221

RESPONSE COEFFICIENT P(5)= 0.357771

OVERALL RESPONSE COEFFICIENT PN(5)= 0.655900

EQUILIBRIUM PROFIT FOR EACH OF 2. IDENTICAL COMPETITORS IS 68415.64

APPENDIX D

CALCULATION OF EQUILIBRIUM ADVERTISING EXPENDITURE FOR EACH OF TWO IDENTICAL COMPETITORS USING MODEL II

The program listed in this appendix will calculate the equilibrium advertising expenditure for any number, Z, of identical competitors. The basis for this program is the Model of Chapter V, Model II. The underlying equations of interest are (5.3), (5.4), and (5.5).

As described in Chapter V, the past and future "equilibrium" advertising expenditures needed in this model when $b_h > 0.0$ are merely initial guesses or estimates. Since the competitors (Z = 2.0 is used in this research) are identical, one set of parameters will suffice as program inputs.

In order to approach a cycling equilibrium pattern of expenditures it may be necessary to go through the twelve month cycle several times. The number of time periods considered is controlled by the statement

```
IF(II.GE.48) GO TO 115
```

near the end of the program. If more than four cycles are desired, increase the decision level of II to, say, 72 for six complete cycles. The precision of calculation is controlled by card 150. A value of M = 4 will calculate the equilibrium advertising to the nearest ten dollars; M = 5 to the nearest dollar, etc.

The input data for this program involves a total of 22 cards plus the same number of DEMAND cards as periods to be considered. They are organized as follows:

Cards 1-4: Past equilibrium expenditure estimates in chronological order, $a_h(t-4)$ through $a_h(t-1)$; cols. 1-10 with decimal point.

Card 5: An initial value for the advertising expenditure to be determined, $a_h(t)$; cols. 1-10 with decimal point.

Cards 6-12: Future equilibrium expenditure estimates in chronological order, $a_h(t+1)$ through $a_h(t+7)$; cols. 1-10 with decimal point.

Card 13: Value of carry-over, b_h; cols. 1-5 with up to 4 decimal places.

Card 14: Value of Gompertz parameter, D; cols. 1-5 with up to 4 decimal places.

Card 15: Value of Gompertz parameter, D'; cols. 1-5 with up to 4 decimal places.

Card 16: Value determining precision of calculation of $a_h(t)$. This variable is overriden by card 150 in the program as shown, M; col. 1, with no decimal places.

Card 17: Value of each dollar of sales attributed to cost, r_h; cols. 1-5 with up to 4 decimal places.

Card 18: Value of Gompertz parameter, S; cols. 1-5 with up to 4 decimal places.

Card 19: Value of initial increment in determining $a_h(t)$. This variable is overridden by STEP = 10000. in the program as shown, STEP; cols. 1-10 with decimal point.

Card 20: Value of Gompertz parameter, U; cols. 1-9, with up to 8 decimal places.

Card 21: Number of identical competitors, Z; cols 1-5 with decimal point.

Card 22: Value of retention for all brands, q_h; cols. 1-5 with up to 4 decimal places.

Card 23 - : Month and value of DEMAND terms, MONTH, E(5), E(6), E(7), E(8), E(9); cols. 1-5, right justify the month number; Cols. 6-20, 21-35, 36-50, 51-65, 66-80, E(5) through E(9), respectively, with decimal point. See Appendix B for determining E(5) through E(9) values.

The program as described is listed starting on the next page. It is followed by a sample output.

```
$JOB 10322,444-42-7413,TIME=25          KEN CASE
      C     PROGRAM TO DETERMINE EQUILIBRIUM ADVERTISING EXPENDITURE FOR EACH
      C     OF Z IDENTICAL COMPETITORS USING MODEL 2
      1     DOUBLE PRECISION A(20),B,D,E(10),Q,R,S,STEP,U,Z,Y(10),GOMEXP(10,10
           1),ADNTSB,PRLAST,PROFIT,P(10,10),F(10),ALAST,PN(10,10),AA,BB,YY,X,D
           2ABS,PRONOW
      C
      C     READ INPUT DATA AS EXPLAINED IN TEXT
      2     READ (5,3) (A(J),J=1,12),B,D,DP,M,R,S,STEP,U,Z,Q
      3     FORMAT (F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.
           10/F10.0/F10.0/F5.0/F5.0/F5.0/I1/F5.0/F5.0/F10.0/F10.0/F5.0/F5.0)
            II=0
            JJ=6
            KK=7
            LL=8
            MM=9
            NN=10
      C
      C     THE INPUT DATA TO BE READ NOW IS OUTPUT DATA FROM THE PROGRAM OF
      C     APPENDIX B
      170   READ (5,130) MONTH,(E(J),J=5,9)
      130   FORMAT (I5,5F15.0)
      150   M=4
            STEP=10000.
      C
      C     CALCULATE POTENTIAL PROFIT OF 'PRESENT' AND 'FUTURE' PERIODS
      43    DO 44 J=5,9
      44    Y(J)=(1.-R)*(1.-Q)*E(J)
      C
      C     THE FOLLOWING EIGHT STATEMENTS DETERMINE HOW MUCH EACH OF THE FOUR
      C     PREVIOUS ADVERTISING EXPENDITURES CONTRIBUTES TO THE RESPONSE CURVE
            GOMEXP(5,1)=Z*(B**4)*A(1)
            GOMEXP(5,2)=GOMEXP(5,1)+Z*(B**3)*A(2)
            GOMEXP(5,3)=GOMEXP(5,2)+Z*(B**2)*A(3)
            GOMEXP(5,4)=GOMEXP(5,3)+Z*B*A(4)
            P(5,1)=((D**(S**(U*GOMEXP(5,1))))-DP)/(1.-DP)
            P(5,2)=((D**(S**(U*GOMEXP(5,2))))-(D**(S**(U*GOMEXP(5,1)))))/(1.-D
           1P)
            P(5,3)=((D**(S**(U*GOMEXP(5,3))))-(D**(S**(U*GOMEXP(5,2)))))/(1.-D
           1P)
            P(5,4)=((D**(S**(U*GOMEXP(5,4))))-(D**(S**(U*GOMEXP(5,3)))))/(1.-D
           1P)
      C
      C     INITIALIZE LOGIC VARIABLES TO BE USED IN THE EQUILIBRIUM SEARCH PORTION
      C     OF THE PROGRAM
      52    N=0
            AA=0.0
            ALAST=0.0
      54    BB=AA
            AA=ALAST
            ALAST=A(5)
            YY=BB-ALAST
            IF(DABS(YY).LE..001)GO TO 100
      154   PRLAST=-10000.
      55    CONTINUE
      58    N=N+1
      C
      C     BEGIN CALCULATION OF 'PRESENT' PERIOD PROFIT BEFORE ADVERTISING
            GOMEXP(5,5)=GOMEXP(5,4)+A(5)+(Z-1.)*ALAST
            PN(5,5)=((D**(S**(U*GOMEXP(5,5))))-DP)/(1.-DP)
```

```
      P(5,5)=PN(5,5)-(((D**(S**(U*GOMEXP(5,4))))-DP)/(1.-DP))
      F(5)=(A(5)+B*A(4)+(B**2)*A(3)+(B**3)*A(2)+(B**4)*A(1))/(A(5)+(Z-1.
     1)*ALAST+Z*(B*A(4)+(B**2)*A(3)+(B**3)*A(2)+(B**4)*A(1)))
      ADNTSB=Y(5)*PN(5,5)*F(5)
      PRONOW=ADNTSB-A(5)
C
C     BEGIN CALCULATION OF 1ST 'FUTURE' PERIOD PROFIT BEFORE ADVERTISING
      GOMEXP(6,6)=Z*A(6)+B*A(5)+B*(Z-1.)*ALAST+Z*((B**2)*A(4)+(B**3)*A(3
     1)+(B**4)*A(2))
      PN(6,6)=((D**(S**(U*GOMEXP(6,6))))-DP)/(1.-DP)
      F(6)=(A(6)+B*A(5)+(B**2)*A(4)+(B**3)*A(3)+(B**4)*A(2))/(Z*A(6)+B*A
     1(5)+B*(Z-1.)*ALAST+Z*((B**2)*A(4)+(B**3)*A(3)+(B**4)*A(2)))
      ADNTSB=ADNTSB+Y(6)*PN(6,6)*F(6)
C
C     BEGIN CALCULATION OF 2ND 'FUTURE' PERIOD PROFIT BEFORE ADVERTISING
      GOMEXP(7,7)=Z*(A(7)+B*A(6))+(B**2)*(A(5)+(Z-1.)*ALAST)+Z*((B**3)*A
     1(4)+(B**4)*A(3))
      PN(7,7)=((D**(S**(U*GOMEXP(7,7))))-DP)/(1.-DP)
      F(7)=(A(7)+B*A(6)+(B**2)*A(5)+(B**3)*A(4)+(B**4)*A(3))/(Z*(A(7)+B*
     1A(6))+(B**2)*(A(5)+(Z-1.)*ALAST)+Z*((B**3)*A(4)+(B**4)*A(3)))
      ADNTSB=ADNTSB+Y(7)*PN(7,7)*F(7)
C
C     BEGIN CALCULATION OF 3RD 'FUTURE' PERIOD PROFIT BEFORE ADVERTISING
      GOMEXP(8,8)=Z*(A(8)+B*A(7)+(B**2)*A(6))+(B**3)*(A(5)+(Z-1.)*ALAST)
     1+Z*(B**4)*A(4)
      PN(8,8)=((D**(S**(U*GOMEXP(8,8))))-DP)/(1.-DP)
      F(8)=(A(8)+B*A(7)+(B**2)*A(6)+(B**3)*A(5)+(B**4)*A(4))/(Z*(A(8)+B*
     1A(7)+(B**2)*A(6))+(B**3)*(A(5)+(Z-1.)*ALAST)+Z*(B**4)*A(4))
      ADNTSB=ADNTSB+Y(8)*PN(8,8)*F(8)
C
C     BEGIN CALCULATION OF 4TH 'FUTURE' PERIOD PROFIT BEFORE ADVERTISING
      GOMEXP(9,9)=Z*(A(9)+B*A(8)+(B**2)*A(7)+(B**3)*A(6))+(B**4)*(A(5)+(
     1Z-1.)*ALAST)
      PN(9,9)=((D**(S**(U*GOMEXP(9,9))))-DP)/(1.-DP)
      F(9)=(A(9)+B*A(8)+(B**2)*A(7)+(B**3)*A(6)+(B**4)*A(5))/(Z*(A(9)+B*
     1A(8)+(B**2)*A(7)+(B**3)*A(6))+(B**4)*(A(5)+(Z-1.)*ALAST))
      ADNTSB=ADNTSB+Y(9)*PN(9,9)*F(9)
C
C     CALCULATE PROFIT
      PROFIT=ADNTSB-A(5)-A(6)-A(7)-A(8)-A(9)
C
C     BEGIN EQUILIBRIUM SEARCH PORTION OF PROGRAM
   86 IF(N.LE.2) GO TO 88
   87 GO TO 92
   88 IF(PROFIT.LE.PRLAST) GO TO 90
   89 GO TO 97
   90 A(5)=STEP/10.0
   91 GO TO 100
   92 IF(PROFIT.LE.PRLAST) GO TO 94
   93 GO TO 97
   94 A(5)=A(5)-(2.*STEP)
   95 IF(A(5).LE.0.0) GO TO 120
      X=A(5)-ALAST
      IF(DABS(X).GE..001) GO TO 54
   96 GO TO 100
   97 PRLAST=PROFIT
   98 A(5)=A(5)+STEP
   99 GO TO 55
  120 A(5)=STEP/10.0
      ALAST=A(5)
```

```
100   M=M-1
101   IF(M.LE.0) GO TO 103
      STEP=STEP/10.0
102   GO TO 154
103   A(5)=A(5)+STEP
      WRITE (6,104)
104   FORMAT (1H1)
C
C     AFTER COMPETITIVE EQUILIBRIUM IS DETERMINED, WRITE PARAMETERS, INPUTS,
C     ADVERTISING EXPENDITURES, RESPONSE COEFFICIENTS, AND PROFIT
105   WRITE (6,106)
106   FORMAT ((20H SINE WAVE POTENTIAL)/)
107   WRITE (6,108) MONTH
108   FORMAT ((6H MONTH,I3)/)
      WRITE (6,30) D,DP,S,U
30    FORMAT ((27H GOMPERTZ PARAMETERS ARE D=,F5.2,3X,3HDP=,F5.2,3X,2HS=
     1,F5.2,3X,2HU=,F10.7)/)
      WRITE (6,160) B
160   FORMAT ((33H ADVERTISING CARRY-OVER FACTOR B=,F5.2)/)
113   WRITE (6,114) Q
114   FORMAT ((20H RETENTION FACTOR Q=,F5.2)/)
15    WRITE (6,16)
16    FORMAT (48H DEMAND DISCOUNTED IN TERMS OF RETENTION AND TVM)
17    DO 20 J=5,9
18    WRITE (6,19) J, J, E(J)
19    FORMAT ((8H PERIOD ,I1,5H   E(,I1,2H)=,F12.2)/)
20    CONTINUE
41    WRITE (6,42)
42    FORMAT (72H POTENTIAL PROFIT VALUE FROM TIME J AS SEEN BY EACH IDE
     1NTICAL COMPETITOR)
      DO 47 J=5,9
45    WRITE (6,46) J, Y(J)
46    FORMAT ((3H Y(,I1,2H)=,F16.8)/)
47    CONTINUE
      DO 164 J=1,4
      WRITE (6,163) J,A(J)
163   FORMAT ((32H PAST ADVERTISING EXPENDITURE A(,I1,2H)=,F12.2)/)
164   CONTINUE
109   WRITE (6,110) A(5)
110   FORMAT ((42H EQUILIBRIUM ADVERTISING EXPENDITURE A(5)=,F12.2)/)
      DO 200 J=6,9
      WRITE (6,199) J,A(J)
199   FORMAT ((34H FUTURE ADVERTISING EXPENDITURE A(,I1,2H)=,F12.2)/)
200   CONTINUE
      DO 167 J=1,5
      WRITE (6,166) J,P(5,J)
166   FORMAT ((26H RESPONSE COEFFICIENT P(5,,I1,2H)=,F9.6)/)
167   CONTINUE
      DO 169 J=5,9
      WRITE (6,168) J,J,PN(J,J)
168   FORMAT ((33H OVERALL RESPONSE COEFFICIENT PN(,I1,1H,,I1,2H)=F9.6)/
     1)
169   CONTINUE
117   WRITE (6,112) Z,PRONOW
112   FORMAT (27H OPTIMAL PROFIT FOR EACH OF,F3.0,25H IDENTICAL COMPETIT
     1ORS IS,F12.2)
C
C     PROCEED TO THE NEXT PERIOD
      II=II+1
      IF(II.GE.48) GO TO 115
```

```
          A(13)=A(1)
          DO 180 J=1,12
    180   A(J)=A(J+1)
          A(5)=10000.
          GO TO 170
    115   STOP
    116   END

    $ENTRY
```

SINE WAVE POTENTIAL

MONTH 1

GOMPERTZ PARAMETERS ARE D= 0.30 DP= 0.30 S= 0.60 U= 0.0000124

ADVERTISING CARRY-OVER FACTOR B= 0.50

RETENTION FACTOR Q= 0.00

DEMAND DISCOUNTED IN TERMS OF RETENTION AND TVM
PERIOD 5 E(5)= 1061409.43

PERIOD 6 E(6)= 1167773.69

PERIOD 7 E(7)= 1229183.12

PERIOD 8 E(8)= 1229183.12

PERIOD 9 E(9)= 1167773.69

POTENTIAL PROFIT VALUE FROM TIME J AS SEEN BY EACH IDENTICAL COMPETITOR
Y(5)= 424563.77200000

Y(6)= 467109.47600000

Y(7)= 491673.24800000

Y(8)= 491673.24800000

Y(9)= 467109.47600000

PAST ADVERTISING EXPENDITURE A(1)= 85000.00

PAST ADVERTISING EXPENDITURE A(2)= 93270.00

PAST ADVERTISING EXPENDITURE A(3)= 108820.00

PAST ADVERTISING EXPENDITURE A(4)= 119240.00

EQUILIBRIUM ADVERTISING EXPENDITURE A(5)= 133410.00

FUTURE ADVERTISING EXPENDITURE A(6)= 142320.00

FUTURE ADVERTISING EXPENDITURE A(7)= 144600.00

FUTURE ADVERTISING EXPENDITURE A(8)= 139660.00

FUTURE ADVERTISING EXPENDITURE A(9)= 128340.00

RESPONSE COEFFICIENT P(5,1)= 0.034935

RESPONSE COEFFICIENT P(5,2)= 0.077470

RESPONSE COEFFICIENT P(5,3)= 0.177021

RESPONSE COEFFICIENT P(5,4)= 0.315987

RESPONSE COEFFICIENT P(5,5)= 0.311877

```
OVERALL RESPONSE COEFFICIENT PN(5,5)= 0.917290

OVERALL RESPONSE COEFFICIENT PN(6,6)= 0.936219

OVERALL RESPONSE COEFFICIENT PN(7,7)= 0.945414

OVERALL RESPONSE COEFFICIENT PN(8,8)= 0.945987

OVERALL RESPONSE COEFFICIENT PN(9,9)= 0.937919

OPTIMAL PROFIT FOR EACH OF 2. IDENTICAL COMPETITORS IS     61312.21
```

APPENDIX E

CALCULATION OF OPTIMAL ADVERTISING EXPENDITURE USING MODEL I

The program listed in this appendix will calculate the optimal advertising expenditure for one of two competitors. As shown, this program will find the optimal spending pattern over a complete cycle of potential demand. The basis for this program is the advertising model of Chapter IV, Model I. The underlying equations of interest are (4.5), (4.6), and (4.7).

As described in Chapter VI, the past advertising expenditures are needed for both firms in this model when $b_1 > 0.0$ and $b_2 > 0.0$. Also, the competitor's expected advertising at the period under consideration is needed. Assuming, as in Chapter VI, that the competitor's expenditure pattern is known, this program will determine the optimal Brand 1 advertising over time. Since the two competitors are non-identical, two sets of parameters are necessary program inputs.

In order to approach a cycling optimal pattern of advertising expenditures it may be necessary to go through the twelve month cycle several times. The number of time periods is controlled by the statement

IF(II.GE.12)GO TO 115

near the end of the program. If more than one cycle is desired, increase the decision level of II to, say, 36 for three complete cycles. The precision of calculation is controlled by card 150. A value of M = 4 will calculate the optimal advertising to the nearest ten dollars; M = 5 to the nearest dollar, etc.

The input data for this program involves a total of 34 cards plus the same number of DEMAND cards as periods to be considered. They are to be organized as follows:

Cards 1-4: Past optimal expenditure estimates in chronological order, $a_1(t-4)$ through $a_1(t-1)$; cols. 1-10 with decimal point.

Card 5: An initial value for the advertising expenditure to be determined, $a_1(t)$; cols. 1-10 with decimal point.

Cards 6-12: Dummy values for future advertising. These values have no bearing on the optimal advertising value determined. They merely assign values to the variables, $a_1(t+1)$ through $a_1(t+7)$; cols. 1-10 with decimal point.

Cards 13-16: Past four competitor expenditures in chronological order, $a_2(t-4)$ through $a_2(t-1)$; cols. 1-10 with decimal point.

Card 17: Estimate of competitor's present period expenditure, $a_2(t)$; cols. 1-10 with decimal point.

Cards 18-24: Estimates of competitor's future period expenditures, $a_2(t+1)$ through $a_2(t+7)$; cols. 1-10 with decimal point.

Card 25: Value of each brand's advertising carry-over factor, b_1 and b_2; cols. 1-5 and 6-10, respectively, with up to 4 decimal places.

Card 26: Value of each firm's brand-share at the end of last period, $c_1(t)$ and $c_2(t)$; cols. 1-5 and 6-10, respectively, with up to 4 decimal places.

Card 27: Value of Gompertz parameter, D; cols. 1-5 with up to 4 decimal places.

Card 28: Value of Gompertz parameter, D'; cols. 1-5 with up to 4 decimal places.

Card 29: Value determining precision of calculation of $a_1(t)$. This variable is overriden by card 150 in the program as shown, M; col. 1 with no decimal places.

Card 30: Value of each dollar of sales attributed to cost, r_h; cols. 1-5 with up to 4 decimal places.

Card 31: Value of Gompertz parameter, S; cols. 1-5 with up to 4 decimal places.

Card 32: Value of initial increment in determining $a_1(t)$. This variable is overriden by STEP = 10000. in the program as shown, STEP; cols. 1-10 with decimal point.

Card 33: Value of Gompertz parameter, U; cols. 1-9, with up to 8 decimal places.

Card 34: Value of each firm's retention factor,

q_1 and q_2; cols. 1-5 and 6-10, respectively, with up to 4 decimal places.

Cards 35 - : Month and value of DEMAND term, MONTH, E(5); cols. 1-5, right justify the month number; cols. 6-20, E(5), with decimal point. See Appendix B for determining E(5) value.

(Use same number as periods to be considered)

The program as described is listed starting on the next page. It is followed by a sample output.

It should be noted that this program can be used for optimizing a single period's advertising. In such a case, the past four expenditures of both firms must be known. Also, the expected present period advertising of the competition must be known. The input slots for each firm's future expenditures may be filled with dummy values as they are not needed when only one (the present) period is considered. If the number of periods is then reduced to 1 by

```
IF(II.GE.01)GO TO 115
```

the optimal advertising $a_1(t)$ will be determined. In a similar manner, any number of periods may be considered.

```
$JOB 10322,444-42-7413,TIME=15        KEN CASE
C     PROGRAM TO DETERMINE OPTIMAL ADVERTISING EXPENDITURE FOR ONE OF
C     TWO COMPETITORS USING MODEL 1
 1    DOUBLE PRECISION A(2,20),B(2),E(10),Q(2),R,S,STEP,U,Y(10),GOMEXP(1
     10),C(2),ADNTSB,PRLAST,PROFIT,P(10),F(10),PN(10),DABS,NUM,PCTINF,D,
     2DP,PRONOW,CN(2)
C
C     READ INPUT DATA AS EXPLAINED IN TEXT
 2    READ (5,3) (A(1,J),J=1,12),(A(2,J),J=1,12),(B(I),I=1,2),(C(I),I=1,
     12),D,DP,M,R,S,STEP,U,(Q(I),I=1,2)
 3    FORMAT (F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.
     10/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.
     20/F10.0/F10.0/F10.0/2F5.0/2F5.0/F5.0/F5.0/I1/F5.0/F5.0/F10.0/F10.0
     3/2F5.0)
      II=0
C
C     THE INPUT DATA TO BE READ NOW IS OUTPUT DATA FROM THE PROGRAM OF
C     APPENDIX B
170   READ (5,130) MONTH,E(5)
130   FORMAT (I5,F15.0)
      M=4
      STEP=10000.
C
C     CALCULATE POTENTIAL PROFIT OF 'PRESENT' PERIOD
131   PCTINF=(1.-Q(1))*C(1)+(1.-Q(2))*C(2)
      Y(5)=(1.-R)*PCTINF*E(5)
C
C     THE FOLLOWING SEVEN STATEMENTS DETERMINE HOW MUCH EACH OF THE FOUR
C     PREVIOUS ADVERTISING EXPENDITURES CONTRIBUTES TO THE RESPONSE CURVE
      GOMEXP(1)=(B(1)**4)*A(1,1)+(B(2)**4)*A(2,1)
      GOMEXP(2)=GOMEXP(1)+(B(1)**3)*A(1,2)+(B(2)**3)*A(2,2)
      GOMEXP(3)=GOMEXP(2)+(B(1)**2)*A(1,3)+(B(2)**2)*A(2,3)
      GOMEXP(4)=GOMEXP(3)+B(1)*A(1,4)+B(2)*A(2,4)
      P(1)=((D**(S**(U*GOMEXP(1))))-DP)/(1.-DP)
      DO 63 J=2,4
63    P(J)=((D**(S**(U*GOMEXP(J))))-(D**(S**(U*GOMEXP(J-1)))))/(1.-DP)
C
C     INITIALIZE LOGIC VARIABLES TO BE USED IN OPTIMALITY SEARCH
52    N=0
154   PRLAST=-10000.
55    CONTINUE
58    N=N+1
C
C     BEGIN CALCULATION OF PROFIT
      GOMEXP(5)=GOMEXP(4)+A(1,5)+A(2,5)
      PN(5)=((D**(S**(U*GOMEXP(5))))-DP)/(1.-DP)
      P(5)=PN(5)-(((D**(S**(U*GOMEXP(4))))-DP)/(1.-DP))
      NUM=A(1,5)+B(1)*A(1,4)+(B(1)**2)*A(1,3)+(B(1)**3)*A(1,2)+(B(1)**4)
     1*A(1,1)
      F(5)=NUM/(NUM+A(2,5)+B(2)*A(2,4)+(B(2)**2)*A(2,3)+(B(2)**3)*A(2,2)
     1+(B(2)**4)*A(2,1))
      ADNTSB=Y(5)*PN(5)*F(5)
      PROFIT=ADNTSB-A(1,5)
C
C     BEGIN OPTIMALITY SEARCH TO DETERMINE A(1,5)
86    IF(N.LE.2) GO TO 88
87    GO TO 92
88    IF(PROFIT.LE.PRLAST) GO TO 90
89    GO TO 97
90    A(1,5)=STEP/10.0
```

```
91    GO TO 100
92    IF(PROFIT.LE.PRLAST) GO TO 94
93    GO TO 97
94    A(1,5)=A(1,5)-(2.*STEP)
95    IF(A(1,5).LE.0.0) GO TO 120
96    GO TO 100
97    PRLAST=PROFIT
98    A(1,5)=A(1,5)+STEP
99    GO TO 55
120   A(1,5)=STEP/10.0
      ALAST=A(1,5)
100   M=M-1
101   IF(M.LE.0) GO TO 103
      STEP=STEP/10.0
102   GO TO 154
103   A(1,5)=A(1,5)+STEP
      CN(1)=C(1)*(Q(1)+(1.-Q(1))*F(5))+C(2)*(1.-Q(2))*F(5)
      CN(2)=1.-CN(1)
C
C     AFTER OPTIMAL ADVERTISING IS DETERMINED, WRITE PARAMETERS, INPUTS,
C     ADVERTISING EXPENDITURES, RESPONSE COEFFICIENTS, AND PROFIT
135   WRITE (6,104)
104   FORMAT (1H1)
105   WRITE (6,106)
106   FORMAT ((20H SINE WAVE POTENTIAL)/)
107   WRITE (6,108) MONTH
108   FORMAT ((6H MONTH,I3)/)
      WRITE (6,30) D,DP,S,U
30    FORMAT ((27H GOMPERTZ PARAMETERS ARE D=,F5.2,3X,3HDP=,F5.2,3X,2HS=
     1,F5.2,3X,2HU=,F10.7)/)
      WRITE (6,150) B(1),B(2)
160   FORMAT ((36H ADVERTISING CARRY-OVER FACTOR B(1)=,F5.2,5X,5HB(2)=,F
     15.2)/)
113   WRITE (6,114) Q(1),Q(2)
114   FORMAT ((23H RETENTION FACTOR Q(1)=,F5.2,5X,5HQ(2)=,F5.2)/)
      WRITE (6,175) C(1),C(2)
175   FORMAT ((39H BRAND SHARE OF EACH COMPETITOR C(1,5)=,F5.2,5X,7HC(2,
     15)=,F5.2)/)
      WRITE (6,176) CN(1),CN(2)
176   FORMAT ((39H BRAND SHARE OF EACH COMPETITOR C(1,6)=,F5.2,5X,7HC(2,
     16)=,F5.2)/)
      WRITE (6,16) E(5)
16    FORMAT ((57H POTENTIAL $ SALES DISCOUNTED IN TERMS OF RETENTION E(
     15)=,F12.2)/)
      WRITE (6,42) Y(5)
42    FORMAT ((29H POTENTIAL PROFIT VALUE Y(5)=,F16.8)/)
142   DO 164 J=1,4
      WRITE (6,163) J,A(1,J),J,A(2,J)
163   FORMAT ((34H PAST ADVERTISING EXPENDITURE A(1,,I1,2H)=,F12.2,5X,4H
     1A(2,,I1,2H)=,F12.2)/)
164   CONTINUE
109   WRITE (6,110) A(1,5)
110   FORMAT ((44H RECOMMENDED ADVERTISING EXPENDITURE A(1,5)=,F12.2)/)
      WRITE (6,169) A(2,5)
169   FORMAT ((54H COMPETITOR'S EXPECTED ADVERTISING EXPENDITURE A(2,5)=
     1,F12.2)/)
      DO 167 J=1,5
      WRITE (6,166) J,P(J)
166   FORMAT ((24H RESPONSE COEFFICIENT P(,I2,2H)=,F9.6)/)
167   CONTINUE
```

```
        WRITE (6,168) PN(5)
  168   FORMAT ((36H OVERALL RESPONSE COEFFICIENT PN(5)=,F9.6)/)
  117   WRITE (6,112) PRLAST
  112   FORMAT (18H OPTIMAL PROFIT IS,F12.2)
C
C       PROCEED TO THE NEXT PERIOD
        II=II+1
        IF(II.GE.12)GO TO 115
        A(1,13)=A(1,1)
        A(2,13)=A(2,1)
        DO 180 J=1,12
        A(1,J)=A(1,J+1)
  180   A(2,J)=A(2,J+1)
        C(1)=CN(1)
        C(2)=CN(2)
        A(1,5)=10000.
        GO TO 170
  115   STOP
  116   END

$ENTRY
```

SINE WAVE POTENTIAL

MONTH 1

GOMPERTZ PARAMETERS ARE D= 0.65 DP= 0.30 S= 0.60 J= 0.0000124

ADVERTISING CARRY-OVER FACTOR B(1)= 0.25 B(2)= 0.25

RETENTION FACTOR Q(1)= 0.00 Q(2)= 0.50

BRAND SHARE OF EACH COMPETITOR C(1,5)= 0.16 C(2,5)= 0.84

BRAND SHARE OF EACH COMPETITOR C(1,6)= 0.18 C(2,6)= 0.82

POTENTIAL $ SALES DISCOUNTED IN TERMS OF RETENTION E(5)= 1061409.43

POTENTIAL PROFIT VALUE Y(5)= 246246.98776000

PAST ADVERTISING EXPENDITURE A(1,1)= 16930.00 A(2,1)= 57811.27

PAST ADVERTISING EXPENDITURE A(1,2)= 18050.00 A(2,2)= 57811.27

PAST ADVERTISING EXPENDITURE A(1,3)= 22220.00 A(2,3)= 62416.97

PAST ADVERTISING EXPENDITURE A(1,4)= 28790.00 A(2,4)= 70394.29

RECOMMENDED ADVERTISING EXPENDITURE A(1,5)= 36070.00

COMPETITOR'S EXPECTED ADVERTISING EXPENDITURE A(2,5)= 79605.71

RESPONSE COEFFICIENT P(1)= 0.500739

RESPONSE COEFFICIENT P(2)= 0.002994

RESPONSE COEFFICIENT P(3)= 0.013203

RESPONSE COEFFICIENT P(4)= 0.058454

RESPONSE COEFFICIENT P(5)= 0.201867

OVERALL RESPONSE COEFFICIENT PN(5)= 0.777258

OPTIMAL PROFIT IS 22430.77

APPENDIX F

CALCULATION OF OPTIMAL ADVERTISING EXPENDITURE USING MODEL II

The program listed in this appendix will calculate the optimal advertising expenditure for one of two competitors. As shown, this program will find the optimal spending pattern over a complete cycle of potential demand. The basis for this program is the advertising model of Chapter V, Model II. The underlying equations of interest are (5.3), (5.4), and (5.5).

As described in Chapter VI, the past and future expenditures are needed for both firms when this model is used. Also, the competitor's expected advertising at the period under consideration is needed. For the purpose of the example as described in Chapter VI - determining the optimal expenditures over time against a competitor whose cyclic spending pattern is known - only estimates of past and future optimal spending are necessary. The program will eventually approach a repeating cycle of optimal expenditures. Since the two competitors are non-identical, two sets of parameters are necessary program inputs.

In order to approach a cycling optimal pattern of advertising expenditures it may be necessary to go through the twelve month cycle several times. The number of time periods is controlled by the statement

IF(II.GE.48)GO TO 115

near the end of the program. If more than four cycles are desired, increase the decision level of II to, say, 72 for six complete cycles. The precision of calculation is controlled by card 150. A value of M = 4 will calculate the optimal advertising to the nearest ten dollars; M = 5 to the nearest dollar, etc.

The input data for this program involves a total of 34 cards plus the same number of DEMAND cards as periods to be considered. They are to be organized as follows:

Cards 1-4: Past optimal expenditure estimates in chronological order, $a_1(t-4)$ through $a_1(t-1)$; cols. 1-10 with decimal point.

Card 5: An initial value for the advertising expenditure to be determined, $a_1(t)$; cols. 1-10 with decimal point.

Cards 6-12: Future optimal expenditure estimates in chronological order, $a_1(t+1)$ through $a_1(t+7)$; cols. 1-10 with decimal point.

Cards 13-16: Past four competitor expenditures in chronological order, $a_2(t-4)$ through $a_2(t-1)$; cols. 1-10 with decimal point.

Card 17: Estimate of competitor's present period expenditure, $a_2(t)$; cols. 1-10 with decimal point.

Cards 18-24: Estimates of competitor's future period expenditures, $a_2(t+1)$ through $a_2(t+7)$; cols. 1-10 with decimal point.

Card 25: Value of each brand's advertising carry-over factor, b_1 and b_2; cols. 1-5 and 6-10, respectively, with up to 4 decimal places.

Card 26: Value of each firm's brand-share at the end of last period, $c_1(t)$ and $c_2(t)$; cols. 1-5 and 6-10, respectively, with up to 4 decimal places.

Card 27: Value of Gompertz parameter, D; cols. 1-5 with up to 4 decimal places.

Card 28: Value of Gompertz parameter, D'; cols. 1-5 with up to 4 decimal places.

Card 29: Value determining precision of calculation of $a_1(t)$. This variable is overridden by card 150 in the program as shown, M; col. 1 with no decimal places.

Card 30: Value of each dollar of sales attributed to cost, r_h; cols. 1-5 with up to 4 decimal places.

Card 31: Value of Gompertz parameter, S; cols. 1-5 with up to 4 decimal places.

Card 32: Value of initial increment in determining $a_1(t)$. This variable is overridden by STEP = 10000. in the program as shown, STEP; cols. 1-10 with decimal point.

Card 33: Value of Gompertz parameter, U; cols. 1-9 with up to 8 decimal places.

Card 34: Value of each firm's retention factor, q_1 and q_2; cols. 1-5 and 6-10, respectively, with up to 4 decimal places.

Cards 35 - : Month and value of DEMAND terms, MONTH, E(5), E(6), E(7), E(8), E(9); cols. 1-5, right justify the month number. cols. 6-20, 21-35, 36-50, 51-65, 66-80, E(5) through E(9), respectively, with decimal point. See Appendix B for determining E(5) through E(9) values.

The program as described is listed starting on the next page. It is followed by a sample output.

It should be noted that this program can be used for optimizing a single period's advertising. In such a case, the past four expenditures of both firms must be known. Also, the expected present period advertising of the competition must be known. Finally, the future four expenditures of each firm must be estimated. If the number of periods is then reduced to 1 by

```
IF(II.GE.01)GO TO 115
```

the optimal advertising $a_1(t)$ will be determined. In a similar manner, any number of periods may be considered.

```
$JOB 10322,444-42-7413,TIME=25          KEN CASE
C      PROGRAM TO DETERMINE OPTIMAL ADVERTISING EXPENDITURE FOR ONE OF
C      TWO COMPETITORS USING MODEL 2
1      DOUBLE PRECISION A(2,20),B(2),E(10),Q(2),R,S,STEP,U,Y(10),GOMEXP(1
      10,10),C(2,10),ADNTSB,PRLAST,PROFIT,P(10,10),F(10),PN(10,10),DABS,N
      2UM(10),PCTINF(10),D,DP,PRONOW
C
C      READ INPUT DATA AS EXPLAINED IN TEXT
2      READ (5,3) (A(1,J),J=1,12),(A(2,J),J=1,12),(B(I),I=1,2),(C(I,5),I=
      11,2),D,DP,M,R,S,STEP,U,(Q(I),I=1,2)
3      FORMAT (F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.
      10/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.0/F10.
      20/F10.0/F10.0/F10.0/2F5.0/2F5.0/F5.0/F5.0/I1/F5.0/F5.0/F10.0/F10.0
      3/2F5.0)
       II=0
C
C      THE INPUT DATA TO BE READ NOW IS OUTPUT DATA FROM THE PROGRAM OF
C      APPENDIX B
170    READ (5,130) MONTH,(E(J),J=5,9)
130    FORMAT (I5,5F15.0)
       M=3
       STEP=10000.
C
C      CALCULATE POTENTIAL PROFIT OF 'PRESENT' PERIOD
       PCTINF(5)=(1.-Q(1))*C(1,5)+(1.-Q(2))*C(2,5)
       Y(5)=(1.-R)*PCTINF(5)*E(5)
C
C      THE FOLLOWING EIGHT STATEMENTS DETERMINE HOW MUCH EACH OF THE FOUR
C      PREVIOUS ADVERTISING EXPENDITURES CONTRIBUTES TO THE RESPONSE CURVE
       GOMEXP(5,1)=(B(1)**4)*A(1,1)+(B(2)**4)*A(2,1)
       GOMEXP(5,2)=GOMEXP(5,1)+(B(1)**3)*A(1,2)+(B(2)**3)*A(2,2)
       GOMEXP(5,3)=GOMEXP(5,2)+(B(1)**2)*A(1,3)+(B(2)**2)*A(2,3)
       GOMEXP(5,4)=GOMEXP(5,3)+B(1)*A(1,4)+B(2)*A(2,4)
       P(5,1)=((D**(S**(U*GOMEXP(5,1))))-DP)/(1.-DP)
       P(5,2)=((D**(S**(U*GOMEXP(5,2))))-(D**(S**(U*GOMEXP(5,1)))))/(1.-D
      1P)
       P(5,3)=((D**(S**(U*GOMEXP(5,3))))-(D**(S**(U*GOMEXP(5,2)))))/(1.-D
      1P)
       P(5,4)=((D**(S**(U*GOMEXP(5,4))))-(D**(S**(U*GOMEXP(5,3)))))/(1.-D
      1P)
C
C      INITIALIZE LOGIC VARIABLES TO BE USED IN OPTIMALITY SEARCH
52     N=0
154    PRLAST=-10000.
55     CONTINUE
58     N=N+1
C
C      BEGIN CALCULATION OF 'PRESENT' PERIOD PROFIT BEFORE ADVERTISING
       GOMEXP(5,5)=GOMEXP(5,4)+A(1,5)+A(2,5)
       PN(5,5)=((D**(S**(U*GOMEXP(5,5))))-DP)/(1.-DP)
       P(5,5)=PN(5,5)-(((D**(S**(U*GOMEXP(5,4))))-DP)/(1.-DP))
       NUM(5)=A(1,5)+B(1)*A(1,4)+(B(1)**2)*A(1,3)+(B(1)**3)*A(1,2)+(B(1)*
      1*4)*A(1,1)
       F(5)=NUM(5)/(NUM(5)+A(2,5)+B(2)*A(2,4)+(B(2)**2)*A(2,3)+(B(2)**3)*
      1A(2,2)+(B(2)**4)*A(2,1))
       ADNTSB=Y(5)*PN(5,5)*F(5)
       PRONOW=ADNTSB-A(1,5)
C
C      CALCULATE POTENTIAL PROFIT OF 1ST 'FUTURE' PERIOD
       C(1,6)=C(1,5)*(Q(1)+(1.-Q(1))*F(5))+C(2,5)*(1.-Q(2))*F(5)
```

```
31          C(2,6)=1.-C(1,6)
32          PCTINF(6)=(1.-Q(1))*C(1,6)+(1.-Q(2))*C(2,6)
33          Y(6)=(1.-R)*PCTINF(6)*E(6)
      C
      C     BEGIN CALCULATION OF 1ST 'FUTURE' PERIOD PROFIT BEFORE ADVERTISING
34          GOMEXP(6,6)=A(1,6)+A(2,6)+B(1)*A(1,5)+B(2)*A(2,5)+(B(1)**2)*A(1,4)
           1+(B(2)**2)*A(2,4)+(B(1)**3)*A(1,3)+(B(2)**3)*A(2,3)+(B(1)**4)*A(1,
           22)+(B(2)**4)*A(2,2)
35          PN(6,6)=((D**(S**(U*GOMEXP(6,6))))-DP)/(1.-DP)
36          NUM(6)=A(1,6)+B(1)*A(1,5)+(B(1)**2)*A(1,4)+(B(1)**3)*A(1,3)+(B(1)*
           1*4)*A(1,2)
37          F(6)=NUM(6)/(NUM(6)+A(2,6)+B(2)*A(2,5)+(B(2)**2)*A(2,4)+(B(2)**3)*
           1A(2,3)+(B(2)**4)*A(2,2))
38          ADNTSB=ADNTSB+Y(6)*PN(6,6)*F(6)
      C
      C     CALCULATE POTENTIAL PROFIT OF 2ND 'FUTURE' PERIOD
39          C(1,7)=C(1,6)*(Q(1)+(1.-Q(1))*F(6))+C(2,6)*(1.-Q(2))*F(6)
40          C(2,7)=1.-C(1,7)
41          PCTINF(7)=(1.-Q(1))*C(1,7)+(1.-Q(2))*C(2,7)
42          Y(7)=(1.-R)*PCTINF(7)*E(7)
      C
      C     BEGIN CALCULATION OF 2ND 'FUTURE' PERIOD PROFIT BEFORE ADVERTISING
43          GOMEXP(7,7)=A(1,7)+A(2,7)+B(1)*A(1,6)+B(2)*A(2,6)+(B(1)**2)*A(1,5)
           1+(B(2)**2)*A(2,5)+(B(1)**3)*A(1,4)+(B(2)**3)*A(2,4)+(B(1)**4)*A(1,
           23)+(B(2)**4)*A(2,3)
44          PN(7,7)=((D**(S**(U*GOMEXP(7,7))))-DP)/(1.-DP)
45          NUM(7)=A(1,7)+B(1)*A(1,6)+(B(1)**2)*A(1,5)+(B(1)**3)*A(1,4)+(B(1)*
           1*4)*A(1,3)
46          F(7)=NUM(7)/(NUM(7)+A(2,7)+B(2)*A(2,6)+(B(2)**2)*A(2,5)+(B(2)**3)*
           1A(2,4)+(B(2)**4)*A(2,3))
47          ADNTSB=ADNTSB+Y(7)*PN(7,7)*F(7)
      C
      C     CALCULATE POTENTIAL PROFIT OF 3RD 'FUTURE' PERIOD
48          C(1,8)=C(1,7)*(Q(1)+(1.-Q(1))*F(7))+C(2,7)*(1.-Q(2))*F(7)
49          C(2,8)=1.-C(1,8)
50          PCTINF(8)=(1.-Q(1))*C(1,8)+(1.-Q(2))*C(2,8)
51          Y(8)=(1.-R)*PCTINF(8)*E(8)
      C
      C     BEGIN CALCULATION OF 3RD 'FUTURE' PERIOD PROFIT BEFORE ADVERTISING
52          GOMEXP(8,8)=A(1,8)+A(2,8)+B(1)*A(1,7)+B(2)*A(2,7)+(B(1)**2)*A(1,6)
           1+(B(2)**2)*A(2,6)+(B(1)**3)*A(1,5)+(B(2)**3)*A(2,5)+(B(1)**4)*A(1,
           24)+(B(2)**4)*A(2,4)
53          PN(8,8)=((D**(S**(U*GOMEXP(8,8))))-DP)/(1.-DP)
54          NUM(8)=A(1,8)+B(1)*A(1,7)+(B(1)**2)*A(1,6)+(B(1)**3)*A(1,5)+(B(1)*
           1*4)*A(1,4)
55          F(8)=NUM(8)/(NUM(8)+A(2,8)+B(2)*A(2,7)+(B(2)**2)*A(2,6)+(B(2)**3)*
           1A(2,5)+(B(2)**4)*A(2,4))
56          ADNTSB=ADNTSB+Y(8)*PN(8,8)*F(8)
      C
      C     CALCULATE POTENTIAL PROFIT OF 4TH 'FUTURE' PERIOD
57          C(1,9)=C(1,8)*(Q(1)+(1.-Q(1))*F(8))+C(2,8)*(1.-Q(2))*F(8)
58          C(2,9)=1.-C(1,9)
59          PCTINF(9)=(1.-Q(1))*C(1,9)+(1.-Q(2))*C(2,9)
60          Y(9)=(1.-R)*PCTINF(9)*E(9)
      C
      C     BEGIN CALCULATION OF 4TH 'FUTURE' PERIOD PROFIT BEFORE ADVERTISING
61          GOMEXP(9,9)=A(1,9)+A(2,9)+B(1)*A(1,8)+B(2)*A(2,8)+(B(1)**2)*A(1,7)
           1+(B(2)**2)*A(2,7)+(B(1)**3)*A(1,6)+(B(2)**3)*A(2,6)+(B(1)**4)*A(1,
           25)+(B(2)**4)*A(2,5)
62          PN(9,9)=((D**(S**(U*GOMEXP(9,9))))-DP)/(1.-DP)
```

```
         NUM(9)=A(1,9)+B(1)*A(1,8)+(B(1)**2)*A(1,7)+(B(1)**3)*A(1,6)+(B(1)*
        1*4)*A(1,5)
         F(9)=NUM(9)/(NUM(9)+A(2,9)+B(2)*A(2,8)+(B(2)**2)*A(2,7)+(B(2)**3)*
        1A(2,6)+(B(2)**4)*A(2,5))
         ADNTSB=ADNTSB+Y(9)*PN(9,9)*F(9)
C
C        CALCULATE 'FIVE-PERIOD' PROFIT
         PROFIT=ADNTSB-A(1,5)-A(1,6)-A(1,7)-A(1,8)-A(1,9)
C
C        BEGIN OPTIMALITY SEARCH TO DETERMINE A(1,5)
   86    IF(N.LE.2) GO TO 88
   87    GO TO 92
   88    IF(PROFIT.LE.PRLAST) GO TO 90
   89    GO TO 97
   90    A(1,5)=STEP/10.0
   91    GO TO 100
   92    IF(PROFIT.LE.PRLAST) GO TO 94
   93    GO TO 97
   94    A(1,5)=A(1,5)-(2.*STEP)
   95    IF(A(1,5).LE.0.0) GO TO 120
   96    GO TO 100
   97    PRLAST=PROFIT
   98    A(1,5)=A(1,5)+STEP
   99    GO TO 55
  120    A(1,5)=STEP/10.0
         ALAST=A(1,5)
  100    M=M-1
  101    IF(M.LE.0) GO TO 103
         STEP=STEP/10.0
  102    GO TO 154
  103    A(1,5)=A(1,5)+STEP
C
C        AFTER OPTIMAL ADVERTISING IS DETERMINED, WRITE PARAMETERS, INPUTS,
C        ADVERTISING EXPENDITURES, RESPONSE COEFFICIENTS, AND PROFIT
  189    WRITE (6,104)
  104    FORMAT (1H1)
  105    WRITE (6,106)
  106    FORMAT ((20H SINE WAVE POTENTIAL)/)
  107    WRITE (6,108) MONTH
  108    FORMAT ((6H MONTH,I3)/)
         WRITE (6,30) D,DP,S,U
   30    FORMAT ((27H GOMPERTZ PARAMETERS ARE D=,F5.2,3X,3HDP=,F5.2,3X,2HS=
        1,F5.2,3X,2HU=,F10.7)/)
         WRITE (6,160) B(1),B(2)
  160    FORMAT ((36H ADVERTISING CARRY-OVER FACTOR B(1)=,F5.2,5X,5HB(2)=,F
        15.2)/)
  113    WRITE (6,114) Q(1),Q(2)
  114    FORMAT ((23H RETENTION FACTOR Q(1)=,F5.2,5X,5HQ(2)=,F5.2)/)
         WRITE (6,175) C(1,5),C(2,5)
  175    FORMAT ((39H BRAND SHARE OF EACH COMPETITOR C(1,5)=,F5.2,5X,7HC(2,
        15)=,F5.2)/)
         WRITE (6,176) C(1,6),C(2,6)
  176    FORMAT ((39H BRAND SHARE OF EACH COMPETITOR C(1,6)=,F5.2,5X,7HC(2,
        16)=,F5.2)/)
   15    WRITE (6,16)
   16    FORMAT (48H DEMAND DISCOUNTED IN TERMS OF RETENTION AND TVM)
   17    DO 20 J=5,9
   18    WRITE (6,19) J, J, E(J)
   19    FORMAT ((8H PERIOD ,I1,5H   E(,I1,2H)=,F12.2)/)
   20    CONTINUE
```

```
 41     WRITE (6,42)
 42     FORMAT (23H POTENTIAL PROFIT VALUE)
        DO 47 J=5,9
 45     WRITE (6,46) J, Y(J)
 46     FORMAT ((3H Y(,I1,2H)=,F16.8)/)
 47     CONTINUE
        DO 164 J=1,4
        WRITE (6,163) J,A(1,J),J,A(2,J)
 163    FORMAT ((34H PAST ADVERTISING EXPENDITURE A(1,,I1,2H)=,F12.2,5X,4H
       1A(2,,I1,2H)=,F12.2)/)
 164    CONTINUE
 109    WRITE (6,110) A(1,5)
 110    FORMAT ((44H RECOMMENDED ADVERTISING EXPENDITURE A(1,5)=,F12.2)/)
        WRITE (6,169) A(2,5)
 169    FORMAT ((54H COMPETITOR'S EXPECTED ADVERTISING EXPENDITURE A(2,5)=
       1,F12.2)/)
        DO 200 J=6,9
        WRITE (6,199) J,A(1,J),J,A(2,J)
 199    FORMAT ((36H FUTURE ADVERTISING EXPENDITURE A(1,,I1,2H)=,F12.2,5X,
       14HA(2,,I1,2H)=,F12.2)/)
 200    CONTINUE
        DO 167 J=1,5
        WRITE (6,166) J,P(5,J)
 166    FORMAT ((26H RESPONSE COEFFICIENT P(5,,I1,2H=,F9.6)/)
 167    CONTINUE
        DO 179 J=5,9
        WRITE (6,168) J,J,PN(J,J)
 168    FORMAT ((33H OVERALL RESPONSE COEFFICIENT PN(I1,1H,,I1,2H)=,F9.6)/
       1)
 179    CONTINUE
 117    WRITE (6,112) PRONOW
 112    FORMAT (18H OPTIMAL PROFIT IS,F12.2)
C
C       PROCEED TO THE NEXT PERIOD
 190    II=II+1
        IF(II.GE.48) GO TO 115
        A(1,13)=A(1,1)
        A(2,13)=A(2,1)
        DO 180 J=1,12
        A(1,J)=A(1,J+1)
 180    A(2,J)=A(2,J+1)
        C(1,5)=C(1,6)
        C(2,5)=C(2,6)
        A(1,5)=10000.
        GO TO 170
 115    STOP
 116    END

$ENTRY
```

SINE WAVE POTENTIAL

MONTH 1

GOMPERTZ PARAMETERS ARE D= 0.65 DP= 0.30 S= 0.60 U= 0.0000124

ADVERTISING CARRY-OVER FACTOR B(1)= 0.25 B(2)= 0.25

RETENTION FACTOR Q(1)= 0.00 Q(2)= 0.50

BRAND SHARE OF EACH COMPETITOR C(1,5)= 0.42 C(2,5)= 0.58

BRAND SHARE OF EACH COMPETITOR C(1,6)= 0.42 C(2,6)= 0.58

DEMAND DISCOUNTED IN TERMS OF RETENTION AND TVM
PERIOD 5 E(5)= 1061409.43

PERIOD 6 E(6)= 1167773.69

PERIOD 7 E(7)= 1229183.12

PERIOD 8 E(8)= 1229183.12

PERIOD 9 E(9)= 1167773.69

POTENTIAL PROFIT VALUE
Y(5)= 301440.27812000

Y(6)= 332412.31300212

Y(7)= 349218.09624604

Y(8)= 347489.88692210

Y(9)= 327897.85797683

PAST ADVERTISING EXPENDITURE A(1,1)= 71100.00 A(2,1)= 57811.27

PAST ADVERTISING EXPENDITURE A(1,2)= 77200.00 A(2,2)= 57811.27

PAST ADVERTISING EXPENDITURE A(1,3)= 89400.00 A(2,3)= 62416.97

PAST ADVERTISING EXPENDITURE A(1,4)= 104200.00 A(2,4)= 70394.29

RECOMMENDED ADVERTISING EXPENDITURE A(1,5)= 117700.00

COMPETITOR'S EXPECTED ADVERTISING EXPENDITURE A(2,5)= 79605.71

FUTURE ADVERTISING EXPENDITURE A(1,6)= 125700.00 A(2,6)= 87583.03

FUTURE ADVERTISING EXPENDITURE A(1,7)= 127000.00 A(2,7)= 92188.73

FUTURE ADVERTISING EXPENDITURE A(1,8)= 121200.00 A(2,8)= 92188.73

FUTURE ADVERTISING EXPENDITURE A(1,9)= 109600.00 A(2,9)= 87583.03

RESPONSE COEFFICIENT P(5,1=, 0.501275

RESPONSE COEFFICIENT P(5,2=, 0.005315

```
RESPONSE COEFFICIENT P(5,3=, 0.023401

RESPONSE COEFFICIENT P(5,4=, 0.096983

RESPONSE COEFFICIENT P(5,5=, 0.254432

OVERALL RESPONSE COEFFICIENT PN(5,5)= 0.881406

OVERALL RESPONSE COEFFICIENT PN(6,6)= 0.897070

OVERALL RESPONSE COEFFICIENT PN(7,7)= 0.904183

OVERALL RESPONSE COEFFICIENT PN(8,8)= 0.902477

OVERALL RESPONSE COEFFICIENT PN(9,9)= 0.891853

OPTIMAL PROFIT IS    40594.46
```

VITA

Kenneth Eugene Case

Candidate for the Degree of

Doctor of Philosophy

Thesis: DEVELOPMENT AND ANALYSIS OF AN ADVERTISING MODEL CONCEPT

Major Field: Industrial Engineering and Management

Biographical:

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Education: Graduated from Thomas A. Edison High School, Tulsa, Oklahoma, May, 1962; entered Oklahoma State University September, 1962, and received the Bachelor of Science degree in Electrical Engineering, May, 1966; received the Master of Science degree from Oklahoma State University in July, 1967, as a National Science Foundation Graduate Trainee with a major in Industrial Engineering and Management; completed requirements for the Doctor of Philosophy degree at Oklahoma State University in May, 1970, as a National Aeronautics and Space Administration Trainee.

Professional Experience: Technician, Aerotron Radio Company, Tulsa, Summers, 1959-1963; Summer College Employment Program, Humble Oil and Refining Company, Oklahoma City, Summer, 1965; Summer College Employment Program, Collins Radio Company, Cedar Rapids, Iowa, Summer, 1966; Graduate Research Assistant, School of Industrial Engineering, Oklahoma State University, 1966-67; Graduate Teaching Assistant, School of Industrial Engineering, Oklahoma State University, 1967-68; Assistant Professor of Industrial Engineering, Virginia Polytechnic Institute, Blacksburg, Virginia, effective September, 1969.

Professional Activities: Engineer-in-Training; Member of: American Institute of Industrial Engineers, Alpha Pi Mu, Phi Kappa Phi, Sigma Tau, Eta Kappa Nu, Pi Mu Epsilon; Selected "Outstanding Graduate Teaching Assistant at Oklahoma State University, 1968."