METRIZATION OF MOORE SPACES

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1963

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Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the Degree of
DOCTOR OF EDUCATION
May, 1970

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ACKNOWLEDGEMENTS

At this time it would seem appropriate to express my appreciation to those who have assisted me in the pursuit of the degree for which this paper is written.

Deepest gratitude is due Professor John Jobe for his encouragement during my years at Oklahoma State University, and particularly for the invaluable assistance he has rendered toward the preparation of this paper. Appreciation is also due Professors E. K. McLachlan and W. Ware Marsden for their assistance in determining my plan of study and eventual goal. Particular thanks is to be given to the Von Kayes, for their undaunted faith and encouragement at crucial periods of this endeavor, and to Ronnie Reb for his ever-vocal expressions of confidence in the successful outcome of this undertaking.

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CHAPTER I

INTRODUCTION

In the 1910's and 1920's there was considerable interest in the metrization problem for a topological space. Mathematicians strove to determine under what conditions a topological space is metrizable. A short history of the attempts of this period aimed at solving the metrization problem may be found in Chittenden [6]. This interest led to a solution of this problem by Alexandroff and Urysohn (Theorem 3.4) in 1923. This solution [9] stated that a necessary and sufficient condition that a topological space be metrizable is there exist a sequence $G = \{G_i\}$ of open coverings such that (a) G_{i+1} is a refinement of G_i, (b) for each point p and for each domain D containing p there is an integer n such that every element of G_n which contains p is a subset of D, and (c) the sum of each pair of intersecting elements of G_{i+1} is a subset of an element of G_i. Kindred theorems to Alexandroff and Urysohn's theorem were proved by E. W. Chittenden [6] in 1927, R. L. Moore [22] in 1935, and C. W. Vickery [33] in 1940. Later in 1947, R. H. Bing [3] proved a generalization of Alexandroff and Urysohn's theorem in that G_i did not have to be an open covering.

The solution of the general metrization problem then led mathematicians to ask, when is a regular topological space metrizable? A partial solution came in 1925 when P. Urysohn [12] proved his well-known metrization theorem, that is, a regular second countable topological space is metrizable. Following Urysohn's result little progress was made on this problem for twenty-five years. Then in 1951, two independent, but similar, results were published. One, by Nagata and Smirnof [1], stated that a regular topological space is metrizable if and only if it has a σ-locally finite base (Theorem 3.30). The other, by R. H. Bing [4], stated that a regular topological space is metrizable if and only if it is prefectly screenable (Theorem 3.27).

In the early stages of the exploration of the metrizability of topological spaces, attempts were also made to generalize metric spaces. Most of these attempts were done by replacing the triangle inequality with a more general condition. For examples, see Chittenden [5] and [6], Niemytzki [24], and Wilson [34]. One such attempt (which was not a generalization of the triangle inequality) was made by R. L. Moore [23] in the 1920's. The result of this attempt is the now-familiar Moore space (Definition 2.33). This particular space is a regular Hausdorff topological space which has defined on it a development. It was at first speculated that all Moore spaces were metrizable, but R. L. Moore himself produced an example which showed that this was not the case. This discovery led to the question: Under what conditions are Moore spaces metrizable?

In 1937, F. B. Jones [18] published an answer to this question, in his theorem which stated that if $2^{\aleph_1} > 2^{\aleph_0}$, then every separable Moore space is metric and second countable (Theorem 3.21). In this same article, Jones posed the new question: Is every normal Moore space metrizable? Jones' theorem has since also raised the question of whether $2^{\aleph_1} > 2^{\aleph_0}$ is a necessary part of the hypothesis. These two questions are of this date unanswered even though progress has been made on the question of the metrizability of Moore spaces.

R. H. Bing [4], in his 1951 paper, stated and proved that a Moore space is metrizable if and only if it is perfectly screenable, or strongly screenable, or collectionwise normal, or paracompact, or fully normal. While these results answer the metrizability question, all of these conditions will be shown (in Chapter II) to be stronger than normality. Another approach to Jones' normality question was taken in 1964 by R. W. Heath [14]. Using the idea of a uniform base (Definition 2.42) Heath proved the following: a regular topological space is a pointwise paracompact Moore space if and only if it has a uniform base (Theorem 2.45). In 1956, P. S. Aleksandrov [1] proved that in order that a regular topological space be metrizable, it is necessary and sufficient that it have a uniform base and that it satisfy one of the following conditions: (1) it is paracompact; (2) it is collectwise normal; or (3) each point-finite covering of it has a locally finite refinement. These last two results, when taken together, do not produce any new evidence, since one may ask if every normal topological space which has a uniform base is metrizable. However, these two results do serve to point out the relationships between the general case and the specific case of Moore spaces.

Results in the 1960's have tended to consider properties of the boundaries of domains in a Moore space rather than properties of the Moore space. D. R. Traylor [28] in 1962 showed that a Moore space is metrizable if the boundary of each domain is second countable. A result by Grace and Heath [11] showed that a Moore space is metrizable if the boundary of each domain is strongly screenable. Traylor [29] similarly showed that if the boundary of each domain in a normal Moore space is screenable then the space is metrizable.

The above discussion constitutes basically the results contained in Chapter III of this paper. Chapter II gives a discussion of the terms basic to this paper. Also, the second chapter contains examples illustrating the ideas of this paper. Chapter IV contains a translation of the normal Moore space conjecture to a non-topological setting. Chapter IV also contains a discussion of properties of nonmetrizable normal Moore spaces if, indeed, there does exist a nonmetrizable normal Moore space.

CHAPTER II

FUNDAMENTAL TOPOLOGICAL CONCEPTS

Introduction

In this chapter, the topological concepts basic to this paper are presented. It is assumed that the reader is familiar with the definitions and theorems that occur in a first course in elementary point set topology. In particular, any term appearing in Elementary Topology by D. W. Hall and G. L. Spencer [12] will not be defined in this chapter.

It should be noted that the term "domain" is used in preference to the term "open set". The term "topological space" will refer to a Hausdorff topological space. These two terms are used throughout this paper, hence the reader is advised to become familiar with them.

Certain definitions in this chapter are of such a nature that examples are necessary to clarify their statement. In such a case, an example will generally follow the definition.

Screenable, Strongly Screenable and Perfectly Screenable Spaces

In 1951, R. H. Bing introduced the definitions and theorems of this section in his paper on metrization [4]. They are probably the most important since they lead to one of the first satisfactory characterizations of metrizability in a regular topological space.

Definition 2.1: Let H be a collection of sets. A collection of sets G is a refinement of H if each element of G is a subset of an element of H.

Definition 2.2: A collection of point sets is discrete if the closures of these point sets are mutually exclusive and any subcollection of these closures has a closed sum.

The reader may easily see that any finite collection of disjoint closed sets in the plane is a discrete collection. However, note that a finite collection of disjoint domains in the plane may not be a discrete collection.

Definition 2.3: A topological space is screenable if for each open covering G of the space, there is a sequence $H = \{H_i\}$ such that H_i is a collection of mutually exclusive domains, $\bigcup_{i=1}^{\infty} H_i$ is a refinement of G, and $\bigcup_{i=1}^{\infty} H_i$ is an open covering of the space.

Definition 2.4: A topological space is strongly screenable if for each open covering G of the space there is a sequence $H = \{H_i\}$ such that H_i is a discrete collection of domains, $\bigcup_{i=1}^{\infty} H_i$ is a refinement of G, and $\bigcup_{i=1}^{\infty} H_i$ is an open covering of the space.

Of course, any strongly screenable topological space is a screenable topological space. The converse of this statement is not necessarily true as the following example illustrates.

Example 2.5: A regular screenable topological space which is non-normal and not strongly screenable.

The points of the space S are all the points of the plane on or above the x-axis. A basis G for S is as follows: (1) for each point p above the x-axis, $\{p\} \in G$; (2) for each point $p = (r, 0)$, where r is a rational number, and natural number n, $D(r, n) = \{(r, y) : 0 \leq y \leq 1/n\} \in G$ (a vertical line segment with its lower end point at $(r, 0)$); and (3) for each point $p = (x, 0)$, where x is an irrational number, and natural number n, $D(x, n) = \{(t, y) : t = x + y,\ 0 \leq y \leq 1/n\} \in G$ (a line segment with slope 1 and end point at $(x, 0)$). By using a geometric argument one sees that the space S is a regular topological space. Also a geometric argument shows that each element of the basis is a closed set.

Let H be an open covering of S. Consider the collection $H_1 = \{\{p\} \mid p \text{ is above the x-axis}\}$ and note that H_1 is a collection of mutually exclusive domains. Also we have that H_1 is a refinement of H which covers the portion of S which lies above the x-axis. Let $R = \{(r, 0) : r \text{ is a rational number}\}$, since G is a basis of S we have for each $(r, 0) \in R$ a natural number n such that $D(r, n) \subset h$, for some $h \in H$. Let $H_2 = \{D(r, n) \mid (r, 0) \in R \text{ and } D(r, n) \subset h \text{ for some } h \in H\}$. The definition of G implies the domains of H_2 are mutually exclusive. The definition of H_2 implies that H_2 is a refinement of H which covers R. Upon letting $I = \{(x, 0) \mid x \text{ is an irrational real number}\}$ one sees similarly that $H_3 = \{D(x, n) \mid (x, 0) \in I \text{ and } D(x, n) \subset h \text{ for some } h \in H\}$ is a collection of mutually exclusive domains which covers I, and H_3 is refinement of H. Now $K = \{H_1, H_2, H_3\}$ is a sequence such that H_i $(i = 1, 2, 3)$ is a collection of mutually exclusive domains, $\bigcup_{i=1}^{3} H_i$ covers S, and $\bigcup_{i=1}^{3} H_i$ is a refinement of H. Hence by definition, S is screenable.

Suppose that S is a normal topological space. It is clearly seen that the sets $R = \{(r, 0) : r \text{ is a rational number}\}$ and

$$I = \{(x, 0) : x \text{ is an irrational number}\}$$

are disjoint closed subsets of S. Since S is a normal topological space, there are domains D_R and D_I such that $R \subset D_R$, $I \subset D_I$, and $D_R \cap D_I = \emptyset$. Upon referring to the definition of the base G, one sees that to each point $x = (x, 0) \in I$ we can associate a base element $D(x, n)$ such that $D(x, n) \subset D_I$. Define the set I_m as follows:

$$I_m = \{x \mid x = (x, 0) \in I,\ D(x, n) \subset D_I, \text{ and } m \geq n\}.$$

Now the definition of the base G implies that

$$I = \bigcup_{m=1}^{\infty} I_m .$$

But since I is a set of second category when regarded as a subset of the real line, the closure of some I_j, in the sense of the topology of the real line, must contain an open interval (a, b), as shown in Taylor [26]. Therefore, there is an N such that $(a, b) \subset \overline{I_N}$. Since (a, b) is a subset of the real line, there is a rational number r such that $r \in (a, b)$. Now consider the interval (a, r). Since $(a, r) \subset \overline{I_N}$ there is a sequence $\{x_i\}$ such that x_i is irrational, $x_i \in I_N$, and $\{x_i\}$ converges to r. Now consider the associated base elements $D(x_i, n)$; by definition of I_N and $D(x_i, n)$ we have that the height (using Euclidean geometry) of each $D(x_i, n)$ is greater than or equal to 1/N. Since $D_R \cap D_I = \emptyset$, there is a base element $D(r, M)$ such that $D(r, M) \subset D_R$. Now let $\delta = \min\{r - a, 1/2N\}$ and consider the interval $(r - \delta, r)$. Since the sequence $\{x_i\}$ converges to r, there is an $x_j \in (r - \delta, r)$. But $x_j \in (r - \delta, r)$ implies by a geometric argument that $D(x_j, n)$ intersects $D(r, M)$.

Hence $D_R \cap D_I \neq \emptyset$, which is a contradiction. Therefore, the topological space S is not normal.

Suppose that S is a strongly screenable topological space. Since S is a regular topological space we have by an argument similar to Theorem 2.13, of the next section, that S is a normal topological space. This is a contradiction to the previous paragraph. Hence S is not a strongly screenable topological space.

The reader may wish to know that when referring to the literature the sequence H in the definition of strong screenability is sometimes referred to as a σ-discrete refinement of G.

Definition 2.6: A topological space is perfectly screenable if there exists a sequence $G = \{G_i\}$ such that G_i is a discrete collection of domains and for each domain D and each point p in D there is a natural number n(p, D) such that $G_{n(p, D)}$ contains a domain which lies in D and contains p.

The existence of screenable, strongly screenable, and perfectly screenable topological spaces can quite easily be shown by considering any finite set with the domains defined in an appropriate way. Also the topological space E_2 with the discrete topology is readily shown to satisfy the above mentioned properties. The author will also show in Chapter III that a metric space (the topological space E_2 with the usual topology) is perfectly screenable which, with the theorems of this section, will show that any metric space (in particular E_2) is screenable and strongly screenable.

Theorem 2.7: Let S be a topological space. If S is perfectly screenable, then S is strongly screenable.

Proof: Let H be an open covering of S. Since S is perfectly screenable, there exists a sequence $G = \{G_i\}$ such that G_i is a discrete collection of domains such that for any domain D and for any point p in D there is a natural number n(p, D) such that $G_{n(p, D)}$ contains a domain which lies in D and contains p. Defining $H_i = \{g \mid g \in G_i \text{ and } g \subset h \text{ for some } h \in H\}$, we find that H_i is a discrete collection of domains since G_i is a discrete collection. Consider p, an element of S. Since H covers S there exists some domain D of H which contains p, but S perfectly screenable implies there exists an integer n(p, D) such that $G_{n(p, D)}$ contains a domain g which lies in D and contains p. Therefore, g is in $H_{n(p, D)}$, and p is contained in $\bigcup_{i=1}^{\infty} H_i$. Also we see that the definition of H_i implies $\bigcup_{i=1}^{\infty} H_i$ is a refinement of H. Hence S is a strongly screenable space.

In general the converse of this theorem fails as can be seen by the following example.

Example 2.8: A regular, separable, strongly screenable topological space that is not perfectly screenable.

Points of our space are the points of the positive x-axis. Neighborhoods are closed intervals minus their right hand end points, that is, intervals of the form [a, b). Since our topology is similar to the topology of the real line, our space is seen to be both regular and separable. Let H be any open covering of our space, and consider the point 0. There is an $h \in H$ such that $0 \in h$ since H is an open covering of our space. Consider the set $\{y : [0, y) \subset h\}$. This set is non-empty since h is an open set. Let $b = \sup \{y : [0, y) \subset h\}$, and note that $b \notin \{y : [0, y) \subset h\}$ for if so there would exist an $\epsilon > 0$ such that $[b - \epsilon, b + \epsilon) \subset h$ by definition of sup. Thus we would have $[0, b + \epsilon) \subset h$ which implies $b \neq \sup \{y : [0, y) \subset h\}$, a contradiction. Since $b \notin \{y : [0, y) \subset h\}$ and H is an open covering of our space, we know there is an $h_1 \in H$ such that $b \in h_1$. Now repeating the above argument we see that we can construct an open covering G of the space such that G refines H and no two elements of G intersect each other. Since each g of G contains a rational number, our collection G is countable. Therefore, we can index the elements of G with the set of positive integers. Thus, if we let $H_i = \{g_i\}$ we have that H_i is a discrete collection, $\bigcup_{i=1}^{\infty} H_i$ covers the space, and $\bigcup_{i=1}^{\infty} H_i$ is a refinement of H. Hence our space is strongly screenable.

Let G denote any countable collection of neighborhoods, then there is a point p that does not belong to the left end of any element of G because the reals are uncountable. Therefore, the set $[p, a)$ is not the sum of a subcollection of G for every real number a. Thus, our space is not second countable.

Since our space is separable but not second countable, it is not metrizable, as shown in Hall and Spencer [12], p. 107. Also a regular topological space is metrizable if and only if it is perfectly screenable (see Chapter III). Since our space is a regular non-metrizable topological space, we have it is not perfectly screenable.

Normality, Full Normality, and Collectionwise Normality

This section of the paper will introduce two generalizations of normality. The first, collectionwise normality, Bing [4], will be shown to have a definite relation to the preceding section in regular spaces. The second, first introduced by Turkey in 1940, is the concept of full normality. As will be shown in Chapter III, the concepts of fully normality and collectionwise normality lead to characterizations of metrizability in Moore spaces.

Definition 2.9: A topological space is collectionwise normal if for each discrete collection X of point sets, there is a collection Y of mutually exculsive domains covering X* such that no element of Y intersects two elements of X. We use X* to denote the sum of the elements of X.

Theorem 2.10: Any metric space is a collectionwise normal topological space.

Proof: Let S be a metric space with metric D and $X = \{x_\alpha | \alpha \in \Lambda\}$ a discrete collection of point sets. Since X is a discrete collection we have for each $\alpha \in \Lambda$ that $\overline{x}_\alpha$ and $(\bigcup_{\beta \neq \alpha} \overline{x}_\beta)$ are closed sets such that

$$\overline{x}_\alpha \cap (\bigcup_{\beta \neq \alpha} \overline{x}_\beta) = \emptyset.$$

For each $\alpha \in \Lambda$ let

$$r_\alpha = D(\overline{x}_\alpha, \bigcup_{\beta \neq \alpha} \overline{x}_\beta).$$

Since $\overline{x}_\alpha$, $\bigcup_{\beta \neq \alpha} \overline{x}_\beta$ are closed disjoint sets we have that $r_\alpha \neq 0$. For each $p \in x_\alpha$ consider the sphere $S_{r_{\alpha/4}}(p)$. Let

$$y_\alpha = \bigcup_{p \in x_\alpha} S_{r_{\alpha/4}}(p),$$

then y_α is a domain that contains x_α. Therefore, the collection $Y = \{y|\alpha \in \Lambda\}$ is a collection of domains such that Y covers X*. If $y_\alpha \cap y_\beta \neq \emptyset$ then there exist points p and q; spheres $S_{r_{\alpha/4}}(p)$ and $S_{r_{\beta/4}}(q)$ such that $p \in S_{r_{\alpha/4}}(p)$, $q \in S_{r_{\beta/4}}(q)$ and $S_{r_{\alpha/4}}(p) \cap S_{r_{\beta/4}}(q) \neq \emptyset$. Assume without loss of generality that $r_\beta \leq r_\alpha$. Thus there is a point x such that $D(p,x) \leq r_{\alpha/4}$ and $D(x,q) \leq r_{\beta/4}$. Therefore we have

$$D(p,q) \leq D(p,x) + D(q,x) \leq r_{\alpha/4} + r_{\beta/4} \leq r_{\alpha/2} \leq r_\alpha$$

which is a contradiction, since $D(\overline{x}_\alpha, \overline{x}_\beta) \geq r_\alpha$. Hence the collection Y is a mutually exclusive collection of domains. Suppose y_α contains points from x_α and x_β. Then there is a point $p \in x_\alpha$, and a sphere $S_{r_{\alpha/4}}(p)$, and a point $q \in x_\beta$ such that $q \in S_{r_{\alpha/4}}(p)$. Now $q \in S_{r_{\alpha/4}}(p)$ implies $D(p,q) \leq r_{\alpha/4} < r_\alpha$. Thus $D(x_\alpha, x_\beta) < r_\alpha$ which is a contradiction. Hence Y is a collection of mutually exclusive domains such that Y covers X* and no element of Y contains points of two elements of X. Therefore, by definition, S is a collectionwise normal topological space.

Hence, in particular, the topological spaces E_1 and E_2 with the usual topologies are collectionwise normal topological spaces. The following theorem and example show that collectionwise normality is a generalization of normality, but that every normal space is not necessarily collectionwise normal.

Theorem 2.11: Let S be a topological space. If S is collectionwise normal, then S is normal.

Proof: Let A and B be two mutually exclusive closed subsets of S; that is, $\{A, B\}$ is a discrete collection. The definition of collectionwise normal implies that there exists a collection $Y = \{y_\alpha | \alpha \in \Lambda\}$ of mutually exclusive domains such that for each α, y_α does not intersect both A and B, and $A \cup B \subset Y^*$. Let C be the class of all y_α such that $y_\alpha \cap A \neq \emptyset$. Similarly let D be the class of all y_α such that $y_\alpha \cap B \neq \emptyset$. Since the collection Y is a mutually exclusive collection of domains, the domains C^* and D^* are disjoint. Since $A \cup B \subset Y^*$, the definitions of C and D imply $A \subset C^*$ and $B \subset D^*$. Hence S is a normal topological space.

Example 2.12: A normal topological space which is not a collectionwise normal topological space.

Let P be an uncountable set, $\mathfrak{B}$ the class of all subsets of P, and $F = \{f | f : \mathfrak{B} \to \{0, 1\}\}$. For each $p \in P$ associate the function defined as follows: $f_p(B) = 1$ if $p \in B$ and $f_p(B) = 0$ if $p \notin B$, and let $F_p = \{f_p | p \in P\}$. Now F is topologized as follows: (1) If $f \in F - F_p$ then $\{f\}$ is a neighborhood of f, (2) if $f_p \in F_p$ and $\mathscr{R}$ is a finite subclass of $\mathfrak{B}$ then the $\mathscr{R}$ neighborhood of f_p is defined to be the set of all $f \in F$ such that $f(B) = f_p(B)$ whenever $B \in \mathscr{R}$. Let $D_{\mathscr{R}}$ be the $\mathscr{R}$ neighborhood of f_p and $D_{\mathscr{S}}$ the $\mathscr{S}$ neighborhood of f_q such that $D_{\mathscr{R}} \cap D_{\mathscr{S}} \neq \emptyset$. Let $f_r \in D_{\mathscr{R}} \cap D_{\mathscr{S}}$ and note that if $D_{\mathscr{R} \cup \mathscr{S}}$ is the $\mathscr{R} \cup \mathscr{S}$ neighborhood of f_r then $D_{\mathscr{R} \cup \mathscr{S}}$ is a subset of $D_{\mathscr{R}} \cap D_{\mathscr{S}}$. Thus we have a base for a topology of F.

Let H_1 and H_2 be two disjoint closed subsets of F. Let A_1 be the set of points common to H_1 and F_p and let A_2 be the set of points common to H_2 and F_p. Let Q_1 be the associated subset in P consisting of all p for which f_p belongs to A_1 and let Q_2 be the associated subset in P consisting of all p for which f_p belongs to A_2. Now if $A_1 = \emptyset$, then $H_1 \subset F - F_p$. Hence H_1 is a domain containing H_1. Thus H_1 and $F - H_1$ are mutually exclusive domains containing H_1 and H_2 respectively. Similarly if $A_2 = \emptyset$, we have H_2 and $F - H_2$ are mutually exclusive domains containing H_2 and H_1 respectively. Thus assume that neither A_1 nor A_2 is empty. Now define $D_1 = \{f \in F | f(Q_1) = 1$ and $f(Q_2) = 0\}$. Let $f_p \in A_1$ then $f_p(Q_1) = 1$, since $p \in Q_1$, and $f(Q_2) = 0$ since $p \notin Q_2$. Hence we have $A_1 \subset D_1$. If $f \in D_1$ such that $f \in F - F_p$ then the definition of the base for the topology of F implies f is a neighborhood of f such that $\{f\} \subset D_1$. If $f \in D_1$ and $f \in F_p$ then $f = f_p$ for some $p \in Q_1$. Now consider $\mathcal{R} = \{Q_1, Q_2\}$. The definition of the base for the topology implies the $\mathcal{R}$ neighborhood of f_p is the set $D = \{f \in F | f(Q_i) = f_p(Q_i)$ for $i = 1, 2\}$. Since $f \in D$ implies $f(Q_i) = f_p(Q_i)$ $i = 1, 2$, we have $f(Q_1) = 1$ and $f(Q_2) = 0$. Thus D is contained in D_1. Therefore D_1 is a domain containing A_1. Similarly the set D_2 of all $f \in F$ such that $f(Q_2) = 1$ and $f(Q_1) = 0$ is an open set containing A_2. If $D_1 \cap D_2 \neq \emptyset$, then there is an f such that $f \in D_1 \cap D_2$. Now $f \in D_1$ implies that $f(Q_1) = 1$ and $f(Q_2) = 0$, but $f \in D_2$ implies $f(Q_1) = 0$. This contradiction implies that $D_1 \cap D_2 = \emptyset$. Consider the sets

$$(D_1 - H_2) \cup (H_1 - A_1) \quad \text{and} \quad (D_2 - H_1) \cup (H_2 - A_2).$$

Since $H_1 - A_1 \subset F - F_p$ and $F - F_p$ is discrete, we have that $H_1 - A_1$ is an open set. Therefore, the set $(D_1 - H_2) \cup (H_1 - A_1)$ is an open set. Similarly the set $(D_2 - H_1) \cup (H_2 - A_2)$ is an open set.

Suppose $f \in H_1$ and $f \notin H_1 - A_1$, then $f \in A_1$. Now $f \in A_1$ implies $f \in D_1$. Then $f \in D_1 - H_2$ and $f \in (D_1 - H_2) \cup (H_1 - A_1)$. Otherwise suppose $f \in H_1$ and $f \in H_1 - A_1$; thus $f \in (D_1 - H_2) \cup (H_1 - A_1)$. There fore,

$$H_1 \subset (D_1 - H_2) \cup (H_1 - A_1).$$

Similarly

$$H_2 \subset (D_2 - H_1) \cup (H_2 - A_2).$$

Also we have

$$[(D_1 - H_2) \cup (H_1 - A_1)] \cap [(D_2 - H_1) \cup (H_2 - A_2)] = \emptyset.$$

Therefore we have $(D_1 - H_2) \cup (H_1 - A_1)$ and $(D_2 - H_1) \cup (H_2 - A_2)$ are mutually exclusive domains containing H_1 and H_2 respectively. Thus F is a normal topological space.

Consider the set $F_p = \{f_p | p \in P\}$. Let f_p be any element in F_p. For each $f \in F - F_p$ we have that $\{f\}$ is a neighborhood of f which does not intersect $\{f_p\}$. Let f_q be any point in F_p different from f_p, that is, $p \neq q$. The set $\mathscr{R} = \{\{q\}\}$ is a finite subset of $\mathfrak{B}$. An $\mathscr{R}$ neighborhood of f_q is the set of all $f \in F$ such that $f(\{q\}) = f_q(\{q\}) = 1$. Therefore f_p is not an element of the $\mathscr{R}$ neighborhood of f_q since $f_p(\{q\}) = 0$. Thus $\{f_p\}$ is a closed set. Hence $\{\overline{f_p}\} \cap \{\overline{f_q}\} = \emptyset$ if and only if $p \neq q$. Let K be a subcollection of F_p, and consider $\bigcup_{k \in K} \overline{k}$. If $f \in F - F_p$ then $\{f\}$ is a neighborhood which does not intersect $\bigcup_{k \in K} \overline{k}$. If $f_q \in F_p$ such that $f_q \notin K$ then by an argument similar to the above we have that there is an $\mathscr{R}$ neighborhood of f_q that does not contain any points of K. Thus the set $\bigcup_{k \in K} \overline{k}$ is a closed set. Hence the set F_p is a discrete collection of points.

Suppose there is a collection of mutually exclusive neighborhoods $\{D_p\}$ such that D_p is a neighborhood of f_p. Let D_p be an $\mathcal{R}_p$ neighborhood of f_p. Since $\mathcal{R}_p$ is a finite subset of $\mathfrak{B}$ and P is uncountable there is a natural number n and an uncountable subset W of P such that $\mathcal{R}_p$ has exactly n elements for every $p \in W$; for otherwise P is countable. Let a and b be two elements of W. If $\mathcal{R}_a \cap \mathcal{R}_b = \emptyset$, then consider a function f such that $f(B) = f_a(B)$ for every $B \in \mathcal{R}_a$, and $f(B) = f_b(B)$ for every $B \in \mathcal{R}_b$, and $f(B) = 0$ for every $B \notin \mathcal{R}_a \cup \mathcal{R}_b$. Since $\mathcal{R}_a \cap \mathcal{R}_b = \emptyset$ the function f is well defined and $f \in D_a \cap D_b$, we have a contradiction to the hypothesis that the D_p's were mutually exclusive. Thus for every a and b in W we have $\mathcal{R}_a \cap \mathcal{R}_b \neq \emptyset$. Hence there is an element B_1 of $\mathfrak{B}$ and an uncountable subset W_1' of W such that B_1 belongs to $\mathcal{R}_p$ for every p in W_1'; for if not some $\mathcal{R}_a$ would contain an infinite number of elements. Moreover, there is a t_1 with value 1 or 0 and an uncountable W_1 of W_1' such that $f_p(B_1) = t_1$ for every p in W_1. Using similar reasoning on $\mathcal{R}_p - \{B_1\}$ for $p \in W_1$ we find

$$(\mathcal{R}_a - \{B_1\}) \cap (\mathcal{R}_b - \{B_1\}) \neq \emptyset$$

for every a and b in W_1. Thus there is an element B_2 of $\mathfrak{B}$, $B_2 \neq B_1$, a t_2 with value 1 or 0, and an uncountable subset W_2 of W_1 such that B_2 belongs to $\mathcal{R}_p$ and $f_p(B_2) = t_2$ for every $p \in W_2$. Continuing recursively in this fashion we get B_k, t_k, W_k, for $k = 1, 2, \ldots, n$. Let $\mathcal{R}$ be the set consisting of $B_1, B_2, \ldots, B_n$ and D the set of all f with $f(B_k) = t_k$ for $k = 1, 2, \ldots, n$. Then $\mathcal{R}_p =$ and $D_p = D$ for all p in W_n, since each $_p$ contains exactly n sets. This contradicts the fact that the collection D_p was a mutually exclusive collection. Therefore the space F is not collectionwise normal.

As we have seen in Example 2.5 there is a regular screenable topological space that is not a normal topological space. However, if we replace screenability by strong screenability we obtain the following result.

Theorem 2.13: If S is a regular strongly screenable topological space then S is a collectionwise normal topological space.

Proof: Let $\{X_\alpha | \alpha \in \Lambda\}$ be a discrete collection of point sets. For each α, let A_α denote $\overline{X}_\alpha$; then the collection $\{A_\alpha | \alpha \in \Lambda\}$ is a discrete collection of closed point sets. Let A_β, β some element of Λ, be a set in our collection; since we have a discrete collection we know that A_β and $\bigcup_{\alpha \neq \beta} A_\alpha$ are closed sets. Therefore for each point p in A_β there exists a domain which contains p and does not intersect $\bigcup_{\alpha \neq \beta} A_\alpha$. Since our space is regular there exists a domain $V_{p\beta}$ which contains p and whose closure does not intersect $\bigcup_{\alpha \neq \beta} A_\alpha$. Thus $\{V_{p\beta} | p \in A_\beta\}$ is a collection of domains that cover A_β and whose closures do not intersect $\bigcup_{\alpha \neq \beta} A_\alpha$. Similarly, for each α, there exists a collection $\{V_{p\alpha} | p \in A_\alpha\}$ of domains which cover A_α and whose closures do not intersect $\bigcup_{\gamma \neq \alpha} A_r$. Let

$$K = \bigcup_{\alpha \in \Lambda} \{V_{p\alpha} | p \in A_\alpha\},$$

then K is an open covering of S by domains such that the closure of no element of K intersects two elements of the collection $\{A_\alpha | \alpha \in \Lambda\}$. Since S is strongly screenable, there exists a sequence $G = \{H_i\}$ such that H_i is a discrete collection of domains, $\bigcup_{i=1}^{\infty} H_i$ covers S, and $\bigcup_{i=1}^{\infty} H_i$ is a refinement of K. Now let $W_{i\beta}$ be the sum of the elements of H_i that intersect A_β, where $\beta \in \Lambda$, and $V_{i\beta}$ be the sum of the elements of

H_i that intersect A_α, where $\alpha \neq \beta$. With this in mind, define

$$D_\beta = \bigcup_{i=1}^{\infty} (W_{i\beta} - \bigcup_{j=1}^{i-1} \overline{V}_{j\beta})$$

$$= W_{1\beta} \cup (W_{2\beta} - \overline{V}_{1\beta}) \cup [W_{3\beta} - (\overline{V}_{1\beta} \cup \overline{V}_{2\beta})] \cup \ldots$$

Since $\bigcup_{i=1}^{\infty} H_i$ is a refinement of K we have that each element of $\bigcup_{i=1}^{\infty} H_i$ is a subset of an element of K. Therefore, the closure of each element of $\bigcup_{i=1}^{\infty} H_i$ does not intersect two elements of the collection $\{A_\alpha | \alpha \in \Lambda\}$. Hence for each β, $W_{i\beta}$ does not intersect two elements of the collection $\{A_\alpha | \alpha \in \Lambda\}$ which implies

$$(W_{i\beta} - \bigcup_{j=1}^{i-1} \overline{V}_{j\beta}),$$

for each i, does not intersect two elements of the collection $\{A_\alpha | \alpha \in \Lambda\}$. Therefore, no element of the collection $\{D_\beta | \beta \in \Lambda\}$ intersects two elements of the collection $\{A_\alpha | \alpha \in \Lambda\}$.

Consider p an element in $\bigcup_{\alpha \in \Lambda} A_\alpha$, then we know p is contained in A_β, for some $\beta \in \Lambda$. Since $\bigcup_{i=1}^{\infty} H_i$ is a refinement of K, there exists a domain h in some H_j that contains p and h does not intersect $\bigcup_{\alpha \neq \beta} A_\alpha$. Hence, we have that p is contained in $W_{j\beta}$, and p is not contained in $V_{i\beta}$, for all i. Since H_i, for each i, is a discrete collection of domains we have that p is not contained in $\overline{V}_{i\beta}$, for all i. Therefore, p is contained in D_β, and the collection $\{D_\beta | \beta \in \Lambda\}$, covers $\bigcup_{\alpha \in \Lambda} A_\alpha$. The point set $W_{i\beta}$ is a domain since $W_{i\beta}$ is the sum of domains. Now, since

$$(W_{i\beta} - \bigcup_{j=1}^{i-1} \overline{V}_{j\beta}), \quad i = 1, 2, \ldots,$$

is a domain, we have that D_β is a domain for each β.

By considering the definitions of D_β and D_α, $\alpha, \beta \in \Lambda$; if $D_\beta \cap D_\alpha \neq \emptyset$ then there would have to exist a domain h in $\bigcup_{i=1}^{\infty} H_i$ which would intersect both A_β and A_α. This is impossible by the way we constructed our covering K and the fact that $\bigcup_{i=1}^{\infty} H_i$ is a refinement of K. Hence the collection $\{D_\beta | \beta \in \Lambda\}$ is a collection of mutually exclusive domains covering $\bigcup_{\alpha \in \Lambda} A_\alpha$, such that no element of the collection $\{D_\beta | \beta \in \Lambda\}$ intersects two elements of the collection $\{A_\alpha | \alpha \in \Lambda\}$. Hence the space S is a collectionwise normal topological space.

Now Theorems 2.7, 2.11, and 2.13 imply the following results.

Corollary 2.14: A regular perfectly screenable topological space is a collectionwise normal topological space.

Corollary 2.15: A regular strongly screenable topological space is a normal topological space.

Corollary 2.16: A regular perfectly screenable topological space is a normal topological space.

At this point it is worth mentioning that there is a normal topological space which is not strongly screenable, which may be found in Hodel [16]. Also a strongly screenable topological space which is not normal may be constructed from the example given in Hall and Spencer [12], p. 65.

This section will now be concluded with the introduction of full normality which will be shown to be a generalization of collectionwise normality as well as normality.

Definition 2.17: Let H be an open covering of a topological space S. The star of a point set A with respect to an open covering H is the sum of the elements of H that intersect A. The star of A will be denoted by star (A).

Definition 2.18: A topological space is fully normal if for each open covering H of the space there is an open covering H_1 of the space such that the star of each point with respect to H_1 is a subset of an element of H.

One quickly sees that the topological spaces E_1 and E_2 with the discrete topology are fully normal topological spaces.

Theorem 2.19: If a topological space S is fully normal then S is collectionwise normal.

Proof: Let W be a discrete collection of closed sets. Since W is a discrete collection of closed sets we can construct an open covering H of S such that no element of H intersects two elements of W. Since S is fully normal there is an open covering H_1 of S such that for each point $p \in S$ the star of p with respect to H_1 is a subset of an element of H.

Let us consider the collection $G = \{\text{star}(w) | w \in W^*\}$. Since H_1 covers S and hence W* we have that G covers W*. Let w_1 and w_2 be two different elements in W*. If $\text{star}(w_1) \cap \text{star}(w_2) \neq \emptyset$, then let $p \in \text{star}(w_1) \cap \text{star}(w_2)$. The star of p with respect to H_1 is a subset of an element of H, but the star of p intersects both w_1 and w_2 since $p \in \text{star}(w_1) \cap \text{star}(w_2)$. Hence an element of H intersects both w_1 and w_2 which implies a contradiction. Therefore, we have star $(w_1) \cap$ star $(w_2) = \emptyset$. Thus the collection G is a collection of mutually exclusive domains such that no one of these domains intersects two elements of W. Hence by definition S is a collectionwise normal topological space.

Corollary 2.20: If S is a fully normal topological space then S is a normal topological space.

That the converse of Theorem 2.19 does not hold is shown by the following example.

Example 2.21: A collectionwise normal topological space which is not fully normal.

Let R denote the set of real numbers, and let

$$W_1 = \{x_1, x_2, \ldots, x_\alpha, \ldots\}$$

be a well ordering of R. If W_1 is a well ordering of R then

$$W = \{x_2, x_3, \ldots, x_\alpha, \ldots; x_1\}$$

is a well ordering of R. Hence let * be the first element of W such that * is preceded by an uncountable number of elements of W. Let S be the set of elements of W that precede *. Let the space S be defined to be the set S in the interval topology (see Spencer and Hall, p. 160).

Suppose K is an infinite discrete collection of points in S. Now K contains an infinite sequence $\{k_i\}$ such that k_i precedes k_{i+1}. Denote by p the first element of W which follows this infinite sequence. Let U be any open set such that $p \in U$. There is an interval of the form $(k_n, p]$ contained in U, for otherwise p is not the first element of W which follows the sequence $\{k_i\}$. Thus the point p is a limit point of this infinite sequence $\{k_i\}$. Let $K_i = \{s \mid s \in S$ and s precedes $k_i\}$ and note that K_i is countable for each i. If $p = *$, then $S = \bigcup_{i=1}^{\infty} K_i$, but this is impossible since $\bigcup_{i=1}^{\infty} K_i$ is countable and S is uncountable. Therefore, $p \in S$ and the set K is not a discrete collection because $M = \bigcup_{i=1}^{\infty} \overline{k_i}$ is not a closed set. This contradiction implies there does not exist an infinite discrete collection of points in S. At this point note that the above argument could be used to show that every infinite set of points of S has a limit point in S.

Let A and B be two disjoint closed sets in S. For each $a \in A$, consider the set $\{b \mid b \in B$ and b precedes $a\}$. This set has an upper bound in S, namely a; thus the set $\{b \mid b \in B$ and b precedes $a\}$ has a least upper bound b_a. Since the set B is closed, we have $b_a \in B$. The interval $(b_a, a]$ is an open set such that $a \in (b_a, a]$ and $(b_a, a] \cap B = \emptyset$. Let

$$U = \bigcup \{(b_a, a] \mid a \in A\},$$

and note $A \subset U$. Similarly let

$$V = \bigcup \{(a_b, b] \mid b \in B\}$$

and note $B \subset V$. If $U \cap V \neq \emptyset$, then some $(b_a, a] \cap (a_{\overline{b}}, \overline{b}] \neq \emptyset$. Now without loss of generality we may assume that $\overline{b}$ precedes a. If $\overline{b}$ precedes a, $(b_a, a] \cap (a_{\overline{b}}, \overline{b}] \neq \emptyset$, then the definition of $(b_a, a]$ implies $\overline{b} \in (b_a, a]$. This contradiction implies $U \cap V = 0$. Hence the space S is normal.

Let Y be any discrete collection of point sets. Now suppose that $Y = \{Y_\alpha \mid \alpha \in \Lambda\}$ is an infinite collection. For each $\alpha \in \Lambda$, let $y_\alpha \in Y_\alpha$ and consider the set $Y_1 = \{y_\alpha \mid y_\alpha \in Y_\alpha\}$. Since Y is an infinite collection the set Y is an infinite set. Therefore Y_1 has a limit point p in S. Since Y is a discrete collection $p \notin S - \bigcup \overline{Y}_\alpha$. Also since Y is a discrete collection $p \in \overline{Y}_\alpha$, for each α, because $\overline{Y}_\alpha$ contains only one point of Y_1. This contradiction implies that Y is a finite collection of point sets. Since S is normal there is a collection G of mutually exclusive domains such that G covers Y* and no element of G intersects two elements of Y. Therefore by definition, S is collectionwise normal.

Let H be any collection of open sets such that H covers S. For each $x_2 \in W$, consider the star (x_2) with respect to H. If $S \not\subset$ star (x_2) let y_1 be the first element of W such that $y_1 \notin$ star (x_2). If $S \not\subset$ star $(x_2) \cup$ star (y_1) let y_2 be the first element of W such that $y_2 \notin$ star $(x_2) \cup$ star (y_1). Suppose for $\beta < \alpha$ that y_β has been chosen. If $S \not\subset$ star $(x_2) \cup (\bigcup_{\beta<\alpha}$ star $(y_\beta))$ then let y_α be the first element of W such that $y_\alpha \notin$ star $(x_2) \cup (\bigcup_{\beta<\alpha}$ star $(y_\beta))$. By construction and definition the domain star (y_α), for each α, contains only y_α from the set $\{y_1, y_2, \ldots, y_\beta, \ldots\}$. Therefore if there does not exist a y_σ such that $S \subset$ star $(x_2) \cup (\bigcup_{\beta<\sigma}$ star $(y_\beta))$ then S contains an infinite set of points which does not have a limit point in S. For suppose the contrary, that p is a limit point of $\{y_1, y_2, \ldots, y_\beta, \ldots\}$. Then we have $p \in$ star (y_β); for some β; otherwise p would not be a limit point of $\{y_1, y_2, \ldots, y_\beta, \ldots\}$ Now by construction the star (y_β) contains only y_β from our set. Hence star (y_β) is a domain which contains only y_β from our set. Therefore p is not a limit point of $\{y_1, y_2, \ldots, y_\beta, \ldots\}$. Thus we have a contradiction to the fact that every infinite set of points of S has a limit point in S. Therefore there is a point $p \in S$ such that star (p) with respect to H contains all the points of S that follow p.

The collection $H_1 = \{[x_2, s) \mid s \in S\}$ is a collection of open sets such that H_1 covers S, and no element of H_1 contains all the points that follow some point in S. Hence there does not exist an open covering H_2 of S such that the star of each point with respect to H_2 is a subset of an element of H_1. Thus S is not fully normal.

A Theorem of A. H. Stone and Paracompactness

In 1948, A. H. Stone stated and proved that a topological space is paracompact if and only if it is fully normal [25]. However, as one reads the literature and sees that the concept of paracompactness is preferred over the concept of full normality when speaking of metrizability in Moore spaces. For example, see Jones [19]. Hence this section will be devoted to showing properties of paracompact topological spaces.

Definition 2.22: A collection G of subsets of a topological space S is said to be locally finite if and only if every point of S has a neighborhood which meets at most a finite number of elements of G.

Definition 2.23: A topological space S is paracompact if and only if every open cover H of S has an open refinement H_1 such that H_1 covers S and H_1 is locally finite.

Example 2.24: The topological space E_2 is a paracompact topological space.

Let H be any open covering of the topological space E_2. The set

$$C_n = \{(x, y) \mid n - 1 \le x^2 + y^2 \le n\}, \ n = 1, 2, \ldots,$$

is a compact set in E_2. Therefore there is a finite subcollection K_n of

H which covers C_n. Now the set $S_{n+1} = \{(x, y) | x^2 + y^2 < n + 1\}$ is a domain which contains C_n. Let $M_n = \{k \cap S_{n+1} | k \in K_n\}$ and define H_1 as follows: each element of M_1 is an element of H_1 and if $g \in M_n$ then

$$g - \bigcup_{i=1}^{n-1} C_i$$

is an element of H_1. Let $h \in H_1$, then there is a natural number n such that

$$h = g - \bigcup_{i=1}^{n-1} C_i$$

where $g \in M_n$. Since elements of M_n are contained in K_n and K_n is a finite subcollection of H we have that h is contained in an element of H. Thus H_1 is a refinement of H. Let $x \in S$ then there is a natural number n such that $x \in C_n$ and

$$x \notin \bigcup_{i=1}^{n-1} C_i .$$

Clearly there is an element of H_1 that contains x. Hence H_1 is a covering of S. Now the definition of H_1 implies that each element of H_1 is a domain. Thus H_1 is an open refinement of H. Let $x \in S$ then there is a natural number n such that $x \in C_n$ and

$$x \notin \bigcup_{i=1}^{n-1} C_i .$$

Since M_{n-1}, M_n, and M_{n+1} are finite collections and some element of M_n contains x we have that H_1 is a locally finite open refinement of H. Hence by the definition the topological space E_2 is paracompact.

A somewhat weaker condition than paracompactness is pointwise paracompactness.

Definition 2.25: A topological space is pointwise paracompact if for each open covering H there is an open covering H_1 such that H_1 refines H and no point lies in infinitely many elements of H_1.

The topological space E_2 is clearly pointwise paracompact by the following theorem.

Theorem 2.26: Every paracompact topological space is a pointwise paracompact topological space.

Proof: Let S be a paracompact topological space and H an open covering of S. Since S is paracompact there is an open refinement H_1 of S such that H_1 covers S and H_1 is locally finite. Since H_1 is a locally finite covering of S there is for each point $p \in S$ a domain U_p such that U_p intersects at most a finite number of elements of H_1. Let $H_2 = \{U_p | p \in S\}$. Then H_2 is an open covering of S such that no element of H_2 intersects an infinite number of elements of H_1. Let $x \in S$ and suppose that x is contained in an infinite number of elements of H_1. Since H_2 is an open covering of S there is an $h \in H_2$ such that $x \in h$. Thus h intersects an infinite number of elements of H_1 which is a contradiction. Therefore, no element of S lies in infinitely many elements of H_1. Since the open refinement H_1 covers S we have by definition that the topological space S is pointwise paracompact.

The converse of the above theorem is false as will be shown by Example 2.29. However, first let us show two further properties of paracompact topological spaces.

Theorem 2.27: Every paracompact topological space is a regular topological space.

Proof: Let S be a paracompact topological space and D a domain which contains a point $p \in S$. For each $q \in S - D$ there are domains U_q and V_q such that $q \in U_q$, $p \in V_q$, and $U_q \cap V_q = \emptyset$. The collection

$$G = \{U_q | q \in S - D\} \cup \{D\}$$

is an open cover of S. Since S is a paracompact topological space there is an open cover H such that H refines G and H is locally finite. Since H is locally finite, there is a domain $U_p \subset D$ such that $p \in U_p$ and U_p intersects at most a finite number of elements of H. If $\overline{U}_p \subset D$ then S is regular by definition. If $\overline{U}_p \not\subset D$ then let

$$K = \{k | k \in H,\ k \cap U_p \neq \emptyset\}.$$

Since $\overline{U}_p \subset D$ there is at least one member of K not contained in D. Therefore let $k_1, k_2, \ldots, k_n$ denote the members of K not contained in D. Since H is a refinement of G there are n points $q_1, q_2, \ldots, q_n$ of $S - D$ such that $k_i \subset U_{q_i}$ for $i = 1, 2, \ldots, n$. Let $V = U_p \cap V_{q_1} \cap \ldots \cap V_{q_n}$ and W the union of all members of G which are not contained in D. Then $V \cap W = \emptyset$ since

$$(h \cap V_{q_1} \cap \ldots \cap V_{q_n}) \cap (U_{q_1} \cup \ldots \cup U_{q_n}) = \emptyset$$

and

$$V_{q_i} \cap U_{q_i} = \emptyset$$

for all i. Therefore, $V \subset S - W$ is a closed set. By definition of W we have that $S - W \subset D$. Therefore, $\overline{V} \subset S - W \subset D$ and by definition the space is regular.

Theorem 2.28: Every paracompact topological space is a normal topological space.

Proof: Let S be a paracompact topological space. Then Theorem 2.27 implies that S is a regular topological space. Let A and B be any two disjoint closed subsets of S. For each $p \in A$ we know $p \in S - B$ since A and B are disjoint closed sets. Since S - B is open and S is regular, there is a domain U_p such that $p \in U_p$ and $\overline{U}_p \subset S - B$. The open sets $\{U_p | p \in A\}$ together with the open set S - A form an open cover G of S. Since S is paracompact there is an open cover H such that H refines G and H is locally finite. Now let U denote the union of all the members of H not contained in S - A. Then U is a domain such that $A \subset U$. Let $q \in B$, then since H is locally finite there is a domain V_q such that $q \in V_q$ and V_q intersects finitely many members of H. Now let $k_1, k_2, \ldots, k_n$ denote the members which are not contained in S-A. If V_q intersects no member of H not contained in S-A then let $D_q = V_q$. Otherwise, since H refines G there are points $p_1, p_2, \ldots, p_n$ of A such that $k_i \subset U_{p_i}$ for $i = 1, 2, \ldots, n$. Let

$$D_q = V_q \cap (S - \overline{U}_{p_1}) \cap (S - \overline{U}_{p_2}) \cap \ldots \cap (S - \overline{U}_{p_n}).$$

Then D_q is a domain such that $q \in D_q$. Suppose $U \cap D_q \neq \emptyset$ and let $x \in U \cap D_q$. Now $x \in U$ implies there is an $h \in H$ such that h is not contained in S-A and $x \in h$. Since $x \in D_q \cap U$ we have $x \in D_q$. Hence $x \in V_q$ by the definition of D_q. Since V_q intersects only a finite number of elements of S-A we have D_q intersects only a finite number of elements of S-A. Therefore $x \in k_i = h$, for some $i (1 \leq i \leq n)$. But $k_i \subset U_{p_i}$ which implies $x \in U_{p_i}$. This is a contradiction since $x \in D_q$ implies $x \in (S - \overline{U}_{p_i})$. Thus we have that $U \cap D_q = \emptyset$. Since $D_q \cap U = \emptyset$

for every $q \in B$, we have $U \cap V = \emptyset$. Hence S is a normal topological space.

Example 2.29: A pointwise paracompact topological space that is not a paracompact topological space.

The topological space is the space S of Example 2.5. Since S is a non-normal topological space we have by Theorem 2.28 that S is not paracompact. Let H be any open covering of S. Let $x = (x, 0) \in S$ then since H is an open covering of S there is an element $h \in H$ such that $x \in h$. Since G is a basis for S there is a basis element g_x such that $x \in g_x \subset h$. For each $x = (x, 0) \in S$ associate a basis element g_x and consider the collection $K = \{g_x | x = (x, 0) \in S\}$. The set K^* is a set such that no point of K^* lies in two elements of K. (i.e., consider an arbitrary point (x, y) and use a geometric argument). Now each point of $S\text{-}K^*$ is a basis element by definition; hence consider $L = \{\{p\} | p \in S\text{-}K^*\}$. Then one quickly sees that $H_1 = K \cup L$ is an open covering of S such that H_1 refines H. By our construction of H_1 we have that no point of S lies in more than two elements of H_1. Hence by definition the topological space S is pointwise paracompact.

The remaining portion of this section will be devoted to two theorems by A. H. Stone. Since the first is not an integral part of this paper (and the proof is quite long) it will be stated without proof, but is included to clarify relationships between paracompactness and full normality.

Theorem 2.30: A fully normal topological space is a paracompact topological space.

Stone's second theorem which is the converse of the above can be best shown by use of the following lemma.

Lemma 2.31: Let S be a pointwise paracompact topological space. Then S is normal if and only if for each open covering $G = \{g_\alpha | \alpha \in \Lambda\}$ of S there is an open covering $H = \{h_\alpha | \alpha \in \Lambda\}$ such that $\overline{h}_\alpha \subset g_\alpha$ for each $\alpha \in \Lambda$.

Proof: Suppose S is normal and let $W = \{g_1, g_2, \ldots, g_\alpha, \ldots\}$ be a well ordering of G. Since S is a pointwise paracompact topological space we may assume that no point of S lies in infinitely many elements of G. Since G is an open covering of S the set $F_1 = S - \bigcup_{\beta \neq 1} g_\beta$ is a closed set contained in g_1. Since S is normal there is a domain h_1 such that $F_1 \subset h_1 \subset \overline{h}_1 \subset g_1$. The collection

$$H_1 = \{h_1\} \cup \{g_\beta | \beta \neq 1\}$$

is an open covering of S. Suppose, for $\beta < \alpha$, that h_β and H_β have been chosen such that $\overline{h}_\beta \subset g_\beta$ and

$$H_\beta = \{h_\gamma | \gamma \leq \beta\} \cup \{g_\gamma | \gamma > \beta\}$$

is an open covering of S. The set

$$F_\alpha = S - [(\bigcup_{\gamma < \alpha} h_\gamma) \cup (\bigcup_{\gamma > \alpha} g_\gamma)]$$

is a closed set contained in g_α since H_β is a covering of S for all $\beta < \alpha$. Therefore, there is a domain h_α such that $F_\alpha \subset h_\alpha \subset \overline{h}_\alpha \subset g_\alpha$. The collection

$$H_\alpha = \{h_\gamma | \gamma \leq \alpha\} \cup \{g_\gamma | \gamma > \alpha\}$$

is an open covering of S, since

$$H_\beta = \{h_\gamma \mid \gamma \leq \beta\} \cup \{g_\gamma \mid \gamma > \beta\}$$

is an open covering of S for $\beta < \alpha$ and $F_\alpha \subset h_\alpha$. Hence by trans-finite induction we have defined a collection $H = \{h_\alpha\}$ such that $\overline{h}_\alpha \subset g_\alpha$ for each α. Letting $x \in S$ we know that there are at most a finite number of sets of G containing x. Denote these sets by $g_{\alpha_1}, g_{\alpha_2}, \ldots, g_{\alpha_n}$ and let $\alpha = \sup \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. By construction we have $x \in h_\alpha$. Therefore H is an open covering of S possessing the desired property.

Let A and B be two disjoint closed sets in S. Then $\{S-A, S-B\}$ is an open covering of S; hence there is an open covering $\{U, V\}$ such that $\overline{U} \subset S-A$ and $\overline{V} \subset S-B$. Thus we have that $S-\overline{U}$ and $S-\overline{V}$ are open sets such that $A \subset S-\overline{U}$ and $B \subset S-\overline{V}$. Also we have that

$$(S-\overline{U}) \cap (S-\overline{V}) = S - (\overline{U} \cup \overline{V}) = S - S = \emptyset,$$

and thus the space S is normal.

At this point, before proving our main result let us point out that there does exist a Moore space that is not pointwise paracompact. (See Example 2.44).

Theorem 2.32: Every paracompact topological space is a fully normal topological space.

Proof: Let S be a paracompact topological space. Then Theorems 2.26 and 2.28 imply that S is a normal pointwise paracompact topological space. Let H be an open covering of S. Since S is paracompact there is an open covering $H_1 = \{U_\alpha \mid \alpha \in \Lambda\}$ of S such that H_1 refines H, and H_1 is locally finite. By Lemma 2.31 there is an open covering H_2 of S such that for each $U_\alpha \in H_1$ there exists an $X_\alpha \in H_2$ such that $\bar{X}_\alpha \subset U_\alpha$. For each $x \in S$ there is a domain $U(x)$ such that $x \in U(x)$, and the domain $U(x)$ intersects only a finite number of elements of H_1. Now let

$$A(x) = \{\alpha \in \Lambda \mid U(x) \cap U_\alpha \neq \emptyset\},$$

$$B(x) = \{\alpha \in A(x) \mid x \in U_\alpha\},$$

and

$$C(x) = \{\alpha \in A(x) \mid x \notin \bar{X}_\alpha\} \ ;$$

and note $A(x) = B(x) \cup C(x)$ since $x \in U_\alpha$ or $x \notin U_\alpha$. Define $W(x)$ as follows:

$$W(x) = U(x) \cap [\cap\{U_\alpha \mid \alpha \in B(x)\}] \cap [\cap \{(S - \bar{X}_\alpha) \mid \alpha \in C(x)\}].$$

By definition of $B(x)$ and $C(x)$ we have $W(x)$ is a domain containing x.

Define $H_3 = \{W(x) \mid x \in S\}$; and observe that H_3 is an open covering of S. Let $y \in S$, then there is an $X_\beta \in H_2$ such that $y \in X_\beta$ since H_2 covers S. If $y \in W(x)$ then $W(x) \cap \bar{X}_\beta \neq \emptyset$. Since $y \in W(x)$, we have $y \in U(x)$ by definition of $W(x)$. Also $W(x) \cap \bar{X}_\beta \neq \emptyset$ implies $U(x) \cap U_\beta \neq \emptyset$, thus $\beta \in A(x)$. If $\beta \in C(x)$ then $W(x) \subset S - \bar{X}_\beta$ which implies $W(x) \cap \bar{X}_\beta = \emptyset$, a contradiction. Thus $\beta \notin C(x)$, and $\beta \in A(x)$ which implies $\beta \in B(x)$ since $A(x) = B(x) \cup C(x)$. The definition of $W(x)$ now implies that $W(x) \subset U_\beta$. Thus the star $(y) = \bigcup_{y \in W(x)} W(x)$ with respect to H_3 is contained in an element of H_1. Since H_1 refines H we have that star (y) with respect to H_3 is contained in an element of H. Hence by definition we have that S is a fully normal topological space.

Note that the topological space E_2 is fully normal by the above theorem since E_2 is a paracompact topological space. (see Example 2.24).

Moore Spaces and Uniform Bases

Let S be a set such that there is a sequence $G = \{G_i\}$ which satisfies the following: (1) for each n, G_n is a collection covering S such that each element of G_n is a domain; (2) for each n, G_{n+1} is a subcollection of G_n; (3) if D is a domain such that $x, y \in D$ then there exists a natural number m such that if $g \in G_m$ and $x \in g$ then $\overline{g}$ is a subset of D and unless $y = x$, $\overline{g}$ does not contain y, Moore [23].

Definition 2.33: A Moore space is a set S such that there is a sequence G which satisfies conditions (1), (2), and (3) of the above.

Hence a Moore space is a regular Hausdorff space with a sequence $G = \{G_i\}$ such that for each point p of S, the sequence $H = \{\text{star}(p, G_1), \text{star}(p, G_2), \ldots\}$ is a countable basis for p, where star (p, G_i) means the star of p with respect to the covering G_i.

Clearly then the space S of Example 2.5 is a Moore space with G_n defined in the obvious way. (i.e., G_n is the collection of all points above the x-axis together with all $D(x, m)$ such that $m \geq n$ where x is a point on the x-axis). Another example of a Moore space is the topological space E_2 with G_n the collection of all spheres with diameter less than or equal to $1/n$.

An alternate way of characterizing a Moore space is the following theorem which is stated for information only and hence will not be proved.

Theorem 2.34: A regular Hausdorff topological space S is a Moore space if and only if (1) for each $p \in S$ there is a countable basis $U(p, 1), U(p, 2), \ldots$ of p and (2) for every domain V containing a point

$q \in S$ there is an integer $n(q, V)$ such that $q \in U(r, n(q, V))$ implies $U(r, n(q, V)) \subset V$, where $r \in S$.

Since, for much of this paper, the condition of regularity is not needed let us make the following definitions.

<u>Definition 2.35</u>: Let S be a Hausdorff space. A sequence $G = \{G_i\}$ such that G_i is an open covering of S is called a <u>development</u> of S if for each point p and each domain containing p there is a natural number n such that every element (domain) of G_n containing p is a subset of D.

<u>Definition 2.36</u>: Let S be a Hausdorff space with development $G = \{G_i\}$. If G_{i+1} refines G_i then S is called a <u>developable space</u>.

Of course, any Moore space is a developable space but the converse is not the case as the following examples illustrate.

<u>Example 2.37</u>: A developable space that is not regular. (See Hall and Spencer [12], p. 65).

<u>Example 2.38</u>: A regular topological space that is not a developable space.

The space is the topological space S of Example 2.8. As we have seen the space S is regular, separable and strongly screenable. Since S is strongly screenable we know that S is screenable. However, we also noticed that S was not second countable. Hence by the following theorem the space S is not developable.

Theorem 2.39: A separable screenable developable topological space S is second countable.

Proof: Let the sequence $G = \{G_i\}$ be a development of the topological space S. Since S is a screenable topological space we know that for each G_i, there exists a sequence $\{H_{in}\}$ such that H_{ij} is a collection of mutually exclusive domains and $\bigcup_{j=1}^{\infty} H_{ij}$ is an open covering that refines G_i. Since S is a separable topological space, S does not contain uncountably many mutually exclusive domains. Thus H_{ij} is a countable collection for each i, and $\bigcup_{j=1}^{\infty} H_{ij}$ is a countable open covering that refines G_i. Therefore, we have that $\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} H_{ij}$ is a countable basis for S, which implies that S is a second countable topological space.

Hence not every screenable topological space is developable as seen by Examples 2.8 and 2.38. Also, there is a developable topological space that is not screenable.

Example 2.40: A separable Moore space that is not screenable.

The points of the topological space S are the points of the plane and the open sets are given in terms of a development $\{G_i\}$ defined as follows: Let $\{L_i\}$ be a sequence of horizontal lines whose sum is dense in the plane. A set g is in G_n if and only if either g is the interior of a circle with diameter less than 1/n which does not intersect $\bigcup_{i=1}^{n} L_i$, or $g = \{p\} \cup I_1 \cup I_2$, where $p \in L_j$ for some j, I_1 and I_2 are interiors of circles of diameter less than 1/2n which are tangent to L_j at p on opposite sides of L_j and such that $I_1 \cup I_2$ does not intersect $\bigcup_{i=1}^{n} L_i$. Our development $\{G_i\}$ satisfies Definition 2.33. (Use a geometric argument to show part 3). Hence S is a Moore space. Since the plane is separable with the usual topology one quickly sees that our space S is separable.

Let L_j be one of the given lines and let L_j have equation $y = j$, where j is a real number. Now define $J = \{(x, j) \mid x \text{ is an irrational}\}$ and $K = \{(x, j) \mid x \text{ is a rational}\}$. By our development, there exists an open covering H of S such that no two points of $J \cup K$ belong to the same element of H. Therefore, any open covering of S that refines H contains uncountably many elements. Since S is separable it does not contain an uncountable collection of mutually exclusive domains. Hence S is not a screenable topological space.

In Example 2.40 consider J as a subspace of the topological space S. Then one quickly sees that with the relative topology J is a closed discrete subspace of S. Since the topological space S is separable there exists a dense countable set D such that D is dense in S. Since J is an uncountable set and D is countable we have $\mathfrak{N}(J) \geq 2^{\mathfrak{N}(D)}$. Hence the topological space S of Example 2.40 is not normal by the following theorem.

<u>Theorem 2.41</u>: Let S be a topological space. If S contains a dense set D and a closed discrete subspace K with $\mathfrak{N}(K) \geq 2^{\mathfrak{N}(D)}$ then S is not normal.

Proof: Assume that S is a normal topological space. Since K is a closed discrete subspace of S, every subspace of K is closed in S. Therefore, for each subset A of K we have A and K - A are mutually exclusive closed sets. Since S is normal, there exists domains U(A) and V(K-A) such that $A \subset U(A)$, $K-A \subset V(K-A)$ and $U(A) \cap V(K-A) = \emptyset$. Therefore the map $f: \mathcal{P}(K) \to \mathcal{P}(D)$ defined by $f(A) = D \cap U(A)$ is well defined. Let A and B be elements of $\mathcal{P}(K)$ such that $A \neq B$. If $A \neq B$, then we have either $A - B \neq \emptyset$ or $B - A \neq \emptyset$. Without loss of generality, assume $A - B \neq \emptyset$. Since $A - B \neq \emptyset$, we have that $A \cap (K - B) \neq \emptyset$, which implies that $U(A) \cap V(K - B) \neq \emptyset$, where U(A) and V(K - B) were defined above. Hence $D \cap U(A) \cap V(K - B) \neq \emptyset$ since D is a dense set. Also note $U(A) - U(B) \neq \emptyset$ or $U(A) \neq U(B)$ since $U(A) \cap V(K - B) \neq \emptyset$. Now $D \cap U(A) \cap V(K - B) \neq \emptyset$ and $U(A) \neq U(B)$ implies that $D \cap U(A) \neq D \cap U(B)$ or $f(A) \neq f(B)$. Hence our mapping is a one to one mapping, which implies $\mathfrak{N}(\mathcal{P}(K)) \leq \mathfrak{N}(\mathcal{P}(D))$. This is impossible since $\mathfrak{N}(\mathcal{P}(K)) > \mathfrak{N}(K) \geq \mathfrak{N}(\mathcal{P}(D)) = 2^{\mathfrak{N}(D)}$. Hence our space is not normal.

This section will now be concluded with one further condition for a regular topological space to be a Moore space.

Definition 2.42: Let S be a topological space. A base B of S is uniform if every infinite subcollection of B containing an arbitrary point $p \in S$ is a base at the point $p \in S$.

Example 2.43: The topological space E_2 has a uniform base.

A development for E_2 is $G = \{G_n\}$ where G_n is the collection of all open spheres of radius less than $1/n$. The topological space E_2 is pointwise paracompact by Example 2.24 and Theorem 2.26. Hence for each n there is an open refinement G_n' of G_n which covers E_2 and no point of E_2 lies in infinity many elements of G_n'. Define $B = \bigcup_{n=1}^{\infty} G_n'$ and note that B is a base of E_2 since $\bigcup_{n=1}^{\infty} G_n$ is a base of E_2. Suppose

B' is an infinite subcollection of B such that each element of B' contains p, where $p \in S$. Let D be any domain such that $p \in D$. Since G is a development of E_2 there is a natural number N such that every domain of G_N that contains p lies in D. If there is a $b' \in B'$ such that $b' \in G_N'$ then B' is a base at p. If no element of B' is contained in G_N' then there is an $m \geq N$ such that G_m' contains an element $b' \in B'$ such that $p \in b'$, since B' is infinite and for each n only a finite number of elements of G_n' contain p. Since G_m refines G_N we have b' is contained in an element of G_N. Hence b' is contained in D and B' is a base at p. Thus by definition B is a uniform base.

Example 2.44: A Moore space that does not have a uniform base.

Let S denote the points of the plane on or above the x-axis. Define a base B for S as follows: (1) If a circle lies entirely above the x-axis, its interior is a basis element; and (2) if a circle is tangent to the x-axis (from above) then its interior plus the point of tangency is a basis element. For example,

$$C = \{(x, y) \mid x^2 + (y - 1/2)^2 < 1\} \cup \{(0, 0)\}$$

is a basis element. A development $G = \{G_n\}$ is defined as follows: $g \in G_n$ if and only if $g \in B$ and the diameter of g is less than or equal to $1/n$. By using a geometric argument one sees that $G = \{G_n\}$ satisfies Definition 2.33. Hence S is a Moore space.

Suppose that S has a uniform base B_1 and consider the set $X = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } y = 0\}$ on the x-axis. There is an open covering H of X by elements of B such that no point of X is contained in more than one element of H. Since B_1 is a base there is an open covering H_1 of X by elements of B_1 such that H_1 refines H. Hence since B is a base there is an open covering H_2 of X by elements of B such that H_2 refines H_1. For each $x \in X$ let b_x denote the element of H_2 that contains x. Since X is an uncountable set, there must exist an uncountable number of elements of H_2 with diameter greater than $1/n$ for some n. Denote this collection by C. Now for each element b_x of C replace b_x by an element with diameter equal to $1/n$. Denote this new collection by D. Since D is uncountable and the centers of the associated circles lie on the line $y = 1/2n$ we have that the set of centers has a limit point p. Let b_p denote the element of H_1 that contains p. Then since H_2 refines H_1 there is an infinite collection of elements of H_1 that contain p. But since no element of H contains more than one point of X we have that no element of H_1 contains more than one point of X. Thus there is an infinite collection of elements of H_1 such that no element lies in b_p. Therefore B_1 is not a uniform base.

Since Example 2.44 is a Moore space S which does not have a uniform base the following theorem then implies that S is not pointwise paracompact.

Theorem 2.45: Let S be a regular topological space. The topological space S has a uniform base if and only if S is a pointwise paracompact Moore space.

Proof: Let $G = \{G_n\}$ be a development for S. Now if S is assumed to be pointwise paracompact, then for each natural number n, G_n has a refinement G_n' which covers S such that no point of S lies in infinitely many elements of G_n'. Let $B = \bigcup_{n=1}^{\infty} G_n'$ and suppose B' is an infinite subcollection of B such that each element of B' contains p, where $p \in S$. Now if D is any domain such that $p \in D$ then there is a natural number N such that every domain of G_N containing p lies in D. Since G_N' is a refinement of G_N we know there are only a finite number of elements of G_N' containing p. If no element of G_N' is an element of B' then since B' is an infinite subcollection of B there is an $m \geq n$ such that G_m' contains a domain such that $g \in B'$ and $p \in g$. Since G_m refines G_N we have g is contained in some element of G_N. Hence g is contained in D since every element of G_N which contains p lies in D. Thus B' is a base at the point p which implies that $B = \bigcup_{n=1}^{\infty} G_n'$ is a uniform base.

Conversely, suppose that the regular topological space has a uniform base B. Let $B = \{b_1, b_2, \ldots, b_\alpha, \ldots\}$ be a well-ordering of B. The cover G_1 is defined as follows: (1) Let $b_1 \in G_1$, (2) each subsequent element of G_1 is the first element of B to contain a point not in any preceding element of G_1. Let p be an element of S. Since B is a basis of S, there is an element of B which contains p. Let b be the first element of B to contain p. By construction of G_1 we have that $b \in G_1$. Hence G_1 covers S.

Now define G_2 as follows: Let B_1 be the subcollection of B remaining after removing from B all of the nondegenerate elements of G_1. The collection $B_1 \neq \emptyset$ because either b_1 is degenerate or b_1 is not degenerate. If b_1 is degenerate then $b_1 \in B_1$ which implies $B_1 \neq \emptyset$. If b_1 is not degenerate then there is a $b_\alpha \in B$ such that b_α is properly contained in b_1. Thus we have $b_\alpha \in B_1$ and $B_1 \neq \emptyset$. Let the first element of G_2 be the first term of B_1 (in the well-ordering of B) and each subsequent element of G_2 is the first element of B_1 (in the well-ordering of B) to contain a point not in any preceding element of G_2. Let $p \epsilon S$, if $\{p\}$ is an element of B then $\{p\} \epsilon B_1$. If b is the first non-degenerate element of B which contains p then there is a b_α, b_α follows b, $p \epsilon b_\alpha$, such that b_α is properly contained in b. Thus $b_\alpha \epsilon B_1$ and hence B_1 covers S. Therefore since B_1 covers S we have that G_2 covers S by construction. This process may be continued to define a sequence $G = \{G_n\}$ of open covers of S. Note, be certain not to delete the degenerate elements of G_i in B_{i-1} in producing B_i. Now if $g \epsilon G_{n+1}$ then $g \epsilon B_n$ by definition of G_{n+1}. Hence $g \epsilon B_{n-1}$ since B_n is a sub-collection of B_{n-1}. By construction either $g \epsilon G_n$ or $g \subset g_1$, where $g_1 \epsilon G_n$. Hence G_{n+1} refines G_n.

Let $p \epsilon D$ and D any domain. Suppose for every natural number n there is an element g_n of G_n such that $p \epsilon g_n$ and $g_n \not\subset D$. By construction the g_n's are distinct sets. Thus $\{g_n\}$ is an infinite collection of elements of B such that $p \epsilon g_n$ for each n and $\{g_n\}$ is not a base of p. This contradicts the definition of B; thus there is a natural number n such that every domain of G_n containing p lies in D. Hence $\{G_n\}$ is a development of S. Now by definition S is a developable space. One quickly sees that since S is a regular topological space that $G_n' = \bigcup_{k=n}^{\infty} G_k$ satisfies the conditions (1), (2), and (3); hence S is a Moore space.

Let H be an open covering of S. Since B is a basis there is a subcollection B^1 of B which refines H and covers S. Let $B^1 = \{b_1, b_2, \ldots, b_\alpha, \ldots\}$ be a well-ordering of B^1. An open refinement B^2 of B^1 is defined as follows: (1) $b_1 \epsilon B^2$, (2) each subsequent element of B^2 is the first element of B^1 to contain a point not in any preceding element of B^2. Let $p \epsilon S$ and let b be the first element of B^2 containing p. Suppose the collection, B^3, of all elements of B^2

which contains p is infinite. Then $B^3 - \{b\}$ is an infinite collection which is not a base at p since no element of $B^3 - \{b\}$ is contained in b. Hence the collection of all elements of B^2 which contains p does not form a base for p. Therefore by definition of uniform base the collection of all elements of B^2 which contain p is finite. Therefore by definition, S is a pointwise paracompact topological space.

Pointwise Paracompact and Screenable Moore Spaces

This chapter will be concluded with a relationship that holds in Moore spaces, but which does not hold in general. That is, if a Moore space is screenable then it is pointwise paracompact. The converse is shown to be false by a counterexample. In connection with these concepts there are several unanswered questions. Is every normal Moore space screenable? Is every normal Moore space pointwise paracompact? The answer to these questions would provide an answer to whether every normal Moore space is metrizable. Also in this connection is the following question: Is every normal Moore space collectionwise normal? The answer to these questions would give a solution, as shown by theorems of Chapter III. These questions give one an indirect method of attacking the question of whether a normal Moore space is metrizable.

Before attacking the theorem that every screenable Moore space is pointwise paracompact, let us recall that a set is a G_δ set if and only if it is the intersection of at most countably many domains, Dugundji [7].

Theorem 2.46: Let S be a topological space in which every closed set is a G_δ set. If S is a screenable topological space then S is a pointwise paracompact topological space.

Proof: Let H be an open covering of S. Since S is screenable, there is a sequence $K = \{K_i\}$ such that K_i is a collection of mutually exclusive domains, $\bigcup_{i=1}^{\infty} K_i$ covers S, and $\bigcup_{i=1}^{\infty} K_i$ refines H. For each i, let $M_i = S - K_i^*$. Also let $R_i = \{R_{ij}\}$ be a decreasing sequence of domains such that $M_i = \bigcap_{j=1}^{\infty} R_{ij}$. Let

$$G_1 = \{g \mid g = [\bigcap_{j=1}^{i-1} R_{ji}] \cap k, \quad k \in K_i, \; i=2,3,\ldots\}.$$

Now consider the collection $G_1 \cup K_1 = G$.

Let $p \in S$ and i the smallest natural number such that $p \in k$ for some $k \in K_i$. Now either $i = 1$ or $i > 1$. If $i > 1$, then

$$p \notin \left[\bigcup_{j=1}^{i-1} K_j\right]^*.$$

Thus

$$p \in \bigcap_{j=1}^{i-1} M_j \subset \bigcap_{j=1}^{i-1} R_{ji}.$$

Hence

$$p \in \left[\bigcap_{j=1}^{i-1} R_{ji}\right] \cap k,$$

for some $k \in K_i$, which implies $p \in g \in G_1$. If $i = 1$, then there exists $k \in K_i$ such that $p \in k \in G$. In either case, p is contained in an element of G. Therefore G is an open covering of S. By definition of G_1 and K we have that G refines H.

Let $p \in S$ and i the smallest natural number such that $p \in k$, for some $k \in K_i$. There is a natural number N such that for $j \geq N$ we have $p \notin R_{ij}$, for otherwise $p \in M_i$, a contradiction. Hence for $j > N$ and for any $k \in K_j$,

$$p \notin \left[\bigcap_{n=1}^{j-1} R_{nj}\right] \cap k$$

because

$$p \notin \left[\bigcap_{n=1}^{j-1} R_{nj}\right] .$$

Since p belongs to at most one element of each of the collections $K_1, K_2, \ldots, K_N$, p belongs to at most N elements of G. Thus G is an open covering of S such that G refines H and no point of S lies in infinitely many elements of G. Thus by definition, S is a pointwise paracompact topological space.

Theorem 2.47: Let S be a developable topological space. Every closed point set M of S is a G_δ set.

Proof: Let $G = \{G_n\}$ be a development of S. For each n, let

$$H_n = \{g | g \in G_n,\ g \cap M \neq \emptyset\}.$$

Suppose there is a point $p \in S$ such that $p \in \bigcap_{n=1}^{\infty} H_n^*$ and $p \notin M$. Now $p \notin M$ and M a closed set implies there is a domain D such that $p \in D$ and $D \cap M = \emptyset$. Since G is a development of S there is a natural number N such that every domain of G_N that contains p lies in D. Thus every domain of G_N that contains p does not contain any points of M. Since $p \in \bigcap_{n=1}^{\infty} H_n^*$ there is a domain $h \in H_N$ which contains p. Hence be definition of H_N, $h \cap M \neq \emptyset$, which is a contradiction. Thus $M = \bigcap_{n=1}^{\infty} H_n^*$ which implies M is a G_δ set.

Hence by Theorems 2.44 and 2.45 we have the following corollary.

Corollary 2.48: Every screenable Moore space is pointwise paracompact.

The converse of Corollary 2.48 is false and will be shown with the aid of the following definition and example.

Definition 2.49: A topological space S is said to be a semi-metric topological space provided there is a distance function d defined for S such that (1) $d(x, y) \geq 0$ for each $x, y \in S$, (2) $d(x, y) = d(y, x)$ for each $x, y \in S$, (3) $d(x, y) = 0$ if and only if $x = y$ and (4) the topology of S is invariant with respect to the distance function d.

Example 2.50: A regular, connected, locally connected, separable developable space which is not metrizable, but is semi-metrizable.

Let X denote the x-axis of E_2. Suppose p and q are distinct points of S. Define D on S as follows: (1) If $p, q \notin X$ then $D(p, q) = d(p, q)$ where d is the usual metric for E_2; (2) If $p \in X$ then $D(p, q) = d(p, q) + \alpha$ where α is counterclockwise radian measure between a line L_1 containing $\{p\} \cup \{q\}$ and a vertical line L_2 containing p such that $0 \leq \alpha \leq \pi/2$ (see Figure 1); (3) If $D(q, p)$ is not defined above, then let $D(q,p)=D(p,q)$; and (4) If $p \in X$ then $D(p, p) = 0$.

Clearly, D satisfies parts 1, 2, and 3 of the definition of a semi-metric. Consider the topological space S whose topology is induced by the semi-metric D. Denote

$$U_r(p) = \{q \in S \mid D(p, q) < r, \text{ and } r > 0\}.$$

(See Figure 2 for an open set on X).

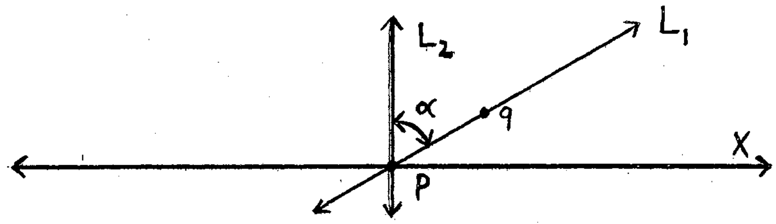

Figure 1. The Angle α

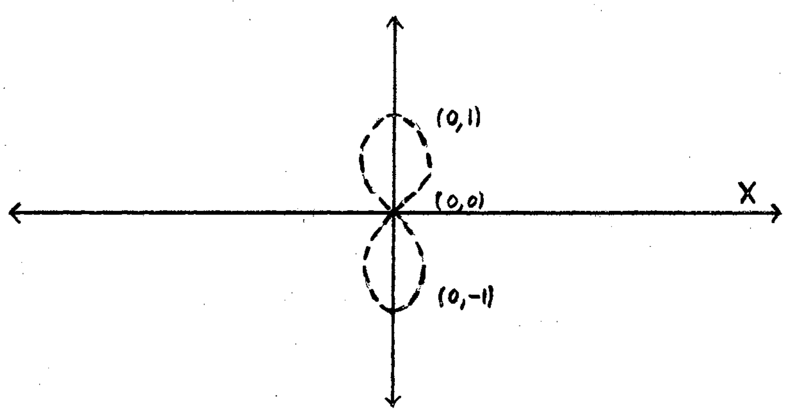

Figure 2. The set $U_1(0, 0)$

With the topology, the topological space S becomes a regular, connected, locally connected, separable semi-metric topological space. Also the definition of $U_{1/n}(p)$ implies that S is developable. Let $I = \{(x, 0) | x \text{ is an irrational number}\}$, then Theorem 2.41 implies that S is not normal. Hence S is not a metrizable topological space. Now suppose that S is screenable. If S is screenable then Theorem 2.39 implies that S is second countable. Thus S is metrizable by Urysohn's Metrization theorem which is a contradiction. Therefore

S is not a screenable topological space.

For more information on semi-metrizable spaces see McAuley [21] and Wilson [34].

Example 2.51: A pointwise paracompact Moore space with a uniform base which is not screenable, not normal, and not metrizable.

The space S consists of all points on or above the x-axis (denoted by X) with a basis G defined as follows: (1) for p above X, $\{p\} \in G$ and (2) for each $x \in X$ and each natural number n,

$$\{(t, y) : t = x + y \text{ or } t = x - y, \ 0 \leq y \leq 1/n\} \in G$$

(every "V" with vertex on X, sides of slope 1 or -1, height 1/n). Clearly, S with the topology generated by G is a topological space (by use of ordinary Euclidean distance).

Now suppose S is a screenable topological space. By definition of G, there is a covering H of S by basis elements, each of diameter less than 1/n (using Euclidean distance), and such that no point of X belongs to two elements of H. Since S is a screenable topological space there is a sequence $K = \{G_n\}$ such that G_n is a collection of mutually exclusive domains, $\bigcup_{n=1}^{\infty} G_n$ refines H, and $\bigcup_{n=1}^{\infty} G_n$ covers S. Hence there is a subsequence $K_1 = \{G_{n_i}\}$ of K such that G_{n_i} is a collection of mutually exclusive domains, $\bigcup_{i=1}^{\infty} G_{n_i}$ refines H, and $\bigcup_{i=1}^{\infty} G_{n_i}$ covers X (x-axis). Since $\bigcup_{i=1}^{\infty} G_{n_i}$ refines H, we have (1) no element of $\bigcup_{i=1}^{\infty} G_{n_i}$ contains more than one point of X and (2) each element of $\bigcup_{i=1}^{\infty} G_{n_i}$ is a basis element of diameter less than 1/n.

For each $g_x \in G_{n_i}$, where $\{x\} = g_x \cap X$, let diameter $(g_x) = 1/N_x$. Referring to Example 2.50, let us associate with g_x the domain $U_{1/N_x}(x)$. That is, there is a collection

$$H_{n_i} = \{U_{1/N_x}(x) \mid g_x \in G_{n_i}\}$$

such that the elements of H_{n_i} and G_{n_i} are in a one to one correspondence. For each g_x, diameter $(g_x) = 1/N_x$, consider the triangle denoted by Δg_x, formed in the plane by g_x and the line $y = 1/2N_x$ (see Figure 3).

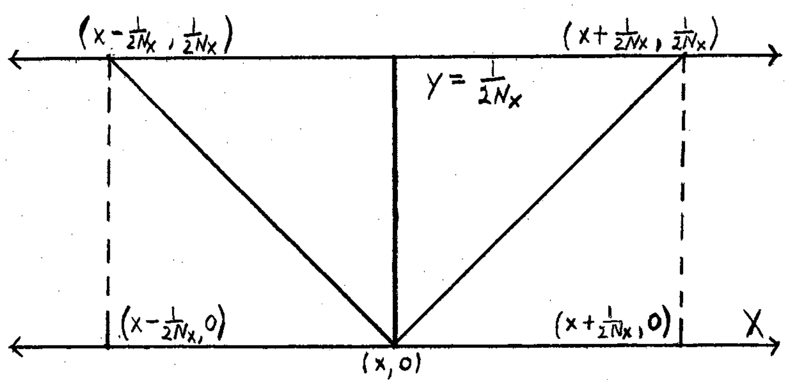

Figure 3. The Triangle g_x

Now consider $[\Delta g_x \cup \text{interior } \Delta g_x] = \blacktriangle g_x$. Let g_x and g_y be two elements of G_{n_i} and consider $\blacktriangle g_x$ and $\blacktriangle g_y$. By use of a distance argument, one sees that $\blacktriangle g_x \cap \blacktriangle g_y = \emptyset$ since elements of G_{n_i} are mutually exclusive. The definition of $U_{1/N_x}(x)$ implies that if $q \neq x$, $q \in S$, and the angle between a line containing $\{q\} \cup \{x\}$ and a vertical line containing x is greater than $\pi/4$, then $q \notin U_{1/N_x}(x)$. Thus if

$U_{1/N_x}(x)$ is constructed at x in the space S then the portion of $U_{1/N_x}(x)$ lying above the x-axis lies in $\blacktriangle g_x$. Therefore if G_{n_i} is a collection of mutually exclusive domains in S, then the associated collection H_{n_i} of Example 2.50 is a collection of mutually exclusive domains in the space of Example 2.50. Hence there is a sequence $K_2 = \{H_{n_i}\}$ in the space of Example 2.50 such that H_{n_i} is a collection of mutually exclusive domains. Since $\bigcup_{i=1}^{\infty} G_{n_i}$ covers X we have $\bigcup_{i=1}^{\infty} H_{n_i}$ covers X. Also no element of $\bigcup_{i=1}^{\infty} H_{n_i}$ contains more than one point of X since no element of $\bigcup_{i=1}^{\infty} G_{n_i}$ contains more than one point of X. Since the set of points on the x-axis is uncountable, there is a natural number n_j such that H_{n_j} contains an uncountable number of domains. Thus H_{n_j} is an uncountable collection of mutually exclusive domains. This is a contradiction, since the space of Example 2.50 is separable. Therefore, the topological space S is not screenable.

For each natural number n, let F_n be the collection of all basis elements of diameter 1/n, or 0 (using ordinary Euclidean distance). Let $B = \bigcup_{n=1}^{\infty} F_n$ and B' any infinite subcollection of B containing a point $x \in X$. The definition of G and the fact that B' is an infinite collection implies that B' is a base at x. Hence, one sees that $B = \bigcup_{n=1}^{\infty} F_n$ is a uniform base of S. Now let

$$D = \{(x, y) : t = x + y \text{ or } t = x - y,\ 0 \leq y \leq 1/n\},$$

n fixed, be any basis element containing a point $x \in X$. By definition of G there is a natural number $m > n$ such that

$$D_1 = \{(x, y) : t = x + y \text{ or } t = x - y,\ 0 \leq y \leq 1/m\}$$

is contained in D. For each $y \in S$, $y \notin D_1$, there is a domain D_y such that $y \in D_y$ and $D_y \cap D_1 = \emptyset$ (use ordinary Euclidean distance to construct such a domain). Thus D_1 is a domain such that $\bar{D}_1 \subset D$. Noting that the plane above the x-axis is a regular topological space, we have that S is a regular topological space. Hence Theorem 2.45 implies that S is a pointwise paracompact Moore space.

Now assume S is a normal topological space. The definition of G implies that the set I of all irrational numbers on the x-axis and the set K of all rational numbers on the x-axis are closed sets. Also the sets I and K are disjoint sets. Since S is normal there is a cover Q of I by basis elements such that $\bar{Q}^* \cap K = \emptyset$ and for each $x \in I$, there is a unique $q \in Q$ such that $x \in q$. Define $F : R \to R$ as follows: (1) if $x \in K$, then $f(x) = 0$; (2) if $x \in I$, then $f(x)$ = diameter of the element of Q containing x. Let $x \in I$ and suppose the diameter of the element of Q containing x is $1/n$. Since each neighborhood $N_\delta(x)$, for each $\delta > 0$, contains a rational number r, we have $|f(r) - f(x)| = 1/n$. Thus if ϵ is a positive real number such that $0 < \epsilon < 1/n$ we have $|f(r) - f(x)| > \epsilon$. Therefore f is not continuous at each irrational number.

Let $x \in K$ and ϵ any positive real number. Now $\bar{Q}^* \cap K = \emptyset$ and the definition of the base imply that there is a basis element b and a natural number N such that $x \in b$, $b \cap \bar{Q}^* = \emptyset$, $1/N < \epsilon$, and the diameter of b is $1/N$. For each

$$y \in (x - \frac{1}{2N}, x) \cup (x, x + \frac{1}{2N}), \quad y \in I,$$

there is an element q of Q such that $q \cap b = \emptyset$ and $y \in q$. Since $q \cap b = \emptyset$, a geometric argument shows that the diameter of q is less than $1/2N$ (see Figure 4). Hence for each

$$y \in (x - \frac{1}{2N}, x + \frac{1}{2N}),$$

we have

$$|f(y) - f(x)| = |f(y)| < \frac{1}{2N} < \epsilon.$$

Thus the function f is continuous at every rational number $x \in K$. This is a contradiction since there does not exist a function which is continuous at each rational point and discontinuous at each irrational point (Gelbaum and Olmsted [10]). Therefore S is not normal. Since S is not normal, S is not a metrizable topological space.

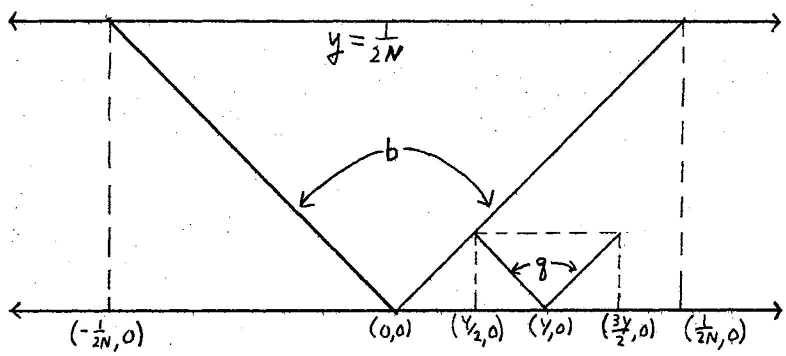

Figure 4. An Example of a Set q

CHAPTER III

METRIZATION

Introduction

The goal of Chapter III is to give a development of the theory of metrizability in Moore spaces. The development will follow a somewhat historical account of the metrizability of Moore spaces. However, since some of the later results are closely related to earlier results, order will not be strictly chronological.

The second section of this chapter will define a metric space and include a discussion of attempts to generalize a metric space. The proof of Alexandroff and Urysohn's metrization theorem will be given in detail. Also, theorems which are similar or whose proofs are based on Alexandroff and Urysohn's theorem will be indluded in this section.

Section three of this chapter includes the proof of F. B. Jones' theorem on the metrization of regular developable spaces. Also in this section is Urysohn's theorem which is stated, but not proved.

The fourth section is devoted to the proof of R. H. Bing's metrization theorem for regular topological spaces. After proving this, the result is applied to Moore spaces to obtain a series of characterizations of metrizability of Moore spaces. The last section of this chapter summarizes the results obtained within the past fifteen years. These results include generalizations of properties of Moore spaces to properties of boundaries of domains in Moore spaces. Then, using these properties of boundaries of domains in Moore spaces, a series of metrization theorems are proved.

Theorem of Alexandroff and Urysohn and Related Results

Before proving the main result of this section let us first define a metric space.

Definition 3.1: A topological space S is a metric space if and only if there is a non-negative real valued function D defined on S such that the following hold for a, b, and c in S:

Condition 1: $D(a,b) = 0$ if and only if $a = b$

Condition 2: $D(a,b) = D(b,a)$

Condition 3: $D(a,c) \leq D(a,b) + D(b,c)$

Condition 4: The function D preserves limit points, that is, p is a limit point of M a subset of S if and only if $D(p,M) = 0$.

In attempts to generalize metric spaces, condition 3 is often replaced by a condition more generalized than condition 3. The following are conditions which often replace condition 3.

Condition 5: If $D(a,b) < \epsilon$ and $D(c,b) < \epsilon$ then $D(a,c) < 2\epsilon$.

Condition 6: For every $\epsilon > 0$ there exists a number $\emptyset(\epsilon) > 0$ such that if $D(a,b) < \emptyset(\epsilon)$ and $D(c,b) < \emptyset(\epsilon)$ then $D(a,c) < \epsilon$. This condition is called uniform regularity.

Condition 7: Given a point a and a number $\epsilon > 0$ there exists a number $\emptyset(a,\epsilon) > 0$ such that if $D(a,b) < \emptyset(a,\epsilon)$ and $D(c,b) < \emptyset(a,\epsilon)$ then $D(a,c) < \epsilon$. This condition is referred to as Niemytski's local axiom of the triangle.

Condition 8: If $D(a, a_n) \to 0$ and $D(b_n, a_n) \to 0$ then $D(a, b_n) \to 0$.

Condition 9: For each point a and each positive number k, there is a positive number r such that if b is a point for which $D(a, b) \geq k$ and c is any point, then $D(a, c) + D(b, c) \geq r$.

In this section it will be shown that if a topological space S has defined upon it a non-negative real valued function D satisfying conditions 1, 2, 4, and 5 then S is a metric space. Frink [9] has shown that if a topological space S has a non-negative real valued function D defined upon it satisfying conditions 1, 2, 4, and 6, then S is a metric space. Niemytski [24] showed that a similar result is true if condition 3 is replaced with either condition 7 or 8. Also W. A. Wilson [34] showed a similar result for condition 9. Hence a topological space S is metric if there exists a non-negative real valued function D defined on S satisfying conditions 1, 2, 4, and any one of the conditions 3, 5, 6, 7, 8, or 9.

Alexandroff and Urysohn's metrizability theorem will now follow the proof (Lemma 3.2 and Lemma 3.3) that a topological space S is a metric space if there exists a non-negative real valued function D satisfying conditions 1, 2, 4, and 5.

Lemma 3.2: If $a, x_1, x_2, \ldots, x_n, b$ are points of a topological space S and d is a non-negative real valued function satisfying conditions 1, 2, and 5, then

$$d(a, b) \leq 2d(a, x_1) + 4d(x_1, x_2) + \ldots + 4d(x_{n-1}, x_n) + 2d(x_n, b) \qquad (1)$$

Proof: Suppose the contrary that the lemma is false. Then there is a natural number n and a set $\{a, x_1, \ldots, x_n, b\}$ such that (1) is false. Let N be the smallest natural number such that

$$d(a,b) > 2d(a,x_1) + 4d(x_1,x_2) + \ldots + 4d(x_{N-1},x_N) + 2d(x_N,b) \tag{2}$$

Hence by definition of N we have that (1) is satisfied for all $n < N$.

Now suppose that $N = 1$. If any pair of the points a, x_1, and b are identical then clearly we have

$$d(a,b) \geq 2d(a,x_1) + 2d(x_1,b),$$

which is a contradiction. Hence suppose that all three points a, x_1, and b are distinct and that

$$d(a,b) > 2d(a,x_1) + 2d(x_1,b).$$

There is a natural number N_1 such that

$$N_1\,[2d(a,x_1) + 2d(x_1,b)] > d(a,b).$$

Hence, we have that

$$d(a,x_1) + d(x_1,b) > d(a,b)/2N_1.$$

Let $\epsilon = d(a,b)/2N_1$. Now since

$$d(a,b) > 2d(a,x_1) + 2d(x_1,b)$$

we have that

$$\epsilon = d(a,b)/2N_1 > d(a,x_1)/N_1 + d(x_1,b)/N_1.$$

Thus we have $d(a,x_1)/N_1 < \epsilon$ and $d(x_1,b)/N_1 < \epsilon$. Since d satisfies condition 5 we have

$$d(a,b) < 2N_1\,\epsilon = 2N_1\,d(a,b)/2N_1 = d(a,b)$$

This contradiction implies $d(a,b) \leq 2d(a,x_1) + 2d(x_1,b)$. Hence we must have $N > 1$ since (2) is not satisfied when $N = 1$.

If a, b and $x_r \in S$ then $d(a,b) \leq 2d(a,x_r)$ or $d(a,b) \leq 2d(x_r,b)$. This can be shown by assuming the contrary and using the method of the previous paragraph to arrive at a contradiction. If $r = 1$ then $d(a,b) \leq 2d(a,x_r)$ does not hold because of (2). If $r = N$ then $d(a,b) \leq 2d(x_r,b)$ does not hold because of (2). Let k be the largest value of r for which $d(a,b) \leq 2d(x_r,b)$. Then $k < N$ and

$$d(a,b) \leq 2d(x_k,b) \tag{3}$$

From the definition of k and the above we have

$$d(a,b) \leq 2d(a,x_{k+1}) \,. \tag{4}$$

Since (1) holds for all $n < N$ we have

$$d(x_k,b) \leq 2d(x_k,x_{k+1}) + 4d(x_{k+1},x_{k+2}) + \ldots + 2d(x_N,b) \tag{5}$$

and

$$d(a,x_{k+1}) \leq 2d(a,x_1) + 4d(x_1,x_2) + \ldots + 4d(x_{k-1},x_k) + 2d(x_k,x_{k+1}). \tag{6}$$

Adding (5) and (6) we obtain

$$d(a,x_{k+1})+d(x_k,b) \leq 2d(a,x_1)+\ldots+4d(x_{k-1},x_k)+4d(x_k,x_{k+1})+\ldots+2d(x_N,b). \tag{7}$$

By (3) and (4) we have a contradiction to (2). This contradiction implies that (1) holds for the points $a, x_1, \ldots, x_n, b$, for any natural number n.

<u>Lemma 3.3</u>: Let S be a topological space such that there is a non-negative real valued function d satisfying conditions 1, 2, 4, and 5.

Then S is a metric space.

Proof: Let $a, b \in S$ and define

$$D(a,b) = \inf \{d(a,x_1) + d(x_1,x_2) + \ldots + d(x_n,b)\}$$

where $\{x_1, x_2, \ldots, x_n\}$ is a finite subset of S; the points $x_1, x_2, \ldots, x_n$ are not necessarily distinct from each other or from a and b. One quickly sees that D satisfies conditions 1, 2, and 3 of Definition 3. 1.

Now Lemma 3. 2 implies $d(a,b)/4 \leq D(a,b)$. The definition of D implies $D(a,b) \leq d(a,b)$. Hence the distance function D leads to the same definition of limit point as the old distance function d and is equivalent to it. Thus the space S is a metric space by Definition 3. 1.

<u>Theorem 3. 4</u>: A topological space S is metrizable provided there exists a sequence $G = \{G_i\}$ such that

(a) for each i, G_i is a collection of domains covering S,

(b) if D is a domain and $p \in D$, there is a natural number n such that every element of G_n containing p is contained in D, and

(c) each pair of intersecting elements of G_{i+1} is a subset of some element of G_i.

Proof: Let $a, b \in S$ and define $d(a,b)$ as follows: (1) if no element of G_n, for each n, contains a and b then $d(a,b) = 1$, (2) if n is the largest natural number such that a and b are both contained in an element of G_n then $d(a,b) = 2^{-n}$, and (3) $d(a,b) = 0$ if $a = b$. Note that d is a non-negative real valued function.

If $a \neq b$ there is a domain D such that $a \in D$ and $b \notin D$. Thus there is a natural number n such that $a \in D$ and $b \notin D$. Thus there is a natural number n such that every element of G_n that contains a lies in

D. Hence no element of G_n contains a and b. Therefore $d(a, b) > 2^{-n} > 0$. Hence $a = b$ if and only if $d(a, b) = 0$.

If a and b are such that $d(a, b) = 1$ then certainly $d(b, a) = 1$. Now if $d(a, b) = 2^{-n}$ then one sees that $d(b, a) = 2^{-n}$. Hence we have $d(a, b) = d(b, a)$.

Now suppose that $a, b, c \in S$ such that $d(a, b) < \epsilon$ and $d(c, b) < \epsilon$ where $\epsilon < 1$; for otherwise the result follows immediately. The only case for which it is not easily seen is when a, b, and c are distinct points. Since $d(a, b) < \epsilon$, there is a largest natural number N_1 such that a and b are both contained in an element of G_{N_1}. Since $d(c, b) < \epsilon$, there is a largest natural number N_2 such that c and b are both contained in an element of G_{N_2}. Let $N = \min \{N_1, N_2\}$ then $N-1$ is a natural number such that a and c are both contained in an element of G_{N-1}. Hence the largest natural number k such that a and c are both in an element of G_k is certainly greater than or equal to $N-1$. Thus $d(a, c) = 2^{-k} \leq 2^{-N+1}$. Since

$$2^{-(N-1)} = 2(2^{-1}2^{-N+1}) = 2\,2^{-N} < 2\,\epsilon$$

we have that $d(a, c) < 2\epsilon$.

Let M be a subset of S, x a limit point of M, and ϵ a positive real number. There is a natural number N such that $2^{-N} < \epsilon$. Since G_N covers S there is a domain $g \in G_N$ such that $x \in g$. Since x is a limit point there is an $m \in M$, $m \neq x$, such that $m \in g$. Now if K is the largest natural number such that x and m are both contained in an element of G_K then $K \geq N$ since $\{x, m\}$ is a subset of $g \in G_N$. Hence $d(x, m) = 2^{-K} \leq 2^{-N} < \epsilon$. Thus $d(x, M)$ is not bounded from 0.

Let $x \in S$ such that x is not a limit point of M. Then there is a domain D such that $x \in D$ and $D \cap (\bar{M} - \{x\}) = \emptyset$. There is a natural number N such that every element of G_N that contains x lies in D. Hence no element of G_N contains x and a point of M. Therefore $d(x, m) \geq 2^{-N} > 0$ for every $m \in M$. Thus we have that x is a limit point of M if and only if d(x, M) is not bounded from 0. That is, d preserves limit points. Thus d satisfies conditions 1, 2, 4 and 5. Hence by Lemma 3.3 we have that S is a metric space.

The method of proof of the above theorem is due to Frink [9]. Since a Moore space satisfies the first two parts of the hypothesis of Theorem 3.4 we have the following corollary.

<u>Corollary 3.5</u>: A Moore space is metrizable if it satisfies the third condition of Theorem 3.4.

That not every Moore space satisfies the third condition of Theorem 3.4 is seen by Example 2.51.

Following the proof of Alexandroff and Urysohn's theorem in 1923, a result due to Chittenden [6] was proved in 1927. Before stating this result let us consider the following condition.

<u>Condition 10</u>: There is a positive valued f of a positive variable such that $\lim_{\epsilon \to 0} f(\epsilon) = 0$, and such that for any three points p, q, and r of the space if $d(p, r) \leq \epsilon$, and $d(q, r) \leq \epsilon$, then $d(p, q) \leq f(\epsilon)$ where d is a non negative real valued function.

Chittenden [5] was able to prove that if a topological space S has a non-negative real valued function d defined upon it satisfying conditions 1, 2, 4 and 10 then S is a metric space. Since the proof of this theorem is long, it will be omitted from this paper. Using the above mentioned result, Chittenden then was able to prove the following.

<u>Theorem 3.6:</u> A topological space S is metrizable provided there exists a sequence $G = \{G_i\}$ such that

(a) for each i, G_i is a collection of domains covering S,

(b) for each i, G_{i+1} is a subcollection of G_i,

(c) if D is a domain and $p \in D$ there is a natural number n such that every element of G_n containing p is contained in D, and

(d) for any positive integer m there exists an integer n such that for any point p there is a $g \in G_m$ such that for every $h \in G_n$ that contains p we have $h \subset g$.

Proof: Let $G_0 = \{S\}$ and then we have that $m = 0, 1, 2, \ldots$. Now from condition (d) there exists for each integer n an integer $m = g(n)$, the greatest value of m for which n is the integer determined by condition (d). The function g(n) is unbounded. Suppose the contrary. Then there exists an integer m' such that $g(n) < m'$ for all n. But there is an integer n' determined by m', contrary to the definition of g(n) as the greatest such integer.

Now define $d(p,q) = 1/2^m$ if m is the largest integer for which $p, q \in g$ where $g \in G_m$. Also define $d(p,p) = 0$. By the definition of d it follows that $d(p,q) = d(q,p)$. Now suppose $p \neq q$ then there is a domain D such that $p \in D$ and $q \notin D$. Therefore there is an integer n(p, D) such that if $p \in h$, $h \in G_{n(p,D)}$ then $h \subset D$. Hence

$$d(p,q) > 1/2^{n(p,D)} > 0.$$

Thus $d(p,q) = 0$ if and only if $p = q$.

Let n be any integer and $p, q, r \in S$ such that $d(p, r) < 1/2^n$ and $d(q, r) < 1/2^n$. If $m = g(n)$ then m is the greatest value of m for which n is the integer determined by condition (d). This implies that p and q are elements of some element of G_m. Therefore $d(p, q) < 1/2^m$. If $\epsilon \leq 1/2^n$ then set $f(\epsilon) = 1/2^m$. Since $g(n) = m$ is unbounded we have $f(\epsilon) \to 0$ as $\epsilon \to 0$ and if $d(p, r) \leq \epsilon$, $d(q, r) \leq \epsilon$ then $d(p, q) \leq f(\epsilon)$. Hence d satisfies condition 10.

Let p be a limit point of M a subset of S and $1/2^n < \epsilon$. Then there exists a $g \in G_n$, $m \neq p \in g$ such that $m, p \in g$. Therefore $d(p, q) = 1/2^k \leq 1/2^n < \epsilon$. Hence $d(p, M) = 0$. Now suppose p is not a limit point of M. Then there is a domain D such that $p \in D$ and $D \cap M = \emptyset$. Thus there is an integer $n(p, D)$ such that every element of $G_{n(p, D)}$ that contains p lies in D. Therefore no element of $G_{n(p, D)}$ contains a point of M and p. Therefore $d(p, M) \geq 1/2^{n(p, D)} > 0$. Therefore p is a limit point of M if and only if $d(p, M) = 0$. Hence we have that S is a metric space.

As one has seen, the above proof is a direct method of showing Theorem 3.6. An alternate method would be to show the above theorem satisfies the hypothesis of Theorem 3.4. This would be done by defining a new sequence $G' = \{G_i'\}$ as follows: (1) let $G_1' = G_1$, (2) let $G_2' = G_{n_1}$ where n_1 is the integer of condition (d) for the integer $m = 1$, and (3) $G_1' = G_{n_i}$ where n_i is the integer of condition (d) for the integer $m = n_{i-1}$. Clearly conditions (a) and (b) of Theorem 3.4 are satisfied by $G' = \{G_i'\}$. Now let g and h be elements of G_{i+1} such that $g \cap h \neq \emptyset$. Then by definition of G_{i+1}' we have that $g, h \in G_{n_{i+1}}$. Since $g \cap h \neq \emptyset$ there is a point $p \in g \cap h$. Thus by condition (d) and the definition of G' we have that there is a $g_1 \in G_{n_i}$ such that $g \cup h \subset g_1$.

But $g_1 \in G_i'$, therefore we have $g \cup h$ is a subset of an element of G_i'. Thus the hypothesis of Theorem 3.4 is satisfied and the space is metrizable.

As another application of Theorem 3.4 let us consider a result due to R. L. Moore [22]. This result proved in 1932 will be shown to satisfy the hypothesis of Alexandroff and Urysohn's theorem. However, a direct argument similar to either Theorem 3.4 or Theorem 3.6 could also be used to prove this theorem.

Theorem 3.7: Let S be a topological space. Let $G = \{G_i\}$ be a sequence such that

(a) for each n, G_n is an open covering of S,

(b) for each n, G_{n+1} is a subcollection of G_n, and

(c) if D is a domain and $x, y \in D$ then there exists a natural number n such that if $h, k \in G_n$ such that $h \cap k \neq \emptyset$ and $x \in h$ then $h \cup k \subset (D - \{y\}) \cup \{x\}$.

Then the topological space S is metrizable.

Proof: Let $G_1' = G_1$. Define G_2' to be the collection of domains $g \in G_m$, $m > 1$, such that if $h \in G_m$ and $g \cap h \neq \emptyset$ then $g \cup h$ is a subset of some domain of G_1'. Let $p \in S$ then since G_1' covers S there is an element g of G_1' such that g contains p. If $g = \{p\}$ then since for each n there is an element h_n of G_n that contains p, there exists a natural number N such that if $h, k \in G_N$ such that $h \cap k \neq \emptyset$ and $p \in h$ then $h \cup k \subset (g - \{p\}) \cup \{p\}$. Since $p \in h_N$ and $h_N \cup h_N \subset (g - p) \cup \{p\}$ we have $h_N = \{p\}$. Therefore $h_N \subset g$ and $h_N \in G_2'$. Thus G_2' covers S. Now suppose $g \neq \{p\}$ and let $q \neq p \in g$. Since, for each n, G_n is an open covering of S there is an element $h_n \in G_n$ such that h_n contains p.

Therefore by hypothesis there is an integer m, $m > 1$, such that if $h_m \cap k \neq 0$, and $k \in G_m$ then $h_m \cup k \subset (g - \{q\}) \cup \{p\}$. Thus $h_m \in G_2'$ and we have that G_2' covers S. Hence in either case the collection G_2' covers S.

Let $h, k \in G_2'$ such that $h \cap k \neq \emptyset$. Now $h \in G_2'$ implies $h \in G_m$, $m > 1$, such that if $g \in G_m$ and $h \cap g \neq \emptyset$ then $h \cup g$ is a subset of some domain of G_1'. Also $k \in G_2'$ implies $k \in G_r$, $r > 1$, such that if $g \in G_r$ and $k \cap g \neq \emptyset$ then $k \cup g$ is a subset of some domain of G_1'. Without loss of generality assume that $m < r$. Then since G_r is a subcollection of G_m we have that $k \in G_m$. Since $h \cap k \neq \emptyset$ we have that $h \cup k$ is contained in an element of G_1'.

Now define G_3' to be the collection of domains $g \in G_m$, $m > 2$, such that if $h \in G_m$ and $g \cap h \neq \emptyset$ then $g \cup h$ is a subset of some domain of G_2'. By using an argument similar to the one above we have that G_3' covers S and if $h, k \in G_3'$ such that $h \cap k \neq \emptyset$ then $h \cup k$ is contained in an element of G_2'. Continuing our process we have that there exists a sequence $G' = \{G_n'\}$ such that (1) $G_1' = G_1$, (2) for each i, G_{i+1}' is the collection of all domains g such that for some $m > i$, $g \in G_m$ and if $h \in G_m$ such that $h \cap g \neq \emptyset$ then $h \cup g$ is contained in an element of G_i', (3) for each i, G_i' covers S, and (4) if $g, h \in G_{i+1}'$ such that $h \cap g \neq \emptyset$ then $g \cup h$ is contained in an element of G_i'.

Let D be any domain and $p \in D$. Then by hypothesis there is an n such that if $p \in g \in G_n$ then $g \subset (D - \{x\}) \cup \{x\}$. Hence every domain of G_n containing p lies in D. Consider the collection G_n'. Let $h \in G_n'$ such that h contains p. Now $h \in G_n'$ implies $h \in G_m$, $m > n$. Since G_m is a subcollection of G_n we have that $g \in G_n$. Hence $g \subset D$ and therefore G' is a development for S. Therefore by Theorem 3.4 the topological space S is metrizable.

Interrupting our historical development for the moment let us prove the following corollaries to Theorem 3.7. Jones [19], in 1966, proved the following results and referred to them as the weak form and the strong form, respectively, of Theorem 3.7.

Corollary 3.8: Let S be a topological space and $G = \{G_i\}$ a sequence such that

(a) for each i, G_i is an open covering of S,

(b) for each i, G_{i+1} is a subcollection of G_i, and

(c) if H is a closed set and $p \in S - H$ then there is an integer n such that star (p) $\cap$ star (H) = $\emptyset$, where the star is taken with respect to G_n.

Then the space S is metrizable.

Proof: Let D be any domain and $x, y \in D$. The set $\{y\}$ is a closed set, hence $S - \{y\}$ is a domain that contains x. Thus there is an integer N_1 such that star (x) $\cap$ star (y) = $\emptyset$ with respect to the covering G_{N_1}. Since D is a domain we have that S - D is a closed set. Also we have $x \in D = S - (S - D)$. Therefore there exists an integer N_2 such that star (x) $\cap$ star (S - D) = $\emptyset$, with respect to the covering G_{N_2}. Letting $N = \max \{N_1, N_2\}$ we have star (x) $\cap$ star (y) = $\emptyset$ and star (x) $\cap$ star (S - D) = $\emptyset$ with respect to the covering G_N. Now if $h, k \in G_N$ such that $x \in h$ and $h \cap k \neq \emptyset$ then $h \cup k \subset (D - \{y\}) \cup \{x\}$ since star (x) $\cap$ star (y) = $\emptyset$ and star (x) $\cap$ star (S - D) = $\emptyset$. Hence by Theorem 3.7 the space S is metrizable.

Corollary 3.9: Let S be a regular topological space and $G = \{G_i\}$ a sequence such that

(a) for each i, G_i is an open covering of S

(b) for each i, G_{i+1} is a subcollection of G_i, and

(c) if H, K are closed disjoint subsets of S, one of which is compact, then there exists an integer n such that star (H) $\cap$ K = $\emptyset$ and H $\cap$ star (K) = $\emptyset$.

Then the space S is metrizable.

Proof: Suppose S is not metrizable then for each n there is a closed set H and a point p $\in$ S - H such that star (p) $\cap$ star (H) $\neq \emptyset$ by the previous corollary. For each n let p_n denote a point in the intersection of star (p) $\cap$ star (H). Let K = $\{p_i\}$ and consider the set $\overline{K}$. Since $\{p\}$ is a closed and compact set the sequence $\{p_i\}$ converges to p. Hence H and $\overline{K}$ - H are disjoint closed sets and $\overline{K}$ - H is compact. Thus there exists an integer n such that star (H) $\cap$ $\overline{K}$ - H = $\emptyset$ and H $\cap$ star ($\overline{K}$ - H) = $\emptyset$. But for some N we have $p_i \in \overline{K}$ - H for i $\geq$ N. Now without loss of generality assume N $\geq$ n and consider p_N. Now $p_N \in \overline{K}$ - H and $p_N \in$ star (p) $\cap$ star (H) implies $p_N \in$ ($\overline{K}$ - H) $\cap$ star (H). This contradiction implies that S is metrizable.

Continuing our development we have in 1940 a result due to C. W. Vickery [33]. As in previous results one should notice the application of Alexandroff and Urysohn's theorem in the proof of this theorem.

Theorem 3.10: Let S be a topological space such that

(a) there exists a sequence G = $\{G_i\}$ such that for each n, G_n is an open covering of S,

(b) if D is a domain and a and b are points of D then there exists a natural number n such that if g $\in$ G_n and a $\in$ g then $\overline{g}$ is a subset of $(D - \{b\}) \cup \{a\}$, and

(c) if G is an open covering of S then there exists an open covering H of S such that if $h, k \in H$ and $h \cap k \neq \emptyset$ then $h \cup k$ is a subset of an element of G.

Then the topological space S is metrizable.

Proof: Let us define a new sequence $G' = \{G_i'\}$ as follows: (1) $G_1' = G_1$, (2) $g \in G_2'$ if and only if $g = g_1 \cap g_2$ where $g_1 \in G_1'$ and $g_2 \in G_2$, and (3) $g \in G_i'$ if and only if $g = g_i \cap g_{i-1}$ where $g_{i-1} \in G_{i-1}'$ and $g_i \in G_i$. Now define $G'' = \{G_i''\}$ by letting

$$G_i'' = \bigcup_{j=i}^{\infty} G_j'.$$

By construction G_n'' is an open covering of S, for each n. By definition of G'' we have for each n that G_{n+1} is a subcollection of G_n''. Now let D be a domain and $a, b \in D$. Then by condition (b) there is a natural number n such that if $g \in G_n$ and $a \in g$ then $\overline{g} \subset (D - \{b\}) \cup \{a\}$. Now consider G_n'' and let $g \in G_n''$ such that $a \in g$. By construction of G' we have that g is contained in an element of G_n. Hence $\overline{g}$ is a subset of $(D - \{b\}) \cup \{a\}$. Hence by definition S is a Moore space.

Since S is a Moore space, to simplify notation let $G = \{G_i\}$ be the sequence of the definition of a Moore space. Now define $G_1' = G_1$. Let H_2 be as in condition (b) for the open covering G_2, and define $G_2' = \{g \mid g \in G_2,\ g \subset h \text{ for some } h \in H_2\}$. Let g_1 and g_2 be elements of G_2' and suppose that $g_1 \cap g_2 \neq \emptyset$. Then $g_1 \subset h_1$ and $g_2 \subset h_2$ for some $h_1, h_2 \in H_2$. Since $g_1 \cap g_2 \neq \emptyset$ we have that $h_1 \cap h_2 \neq \emptyset$, thus $h_1 \cup h_2$ is a subset of an element of G_2. Therefore $g_1 \cup g_2$ is a subset of an element of G_2. But G_2 is a subcollection of G_1 thus

$g_1 \cup g_2$ is a subset of an element of G_1'. Also note that G_2' is an open covering of S and that G_2' is a subcollection of G_1'. Now let G_3'' denote the collection $\{g \mid g \in G_3,\ g \subset g_2 \text{ for some } g_2 \in G_2'\}$. Then G_3'' is an open covering of S. Now let H_3 be as in part (b) for the open covering G_3''. Let $G_3' = \{g \mid g \in G_3'',\ g \subset h \text{ for some } h \in H_3\}$. Let g_1 and g_2 be elements of G_3' such that $g_1 \cap g_2 \neq \emptyset$. Since $g_1, g_2 \in G_3'$, we have $g_1 \subset h_1$, $g_2 \subset h_2$ for some $h_1, h_2 \in H_3$. Hence $h_1 \cap h_2 \neq \emptyset$ since $g_1 \cap g_2 \neq \emptyset$. Therefore $h_1 \cup h_2 \subset g$ where $g \in G_3''$. But $h_1 \cup h_2 \subset g$ implies by definition of G_3'' that $h_1 \cup h_2$ is contained in an element of G_2'. Hence $g_1 \cup g_2$ is contained in an element of G_2'. Also since S is a Moore space we have G_3' is an open covering of S and G_3' is a subcollection of G_2' by the definition of G_3''. Continuing our process we obtain a sequence $G' = \{G_i'\}$ such that G_i' is an open covering of S, G_{i+1}' is a subcollection of G_i' and if $g_1, g_2 \in G_i'$ such that $g_1 \cap g_2 \neq \emptyset$ then $g_1 \cup g_2$ is contained in an element of G_{i-1}'. Thus by Theorem 3.4 the topological space S is metrizable.

The next generalization of Theorem 3.4 was proved in 1947 by Bing [3] and is stated and proved below following a lemma which is similar to Lemma 3.2 and is not proved.

Definition 3.11: A collection G of point sets is coherent provided that each proper subcollection G' of G contains an element which intersects an element of G - G'.

Note that a collection G = $\{g\}$ is a coherent collection.

Lemma 3.12: Suppose that r is a positive integer and H = $\{H_i\}$ is a sequence such that H_i is a collection of sets and each pair of points that can be covered by a coherent collection of r or fewer elements of H_{i+1} can be covered by an element of H_i. If p and q are two points whose sum cannot be covered by any element of H_s but which can be covered by a coherent collection of sets $h_1, h_2, \ldots, h_n$ belonging to $H_{\alpha(1)}, H_{\alpha(2)}, \ldots, H_{\alpha(n)}$, respectively, then

$$2(1/r^{\alpha(1)} + 1/r^{\alpha(2)} + \ldots + 1/r^{\alpha(n)}) > 1/r^s \qquad (8)$$

Theorem 3.13: A topological space S is metrizable provided there exists a sequence $H = \{H_i\}$ such that

(a) for each natural number i, H_i is a collection of sets covering the space S,

(b) a point p is a limit point of the set M if and only if for each natural number n, some element of H_n contains p and intersects $M - \{p\}$, and

(c) each pair of points that is covered by the sum of a pair of intersecting elements of H_{i+1} can be covered by an element of H_i.

Proof: Let p and q be elements of S. Then define d(p, q) to be the minimum of 1 and the greatest lower bound of the collection of sums of the type

$$1/2^{\alpha(1)} + 1/2^{\alpha(2)} + \ldots + 1/2^{\alpha(n)}$$

where $h_1, h_2, \ldots, h_n$ are the elements of a coherent collection of sets covering $p \cup q$ and h_i is an element of $H_{\alpha(i)}$.

Let $p, q, r \in S$. If U is a coherent collection of sets covering $\{p, r\}$. Then we have that $U \cup V$ is a coherent collection of sets covering $\{p, q\}$. Suppose $d(p, q) = 1$ and $d(p, r) + d(r, q) < 1$. Since $d(p, r) + d(r, q) < 1$ we have $d(p, r) < 1$ and $d(r, q) < 1$. Thus there exists coherent collections U and V such that U is a coherent collection of sets covering $\{p, r\}$ and V is a coherent collection of sets covering $\{r, q\}$. Then we have that $U \cup V$ is a coherent collection of sets covering $\{p, q\}$ which implies that $d(p,q) \leq d(p,r) + d(r,q)$. This contradiction implies that $d(p,r) + d(r,q) \geq 1$ and the triangle inequality is true.

If M is a set and x is not a limit point of M then there is an integer s such that no element of H_s contains x and a point of M. Then Lemma 3.12 implies that if $m \in M$ then $d(x,m) > 1/2^{s+1}$. Thus $d(x, M)$ is bounded from 0. Now suppose x is a limit point of M and ϵ is a positive real number. There is a natural number n such that $1/2^n < \epsilon$. There is an element h of H_n containing p and a point of M. Then $\{h\}$ is a coherent collection containing $\{x, m\}$. Thus $d(x, M)$ is less than $1/2^n$. Therefore d preserves limit points.

One quickly sees that $d(p,q) = d(q,p)$ for every $p, q \in S$. Also we have that $d(p,q) = 0$ if and only if $p = q$. Hence S is a metric space.

Theorem of F. B. Jones

Following Alexandroff and Urysohn metrization theorem in 1923, Paul Urysohn in 1925 proved the following well known metrization theorem.

Theorem 3.14: A regular second countable topological space is metrizable.

The proof of this theorem is omitted, but the interested reader will find the proof in Hall and Spencer [12], p. 122. The interested reader should also note the similarity between this proof and the proof of Bing's metrization theorem of the next section.

The above theorem will be applied in this section to give a result due to Jones [18] in 1937. This was one of the first successful attempts to give necessary conditions for a Moore space to be metrizable. However, in providing his theorem Jones questioned whether the hypothesis of the theorem was too strong. The resulting question of whether every normal Moore space is metrizable is still unanswered at this time. The proof of Jones' theorem will constitute the rest of this section.

Lemma 3.15: If S is a separable, completely normal topological space then every subset of power c contains a limit point of itself.

Proof: Suppose, on the contrary, that M is a subset of power c and M does not contain a limit point of M. Since S is separable, let Z denote a countable subset of S such that $\overline{Z} = S$. Let J be a proper subset of M, then by hypothesis J and (M - J) are two mutually separated point sets. Since S is completely normal there is a domain D_J such that $J \subset D_J$ and $\overline{D}_J \cap (M - J) = \emptyset$.

Now let L and K be two proper subsets of M such that $L \neq K$. If $K \not\subset L$ then let $y \in (K - L)$. Define D_K and D_L as in paragraph one. If $y \in Z$ then $Z \cap D_K \neq Z \cap D_L$ since $y \in D_K$ and $y \in (M - L)$. If y is a limit point of Z, $y \in Z$, then every domain containing y contains a point $z \in Z$. Now $y \in (M - L)$ implies $y \notin \overline{D}_L$, hence there is a domain D such that $y \in D$ and $D \cap \overline{D}_L = \emptyset$ since S is closed. Since the domain $D \cap D_K$ contains y a limit point of Z there is a $z \in Z$ such that $z \in D \cap D_K$. Since $D \cap \overline{D}_L = \emptyset$, we have that $z \notin D_K$ and thus $Z \cap D_K \neq Z \cap D_L$. Therefore, if K and L are two different proper subsets of M then

$Z \cap D_K$ and $Z \cap D_L$ are two different subsets of Z. If $K \not\supset L$ then a similar argument applies to give the same conclusion.

Thus, there are at least as many subsets of Z as there are proper subsets of M. However, since M is of power c and Z is only countable, there are more than c proper subsets of M but at most c subsets of Z. This is a contradiction; hence, the set M contains a limit point of M.

The above argument, with slight changes, establishes the following lemma.

Lemma 3.16: If S is a separable, completely normal topological space and $2^{\aleph_1} > 2^{\aleph_0}$, then every uncountable subset of S contains a limit point of itself.

Definition 3.17: A topological space S is said to have the Lindelof property provided that if G is a collection of domains of S covering a point set K, then G contains a countable subcollection G' which covers K.

Lemma 3.18: Let S be a developable topological space. If every uncountable subset M of S has a limit point, then S has the Lindelof property.

Proof: Let $G = \{G_n\}$ be a development of S. Let H be a collection of domains covering a subset M of S. If M is countable, then for each $p \in M$, select one and only one $h_p \in H$ such that $p \in h_p$. The collection $H' = \{h_p | p \in M\}$ is a countable subcollection of H covering M. If M is uncountable, then let $M = \{p_1, p_2, \ldots, p_\alpha, \ldots\}$ denote a well-ordering of M. For each $p_\beta \in M$ there is an $h_{p_\beta} \in H$ such that $p_\beta \in h_{p_\beta}$

since H covers M. Since $G = \{G_n\}$ is a development of S there is a natural number N such that every domain of G_N containing p_β lies in h_{p_β}. Let k denote the smallest natural number such that there is a $p \in M$ and $h \in H$ such that every domain of G_k that contains p lies in h. Note that since G_{n+1} is a refinement of G_n that for all $n \geq k$ there is a $p \in M$ and $h \in H$ such that every domain of G_n that contains p lies in h. Define $G' = \{G_n'\}$ as follows: $G_n' = G_{n+k}$. Hence $G' = \{G_n'\}$ is a development of S.

For each natural number n, let H_n denote a subcollection of H obtained by the following method: let p_{α_1} denote the first element of M such that some element h_1 of H contains every region of G_n' that contains p_{α_1}. Let p_{α_2} denote the first element (if any) of M, not contained in h_1, such that some element h_2 of H contains every region of G_n' that contains p_{α_2}. In general, if $\beta < \mathfrak{z}$, and p_{α_β} and h_β are chosen, then let $p_{\alpha_\mathfrak{z}}$ denote the first point (if any) of M not contained in $\bigcup_{\beta < \mathfrak{z}} h_\beta$ such that some element $h_\mathfrak{z}$ of H contains every region of G_n' that contains $p_{\alpha_\mathfrak{z}}$. Then define $H_n = \{h_1, h_2, \ldots, h_\mathfrak{z}, \ldots\}$.

From this construction, the set $P = \{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_\mathfrak{z}}, \ldots\}$ has no limit point since no region of G_n' contains more than one element of P. For consider the possibility that some region g of G_n' contains p_{α_β} and $p_{\alpha_\mathfrak{z}}$ of P. Assume $\beta < \mathfrak{z}$, then by construction $p_{\alpha_\mathfrak{z}} \notin h_{p_{\alpha_\beta}}$. Now $g \in G_n'$, and $p_{\alpha_\beta}, p_{\alpha_\mathfrak{z}} \in g$ implies that $g \subset h_{p_{\alpha_\beta}}$. Hence $p_{\alpha_\mathfrak{z}} \in h_{p_{\alpha_\beta}}$ which is a contradiction. Therefore P is a countable collection of points since P has no limit point. Thus $H_n = \{h_1, h_2, \ldots\}$ is a countable subcollection of H.

Letting $H' = \bigcup_{h=1}^{\infty} H_n$, then H' is a countable subcollection of H. Let $p \in M$ and suppose $p \notin h$ for every $h \in H'$, and let p be the first point of M with this property. Since H covers M there is an $h \in H$ such that $p \in h$. Now $G' = \{G_n'\}$ is a development of S, hence there is an integer N such that every domain of G_N' containing p lies in h. Since $p \notin h$ for every $h \in H'$ implies that

$$p \notin \bigcup_{h_i \in H_N} h_i ,$$

we have by construction $p \in h \in H_N$. This is a contradiction, thus the collection H' covers M. Hence by definition the space S has the Lindelof property.

Lemma 3.19: Let S be a developable topological space. If every uncountable subset of S has a limit point, then S is second countable.

Proof: Let $G = \{G_n\}$ be a development of S. For each n, Lemma 3.18 implies that the collection G_n contains a countable subcollection G_n' which covers S. Since $G = \{G_n\}$ is a basis for S, we have that $G' = \{G_n'\}$ is a countable basis for S. Thus by definition S is second countable.

Lemma 3.20: Let S be a developable topological space. If S is normal, then S is completely normal.

Proof: Let $G = \{G_n\}$ be a development for S, and H and K be two nonempty subsets of S such that $\overline{H} \cap K = \emptyset$ and $H \cap \overline{K} = \emptyset$. Define H_n and K_n, for each n, as follows:

$$H_n = \{p \mid p \in \overline{H} \text{ and if } g \in G_n,\ p \in g \text{ then } g \cap \overline{K} = \emptyset\}$$

and

$$K_n = \{p \mid p \in \overline{K} \text{ and if } g \in G_n,\ p \in g \text{ then } g \cap \overline{H} = \emptyset\}$$

and without loss of generality assume H_n and K_n are point sets. Let p be a limit point of H_n and $g \in G_n$ such that $p \in g$. Thus there is a $q \in H_n$, $q \neq p$, such that $q \in g$. Now $q \in H_n$ and $q \in g$ implies $g \cap \overline{K} = \emptyset$. Therefore, by definition of H_n, we have $p \in H_n$. Hence H_n is a closed set for every n. Similarly, K_n is a closed set for every n.

Since S is normal and H_1 and K are closed sets such that $H_1 \cap \overline{K} = \emptyset$, there is a domain D_{H_1} such that $H_1 \subset D_{H_1}$ and $\overline{D}_{H_1} \cap \overline{K} = \emptyset$. Also K_1 and $(\overline{H \cup D_{H_1}})$ are closed sets such that $K_1 \cap (\overline{H \cup D_{H_1}}) = \emptyset$ since $K_1 \subset K$, $K \cap \overline{H} = \emptyset$, and $K \cap \overline{D}_{H_1} = \emptyset$; thus there is a domain D_{K_1} such that $K_1 \subset D_{K_1}$ and $\overline{D}_{K_1} \cap (\overline{H \cup D_{H_1}}) = \emptyset$. Now let D_{H_2} be a domain such that $H_2 \subset D_{H_2}$ and $\overline{D}_{H_2} \cap (\overline{K \cup D_{K_1}})$. The above process may be continued by finite induction. Now define

$$D_H = \bigcup_{n=1}^{\infty} D_{H_n} \quad \text{and} \quad D_K = \bigcup_{n=1}^{\infty} D_{K_n},$$

and note that D_H and D_K are domains. Since

$$H \subset \bigcup_{n=1}^{\infty} H_n \subset \bigcup_{n=1}^{\infty} D_{H_n}$$

and

$$K \subset \bigcup_{n=1}^{\infty} D_n \subset \bigcup_{n=1}^{\infty} D_{K_n},$$

we have $H \subset D_H$ and $K \subset D_K$. Suppose $D_H \cap D_K \neq \emptyset$; then there is a point p such that $p \in D_H$ and $p \cap D_K$. If $p \in D_H$ and $p \in D_K$ then $p \in D_{H_n}$ $p \in D_{K_m}$ for some n and m. Without loss of generality, assume $m > n$. If $p \in D_{H_n}$ then by construction the domain D_{K_m} is a domain such that $K_m \subset D_{K_m}$ and

$$\overline{D}_{K_m} \cap \overline{(H \cup D_{H_1} \cup \ldots \cup D_{H_n} \cup \ldots D_{H_m})} = \emptyset.$$

Hence $p \notin D_{K_m}$, which is a contradiction. Thus $D_H \cap D_K = \emptyset$, and S is completely normal.

Theorem 3.21: Let S be a separable normal developable topological space. If $2^{\aleph_1} > 2^{\aleph_0}$, then S is second countable and metrizable.

Proof: Lemma 3.20 implies that S is completely normal. Lemma 3.16 implies that every uncountable subset of S contains a limit point of itself. Thus by Lemma 3.19 we have that S is second countable. Therefore by Urysohn's theorem, we have that S is metrizable.

As we have seen in Example 2.40 and the discussion following Example 2.40 there does exist a separable Moore space that is not metrizable. For efforts to remove the hypothesis that $2^{\aleph_1} > 2^{\aleph_0}$ the reader can consult Heath [14] and [15], McAuley [21], and Traylor [29] and [30]. However, to this date all efforts have failed.

Bing's Metrization Theorem and Related Results

This section will present the results of Bing [4] in his major paper on metrization of topological spaces. First, we will prove Bing's metrization theorem (Theorem 3.27) for a regular topological space, then apply this theorem to obtain a series of results which hold in Moore spaces. These results show that perfect screenability, strong screenability, paracompactness, and full normality are necessary and sufficient conditions for a Moore space to be metrizable. This section will then be concluded with a generalization of Bing's metrization theorem (Theorem 3.27).

Lemma 3.22: Every metric space is a developable topological space.

Proof: Let S be a metric space with metric d. Define a development, $G = \{G_n\}$, for S as follows: G_n is the collection of all spheres with radius less than $1/n$. Clearly G_{n+1} refines G_n. Now let $p \in S$ and D any domain such that $p \in D$. Since S is a metric space there is a natural number n such that $U_{1/n}(p) \subset D$, where $U_{1/n}(p)$ is the sphere of radius $1/n$ about the point p. Consider the collection G_{4n} and let $U_{1/4n}(q)$ be any element of G_{4n} containing p. Let $x \in U_{1/4n}(q)$, then

$$d(p, x) \leq d(p, q) + d(q, x) < 1/4n + 1/4n < 1/n.$$

Thus $x \in U_{1/n}(p)$ and $U_{1/4n}(q) \subset U_{1/n}(p)$. Hence every domain of G_{4n} which contains p lies in D. Therefore the metric space S is a developable topological space.

Lemma 3.23: For each open covering H of a developable topological space S, there is a sequence $\{X_i\}$ such that X_i is a discrete collection of closed sets which is a refinement of both X_{i+1} and H while $\bigcup_{i=1}^{\infty} X_i$ covers S.

Proof: Let W be a well ordering of H and let $\{G_i\}$ be a development of S. For each $h \in H$, let us define $x(h, i)$ as follows: $p \in x(h, i)$ if and only if (1) $p \in h$, (2) there does not exist an h_α in W which precedes h_α and contains p, and (3) for each g in G_i which contains p we have g contained in h. Noting that we may have some $x(h, i)$ empty let us define $X_i = \{x(h, i) \mid h \in H\}$.

Since $x(h, i) \subset h$ we have that X_i is a refinement of H. If $p \in x(h,i)$ then for each g in G_i such that $p \in g$ we have $g \subset h$. Now G_{i+1} being a refinement of G_i tells us every g in G_{i+1} such that $p \in g$ is contained in an element of G_i. Thus every g in G_{i+1} such that $p \in g$ is contained in h. Therefore, $x(h, i) \subset x(h, i+1)$ and X_i is a refinement of X_{i+1}. Let $p \in S$ and h(p) the first element of W containing p. Since $\{G_i\}$ is a development of S there exists an integer n(p) such that every domain of $G_{n(p)}$ which contains p is contained in h(p). Thus $p \in x(h(p), n(p))$ and $\bigcup_{i=1}^{\infty} X_i$ covers S.

Let g be an element of G_i and suppose $g \cap x(h_\alpha, i) \neq \emptyset$, $g \cap x(h_\beta, i) \neq \emptyset$, and without loss of generality assume h_β precedes h_α in W. Now by definition of x(h, i) we know that if $g \cap x(h_\alpha, i) \neq \emptyset$, then g is contained in h_α. Now let $p \in g \cap x(h_\alpha, i)$; then p is contained in h_α. Since $g \cap x(h_\beta, i) \neq \emptyset$, we know that g is contained in h_β, thus p is contained in h_β. This is a contradiction since h_α was the first element of W which contained p. Therefore no element of G_i intersects two elements of X_i.

Let $x(h, i) \in X_i$ and let p be a limit point of x(h, i). Suppose that $p \notin x(h, i)$. Since G_i is a covering of S there is a domain $g \in G_i$ such that $p \in g$. Since p is a limit point of x(h, i) we know that there is a $q \in g \cap x(h, i)$. But since $q \in x(h, i)$ we have by definition of x(h, i) that $g \subset h$ which implies that $p \in x(h, i)$, a contradiction. Hence $p \in x(h, i)$ and x(h, i) is a closed set.

Let $x(h_\alpha, i)$ and $x(h_\beta, i)$ be two elements of X_i. Since $x(h_\alpha, i)$ and $x(h_\beta, i)$ are closed sets we have $x(h_\alpha, i) = \overline{x(h_\alpha, i)}$ and $x(h_\beta, i) = \overline{x(h_\beta, i)}$. Now since no element of G_i intersects two elements of X_i we have that

$$\overline{x(h_\alpha, i)} \cap \overline{x(h_\beta, i)} = \emptyset.$$

Let K_i be any subcollection of X_i and let

$$p \notin \bigcup_{k \in K_i} \overline{k} = \bigcup_{k \in K_i} k.$$

Since G_i is a covering of S there is a $g \in G_i$ such that $p \in g$. If

$$g \cap (\bigcup_{k \in K_i} k) \neq \emptyset$$

then there is a point $q \in k$ for some $k \in K_i$, such that $q \in g \cap k$. The definition of k then implies that $g \subset k$ and hence

$$p \in k \subset \bigcup_{k \in K_i} k.$$

This contradiction tells us that $\bigcup_{k \in K_i} \overline{k}$ is a closed set. Hence the collection X_i is a discrete collection of closed sets.

<u>Lemma 3.24:</u> Let S be a collectionwise normal topological space and H an open covering of S. Let H_i be a discrete collection of closed sets; then there exists a discrete collection of domains W_i such that W_i covers H_i^*, W_i is a refinement of H and each element of W_i contains just one element of H_i.

Proof: Since S is a collectionwise normal topological space there is a collection Y_i of mutually exclusive domains such that Y_i covers H_i^* and each element of Y_i intersects just one element of H_i. For each $h \in H_i$ consider the star (h) with respect to Y_i. The collection $X_i = \{\text{star}(h) | h \in H_i\}$ is a collection of mutually exclusive domains such that X_i covers H_i^* and each element of X_i contains just one element of H_i. Since each $h \in H_i$ is contained in an element $g \in H$ and $h \subset \text{star}(h) \in X_i$ we have $h \subset g \cap \text{star}(h) \subset g \in H$. The collection $Z_i = \{g \cap \text{star}(h) | h \in H_i \text{ and } g \in H\}$ is a collection of mutually exclusive domains such that Z_i covers H_i^*, Z_i is a refinement of H, and each element of Z_i contains just one element of H_i. Theorem 2.11 implies that S is normal since S is collectionwise normal. The set $A = \bigcup_{h \in H_i} h$ is closed since H_i is a discrete collection of closed sets. Since Z_i is a collection of domains the set $B = S - \bigcup_{z \in Z_i} z$ is a closed set. Since $A \cap B = \emptyset$ and S is normal, there are domains D_A and D_B such that $A \subset D_A$, $B \subset D_B$, and $D_A \cap D_B = \emptyset$. The set $D_A \subset \bigcup_{z \in Z_i} z$ since $D_A \cap B = \emptyset$. Therefore $D_A \cap z$, for each $z \in Z_i$, is a domain containing an element of H_i. Thus there is a domain w_h, for each $h \in H_i$, such that $h \subset w_h \subset \bar{w}_h \subset D_A \cap z$. Let $W_i = \{w_h | h \in H_i\}$. Let w_{h_1} and w_{h_2} be two different elements of W_i then $\bar{w}_{h_1} \cap \bar{w}_{h_2}$ since $w_{h_1} \subset D_A \cap z_1$, $w_{h_2} \subset D_A \cap z_2$, and $z_1 \cap z_2 = \emptyset$. Let M_i be a subcollection of W_i and let p be a limit point of $\bigcup_{m \in M_i} \bar{m}$. If $p \notin \bigcup_{m \in M_i} \bar{m}$ and $p \in z$ for some $z \in Z_i$, then there is a domain D such that $p \in D$, $D \subset z$, and $D \cap \bar{w}_h = \emptyset$, where $h \subset z$. Since $D \subset z \in Z_i$, $D \cap \bar{w}_h = \emptyset$ for every $w_h \in W_i$. Therefore, there is a domain which contains p which does not contain any points of $\bigcup_{m \in M_i} m$. Hence p is not a limit point of $\bigcup_{m \in M_i} \bar{m}$, a contradiction to the fact that p is a limit point of $\bigcup_{m \in M_i} \bar{m}$. Thus if p is a limit point of $\bigcup_{m \in M_i} \bar{m}$ and $p \notin \bigcup_{m \in M_i} \bar{m}$ then p must be an element of B. Since $\bar{m} \subset D_A$, for every $m \in M_i$, and $\bar{m} \cap D_B = \emptyset$, for every $m \in M_i$, the domain D_B does not contain any points of $\bigcup_{m \in M_i} m$. Therefore, p is not a limit point of $\bigcup_{m \in M_i} \bar{m}$, a contradiction to the fact that p is a limit point of $\bigcup_{m \in M_i} \bar{m}$. Thus if p is a limit point of $\bigcup_{m \in M_i} \bar{m}$, then $p \in \bigcup_{m \in M_i} \bar{m}$. Therefore W_i is a discrete collection of domains such that W_i covers H_i^*, W_i is a refinement of H, and each element of W_i contains just one element of H_i.

Lemma 3.25: Let S be a developable topological space. Then S is a strongly screenable space if and only if S is a perfectly screenable topological space.

Proof: Let $\{G_n\}$ be a development of the topological space S. If the topological space S is strongly screenable, we have for each G_i a sequence $\{H_{in}\}$ such that H_{in} is a discrete collection of domains and $\bigcup_{n=1}^{\infty} H_{in}$ covers the space and $\bigcup_{n=1}^{\infty} H_{in}$ is a refinement of G_i. Consider the countable collection $\{H_{ij} : i, j = 1, 2, \ldots\}$, and let D be any domain and p any point in D. Since S is developable with development $\{G_n\}$ we know there is an integer n(p, D) such that each domain of $G_{n(p,D)}$ which contains p lies in D. Since $\bigcup_{j=1}^{\infty} H_{n(p,D)j}$ is a refinement of $G_{n(p,D)}$ and covers the space S, there is a j and a domain in $H_{n(p,D)j}$ which contains p and is contained in a domain of $G_{n(p,D)}$, and hence is contained in D. Since the collection $\{H_{ij}\}$ defines a sequence we have by definition that S is a perfectly screenable topological space. If S is a perfectly screenable topological space, then Theorem 2.7 implies that S is a strongly screenable topological space.

Lemma 3.26: (Urysohn's Characterization of Normality). The topological space Y is normal if and only if for each pair of disjoint nonempty closed subsets A and B in Y, there exists a continuous function $f: Y \to \{x \mid x \in \text{real numbers}, 0 \le x \le 1\}$, called a Urysohn function for A, B such that (a) $f(a) = 0$ for each $a \in A$ and (b) $f(b) = 1$ for each $b \in B$.

Proof: Let Y be a topological space which satisfies the condition. Let A and B be two disjoint nonempty closed subsets of Y. Since Y satisfies the condition there is a continuous function $f: Y \to [0, 1]$ for A and B such that (1) $f(a) = 0$ for each $a \in A$, and (2) $f(b) = 1$ for each $b \in B$. Since f is a continuous function, the sets $D = \{y | f(y) < 1/2\}$ and $G = \{y | f(y) > 1/2\}$ are disjoint domains such that $A \subset D$ and $B \subset G$. Hence the topological space Y is normal.

Let Y be a normal topological space and A and B two disjoint nonempty closed sets. Let R be the set of all rational numbers of the form $k/2^n$, $0 \leq k/2^n \leq 1$, where k and n are natural numbers. We will first show that for each $r \in R$ we can associate a domain $U(r) \subset Y$ such that (a) $A \subset U(r)$ and $U(r) \cap B = \emptyset$, and (b) that if $r < r'$ then $\overline{U(r)} \subset U(r')$. We proceed by induction on the exponent of the dyadic fraction, letting $D_m = \{U(k/2^m) | k=0, 1, \ldots, 2^m\}$. The set D_o consists of $U(1) = Y - B$ and a domain $U(0)$ satisfying

$$A \subset U(0) \subset \overline{U(0)} \subset Y - B = U(1)$$

which exists since Y is normal. The set D_1 consists of $U(0)$, $U(1)$, and a domain $U(1/2)$ satisfying

$$\overline{U(0)} \subset U(1/2) \subset \overline{U(1/2)} \subset U(1)$$

which exists since Y is normal. Now assume D_{m-1} has been constructed such that

$$A \subset U(0) \subset \overline{U(0)} \subset U(1/2^{m-1}) \subset \overline{U(1/2^{m-1})} \subset \ldots \subset U(1) = Y - B.$$

Note that only $U(k/2^m)$ for odd k requires definition, since if k is even, the fraction $k/2^m$ can be factored to a number which has already been defined. For each odd k we have from D_{m-1} that

$$\overline{U(k - 1/2^m)} \subset U(k + 1/2^m)$$

since k+1 and k-1 are even. Now define $U(k/2^m)$ to be a domain U satisfying

$$\overline{U(k - 1/2^m)} \subset U \subset \overline{U} \subset U(k + 1/2^m).$$

The domain U exists since Y is normal. By construction,

$$D_m = \{U(k/2^m) \mid k=0, 1, \ldots, 2^m\}$$

satisfies the induction hypothesis. That is,

$$A \subset U(0) \subset \overline{U(0)} \subset U(1/2^m) \subset \overline{U(1/2^m)} \subset \ldots \subset U(1).$$

Now replace U(1) by Y, and let $D = \bigcup_{m=0}^{\infty} D_m$. Define $f: Y \to [0, 1]$ as follows: $f(y) = \inf \{r \mid y \in U(r)\}$. The function f is well-defined since y is always an element of U(1) = Y. Since $0 \leq r \leq 1$ we have $0 \leq f(y) \leq 1$. Furthermore $f(A) = 0$ since each U(r) contains A for every r; and $f(B) = 1$ since Y contains B and $U(r) \subset Y - B$ for every r.

Let $f(y_0) = r_0$ and $W = (r_0 - \epsilon, r_0 + \epsilon)$, where ϵ is a positive real number. If $r_0 \neq 0$, $r_0 \neq 1$, then the definition of infimum implies there exist r_1 and r_2 in R such that $r_0 - \epsilon < r_1 < r_0 < r_2 < r_0 + \epsilon$. Now the set $U(r_2) - \overline{U(r_1)}$ is a domain since $U(r_2)$ is a domain and $\overline{U(r_1)}$ is a closed set contained in $U(r_2)$. Since $r_1 < r_0$ there is a dyadic fraction r' such that $r_1 < r' < r_0$. Thus by construction we have $\overline{U(r_1)} \subset U(r')$. Now if $y_0 \in \overline{U(r_1)}$ then $y_0 \in U(r')$; hence by definition of f we have $f(y_0) \leq r'$. Since $f(y_0) = r_0$ and $r' < r_0$ we have a contradiction; thus $y_0 \notin U(r_1)$. Also the definition of infimum implies $y_0 \in U(r_2)$. Hence we have $y_0 \in U(r_2) - \overline{U(r_1)}$. Now if $y \in U(r_2) - \overline{U(r_1)}$ then $y \in U(r_2)$ and $y \in \overline{U(r_1)}$. Since $y \in U(r_2)$ we have by definition that $f(y) \leq r_2$. Since $y \notin \overline{U(r_1)}$ and $r < r'$, where $r, r' \in R$, we have that $\overline{U(r)} \subset U(r')$. Thus by definition of f we have $f(y) \geq r_1$. Therefore we have $f(U(r_2) - \overline{U(r_1)}) \subset W$. Hence the function f is continuous at y_0. If $r_0 = 0$ then $U(r_2)$ is a domain containing y_0. If $y \in U(r_2)$ then $f(y) \leq r_2 < r_0 + \epsilon$ by definition of the function f. Since f is a non-negative function we have that $f(U(r_2)) \subset W$ and thus f is continuous at y_0. If $r_0 = 1$ then consider the domain $Y - \overline{U(r_1)}$, which contains y_0. If $y \in Y - \overline{U(r_1)}$ then $f(y) \geq r_1 > r_0 - \epsilon$ by definition of f. Since we know that $f(y) \leq 1$ for every $y \in Y$ we have $f(Y - \overline{U(r_1)}) \subset W$. Therefore f is continuous at each point of the topological space Y. Hence f is a continuous function satisfying all the desired properties.

Theorem 3.27: A necessary and sufficient condition that a regular topological space S be metrizable is that it be perfectly screenable.

Proof of necessity: Let S be a metric space and H an open covering of S. Since S is a metric space we know that S is developable by Lemma 3.22. By Lemma 3.23 there is a sequence $X = \{X_i\}$ such that X_i is a discrete collection of closed sets, X_i refines X_{i+1}, X_i refines H, and $\bigcup_{i=1}^{\infty} X_i$ covers S. Since S is a metric space, Theorem 2.10 implies that S is a collectionwise normal topological space. By Lemma 3.24 we know that for each X_i there is a W_i such that W_i is a discrete collection of domains, W_i covers X_i^*, and W_i is a refinement of H. Let $W = \{W_i\}$ and observe that $\bigcup_{i=1}^{\infty} W_i$ covers S since $\bigcup_{i=1}^{\infty} X_i$ covers S and W_i covers X_i^*. Therefore, $W = \{W_i\}$ is a sequence such that W_i is a discrete collection of domains, $\bigcup_{i=1}^{\infty} W_i$ covers S, and $\bigcup_{i=1}^{\infty} W_i$ is a refinement of H. Hence by definition, S is a strongly screenable topological space. Lemma 3.25 now implies S is perfectly screenable.

Proof of sufficiency: Since S is perfectly screenable, there is a sequence $H = \{H_i\}$ such that H_i is a discrete collection of domains and for any domain D and $p \in D$ there is a natural number n(p, D) such that $H_{n(p, D)}$ contains a domain which lies in D and contains p. Let h be any element of H_i and p any point of h. Since S is a regular topological space, there exists a domain U such that $p \in U \subset \overline{U} \subset h$. Since S is perfectly screenable there is a natural number j such that H_j contains a domain which lies in U and contains p. (Note the closure of this domain lies in h). Let K_{ij} denote the union of the elements of H_j whose closures lie in an element of H_i, for instance, K_{12} is the union of the elements of H_2 whose closures lie in an element of H_1. Noting that K_{ij} may be empty for some i and j; let $C = \{K_{ij} | K_{ij} \neq \emptyset\}$ and for the following assume $K_{ij} \in C$. Let $K_{ij} = \bigcup_{\alpha \in \Lambda} h_\alpha$, where $h_\alpha \in H_j$, then since H_j is a discrete collection of domains we have $\overline{K_{ij}} = \overline{\bigcup h_\alpha} = \bigcup \overline{h}_\alpha$ is a closed set. Since $h_\alpha \subset g$, for some $g \in H_i$, for every $\alpha \in \Lambda$ we have $\bigcup \overline{h}_\alpha \subset H_i^*$. Hence K_{ij} and $S - H_i^*$ are disjoint closed subsets of S. The topological space is normal by Corollary 2.16. Therefore by Urysohn's Lemma (Lemma 3.26) there is a continuous function $F_{ij}: S \to [0, 1]$ such that $F_{ij}(x) = 1$ if $x \in \overline{K_{ij}}$ and $F_{ij}(x) = 0$ if $x \in S - H_i^*$. The sequence of functions $\{F_{ij}\}$ leads to a definition of a distance function for the topological space S. Define $D: SxS \to R$, where R is the set of real numbers, by

$$D(x, y) = \Sigma\ \Sigma\ \frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}}$$

where $R_{ij}(x, y) = \pm 1$, depending on whether y does not or does belong to an element of H_i that contains x. Using the above definition of D, the space S is a metric space as will be shown below.

Since the series

$$\Sigma \Sigma \frac{1}{2^{i+j-1}}$$

is a converging series and for each i and j

$$\frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}} \leq \frac{1}{2^{i+j}}$$

the function D is well-defined, that is, the series

$$\Sigma \Sigma \frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}}$$

converges. Since

$$\frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}} \geq 0,$$

for every i and j, we have that $D(x, y) \geq 0$ for every $x, y \in S$. Since $R_{ij}(x, y) = R_{ij}(y, x)$, for every $x, y \in S$, we have $D(x, y) = D(y, x)$ for each $x, y \in S$. Now consider the following three terms:

$$|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|,$$

$$|F_{ij}(x) + R_{ij}(x, z)F_{ij}(z)|,$$

and

$$|F_{ij}(z) + R_{ij}(z, y)F_{ij}(y)|,$$

where x, y, and z are in S. One quickly sees that showing

$$|F_{ij}(x)+R_{ij}(x, y)F_{ij}(y)| \leq |F_{ij}(x) + R_{ij}(x, z)F_{ij}(z)| + |F_{ij}(z)+R_{ij}(z, y)F_{ij}(y)|$$

is equivalent to showing $D(x, y) \leq D(x, z) + D(z, y)$, where $x, y, z \in S$. The above inequality can be shown by considering the possible cases that can arise. Listed below are the basic distinct cases that can happen for $x, y, z \in S$.

1. $x, y, z \in \overline{K_{ij}}$ and $x, y, z \in h$, where $h \in H_i$
2. $x, y, z \in \overline{K_{ij}}$, $x, y \in h$, and $z \in g$, where $h, g \in H_i$
3. $x, y, z \in \overline{K_{ij}}$, $x \in h$, $y \in g$, and $z \in k$, where $h, g, k \in H_i$
4. $x, y \in \overline{K_{ij}}$, $z \in H_i^*$, and $x, y, z \in h$, where $h \in H_i$
5. $x, y \in \overline{K_{ij}}$, $z \in H_i^*$, $x, y \in h$, and $z \in g$, where $h, g \in H_i$
6. $x, y \in \overline{K_{ij}}$, $z \in H_i^*$, $x \in h$, $y \in g$, and $z \in k$, where $h, g, k \in H_i$
7. $x, y \in \overline{K_{ij}}$, $z \in H_i^*$, $x, z \in h$, and $y \in g$, where $h, g \in H_i$
8. $x \in \overline{K_{ij}}$, $y, z \in H_i^*$, and $x, y, z \in h$, where $h \in H_i$
9. $x \in \overline{K_{ij}}$, $y, z \in H_i^*$, $x, y \in h$, and $z \in g$, where $h, g \in H_i$
10. $x \in \overline{K_{ij}}$, $y, z \in H_i^*$, $x \in h$, $y \in g$, and $z \in k$, where $h, g, k \in H_i$
11. $x \in \overline{K_{ij}}$, $y, z \in H_i^*$, $x \in h$, and $y, z \in g$, where $h, g \in H_i$
12. $x, y, z \in H_i^*$, and $x, y, z \in h$, where $h \in H_i$
13. $x, y, z \in H_i^*$, $x, y \in h$, and $z \in g$, where $h, g \in H_i$
14. $x, y, z \in H_i^*$, $x \in h$, $y \in g$, and $z \in k$, where $h, g, k \in H_i$
15. $x, y \in H_i^*$, $z \in S - H_i^*$, and $x, y \in h$, where $h \in H_i$
16. $x, y \in H_i^*$, $z \in S - H_i^*$, $x \in h$, and $y \in g$, where $h, g \in H_i$
17. $x \in H_i^*$, $y, z \in S - H_i^*$, and $x \in h$, where $h \in H_i$
18. $x, y, z \in S - H_i^*$

One can easily verify the above inequality for the above cases. Hence, we have $D(x, y) \leq D(x, z) + D(z, y)$ for every $x, y, z \in S$.

If $x = y$ then we see $|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)| = 0$ since if $x, y \in h \in H_i$ then $R_{ij}(x, y) = -1$; while if $x, y \in S - H_i^*$ then $F_{ij}(x) = F_{ij}(x) = 0$. If $x \neq y$ then since S is a Hausdorff topological space there exist domains U_x and U_y such that $x \in U_x$, $y \in U_y$, and $U_x \cap U_y = \emptyset$. Since S is perfectly screenable, there is a natural number i and an $h \in H_i$ such that $x \in h$ and $h \subset U_x$. Hence by regularity we have that there is a domain D such that $x \in D \subset \overline{D} \subset h \subset U_x$. Now perfect screenability implies there is a natural number j and a $g \in H_j$ such that $x \in g \subset \overline{g} \subset D \subset h \subset U_x$. Therefore, we have $g \subset K_{ij}$ and by definition of F_{ij} that $F_{ij}(x) = 1$. Since $y \in S - h$ and H_i is a discrete collection of domains we have that y does not belong to an element of H_i that contains x. Thus by definition $R_{ij}(x, y) = 1$ and $|F_{ij}(x) + R_{ij}(x,y)F_{ij}(y)| > 1$. Therefore,

$$\Sigma\,\Sigma \ \frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}} \geq \frac{1}{2^{i+j-1}} > 0$$

which implies $D(x, y) > 0$. Hence $x = y$ if and only if $D(x, y) = 0$.

Let M be a subset of S and $x \in S$ such that $D(x, M) = 0$. Let D be any domain such that $x \in D$. Since S is perfectly screenable there is a natural number i and a domain h in H_i such that $x \in h \subset D$. Since S is a regular topological space there is a domain U such that $x \in U \subset \overline{U} \subset h \subset D$. Therefore by perfect screenability there is a natural number j and a domain $g \in H_j$ such that $x \in g \subset U$. Since $\overline{U} \subset h$ we have $g \subset h$; thus the point x is contained in K_{ij}. Now by definition of the function F_{ij} we have $F_{ij}(x) = 1$. Since $D(x, M) = 0$ we have for every positive real number ϵ an $m \in M$ such that $D(x,m) < \epsilon$. Now if there does not exist $m \in M$ such that $m \in h$ let

$$\epsilon = \frac{1}{2^{i+j+1}} \ .$$

Since $m \notin h$ for every $m \in M$ we have by definition $R_{ij}(x,m) = 1$ for every $m \in M$. Therefore

$$D(x, m) = \sum \sum \frac{|F_{ij}(x) + R_{ij}(x, m)F_{ij}(m)|}{2^{i+j}}$$

$$\geq \frac{|F_{ij}(x) + R_{ij}(x, m)F_{ij}(m)|}{2^{i+j}}$$

$$= \frac{|1 + F_{ij}(m)|}{2^{i+j}}$$

$$\geq \frac{1}{2^{i+j}} > \epsilon .$$

This contradiction implies there is an $m \in M$ such that $m \in h$. Therefore there is an $m \in M$ such that $m \in D$. Hence x is a limit point of M.

Let M be a subset of S and x a limit point of M. Now we wish to show that $D(x, M) = 0$, that is, for every positive real number ϵ there is an $m \in M$ such that $D(x, m) < \epsilon$. Since

$$\sum_{i=N}^{\infty} \sum_{j=N}^{\infty} \frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}} \leq \sum_{i=N}^{\infty} \sum_{j=N}^{\infty} \frac{1}{2^{i+j-1}} = \frac{1}{2^{2N-3}} ,$$

we see that

$$\sum_{i=N}^{\infty} \sum_{j=N}^{\infty} \frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}}$$

can be made arbitrary small, that is, there is an N such that

$$\sum_{i=N}^{\infty} \sum_{j=N}^{\infty} \frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}} < \frac{\epsilon}{2}$$

for every $x, y \in S$. Hence we will show there is an $m \in M$ such that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{|F_{ij}(x) + R_{ij}(x,m)F_{ij}(m)|}{2^{i+j}} < \frac{\epsilon}{2}.$$

To this end, let i and j be fixed natural numbers and consider the function F_{ij}. Since $x \in S$, we know that one of the following occurs: (1) $x \in S - H_i^*$, or (2) $x \in K_{ij}$, or (3) $x \in H_i^*$ and $x \notin K_{ij}$. If $x \in S - H_i^*$ then since F_{ij} is continuous at x implies

$$\frac{F_{ij}}{2^{i+j}}$$

is continuous at x there is a domain g_{ij} such that $x \in g_{ij}$ and

$$\frac{|F_{ij}(x) - F_{ij}(y)|}{2^{i+j}} < \frac{\epsilon}{2N^2},$$

where $y \in g_{ij}$, and ϵ is a given positive real number. Now the definition of F_{ij} implies $F_{ij}(x) = 0$; thus

$$\frac{|F_{ij}(y)|}{2^{i+j}} < \frac{\epsilon}{2N^2}$$

where $y \in g_{ij}$ and ϵ is a given positive real number. Hence we have trivially that

$$\frac{|F_{ij}(x) + R_{ij}(x,y)F_{ij}(y)|}{2^{i+j}} < \frac{\epsilon}{2N^2}$$

for $y \in g_{ij}$. If $x \in K_{ij}$ then by definition of K_{ij} there is a domain $g \in H_i$ and a domain $h \in H_j$ such that $x \in \overline{h} \subset g$. Note by definition we have $R_{ij}(x,y) = -1$ for every $y \in g$. Since the function F_{ij} is a continuous function at x the function $\frac{F_{ij}}{2^{i+j}}$

is a continuous function at x. Therefore there is a domain D such that $x \in D$ and

$$\frac{|F_{ij}(x) - F_{ij}(y)|}{2^{i+j}} < \frac{\epsilon}{2N^2}$$

where $y \in D$ and ϵ is a given positive real number. Letting $g_{ij} = g \cap D$ we have $x \in g_{ij}$ and

$$\frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}} < \frac{\epsilon}{2N^2},$$

where $y \in g_{ij}$, since $R_{ij}(x, y) = -1$ for every $y \in g_{ij}$. If $x \in H_i^*$ and $x \notin K_{ij}$ then there is an $h \in H_i$ such that $x \in h$. Since F_{ij} is continuous at x the function

$$\frac{F_{ij}}{2^{i+j}}$$

is continuous at x. Therefore there is a domain D_1 such that $x \in D_1$ and

$$\frac{|F_{ij}(x) - F_{ij}(y)|}{2^{i+j}} < \frac{\epsilon}{2N^2},$$

where $y \in D_1$ and ϵ is a given positive real number. Letting $G_{ij} = h \cap D_1$ we have $x \in g_{ij}$ and

$$\frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}} < \frac{\epsilon}{2N^2},$$

where $y \in g_{ij}$ since by definition $R_{ij}(x, y) = -1$ for $y \in h$. Thus for each i and j $(i, j = 1, 2, \ldots, N)$ we can associate a domain g_{ij} such that $x \in g_{ij}$ and

$$\frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}} < \frac{\epsilon}{2N^2}$$

for $y \in g_{ij}$. The point set $D_2 = \bigcap g_{ij}$, where $i, j = 1, 2, \ldots, N$ is a domain which contains x. Since x is a limit point of M there is an m_0 such that $m_0 \in D_2$ and

$$\frac{|F_{ij}(x) + R_{ij}(x, m_0)F_{ij}(m_0)|}{2^{i+j}} < \frac{\epsilon}{2N^2}$$

for $i, j = 1, 2, \ldots, N$. Thus we have

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{|F_{ij}(x) + R_{ij}(x, m_0)F_{ij}(m_0)|}{2^{i+j}} < \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\epsilon}{2N^2} = \frac{\epsilon}{2N^2} \cdot N^2 = \frac{\epsilon}{2}.$$

Therefore,

$$D(x, m_0) = \sum \sum \frac{|F_{ij}(x) + R_{ij}(x, m_0)F_{ij}(m_0)|}{2^{i+j}} < \epsilon$$

and we have $D(x, M) = 0$.

The above two paragraphs imply that x is a limit point of M, where M is a subset of S, if and only if $D(x, M) = 0$. Thus S is a metric space by Definition 3. 1.

A result similar to Theorem 3. 27 was proved in 1951 by Nagata and Smirnof [1]. Because the proof of this theorem is similar to Theorem 3. 27 it will not be included in this paper. Since this theorem is an important theorem from a historical view, it will be stated below.

<u>Definition 3. 28</u>: A collection of sets H is called <u>σ-locally finite</u> if there is a sequence $G = \{G_i\}$ such that G_i is a locally finite

collection of sets and $H = \bigcup_{i=1}^{\infty} G_i$.

Definition 3.29: Let S be a topological space with base B. If B is σ-locally finite then B is called an NS-base.

Theorem 3.30: (Nagata and Smirnof) A topological space S is metrizable if and only if it is regular and has an NS-base.

Now with the aid of Theorem 3.27, we will prove a series of theorems dealing with the metrizability of Moore spaces.

Theorem 3.31: A Moore space is metrizable if and only if it is perfectly screenable.

Proof: Since a Moore space is regular, Theorem 3.27 gives the desired conclusions.

Theorem 3.32: A Moore space is metrizable if and only if it is strongly screenable.

Proof: Lemma 3.25 shows that in Moore spaces, strong screenability is equivalent to perfect screenability. Hence Theorem 3.27 produces the desired conclusions.

Lemma 3.33: If H is a collection of mutually exclusive domains in a normal developable topological space S then there is a sequence $\{H_i\}$ such that H_i is a discrete collection of domains and $\bigcup_{i=1}^{\infty} H_i$ covers H^* and $\bigcup_{i=1}^{\infty} H_i$ is a refinement of H, where $H^* = \bigcup_{h \in H} h$.

Proof: Let $\{G_i\}$ be a development of the space and let $W = S - H^*$. Since H^* is a domain, we know for each $p \in H^*$ there exists an integer

$n(p, H^*)$ such that each element of $G_{n(p, H^*)}$ which contains p lies in H^*. Let $X_i = \{p \mid \text{no element of } G_i \text{ containing } p \text{ intersects } W\}$, that is, $p \in X_i$ if and only if (1) $p \in H^*$ and (2) if $p \in g \in G_i$ then $g \cap W = \emptyset$. Let $p \notin X_i$, then there is a domain g in G_i such that $p \in g$ and $g \cap W \neq \emptyset$. Hence, by definition of X_i, we have $g \cap X_i = \emptyset$ which implies that X_i is a closed set. Since $X_i \cap W = \emptyset$ and S is a normal space there exists a domain D_i such that $X_i \subset D_i$ and $\overline{D_i} \cap W = \emptyset$. Now define $H_i = \{h \cap D_i \mid h \in H\}$ and observe that H_i is a collection of mutually exclusive domains such that $\bigcup_{i=1}^{\infty} H_i$ is a refinement of H. Since S is developable, we have that each point of H^* is contained in an X_i and hence in a domain D_i; thus we have that $\bigcup_{i=1}^{\infty} H_i$ covers H^*.

For a fixed i, let us consider $h_\alpha \cap D_i$ and $h_\beta \cap D_i$ in H_i. If $x \in (\overline{h_\alpha \cap D_i}) \cap (\overline{h_\beta \cap D_i})$, then $x \in \overline{D_i}$. Now the definitions of h_α, h_β, and D_i imply that $x \in \partial(h_\alpha \cap D_i)$ and $x \in \partial(h_\beta \cap D_i)$. Therefore, there exists a point $y \neq x \in h_\alpha \cap D_i$ and $z \neq x \in h_\beta \cap D_i$ for every domain containing x. Hence every domain containing x contains a point of h_β and a point not in h_β. Thus, $x \in \partial h_\beta$ which implies $x \in W$ since $\partial h_\beta \subset W$. Therefore, $x \in \overline{D_i} \cap W$ which implies $\overline{D_i} \cap W \neq \emptyset$, which contradicts the fact that $\overline{D_i} \cap W = \emptyset$. Thus the sets $\overline{h_\alpha \cap D_i}$ and $\overline{h_\beta \cap D_i}$ are disjoint.

Let K_i be a subcollection of H_i and suppose $x \notin \bigcup_{k \in K_i} \overline{k}$. If $x \in W$ then $S - D_i$ is a domain containing x which does not intersect $\bigcup_{k \in K_i} \overline{k}$. If $x \in H^*$ then $x \in h_\alpha$ for some α. If $x \notin \overline{h_\alpha \cap D_i}$ then by normality there exists a domain D_α such that $x \in D_\alpha \subset \overline{D}_\alpha$ and $\overline{D}_\alpha \cap (\overline{h_\alpha \cap D_i}) = \emptyset$. Since $\{G_i\}$ is a development of the space, we know there is an integer $n(x, h_\alpha)$ such that every domain of $G_{n(x, h_\alpha)}$ containing x lies in h_α. Let g be such a domain and consider $g \cap D_\alpha$. The set $g \cap D_\alpha$ is a domain such that $x \in g \cap D_\alpha$ and $g \cap D_\alpha \subset h_\alpha$. Since H is a mutually exclusive collection of domains then h_α and K_i^* are disjoint. Therefore x is not a limit point of K_i^*, and thus H_i is a discrete collection of domains.

Remembering that $x \notin \bigcup_{k \in K_i} \overline{k}$, let us assume $x \in \overline{h_\alpha \cap D_i}$. Since $h_\alpha \cap \bigcup_{k \in K_i} \overline{k} = \emptyset$, it will be sufficient to show $\overline{h_\alpha \cap D_i} \subset h_\alpha$. Suppose there is a $y \in \overline{h_\alpha \cap D_i}$ such that $y \notin h_\alpha$. Since $h_\alpha \cap D_i \subset h_\alpha$, we must have $y \in \partial(h_\alpha \cap D_i)$, therefore for every domain D containing y there is a $z \in (h_\alpha \cap D_i)$ which lies in D. Thus every domain containing y contains a point of h_α and hence $y \in \partial h_\alpha$. Therefore $y \in W$ since $\partial h_\alpha \subset W$, but y is also in $\overline{D_i}$. Hence $y \in W \cap \overline{D_i}$, a contradiction to the fact $W \cap \overline{D_i} = \emptyset$. Thus we have $\overline{h_\alpha \cap D_i}$ is contained in h_α, and the collection H_i is a discrete collection.

Lemma 3.34: A screenable normal developable topological space S is a strongly screenable topological.

Proof: Let H be an open covering of the space S. Since S is screenable we know there is a sequence $G = \{H_i\}$ such that H_i is a collection of mutually exclusive domains and $\bigcup_{i=1}^{\infty} H_i$ covers S and $\bigcup_{i=1}^{\infty} H_i$ is a refinement of H. Now Lemma 3.33 implies that for each i, there is a sequence $G_i = \{H_{ij}\}$ such that H_{ij} is a discrete collection of domains and $\bigcup_{j=1}^{\infty} H_{ij}$ covers H_i^* and $\bigcup_{j=1}^{\infty} H_{ij}$ is a refinement of H_i. Hence the collection $\{G_i\}$ defines a sequence such that each element of the sequence is a discrete collection of mutually exclusive domains and the union covers S and is a refinement of H. Thus our space S is strongly screenable.

Theorem 3.35: A screenable Moore space is metrizable if it is normal.

Proof: Lemma 3.34 implies that a screenable normal Moore space is strongly screenable. Hence Theorem 3.32 implies that the space is metrizable.

Lemma 3.36: A collectionwise normal Moore space S is screenable.

Proof: Let H be an open covering of S. Lemma 3.23 implies that there is a sequence $\{X_i\}$ such that X_i is a discrete collection of closed sets and X_i is a refinement of X_{i+1} and H and $\bigcup_{i=1}^{\infty} X_i$ covers S. Since S is a collectionwise normal we have for each i, a collection Y_i of mutually exclusive domains, such that Y_i^* covers X_i^* and that no element of Y_i intersects two elements of X_i.

Now for each $x \in X_i$ we know that $x \subset h$ for some $h \in H$. Consider $K_x = \{g \mid g \in Y_i,\ g \cap x \neq \emptyset\}$ then we know K_x^* is a domain and $x \subset K_x^*$. Hence, $K_x^* \cap h$ is a domain contained in h. Because of this reasoning we may assume each element of Y_i is contained in an element of H. Letting $Y = \{Y_i\}$ and noting that Y^* covers S since $\bigcup_{i=1}^{\infty} X_i$ covers S. We have by definition that S is screenable.

Theorem 3.37: A Moore space is metrizable if and only if it is a collectionwise normal topological space.

Proof: If S is a metric space then Theorem 2.10 implies that S is a collectionwise normal topological space. Now let S be a collectionwise normal Moore space. Lemma 3.35 implies that S is a screenable Moore space. Also by Theorem 2.11 we know that S is a normal Moore space. Hence S is metrizable by Theorem 3.35.

Lemma 3.38: Let S be a collectionwise normal topological space and H an open covering of S. If $K = \{H_i\}$ is a sequence such that each H_i is a discrete collection of closed sets, $\bigcup_{i=1}^{\infty} H_i$ is a refinement of H, and $\bigcup_{i=1}^{\infty} H_i$ covers S. Then there is an open covering G of S such that G is a refinement of H and for each point $p \in h$, where $h \in H_i$, there is a domain D containing p such that not more than i elements of G intersect D.

Proof: By Lemma 3.24 we have for each H_i a discrete collection of domains W_i such that W_i covers H_i^*, W_i is a refinement of H, and each element of W_i contains just one element of H_i. Also since S is a collectionwise normal topological space we have that Theorem 2.11 implies S is a normal topological space.

As S is normal, there is a domain D_i containing H_i^* such that $\overline{D_i} \subset W_i^*$. Define G as follows: each element of W_1 is an element of G and if $w \in W_{i+1}$ such that $w \not\subset \bigcup_{j=1}^{i} \overline{D}_j$, then $w - \bigcup_{j=1}^{i} \overline{D}_j$ is an element of G. Since W_i is a refinement of H for each i, we have that G is a refinement of H. Also since W_i covers H_i^* and $\bigcup_{i=1}^{\infty} H_i$ covers S, we have that G covers S. Let $p \in h$, where $h \in H_i$, and assume that H_i^* is the first H_j^*, $j = 1, 2, \ldots, i$ such that $p \in H_i^*$. Since W_i covers H_i^* there is a domain w_i such that $p \in w_i$, also note that w_i is the only element of W_i which contains p. Since W_j is a discrete collection of domains, we have for each j at most one $w_j \in W_j$ such that $p \in \overline{w}_j$. Now for $j = 1, 2, \ldots, i-1$, we have either $p \in \overline{w}_j$ or $p \notin \bigcup_{w \in W_j} \overline{w}$. If $p \in \overline{w}_j$ then since W_j is a discrete collection there is a domain D_p such that $\overline{w}_j \subset D_p$ and

$$\overline{D}_p \cap \Big(\bigcup_{\substack{w \in W_j \\ w \neq w_j}} \overline{w} \Big) = \emptyset .$$

If $p \notin \bigcup_{w \in W_j} \overline{w}$, then there is a domain D_p such that $p \in D_p$ and

$$\overline{D}_p \cap \left(\bigcup_{w \in W_j} \overline{w} \right) = \emptyset.$$

Now define

$$D = \left(\bigcap_{j=1}^{i-1} D_{pj} \right) \cap w_i \cap D_i$$

and note D is a domain containing p. If $g \in G$ and

$$g = w - \bigcup_{k=1}^{i} \overline{D}_k$$

and $j > i$ then $D \cap g = \emptyset$ since $D \subset D_i \subset \overline{D}_i$. Hence be definition of D and W_i we have that D can intersect at most an i number of elements of G.

Theorem 3.39: A Moore space is metrizable if and only if it is paracompact.

Proof: Let S be a paracompact Moore space. Now Theorems 2.32 and 2.19 imply that S is a collectionwise normal Moore space. Hence by Theorem 3.37, S is metrizable. Now let S be a metric space with metric D and H any open covering of S. Theorem 2.10 and Lemma 3.22 imply that S is developable and collectionwise normal. Hence by Lemma 3.23 there is a sequence $X = \{X_i\}$ such that X_i is a discrete collection of closed sets which is a refinement of both X_{i+1} and H while $\bigcup_{i=1}^{\infty} X_i$ covers S. Thus Lemma 3.38 implies there is an open covering G of S such that G is a refinement of H and for each point $p \in g$, where $g \in X_i$, there is a domain D containing p such that not more than i elements of G intersect D. Therefore we have by

Definition 2.23 that S is a paracompact topological space.

Hence by Theorems 3.39, 2.30, and 2.32 we have the following theorem.

Theorem 3.40: A Moore space S is metrizable if and only if S is a fully normal Moore space.

This section will now be concluded with a generalization of Theorem 3.27.

Theorem 3.41: A regular topological space S is metrizable if there is a sequence $G = \{G_i\}$ such that

(a) G_i is a collection of domains such that the sum of the closures of any subcollection of G_i is closed and

(b) if $p \in S$ and D is a domain such that $p \in D$ then there is an integer $n(p, D)$ such that an element of $G_{n(p,D)}$ contains p and every element of $G_{n(p,D)}$ containing p lies in D.

Proof: Let D be any domain such that $D \subset S$. First, we will show that D is strongly screenable. Let $H = \{h_1, h_2, \ldots, h_\alpha, \ldots\}$ be a well ordered collection of domains which covers D. Let $V_{\alpha i}$ be the sum of the elements of G_i whose closures lie in h_α. Note that some of the $V_{\alpha i}$ may be empty. Now if $U_{\alpha ij}$ denotes the sum of the elements of G_j whose closures lie in $V_{\alpha i}$ but do not intersect $\bigcup_{\beta<\alpha} \overline{V}_{\beta i}$ then let $W_{ij} = \{U_{\gamma ij}: \gamma = 1, 2, \ldots, \alpha, \ldots\}$. Now if $U_{\alpha ij}$ and $U_{\beta ij}$ $(\beta < \alpha)$ are elements of W_{ij}, then by condition (a) we have that $\overline{U_{\alpha ij}}$ is a closed set that lies in $V_{\alpha i}$; also $\overline{U_{\beta ij}}$ is a closed set that lies in $V_{\beta i}$. Since $\overline{V_{\beta i}}$ is a closed set by condition (a) we have

$$\overline{U_{\alpha ij}} \cap (\bigcup_{\beta<\alpha} \overline{V_{\beta i}}) = \emptyset,$$

and

$$\overline{U_{\alpha ij}} \cap \overline{U_{\beta ij}} = \emptyset.$$

Also condition (a) implies that the sum of the closures of any sub-collection of W_{ij} is closed. Hence W_{ij} is a discrete collection of domains. One quickly sees by definition that W_{ij} is a refinement of H. Now let $p \in D$ and h_β be the first element of H to contain p. Then p belongs to some $V_{\beta k}$ but does not belong to $\bigcup_{\alpha<\beta} \overline{V_{\alpha k}}$. Then for some integer m, p lies in an element of G_m whose closure lies in $V_{\beta k}$ but does not intersect $\bigcup_{\alpha<\beta} \overline{V_{\alpha k}}$. Hence $p \in U_{\beta km}$ and $\bigcup \bigcup W_{ij}$ covers D. Hence D is strongly screenable.

For each positive integer k let $X_k = \{X_{ki}\}$ be a sequence of discrete collections of domains such that each X_{ki} is a refinement of G_k and $\bigcup_{i=1}^{\infty} X_{ki}$ covers G_k^*. That S is perfectly screenable follows from the fact that the elements of $\{X_{ki} : i, k = 1, 2, \ldots\}$ may be ordered in a sequence fulfilling the conditions to be satisfied by the sequence $G = \{G_i\}$ mentioned in the definition of a perfectly screenable topological space. Hence by Theorem 3.27, S is metrizable.

Some Recent Results

This chapter will be concluded with some results obtained within the past fifteen years. The first such result is due to P. S. Aleksandrov [1]. This theorem, proved in 1956, will be shown to follow other results which have been previously proved in this paper.

Theorem 3.42: In order that a regular topological space be metrizable, it is necessary and sufficient that it have a uniform base and that it satisfy any one of the following conditions:

(a) it is paracompact

(b) it is collectionwise normal

(c) each point-finite covering of it has a locally finite refinement.

Proof: If S is a metric space then an argument similar to Example 2.43 implies that S has a uniform base. Since a metric space is a Moore space we have by Theorems 3.37 and 3.39 that conditions (a), (b), and (c) are satisfied. If the regular topological space S has a uniform base then by Theorem 2.45 it is a pointwise paracompact Moore space. If (a) is true then Theorem 3.39 implies that S is metrizable. If (b) is true then Theorem 3.37 implies that S is metrizable. If (c) is true then since S is pointwise paracompact we have that S is paracompact, and hence metrizable.

In light of Theorems 3.42 and 2.45 the following collection of theorems on metrization of pointwise paracompact Moore spaces will be of interest. The proofs of these theorems will be omitted, but may be found in Traylor [29], and Heath and Grace [11]. However, before stating these theorems, all definitions needed to read these theorems will be given.

Definition 3.43: A topological space is locally separable if each domain D contains a domain D' which is separable.

Definition 3.44: A topological space S is locally peripherally separable if for each point p and domain D containing p there is a domain D' containing p such that D' is a subset of D and the boundary of D' is separable.

Definition 3.45: Let S be a topological space and D a domain in S. If B is the boundary of D then the statement that B is accessible means that if p is a point of B and R is a region containing p then there exist points q and q' such that q is in D, q' is in R $\cap$ B, and there is an arc with end points q and q' which, except for q', lies wholly in D.

Definition 3.46: A topological space is locally arcwise connected if each domain D contains a domain D' which is arcwise connected.

Definition 3.47: A topological space S is locally peripherally connected if p is a point of S and D is a domain of S containing p, there is a domain D' containing p such that D' is a subset of D and the boundary of D' is connected.

Following are the theorems dealing with the metrization of pointwise paracompact Moore spaces.

Theorem 3.48: A locally separable Moore space is metrizable if and only if it is pointwise paracompact.

Theorem 3.49: If S is a locally peripherally separable Moore space such that the boundary of each domain is accessible, then S is metrizable if and only if it is pointwise paracompact.

Theorem 3.50: A locally peripherally separable, locally arcwise connected Moore space is metrizable if and only if it is pointwise paracompact.

Theorem 3.51: A connected, locally connected, locally peripherally separable, pointwise paracompact Moore space is separable and hence metrizable.

Theorem 3.52: A connected, locally peripherally connected, locally peripherally separable, pointwise paracompact Moore space is separable and hence metrizable.

Theorem 3.53: Suppose that (1) X is a connected pointwise paracompact Moore space, (2) X has only one cut point p and (3) for all p, q in X, $p \neq q$, and every open set R containing p, there exists a closed connected separable set N such that $N \subset R$ and N separates p from q in X. Then X is metrizable.

Theorem 3.54: Suppose that X satisfies (1) and (3) of the previous theorem and that X has a finite number of cut points. Then X is metrizable.

Theorem 3.55: Suppose that (1) S is a connected, locally connected, pointwise paracompact Moore space, (2) there is a separable closed set which separates S, and (3) each non-degenerate separable closed set which separates S contains two points which are separated by a separable closed set. If S has only a finite number of cut points, then S is metrizable.

Following Aleksandrov's metrization theorem in 1956 the next result was in 1964 by Heath [15]. This result depended upon a generalization of pointwise paracompactness and is called property P.

Definition 3.56: A topological space S has property P provided that every open covering G of S has an open refinement H such that H covers S and no point of S belongs to more than countably many members of H.

Definition 3.57: A topological space S is $\aleph_1$-compact provided every uncountable subset of it has a limit point.

Theorem 3.58: Every separable topological space S having property P is $\aleph_1$-compact, and hence if S is also a Moore space then S is metrizable.

Proof: Suppose that S is not $\aleph_1$-compact and let M be an uncountable subset of S having no limit point. Since S is separable, let K be a countable dense subset of S and, for each point $x \in M$, let R(x) be a domain containing x and no point of $M - \{x\}$.

Then, if G is an open covering of S consisting of S - M and $\{R(x) | x \in M\}$, any refinement of G must include a subset Q such that, for each x and y in M, $x \neq y$, there are members $g, h \in Q$ such that $x \in g$, $y \in h$, and $g \neq h$.

Then since every member of Q must contain a point of the countable set K there is a point z in K such that z belongs to uncountably many members of Q contrary to the assumption that S has property P.

If S is also a Moore space then Lemmas 3.18 and 3.19 imply that S is a second countable topological space. Hence by Urysohn's Theorem we have that S is metrizable.

Corollary 3.59: A separable pointwise paracompact Moore space is metrizable.

This chapter will now be concluded with three theorems which are generalizations of previous theorems. These theorems deal with properties of the boundary of each domain rather than with properties of the space. The first, proved by Traylor [27] in 1964, is stated and proved below.

Theorem 3.60: A normal Moore space is metrizable if the boundary of each domain is screenable.

Proof: Suppose that S is a normal Moore space and H is an open covering of S. By an argument similar to that of Theorem 3.41 there is a collection H' of mutually exclusive domains such that H'^* is dense in S and H' refines H. Now H'^* is a domain so we have $S - H'^*$ is screenable by hypothesis. Hence there is a sequence $K = \{H_i\}$ such that H_i is a collection of mutually exclusive domains, $\bigcup_{i=1}^{\infty} H_i$ is a refinement of H, and $\bigcup_{i=1}^{\infty} H_i$ covers $S - H'^*$. Therefore letting $H' = H_0$ we have that $K_1 = \{H_0, H_1, H_2, \ldots\}$ can be ordered in a sequence satisfying the conditions of the sequence mentioned in the definition of a screenable topological space. Hence S is a normal screenable topological space. Therefore by Theorem 3.35 the space S is metrizable.

The following result is due to Grace and Heath [11].

Theorem 3.61: A Moore space is metrizable if the boundary of each domain is strongly screenable.

Proof: Let S be a Moore space and H an open covering of S. Using a construction similar to Theorem 3.41 there is a collection of domains G whose closures are mutually exclusive, G^* is dense in S, and G is a refinement of H. Then $\overline{G^*}$ is S and $\overline{G^*} - G^*$ is strongly screenable since it is the boundary of G^*. Thus there is a sequence $K = \{H_i\}$ such that H_i is a discrete collection of domains, $\bigcup_{i=1}^{\infty} H_i$ refines H, and $\bigcup_{i=1}^{\infty} H_i$ covers $\overline{G^*} - G^*$.

Denote by G' the collection to which a domain d belongs if and only if there is a domain $g \in G$ such that

$$d = g - (g \cap \overline{\bigcup_{i=1}^{\infty} H_i^*}).$$

Then each point in $\overline{G'^*}$ must belong to the closure of some element of G' since no point of $\overline{G^*} - G^*$ is a limit point of G'^*. Thus G' is a discrete collection of domains.

Suppose that M is the boundary of $\bigcup_{i=1}^{\infty} H_i^*$. Then each point of S belongs to either G'^*, M, or $\bigcup_{i=1}^{\infty} H_i^*$. But M is strongly screenable since it is the boundary of a domain. Thus there is a sequence $K' = \{H_i'\}$ satisfying the notion of strong screenability with respect to H and M. Clearly, the sequence $G', H_1, H_1', H_2, H_2', \ldots$ satisfies the notion of strong screenability with respect to H and S. Thus S is strongly screenable. Hence Theorem 3.32 implies that S is metrizable.

The final theorem of this chapter deals with still another property of the boundary of a domain. This theorem by Traylor [28] will conclude this chapter.

<u>Theorem 3.62</u>: Let S be a Moore space, such that if B is the boundary of a domain in S and G is a collection of domains covering B, then some countable subcollection of G covers B. Then S is strongly

screenable, and thus metrizable.

Proof: Let H be an open covering of S, $K = \{x \mid \{x\} \text{ is a region}\}$, $L = \{x \mid x \text{ is a limit point of } K\}$, and $M = \{x \mid x \in S - (K \cup L)\}$. Then no point of M is a limit point of L, otherwise that point is a limit point of K and not in M. Furthermore, K is a domain and L is the boundary of K. Thus some countable subcollection H_1 of H covers L.

Now suppose there is an uncountable subset T of M such that T has no limit point. Each point of T is a limit point of M since T contains no degenerate region and no point of M is a limit point of either K or L. So M - T is a domain, and T is an uncountable subset of the boundary B of M - T. But the boundary B of M - T is second countable and T is an uncountable subset of B. Hence, T has a limit point. This contradiction implies that every uncountable subset of M has a limit point. Therefore by Lemma 3.19 we have that M is second countable. Denote by H' a countable subcollection of H such that H' covers M.

Denote by H_2 the collection to which the region R belongs if and only if the only element of R is a point of $(K - H_1^* \cap K)$. Clearly, H_2 is a discrete collection of mutually exclusive domains. Denote by $F_1, F_2, \ldots$ a sequence such that each F_i is a collection whose only element is some domain of $H_1 \cup H'$ and each domain of $H_1 \cup H'$ is the only domain of some F_i. Clearly, by the definitions of H_1 and H' this sequence is at most countable. Then the sequence $H_2, F_1, F_2, \ldots$ is a countable sequence such that H_2 and F_i are discrete collections of domains. By definition of H_2 and F_i we have that $H_2 \cup (\bigcup_{i=1}^{\infty} F_i)$ is a refinement of H and $H_2 \cup (\bigcup_{i=1}^{\infty} F_i)$ covers S. Thus by definition we have that S is strongly screenable. Hence by Theorem 3.32, the Moore space S is metrizable.

CHAPTER IV

PROPERTIES OF NONMETRIZABLE NORMAL MOORE SPACES

Introduction

The goal of this chapter is to give the reader who seeks a counterexample to the normal Moore space conjecture some insight into results which should be of help. This chapter includes a translation of the conjecture into a nontopological setting, which should enable a greater audience to pursue a solution to this evasive problem. The paper concludes with a presentation of some of the properties of nonmetrizable normal Moore spaces.

A Translation of the Normal Moore Space Conjecture

This section of the paper, Bing [2], will translate the yet unsolved problem in topology of whether a normal Moore space is metrizable into a nontopological setting. The advantage of this translation is that it will allow a wider audience to examine the problem.

Let X be a set and R the cartesian product of X with itself, that is, $R = X \times X$. Let L denote the diagonal of R, $L = \{(x, x) | x \in X\}$. For a visual aid one could consider $X = [0, 1]$ on the real line. Then $R = [0, 1] \times [0, 1]$, the unit square in the Euclidean plane, and with diagonal L from (0, 0) to (1, 1) as shown in Figure 5. However, our discussion will not insist that X have the cardinality of the continuum.

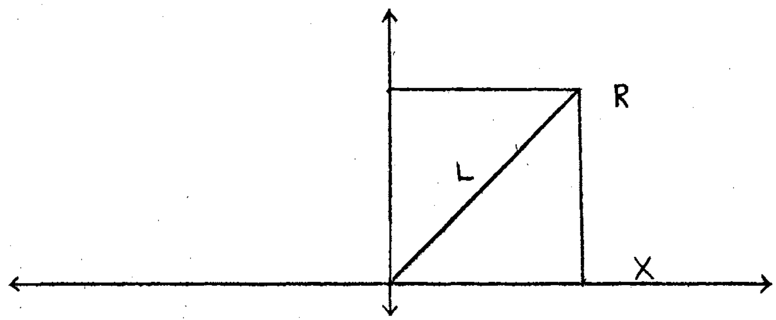

Figure 5. The Sets X, R, and L

Now let $h(x, y) = (y, y)$ be the horizontal projection of a point in R into a point on L. Also let $v(x, y) = (x, x)$ be the vertical projection. We will be concerned with sets W such that $h(W) \cap v(W) = \emptyset$. For example, in Figure 5, W could be $\{(x, y) | 3/4 \leq x \leq 1 \text{ and } 0 \leq y \leq 1/4\}$, then we have $h(W) = \{(y, y) | 3/4 \leq y \leq 1\}$ and $v(W) = \{(x, x) | 0 \leq x \leq 1/4\}$.

Let $f : (R - L) \rightarrow \{0, 1, 2, \ldots\}$ be a transformation of R - L into the non-negative integers. The transformation f is not necessarily continuous. The question to be answered is, do the following possible properties of f imply each other?

(a) There is a transformation $F : X \rightarrow \{0, 1, 2, \ldots\}$ such that $\max [F(x), F(y)] > f(x, y)$ for each $(x, y) \in R - L$.

(b) For each subset W of R with $h(W) \cap v(W) = \emptyset$ there is a transformation $F_W : X \rightarrow \{0, 1, 2, \ldots\}$ such that $\max [F_W(x), F_W(y)] > f(x, y)$ for each $(x, y) \in W$.

If (a) is true then by letting $F_W = F|_W$ we have that (b) is true. The question of whether (b) implies (a) is not obvious. Whether or not there is such an implication is related to the following conjecture.

<u>Conjecture 4.1</u>: Each normal Moore space is metrizable.

As we have seen by Theorem 3.36 that if a Moore space is collectionwise normal then it is metrizable. Thus if there is a counterexample S to Conjecture 4.1 then S contains a discrete collection of closed sets such that S is not collectionwise normal with respect to the collection. We say that S is a counterexample of Type D if it has the additional property that it contains a discrete collection of points with respect to which it is not collectionwise normal.

<u>Theorem 4.2</u>: A necessary and sufficient condition that there be a counterexample of Type D is that there be an X, R, L, f(x, y) satisfying condition (b) but not condition (a).

The proof of the theorem is omitted, but the interested reader may find this proof in Bing [2].

Properties of Nonmetrizable Normal Moore Spaces

The last section of this paper will include some properties of nonmetrizable normal Moore spaces. Of course, this is based on the assumption that there does exist a nonmetrizable normal Moore space. The first property, established by D. R. Traylor [32] in 1964, is as follows:

<u>Theorem 4.3</u>: If there exists a nonmetrizable, normal, separable Moore space then there is a nonmetrizable, normal, separable, arcwise connected, locally arcwise connected Moore space.

Another property also established by Traylor [31] in 1966 depends upon the following definition. The reader should refer back to the definition of collectionwise normal.

Definition 4.4: The discrete collection G is collectionwise abnormal if no collection of domains in the space covering G* satisfies the notion of collectionwise normality.

Using this definition Traylor proved the following property.

Theorem 4.5: If S is a normal, nonmetrizable Moore space and H is an open covering of S then there is an uncountable discrete collection G of mutually exclusive, closed point sets such that G refines H and G is collectionwise abnormal.

Also in 1966, Ben Fitzpatrick and D. R. Traylor [8] proved the following two results.

Theorem 4.6: If there is a normal Moore space which is not metrizable, there is one which is not locally metrizable at any point, that is, if p is a point of the space then there does not exist a domain D containing p such that D is metrizable.

Theorem 4.7: If there is a normal, separable Moore space which is not metrizable, then there is one which is not locally metrizable at any point.

CHAPTER V

SUMMARY

This paper has provided a historical account of the basic theories concerning the metrization of Moore spaces, from their earliest beginnings to their present status in mathematics. The author has included such proofs and examples as were deemed necessary for best comprehension.

In preparing the paper, the most recent guides to mathematical literature were consulted; due to the time lapse between publication and inclusion in these indices, any work which may have been done after 1966 was not included. The bibliographies of the indexed articles were fully explored; any applicable material from this source has been included. For these reasons, the author feels that this paper, coupled with the material now completed but not yet available, would provided a comprehensive and usable reference tool to any who may wish to pursue this area of topology in the future.

The material contained in this paper would serve to clarify many of the questions relating to the metrizability of Moore spaces; among those of major consequence would be: Is every normal pointwise paracompact Moore space metrizable? Does there exist a Moore space which is neither screenable nor collectionwise normal? Is every separable normal Moore space metrizable? Can Example 2.12 be modified so as to obtain a normal developable topological space which is not metrizable? What is a sufficient condition for a pointwise para-compact Moore space to be screenable? Is every normal topological space with a uniform base metrizable?

The author wishes to take the liberty of suggesting the above as worthwhile areas of investigation to those interested in the metrization of Moore spaces.

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