CAPITAL BUDGETING PROJECT ANALYSIS
AND SELECTION WITH COMPLEX
UTILITY FUNCTIONS

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PREFACE

The principal thrust of this research is toward a better understanding of the complex nature of business decision-making behavior, especially under conditions of uncertainty and in those situations in which "risks" are apparently sought rather than avoided. Most, if not all, of the prior research in this area has assumed that decision-nakers are inherently risk-avoiders. But, it is often observed, and particularly in "growth" companies, that executives near the apex of the organizational hierarchy do not consistently act in accordance with a strict risk-avoidance criterion. Rather, they take seemingly irrational risks under some conditions--that is, they sometimes accept alternative courses of action that, on the face of it, do not require an increased prospective return for increased uncertainty of payoff. While such decisions may lead to minor disaster, it seems that just as often they lead to major gains for the firm. Why, then, do some executives seem to <u>seek</u> risk under certain conditions (perhaps in the hope of handsome returns), while at the same time they purchase insurance against loss in other aspects of their business? It is to this apparent paradox that this research is addressed, with the principal objective of formulating at least one definitive explanation for this type of complex decision behavior.

The genesis of this investigation originally lay in some first-hand observations of decision-making in a group of small, growing corporations. Over a period of more than ten years, I was privileged to participate in--and to observe--the making of capital budgeting and investment decisions for these companies at a high level in the organization. As a generalization, the long-range goal of the principal executives seemed to be one of risk conversion--that is, decisions were made that would enhance long-range organizational control of the environmental determinants of outcome, and thereby reduce the inherent overall risks taken. But short-run goals often fluctuated. In many instances, especially when "important money" was at stake, risky alternatives were avoided. In other instances, when the opportunities seemed somewhat better or if an appreciable part of the firm's net worth were not at stake, then risks were often accepted. And, in all instances appropriate insurance against large losses was consistently carried at considerable cost. What was happening, apparently, was that "growth leverage" was being generated at opportune times by the selective acceptance of risk opportunities. While such behavior, on the surface, seemed to be "irrational," nevertheless, there was an underlying thread woven through the decisions that strongly implied a rather consistent form of behavior, especially if the riddle could be unraveled. The present research is a first step toward such an "unraveling." It lays a theoretical base for the complex utility function as one way of explaining, and therefore, predicting simultaneous risk-avoidance and risk-seeking.

To the graduate student, the dissertation is somewhat of a pinnacle of accomplishment and progress. But no pinnacle can be reached alone; rather, such attainment requires cooperative effort. So it is with this dissertation. Little could have been accomplished without the assistance, support, aid, comfort, encouragement and actual labor of those who helped make it all possible. Chester Barnard once said that the weakest link in the chain of cooperative effort is the "will to co-operate." But, in those who have been so instrumental in the accomplishment of this work, there has been no lack of will to cooperate. Support for my graduate work and for this research has been provided principally by the School of Industrial Engineering and Management of Oklahoma State University, and without this assistance the task would have been considerably more difficult.

Yet, from a subjective standpoint, the contributions of the members of my Doctoral Committee are valued more highly than organizational support. Special remarks of gratitude and appreciation, therefore, are addressed to those individuals who have so significantly given of themselves in this endeavor. Professor Wilson J. Bentley, the chairman of my committee, had a great deal to do with this undertaking. It was he who first offered the chance to do graduate work, after my having been away from academic pursuits for many years. It was also he who consistently displayed imperturbable faith that the work could be done. For the "risk" he took several years ago in admitting me to the graduate program, and for his confidence, I am most grateful. Likewise, Professor G. T. Stevens has been a most patient and understanding thesis adviser. From the beginning he has offered encouragement, precise definition of objectives, and unlimited amounts of time in my behalf. Not only am I appreciative of these gifts, but probably more so for his fundamental understanding of and empathy for human worth and human work. It has been a real and genuine pleasure to work for, and with, Dr. Stevens.

In a similar vein, I owe debts of sincere thanks to the other members of my Doctoral Committee. Professor James E. Shamblin is responsible for challenging me, both by his keen sense of humor and his superior professional ability, to strive for a standard of excellence in the discipline of operations research. For this I am most appreciative. Professor William W. Rambo, with his incisive empirical bent, first made me aware of the potentialities of psychology as a means of investigating decision-making behavioral problems. As a result of his guidance, the field of social psychology was partially opened to me, and this offers a fertile field for future research into decision-making behavior in executive groups. This is a major contribution, and I am grateful for it. And to Professor Kent Mingo goes a great measure of thanks for long hours spent in discussion, for his encouragement, and especially for his infectious curiosity about organizational behavior, which he has conveyed so well to me.

But friends and counselors do not necessarily a dissertation make. A wife does. I want to share this milestone with my dear wife, Virginia; for, in a sense, it is as much hers as it is mine. It is she who saw to it that the environment in which I worked was cheerful, and it is she who supplied the innumerable cups of coffee on which I subsisted. But, more importantly, it is she who also supplied the encouragement that rendered minor defeats meaningless, and it is she who supplied the essential faith that conquered seeming despair at times. Quite simply, this work could not have been done without her. For all of this, and more, I am deeply grateful. And so am I grateful for the continued patience, good humor, and encouragement from our children, Linda and Lewis.

While most of these persons have, at one time or another in the

stages of preparation of this dissertation, read the manuscript and offered valuable suggestions for its improvement, none is responsible for any of my errors of omission or commission that may remain. These are solely mine.

Finally, it is my desire that this dissertation bear a dedication. While this is an uncommon act, it arises from some uncommon circumstances some years ago--circumstances that would have prevented the accomplishment of this work had not a special person intervened. Therefore, it is with deep gratitude and profound thanks that I dedicate this work as a memorial to Bernard Alan Cruvant--physician, humanitarian, and great friend--without whose assistance and labor on my behalf and in a time of need this work neither could have been undertaken nor completed.

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CHAPTER I

INTRODUCTION

The principal objective of this research is to extend the existing theory underlying solutions to the probabilistic capital budgeting problem, particularly into areas where the decision-maker does not necessarily display risk-avoiding behavior. In addition, secondary objective are (1) to correct a portion of the existing theory from the standpoint of correctness of interpretation, (2) to present a rigorous formulation of the complete non-sequential capital budgeting problem, and (3) to develop a classification scheme by which existing research can be placed into perspective, in relation to the complete problem of probabilistic capital budgeting.

Until quite recently, much of the literature (in fact, nearly all of it) on the allocation of scarece resources to investment alternatives has been concerned with deterministic models. Such models are based upon the tenet of assumed certainty concerning all the operational parameters of the models. While such an assumption may be warranted in some highly specialized situations which are repetitive or do not display significant variation in costs or incomes, the assumption of "certainty" is <u>not</u> warranted in most situations in which a capital budgeting decision is required. Since the capital budgeting problem is usually stated in terms of <u>future</u> costs and cash flows, probably the only factor that is certain is the <u>un</u>certainty attached to the parameters of the problem.

Typically, in making a decision concerning the allocation of scarce funds, the decision-maker (or, in a corporate context, the firm) must select from a list of "candidate" alternative possibilities a subset of projects to which funds will be allocated for ultimate execution. In a realistic capital budgeting decision environment, some or all of the following factors are present: (1) costs and cash flows are not known with certainty, (2) there is a limitation on the amount of investment funds available in succeeding fiscal periods for executing projects, (3) some or all of the alternative projects may be interrelated either functionally or fiscally, (4) some or all of the projects may display interrelationships within their own cash flow streams, and (5) the desirability of undertaking project alternatives depends not only on the prospective net revenues from the alternatives, but also upon the uncertainties associated with the net revenues, and more importantly, upon the attitudes of the decision-maker toward such uncertainties.

Thus, when all of these factors are assumed to be "known" with "certainty," as they are in the deterministic formulation of the capital budgeting problem, virtually all correspondence between the decision model and its real-life counterpart is lost. A closer approach to reality is attained when these factors are permitted to play their full roles and to interact with one another, as they do in the probabilistic formulation of the capital budgeting decision models. In such a formulation, the solution to the problem is based on a joint consideration of all of these factors and their interactions.

The Problem

Formulations of the probabilistic capital budgeting problem have heretofore generally been approached from one of two viewpoints. The first is based on the tenet that the capital budgeting problem can be "solved" at one point in time by considering a finite set of alternative projects or actions, and making the decision by selecting from this set a subset of projects for execution. The criterion of selection is usually some form of benefit to the decision-maker of the firm. Formulations of the capital budgeting problem in this fashion are termed non-sequential ones. The second viewpoint is based on the tenet that the problem is better solved by a sequence of related decisions over a "horizon" of time. Decision models based on this viewpoint use a Bayesian approach, in which probabilistic "states of nature" are assumed to exist which act as a priori conditions for subsequent decisions. A succession of such states and decisions over several periods of time, termed the "planning horizon," comprises the formulation. The optimum decision is that one which maximizes some expected "payoff"[1] to the decision-maker. Such formulations of the capital budgeting problem are termed sequential ones.[2]

Both of these viewpoints possess advantages and disadvantages, which will not be discussed here. The two forms of the capital budgeting problem are mentioned, however, to delineate the particular type of formulation developed here. The present research is concerned only with non-sequential decision-making, and no arguments are advanced here for the superiority of this particular viewpoint over the other, although it appears that more research has been reported in the literature on the non-sequential formulation of the problem.

Perhaps without exception, the known non-sequential formulations of the probabilistic capital budgeting problem have assumed, either tacitly or implicitly, some form of risk-averse behavior on the part of the decision-maker. That is, the known models reported in the literature rest upon an assumption that the decision-maker is averse in his attitude toward uncertainty of payoff. This is a peculiar direction for capital budgeting research to take. First, it is almost a matter of common knowledge that many businesses often succeed because of the risks they do take; and second, there is empirical evidence in the literature to show that some decision-makers are not risk-averse, at least for some combinations of payoff and uncertainty associated with payoff.

There are at least two reasons that non-sequential probabilistic capital budgeting research has followed the risk-aversion path. One is that virtually all of the models are based on the pioneering and highly imaginative work of Harry Markowitz (42), who used a risk-averse objective function in his model for the selection of optimal portfolios of marketed securities. The other reason is that the mathematics used in the formulation of the problem are considerably simpler for the risk-averse case.

Notwithstanding these reasons, the compelling consideration here is that not all decision-makers are averse to risk. Many do, indeed, seek opportunities with uncertainty attached to payoff, in accordance with a belief from time immemorial that greater payoffs are associated with situations involving uncertainty of outcome. While it is no doubt true that such situations also involve substantial losses sometimes, the risk-seeking decision-maker is not only willing to accept this possibility, but in doing so seems to mentally minimize the probability of its occurrence; whereas, the risk-avoiding decision-maker appears to be always cognizant of the effect of a probable loss. The risk-averse decision-maker always avoids a "fair bet," since he prefers certainty to uncertainty. On the other thand, the risk-seeker will often pay a premium to obtain a bet, presumably because he mentally visualizes the possibility of a larger payoff than for an outcome that is certain. It would seem, therefore, that risk-seeking — at least for some combinations of payoff and uncertainty — it not only an acceptable form of business decision-making behavior, but is a fairly typical one also. This is particularly true for some types of businesses that operate in an entrepreneurial environment, such as petroleum exploration and development, where little control can be exercised over the environmental determinants of outcome but where payoffs can be exceptional as well.

Therefore, the problem that is of immediate concern in this research is the formulation of a theory of probabilistic capital budgeting that will, to some acceptable degree, model and explain the behavior of a decision-maker who is <u>not</u> risk-averse, at least with respect to some combinations of payoff and uncertainty. Specifically, the problem is one in which all of the factors mentioned earlier are present and interacting: (1) project costs and cash flows are not know with certainty, (2) some or all of the projects may be interrelated functionally or financially, (3) some or all of the projects may have interrelationships in their respective cash flow streams, and (5) limitations in succeeding years exist in the amount of investment funds available. Additionally, the desirability of undertaking projects is assumed to depend not only upon the net revenues of the projects selected, but also upon the uncertainty attached to those net revenues and especially upon the risk-seeking attitude of the decision-maker toward certain combinations of net revenue and uncertainty. What is sought in this research is a mathematical model of this situation that can be used for predictive purposes, not from a normative standpoint of what the decision-maker ought to do, but rather from a descriptive standpoint of what he does or would do under the given circumstances.

This type of capital budgeting problem is not an uncommon one in growing companies. While it is undoubtedly true that business decisin-makers in the long-run are risk-averse in a gross sense — as is evidenced by almost continual attempts to convert uncertain situations into certain ones via vertical and horizontal integration or the exercise of control over the environmental factors affecting production and market instability — nevertheless, the potential for risk conversion arises from the very act of taking risks and finding the opportunities for exceptional payoffs during the growth period. It is only by taking risk that growing enterprises are able to generate the capacity for integrating factors of production and exercising control over their markets. But even in growing enterprises, not all risk situations are equally attractive. "He who has little to lose will risk much, but he who has much to lose will risk little" is an adage that describes the decision behavior of many growth enterprises. If, for example, the amount of investment risked in a prospective venture is small compared to, say, the total net worth of the firm then the prospect of loss (if it occurs) is not of great moment. On the other hand, if the amount risked is great in comparison to the firm's resources, then the prospect of loss assumes much more importance in the mind of the decision-maker. This is merely another way of saying that under some circumstances decision-makers seek and accept risk as a means of generating the leverage necessary for growth, yet under other circumstances (where the "stakes" become too "great") the same decision-makers will act as risk-avoiders. This type of decision behavior cannot be explained by an assumption of overall risk aversion on the part of the decision-maker, as is done in virtually all probabilistic formulations of the capital budgeting problem reported in the literature.

Thus, while existing formulations of the problem do explain and predict decision behavior for decision-makers who are risk-averse — and, by implication, this category includes firms that have reached the "age of maturity" in their growth cycle — they do not explain nor predict the types of decisions that will be made by growth firms or by decision-makers who are not risk-averse, and who are looking for opportunities for creating growth leverage. It is to the latter problem that this research is addressed.

Review of the Literature

Harvey (28) has conceptualized the "capital investment decision process" as one which includes the actions of searching for new and profitable investment opportunities, investigating the technical and financial aspects of potential projects, estimating cash flows, and selecting from the set of eligible investment opportunities the subset of feasible projects most likely to satisfy the goals of the firm. Three stages of the "decision process" are recognized: (1) the analysis of investment opportunities, (2) the specification of the firm's requirements and constraints, and (3) the selection of an optimal portfolio of executable projects. While this is an apt description of the decision process as it is actually undertaken, it is possible to combine the second and third stages into what might be called the selection problem, and thereby gain a more accurate description of how capital budgeting decision models are formulated.

To explain further, project analysis models generally have as their objective the development of a mathematical evaluative description on one project, whereas the selection model provides the criterion for selection among projects. Project analysis models generally do not contain, within their mathematical formulation, the criteria by which one project may be compared with another or with some external standard. Project selection models, on the other hand, exactly specify the selection criterion by which projects are compared with each other or with some external standard. In general, project analytic models merely formulate the information necessary for use as inputs to the selection model, which is then used to specify the firm's requirements and constraints as well as the criterion by which selection is to be made. It is necessary, therefore, to consider briefly the general form of the project selection model before proceeding with a review of the literature, so that the latter will appear in the proper contextual framework.

The Basic Selection Model

The "maximization of net present value" model has been chosen here as the basic selection model, since it relates all of the variables that are of interest in the non-sequential capital budgeting problem.[3] This model can be formulated (at least for descriptive purposes if not for computational purposes) as follows. Let Y_{tj} be the net cash inflow from project j (j = 1, 2, ..., m) at the end of year (period) t (t = 0, 1, ..., n). Let i be a discount rate for money, assumed invariate among periods. Then, the Net Present Value for the j^{th} project is

$$NPV_j = \sum_{t=0}^{n} \frac{Y_{tj}}{(1+i)^t} \quad . \tag{1}$$

Now, let B_t be the undiscounted capital expenditure limit (the "budget") in period t, and c_{tj} be the capital consumption ("cost") of project j in period t. Also, let $\theta[NPV]$ be some (as yet, undefined) function of equation (1). Then, the general non-sequential capital budgeting problem is to

$$\underset{\forall j,t}{\text{Max}} \quad \sum_{j=1}^{m} \theta\left[\sum_{t=0}^{n} \frac{Y_{tj}}{(1+i)^t} \right] x_j \ ; \quad (t = 0, 1, \ldots, n; \ j = L, \ldots, m) \tag{2}$$

subject to: (a) $$\sum_{t=0}^{n} \sum_{j=1}^{m} c_{tj} x_j \leq B_t \tag{3}$$

(b) $$\text{All } x_j = 0,1 \quad . \tag{4}$$

Equations (2), (3) and (4) comprise a mathematical programming statement of the non-sequential capital budgeting model, which in the completely general case might contain more constraints than equations (3) and (4), so as to express dependencies and contingencies among projects. However, the foregoing formulation is sufficient to define the selection problem for purposes of reviewing the existing literature. Note that the objective function, equation (2), requires the maximization of sums of functions of net present value, and that the constraint of equation (4) requires indivisibility of projects (fractional projects cannot be executed). Thus formulated, the non-sequential capital budgeting problem is solved as a special case (0/1) of the integer programming problem or by special branch-and-bound algorithms, which permit an efficient search among possible combinations of the j projects to find the combination which satisfies the objective function and the constraints. With this form of statement for the selection problem, it is possible to compare previous research, as will be outlined briefly below.

Selection Under Assumptions of Certainty

A capital budgeting decision can be said to be made under conditions of "certainty" when all possible outcomes are known for all possible alternative actions. Thus, the decision-maker would possess perfect information under these circumstances. Seldom are these conditions complied with under actual business conditions, and certainly almost never when future contingencies are the basis for judgment. However, if these conditions can be approximated without significantly altering the decision environment, then an assumption of certainty may be appropriate. That is, if the variables that appear in equations (2), (3) and (4) can be assumed to be invariate, and if it can be assumed that a sufficient number of projects, m, are subjected to investigation so as to approximate all alternative courses of action, then one is justified in assuming "certainty" and proceeding as if all possible outcomes were known. However, even under these circumstances the selection problem is far from trivial, especially if there are a large number of projects to be considered, since the number of combinations potentially subject to inspection by equation (2) is $2^m - 1$, if the combination "do nothing" (all $x_j = 0$) is ignored.

<u>The Winegartner Model</u>. The potentially large number of combinations of projects subject to maximization, in part, led Winegartner (63) to apply the 0/1 case of integer programming to the selection problem. Other (and probably more important) motivations were to correct the rate-of-return ranking solution proposed earlier by Dean (15), and to demonstrate the existence of a discrete optimum solution in answer to the problems proposed by Lorie and Savage (39).

Winegartner conceived of the non-sequential capital budgeting problem as one in which the cash flows and cash outlays of the available projects are known with certainty. Although he recognized that such assumptions are not realistic, he justified them on the grounds that additional insights into the problem could be obtained from examining an exact model formulated on a theoretical basis. Thus, he formulated a model equivalent to the non-sequential model presented in equations (2), (3) and (4), with the objective function in equation (2) being the maximization of net present value itself. That is, $\theta[NPV]$ is equal to NPV itself in Winegartner's model.

At the time (1963), this was a considerable advance in capital budgeting theory, for Winegartner's formulation disposed of Dean's incorrect conclusions in the rate-of-return ranking method, as well as solving the Lorie-Savage problems of multiple rates of return and project dependencies. However, since that time, it has come to be recognized that an assumption of maximization of net present value itself (the deterministic or "certainty" case) implies the existence of a linear utility-of-money function for the decision-maker, which might not represent the actual decision function that obtains in a specific instance. This point will be discussed more fully in Chapter IV. Nevertheless, the importance of the Winegartner model lies in its demonstration that a discrete optimum solution can be guaranteed for the objective function, equation (2), using methods of mathematical programming, and that such an optimum is not the one proposed by Dean. Further, it disposes of the Lorie-Savage problems of project dependencies and multiple rates of return.

Selection Under Uncertainty and Risk

When any of the variables in the selection problem becomes other than certain — for example, the increments comprising the cash flow stream, the project life, the investment "costs," or the money discounting rate — then the selection problem immediately becomes one involving undertainty or risk, since the concept of random variation comes into play. Another way of stating this tenet is that when imperfect information about the capital budgeting parameters is possessed, then the selection problem becomes probabilistic. Whether or not the problem involves "uncertainty" or "risk" (as differentiable semantic terms) is largely a problem in interpretation of the meaning of probability.

Many authors make a clear-cut distinction between "uncertainty" and "risk." Objective probabilists hold that if a decision leads to several outcomes, then the decision is made under risk only when all of the possible outcomes are known and the likelihoods or probabilities of each of these outcomes are known or can be estimated from a series of observable events. According to this construction of the probability concept, then, a decision under uncertainty is one in which either all of the possible outcomes are not known or the probabilities cannot be established from objective data. An example of a decision under "risk" would be one made by an insurance carrier concerning the level of premium to be charged for an insurable event, where similar events had occurred in the past a sufficient number of times to permit the construction of an actuarial mortality table.

The capital budgeting decision-maker usually has no such information on which to base an estimate of probabilities. His "events" are usually one-of-a-kind, and they lie in the future. That is, "projects" under consideration for possible execution often represent new facilities, new processes, or untried changes in operating policies. At best, the decision-maker can adduce only partially complete information concerning his alternatives. Thus, under the strict construction of the probability concept, his decisions would be characterized as being made under "uncertainty."

However, another "school" of probabilists, the subjective probabilists, maintains that if the decision-maker or some other knowledgeable person who is an expert in the field can adduce <u>subjective</u> information concerning the projects and their associated probabilities, then these probabilities can be used in the same manner as objectively determined probabilities. Raiffa (49) reports that James Bernouilli, in his <u>Ars Conjectandi</u> (1713) first formulated the subjective alternative to objective probabilities. Bernouilli suggested that probability is a "degree of confidence" (later writers state degree of "belief") that an individual <u>attaches to an uncertain event</u>, and that this degree depends upon the individual's knowledge and can vary from individual to individual. Later writers include Laplace (37) and De Morgan (17), but the formal concept of subjective probability as an operational <u>theory</u>

of action was first formulated by Ramsey (50), Raiffa reports. In Raiffa's words:[4]

> To Ramsey, probability is not the expression of a logical, rational, or necessary degree of belief, the view held by Keynes and Jeffreys, but rather an expression of a subjective degree of belief interpreted as operationally meaningful in terms of willingness to act or of overt betting behavior.

Thereafter, Raiffa states, De Finetti (16) was able to assess a person's degree of belief by observing his overt betting behavior, and by insisting on an assumption that a series of bets be internally consistent, was able to demonstrate that a person's degrees of belief — his subjective probability assignments — obey the usual laws of objective probability. Even more extensive discussions of the concepts of subjective probabilities can be found in Pratt, Raiffa and Schlaifer (48) and in Schlaifer (54). Thus, the subjective probabilist maintains that, in the absence of probabilities obtained from objective sources, then subjective estimates of the probabilities of events are preferable to no estimates at all, and may be used in the same manner as objective probabilities. Indeed, Ackoff, Gupta and Minas ((1), pp. 53-55) maintain that the decision-maker possesses more information regarding the decision environment than an assumption of uncertainty would require, merely by being able to specify the probable outcomes of a prospective action.

The research in this dissertation is based on the generality and legitimacy of subjective probabilities. From a practical and realistic point of view, the information necessary for the specification of objective probabilities is almost never possessed by decision-makers in an industrial or business context. If this research is to provide an interpretation of the behavior of decision-makers who are not necessarily risk-averse, then it must do so in terms of the data they are able to provide; namely, subjective probability estimates. For this purpose, the terms "risk" and "uncertainty" are considered to be synonymous, with "uncertainty" being preferred to describe incompletely known outcomes and their associated subjective probabilities, and "risk" to describe a belief or attitude on the part of the decision-maker. Thus, a "risk-seeker" is a person who seeks opportunities for investment in "uncertain" projects (he believes that uncertainty provides an opportunity for greater payoffs), and a "risk-avoider" is one who avoids opportunities for investment in "uncertain" projects or, alternatively, requires a premium in the expected payoff to compensate for the assumption of risk.

Von Neumann-Morgenstern Utility Measure

Completely apart from the subjective probabilists, and almost incidentally as an adjunct to their monumental work *Theory of Games and Economic Behavior*, John von Neumann and Oskar Morgenstern (46) presented a series of axioms of rational behavior, the object of which is to assign a numerical measure (utility) to the worth of monetary payoffs. It is not necessary here to go into detail concerning the von Neumann-Morgenstern theory (since this will be dealt with in more detail in Chapter IV), but in essence, this theory permits a numerical measure of worth or utility to be attached to monetary payoffs which have varying degrees of uncertainty. By a series of alternative "gambles" a utility function for a particular decision-maker can be defined, the effect of which is to specify the subjective preferences the decision-maker has for varying degrees of payoff and risk.

The von Neumann-Morgenstern utility index should be carefully distinguished from the concept of cardinal utility enunciated by Marshall (43) and the concept of ordinal utility enunciated by Hicks and Allen (32). Mao (41) makes this disctinction nicely in these words:[5]

> To Marshall and other cardinal utility theorists, utility was a psychic quantity measurable and quantifiable just as one's body temperature and weight are measurable and quantifiable. Thus a person was supposed to be able to feel that one banana gave him x units of satisfaction and each additional banana gave him successively fewer units of satisfaction. In fact, Marshall derived the negative slope of the consumer demand curve from the law of diminishing marginal utility. This cardinal, hedonistic interpretation of utility gave way to the ordinal behavioristic interpretation of utility during the mid-1930's. In 1934 Allen and Hicks constructed a theory of consumer behavior without assuming that utility was a measurable quantity. They based their theory merely on the assumption that a consumer had a scale of preferences on which he ranked the desirability of different collections of goods.
>
> . . . .
>
> The fact that the N-M utility is measurable distinguishes it from the ordinal utility of Allen and Hicks. But, although it is measurable, the N-M utility has little in common with the cardinal utility of Marshall, since whereas Marshall's concept is a psychological quantity for measuring pleasure and pain, the N-M concept is a numerical index for evaluating risky transactions.

What is measured by the utility function of von Neumann-Morgenstern, then, is simply a preference for or an aversion to risk in connection with monetary payoffs.

Different persons exhibit different risk attitudes, some being risk-seeking and many risk-avoiding. Almost none are indifferent to risk. The usefulness of the von Neumann-Morgenstern utility function lies in its ability to distinguish persons on the basis of their risk attitudes. The utility function, when evaluated numerically, is convex (or, concave upward) for risk-seekers, concave downward for risk-avoiders, and linear for risk-indifferents.

The question arises as to how "risk" is quantified. The answer lies in some mathematical manipulations of the utility function and its argument, the payoff or net present value of a project. Since, for the probabilistic case, net present value is a random variable, then it will have a mean and variance (and possibly higher moments also). Considering these parameters, it can be shown (Chapter IV) that the <u>expected</u> utility of the net present value, $E[U(NPV)]$, is a function of the <u>mean</u> net present value and its variance (and possibly higher moments), thus:

$$E[U(NPV)] = A(\mu) \pm B(\sigma^2 + \mu^2) \quad (5)$$

if the utility function is expressible as a quadratic of the form $U(X) = AX \pm BX^2$; or as

$$E[U(NPV)] = A(\mu) \pm B\sigma^2 \quad (6)$$

if the utility function is expressible as an exponential function of the form $U(X) = 1 - e^{f(X)}$; all where

A, B = constants

σ^2 = variance of the NPV distribution

e = natural logarithm base

$f(X)$ = $-(aX \pm b)$, a negative linear function of X.

Further, it can be shown (Chapter IV) that the rate of change of expected utility with respect to variance, $dE[U(NPV)]/d\sigma^2$, is negative for risk-avoiders, positive for risk-seekers, and zero for risk-indifferents. Thus, it is inferred that the net present value variance (or, alternatively, the standard deviation) is a measure of risk associated with the net present value itself. Projects with large net present value variances are said to be "risky," whereas, those with small (or zero) variances are said to be relatively "safe." Obviously, if the net present value variance is zero, then the net present value itself becomes "assumed certain," and one has the exact equivalent of Winegartner's certainty model described above. This accounts for the earlier statement that Winegartner's model implies the assumption of a linear utility function for his decision-maker.

To sum up, subjective probabilities permit the construction of net present value functions for projects when all of the possible outcomes (net present values) are not known and when all that is available is expert subjective judgment about cash flows, costs, lives, and the other component elements of the cash flow stream. The von Neumann-Morgenstern utility theory permits a decision-maker's preference for or avoidance of risk to be stated in such a manner that his attitude toward risk (the variance of the project net present value) can be taken into account in the project selection problem. Subjective probabilities are at the heart of the analysis problem, and the utility function is at the heart of the selection problem.

The Markowitz Model of Security Portfolio Selection

The precursor of all modern models of capital budgeting under risk conditions is the security selection model devised by Markowitz (42). Markowitz was interested in an explanation for the phenomenon of diversification practiced by investors in securities (stocks, bonds, debentures, etc.). Diversification has long been practiced in order to lessen the risk of loss, but a satisfactory explanation cannot be found if an investor simply maximizes net present value. That is, if an investor were to allocate his funds solely on the basis of maximization of net present value, then he would invest all of his funds in the security which he expected would grant him the largest return. Quite evidently, then, either the practice of diversification to lessen overall risk is an irrational one, or the criterion of maximization of net present value is erroneous.

In a tightly reasoned sequence beginning with the von Neumann-Morgenstern axioms of rational behavior, Markowitz deduced an "expectation-variance" criterion function of the form $E[U(R)] = \mu - B\sigma^2$ (similar to equation (6)), where $E[U(R)]$ is the expected utility of the return from a given portfolio of investments, μ is the mean return of the portfolio, σ^2 is the variance of the portfolio returns, and A is a coefficient called the "coefficient of risk aversion," which states the extent to which the decision-maker is averse to the risk as measured by the variance. While this form of criterion function is recognized (later) as derivable from the parent utility function of the form $U(R) = 1 - e^{-aR}$, Markowitz was careful <u>not</u> to specify any particular form of the utility function and thus avoided the error which Farrar (19) committed later.[6] When the criterion function is in the form of equation (6), it is easy to verify that for any given level of variance a portfolio with large mean return will have greater expected utility than one with a smaller mean return, and that for any given level of mean return, a portfolio with greater variance (risk) will have a lower expected utility than one with lesser variance, if the sign in the expression is negative (indicating a risk-avoiding utility function). Maximization of the "expectation-variance" function leads to the selection of what Markowitz calls an "efficient" portfolio which has maximum expected return for any given level of variance (risk) or minimum variance (risk) for any given level of expected return. Thus, an efficient portfolio dominates all others with inferior combinations of mean and variance, and by successively solving the maximization (selection) problem for various coefficients of risk aversion, A, all the feasible sets of efficient portfolios are obtained.

The Markowitz model is written as a quadratic programming problem, thus:

$$\max_{\forall j} \; E[U(R)] = \mu - A\sigma^2 = \sum_{j=1}^{N} \mu_j x_j - \sum_{j=1}^{N} \sum_{k=1}^{N} \sigma_{j,k} x_j x_k \qquad (7)$$

subject to

$$\text{(a)} \qquad \sum_{j=1}^{N} x_j = 1 \; ; \qquad (8)$$

$$\text{(b)} \qquad \text{All } x_j \geq 0 \; ; \; (j = 1, 2, \ldots, N) \qquad (9)$$

where N = the number of securities to which funds could be allocated;

x_j = the proportion of available funds invested in the j^{th} security;

μ_j = expected return for the j^{th} security;

$\sigma_{j,k}$ = covariance between the returns of securities j and k if $j \neq k$; otherwise, the variance σ_j^2 of security j if $j = k$.

Markowitz solved this quadratic programming model to obtain the feasible sets of efficient portfolios, but how he did it is not the important point here, since more efficient solution methods have been devised since the publication of his report (1959). What is important, however, is that the Markowitz formulation, using the "expectation-variance" selection criterion, _is_ able to explain the phenomenon of diversification of investments to reduce risk, particularly when the securities comprising the portfolio are both negatively and positively correlated in their mean returns. That is, a portfolio of investments, in which the expected returns are (particularly) negatively correlated, will display _less_ variance overall than a portfolio otherwise constituted. This can be seen by expressing the portfolio variance in terms of the constituent security variances and covariances:

$$\sigma^2 = \sum_{j=1}^{N} \sigma_j^2 x_j + 2 \sum_{\substack{j=1 \\ j<k}}^{N} \sum_{k=1}^{N} \sigma_{j,k} \, x_j x_k \quad . \tag{10}$$

Thus, when the security covariance, $\sigma_{j,k}$, is negative the overall portfolio variance is reduced, and when it is positive the portfolio variance is increased. A decision-maker with large coefficient of risk aversion, A, therefore, will tend to select portfolios with small variance — those composed of securities whose mean returns are negatively correlated and/or composed of securities with small independent variances, such as high-grade bonds — and decision-makers with small coefficients of risk aversion will tend to pick portfolios with larger variances.

A second conclusion emphasized by Markowitz is that the model is _not_ a normative one. It does not specify what a decision-maker _ought_ to do in selecting a portfolio of securities; it merely specifies what he would do if: (1) he possesses a risk-avoiding criterion function of the form $E[U(R)] = \mu - A\sigma^2$; (2) he has a fixed coefficient of risk aversion, **A**, and (3) the securities available for investment are correlated among their returns.

While the Markowitz model successfully explains investor behavior in terms of risk attitudes and quantified measures of uncertainty, it is not directly applicable to the capital budgeting problem. In the Markowitz formulation of the investment problem, all, or none, or any portion of the available funds may be allocated to a security, in accordance with equation (8). In the capital budgeting problem, this is equivalent to permitting fractional parts of projects to be executed, which is (frequently) an impossibility. Hence, the Markowitz formulation does not solve the special 0/1 case of the mathematical programming problem that is unique to the capital budgeting formulation. The importance of the Markowitz research, however, lies in the fact that it first demonstrated that the maximization of net present value is an insufficient criterion for selection when risk or uncertainty is a factor.

The Farrar Model

Subsequent to the Markowitz research, Farrar (19) subjected the Markowitz hypotheses (concerning the "expectation-variance" selection criterion) to an empirical test. Using data obtained from actual portfolios of investments held by 23 mutual investment "funds," Farrar demonstrated that the "funds" could be distinguished in their risk attitudes by the types of risk portfolios that they held. That is, on the basis of the variances of the portfolio investments, "funds" were determined to be "risk-avoiders" with larger or smaller coefficients of risk aversion. Since Farrar used as a selection criterion the Markowitz expected utility function, $E[U(R)] = \mu - A\sigma^2$, the empirical test by Farrar confirmed the Markowitz hypotheses that maximization of net present value is not sufficient when risk is a consideration.

For the present purposes, however, this is not the important result of the Farrar work. The thing about the Farrar research that is important here is that Farrar assumed the existence of a quadratic utility function of the form $U(R) = AR - BR^2$, and then errouneously derived a form of the expected utility, $E[U(R)] = \mu - B\sigma^2$, from it. Now, in and of itself, this form of expected utility is not necessarily wrong when used as a selection criterion function; it is, however, grossly wrong IF the assumption is made that it comes from a parent utility function of the quadratic form assumed by Farrar (this will be demonstrated in Chapter IV). Through an unfortunate series of coincidences, the Farrar error was not discovered until 1967, six years after Farrar's research was first published in book form (19). In the meantime, the Farrar report (a Ford Foundation Doctoral Dissertation Award-Winning Publication) went out of print, and at least two other research works in the field of capital budgeting were done, using the erroneous form of the Farrar derivation as the basis of the work. Since these two research works (one by Harvey (28) and the other by Watters (62)), however, a second printing of the Farrar dissertation by another publisher has been made available, in which Farrar corrects, or at least notes, the error by footnote. At the present writing, however, both the Harvey and the Watters researches remain uncorrected, and the task of correcting these two works will be undertaken in Chapter IV.

Summary of Chapter I

The capital budgeting decision is concerned with the allocation of scarce resources (funds) by the "firm" (the decision-maker) to alternative opportunities for productive investment (projects) which, it is hoped and planned, will generate a series of annual returns (cash inflows) over a finite length of time, such that the total net present value to the firm, as measured by the investments, cash flows, and the time value of money, will be maximized. This problem has been solved for the deterministic case in which all of the input variables are assumed to be "known with certainty."

For the probabilistic case, in which some of the input variables are random variables, previous research indicates that the criterion of maximization of net present value is an inadequate one. Prior research based on decision-making behavior in the purchase of securities portfolios indicates that the risk attitudes of the decision-maker must be considered as well. A measure of worth (utility) to the decision-maker of various combinations of return and risk associated with return can be determined by employing the von Neumann-Morgenstern utility theory. Based on the von Neumann-Morgenstern utility function thus determined, additional selection criteria can be specified.

Because the elements of the project cash flow streams (income, expense, etc.), which generate net present value, are fixed only by <u>future</u> events when they happen, the values of those elements used for estimation and project selection are random variables and give rise to random variation in net present value. The use of <u>subjective</u>

probabilities is postulated in order to approximate the distributions of project net present values. The generation of a project net present value (a random variable) and its distribution, by the use of subjective probabilities and estimates of the cash flow increments, is termed the analysis problem in probabilistic capital budgeting. Selection of a feasible subset of projects from a larger set of "candidate" projects, such that the decision-maker's expected utility is maximized according to come selection criterion function (obtained from the utility function and expressed in terms of the parameters of the net present value distribution), is termed the selection problem in probabilistic capital budgeting.

Virtually all prior research on this topic has assumed that the decision-maker is risk-averse — that is, a risk-averse form of the decision-maker's utility function has been used to obtain the selection criterion function. Empirical data and direct observation indicate that not all decision-makers are risk-averse, particularly in certain types of industry and in "growth" companies.

The principal problem attacked by the present research is the one presented by the non-risk-averse decision-maker. The objective is to provide a theoretical solution to the capital budgeting problem under conditions of uncertainty, where the decision-maker is not risk-averse, at least for some combinations of return and uncertainty. Secondary objectives are to correct an interpretational error in prior research that has affected solutions to the problem up to the present time, to present a rigorous statement of the probabilistic capital budgeting problem, and to provide a classification scheme by which prior research can be integrated into an overall framework.

FOOTNOTES

[1]To avoid confusion, the term "payoff" in this research means simply some return of value to the decision-maker. It is not to be confused with "payback" or "payout," which are terms sometimes used to describe certain methods of project selection (e.g., by computing the "Payback" period).

[2]For a discussion of the formulation of a sequential probabilistic capital budgeting model, see Massé (44).

[3]The internal rate of return model, if properly formulated to compute the rates on an incremental basis, could be used with equal validity (see, for example, Fleischer (22)). In using the IRR model, however, multiple and indeterminate rates must be detected and properly handled. Moreover, the NPV model is superior in other respects: (1) it is easier to handle computationally, (2) when budget or logical constraints are "tight," it leads to shorter solutions, and (3) for the deterministic case it recognizes explicitly a rational economic goal of the firm, namely, maximization of the net present value of stockholders' equity.

[4]Raiffa (49), p. 274.

[5]Mao (41), pp. 53-54.

[6]See page 23.

CHAPTER II

THE GENERAL NON-SEQUENTIAL PROBABILISTIC CAPITAL BUDGETING MODEL

The selection of a subset of projects from a larger list of possible alternatives must be done in accordance with some selection criterion. For the deterministic case, Dean (15) advocates that the selection procedure be done by computing the internal rate of return for each project, where the internal rate of return is that value of i which makes the following identity true:

$$\sum_{t=0}^{n} \frac{Y_{tj}}{(1+i)^t} - \sum_{t=0}^{n} \frac{C_{tj}}{(1+i)^t} = 0 \qquad (11)$$

and then rank the possible alternative projects in descending order of internal rate of return. Selection, according to Dean, is accomplished by selecting projects in order of descending rates of return, stopping the process when the available amounts of capital for investment (the "budgets") have been exhausted. Dean's criterion, then, is one of maximization of internal rates of return of individual projects.

When Dean published this selection procedure in 1951, the emphasis in capital budgeting research was focused on the deterministic case and a normative selection procedure. Thus, Dean was advocating (in effect) that the firm should maximize internal rate of return subject to funds availability, as a representative method of solving the capital budgeting problem. Thereafter, one of Dean's associates, James Lorie, together with L. J. Savage, show that if a firm had as its objective the maximization of net present value of stockholders' equity (a rational economic goal), then a <u>different</u> set of projects will be selected than those selected by the Dean ranking procedure (Lorie and Savage (39)). Moreover, Lorie and Savage showed that it is possible, under certain conditions of cash flow and cost, for a project to have more than one internal rate of return -- what is now known as the "multiple rate of return case" -- which further complicates the interpretation of the Dean ranking procedure. Thus, the battle was joined over the proper normative selection criterion to use: the Dean ranking method, or the Lorie-Savage maximization of net present value.

It remained for Fleischer (22) to show that if the ranking procedure were done on an <u>incremental</u> basis -- that is, where internal rates of return are computed incrementally on additional cash flows and additional investments -- then the two procedures result in the selection of identical projects when capital rationing is involved. It also remained for Winegartner (63) to show that the selection problem can be solved by mathematical programming, using as an objective function the maximization of net present value, thus leading to an optimum subset of projects. The search for a normative selection criterion, in the deterministic case, seems to have been won by those who would "maximize net present value"; and their case rests upon three well-founded reasons: (1) this criterion is one which expresses a rational economic goal of the firm, (2) it obviates the necessity to resolve the multiple rate-of-return problem, and (3) it permits the selection of a discrete and unambiguous optimum subset of projects (by the Winegartner

0/1 mathematical programming procedure) that will, in fact, maximize the net present value of the projects to the firm. Its weakness, however, lies in the necessity for one to know in advance what time value of money (the discounting rate) will be used to calculate project net present values, as the optimum subset of projects will change depending upon this rate.

For the probabilistic capital budgeting problem, the search for a normative criterion has not fared so well. In the related securities market problem, Markowitz (42) demonstrated that the maximization of net present value is not a sufficient criterion; indeed, to ignore the uncertainty aspects of net present value is to reduce the problem to the deterministic case, which does not satisfactorily explain the "hedging" behavior of investors who diversify their investments in order to avoid excessive risk. What emerges from the Markowitz research, therefore, is not a normative criterion which prescribes what an investor <u>ought</u> to do, but rather a descriptive criterion which more accurately describes what he <u>would</u> do when uncertainty is a factor in his decision-making. Thus, according to the Markowitz conceptualization, there is no "best" decision for all firms; merely one that is appropriate for a given decision-maker when his risk preferences are taken into account. Virtually all of the subsequent research, even in applications to the capital budgeting problem where projects are assumed to be indivisible, have followed Markowitz' lead in using descriptive selection criteria, rather than attempting to formulate normative criteria. Moreover, in the final analysis, it may not be conceptually correct to specify a normative criterion for all firms.

The present research, therefore, follows (somewhat) the lead of prior investigators in specifying descriptive selection criteria rather than normative criteria. As will be evident from the development in Chapter IV, selection criteria that depend upon the von Neumann-Morgenstern utility function require a knowledge of the random net present value and its distribution for each project and combination of projects, from which means, variances, skewnesses and fourth moments can be ascertained. Since these statistics all depend upon the random variate, net present value, it is required here only that the selection criterion be some unspecified function of net present value. With these introductory remarks, the project selection model can be formulated for descriptive purposes, if not for computational purposes, as follows.

Let Y_{tj} be the net cash inflow from project j (j = 1, 2, ..., m) at the end of period t (t = 0, 1, ..., n). Let i be a discount rate for money, assumed to be invariate among periods. Then, when Y_{tj} is a random variable, the random Net Present Value (NPV) for the j^{th} project is

$$NPV_j = \sum_{t=0}^{n} \frac{Y_{tj}}{(1+i)^t} . \tag{12}$$

Now, let B_t be the undiscounted capital expenditure limit (the "Budget") in period t, and c_{tj} be the capital consumption ("cost") of project j in period t. Also let $\theta[NPV_j]$ be some (as yet, undefined) function of (12).

Then, the general non-sequential capital budgeting problem is to

$$\underset{\forall j,t}{\text{Max}} \quad \sum_{j=1}^{m} \theta \left[\sum_{t=0}^{n} \frac{Y_{tj}}{(1+i)^t} \right] x_j \ ; \ (t = 0,1,\ldots, n; \ j = 1,\ldots, m) \qquad (13)$$

subject to:

$$\text{(a)} \quad \sum_{t=0}^{n} \sum_{j=1}^{m} c_{tj} x_j \leq B_t \qquad (14)$$

$$\text{(b)} \quad c_{tj} \geq 0 \qquad (15)$$

$$\text{(c)} \quad x_{k1} + x_{k2} + \ldots + x_{ka} \leq 1 \ ; \ (k \in j; \ a \leq m-1) \qquad (16)$$

$$\text{(d)} \quad \left.\begin{array}{l} x_{k1} - x_{k2} \leq 0 \\ \vdots \qquad \vdots \\ x_{ka} - x_{km} \leq 0 \end{array}\right\} ; \ (k1, k2, \ldots \in j) \qquad (17)$$

$$\text{(e)} \quad \left.\begin{array}{l} x_{k1} - x_{k2} = 0 \\ \vdots \qquad \vdots \\ x_{ka} - x_{km} = 0 \end{array}\right\} ; \ (k1, k2, \ldots \in j) \qquad (18)$$

$$\text{(f)} \quad \left.\begin{array}{l} x_d + x_{de} \leq 1 \\ x_e + x_{de} \leq 1 \end{array}\right\} ; \ (d \in j; \ e \in j; \ d \neq e) \qquad (19)$$

$$\text{(g)} \quad \text{All } x_j = 0,1 \ . \qquad (20)$$

Thus, the problem is one of selection of a subset of projects from all candidate projects such that some function of the Net Present Value to the firm is maximized, in accordance with (13), but also such that none of the feasibility constraints (14) through (20) is violated. Further, by constraint (20), indivisibility of projects is required. That is, only project entities may be selected or rejected; portions of candidate projects cannot be executed.

Constraints

The first set of constraints of interest in the selection problem is represented by inequality (14), which expresses the limitations of capital availability for project development. These are the budget constraints for each of the t periods in which capital availability is limited. A corollary of the budget limitation is that all project capital consumptions ("costs"), c_{tj}, be zero or positive by the inequality in (15).

The relationships expressed by constraints (16) through (19) are termed the "technical" (or "logical") dependency constraints. Inequality (16), for example, expresses a mutual exclusivity requirement among projects included in it. This constraint is written when several projects provide alternative ways of accomplishing the same end. As an example, suppose (in a series of 25 projects) that Projects 9, 16 and 22 are alternative ways, say, of developing a single tract of land. Only one can be done. Therefore, this constraint would be $x_9 + x_{16} + x_{22} \leq 1$, which permits only one of the three x_j ($j = 9, 16, 22$) to take on a value of one. Note, however, that all three x_j may take on a value of zero and still satisfy the constraint. Thus, the mutual exclusivity constraint permits at most one, or alternatively none, of the constrained projects to enter the optimal solution to the problem.

Inequality (17) expresses a conditional dependency between two projects. A project is said to be conditionally dependent on another when its execution, although optional in itself, is operationally or functionally dependent on the execution of the other. As an example, take two projects numbered 8 and 7. If Project 8 were an optional project (that is, it could or could not be done), but if selected it could be executed if and only if Project 7 were also executed, then Project 8 is said to be conditionally dependent upon Project 7. Such dependencies are formulated in the form of (17). For the example given, the constraint would be $x_8 - x_7 \leq 0$. Thus, if Project 7 were accepted ($x_7 = 1$) then Project 8 could either be accepted ($x_8 = 1$) or rejected ($x_8 = 0$); however, if Project 7 were rejected ($x_7 = 0$) then Project 8 also would have to be rejected ($x_8 = 0$) in order to satisfy the constraint.

Constraints in the form of equation (18) express strict conjunctional dependencies between projects. Two projects are strictly dependent if the accomplishment of one also requires the accomplishment of the other. An example would be the situation where one project, if executed, would increase the productive capacity of a particular process which, in turn, would require the increase in a preceding raw-material supply capability at another location.

Constraints in the form of the two inequalities of (19) are designed to express covariant (or, disjunctional) dependencies. Suppose, for example, Project "d" can be executed alone, or Project "e" can be executed alone, or both "d" and "e" can be combined as one project with concommitant savings in first cost or increase in profitability. Disjunctional projects display covariation in either costs or cash inflows; hence, the adjective covariant. The existence of any two such projects is recognized by a set of two constraints in the form of (19). In effect, a new project "de" is created for the covariant possibility, which is then treated simply as a mutually exclusive case along with "d" in one constraint and "e" in another.

Thus formulated, the non-sequential capital budgeting problem is solved as a special case (0/1) of the integer programming problem. For simple problems, a manual solution may be possible. For large problems, however, the digital computer offers a practical (and perhaps only) method of finding the discrete optimum set required by expression (13) and the several constraints.

Deterministic Variant of the Basic Model

Much of the published literature deals with the formulation and solution of the "certainty" or "expected value" model. While this special case is not of major interest here, nevertheless, it is important to review the assumptions that are made when "certainty" is the mode of formulation. Typically, each of the variables in the basic model is either tacitly (or perhaps explicitly) assumed to be "known with certainty." That is, even though one is working with parameter values that all lie in the future (and therefore unknown, or at least uncertain), one assumes that these parameter values are known. Alternative to "certainty," these values are sometimes assumed to be random variables with means, but with zero variances. This is merely an exercise is semantics, as the latter view is exactly equivalent to the certainty assumption.

The assumption of certainty is not harmful *per se*, if one is careful to delimit the interpretations of the solution results. What *is* harmful, however, is a practice that has been noted in the literature in some instances; namely, to "jump" from a probabilistic formulation of the analysis problem to a deterministic formulation of the selection problem. Moreover, some of the models that have appeared in the literature are based on assumptions of technical independence among projects (constraints (16) through (19) are omitted), although the authors do not always make this fact explicit. Budget constraints also are sometimes omitted without qualification, thereby considerably simplifying the problem and removing it far from reality. But perhaps the most serious error of commission, from the standpoint of uncertainty and probabilistic modeling, is the unstated assumption by some authors that the objective function to be maximized, $\theta[NPV_j]$, is equal to NPV_j. The implications of such an assumption are not immediately obvious. When such an assumption is made for probabilistic models, even implicitly, it is tantamount to an assumption of a linear utility function for the decision-maker, which may not faithfully represent the condition being modeled. The burden of this paragraph is essentially a plea for consistency and explicit disclosure of assumptions when dealing with probabilistic modeling of the capital budgeting problem.

Therefore, one of the assumptions that must be carried over from the certainty case to the probabilistic case is made explicit here. This assumption is that the firm's discount rate for money be "known." If the assumed discount rate is approximately a default-free rate, then not much of a problem exists. But if the discount rate is the "cost of capital" (as many authors urge), the solution to the project selection problem depends upon the value of the cost of capital, and this is determined by a solution of the financing problem. However, in the probabilistic case, the financing problem is itself dependent upon a solution to the project selection problem, and so forth. Thus, the two problems are highly interrelated, and the present "state of the art" does not permit their simultaneous solution. One must assume, therefore, that the discount rate for money is "known" for the probabilistic models that follow.

Probabilistic Variants

With the exception of the discounting rate, any of the remaining variables in the capital budgeting problem may be looked upon as a random variable with a mean, variance, and third and fourth moments. For example, the cash flow stream for a project may be assumed to be composed of variables that are randomly valued, such as price, output level, fixed costs, depreciation and tax rate. Similarly, the life of a project and its capital consumption may be assumed to be random variables. Assumptions such as these lead to different special cases of the general probabilistic capital budgeting model, some of which have been reported in the literature. Most of the special cases can be distinguished on the basis of the dependence-independence assumptions that are made. In the selection problem, for example, if projects are technically independent then constraints (16) through (19) are not applicable. Furthermore, probabilistic assumptions lead to a form of the selection criterion that takes into account the risk attitudes of the decision-maker. For these reasons the formulation of the probabilistic capital budgeting problem is considerably more complex and comprehensive than the simple deterministic case, but the general formulation is still a maximization of some function of the random net present values of the projects, subject to the necessary budgetary and technical constraints.

Because the project selection problem embodies a random variable criterion function, a question arises when the problem is constrained by deterministic budget constraints. As Mao ((41), page 290) points out, such a procedure may not be consistent. Also, it is not necessarily meaningful. However, this practice is currently followed in probabilistic capital budgeting formulations, simply because the theory of chance-constrained programming is still in its infancy.[1] For this reason when budget constraints are used or implied in this research, they will be assumed to be deterministic. This assumption permits currently available computer optimization techniques to be used to solve most of the probabilistic selection problems.

FOOTNOTE

[1]For an example of a theoretical formulation of a chance-constrained capital budgeting problem, see R. Byrne et al (9).

CHAPTER III

PROJECT ANALYSIS MODELS

While the project selection problem is concerned with the determination of an optimum subset of projects that will maximize some function of the random net present values of the included projects, the project analysis problem must be solved before the selection problem can be formulated. Recalling that the analysis problem is concerned with the formulation of a model for synthesizing data about one project (for inclusion as an input to the selection model), then there are additional opportunities for dependence-independence relationships other than those that may occur among projects.

The additional opportunities for dependence relationships occur within the cash flow stream for a particular project, and may exist whether or not there are dependencies (of any kind) between or among projects. For example, the cash flow stream for a particular project may be a linear combination of independent random cash flow increments, in which case the random Net Present Value for the project is a sum with simply computed mean and variance. However, if the periodic cash flow increments are covariant time-wise (among periods), then the time-independence assumption must be abandoned even though there may be technical independence among projects, which may be retained. Thus, special-case solutions to the analysis problem can be distinguished (approximately) by the types of dependence-independence assumptions made within the project cash flow stream, apart from any dependence-independence assumptions that may exist in the selection problem among projects. The probabilistic analysis models that are analyzed below, and in some cases extended, are all formulations of special cases of the analysis problem. They are inputs to the selection problem and thus one need not be concerned at this time with either technical dependencies among projects or budget constraints. This point needs to be understood clearly, to guard against incorrect interpretation of the project NPV models. Thus, in the analysis problem, one is concerned principally with the cash flow stream and the types of dependencies that can occur within it, and not with the dependencies that can occur between or among projects.

Cash Flow Stream Relationships

Under this heading, four dependency cases can be identified when any of the variables comprising the cash flow stream of one or more projects becomes a random variable. These four cases arise in the following manner. Assume that each periodic cash flow increment, Y_{tj}, is composed of several elements (such as price, volume, cost, and so forth), which combine in a linear or nonlinear manner to give the period cash flow increment. Then, if some of the unlike elements[1] are correlated within a particular period, there is a functional dependency among the elements which must be accounted for in the NPV formulation. Also, if some of the like elements are correlated among periods (across time), then there is a time (or period) dependency that must be accounted for. Thus, the four cases correspond to the four combinations of the two types of dependency described.

Case I -- Functional Independence, Time Independence

This is the simplest of the probabilistic analytic models, and was originated by Hillier (33, 34).[2] With minor modifications of Hillier's notation, the model can be formulated as follows. Let Y_{tj} be the net cash flow increment in year t (t = 0, 1, ..., n) for project j (j = 1, 2, ..., m); further, let each Y_{tj} be a random variable with mean $\overline{Y}_{tj}$ and variance σ^2_{tj}. Then, the Net Present Value for project j is also a random variable whose mean and variance are denoted by $\overline{PV}_j$ and $\sigma^2_{PV_j}$, respectively. The mean of the project net present value is then simply

$$E[NPV_j] = \overline{PV}_j = \sum_{t=0}^{n} \frac{Y_{tj}}{(1+i)^t} \quad . \tag{21}$$

The variance of the project net present value is found as follows. Let $\hat{\sigma}^2_{Y_{tj}}$ be a present-time (t = 0) unbiased estimate of the distributed cash flow variance in any year t, and let v be the sample size of the unbiased estimator. Then, if Y'_{tj} and $\overline{Y}'_{tj}$ are the discounted values of Y_{tj} and $\overline{Y}_{tj}$, respectively:

$$\hat{\sigma}^2_{Y_{tj}} = \frac{\sum_v (Y'_{tj} - \overline{Y}'_{tj})^2}{v - 1} = \frac{\sum_v \left[\frac{Y_{tj}}{(1+i)^t} - \frac{\overline{Y}_{tj}}{(1+i)^t}\right]^2}{v - 1}$$

$$= \frac{1}{(1+i)^{2t}} \left[\frac{\sum_v (Y_{tj} - \overline{Y}_{tj})^2}{v - 1}\right]$$

$$= \frac{1}{(1+i)^{2t}} \sigma^2_{tj} \tag{22}$$

and, if the Y_{tj} are independent among years, then the variance of the project Net Present Value is

$$V[NPV_j] = \sigma^2_{PV_j} = \sum_{t=0}^{n} \hat{\sigma}^2_{Y_{tj}} = \sum_{t=0}^{n} (1 + i)^{-2t} \sigma^2_{tj} \tag{23}$$

Three observations should be made about the use of this model. First, Hillier (33) states that, even though NPV_j (in the present notation) is a random variable, if the expected value of the net present value is greater than zero "... the investment would be made since this would increase the total wealth of the firm ...". This statement is not factually true without further qualification. It is true only if the objective function in the selection problem (expression (13)) were to maximize <u>expected net present value</u>. If the objective function were other than this, then consideration of the project expected net present value is not sufficient. For example, suppose that the selection criterion in expression (13) were defined to be $\theta[NPV_j] = E[NPV_j] - A\sigma^2_{PV_j}$, where A is a constant. Then, omission of the variance term (as Hillier's statement might imply) would, in effect, convert the selection problem into the certainty model, with a concommitant (implicit) assumption of a <u>linear</u> utility function for the decision-maker; something not at all implied by $E[NPV_j] - A\sigma^2_{PV_j}$. This is an example of "jumping" from a probabilistic analysis problem to a deterministic selection problem. More to the point -- and this applies to <u>all</u> cases of non-sequential probabilistic analysis -- the project <u>selection</u> procedure can be performed <u>only after the form of the function</u> $\theta[NPV_j]$ <u>is specified</u>.

The second observation concerning Case I problems follows from the first. If $\theta[NPV_j]$ is not specified as to form, then project j cannot be selected or rejected by comparison with any external criterion or with any other project. The most that can be said is that its net present value is a random variable with a mean and variance. From these two statistics, however, probability statements can be made concerning the project Net Present Value. As Hillier states, if the Y_{tj} are normally distributed, then the Net Present Value is also normally distributed, thus leading to an ability to make probability statements about values of NPV_j. Or, if the distribution of the Y_{tj} is not normal, then certain weak probability statements can be made by Chebyshev's inequality, or if the Y_{tj} are known to be unimodal with mode equal to the mean, then somewhat stronger statements can be made by the Camp-Meidell inequality.

The third observation concerns the composition of the periodic cash flow increments. Note not only that these increments may be synthesized from component elements (price, volume, etc.) such that $Y_{tj} = \sum_p(X_{tjp})$ -- where X_{tjp} is the p^{th} element, $p = 1, 2, \ldots$) -- but also that all X_{tjp} are required to be statistically independent in both time and across function, in order for the Y_{tj} to be independent, as the model requires.

For a numerical example of Case I, see Problem 1 in Appendix A.

Case II - Functional Independence, Time Dependence

This model also was first proposed by Hiller (33), who considered two cases: (a) one in which the periodic cash flows, Y_{tj}, are perfectly correlated among periods, and (b) one in which the cash flows are partially correlated among periods. Case (a) is a special case of (b).

The formulation for (b) is given here, with the understanding that the parameters for (a) can be derived directly from the results of (b) by setting some of the terms in equations (26) and (30) to zero. In case (b), to accomplish the partial correlation the Y_{tj} are separated into two vectors, $[X_{tj}]$ and $[W_{1j}^{(1)}, W_{1j}^{(2)}, \ldots, W_{tj}^{(p)}]$, that are, respectively, mutually independent and perfectly correlated. The cash flow increment for any period t is the random variable:

$$Y_{tj} = [X_{tj}] + [W_{tj}^{(1)} + W_{tj}^{(2)} + \ldots + W_{tj}^{(p)}] \; . \qquad (24)$$

Omitting the subscript j for clarity but understanding that the formulation applies to a single project, the expected net present value for a project is found by summing present values of the incremental cash flows over time and applying the expected value operator:

$$NPV = X_0 + \frac{X_1}{(1+i)} + \frac{X_2}{(1+i)^2} + \ldots + \frac{X_n}{(1+i)^n} + \ldots + W_0^{(1)} + \frac{W_1^{(1)}}{(1+i)} + \frac{W_2^{(1)}}{(1+i)^2} + \ldots + W_0^{(p)} + \frac{W_1^{(p)}}{(1+i)} + \frac{W_2^{(p)}}{(1+i)^2} + \ldots + \frac{W_n^{(p)}}{(1+i)^n} ; \qquad (25)$$

or,

$$E[NPV] = \sum_{t=0}^{n} \frac{E[X_t]}{(1+i)^t} + \sum_{t=0}^{n} \left[\frac{\sum_p E[W_t^{(p)}]}{(1+i)^t} \right] ; \qquad (26)$$

where the first term on the right-hand side of (26) is the expected NPV of the <u>independent</u> elements and the second is the expected NPV of the <u>perfectly correlated</u> elements.

The variance, V[NPV], is found as follows. For the independent elements, the variance component (by equation (23)) is simply

$$V[NPV]_{X_t} = \sum_{t=0}^{n} (1+i)^{-2t} \sigma_t^2 \quad . \tag{27}$$

For the perfectly correlated elements, the variance component for the p^{th} element is

$$V[NPV:W_t^{(p)}]_{p=const.} = \sigma_{W0}^2 + \frac{\sigma_{W1}^2}{(1+i)^2} + \frac{\sigma_{W2}^2}{(1+i)^4} + \dots + \frac{2}{(1+i)}Cov(W0,W1) + \frac{2}{(1+i)^2}Cov(W0,W2) + \frac{2}{(1+i)^3}Cov(W1,W2) + \dots \tag{28}$$

or, since $\rho = Cov(W\delta, W\theta)/\sigma_{W_\delta}\sigma_{W_\theta} = 1$ for perfect positive correlation, then

$$V[NPV:W_t^{(p)}]_{p=const.} = \sigma_{W0}^2 + \frac{\sigma_{W1}^2}{(1+i)^2} + \frac{\sigma_{W2}^2}{(1+i)^4} + \dots + \frac{2\sigma_{W0}\sigma_{W1}}{(1+i)} + \frac{2\sigma_{W0}\sigma_{W2}}{(1+i)^2} + \frac{2\sigma_{W1}\sigma_{W2}}{(1+i)^3} + \dots$$

$$= \left[\sigma_{W0} + \frac{\sigma_{W1}}{(1+i)} + \frac{\sigma_{W2}}{(1+i)^2} + \dots + \frac{\sigma_{Wn}}{(1+i)^n}\right]^2 ; \tag{29}$$

and hence, the variance of the <u>partially</u> correlated NPV is the sum of (27) and (29):

$$V[NPV_j] = \sum_{t=0}^{n} \frac{\sigma_{Xt}^2}{(1+i)^{2t}} + \sum_{p=1}^{P} \left[\sum_{t=0}^{n} \frac{\sigma_{Wtp}}{(1+i)^t}\right]^2 \tag{30}$$

Note that case (a), mentioned above, can be derived from equations (26) and (30) by setting the first set of summations in each to zero.

At this point, some comments are in order concerning this model. Not only are the expressions for the mean (26) and the variance (30) somewhat difficult to handle computationally, but they also require that the cash flow elements be separated into two sets, one that is completely independent and the other that is perfectly correlated. From a practical standpoint, one does not find such idealism in real-life situations, and it would be difficult indeed to dissociate cash flow elements into these dichotomous categories. What is more likely, from a realistic standpoint, is that historical data or typical data from a similar process or plant would be available, upon which predictive estimates could be based. Given the existence of such data, then it would be an easier task to derive a matrix of simple correlation coefficients (both positive and negative) directly from the data.

If such a matrix of correlation coefficients could be obtained, then a much simpler analysis of the Case II problem is possible. Hillier (33) suggests this procedure, but does not develop it. The following development of the Case II problem, using a correlation coefficient matrix, is offered as an extension of Hillier's work.

Dropping the project subscript j for clarity but recognizing that the development is for a single project, let Y_t be the distributed random variable cash flow increment in year t ($t = 0, 1, \ldots, n$), with mean $\overline{Y}_t$ and variance σ_t^2. Let Y be the project net present value (a distributed random variable) with mean $\overline{Y}$ and variance V(Y). Let Y_δ and Y_θ ($\delta \in t$; $\theta \in t$; $\delta \neq \theta$) be correlated cash flows in years δ and θ, such that $\mathrm{Cov}(Y_\delta, Y_\theta) = \rho_{\delta\theta}\sigma_\delta\sigma_\theta$, where $\rho_{\delta\theta}$ is the simple correlation coefficient ($-1 \leq \rho \leq +1$). Then, the expected value of the project net present value is:

$$\bar{Y} = \bar{Y}_0 + \frac{\bar{Y}_0}{(1+i)} + \frac{\bar{Y}_2}{(1+i)^2} + \dots + \frac{\bar{Y}_n}{(1+i)^n} \tag{31}$$

or,

$$\bar{Y} = \sum_{t=0}^{n} \frac{\bar{Y}_t}{(1+i)^t} \ . \tag{32}$$

The variance, V(Y), is found as follows:

$$V(Y) = V(Y_0) + \frac{V(Y_1)}{(1+i)^2} + \frac{V(Y_2)}{(1+i)^4} + \dots + \frac{V(Y_n)}{(1+i)^{2n}} + \frac{2\ Cov(Y_0Y_1)}{(1+i)} + \frac{2\ Cov(Y_0Y_2)}{(1+i)^2} + \frac{2\ Cov(Y_1Y_2)}{(1+i)^3} + \dots + \frac{2\ Cov(Y_\delta Y_\theta)}{(1+i)^{\delta+\theta}} \ ; \tag{33}$$

or, substituting $\rho_{\delta\theta}\sigma_\delta\sigma_\theta = Cov(Y_\delta Y_\theta)$ and $\sigma_t^2 = V(Y_t)$:

$$V(Y) = \sum_{t=0}^{n} \frac{\sigma_t^2}{(1+i)^{2t}} + 2 \sum_{\substack{\delta=0 \\ \delta<\theta}}^{n} \sum_{\theta=0}^{n} \frac{\rho_{\delta\theta}\sigma_\delta\sigma_\theta}{(1+i)^{\delta+\theta}} \ . \tag{34}$$

Thus, the project expected net present value is obtained straightforward from equation (32), and the variance can be obtained by equation (34) in conjunction with a "known" or assumed correlation coefficient matrix $[\rho_{\delta\theta}]$. The only substantial amount of work lies in obtaining the correlation coefficient matrix. To do so requires first that "samples" of the random cash flows in each of the years be extracted from reliable estimating models, and second, that the sample correlation coefficients be obtained (e.g., by a matrix inversion procedure).

Two observations are necessary for this model. First, the use of correlation coefficients for the estimation of variance components implies a linear relationship between all Y_δ and Y_θ, and further, if the sample correlation coefficients (r) are used, then they are normally distributed (that is, all Y_δ and Y_θ are random variables from a multivariate normal distribution). This follows from the assumptions underlying the sampling procedure from which the correlation coefficients are obtained. Second, the sample size used to obtain the correlation coefficient matrix should be large enough so that sample standard deviations are good estimates of the "population" standard deviations used in the model. For a numerical example of this model, see Problem 2 in Appendix A.

Case III - Function Dependence, Time Independence

In this case, "unlike" elements of each periodic cash flow increment are correlated within a particular period, but there is complete independence between "like" elements in different periods. For example, income and variable expense may be correlated in Period 1, and again in Period 2, and so forth; but income in the first period is not correlated with income in the second period, nor is expense in the first period with expense in the second period, and so forth.

To the author's knowledge, no model of Case III has been reported in the literature. The following simple linear formulation, therefore, is offered as a descriptive model of the Case III problem.

Consider the following simplified cash flow function for any period t:

$$Y_t = (G_t - V_t - F - D_t)(1 - T) + D_t - C_t \qquad (35)$$

where Y_t = cash flow increment in period t,

G_t = "gross" income in period t,

V_t = variable costs in period t,

F = fixed costs per period,

D_t = depreciation expense in period t,

T = effective tax rate, and

C_t = cash outflow (investment) in period t.

For simplicity, assume that fixed costs, depreciation expense and the tax rate are "known" and constant. Assume also that gross income, G_t, and variable costs, V_t, are correlated random variables. This is a common practical situation, in which income and variable expense exhibit covarying behavior, but not necessarily in a directly proportional manner. Also assume that the depreciation expense, D_t, arises from a single initial investment, C_0, and a salvage value, S_n, in conjunction with a straight-line depreciation policy, where n is the project life. Further, let C_0 and S_n be correlated positively. Then, the depreciation expense, D_t, is a random variable and can be expressed in terms of C_0, S_n, and n, as follows:

$$D_t = \frac{C_0 - S_n}{n}, \tag{36}$$

and the cash flow for any period t becomes

$$Y_t = (G_t - V_t - F - \frac{C_0 - S_n}{n})(1 - T) + \frac{C_0 - S_n}{n} \tag{37}$$

if it is assumed that there is only one investment, C_0, at t=0.

Observing that G_t, V_t, C_0, and S_n are linearly related in the cash flow equation, and assuming (again for simplicity) that the joint density functions $f(G_t, V_t)$ and $g(C_0, S_n)$ are dependent bivariate normal distributions, then the simple correlation coefficients, given by the expressions $\rho_{GV} = \mathrm{Cov}(G_t, V_t)/\sigma_G \sigma_V$ and $\rho_{CS} = \mathrm{Cov}(C_0, S_n)/\sigma_C \sigma_S$, can be used to define the dependent relationships.

Under these assumptions, then, the mean cash flow for any year t is simply

$$\bar{Y}_t = (\bar{G}_t - \bar{V}_t - F - \frac{\bar{C}_0 - \bar{S}_n}{n})(1 - T) + \frac{\bar{C}_0 - \bar{S}_n}{n} \qquad (38)$$

and the variance of Y_t is

$$V(Y_t) = [\sigma^2_{G_t} + \sigma^2_{V_t} + 2\,\mathrm{Cov}(G_t, V_t)](1 - T)^2 + [\sigma^2_{C_0} + \sigma^2_{S_n} + 2\,\mathrm{Cov}(C_0, S_n)](\tfrac{T}{n})^2 \; ;$$

or, substituting the correlation coefficient equivalents, then

$$V(Y_t) = [\sigma^2_{G_t} + \sigma^2_{V_t} + 2\,\rho_{GV}\sigma_{G_t}\sigma_{V_t}](1 - T)^2 + [\sigma^2_{C_0} + \sigma^2_{S_n} + 2\,\rho_{CS}\sigma_{C_0}\sigma_{S_n}](\tfrac{T}{n})^2 \; ; \qquad (39)$$

where $\bar{G}_t$, $\bar{V}_t$ = means of gross income and variable costs, respectively;

$\sigma^2_{G_t}, \sigma^2_{V_t}$ = variances of gross income and variable costs, respectively;

$\overline{C}_0$, $\overline{S}_n$ = means of initial investment and salvage value, respectively; and

$\sigma^2_{C_0}, \sigma^2_{S_n}$ = variances of investment and salvage value, respectively.

Case III also requires that all Y_t be independent among periods; thus, this requires that all G_t and V_t be time-independent. In other words, the correlation between G_t and V_t in each period must be estimated for each period independently of any other period.

With these assumptions, the expected project net present value is

$$E[NPV] = \sum_{t=0}^{n} \frac{[\,\overline{G}_t - \overline{V}_t - F - (\overline{C}_0 - \overline{S}_n)/n](1 - T) + (\overline{C}_0 - S_n)/n}{(1 + i)^t} , \quad (40)$$

and the variance of the project net present value is

$$V[NPV] = \sum_{t=0}^{n} \Big\{ [\sigma^2_{G_t} + \sigma^2_{V_t} + 2\, \rho_{GV}\sigma_{G_t}\sigma_{V_t}](1 - T)^2$$

$$+ [\sigma^2_{C_0} + \sigma^2_{S_n} + 2\, \rho_{C_0 S_n}\sigma_{C_0}\sigma_{S_n}](\tfrac{T}{n})^2 \Big\} (1 + i)^{-2t} . \quad (41)$$

Because of the normality assumptions for G_t, V_t, C_0 and S_n, the project net present value will be normally distributed also.

A more complicated model can be constructed from the Horowitz model discussed under the Case IV category and in Problem 3 of Appendix A. Such a model would result in a non-normal distribution of net present value, and could be developed by assuming different and independent forms of the dependency generating function (equation (43)) for each of the t periods.

Further, even more complicated models can be constructed and solved by the simulation methods discussed under Case IV, the only requirement for their classification as Case III models being that the cash flows among periods be statistically independent. However, it should be apparent that project analysis models grow complicated quite rapidly when the strict independence assumptions in the cash flow stream are relaxed.

Case IV - Functional Dependence, Time Dependence

In this case two or more of the elements composing the periodic cash flow increments are correlated, not only among "unlike" functional elements within a particular period, but also among "like" elements among periods. No rigorous formulation of this case is known to exist, at least in discrete periodic form.

The principal difficulty in the formulation of a general case of this model arises from the many dependencies possible, which can be intuitively demonstrated as follows. Consider the case of a series of cash flows, Y_t, that are composed of elements X_{t1}, X_{t2}, ..., X_{tk}. Then, if all elements are correlated in some fashion, the tabular scheme of the possible relationships can be represented, as in Table I, where the sign $\leftrightarrow$ implies a correlational relationship.

Obviously, a solution for the parameters of the net present value distribution rests upon the simultaneous solution of all dependencies. In this case, a simultaneous solution is not possible unless severely restrictive assumptions are made; for example, that all functions of the X_{tk} are linear in form and related by a multivariate normal distribution. Thus, only special cases of this problem have been reported.

TABLE I

REPRESENTATION OF CORRELATED DEPENDENCIES

End of Period	Cash Flow Function
0	$Y_0 = \theta_0[X_{01} \leftrightarrow X_{02} \leftrightarrow \ldots \leftrightarrow X_{0k}]$
	$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$
1	$Y_1 = \theta_1[X_{11} \leftrightarrow X_{12} \leftrightarrow \ldots \leftrightarrow X_{1k}]$
	$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$
2	$Y_2 = \theta_2[X_{21} \leftrightarrow X_{22} \leftrightarrow \ldots \leftrightarrow X_{2k}]$
	$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$
⋮	$\vdots \quad \vdots \quad \vdots \quad \vdots$
	$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$
t	$Y_t = \theta_t[X_{t1} \leftrightarrow X_{t2} \leftrightarrow \ldots \leftrightarrow X_{tk}]$

Canada and Wadsworth (10) have presented a semi-graphical method of formulating a special case of Case IV, in which some of the functional cash flow elements may be partially correlated within a period, but perfectly correlated as to like elements among periods. In this model, the random variables are investment, P, cash flow increment, D, project salvage value, S, and project life, T. These variables may be correlated pair-by-pair, or independence may be assumed. Each is a random variable.

A continuous compounding of the discount factor over the life of the project is used to permit differentiation of the NPV function, which leads to the calculation of variance components. The concept of compounding over the entire life of the project, however, forces the random variables P, D, S and T to remain constant in the integrated discounting function. While correlations are permitted among these specified variables, they are constant with respect to time in the net present value formulation. This is not particularly damaging insofar as initial investment and salvage value are concerned, but a time-invariant cash flow stream seems implausible. This seems to be a major disadvantage of the model. A further limitation is the assumption of only one lump investment at the beginning of the project life.

A considerably different formulation of another special case has been reported by Horowitz (35). In this model, the periodic cash flows are also perfectly correlated among periods, but the Horowitz model differs from the Canada-Wadsworth model in the method of synthesizing the cash flow stream from the correlated elements. Horowitz' formulation will not be presented in detail, since it is lengthy, and the reader is referred to the original source for a complete description.

Briefly, however, Horowitz formulates a project net present value function in terms of the elements of the cash flow stream,

$$NPV_j = \sum_{t=0}^{n} \theta_t [(P_f, Q_f, P_m, Q_m, W, F, V', D, T)(1 + i)^{-t}] \qquad (42)$$

where
P_f = final product price,
Q_f = quantity of final product produced
P_m = raw material price,
Q_m = quantity of raw material used,
W = total wages paid,
F = fixed costs,
V' = other variable costs,
D = depreciation, and
T = tax rate,

ann in year t (t = 0, 1, ..., n).

Final product price, P_f, is assumed to be a function of production costs and the price elasticity of demand, e. With the exception of P_m, F, D and T, the remaining variables are related to e through Q_f, principally through an elasticity relationship that relates Q_f to P_f:

$$Q_f = A\ P_f^{-e}\ u \qquad (43)$$

where A is a constant, e is the price elasticity of demand, and u is a random error term. Price elasticity of demand is assumed to be normally and independently distributed, as is the logarithm of the error term, u. Raw material price, P_m, fixed costs, F, depreciation, D, and tax rate, T, are assumed to be constant from period to period. Thus, some of the within-period unlike variables are related, but there is complete dependence (perfect correlation) between cash flow increments among periods.

The assumptions of a normally distributed elasticity and a random market price force the project NPV to become a random variable, but in this model the NPV for the project is not normally distributed. This is the principal conclusion of the Horowitz model, in contrast to other models (for other cases) that either assume normality of the NPV variable outright or indicate that normality somehow may be approached as a result of the linear additive properties of the cash flow stream. Furthermore, Horowitz correctly stresses (in effect) that this model cannot be used for selection purposes unless the form of $\theta[NPV_j]$ is first specified (see (35), p. 417).

For a numerical example of the Horowitz formulation, see Problem 3 in Appendix A.

What is needed in the functionally dependent, time dependent model is more flexibility together with a rather simple means of formulating the problem for computational purposes. Rigorous mathematical formulations do not lend themselves to these conflicting goals. This impasse suggests the use of computer simulation to develop flexible, yet computationally efficient Case IV models.

Hess and Quigley (31) use a simple Monte Carlo simulation method for evaluating a return function in terms of independent variables (demand, price, variable cost, investment, and plant capacity). Hertz (29, 30) gives considerable insight into the simulation method of formulating the NPV distribution. While he indicates that the cash flow parameters are not necessarily independent, no specific information is provided on the actual formulation of the simulation model to account for dependencies in the cash flow streams.

Bussey (8), using a computer simulation language, has also formulated a special case of the Case IV problem. In this model, periodic cash flows are not perfectly correlated, but are time-related through an exponential growth function for nominal plant output (demand). The random variable actual output (demand) of a plant, Q, in a particular period, is related both to the exponential growth function and to an independently distributed market price, P, by the relationship

$$\frac{Q}{Q_t} = A\ P^{-e} \tag{44}$$

where Q_t is determined by the time-growth curve, and e is the price elasticity of demand (assumed to be $NID(\overline{e},\ \sigma_e^2)$). Market price is assumed to be an independent random variable, as is project life. Both are arbitrarily distributed. Periodic depreciation, however, is a function of the random project life. Another feature is the truncation of the plant output if market price and elasticity force demand to fall below 10% or exceed 130% of plant capacity. Annual variable costs are related to output, and fixed costs are assumed constant in each discounting period. Discrete end-of-period discounting is used with a constant and "known" rate.

The project net present value distribution is obtained directly by repeated synthetic "sampling" of the assumed random variable distributions and their interrelationships. The mean and variance of the NPV are found from a histogram tabulation in the computer printout, and third and fourth moments of the distribution can be calculated if needed.

For the model described, the NPV distribution is not significantly different from a normal distribution, in spite of the many arbitrary decisions in the model (such as truncation of demand, adjustment of market price, and probabilistic life). This, however, is a purely coincidental result. A Kolmogorov-Smirnov goodness-of-fit test fails at $\alpha = 0.01$ to reject the assumption of a normal distribution.

For Case IV models, the fundamental advantage is a closer approach to "reality" in formulating the model. To use this advantage fully, the model must be made as flexible as possible while at the same time retaining computational simplicity. The simulation method makes possible the attainment of these goals.

Summary of Chapter III

The probabilistic non-sequential capital budgeting problem is fundamentally one of project _selection_--that is, the selection of an optimal subset of projects from a set of candidate projects that will maximize some function of net present value to the firm, subject to a series of technical dependency and budget constraints. The optimization is accomplished by first evaluating the net present value functions for each candidate project, where it is understood that NPV is a random variable with a mean and variance (and possibly higher moments). The formulation and evaluation of the NPV function for a candidate project is termed the _analysis_ problem.

Several papers on the theory of the analysis problem have contributed to an understanding of the dependence-independence relationships that may occur when project cash flow streams are considered to be random variables. It has been shown here that the previous research can be classified according to the dependence-independence relationships assumed in the various models, and that "reality" is approached more closely when independence assumptions are relaxed and dependency relationships among the cash flow elements are permitted. It has also been shown that model complexity and computational difficulties rapidly result when the simplifying assumptions are relaxed.

A final conclusion is that project modeling--that is, the formulation of the specific probabilistic model for the net present value of a project--has meaning only when it is viewed as an <u>input</u> to the selection problem. A warranted conclusion is that nothing more than probability statements about a particular project can be made, unless the <u>form</u> of the objective function in the selection model is specified.

FOOTNOTES

[1]Elements that are "unlike" are, for example, price, volume, expense, and so forth. "Like" elements are Price(1), Price(2), Price(3), and so forth, where the subscripts within parentheses denote different time periods.

[2]For a simplified version, see Bierman et al (6), pp. 355-358.

CHAPTER IV

PROJECT SELECTION UNDER UNCERTAINTY

Once the probabilistic net present values and their distributions have been specified for a set of candidate projects by an appropriate solution to the analysis problem, the solution to the non-sequential capital budgeting problem then rests upon the choice of a suitable criterion for selection among projects, the specification of the necessary technical and logical constraints, and an efficient method for comparing the necessary computations that result from the selection criterion. This chapter is concerned with the first of these steps: the choice of a suitable selection criterion.

The research by Markowitz (42) and Farrar (19) already indicates that a simple maximization of net present value is not a sufficient criterion for the explanation of investor behavior when outcomes are uncertain. As many authors have stated, what seems to occur when decision-makers are confronted with uncertainty is overt behavior that suggests the decision-makers attach differing degrees of importance or value to differing degrees of uncertainty and payoff. This can be illustrated by a simple example, in which the reader can exercise his own "preferences" for risk.

Consider a situation involving two alternative choices, Alternative I and Alternative II. Each alternative involves the investment of some of the reader's own funds, and as a result of the investment the reader is then entitled to select one of the alternative payoffs. Alternative I requires the investment of $10, which entitles the reader to select either Payoff Ia or Payoff Ib, as follows:

<u>Payoff Ia</u>: Receive $11 with certainty;

OR

<u>Payoff Ib</u>: On the flip of a fair coin, receive $100 if the coin comes up "heads" or risk a loss of the $10 investment if the coin comes up "tails."

Alternative II, however, involves much larger sums of money. Here, the reader must "invest" $10,000, which entitles him to select one of the following payoffs:

<u>Payoff IIa</u>: Receive $11,000 with certainty;

OR

<u>Payoff IIb</u>: On the flip of a fair coin, receive $22,500 if the coin comes up "heads" or risk a loss of the $10,000 investment if the coin comes up "tails."

All of the investments and payoffs are assumed to occur in "present time" so that the time value of money is not involved.

For Alternative I, which choice of payoff (Ia or Ib) should be made? And for Alternative II, would Payoff IIa or IIb be chosen? The answer, of course, is an individual one and is based on the personal attitudes of the decision-maker toward risk and uncertainty. Most people do not mind risking $10 to obtain a payoff of $100 with a probability of 0.5, and would probably pick Payoff Ib. Moreover, the decision is consistent with the criterion of maximizing expected value, since the expectation of Payoff Ib is 0.5($100) - $10 = $40, which results in an expected gain of this amount to the decision-maker, which is greater than the expected gain of $11 - $10 = $1 for Payoff Ia.

The decision is Alternative II is a bit more controversial. Most people do not have $10,000 with which to gamble in this fashion, since this sum of money represents an appreciable portion of their assets. Those who are definitely risk-avoiders will, without doubt, choose Payoff IIa, the certainty case. Moreover, they will do so in spite of the fact that Payoff IIb results in a higher net expected gain to them:

$$\text{Expected Net Gain (IIb)} = 0.5(\$22{,}500) - \$10{,}000 = \$1{,}250$$

$$\text{Expected Net Gain (IIa)} = 1.0(\$11{,}000) - \$10{,}000 = \$1{,}000 .$$

Thus, the decision-maker's choices are dependent upon the level of the investments he is required to make, the payoffs anticipated from the investments, and upon the uncertainties connected with the payoffs.

Translated into a business situation, suppose the returns from investment opportunities available to the firm can be obtained in one of two ways:

Alternative A: Receive $1,000,000 with certainty;

OR

Alternative B: Risk a loss of $2,000,000 with probability 0.5, or obtain a payoff of

$4,000,200 with probability 0.5.

Suppose also that the firm's financial position and liquidity would be materially affected by a loss of $2,000,000 or gains of $1,000,000 or $4,000,200. The firm's decision-maker, when faced with these alternatives, might very likely conclude that the greater risks associated with Alternative B do not warrant subjecting the firm to those risks, even though the expected return, 0.5($4,000,200) - 0.5($2,000,000) = $1,000,100, is greater than the expected return of $1,000,000 associated with Alternative A. Those decision-makers who choose Alternative A do not act in accordance with the criterion of maximization of expected net present value. Rather, they attach some value, or utility, to the combination of payoff and associated risk in Alternative B that is <u>less</u> than the value or utility attached to Alternative A. That is, to decision-makers who choose Alternative A, the utility of $1,000,000 certain is greater than the utility of a gamble in which $2,000,000 can be lost with a probability of 0.5 or a gain of $4,000,200 can be realized with a probability of 0.5. The question is, then, on what basis does a decision-maker ascribe value or utility to uncertain outcomes? One answer lies in the von Neumann-Morgenstern utility theory, which will be briefly outlined below.

Von Neumann-Morgenstern Utility Theory

Von Neumann and Morgenstern (46) have shown that a preference ordering among alternatives involving risk can be obtained for a decision-maker who consistently follows certain "axioms of rational behavior." The axioms themselves will not be presented here, nor will the method for developing the utility function from them, since these are both reported in detail elsewhere.[1] Briefly, however, von Neumann and Morgenstern have shown that the rational behavior axioms provide a basis for obtaining an interval-scale of utility for monetary returns. The utility scale has neither an absolute (true) zero nor a unique unit of measure, and can, therefore, be transformed by any linear transformation (e.g., by multiplying or dividing every value by a constant, or by adding or subtracting a constant). Stronger measures of utility (for example, a proportional scale with rational zero) have been proposed by Shuford, Jones and Bock (56) and by Restle (51), but the decision environment considered here does not require absolute measurement. For the capital budgeting problem, all that is required is that one be able to determine whether or not one project (or set of projects) has a greater utility to the decision-maker than another; it is not necessary to know how much greater. Thus, a proportional (or ratio) scale utility with rational zero is a refinement not required by the selection model.

While the von Neumann-Morgenstern axioms have been subjected to a considerable number of attacks and discussion in the literature,[2] it is assumed here that valid interval-scale utility functions can be obtained by the proper application of the rational behavior axioms. This assumption is made in order to provide a basis for the development of project selection criteria for both risk-seekers and risk-seeking decision-makers in this and succeeding chapters, and to avoid a discussion of the validity of the utility function itself, which is considered to be beyond the scope of this research.

With the foregoing assumption, and when the rational behavior axioms have been applied properly through the "standard lottery" method (with personal inconsistencies resolved), then a utility function for a particular person can be obtained, perhaps similar to the one illustrated in Figure 1. That is, from the person's responses to the standard lotteries posed to him, a set of data points can be obtained, through which some form of mathematical function can be fitted which is then defined as the person's utility function.

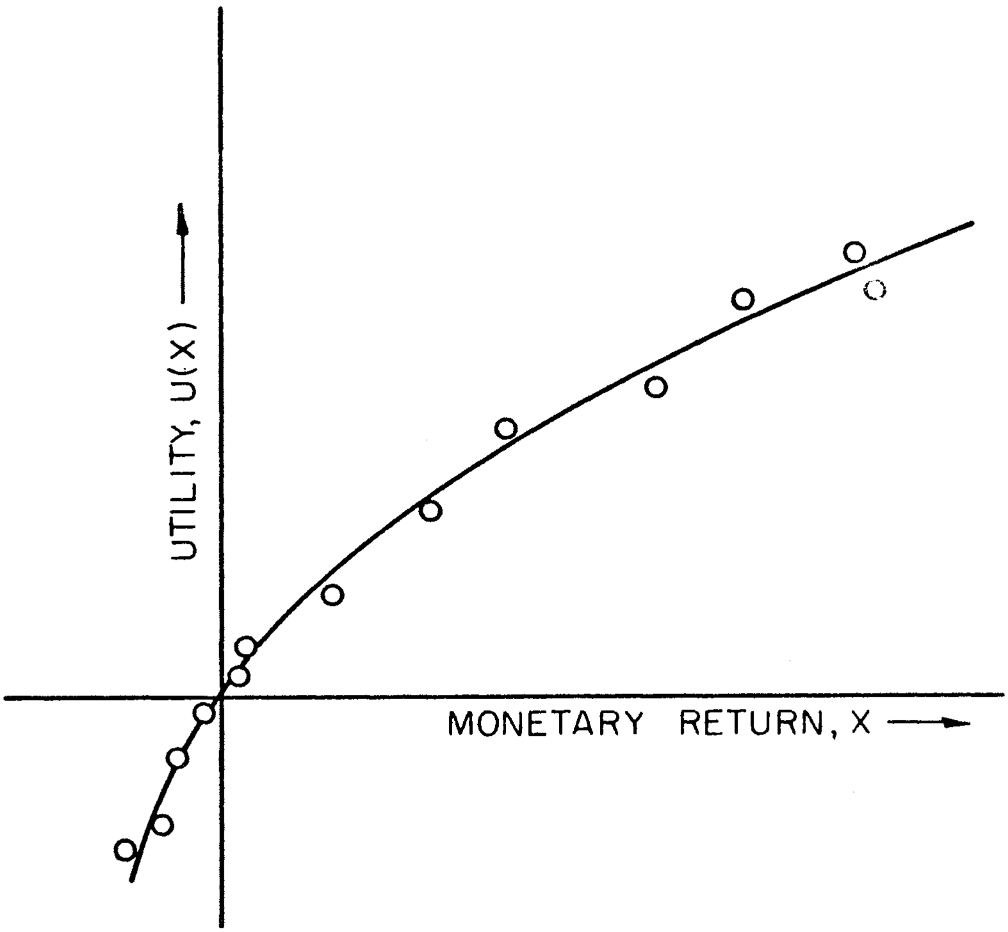


Figure 1. Typical Utility Function

Empirical Determinations of Utility Functions

The von Neumann-Morgenstern utility theory has been utilized by several investigators as a basis for the empirical determination of utility functions of individuals. Mosteller and Nogee (45) were the first investigators (1951) to establish utility functions for individuals, using overt betting behavior in a gambling game to gather data for the construction of the utility curves. Davidson, Suppes and Siegel (14) made additional determinations in 1957. Grayson (25), using the standard lottery technique, determined the utility functions of eleven principal decision-makers in the petroleum exploration and development business, several of which are shown in Figures 2 and 3. Green (26) reported (1963) the determination of utility functions of 16 middle management personnel in a large chemical company, representing the four major divisions of the firm (production, sales, finance and research). Swalm (59) reported (1966) the empirically determined utility functions of 13 executives, 12 of whome were from one company. Several of Swalm's functions are illustrated in Figure 4. Cramer and Smith (13), also using the standard lottery method, reported (1964) the determination of the utility functions of 8 executives of a leading U. S. corporation, some of whom were from the research department and others from the manufacturing department.

Now, one purpose in illustrating several types of empirically determined utility functions is to demonstrate that there is, in general, no one form of mathematical function that will "fit" the data exactly. To be even more precise, one cannot specify from theory any particular mathematical function that is, in general, a utility function. The best that can be done is to assume some form of mathematical function that

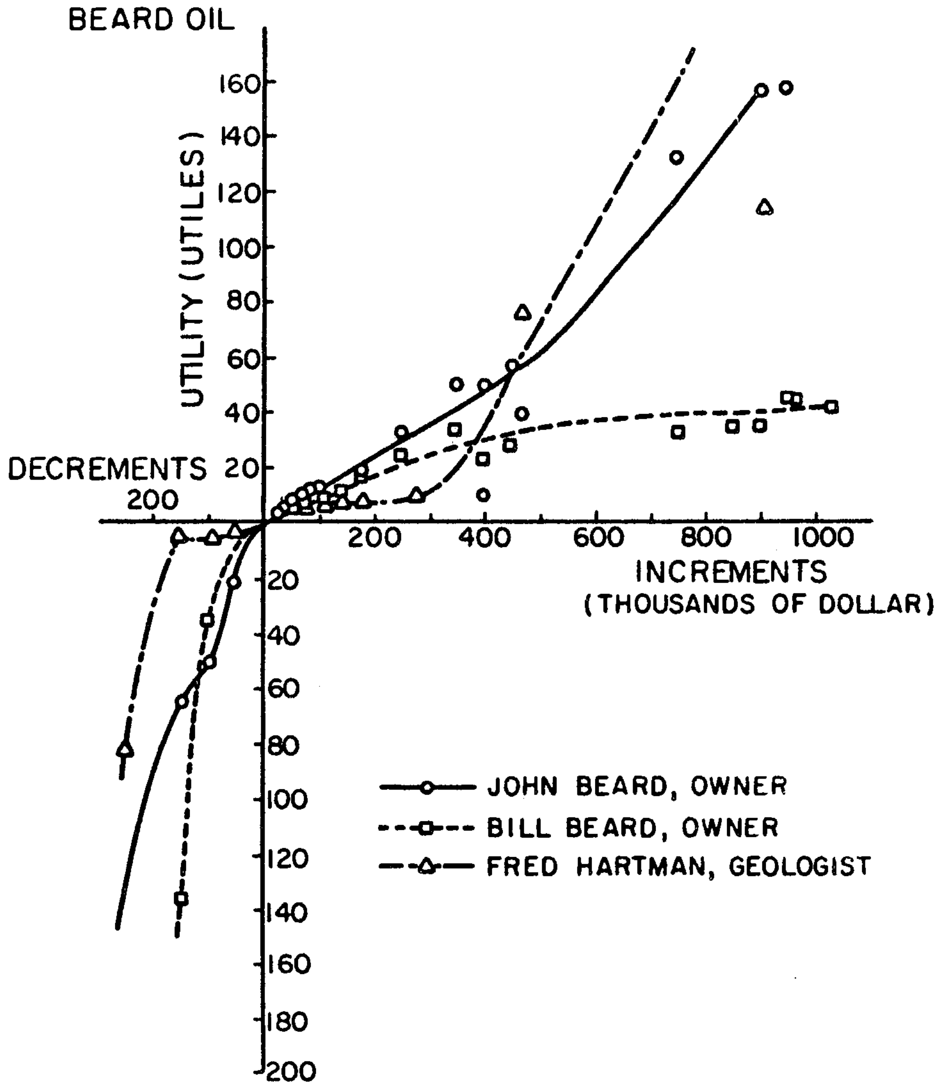


Figure 2. Utility Functions of Three Individuals in the Petroleum Exploration Industry (Reproduced from Reference (25), by Permission)

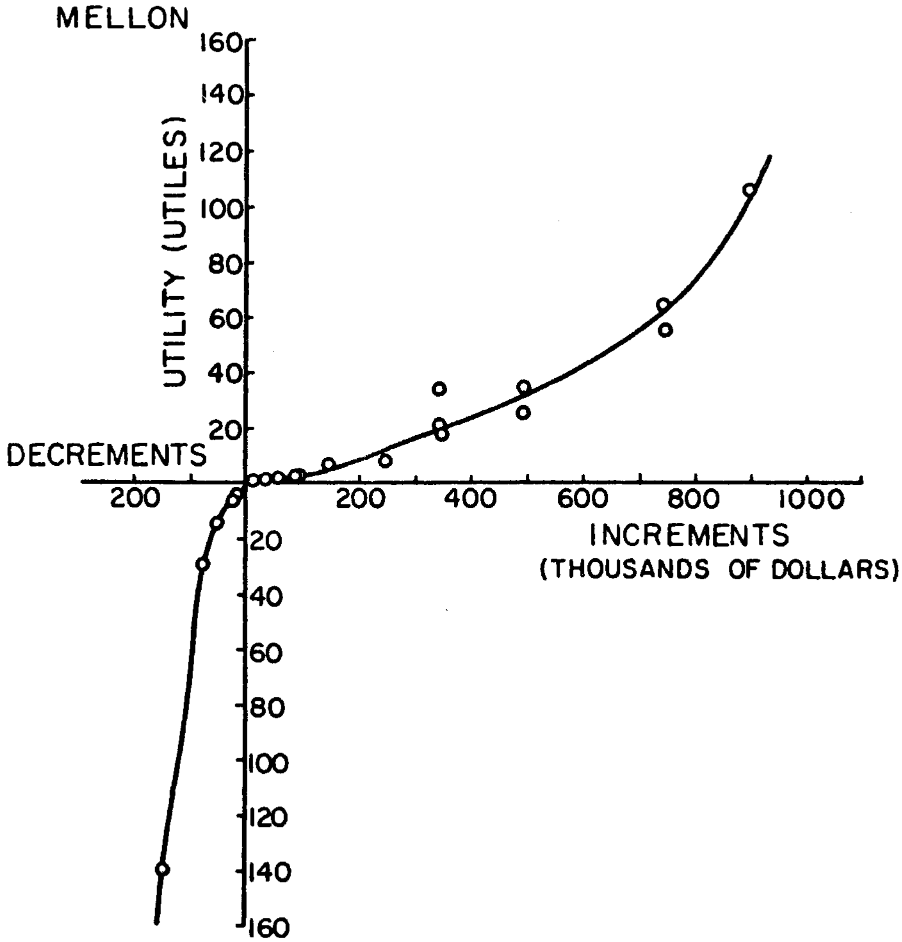


Figure 3. Utility Function of an Owner of a Petroleum Exploration Company (Reproduced from Reference (25), by Permission)

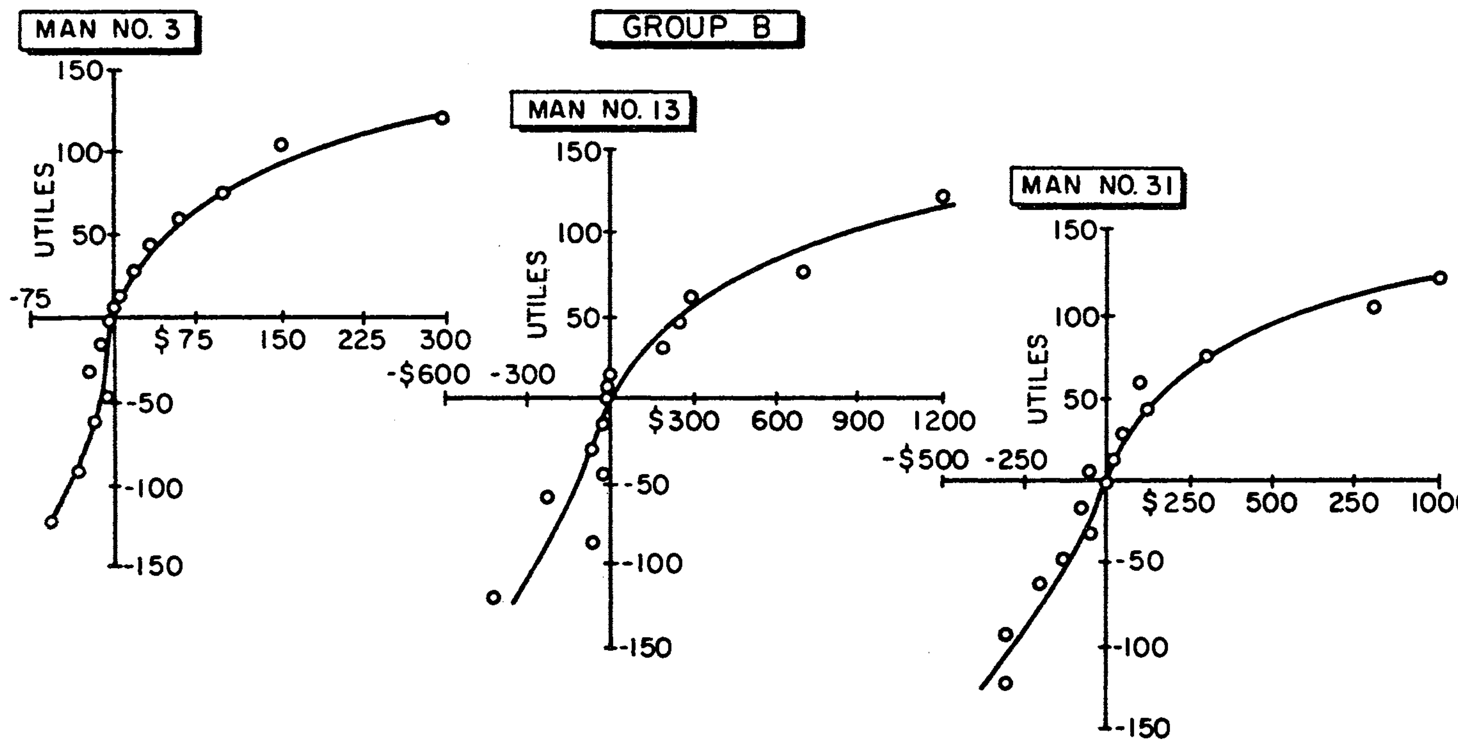


Figure 4. Three Risk-Avoiding Utility Functions
(Reproduced from Reference (59),
by Permission)

seems best to fit the data. For those empirical data that indicate a concave downward trend, functions of quadratic polynomial or negative exponential form have been assumed. Nevertheless, one is never assured that a given mathematical expression is theoretically correct, and this poses a dilemma. The dilemma arises because the criterion function, $\theta[NPV]$, which is used in the optimization procedure for the selection of an optimum subset of projects, is derived from the mathematical expression of the utility function. Thus, the criterion function in the selection problem rests upon the weakness of an assumption as to the form of the utility function that seems best to fit the decision-maker's responses (data points).

All of the published research on the empirical determination of individual utility functions relies on what amounts to "free-hand" curve fitting to fit the data points. This is not an entirely satisfactory procedure. If one looks at the individual data points, which are derived from the decision-maker's responses to the standard lottery procedure, as being repeated samples of the individual's behavior, then a more powerful statistical tool (multiple regression) can be used to assist in the definition of the individual's mathematical utility function. If the standard lotteries posed to the individual are permitted to "overlap" with several data points being taken common to several lotteries, then it is possible to make independent determinations of the experimental (sampling) error, which in turn permits one to perform an analysis of variance on the regression and thereby state with some degree of confidence whether or not the assumed form of the regression equation "fits" the data. The application of multiple regression techniques to the determination of utility functions will be developed more fully in Chapter V. While it is somewhat puzzling why the sampling nature of the decision-maker's responses has not been recognized before, nevertheless, the present concern is with the mathematical form of the equation used to fit the data, and with the criterion for project selection derived from that equation.

Derivation of Selection Criteria From Utility Functions

There are at least two reasons why the maximization of expected money value is an incorrect project selection criterion. The first is due to a classical problem, the St. Petersburg paradox,[3] which first convinced some eighteenth century scholars that something was incorrect about the maxim that the individual maximizes expected monetary returns in risk situations. The second reason is due to Markowitz ((42), pages 207-210). In essence, Markowitz says that if the maxim is true, then an investor would never diversify his investments. Instead, he would merely choose the project (investment) that has the greatest expected return, and invest all his funds in that project. Since empirical evidence is to the contrary, namely, that investors do diversify their investments, then, if one accepts diversification as a sound principle of investment, one must reject the maxim of simply maximizing expected return.

The von Neumann-Morgenstern utility theory (and the implications developed from it) substitute an expected utility rule for the expected return rule. Thus, it is argued, a return of $50 is not necessarily twice as good as a $25 return; nor is a loss of $50 necessarily twice as bad as one of $25. Perhaps there is some function, such as that in

Figure 1, relating the value or utility of a monetary return to the return itself, such that a rational man would maximize the expected value of the utility rather than the expected value of the return. Thus, if two alternative actions were presented to a decision-maker, he would act so as to maximize his expected utility by choosing the alternative with the higher expected utility of return.

Now, mathematical expressions for expected utility are derived from mathematical expressions for the utility function, the form of which (in reality) is unknown. However, one can proceed on that basis, if necessary. The method of deriving a criterion function for a non-complex utility function (as in Figure 5) will be demonstrated below.

If the "true" form of the utility function in Figure 5 is not known, it can be approximated by a Taylor series expansion if it is assumed to be a continuous function and at least twice differentiable. Further, it is assumed that the "true" function is a quadratic in X, the return. Now, consider some value of return, say X*, and expand by a Taylor series about X*:

$$U(X^* + h) = U(X^*) + h \left.\frac{\partial U}{\partial X}\right|_{X=X^*} + \frac{h^2}{2!} \left.\frac{\partial^2 U}{\partial X^2}\right|_{X=X^*} + \ldots \qquad (45)$$

where h is the perturbation away from X*. Now, let h = -X*, so that the function U(X* + h) will be evaluated at X = 0. Then, using the first three terms of the expansion to approximate the quadratic (the remainder will then be zero), the expansion becomes

$$U(X^*+h) = U(X^*-X^*) = U(0) = U(X^*) - X^* \left.\frac{\partial U}{\partial X}\right|_{X=X^*} + \frac{X^{*2}}{2!} \left.\frac{\partial^2 U}{\partial X^2}\right|_{X=X^*} . \qquad (46)$$

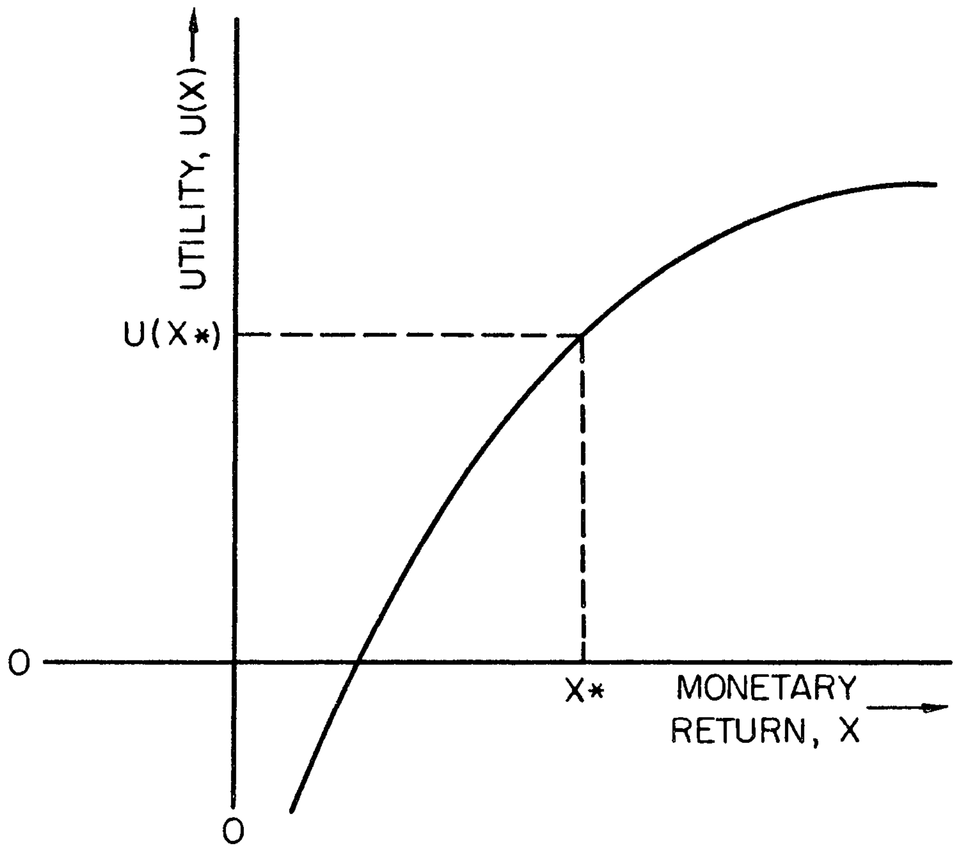


Figure 5. General Quadratic Utility Function

Now, define U(0) = 0. This forces the function to pass through the point [U(0) = 0, X = 0], since the values of h and U(0) were chosen to accomplish this end.[4] From the foregoing, then

$$U(0) = 0 = U(X^*) - X^* \left.\frac{\partial U}{\partial X}\right|_{X=X^*} + \frac{X^{*2}}{2!} \left.\frac{\partial^2 U}{\partial X^2}\right|_{X=X^*} . \qquad (47)$$

Now, letting $A = \left.\frac{\partial U}{\partial X}\right|_{X=X^*}$ and $B = -\frac{1}{2} \left.\frac{\partial^2 U}{\partial X^2}\right|_{X=X^*}$ (since a concave downward function requires a negative second derivative), then both A and B become constants and the utility function U(X*) can be found by solving equation (47) for $U(X^*)$:

$$U(X^*) = A(X^*) - B(X^*)^2 \quad . \tag{48}$$

To find the <u>expected</u> utility, take expectations of both sides of equation (48), thus:

$$E[U(X^*)] = A\ E[X^*] - B\ E[X^{*2}] \quad ;$$

but
$$E[X^*] = \mu$$
and
$$E[X^{*2}] = V[X^*] + (E[X^*])^2 = \sigma^2_{X^*} + \mu^2 \ ;$$
hence,
$$E[U(X^*)] = A\mu - B(\sigma^2_{X^*} + \mu^2) \ ; \tag{49}$$
where
μ = the mean of the random variable X*,

$\sigma^2_{X^*}$ = the variance of X*.

Equation (49) is the expression for the <u>expected</u> <u>utility</u> of a random variable X*, <u>when the utility function is of the form of equation</u> (48). Note that the expected utility is stated in terms of the mean and the variance of the random variable, when the utility function is of quadratic form.

It follows, then, that if the random variable X* in equation (49) is the distributed net present value of a project (the NPV), then the expected utility of the net present value is merely

$$E[U(NPV)] = A\mu_{NPV} - B(\sigma^2_{NPV} + \mu^2_{NPV}) \quad . \tag{50}$$

Equation (50), therefore, is the <u>selection criterion</u> by which the subset of candidate projects is chosen from the set of candidate projects in accordance with the selection model (equation (13), Chapter II), <u>when it is assumed</u> that the decision-maker possesses a utility function of quadratic form. Note, however, that there is an upper limit to the applicability of this criterion. That is, since the utility function is a quadratic polynomial, it possesses a relative maximum, U(NPV*), at some value of NPV = NPV*, and the utility function is invalid when NPV $\geq$ NPV*. This can be shown as follows.

Since the utility function was assumed to be continuous and at least twice differentiable, the necessary and sufficient conditions for a relative maximum are obtained, thus:

$$\frac{dU(NPV)}{d(NPV)} = A - 2B(NPV) = 0 \; ; \tag{51}$$

$$\frac{d^2U(NPV)}{d(NPV)^2} = -2B \tag{52}$$

from which NPV* = A/2B. Now, the marginal utility, $U'(NPV) = \frac{dU(NPV)}{d(NPV)}$, is required to be everywhere positive for a valid utility function, and this requirement restricts the valid range of the quadratic utility function to values of NPV < A/2B. In realistic situations, however, this is not a seriously limiting constraint, as the value of A/2B is generally greater than the range of applicability of the data. From a theoretical standpoint, nevertheless, this is an undesirable limitation since marginal utility (from a classical economic standpoint) is usually assumed to be everywhere positive up to NPV $\leq +\infty$.

The expected utility, E[U(NPV)], serves as a basis for making inferences about the risk attitudes of the decision-maker who possesses a quadratic utility function, as well as acting as a project selection criterion. Referring to equation (50), if E[U(NPV)] is assumed to remain constant, then the right-hand side of (50) describes a family of indifference curves in the parameters μ and σ, the shape of which can be inferred by differentiating equation (50). Thus,

$$0 = A \frac{d\mu}{d\mu} - 2B\sigma \frac{d\sigma}{d\mu} - 2B\mu \frac{d\mu}{d\mu}$$

or, solving for $d\sigma/d\mu$:

$$\frac{d\sigma}{d\mu} = \frac{A - 2B\mu}{2B\sigma} . \qquad (53)$$

The expression $2B\sigma$ is clearly positive. Since NPV, and therefore μ_{NPV}, must be less than A/2B for a valid quadratic utility function, then the numerator of equation (53) is positive. Hence, $d\sigma/d\mu > 0$. Also, the second derivative of (50) is

$$\frac{d^2\sigma}{d\mu^2} = -\frac{1}{\sigma} - \frac{(\sigma')^2}{\sigma} \qquad (54)$$

where $\sigma' = d\sigma/d\mu = (A - 2B\mu)/2B\sigma$, from equation (53). Since both σ' and σ are positive, then $d^2\sigma/d\mu^2 < 0$ by equation (54). Thus, equations (53) and (54) indicate that the indifference curves, $E[U(NPV)]$ = constant, are both upward sloping and downward concave on σ-μ coordinates, which corresponds to the classical economic interpretation of indifference analysis. Moreover, it should be observed that $A - 2B\mu = \frac{\partial E[U(NPV)]}{\partial \mu} > 0$ and $-2B\sigma = \frac{\partial E[U(NPV)]}{\partial \sigma} < 0$ for valid values of NPV and μ, thus indicating that the decision-maker's expected utility varies directly with μ and inversely with risk (σ). It can thus be inferred that the decision-maker whose utility function is quadratic is risk-avoiding for all valid values of μ_{NPV}. A typical set of risk-avoiding indifference curves is illustrated in Figure 6.

The role of the indifference curves and the expected utility as a project selection criterion can be illustrared by reference to Figure 6.

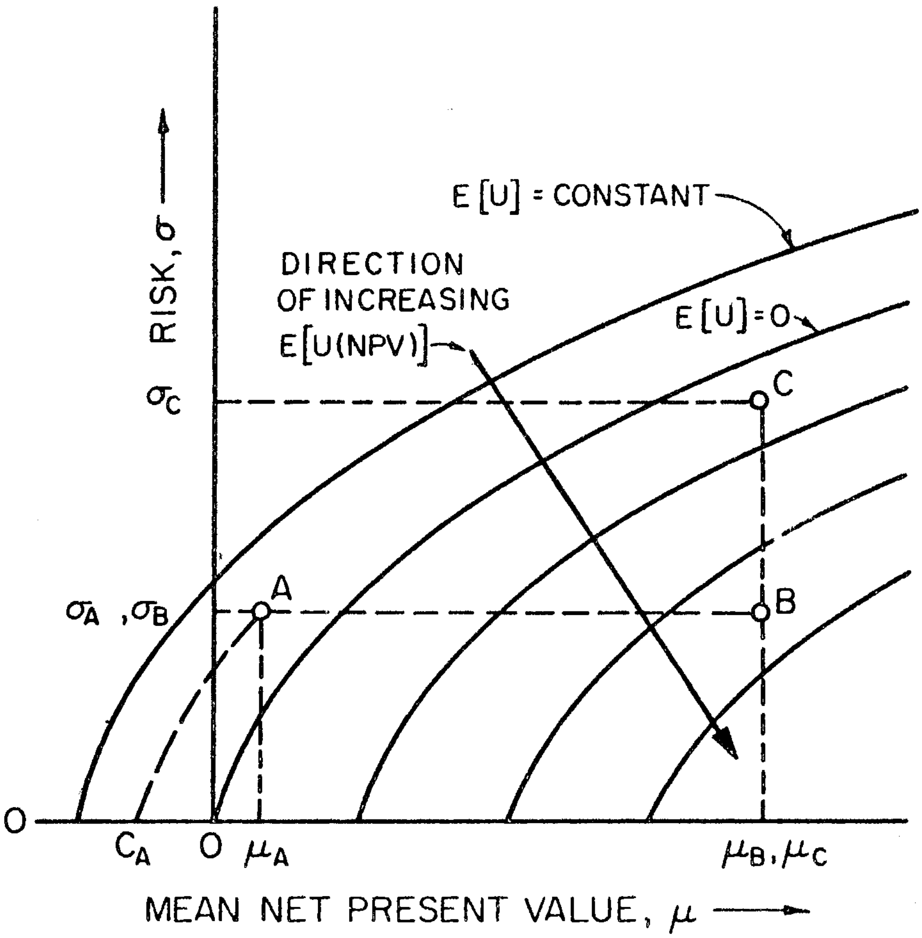


Figure 6. Indifference Curves for a Risk-Avoiding Decision-Maker

The means and standard deviations of net present values for three hypothetical projects are superimposed on the indifference curves. Since the decision-maker's expected utility increases with μ and decreases with increasing risk, σ, then he prefers projects in the following manner. For two projects, A and B, that have the same variance (or standard deviation) of net present value, he prefers the project, B, with the greater expected net present value. For two projects, B and C, that have the same expected net present value, he prefers the one with the smaller risk (B), since it has greater utility for him.

If investment funds were limited, so that only one project could be undertaken, then the decision-maker would choose Project B because it has the highest expected utility of the three. If two projects could be undertaken, then he would (rationally) choose Projects B and C, since these two projects yield the greatest expected utility. If, however, there were no capital limitation then the decision-maker might still choose only Projects B and C. The rationale for this decision can be demonstrated by the method of "certainty equivalents." The certainty equivalent of an available opportunity is that value of NPV which has the <u>same expected utility</u> as the opportunity, but with zero variance. Thus, the certainty equivalent for Project A is the intersection of the iso-utility curve with the abscissa, or point C_A. Now, the NPV of "cash" is zero, since an investment of cash into cash provides no return. Likewise, the investment of cash into cash is (presumably) without risk, thus the variance of "cash" is zero. Hence, the certainty equivalent of "cash" is the point ($\mu = 0$, $\sigma = 0$). Because the certainty equivalent of Project A, C_A, is less than the certainty equivalent of cash, Project A would not be undertaken. An alternative way of demonstrating the same result is to examine equation (50); if both μ and σ are zero-valued, then $E[U(NPV)]$ is zero also. Thus the iso-utility curve, $E[U(NPV)] = 0$, passes through the origin on the indifference curve plot, and by inspection Project A has a negative expected utility. Projects with negative expected utilities are not undertaken. Even though such projects may have a positive expected net present value, their utility becomes negative because of excessive risk to the risk-avoiding decision-maker, or conversely, because of insufficient expected net present value to offset the risk inherent in the project. This type of risk-avoiding decision-making and project selection would be expected from individuals whose utility functions are concave downward--for example, the three executives whose utility functions are illustrated in Figure 4.

In summary, the following inferences can be made for the quadratic utility function:

(1) The quadratic formulation of a utility function for concave downward data points is valid when relevant values of mean project net present values, μ_{NPV}, are less than the constant A/2B.

(2) The decision-maker who possesses a quadratic utility function is risk-averse. That is, for projects with equal risk, he prefers the one with the highest expected net present value; and for projects with equal expected net present value, he prefers the one with the least risk. Furthermore, he will not accept projects with negative expected utility, since cash has a greater utility for him.

(3) Between projects with equal expected utility, the decision-maker with a quadratic utility function is indifferent.

(4) The selection criterion, $E[U(NPV)] = A\mu - B(\sigma^2 + \mu^2)$, is

the correct function of NPV to use as a selection criterion in the selection problem (equation (13)), when the decision-maker possesses a concave-downward quadratic utility function. The constants A and B in the expected utility expression may be evaluated from a linear regression of the decision-maker's responses to the standard lottery method, assuming a utility function of the form $U(X) = AX - BX^2$.

The Risk-Avoidance Criterion Models

The Freund Model

In 1956, six years before Farrar attempted (incorrectly) to derive the selection criterion just presented above, Freund (24) formulated a utility function of the form

$$U(X) = 1 - e^{-BX} \tag{55}$$

where X is what Freund calls the "net revenue." While Freund does not explicitly include the time value of money in his concept, so that X would be a net present revenue, neither does he exclude it and no generality is lost by assuming that X could be a net present value.

The function in equation (55) is concave downward for positive values of B (a constant), and indicates a risk-avoiding decision-maker. Then, Freund says, if X is normally distributed (with mean μ, and standard deviation, σ), the expected utility is

$$E[U(X)] = \int_{-\infty}^{\infty} (1 - e^{-BX}) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} dX \tag{56}$$

which is the selection function to be maximized in the selection problem (equation (13)). This form of the selection function is somewhat intractable, however, and can be simplified as follows.[5]

Expanding equation (56) and separating the resulting integrals, the problem becomes

$$\text{Max } E[U(X)] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2} dX - \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2 - BX} dX \; ; \quad (57)$$

from which one recognizes that the first integral is merely the cumulative normal distribution which integrates to unity. Now, make the substitution $z = \frac{X - \mu}{\sigma}$, and equation (57) becomes

$$\text{Max } E[U(X)] = 1 - \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma e^{-\frac{1}{2}(z)^2 - B\sigma z - B\mu} dz \quad . \qquad (58)$$

Complete the square in the exponent by adding and subtracting the quantity $B^2\sigma^2/2$, obtaining

$$\text{Max } E[U(X)] = 1 - \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma e^{-\frac{1}{2}(z + B\sigma)^2} e^{-\frac{1}{2}(2B\mu - B^2\sigma^2)} dz \; . \qquad (59)$$

Now, the second exponential is a constant and can be brought outside the integral sign, and by back-substituting for z its equivalent $\frac{X - \mu}{\sigma}$ one obtains

$$\text{Max } E[U(X)] = 1 - e^{-\frac{1}{2}(2B\mu - B^2\sigma^2)} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{X+B\sigma^2-\mu}{\sigma}\right]^2} dX.$$

Now, the quantity $B\sigma^2$ in the exponent of the second exponential is a constant, and by defining a new random variable $X^* = X + B\sigma^2$, one obtains $dX^* = dX$, and thus an equivalent statement of the problem is:

$$\text{Max } E[U(X^*)] = 1 - e^{-\frac{1}{2}(2B\mu - B^2\sigma^2)} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{X^* - \mu}{\sigma}\right]^2} dX^*. \qquad (60)$$

The second exponential, under the integral in equation (60), is recognized as being merely the cumulative normal distribution of the random variable X^*, which integrates to unity. Hence, the equivalent selection problem is to

$$\text{Max } E[U(X^*)] = 1 - e^{-\frac{1}{2}(2B\mu - B^2\sigma^2)} \qquad (61)$$

which can be done by <u>minimizing</u> the exponential term. This is accomplished by <u>maximizing</u> the quantity $\frac{1}{2}(2\mu - B\sigma^2)$, since B can be factored and the exponent is negative. Hence, if a new variable $Y^* = \frac{1}{2}(2\mu - B\sigma^2)$ is defined, then the equivalent selection problem is to

$$\text{Max } E[U(X^*)] \approx \text{Max } [Y^*] = \mu - \frac{B}{2}\sigma^2 , \qquad (62)$$

which yields the same result (the same set of optimal projects) as if equation (56) had been used as the selection criterion.

Thus — and this is an important conclusion — if the form of the selection criterion is that of equation (62), then the precedent <u>utility function is an exponential</u> (equation (55)) and <u>an assumption of normality</u> has been made for project net present values. This is considerably different from Farrar's incorrect derivation, wherein he obtained equation (62) from a <u>quadratic</u> utility function (which is impossible).

While Freund does not carry his analysis past this point, one can go on to an indifference curve analysis (as was done for the quadratic utility function) and show that the indifference curves obtained from equation (62) are similar in form to those obtained from the quadratic function. Thus, if $E[U(X^*)]$, and hence Y^* also, is a constant, then by differentiation of (62) one obtains

$$0 = 1 - B\sigma \frac{d\sigma}{d\mu}$$

from which

$$\frac{d\sigma}{d\mu} = \frac{1}{B\sigma} > 0 \; ; \tag{63}$$

and

$$\frac{d^2\sigma}{d\mu^2} = -\frac{1}{B^2\sigma^3} < 0 \; . \tag{64}$$

Thus, the indifference curves on $\sigma-\mu$ coordinates will slope upward and to the right, and will be concave downward (similar to Figure 6). Moreover, since

$$\frac{dY^*}{d\mu} = 1$$

and

$$\frac{dY^*}{d\sigma} = -B\sigma \; ,$$

the decision-maker's expected utility increases with increasing expected net present value and decreases with increasing risk, σ. Hence, the decision-maker with an exponential utility function is also a risk-avoiding one. The same risk attitudes can be inferred for such a person as for one who possesses a risk-avoiding quadratic utility function, except in the exponential case, there is no upper limit on the applicability of the utility function, since the first derivative of equation (55) merely approaches zero utility as an asymptote in the limit as net present value approaches infinity.

The Farrar Model

Adequate mention of the Farrar (19) model, which follows the Freund model by six years, has already been made. In recapitulation, however, what Farrar did is important in that he put Markowitz' hypothesis to an empirical test and substantiated it, thus paving the way for subsequent risk-analysis studies along the same lines.

The Watters 0/1 Capital Budgeting Model

All of the models considered heretofore--the Markowitz portfolio selection model, the Freund risk model, and the Farrar portfolio selection model--have been concerned with optimizing a set of alternative investments, in which the decision variable is the proportion of available funds to be invested in each alternative. The decision variable in the project selection problem of capital budgeting, on the other hand, represents the acceptance or rejection of integral projects. This accounts for the requirement in the project selection problem for the decision variable, x_j, to be integer-valued, 0 or 1. The output of the Markowitz, Farrar and Freund models is a set of funds allocations to alternative investments, whereas the output of the project selection model is a subset of projects which optimizes the objective function (equation (13)).

Watters (62) is the first investigator (1967), to the author's knowledge, who attempted a 0/1 optimization of a probabilistic capital budgeting model. Using a project selection criterion of the form

$$E[U(NPV)] = \mu - B\,\sigma^2 \qquad (65)$$

Watters shows that an indifference curve of the form

$$\sigma^2 = \frac{\mu}{B} - \frac{1}{B}E[U(NPV)] \quad (66)$$

can be obtained by holding E[U(NPV)] constant in the same manner as for the quadratic utility function in the previous section. Equation (66) is linear in μ, and hence is a straight line on σ^2-μ coordinates, as in Figure 7. The slope of each indifference "curve" is 1/B, the reciprocal of the decision-maker's "coefficient of risk aversion," which is determined from the utility function.

Watters then constructs a set of "feasible, efficient, maximum expected utility" (FEMU) portfolios of projects by examining all undominated combinations of projects in relation to equation (66), by assuming various values of B. In the same manner as Markowitz, he is then able to specify a particular set of FEMU projects for a particular decision-maker whose utility function yields a unique value of the risk-aversion coefficient, B.

Watters also investigates a special case of probabilistic capital budgeting constraints: one in which total project cost (that is, the sum of the costs of the selected projects) is assumed to be normally distributed. For this case, additional budget constraints are written in which total project cost is required to be less than some specified amount, with a specified (normal) probability. However, this constraint in and of itself does not prohibit projects being selected that might violate the <u>actual</u> funds available (the budget); thus, Watters also includes the conventional deterministic budget constraints (equation (14)) as well. As a comment, the probabilistic constraint appears to be a refinement that might, on occasion, be required; but it appears to be not very useful from a practical standpoint.

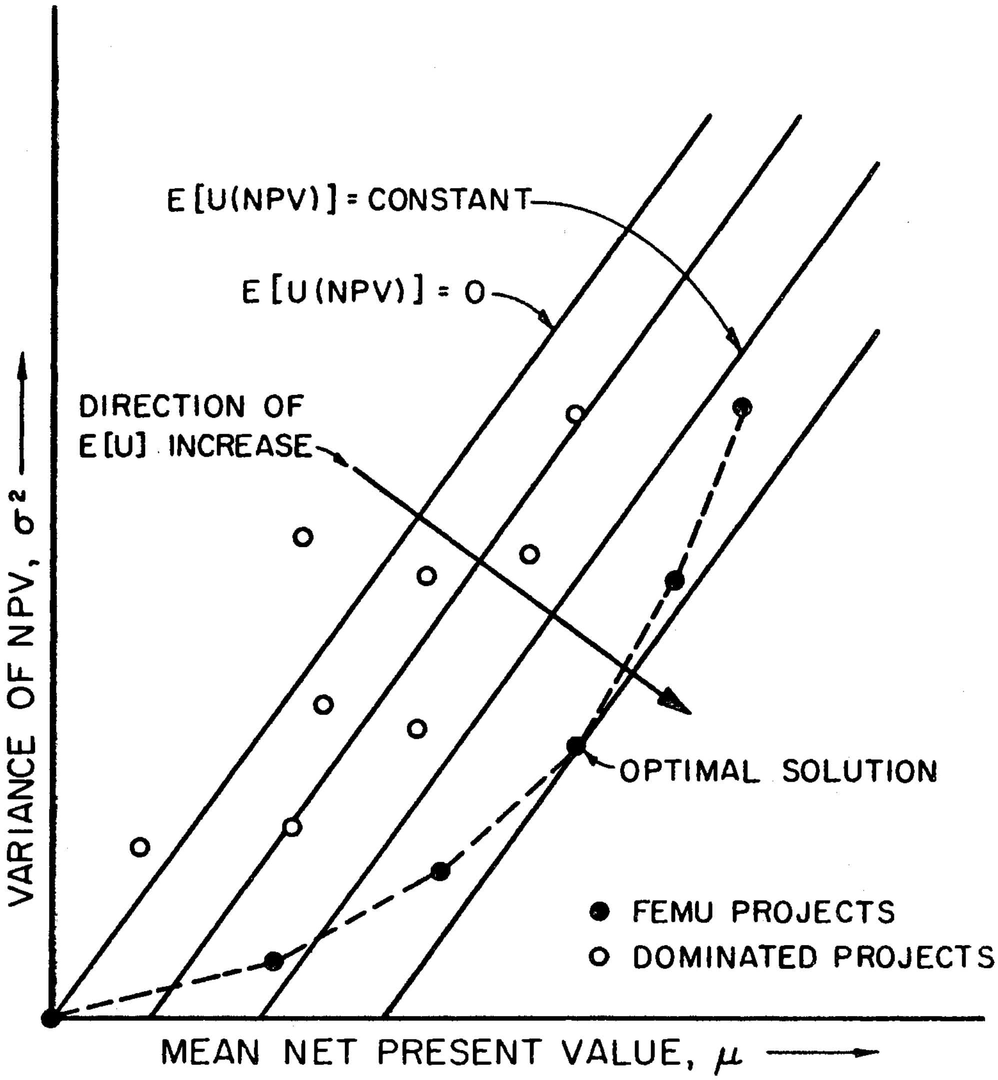


Figure 7. Watters' Project Selection Indifference Curves

Watters' principal contributions are two: (1) he applied a risk-avoidance selection criterion to the capital budgeting (0/1) problem, and (2) he developed a computer program for the solution of the 0/1 selection problem for both independent and dependent projects. At the time, both of these were significant contributions. However, as has been noted earlier, Watters erred in deriving his selection criterion function in the same fashion that Farrar did, apparently because he relied on an uncorrected version of Farrar's dissertation. With regard to the computer selection program, there was another selection algorithm (the Lawler-Bell (38) algorithm) available at the time that is at least as efficient as Watters', although Watters is not to be faulted for not being aware of its implications, as it appeared in a completely different context and had not been programmed for computer usage in 1967.

Other Non-Utilitarian Models

All of the project selection criteria that have been considered up to this point depend upon and are derivable from utility functions that are grounded in the von Neumann-Morgenstern utility theory. Utility theory, however, is not the only basis for project selection. Two of the more important non-utilitarian approaches are based on entirely different criteria. One, exemplified by the Roy model (below), is based upon a maxim of catastrophe avoidance, where catastrophe is defined to be the occurrence of an undesirable net present value. Another, the Harvey model (below), is based on a maxim that the firm's net asset value (both financial and capital assets--i.e., financial and physical) as of some future time called the "horizon," should be maximized.

The idea of catastrophe avoidance as a business decision criterion has strong intuitive appeal. As Barnard ((4), p. 16) indicates, it may be fear of precipitous loss, not profit gain, that dominates the business complex. Certainly, from an empirical standpoint, Swalm's research (59) on utility functions and the risk attitudes of middle-management personnel indicates a pronounced tendency toward avoidance of any kind of loss. It is this mode of thinking that prompted Roy to formulate his "disaster avoidance" model.

The Roy Model

In the introduction to the report of his research, Roy (52) states:

> In the economic world, disasters may occur if an individual makes a net loss as the result of some activity, if his resources are eroded by the process of inflation to, say, 70 percent of their former worth, or if his income is less than what he would almost certainly obtain in some other occupation. For large numbers of people some such idea of a disaster exists, and the principle of Safety First asserts that it is reasonable, and probable in practice, that an individual will seek to reduce as far as is possible the chance of such a catastrophe occurring...
>
> . . . .
>
> From a formal standpoint, the minimisation of the chance of disaster can be interpreted as maximising expected utility if the utility function assumes only two values, e.g., one if disaster does not occur, and zero if it does. It would appear, however, that this formal analogy is scarcely helpful, since in the one case an individual is trying to make the expected proportion of occurrences of disaster as small as possible, while in maximising expected utility he is operating on a level of satisfaction. Readers, however, are open to interpret the principle in this way if they so desire, but the purpose of this discussion is not to suggest that individuals may possess a utility function of peculiar form but rather to find out the implication of a certain mode of behavior, which appears both plausible and simple.

Thus, while the model can be based on a utility function of "peculiar" shape, it does not have to be, but rather it requires only an acceptance of the "disaster avoidance" maxim of behavior.

In applying the Safety First principle to the selection of assets, it is assumed that one is concerned that his gross return (i.e., net present value) should not be less than some quantity D. With every possible outcome is associated μ, the expected (mean) net present value. Since the outcome is not certain, there is coupled with μ a variance, σ^2, of the possible outcomes. It is assumed that both σ^2 and μ are known, although in practice they may have to be estimated by the use of subjective probabilities and a solution to the analysis problem (see Chapter III).

Given the values of μ and σ^2, for all possible combinations of projects, there exists a set of efficient combinations (in the same sense that Watters specifies FEMU projects), the envelope of which will be denoted $f(\sigma, \mu) = 0$, as in Figure 8. Since it is not possible to determine, with this information only, the unique probability of the final net present value being D or less for a given combination of μ and σ^2, the only alternative is to calculate an upper bound of this probability, which can be done by Chebyshev's inequality.

Let the random variable, X, be the distributed net present value with mean μ and variance σ^2. Then, by Chebyshev's inequality

$$P\left[|X - \mu| \geq \mu - D\right] \leq \left(\frac{\sigma}{\mu - D}\right)^2 ; \quad (D < \mu) . \qquad (67)$$

Now, the Chebyshev inequality expresses a symmetric or "two-tailed" probability (that is, $P[-(\mu - D) \leq |(X - \mu)| \leq +(\mu - D)]$), but if it is assumed that the entire probability of the event is concentrated in

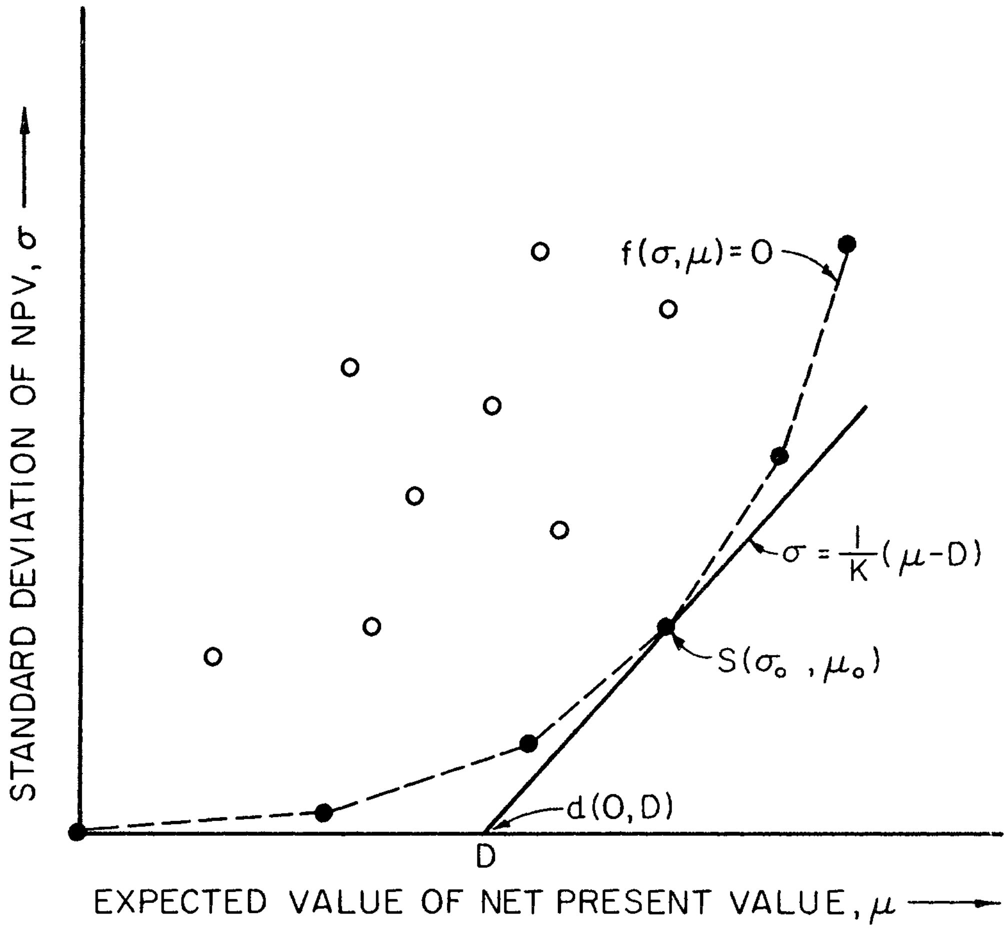


Figure 8. Roy's "Disaster Avoidance" Criterion Selection Model

the event $-(X - \mu)$, then

$$P[-(X - \mu) \geq (\mu - D)] = P[X \leq D] \leq \left(\frac{\sigma}{\mu - D}\right)^2; \quad (D < \mu) . \qquad (68)$$

Then, by defining $P[X \leq D] = 1/k^2$ (a constant), and by assuming the worst possible case (a strict equality in equation (68)), the form of equation (68) becomes

$$P[X \leq D] = \left(\frac{\sigma}{\mu - D}\right)^2 = \frac{1}{k^2} \qquad (69)$$

and the probability of disaster, $P[X \leq D]$, is minimized by maximizing the quantity $(\mu - D)/\sigma$.

By solving equation (69) for σ, one obtains

$$\sigma = \frac{1}{k}(\mu - D) \qquad (70)$$

which is the equation of a straight line on σ-μ coordinates, whose slope is $1/k$ and whose intercept on the abscissa (with $\sigma = 0$) is D. Thus, in Figure 8, the criterion function is the straight line emanating from the point D, and the project combination selected by this criterion is the subset $S(\sigma_0, \mu_0)$ whose mean is μ_0 and whose variance is σ_0^2. Although Roy does not develop the model (as above) by assuming that the probability of disaster becomes a constant evaluated under the worst possible conditions, such an assumption is required for the selection criterion to become a straight-line function with maximum slope, $1/k$. Moreover, as presented here, the constant, k, can be visualized as a pseudo-coefficient of risk aversion for the decision maker, in the same manner that the constant, B, was thus recognized in the Freund model.

The selection process is manifested as follows (referring to Figure 8). If one desires to avoid a net present value outcome of D or worse, the point d(0,D) is plotted on the μ-axis. A tangent of positive slope is then drawn from point d to touch the $f(\sigma, \mu) = 0$ convex set at $S(\sigma_0, \mu_0)$. Then, if one adopts the course of action (selects the projects) that produces the estimated outcome, μ_0, he will have made the upper bound of the probability of D or worse happening as small as possible, given the opportunities (FEMU projects) available.

As Roy points out, an obvious extension of the model occurs if the net present value is normally distributed, in which case normal probabilities instead of Chebyshev upper bounds can be used with considerable reduction in the probability of disaster.

The Harvey 0/1 Planning Horizon Model

As an alternative to the maximization of net present value model, Winegartner (63) formulated a "basic horizon model" for the deterministic case, in which the net assets of the firm (in lieu of net present value) are maximized as of some future time, called the "horizon." In this case, the financial transactions of the firm, which include cash "throwoffs" of projects initiated between present time (t = 0) and the horizon (t = T), borrowings, and lendings, are all converted to equivalent assets as of time T. Likewise, cash inflows from projects initiated at $t \geq T$ are converted by discount factors to equivalent assets at time T. The net assets (i.e., the sum of cash "throwoffs," borrowings, lendings, and net horizon values of future cash inflows) are then maximized at time T by a mathematical programming solution. In Winegartner's formulation, fractional projects were allowed and the solution was obtained by a mixed-integer linear programming formulation.

Harvey (28), in 1967, added some variance-covariance terms to Winegartner's formulation to take into account the uncertainty of net horizon values, and also provided for project indivisibility by requiring the decision variable to be integer-valued, 0 or 1. As with the original Winegartner formulation, which was adapted from prior work by Charnes, Cooper and Miller (11), the basic horizon model has expositional value from an economic theoretic standpoint when assumptions of perfect capital markets (known cost of capital) and independent projects are abandoned. With (assumed) known discount rate and independent projects, however, Winegartner demonstrates that the basic horizon model is exactly equivalent to the maximization of net present value model, and this presumably carries over to the Harvey model also (i.e., it is presumed to be equivalent to the maximization of expected utility model).

Summary of Chapter IV

The material presented in Chapter IV is concerned fundamentally with the choice of a project selection criterion. Based on the von Neumann-Morgenstern utility theory, which is assumed here to be *prima facie* valid, it has been demonstrated that several selection criteria can be derived. In particular, if a quadratic utility function is the precursor, then maximization of expected utility of net present value, $E[U(NPV)]$, of the form $E[U(NPV)] = A\mu - B(\sigma^2 + \mu^2)$ is the correct selection criterion. Such a criterion leads to the stipulation of a family of risk-avoiding indifference curves which can be used to specify which of a set of candidate projects will be chosen by a risk-avoiding decision-maker so as to maximize his expected utility.

Likewise, if a negative exponential utility function is the precursor, then maximization of expected utility of net present value is the correct criterion, where $E[U(NPV)] = \mu - B\sigma^2$ is the proper criterion to use. Such a utility function leads to a similar set of indifference curves and to similar decisions, as for the quadratic utility function.

In addition, two non-utilitarian risk-avoidance models have been examined, to show that risk-avoidance criteria do not necessarily stem from *a priori* utility functions. This leads to one of the more important conclusions concerning decision criteria: at the present time, there is no single criterion that one can assume that will perforce describe all aspects of even risk-avoiding decision behavior.

In spite of this rather crippling conclusion, the utility function models seem to offer the best hope. This is particularly true when one considers *non*-risk-avoiding behavior--for example, the behavior implied by the utility curves of John Beard and Fred Hartman in Figure 2 and R. F. Mellon in Figure 3. The development of non-risk-avoiding selection criteria is the subject of the next chapter.

FOOTNOTES

[1]Savage (53) gives a detailed discussion of the von Neumann-Morgenstern utility theory. Implications of the theory are discussed by Luce and Raiffa (40), Fishburn (21), and Alchian (3). Virtually every capital budgeting text published and dissertation undertaken since these original works were published contains a presentation of the same material, in more or less detail. The subject is overworked.

[2]See Savage (53), pp. 101-103 for a discussion of Allais' criticisms of the rational behavior axioms, and Alchian (3) for an impartial discussions of the merits and demerits of the N-M utility theory.

[3]The St. Petersburg paradox will not be reproduced here, since it is already stated in many places in the literature. See Savage (53), page 93.

[4]The linear transformation just demonstrated--namely, choosing $h = -X^*$ and $U(0) = 0$--was incorrectly performed by Farrar (19), and subsequently was used in its incorrect form by Watters (62). Farrar's error led him from a form of the utility function, $U(X) = AX - BX^2$, to an expected utility function of the form $E[U(X)] = A\mu - B\sigma^2$. This form of the expected utility function <u>cannot</u> be derived from the quadratic utility function. Had Farrar correctly made the linear transformation, he would have derived the expected utility in substantially the same form as equation (49). While both Farrar and Watters formulated their maximization <u>models</u> correctly, on the basis of the criteria they used, the results obtained by the <u>models</u> do not follow from their <u>basic</u> assumption of a quadratic utility function. Farrar's error was first pointed out by Adelson (2) in 1967, and corrected by Schoner (55) later the same year. In the second printing (the Markham edition) of Farrar's dissertation, Farrar credits William F. Sharpe with the detection of the error, but apparently Sharpe's notation was privately conveyed to Farrar, as it does not appear in the literature. Farrar's second (Markham) edition is Reference (20).

[5]Freund omits all of the subsequent details for obtaining the equivalent selection criterion (equation (62)), particularly the technique of the transformation to a new variable. He does, however, mention the skeleton procedure for deriving the criterion in a footnote ((24), page 255).

CHAPTER V

PROJECT SELECTION WITH NON RISK-AVOIDING BEHAVIOR

Introduction to Non Risk-Avoiding Decision Behavior

Many decision-makers do not always behave strictly in accordance with the rational risk-avoiding behavior that is implied by the concave downward utility functions examined in Chapter IV. A simple example is the business executive who "gambles" in prospecting for petroleum and simultaneously buys fire and casualty insurance to avoid loss of his physical assets. If he were strictly a risk-avoider, he would not necessarily engage in a business activity where the uncertainties are great (the ratio of "dry holes" to successful wells varies from about 10:1 to perhaps 30:1 in "unproven" territory), and where payoffs, if attained, sometimes may be disappointingly small in relation to the sums expended for development. On the other hand, if he were a risk-seeker, then why would he purchase insurance against a loss (of physical assets) that could very well result in financial ruin the same as if he drilled too many "dry holes" in succession?

The same sort of question arises when one considers the decisions sometimes made by managers of active, growing and aggressive small businesses. A typical situation here is one in which the management consists of a dedicated nucleus of individuals, probably with diverse talents and capabilities, a financial structure characterized by a small equity (probably owned mostly by the management) and as much borrowed funds as can be levered, and a market structure that is far from certain. If these men, the owner-managers, were completely risk-averse, then they would consistently pass up investment opportunities for product or market development that did not offer increased expected net present value for increased uncertainty. But this is not the modal form of behavior for business executives in such a situation. Somehow, business executives in active and growing businesses are able to "sense" and "size up" unusual opportunities for levering growth capability--that is, they do not strictly require an increased expected net present value for an increased risk. Somehow, they are able to "see" how a given risk situation can be influenced to the firm's advantage by subsequent attempts on their part to control environmental variables that affect outcome. Perhaps they have an introspective "feel" for a skewed net present value distribution that would indicate a higher probability of an outcome greater than the expected net present value--or, perhaps they somehow perceive that they might be able, by controlling the environmental determinants, to introduce such a skewness into the distribution even after the project is undertaken. Regardless of the motivation, however, decisions that do not require an increased expected net present value in exchange for acceptance of increased uncertainty cannot be explained by risk-avoidance project selection criteria. The question is, how is such behavior to be analyzed?

Questions such as this were used by opponents of the von Neumann-Morgenstern utility theory to discredit the theory shortly after it was proposed. Such behavior, the critics said, is "irrational" and shows that the theory is sterile and incapable of predicting the choices of a decision-maker. To counter these allegations, Friedman and Savage (23) demonstrated that a decision-maker with a complex utility function--that is, one that is concave downward (risk-avoiding) over certain ranges of payoff and concave upward (risk-seeking) over others--would explain the simultaneous acceptance of an "unfair" gamble and the purchase of insurance to avoid loss in another quarter. To understand how such behavior can occur, consider the utility function in Figure 9.

First, consider a case in which insurance would be purchased by the decision-maker in order to avoid a large loss. Suppose, for example, that the one-time premium for insurance to insure against a loss of $10,000 is $550, and that the actuarial probability of such a loss is 0.005. Now, from Figure 9, the utility of a payment of -$550 is -3 utiles, and the utility of a loss of -$10,000 is -800 utiles. The alternative actions available are (1) purchase the insurance, and (2) self-insure (carry no insurance). Thus, the expected monetary value (EMV) and the expected utility of each of the alternatives are:

<u>Carry Insurance</u>:

$$\text{EMV} = .005(-\$550) + .995(-\$550) = -\$550$$

$$E(U) = .005(-3) + .995(-3) = -3 \text{ utiles}$$

<u>Self-Insure</u>:

$$\text{EMV} = .005(-\$10{,}000) + .995(0) = -\$500$$

$$E(U) = .005(-800) + .995(0) = -4 \text{ utiles.}$$

Thus, if the decision-maker maximized expected utility, he would purchase the insurance, since this alternative has greater utility (-3 utiles) than the self-insurance alternative (-4 utiles). (Incidentally, note that this is the opposite decision that would have been made had

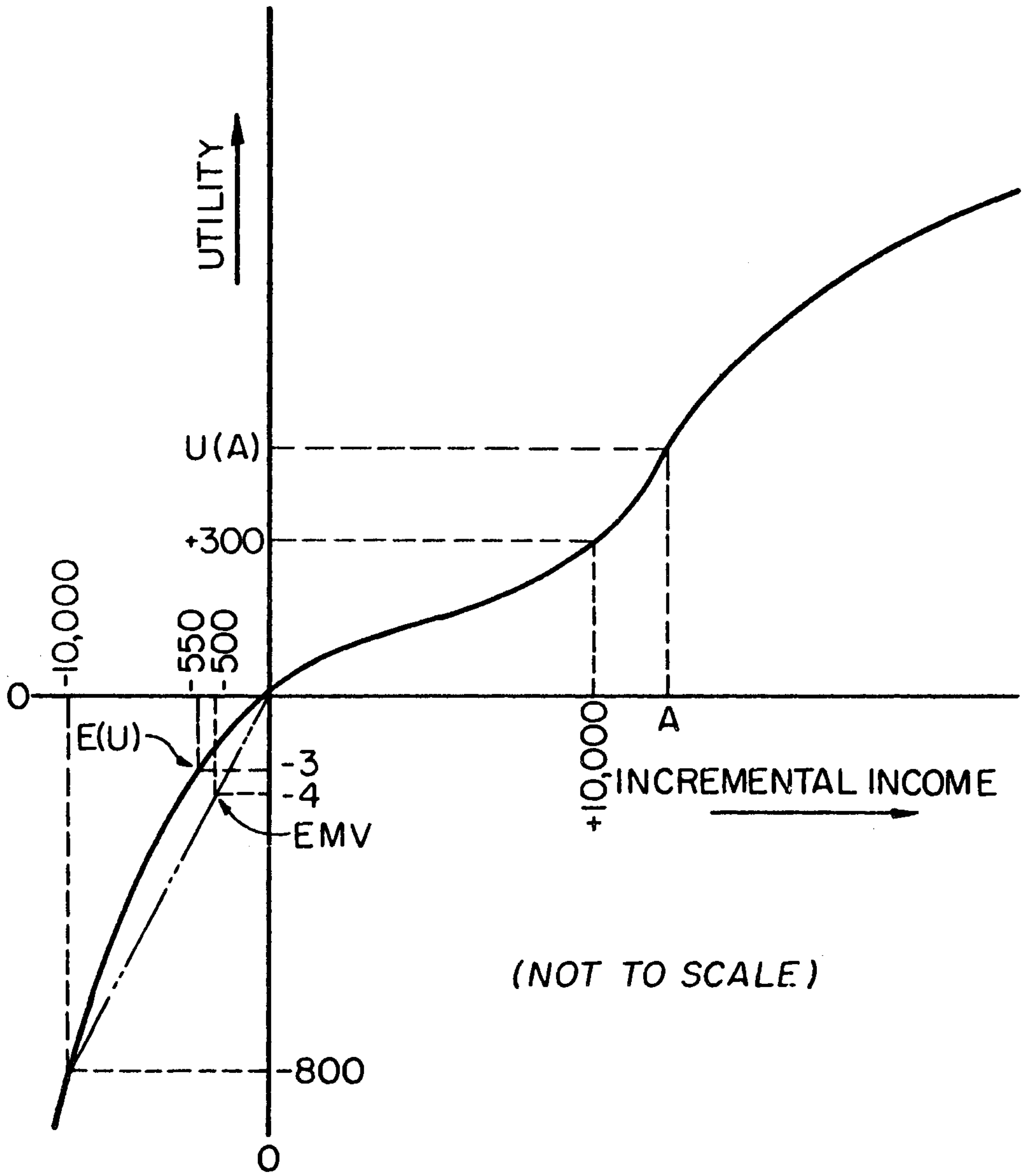


Figure 9. Hypothetical Utility Function

he maximized expected monetary value.

Next, consider that the same decision-maker has been offered an opportunity to invest in an oil-drilling venture in which the geologist has estimated that there is only one chance in 200 ($p = 0.005$) of striking a commercial well and a probability of 0.995 of getting a dry-hole and losing the investment of $550. If the venture is successful, the decision-maker can anticipate a payoff of $10,000. Suppose also that the utility of a payoff of $10,000 is +300 utiles. The alternative actions available are (1) invest in the venture, and (2) do not invest. The expected monetary value and expected utilities of the alternatives are:

<u>Invest in the Venture</u>:

$$\text{EMV} = .995(-550) + .005(10{,}000 - 550) = -\$500.00$$

$$E(U) = .995(-3) + .005(300 - 3) = 12.0 \text{ utiles}$$

<u>Do Not Invest</u>:

$$\text{EMV} = .995(0) + .005(0) = \$0$$

$$E(U) = .995(0) + .005(0) = 0 \text{ utiles.}$$

Again, the decision-maker who maximizes expected utility would invest in the venture--and note, that his cost of accepting the alternative is $550, exactly the same as his cost for the insurance which he also purchased. Note also that had he maximized expected monetary value, he would not have invested in the risk-taking venture.

Markowitz ((42), page 218) also points out that the complex utility function is consistent with the investor's behavior in diversifying his investments, when "important money" is at stake. That is, such an investor will insure against large losses, take small bets, and diversify his portfolio of important money investments. Even on an intuitive basis, such behavior seems more realistic for an entrepreneur than complete risk avoidance under all conditions of payoff and uncertainty.

Even more theoretical evidence for the existence of complex utility functions is added by the work of Sidney Siegel (57), from the field of psychology. Working from Lewin's level of aspiration theory, Siegel showed (1957) that, with very little difference in terminology, the psychological "level of aspiration" is equivalent to the inflection point (A, in Figure 9) of the utility function, and that the individual will take risks in order to obtain a return of at least the utility of A. Siegel substantiated his derivation by a simple experiment involving 20 students and their semester grades, in which the levels of aspiration for the "utility" of a grade were measured and correlated with their risk preferences.

Chernoff and Moses (12) also used, but did not explain nor amplify their choice, utility curves of "sigmoid shape" in explaining insurance purchase behavior (1959). Karl Borch (7), in an article on the derivation of the necessary and sufficient conditions for a quadratic utility function, also reported that cubic and fourth-degree utility functions were to be discussed fully in a forthcoming article in <u>Skandinavisk Aktuarietidskrift</u> (Scandinavian Actuarial Journal), but the latter article has not been located.[1] Apart from the foregoing references to theoretical developments, no other work is known to exist that deals with the interpretation of complex decision behavior by higher-degree utility functions. However, there is some empirical evidence that such functions do exist, aside from Siegel's small experiment, and the implications of this evidence will be examined in the next section.

Empirical Evidence of Complex Utility Functions

Virtually all of the empirical determinations of utility functions for individuals, that have appeared in the literature, have been made by the investigators named in Chapter IV--Mosteller and Nogee (45), Davidson, Suppes and Siegel (14), Green (26), Grayson (25), Cramer and Smith (13), and Swalm (59). Of all of these, with the notable exception of some reported by Grayson and one reported by Swalm, most are of the concave downward (risk-avoiding) type. Only one of Swalm's is an essentially linear utility function. The concern here, however, is with those few functions reported by Grayson that indicate non-risk-avoiding behavior, particularly those of John Beard and Fred Hartman (Figure 2) and R. F. Mellon (Figure 3). These men display utility functions that are, over some portion of the monetary return scale, concave upward and therefore indicate some connate preference for risk on the part of these decision-makers. However, this is not the only reason why our attention is focused on these curves, particularly Mellon's function in Figure 3.

The thing that stands out most noticeably is the <u>variation</u> that is quite apparent in the data at 350, 500 and 750 thousand dollars (in Figure 3), and that an essentially "free-hand' curve has been fitted to the data points elsewhere (in the negative quadrant and in the vicinity of the origin) with such precision as to virtually ignore the statistical meaning of the variation present in the data. This is not to criticize Dr. Grayson, but merely to question whether or not such precision in the mapping of the function itself is warranted. Proceeding further, it was reasoned that if inherent variation in the individual's responses to the standard lottery were present, then Mellon's data offered an opportunity for a curve of best fit to be determined by multiple regression techniques, which could then be analyzed by analysis of variance methods for adequacy of fit. Specifically, if the sampling error could be estimated (which would require repeated data points, as at 350, 500 and 750 thousand dollars), then by analysis of variance one could determine whether or not a linear, quadratic, cubic or higher-order polynomial regression equation fits the data "best," by testing the significance of the variance removed by each of the higher-ordered terms in the regression function.

Fortunately, since Dr. Grayson "overlapped" his standard lotteries (i.e., made multiple determinations of response at repeated (constant) levels of payoff), it is possible to estimate the inherent sampling error for an individual and to perform the regression analysis of variance. However, since Grayson did not report the actual utilities of payoff that he used in plotting his utility functions, his raw data are again manipulated here so as to obtain the input information for the regression analysis. Essentially the same data manipulations as Grayson's are performed here, except that the present assumption is that -150 utiles is associated with a loss of $150,000, for the Mellon data. This assumption does not affect the shape of the utility function; merely the scale, since the utility function is by necessity assumed to be capable of linear transformation. Then, by using the Grayson data ((25), page 300) as raw data, the utilities in Table II were calculated by writing the expected utility identity for each data point. That is, if Mellon's response to the standard lottery were,

TABLE II

UTILITY FUNCTION DATA - R. F. MELLON

Investment (thousands) (I)	Utility of Investment U(I)	Net Payoff (thousands) (NPV)	Indifference Probability P(gamble)	Utility of Payoff U(NPV)
10	- 2.2	20	0.70	0.95
		30	.50	2.2
		50	.40	3.3
		90	.25	6.6
20	- 4.6	20	.90	0.5
		30	.90	0.5
		100	.60	3.1
		150	.40	6.9
50	- 27.5	50	.90	3.1
		100	.90	3.1
		350	.40	41.3
		750	.20	110.
75	- 56.2	90	.90	6.2
		350	.60	37.5
		750	.30	131.
		925	.20	225.
100	-112.5	250	.90	12.5
		350	.80	28.1
		500	.75	37.5
		900	.60	75.0
150	-150.	150	No	--
		350	.80	37.5
		500	.80	37.5
		850	.70	64.3

Source (Columns 1, 3 and 4): Grayson (25), page 300. The data in Column 2 were assumed so that best overlap of individual curves resulted.

for example, an indifference probability of 0.70 between a net payoff of \$850,000 and an investment of \$150,000 to obtain the gamble, then the expected utility of the net gamble is zero and the utility of \$850,000 can be determined, thus:

$$E(U) = 0 = 0.70\ U(850{,}000) + 0.30\ U(-150{,}000)$$

$$0 = 0.70\ U(850{,}000) + 0.30\ (-150)$$

from which

$$U(850{,}000) = 64.3 \text{ utiles} .$$

The remaining data points in Table II were obtained in the same manner.

Using these data, a multiple linear regression was performed, in which the regression equation was assumed to be of the form

$$\hat{U} = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \beta_4 X^4 ; \qquad (71)$$

where $\hat{U}$ = the regressed response variable (utility),

β_i = the regression constants ($i = 0, \ldots, 4$)

X = the independent variable, payoff (NPV) .

The regression constants were evaluated by a computer-programmed abbreviated Doolittle method, yielding a function of least-squares approximation to the data in Table II. The resulting regression estimate is

$$\hat{U} = -17.63 + 48.74\ X - 17.967\ X^2 + 2.721\ X^3 - 0.123\ X^4 , \qquad (72)$$

where X has units of Dollars x 10^{-5}. This equation is exactly analogous to a utility function of the same form, namely

$$U(X) = K + AX - BX^2 + CX^3 - DX^4$$

where the coefficients K, A, B, C, and D are, for the Mellon data, those of equation (72).

Now, the question is, how well does equation (72) fit and describe the data of Table II? This question can be answered by performing an analysis of variance on the regression, which first requires an estimate of the experimental (sampling) error. Noting in Table II that there are two observations of payoff utility, U(NPV), for each of the payoffs of 90, 100, 500 and 750, and four observations for the payoff of 350, a pooled error variance can be calculated, as in Table III.

TABLE III

CALCULATION OF POOLED ERROR VARIANCE

	X =					
	90	100	350	500	750	
U =	6.6 6.2	3.06 3.06	37.5 28.1 37.5 41.3	37.5 37.5	131. 110.	
$\Sigma U^2 =$	82.0	19.7	5294.	2812.	29,261.	
$\frac{(\Sigma U)^2}{N} =$	81.9	19.7	5210.	2812.	29,041.	
$d_u^2 =$	0.1	0	84.	0	220.	$\Sigma d_u^2 = 304.1$
d.f. =	1	1	3	1	1	Σd.f. = 7

$$\sigma_e^2 = \frac{\Sigma d_u^2}{\Sigma d.f.} = \frac{304.1}{7} = 43.5$$

$$\sigma_e = \sqrt{43.5} = 6.6$$

In Table III, d_u^2 is the mean square deviation, or $\Sigma U^2 - (\Sigma U)^2/N$; d.f. is the degrees of freedom per cell; N is the number of observations per cell; and σ_e^2 is the error (sampling) variance.

The analysis of variance for the quartic utility function, equation (72), appears in Table IV.

TABLE IV

ANALYSIS OF VARIANCE: LINEAR REGRESSION OF QUARTIC UTILITY FUNCTION FOR R. F. MELLON

Source of Variation	Sums of Squares	Degrees of Freedom	Mean Square	$F_{calc.}$	Significance
Total	137189.	28	---	---	---
$R(\beta_0)$	9740.	1	9740.	224.	< .01
$R(\beta_1/\beta_0)$	91800.	1	91800.	2100.	< .01
$R(\beta_2/\beta_1,\beta_0)$	1690.	1	1690.	38.8	< .01
$R(\beta_3/\beta_2,\beta_1,\beta_0)$	12550.	1	12550.	288.	< .01
$R(\beta_4/\beta_3,\beta_2,\beta_1,\beta_0)$	2105.	1	2105.	48.4	< .01
Residual	19304.	23	---	---	---
Lack of Fit	(19000.)	(16)	(1188.)	27.3	< .01
Error (Sampling)	(304.)	(7)	(43.5)	---	---

It is quite evident from the analysis of variance that the variation removed by each of the terms $R(\beta_i)$, in the quartic function, is highly significant. Indeed, from the significance of the "lack of fit" remainder, it is also evident that the quartic function could be improved upon by the assumption of a quintic polynomial or one of higher degree, but to do so might be unwarranted because of the relative smallness of the sampling error variance.

Since all of the mean squares corresponding to the coefficients β_i ($i = 0, 1, \ldots, 4$) are highly significant, then all of these coefficients are significantly different from zero.[2] The only one that poses a theoretical problem is the intercept coefficient, $\beta_0 = -17.63$. This value indicates that R. F. Mellon <u>might</u> have a disutility for any payoff less than about $40,000, but this seems hardly plausible. What is more probable is that the standard lottery, in this case, did not measure Mellon's utility with the indicated small sampling error. Had this error been somewhat larger, as would be indicated by the variation in the data points near the upper limit of payoff, then the intercept coefficient might not be significantly different from zero. The regressed utility function for Mellon is illustrated in Figure 10, together with the response data from the standard lottery.

The following conclusions appear to be justified by the regression and analysis of variance. Since the hypothesis that Mellon's utility function is at least a quartic polynomial cannot be rejected, it is assumed that such a function (equation (72)) is a valid approximation to the true utility function. Certainly, to omit the fourth-degree term in the regression equation would make matters worse; this was, in fact, attempted and a worse fit (overall) was obtained. It is concluded, therefore, that there is evidence presented here to show that a complex utility function--in this case, a quartic polynomial--does, in fact, exist. Furthermore, it is concluded that the intercept coeffici-

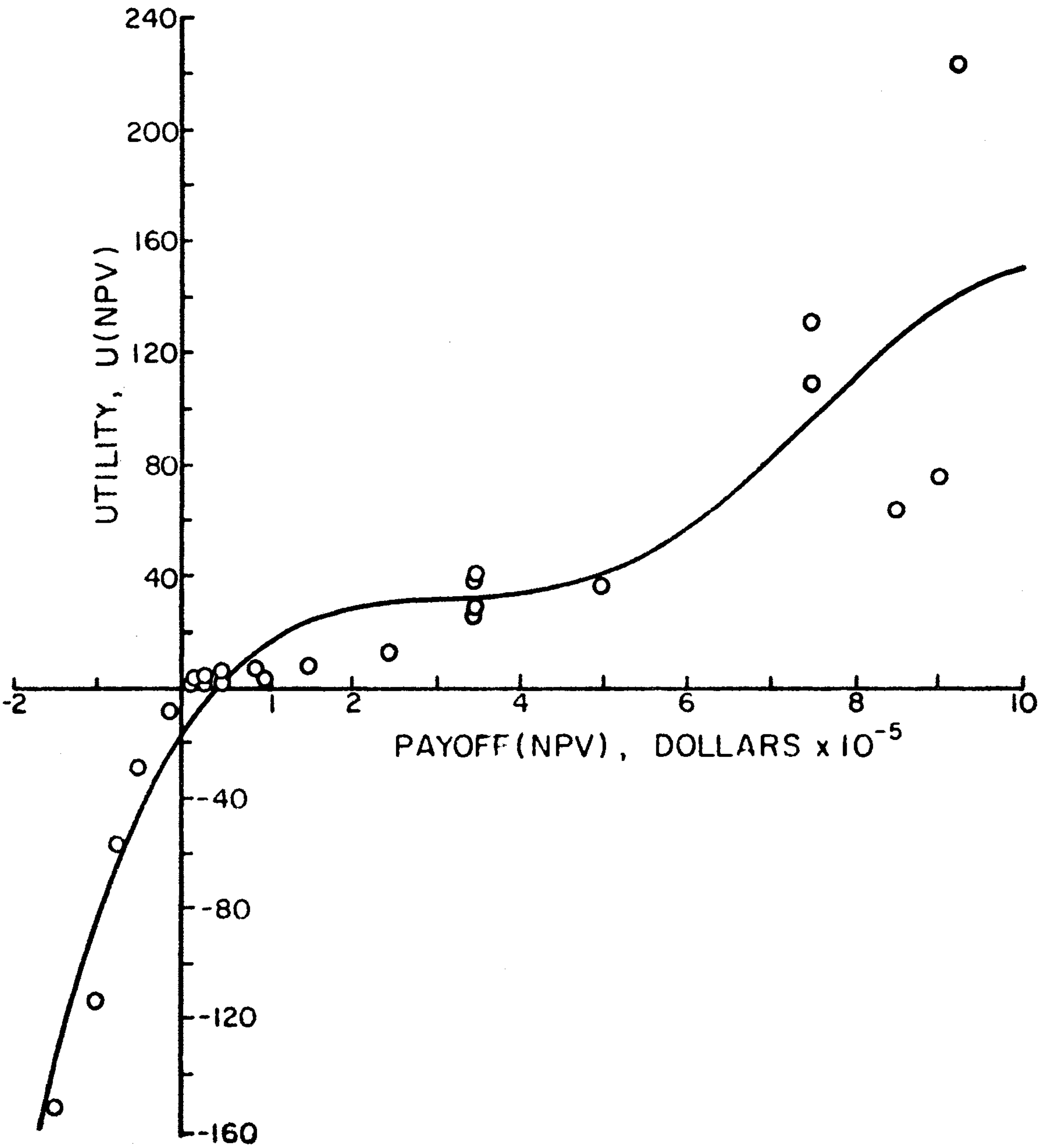


Figure 10. Regressed Utility Function
for R. F. Mellon

ent, β_0, while significantly different from zero, does not pose an insurmountable problem in the analysis of the utility function, since it disappears in the derivatives of the utility function that are used to analyze complex risk attitudes, as will be seen.

Regression equations have also been fitted to the utility data of John Beard and Fred Hartman (Figure 2). The regression equations obtained by least-squares fitting are, for John Beard:

$$\hat{U} = -18.90 + 62.86X - 9.125X^2 + 1.004X^3 - 0.036X^4 \tag{73}$$

and for Fred Hartman:

$$\hat{U} = -1.939 + 108.32X - 40.269X^2 + 5.589X^3 - 0.226X^4 . \tag{74}$$

While these equations both seem to indicate that the utility functions are quartic polynomials, one cannot say so with any degree of confidence. The sampling error could not be estimated reliably in either case, and no meaningful analysis of variance could be obtained. Nevertheless, it is apparent from Figure 2 that both Beard's and Hartman's utility functions are anything but risk-avoiding; and it is, therefore, concluded that these two utility functions are further evidence of non-risk-averse behavior.

Since empirical evidence indicates that non-risk-averse behavior does occur, the question then is, what is the theory behind the utility functions that seem to describe such behavior? Or, what implications can be drawn from a theory of non-risk-avoiding behavior? Answers to these questions will be sought in the next two sections of this Chapter. The first such section is concerned with outright risk-avoiding/risk-seeking behavior (as implied by a <u>cubic</u> utility function). The second investigates more complex behavior, where the range of risk-seeking is limited. The general case, if it exists, is that investigated in the second section.

Theoretic Development of Risk-Seeking Decision Criteria

Here the concern is with the utility function of a decision-maker who simultaneously purchases insurance to avoid large losses and, beyond a certain level of positive payoff, will seek risks regardless of the payoff level. While this is not necessarily a "realistic" form of behavior for a real-life decision-maker, the utility function of Fred Hartman (Figure 2) indicates that he might actually be this kind of decision-maker. The analysis of this kind of utility function also serves as an introduction to more complex behavior, and is worthy of pursuit for that purpose alone.

The form of this type of utility function is illustrated in Figure 11, and here the general case is assumed (the function U(X) does not pass through the origin). The "true" function can be approximated by a Taylor series expansion about the point X = 0, U(0), as follows:

$$U(0 + h) = U(0) + h \left.\frac{\partial U}{\partial X}\right|_{X=0} + \frac{h^2}{2!} \left.\frac{\partial^2 U}{\partial X^2}\right|_{X=0} + \frac{h^3}{3!} \left.\frac{\partial^3 U}{\partial X^3}\right|_{X=0} + \cdots \qquad (75)$$

The assumption is now made that the "true" utility function is a third-degree polynomial. One cannot assume less, because to do so would ignore the compound curvature; and one cannot assume more without commiting himself further as to its shape. Under this assumption the Taylor series remainder becomes zero, and the "true" function is fitted exactly

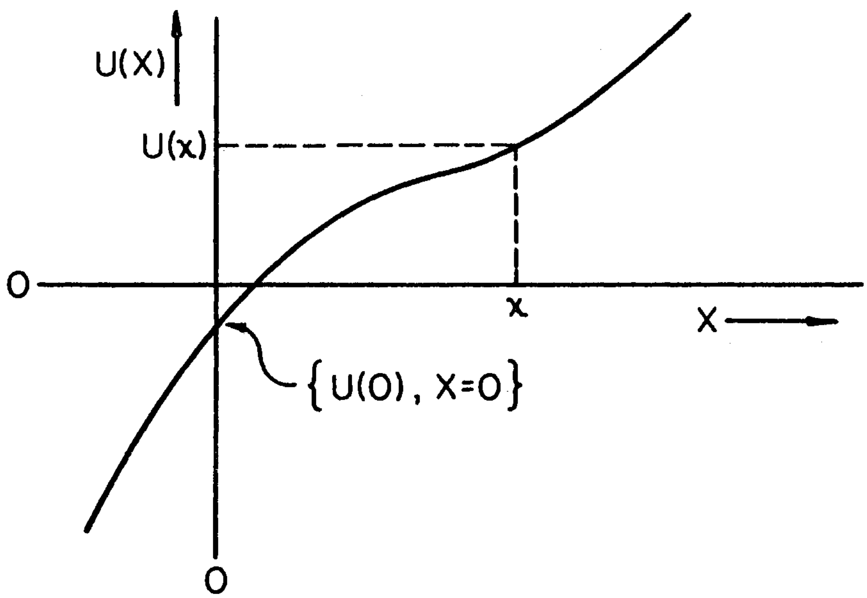


Figure 11. Cubic Utility Function

by the first four terms of the expansion. Further, define the perturbation, h, as some value of X, say h = x. Because U(X) is an arbitrary scale without rational zero point, the value U(0) can be defined to be zero (although it was taken as a negative value in Figure 11), thus causing the expansion to pass through the origin. With h = x, and U(0) = 0, the expansion becomes

$$U(0 + x) = 0 + x \frac{\partial U}{\partial X}\bigg|_{X=0} + \frac{x^2}{2!}\frac{\partial^2 U}{\partial X^2}\bigg|_{X=0} + \frac{x^3}{3!}\frac{\partial^3 U}{\partial X^3}\bigg|_{X=0} . \tag{76}$$

Now, let $A = \frac{\partial U}{\partial X}\Big|_{X=0}$; $B = \frac{1}{2}\frac{\partial^2 U}{\partial X^2}\Big|_{X=0}$; $C = \frac{1}{6}\frac{\partial^3 U}{\partial X^3}\Big|_{X=0}$;

then:

$$U(x) = Ax + Bx^2 + Cx^3 . \tag{77}$$

Characteristics of the Cubic Utility Function

Without qualification, equation (77) is not a utility function. In order to qualify it as a utility function, four conditions must be satisfied simultaneously. These conditions are:

(1) For all values of $x > 0$, $U(x) > 0$;

(2) Equation (77) must possess at most one positive root;

(3) The marginal utility, $U'(x) = dU/dx$, must be everywhere positive; and

(4) The inflection (stationary) point must lie in the first quadrant.

These conditions will now be investigated.

Condition 1: $U(x) > 0$ for All $x > 0$. From the theorem known as the "Fundamental Theorem of Algebra" (in the Theory of Equations), it can be shown that if the coefficient of x^3 is positive and if the highest degree of x is odd, then the function U(x) slopes upward to the right and downward to the left when plotted on rectangular coordinates. This condition, in conjunction with the proof of Condition 2, below, will permit the finding that $U(x) > 0$ for all $x > 0$.

Condition 2: U(x) Has at Most One Positive Root. Factor equation (77) into the following form:

$$\begin{aligned} U(x) &= x(Cx^2 + Bx + A) \\ &= x(x - R_1)(x - R_2) \end{aligned} \qquad (78)$$

where R_1 and R_2 are the roots of the quadratic form $Cx^2 + Bx + A$:

$$R_1, R_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C} .$$

Now, let $U(x) = 0$; then the roots of equation (78) are simply x, $x - R_1$, and $x - R_2$. Two of these roots can be made to be imaginary if the discriminant of the quadratic form, $B^2 - 4AC$, is made negative, thus leaving only one real root, x. Hence, for $U(x) = 0$, then $x = 0$, and

$$B^2 - 4AC < 0$$

or, the equivalent:

$$B^2 < 4AC \quad . \qquad (79)$$

Since B^2 has already been assumed positive in the Taylor series expansion, then A and C must have like signs for inequality (79) to hold. When A and C are of like sign, and when $B^2 < 4AC$, then the function U(x) has only one real root, $x = 0$.

Thus, two of the conditions for equation (77) to be a utility function are that A and C have like sign, and that inequality (79) hold among the coefficients A, B and C. If the coefficient C is assumed to be positive (as in Condition 1, above), then A must be positive also. This requirement, together with inequality (79) and Condition 1, is sufficient to show that $U(x) > 0$ for all $x > 0$.

Condition 3: The Marginal Utility, $U'(x)$, is Positive. Let the marginal utility of $U(x)$ be defined as

$$U'(x) = \frac{\partial U(x)}{\partial x} = A + 2Bx + 3Cx^2 \quad . \tag{80}$$

The form of this quadratic polynomial is such that if C is positive, as has been assumed above (Conditions 1 and 2), then equation (80) is a concave-upward parabola. From the "Fundamental Theorem of Algebra," if the highest degree of the polynomial is even, and if the coefficient, 3C, of the highest degree term is positive, then the function $U'(x)$ slopes upward to the right and upward to the left when plotted on rectangular coordinates. Thus, equation (80) is a concave-upward parabola. Such a quadratic will be everywhere positive, that is

$$A + 2Bx + 3Cx^2 > 0 \tag{81}$$

when the roots of the quadratic form are imaginary. The roots of the quadratic are

$$R_3, R_4 = \frac{-2B \pm \sqrt{4B^2 - 12AC}}{6C} = \frac{-B \pm \sqrt{B^2 - 3AC}}{3C} \tag{82}$$

and the discriminant of the roots is the quantity, $B^2 - 3AC$. If the discriminant is defined to be negative, then the roots are conjugate imaginary numbers. This fact, coupled with the assumption of $C > 0$, then causes the function $U'(x)$ to be everywhere positive. Thus, a condition for the marginal utility to be everywhere positive is that

$$B^2 - 3AC < 0 \ ,$$

or, the equivalent: $$B^2 < 3AC \ . \qquad (83)$$

Note that this restriction is of the same form as inequality (79), but is "tighter" than (79). Thus, compliance with inequality (83) will also cause compliance with (79), and no loss of generality results.

<u>Condition 4: The Inflection (Stationary) Point of U(x) Lies in the First Quadrant</u>. The stationary, or inflection, point of U(x) is defined by the second derivative

$$U''(x) = \frac{d^2U(x)}{dx^2} = 2B + 6Cx \qquad (84)$$

which is also the equation of the tangent of the quadratic in equation (80). Having shown that (80), the expression for the marginal utility, is a concave upward parabola, then its tangent will be minimum-valued when

$$U''(x) = 0 = 2B + 6Cx$$

or, solving for B:

$$B = \frac{-6Cx}{2} = -3Cx \ . \qquad (85)$$

Since it was assumed that $C > 0$, and because x is also a positive number in the first quadrant, then, from equation (85) <u>the value of B must always be negative</u> for the inflection (stationary) point of U(x) to lie in the first quadrant.

Summary of Conditions for the Cubic Utility Function

For a valid cubic utility function to exist, the following conditions must be met, with respect to the coefficients:

(1) The value of C must be positive;

(2) The values of A and C must have like sign; that is, positive;

(3) The coefficients A, B, and C must be related by the inequality $B^2 < 3AC$; and

(4) The value of B must be negative.

If the coefficients of the cubic polynomial of the form of equation (77) meet these tests, then the polynomial may serve as a valid representation of a cubic utility function.

Shape of the Defined Cubic Utility Function

The convexity or concavity of the cubic utility function is determined by its second derivative, equation (84). If $U''(x) < 0$, then the utility function is concave downward, and this occurs in the region

$$U''(x) = 2B + 6Cx < 0 ,$$

or, when

$$x < -\frac{B}{3C} ; \quad (B < 0) . \tag{86}$$

Likewise, if $U''(x) > 0$, then the utility function is convex, which occurs in the region $x > -B/3C$ (for $B < 0$), and the stationary point occurs at $U''(x) = 0$, or at $x = -B/3C$ (for $B < 0$). Hence, the fully defined cubic utility function may now be graphically illustrated, as in Figure 12.

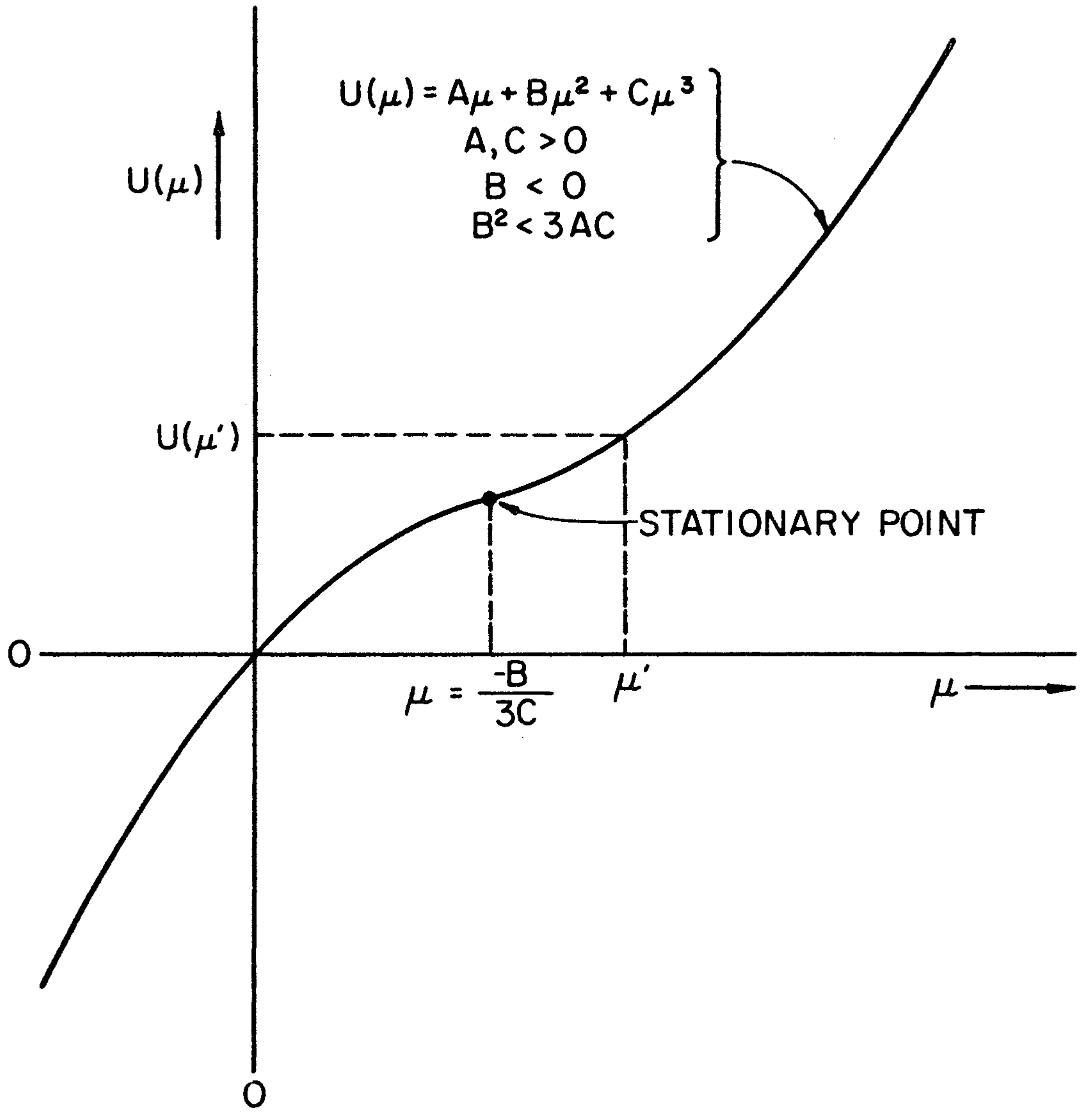


Figure 12. Defined Cubic Utility Function

Risk Attitudes From Indifference Curve Analysis

Let X now be a distributed random variable with mean μ, variance σ^2, and third moment about the mean, m_3. Also, define a general form of the cubic utility function (from equation (77)) with the necessary signs of the coefficients now explicitly incorporated in the function:

$$U(X) = AX - BX^2 + CX^3 \; ; \quad (A, B, C > 0; \; B^2 < 3AC) \; . \tag{87}$$

Taking expected values,

$$E[U(X)] = A\,E(X) - B\,E(X^2) + C\,E(X^3) \; . \tag{88}$$

But E(X) is simply the mean of the distributed variable X, or $E(X) = \mu$. Also, from the definition of variance, $V(X) = E(X^2) - [E(X)]^2$, one can determine that

$$E(X^2) = V(X) + [E(X)]^2 = \sigma^2 + \mu^2 \; . \tag{89}$$

From the definition of the third moment about the mean, one can evaluate $E(X^3)$ in the following manner:

$$\begin{aligned} m_3 &= \int_{-\infty}^{\infty} (X - \mu)^3 f(X)\, dX \\ &= E(X^3) - 3E(X^2)E(X) + 2[E(X)]^3 \\ &= E(X^3) - 3(\sigma^2 + \mu^2)\mu + 2\mu^3 \; ; \end{aligned}$$

or, solving for $E(X^3)$,

$$E(X^3) = 3\sigma^2\mu + \mu^3 + m_3 \; . \tag{90}$$

Making the substitutions of equations (89) and (90), and $E(X) = \mu$, into equation (88), the expected utility is obtained:

$$E[U(X)] = A\mu - B(\sigma^2 + \mu^2) + C(3\sigma^2\mu + \mu^3 + m_3) . \qquad (91)$$

For any particular distribution of X, the parameters μ, σ^2 and m_3 are independent of each other and, therefore, orthogonal in a system of rectangular coordinates, which leads to a three-dimensional system for representing expected utility graphically.

Equation (91) is the expected utility of a random variable whose distribution is sufficiently defined to obtain the parameters μ, σ^2 and m_3. If X is now the distributed <u>net present value</u> of a project, then equation (91) expresses the expected utility of that net present value in terms of the mean, the variance, and the third moment of the distribution. Since this equation was derived from the cubic utility function, a decision-maker with such a utility function who maximizes expected utility will do so on the basis of equation (91). Thus, if two or more projects have the same expected utility by equation (91), then the decision-maker is said to be indifferent between the projects. In this case, $E[U(X)]$ is a constant, and equation (91) describes a family of <u>indifference surfaces</u> in the orthogonal parameters μ, σ^2 and m_3. The behavior of the decision-maker can be inferred from the shape of such surfaces.

To investigate these inferences for a constant expected utility surface, let $E[U(X)]$ be constant and differentiate equation (91) with respect to μ:

$$0 = A - 2B\sigma\frac{d\sigma}{d\mu} - 2B\mu + 6C\sigma\mu\frac{d\sigma}{d\mu} + 3C\sigma^2 + 3C\mu^2 + C\frac{dm_3}{d\mu}$$

and solve for $d\sigma/d\mu$:

$$\frac{d\sigma}{d\mu} = \frac{A - 2B\mu + 3C(\sigma^2 + \mu^2) + C\frac{dm_3}{d\mu}}{2\sigma(B - 3C\mu)} . \tag{92}$$

Now, assume (for the time being) that m_3 is constant; then $dm_3/d\mu = 0$, and hence

$$\frac{d\sigma}{d\mu} = \frac{A - 2B\mu + 3C(\sigma^2 + \mu^2)}{2\sigma(B - 3C\mu)} . \tag{93}$$

Now, the first objective is to show that $d\sigma/d\mu$ is either positive or negative. To do this, consider the numerator of equation (93). Here, it is known that A and C are positively valued, and (now) B is also a positively valued constant. Recalling that only projects with positive mean net present values are considered for evaluation in the capital budgeting problem, then one can assume that $\mu > 0$, and hence $\mu^2 > 0$ also. Recalling again the sign theorem derived from the "Fundamental Theorem of Algebra," if the exponent of μ is even and if $C > 0$, then the numerator of equation (93) is a concave-upward parabola in terms of σ^2 and μ. Furthermore, the quadratic portion of the numerator, $A - 2B\mu + 3C\mu^2$, will always be positive if the two μ-roots are both imaginary. The roots of the quadratic portion are

$$\mu_1, \mu_2 = \frac{B \pm \sqrt{B^2 - 3AC}}{3C}$$

and if the discriminant $B^2 - 3AC < 0$, then μ_1 and μ_2 are conjugate imaginary roots and the quadratic $A - 2B\mu + 3C\mu^2$ is always positive when $C > 0$. This condition exists when $B^2 - 3AC < 0$, which holds for all values of $\mu > 0$ since $B^2 < 3AC$ is a requirement for the existence of the utility function itself. Since $\sigma^2 > 0$ always, then $3C\sigma^2 > 0$, and it has thus been shown that the numerator of equation (93) is always positive for $\mu > 0$.

Now, the denominator of equation (93) will be either positive or negative depending upon the sign associated with the factor $B - 3C\mu$, since $\sigma^2 > 0$ always. Now, when $B - 3C\mu < 0$, then $\mu > B/3C$, and when $B - 3C\mu > 0$, then $\mu < B/3C$. By this reasoning, it has been shown that the signs associated with the terms of equation (93) take on the following significances:

$$\frac{d\sigma}{d\mu} = \frac{\overbrace{A - 2B\mu + 3C\mu^2}^{+\text{ always}} + \overbrace{3C\sigma^2}^{+\text{ always}}}{\underbrace{2\sigma}_{+\text{ always}}\ \underbrace{(B - 3C\mu)}_{\substack{-\text{ if } \mu > B/3C \\ +\text{ if } \mu < B/EC}}} \quad ; \qquad (\mu > 0).$$

Hence, it can now be said that

$$\left.\frac{d\sigma}{d\mu}\right|_{m_3 = \text{const.}} > 0 \; ; \qquad \left(\mu < \frac{B}{3C}\right) \tag{94}$$

and that

$$\left.\frac{d\sigma}{d\mu}\right|_{m_3 = \text{const.}} < 0 \; ; \qquad \left(\mu > \frac{B}{3C}\right) \tag{95}$$

That is, in terms of the parameters σ and μ (with m_3 constant), the indifference surface, $E[U(X)]$, slopes positively (upward) in the region $\mu < B/3C$ and negatively (downward) in the region $\mu > B/3C$. This is already a radical departure from the risk-avoiding indifference curve, where $d\sigma/d\mu > 0$ everywhere.

The next objective logically should be to investigate the second derivative of the indifference function in order to specify its shape. In the first derivative (equation (93)), however, note that as the quantity, $B - 3C\mu$, in the denominator approaches zero (as $\mu \rightarrow B/3C$), then the derivative itself approaches infinity in the limit. Thus, equation (93) is undefined when $\mu = B/3C$ and, therefore, is not continuous for all values of $\mu > 0$. For this reason, the second derivative of the indifference function cannot be taken, and no general statement concerning the concavity or convexity of the indifference surface can be made. The most that can be said is that in the vicinity of $\mu = B/3C$ the slope $d\sigma/d\mu$ becomes very large.

Additional information can be adduced, however, by differentiating equation (91) when $E[U(X)]$ is not constant. When two of the three parameters are held constant, then the directional derivative of the third is obtained. Thus, when σ and m_3 are constant, then

$$\left.\frac{\partial E[U(X)]}{\partial \mu}\right|_{\sigma, m_3} = A - 2B\mu + 3C(\sigma^2 + \mu^2) \; , \tag{96}$$

which is always positive, as has been shown on pages 122-123. Moreover, the derivative with respect to σ is

$$\left.\frac{\partial E[U(X)]}{\partial \sigma}\right|_{\mu, m_3} = -2B\sigma + 6C\sigma\mu, \tag{97}$$

which is negative when $\mu < B/3C$ and positive when $\mu > B/3C$. Further,

$$\left.\frac{\partial E[U(X)]}{\partial m_3}\right|_{\sigma, \mu} = +C \; , \tag{98}$$

which is always positive under the assumed convention of $C > 0$.

With the information developed in equations (94) through (98), the skeleton indifference surface for expected utility can be sketched, as is illustrated in Figure 13.

Risk Attitudes

From equations (94) through (98), and with the aid of Figure 13, the risk attitudes of a rational decision-maker with a cubic utility function can now be summarized. Assuming that σ is a measure of the risk (uncertainty) attached to a particular random net present value, the rational decision-maker who maximized expected utility will exhibit the following characteristics:

(1) In the region $\mu < B/3C$, he will be a risk-avoider, since he attaches greater utility to decreased uncertainty. That is, for two projects with the same expected net present value, he attaches greater utility to the one with the lesser variance, via equation (97). Similarly, for two projects with the same uncertainty, he attaches greater utility to the one with the larger expected net present value. This behavior corresponds to the portion of the utility function, $U(\mu)$, that is concave downward. Thus, it is said that a concave-downward utility function is a "risk-avoider's curve."

(2) In the region $\mu > B/3C$, the decision-maker attaches greater utility to net present values with greater uncertainty. Between two projects with the same uncertainty, he will still choose the one with the greater expected net present value; but, contrary to his behavior in the risk-avoiding portion of the utility function, he will now choose the project with the greater uncertainty if two projects have the same expected net present value. Thus, in the region $\mu > B/3C$, the

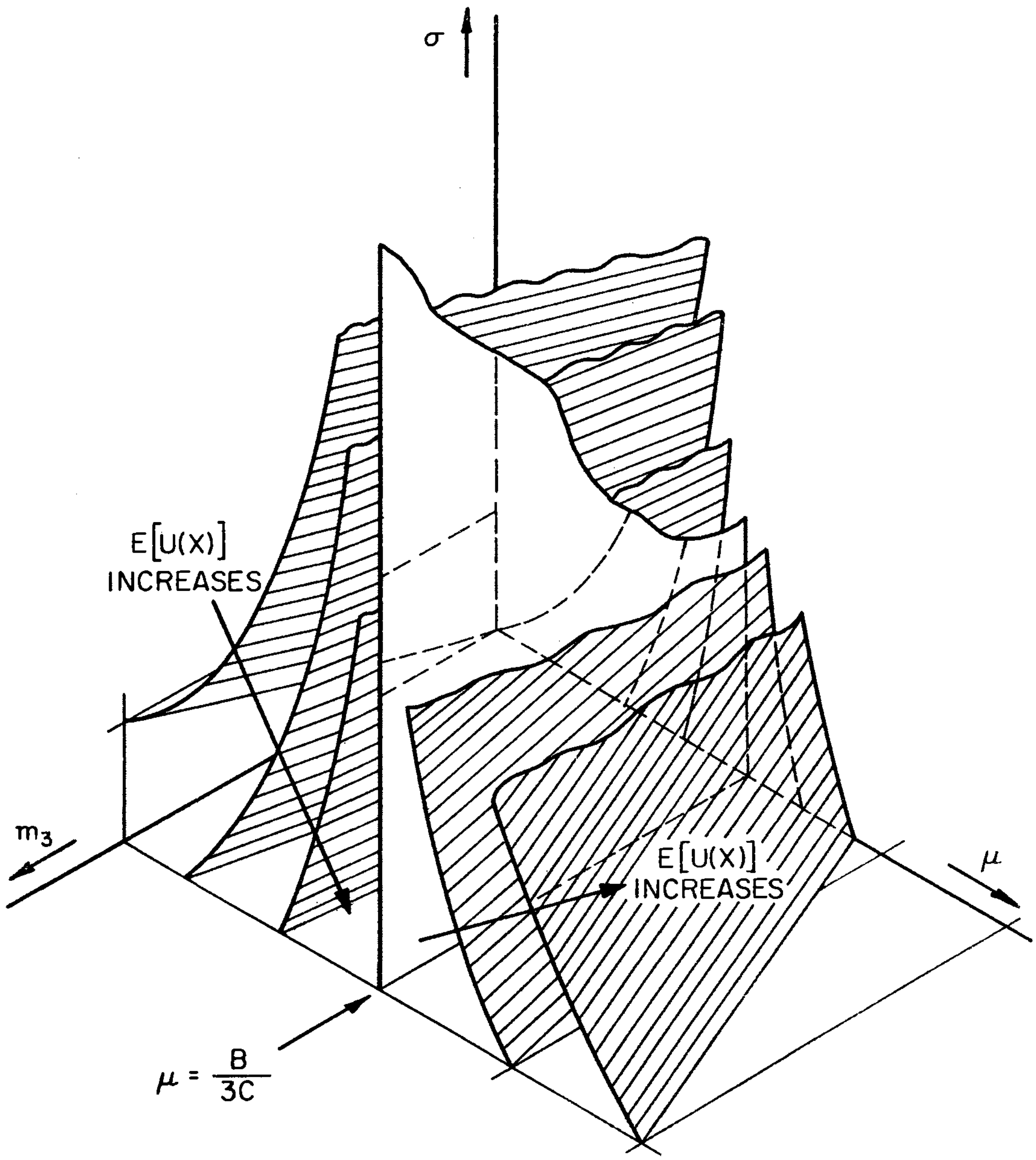


Figure 13. Cubic Expected Utility Indifference Surfaces

decision-maker is said to exhibit risk-seeking behavior, and this corresponds to the convex (concave upward) portion of the cubic utility function.

(3) While the preceding results were to be expected and come as no great surprise, the following conclusion is a most important one. That is, in the development of the equation for the indifference surface (equation (91)), the derivation forced one to take into account the third moment of the net present value distribution. This was not an accident; it is required as a result of taking the expected value of the utility function. The third moment of a distribution expresses the skewness of the distribution about the mean. A symmetrical distribution such as the normal has a zero third moment about the mean, and is not skewed. A distribution with positive third moment will be skewed toward higher values of the random variable--that is, with the right "tail" of the density function longer than the left one.

Statistically, a unimodal distribution with a positive third moment about the mean will exhibit a greater probability for values of the variate greater than the mean, and a lesser probability for values smaller than the mean. That is, $P(X > \mu) > P(X \leq \mu)$, when $m_3 > 0$. See Figure 14 for an illustration of this statement.

For the decision-maker with a cubic utility function, it can be said that he "prefers" (attaches greater utility to) projects that are positively skewed (by equation (98)). Furthermore, one can infer that his reason might be that he desires a greater probability of obtaining net present values greater than the mean, μ, than could be obtained if the distribution were symmetrical. This preference holds, moreover, for all values of $\mu > 0$, regardless of his risk-avoidance or risk-

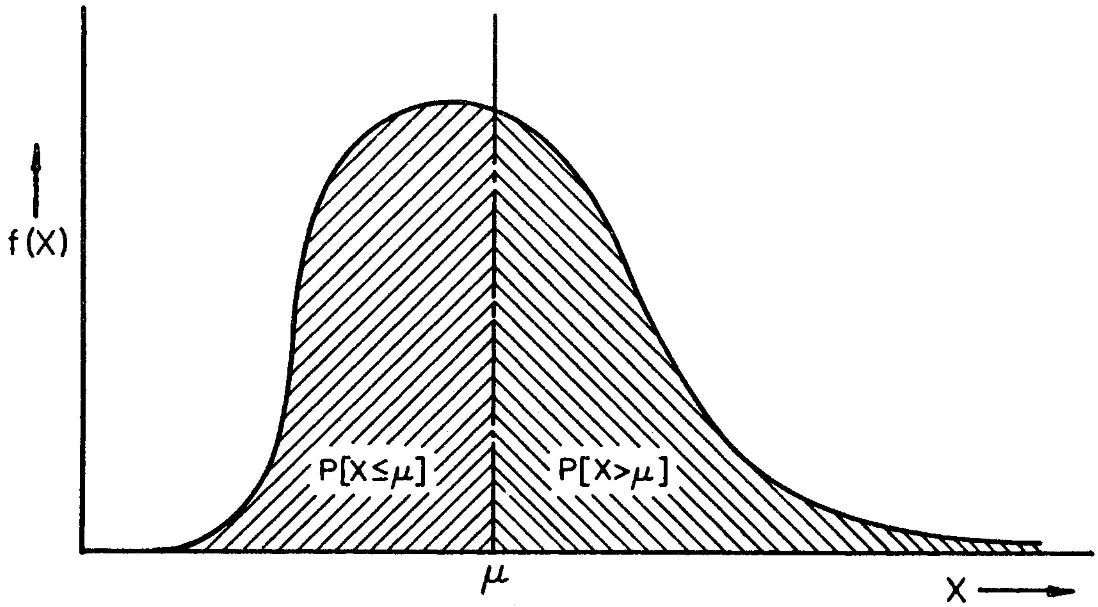


Figure 14. Skewed Distribution With Positive Third Moment

seeking tendencies. These are the important conclusions that were developed from the indifference function analysis; namely, that the third moment of the NPV distribution must be considered for a cubic utility function, and that the decision-maker "prefers" projects with positively skewed NPV distributions. Equation (91), therefore, is the correct project selection criterion to use in a solution of the selection problem, when the precursor utility function is a cubic polynomial.

Theoretic Development of Complex Risk
Behavior Selection Criteria

Risk-seeking behavior beyond some critical value of net present value, as was developed for the cubic utility function, is not a reasonable behavior for most decision-makers. While some persons will take risks (and seek them) over a portion of the range of expected return, it is not likely that they will seek risks when the returns become very great--when "important money" is at stake. To account for this behavior, the quartic utility function is of use, for it describes risk-avoidance both when losses and important money are at issue, and risk-seeking over a portion of the expected return range. Thus, the quartic utility function is yet one more approximation to an interpretation of complex risk behavior.

The derivation of the conditions under which a fourth-degree utility function exists is considerably more difficult than for the third-degree case. The utility function itself is a quartic polynomial of the form

$$U(X) = AX - BX^2 + CX^3 - DX^4 , \qquad (99)$$

where A, B, C and D are constants.[3] Thus, the marginal utility, $U'(X) = d[U(X)]/dX$, is a third-degree polynomial and the shape function, $U''(X)$, is a quadratic. Typically, the fourth-degree utility function and its derivatives can be represented in graphical form, as in Figure 15. From the sign theorem derived from the "Fundamental Theorem of Algebra," the shape of the fourth-degree utility function can be hypothesized. That is, for a fourth-degree polynomial, if the coefficient D is negative, then the trace of U(X) will be downward and to the

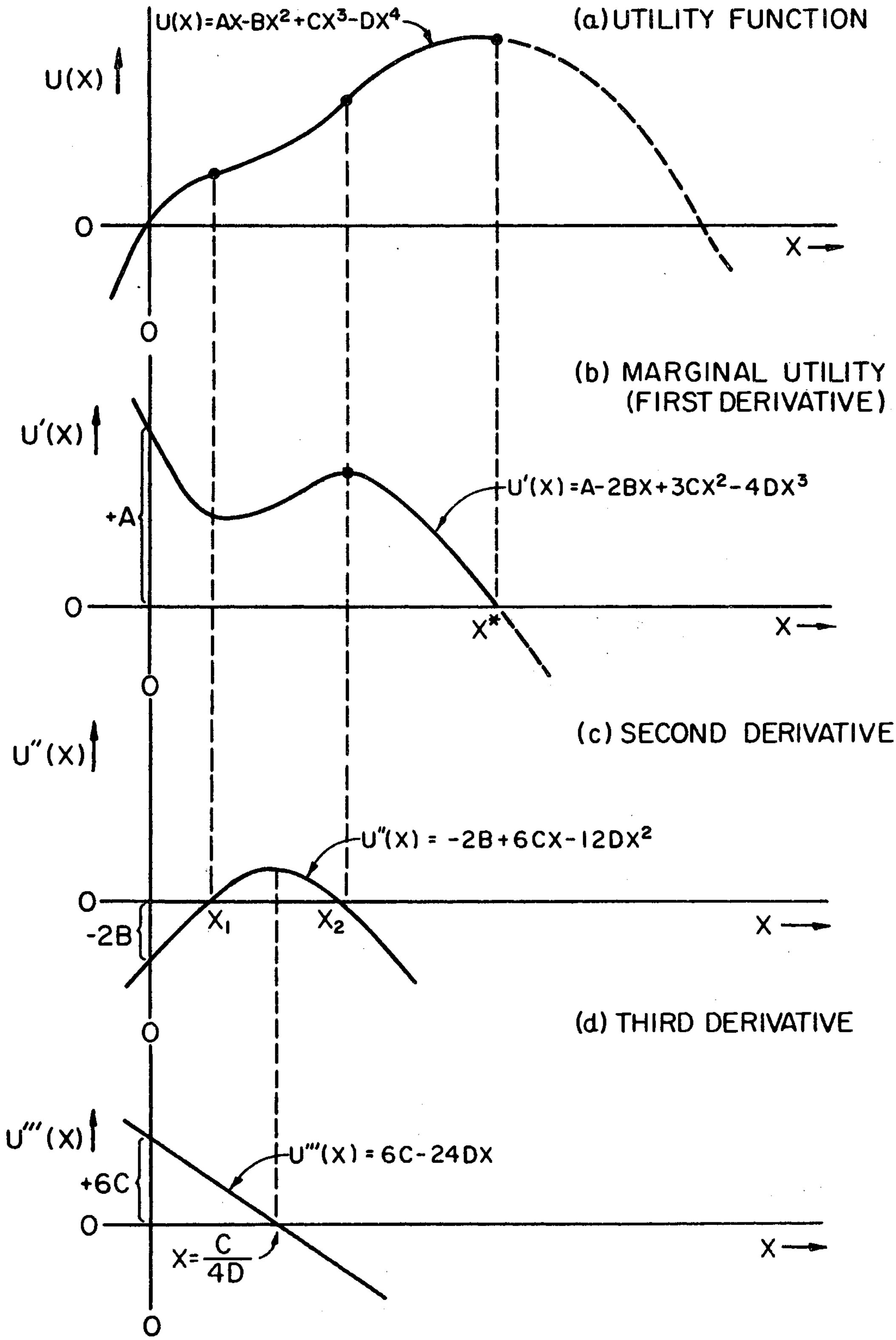


Figure 15. Fourth-Degree Utility Function and Derivatives

right for large values of X, and downward and to the left for large negative values of X. For small values of X (i.e., for $-\epsilon < X < +\epsilon$), the sign of the term AX will dominate, leading to the hypothesis of a positive slope in the vicinity of $X = 0$, for $A > 0$. As will be seen below, the two stationary (inflection) points are provided by the coefficients $-B$ and $+C$. The trace of U(X) passes through the origin since there is no term independent of X in equation (99).

The first derivative of U(X) is the marginal utility:

$$U'(X) = \frac{dU(X)}{dX} = A - 2BX + 3CX^2 - 4DX^3 \quad . \tag{100}$$

For the fourth-degree utility function, it is not possible to specify that the marginal utility be everywhere positive, as was done for the third-degree case, because the marginal utility function here is a cubic polynomial which requires that at least one root be real (all three roots cannot be imaginary). However, the shape of the marginal utility function can be hypothesized from a knowledge of (or assumptions about) the coefficients. If $A > 0$ and $D < 0$, as was hypothesized in equation (99), then $U'(X)$ will intersect the ordinate at a value of $U'(X) = +A$, and from the sign theorem the trace of $U'(X)$ will slope downward to the right and upward to the left for $D < 0$. Moreover, if $B < 0$ is assumed (as in equations (99) and (100)), then for small values of X the slope of the marginal utility function in the vicinity of $X = 0$ will be approximately $-2B$, since the term $-2B$ in equation (100) dominates the other terms in X when X is small. If $C > 0$, then the second-derivative quadratic, $U''(X)$, will have two <u>positively</u> valued real roots, X_1 and X_2, where the values of the roots in terms of the coefficients are:

$$X_1 = \frac{-6C + \sqrt{36C^2 - 96BD}}{-24D} = \frac{3C - \sqrt{9C^2 - 24BD}}{12D}$$

$$X_2 = \frac{-6C - \sqrt{36C^2 - 96BD}}{-24D} = \frac{3C + \sqrt{9C^2 - 24BD}}{12D}$$

provided that $3C > \sqrt{9C^2 - 24BD}$ and $9C^2 - 24BD \geq 0$. That is, the extreme points of the marginal utility function (100) correspond to the roots of the shape function, $U''(X)$, and these roots will be positive roots when the discriminant of the quadratic, $9C^2 - 24BD \geq 0$, and when $3C > \sqrt{9C^2 - 24BD}$. These requirements assure the location of the extreme points of $U'(X)$ in the first quadrant. (If $C < 0$ had been assumed, then both of the extreme points of $U'(X)$ would have been located in the third quadrant, which would invalidate $U(X)$ as a utility function).

Thus, it has been informally demonstrated that the coefficient A must be positive and D must be negative for $U(X)$ to take on a generally concave downward shape; further, B must be negative and C positive for the marginal utility function to be positive in the first quadrant and also have the inflection points of $U(X)$ in the first quadrant (i.e., for any $X > 0$).

Now, under these sign conventions, it will also be apparent that the cubic marginal utility function (equation (100)) can have only one real root (the other two being imaginary), in order for $U'(X)$ to be positive in the range $0 \leq X \leq X^*$. The conditions under which $U'(X)$ has only one real root can be derived as follows.[4] Consider the equation of the marginal utility, $U'(X)$, which has a root at the point $X = X^*$; when this is so, then $U'(X) = 0$, or

$$U'(X) = 0 = A - 2BX + 3CX^2 - 4DX^3; \quad (X = X^*). \qquad (101)$$

Dividing through by the coefficient $-4D$, equation (101) can be placed in the following form:

$$X^3 - \frac{3C}{4D} X^2 + \frac{B}{2D} X - \frac{A}{4D} = 0 . \qquad (102)$$

Now, let $b = -\frac{3C}{4D}$; $c = \frac{B}{2D}$; $d = -\frac{A}{4D}$; and then equation (102) becomes

$$X^3 + bX^2 + cX + d = 0. \qquad (103)$$

Equation (103) can be placed in "reduced" form by applying the transformation

$$X = y - \frac{b}{3}$$

where y is a new variable and the coefficient b is that just defined above. With this transformation, equation (103) in reduced form is

$$(y^3 - by^2 + \frac{b^2 y}{3} - \frac{b^3}{27}) + b(y^2 - \frac{2by}{3} + \frac{b^2}{9}) + c(y - \frac{b}{3}) + d = 0,$$

which simplifies to

$$y^3 + py + q = 0 ; \qquad (104)$$

where

$$p = c - \frac{b^2}{3} = \frac{B}{2D} - \frac{3C^2}{16D^2} \qquad (105)$$

$$q = d - \frac{bc}{3} + \frac{2b^3}{27} = -\frac{A}{4D} + \frac{BC}{8D^2} - \frac{C^3}{32D^3} . \qquad (106)$$

Equation (104) is in reduced form and is the equivalent of equation (102), except that the roots of (102) are related to those of (104) by the transformation, $X = y - b/3$.

Now, the discriminant of the reduced cubic (equation (104)) is the quantity

$$\Delta_3 = 4p^3 + 27q^2 .$$

The reduced cubic has the properties (in terms of its coefficients) that if $\Delta_3 < 0$, then all three roots are real and distinct; if $\Delta_3 = 0$, then all roots are real and two are equal; and if $\Delta_3 > 0$, then one root is real and two are imaginary. Thus, for our case in which only one root can be real, then it is required that $\Delta_3 > 0$.

It can be shown (see Appendix B) that Δ_3 is in fact positive when $B^2 \geq 9AC/4$ <u>and</u> either $D > R_1$ or $D < R_2$, where R_1 and R_2 are the roots of a quadratic form given by equations (B-6) and (B-7) in Appendix B. In other words, when the two conditions above are met, then $\Delta_3 > 0$ and the reduced cubic equation (104) will have one real root and two imaginary roots. This is also the condition for equation (101), the marginal utility function, to have only one real root; and hence, the marginal utility will be positive up to the value of that root, $X = X^*$.

It would be desirable to show that the real root of $U'(X)$ could be stated simply in terms of the coefficients, A, B, C and D, so that the limit of applicability of the function $U(X)$ (at $X = X^*$) could be defined. (That is, since the marginal utility becomes zero at $X = X^*$ (see Figure 15), the quartic utility function is <u>not</u> valid in the range $X > X^*$). Unfortunately, it is not possible to express the real root, X^*, of equation (101) in simple terms of the literal coefficients, since the expression for the real root involves taking the cube roots of two literal (non-numerical) expressions that, in turn, require the extraction of a square root of a sixth-degree polynomial that is not a perfect square. However, for specific instances in which the <u>values</u>

of the coefficients A, B, C and D are known, then the real root $X = X^*$ can be calculated as follows:

(1) Find the surrogate root, y_1:

$$y_1 = \sqrt[3]{\alpha} + \sqrt[3]{\beta}$$

where

$$\alpha = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$\beta = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

$$p = \frac{B}{2D} - \frac{3C^2}{16D^2}$$

$$q = -\frac{A}{4D} + \frac{BC}{8D^2} - \frac{C^3}{32D^3} \quad ;$$

(2) Find the real root:

$$X^* = y_1 + \frac{C}{4D} \quad .$$

Since the signs are explicitly stated correctly in the quartic utility function (99), then the numerical values of the coefficients A, B, C and D, to be used in the root extraction relationships above, must all be the absolute values.

To summarize, the necessary and sufficient conditions for the existence of a quartic utility function of the form of equation (99) are:

(1) The constants A and C are positive and the constants B and D are negative, as in equation (99).

(2) The values of the constants A, B, C and D satisfy the

following relationships:

(a) $$B^2 \geq \frac{9AC}{4} \; ; \tag{107}$$

(b) $$D > R_1 \; , \; \underline{\text{OR}} \; \; D < R_2 \tag{108}$$

where

$$R_1 = \frac{(27ABC - 8B^3) + \sqrt{(27ABC - 8B^3)^2 - 27A^2(27AC^3 - 9B^2C^2)}}{54\,A^2}$$

$$R_2 = \frac{(27ABC - 8B^3) - \sqrt{(27ABC - 8B^3)^2 - 27A^2(27AC^3 - 9B^2C^2)}}{54\,A^2} \; ;$$

(c) $$3C > \sqrt{9C^2 - 24BD}$$

which reduces to $$BD > 0 \; ; \tag{109}$$

(d) $$9C^2 - 24BD \geq 0$$

which reduces to $$C^2 \geq 8BD/3 \; . \tag{110}$$

<u>Risk Attitudes From Indifference Curve Analysis</u>

Let X be the net present value of a project, which is a distributed random variable with mean μ, variance σ^2, third moment m_3, and fourth moment about the mean, m_4. Taking expected values of the quartic utility function, equation (99) becomes

$$E[U(X)] = A\,E(X) - B\,E(X^2) + C\,E(X^3) - D\,E(X^4). \tag{111}$$

However, it has already been shown for the cubic utility function that

$$\left.\begin{aligned} E(X) &= \mu \\ E(X^2) &= \sigma^2 + \mu^2 \\ E(X^3) &= 3\sigma^2\mu + \mu^3 + m_3 \end{aligned}\right\} \tag{112}$$

and one can derive $E(X^4)$ from the definition of the fourth moment about the mean:

$$m_4 = \int_{-\infty}^{\infty} (X - \mu)\; f(X)\; dX$$

$$= E(X^4) - 4\,E(X^3)\,E(X) + 6\,E(X^2)[E(X)]^2 - 3[E(X)]^4\;;$$

or, solving for $E(X^4)$, one obtains

$$E(X^4) \;=\; \mu^4 + 6\sigma^2\mu^2 + 4\,m_3\mu + m_4\;. \tag{113}$$

Substituting equations (112) and (113) into equation (111), one obtains the expression for the expected utility of the net present value in terms of the parameters of the NPV distribution:

$$E[U(X)] = -\,D\,\mu^4 + C\,\mu^3 - (B + 6D\sigma^2)\,\mu^2 +$$

$$+ (A + 3C\sigma^2 - 4Dm_3)\mu \;-\; (B\sigma^2 - Cm_3 + Dm_4)\,. \tag{114}$$

For a decision-maker with a quartic utility function who chooses among projects on the basis of expected utility, one can draw some inferences by examining the directional derivatives of equation (114). First, take the partial derivative of (114) with respect to the mean, μ , holding all other parameters constant:

$$\left.\frac{\partial E[U(X)]}{\partial\mu}\right|_{\sigma,m_3,m_4} = -\,4D\,\mu^3 + 3C\,\mu^2 - 2(B + 6D\sigma^2)\mu\; +$$

$$+ (A + 3C\sigma^2 - 4Dm_3)\;. \tag{115}$$

Rearrange (115) in the form

$$\left.\frac{\partial E[U(X)]}{\partial\mu}\right|_{\sigma,m_3,m_4} = (A - 2B\mu + 3C\,\mu^2 - 4D\,\mu^3) + 3\sigma^2(C - 4D\,\mu)$$

$$-\; 4Dm_3\;. \tag{116}$$

One recognizes the first four terms of (116) as being merely the marginal utility of the mean, $U'(\mu)$, which follows from evaluating equation (100) at $X = \mu$. Thus, with $\sigma^2 = 0$ and $m_3 = 0$, it can be said that equation (116) is positive wherever equation (100) is positive, and this occurs at all values of $\mu = X \leq X^*$. Hence, the decision-maker will assign greater utility to projects with larger mean net present values over the entire valid range of the utility function, when $\sigma^2 = 0$ and $m_3 = 0$.

However, for the more realistic case in which $\sigma^2 > 0$, this behavior is modified. The term $3\sigma^2(C - 4D\mu)$ in equation (116) is positive for all values of $\mu < C/4D$, and negative for all values of $\mu > C/4D$. The transition value, $\mu = C/4D$, occurs in the convex portion of the utility function (see Figure 15), and is well within the valid range of the utility function. Thus, when $\sigma^2 > 0$ and $\mu > C/4D$, as would be the case for many "real" projects, the rate of change of expected utility with respect to μ decreases even faster than $U'(\mu)$--by an amount $3\sigma^2(C - 4D\mu)$--and eventually becomes zero when $U'(\mu) = 3\sigma^2(C - 4D\mu)$. From this reasoning, one can infer that the decision-maker "prefers" (attaches greater utility to) projects with greater mean net present values up to the point where his marginal utility of the mean, $U'(\mu)$, is equal to the quantity $3\sigma^2(C - 4D\mu)$. Thereafter, when $U'(\mu) < 3\sigma^2(C - 4D\mu)$, his expected utility actually decreases with increasing μ, and he attaches less preference to increasing mean net present value.

The same sort of phenomenon occurs with respect to the skewness, m_3, of the net present value distribution. For positive values of m_3, the term $4Dm_3$ is a decrement in equation (116), and for $m_3 < 0$, it is an increment. Thus, the decision-maker "prefers" projects with greater mean net present values "up to a point." When m_3 becomes too great, his preference wanes, and eventually becomes zero when $U'(\mu) = 4Dm_3$.

Basically, then, the equation for the rate of change of expected utility with respect to the project mean NPV (equation (116)), is the same as the equation for marginal utility (equation (100)), which is positive but generally decreasing over a range of valid values of μ. However, both project variance, σ^2, and skewness, m_3, generally modify (and, in most cases, reduce) the range of μ in which expected utility increases with respect to μ. At some point where $U'(\mu) = 3\sigma^2(C - 4D\mu) - 4Dm_3$, the expected utility no longer increases with μ; thereafter, when $U'(\mu) < 3\sigma^2(C - 4D\mu) - 4Dm_3$, the expected utility of NPV decreases with increasing μ and the decision-maker actually attaches greater utility to projects with smaller mean net present values. This suggests the possibility of an "optimal" value of $E[U(X)]$, which will be investigated (below) after the discussion on directional derivatives is completed.

Now, consider the second directional derivative:

$$\left.\frac{\partial E[U(X)]}{\partial (\sigma^2)}\right|_{\mu, m_3, m_4} = -6D\mu^2 + 3C\mu - B$$

$$= \tfrac{1}{2} U''(\mu) \; ; \qquad (117)$$

which is obtained by differentiating equation (114) while holding μ, m_3 and m_4 constant, and recognizing that $-2B + 6C\mu - 12D\mu^2 = U''(\mu)$, from Figure 15. Thus, with respect to the variance, σ^2, of the project net present value, the change in expected utility is positive wherever $U''(\mu)$ is positive, and negative elsewhere. All other factors being constant, expected utility bears the same relative relationship to the project variance, σ^2, as the marginal utility (equation (100) and Figure 15(b)) bears to the project mean net present value, μ, since the first derivative of expected utility, $\partial E[U(X)]/\partial(\sigma^2)$, is one-half the second derivative, $U''(\mu)$, of the utility function itself.

Translated into risk attitudes, the foregoing means that the decision-maker's expected utility increases with positive variance (of NPV) up to some point; thereafter, his preferences turn in the opposite direction and as variance continues to increase his expected utility decreases for projects that have large variances in the NPV distribution. The maximum expected utility, $E[U(X)]$, is reached when $\partial E[U(X)]/\partial(\sigma^2) = 0$, which is the same as $U''(\mu) = 0$, or at the point where $\mu = X_2$ in Figure 15(c). Not coincidentally, this maximum occurs at the same NPV mean at which the decision-maker's marginal utility is a local maximum--Siegel's "level of aspiration."

The third directional derivative of $E[U(X)]$ provides even more information. With μ, σ and m_4 constant, take the derivative of equation (114) with respect to m_3, thus:

$$\left.\frac{\partial E[(X)]}{\partial m_3}\right|_{\mu,\sigma,m_4} = -4D\mu + C \, . \tag{118}$$

When $-4D\mu + C > 0$, which is the same as $\mu > C/4D$, then expected utility increases as m_3 increases; when $\mu < C/4D$, then expected utility decreases as m_3 becomes greater. At the transition point, $\mu = C/4D$, expected utility is constant with respect to m_3 since $\partial E[U(X)]/\partial m_3 = 0$. Note that the second derivative, $\partial^2 E[U(X)]/\partial m_3^2 = 0$; therefore, $E[U(X)]$ does not possess an extreme point at $\mu = C/4D$.

Translated into risk attitudes, the decision-maker prefers projects whose NPV distributions are positively skewed ($m_3 > 0$) when the mean, μ, of the distribution is less than C/4D; on the other hand, when $\mu > C/4D$, he prefers negatively skewed project NPV distributions.

The last directional derivative is simple. For μ, σ, and m_3 constant, then

$$\left. \frac{\partial E[U(X)]}{\partial m_4} \right|_{\mu,\sigma,m_3} = -D. \qquad (120)$$

The interpretation is straightforward. Since $m_4 \geq 0$ for any distribution, then the decision-maker's expected utility decreases for all values of m_4 proportional to the value of his risk-aversion coefficient, D, which is most effective at large values of m_4. In short, the coefficient, D, expresses an overall aversion to risk, which is most effective when "important money" is at stake (for large values of NPV) and when large losses are a distinct possibility.

In the investigation of some of the directional derivatives of expected utility, it was suggested that a maximum value of E[U(X)] might exist, which would cause the decision-maker to choose projects whose parameters (μ, σ^2, m_3 and m_4) yielded a maximum or near-maximum value for E[U(X)]. It can be shown that such a maximum does not exist. To demonstrate this fact, take the second partial derivatives of equation (114) with respect to each of the parameters. Now, these partials in matrix form define the Hessian of equation (114), which is illustrated in Figure 16.

For E[U(X)] to have an extreme point which can be interpreted as a maximum, then $\underset{\sim}{H}$ (the Hessian) must be negative definite. However, since the determinant of $\underset{\sim}{H}$ is zero by inspection (one row is zero, or

$$
\underset{\sim}{H} = \begin{bmatrix}
\frac{\partial^2 E}{\partial \mu^2} = -12D\mu^2 + 6C\mu - 2(B + 6D\sigma^2); & \frac{\partial^2 E}{\partial \mu \partial \sigma} = -24D\mu\sigma \; ; & \frac{\partial^2 E}{\partial \mu \partial m_3} = -4D \; ; & \frac{\partial^2 E}{\partial \mu \partial m_4} = 0 \\
\frac{\partial^2 E}{\partial \sigma \partial \mu} = -24D\mu\sigma + 6C\sigma \; ; & \frac{\partial^2 E}{\partial \sigma^2} = -12D\mu^2 + 6C\mu - 2B; & \frac{\partial^2 E}{\partial \sigma \partial m_3} = 0 \; ; & \frac{\partial^2 E}{\partial \sigma \partial m_4} = 0 \\
\frac{\partial^2 E}{\partial m_3 \partial \mu} = -4D \; ; & \frac{\partial^2 E}{\partial m_3 \partial \sigma} = 0 \; ; & \frac{\partial^2 E}{\partial m_3^2} = 0 \; ; & \frac{\partial^2 E}{\partial m_3 \partial m_4} = 0 \\
\frac{\partial^2 E}{\partial m_4 \partial \mu} = 0 \; ; & \frac{\partial^2 E}{\partial m_4 \partial \sigma} = 0 \; ; & \frac{\partial^2 E}{\partial m_4 \partial m_3} = 0 \; ; & \frac{\partial^2 E}{\partial m_4^2} = 0
\end{bmatrix}
$$

Figure 16. Hessian of Equation (114)

alternatively, one column is zero), then $\underset{\sim}{H}$ cannot be negative definite. Thus, even though some combination of values of the parameters might be found that would satisfy the necessary conditions for a maximum of $E[U(X)]$, no such maximum exists, by the above proof that a <u>sufficient</u> condition does not exist.

In summary, the decision-maker with a quartic utility function, who maximizes expected utility, will choose among projects on the basis of their net present value distributions, as follows:

(1) For projects that have equal variances, equal third moments, and equal fourth moments, he will choose the project with the highest mean NPV up to a point; thereafter, as μ increases, he loses interest. He prefers (ascribes greatest utility to) projects with a mean net present value which satisfies the equation $U'(\mu) = 3\sigma^2(C - 4D\mu) - 4Dm_3$.

(2) For projects that have equal μ, m_3, and m_4, he will choose the project with the largest variance when $X_1 \leq \mu \leq X_2$, where X_1 and X_2 are given by equation (101). He tends to maximize expected utility at $\mu = X_2$, where his marginal utility is also a maximum. Thus, he "seeks" risk in the range $X_1 \leq \mu \leq X_2$, and if offered the opportunity would choose the project whose mean net present value is equal to X_2 and whose variance is largest.

(3) For projects that have equal μ, σ^2 and m_4, he will choose those projects that have positively skewed NPV distributions if their means, μ, are less than $C/4D$; otherwise, he prefers projects with negatively skewed NPV distributions.

(4) For projects that have equal μ, σ^2, and m_3, he increasingly avoids projects with large fourth moments. This expresses an overall risk aversion when "important money" or large losses are at stake.

Finally, when a quartic utility function is implied by the decision-maker's responses to the standard lottery method of determining his utilities for prospective returns, then the proper selection criterion to use in solving the selection problem is the expected utility expression in equation (114), which requires consideration of the first four moments of each project net present value distribution.

FOOTNOTES

[1]At the request of this author, the reference librarians of the Iowa State University (Ames, Iowa) and the Linda Hall Scientific Reference Library (Kansas City, Missouri) made searches in Skandinavisk Aktuarietidskrift for the years 1963, 1964 and 1965. These are the years in which the Borch article might have appeared. (This action was taken because that journal is not available in the Oklahoma State University library). In addition, a search was made covering the same and succeeding years to the present, in the Author Index, an index that lists scientific publications by author. No such article as Borch reports "forthcoming" could be found in any instance. The presumption, therefore, is that the article referred to by Borch was not published.

[2]A two-tailed Student "t" test of the hypotheses $H_0: \beta_0 = 0$, $H_a: \beta_0 \neq 0$, is equivalent to an F-test on the mean square corresponding to $R(\beta_0)$, since the two tests are related by the equation

$$t_n^2 = F_{1,n}$$

Similar reasoning applies to the other β_i.

[3]That is, the constants are real numbers whose values are, respectively, the first derivative, one-half the second derivative, one-sixth the third derivative, and one-twenty fourth the fourth derivative of $U(X)$ evaluated at $X = 0$, which follows via the Taylor series expansion used to express $U(X)$, as in the third-degree case.

[4]See any good text on the Theory of Equations, such as J. V. Uspensky (61).

CHAPTER VI

FORMULATION OF A COMPLEX UTILITY PROJECT SELECTION PROBLEM

This Chapter is concerned with specific numerical solutions to the selection problem and how they may be obtained. For many of the simpler models, solution methods have been developed by prior investigators, and these appear in the literature. Appropriate references will be given for these cases. But for the more complex cases of the selection problem, no compact solution method exists. For these cases, the only solution known at present is that of complete enumeration, and a numerical example will be presented to illustrate the mode of formulation of this type of problem and its solution.

A solution to the project selection problem for the deterministic case, in which project net present values are assumed to be known and constant-valued, is quite simple. By equation (13), the selection criterion is merely that of maximization of net present value, subject to the required constraints in equations (14) through (20), in Chapter II. This results in a simple linear maximization model which can be formulated as

$$\underset{\forall j}{\text{Max}} \sum_{j=1}^{m} (NPV)_j x_j \; ; \qquad (j = 1, 2, \ldots, m) \tag{121}$$

subject to (at least) all $x_j = 0,1$. This problem can be solved by any of the 0/1 optimization methods; for example, the integer programming technique reported by Winegartner (63) or by the Lawler-Bell algorithm (38). Mao reports a computer method ((41), pp. 248-257) based on the Lawler-Bell algorithm, which he states is more efficient than the Gomory method of cutting planes used by Winegartner. When one or more of the technical constraints becomes "tight," so that the optimal solution depends upon the constraint, then either of these solution methods will lead to a discrete optimum in a finite number of steps, whereas some other methods (such as Lagrangian multiplier techniques) may not.[1]

When considerations of uncertainty must be recognized--that is, when net present values are distributed random variables with (at least) means and variances--then the form of the selection problem assumes a more complicated form due to the greater complexity of the selection criterion, which, as has been demonstrated in a previous chapter, depends upon the assumed utility function of the decision-maker. As an example, if the decision criterion is $E[U(NPV)] = \mu - A\sigma^2/2$ (which is the selection criterion derived from the negative exponential utility function in Chapter IV), then the selection problem for independent projects is formulated as a modified quadratic programming model in the following form:

$$\underset{\forall j}{\text{Max}} \quad \sum_{j=1}^{m} (\mu_{NPV})_j \, x_j \; - \; \frac{A}{2} \sum_{j=1}^{m} (\sigma^2_{NPV})_j \, x_j \; ; \quad (j = 1, 2, \ldots, m); \quad (122)$$

subject to (at least) all $x_j = 0,1$; where μ_{NPV} and σ^2_{NPV} are the mean and variance, respectively, of the j^{th} project. If all the projects are not independent, then they must be made so by constructing pseudo-projects from the covariant ones and introducing constraints of the form of equations (19) in Chapter II, or else the covariant relationship must be recognized in the objective function. In the latter case, and if the constraints that bind the problem are <u>linear</u>, then the selection problem can be formulated as a strict quadratic programming problem, as follows:

$$\underset{\forall j}{\text{Max}} \sum_{j=1}^{m} (\mu_{NPV})_j \, x_j - \frac{A}{2}\left[\sum_{j=1}^{m} (\sigma^2_{NPV})_j x_j + 2 \sum_{\substack{j=1 \\ j<k}}^{m-1} \sum_{k=2}^{m} \sigma_j \sigma_k x_j x_k\right]; \quad (123)$$

subject to (at least) x_j, $x_k = 0,1$.

Further, if the budget constraints (equation (14)) involve random variation in either the coefficients or the budget limits themselves, as Watters has suggested they might, then the formulation is no longer a strict quadratic programming problem with linear constraints, but rather a hybrid problem. Watters (62) gives a computer solution method for both the 0/1 quadratic problem and the 0/1 hybrid problem, and Mao (41) gives a modified partial enumeration method, based on the Lawler-Bell algorithm, for solving the 0/1 quadratic problem subject to either linear or <u>non</u>-linear constraints.

When the selection criterion in the objective function is one that is derived from either a cubic or quartic utility function, then matters become worse. In either case, the formulation of the problem is no longer of quadratic form, because not only must linear combinations of variances and covariances be considered, but also linear combinations of third and fourth moments about the means. Linear combinations of these moments do not result in a strictly additive form of the moments themselves, even for independent projects. Since there is no known research that deals with the solution of a problem of this kind, even in the literature that investigates the 0/1 multidimensional "knapsack" problem, then some form of complete enumeration is the only available approach at this time. This severely restricts the number of projects that can be considered when a cubic or quartic utility function is the basis of optimization, but no doubt this problem will be attacked and solved in time. Nevertheless, to illustrate the method of complete enumeration when a cubic or quartic utility function is the basis of selection, a numerical example will be presented step-by-step below. However, before that is done, it will be necessary to investigate and specify the manner in which the third and fourth moments of random variable distributions combine, which is the subject of the next section.

Third and Fourth Moments of a Linear Combination of Distributed Random Variables

Regardless of the distributions of the component net present value random variables, the moments of the resultant sum of two or more project net present values can be estimated by a method known as the "generation of system moments," a term used by Hahn and Shapiro (27).[2] This method is equivalent to the methods of "statistical error propagation" and "delta method" used by other authors. Hahn and Shapiro's method, which is the basis for the summary that follows below, is based on original work by Tukey (60).

Consider a "system" of random variables, x_1, x_2, ..., x_m, which combine in some fashion to result in a "system" resultant, z, such that

$$z = h(x_1, x_2, \ldots, x_m) \tag{124}$$

where x_j = the component random variable ($j = 1, 2, \ldots, m$).

Let $E(x_j)$ denote the mean of the j^{th} component variable, and $m_k(x_j)$ denote the k^{th} central moment (about the mean) for the j^{th} component. Similarly, let $E(z)$ and $m_k(z)$ denote the expected value and the k^{th} moment about the mean, respectively, for the system response variable. The problem is to obtain estimates for $E(z)$ and $m_k(z)$, for $k = 2, 3, 4$; based upon (a) the available data concerning $E(x_j)$ and $m_k(x_j)$, and (b) knowledge of the system logic, $h(x_1, x_2, \ldots, x_m)$.

The method of estimating $E(z)$ and $m_k(z)$ consists of expanding $h(x_1, x_2, \ldots, x_m)$ about the point $[E(x_1), E(x_2), \ldots, E(X_m)]$ at which z (and hence, each of the component variables) takes on its expected value, by a multivariable Taylor series. Thus, for <u>un</u>correlated component variables, the general expression for an estimate of the mean system response is

$$E(z) = h[E(x_1), \ldots, E(x_m)] + \frac{1}{2} \sum_{j=1}^{m} \left(\frac{\partial^2 h}{\partial x_j^2} \bigg|_{x_j = E(x_j)} \right) \mathrm{Var}(x_j). \quad (125)$$

For the special case in which $h(x_j)$ is a linear combination of variables, $z = x_1 + x_2 + \ldots + x_m$, the partials $\partial z/\partial x_j = 1$, and the second partials $\partial^2 z/\partial x_j^2 = 0$; thus, for this case the expression

$$E(z) = E(x_1) + E(x_2) + \ldots + E(x_m) \quad (126)$$

provides an exact result rather than an estimate.

The Taylor series expansion for the <u>variance</u> of the system response, as derived by Hahn and Shapiro, reduces to

$$\mathrm{Var}\,(z) = \sum_{j=1}^{m} \left(\frac{\partial z}{\partial x_j}\right)^2 \mathrm{Var}(x_j) + \sum_{j=1}^{m} \left[\left(\frac{\partial z}{\partial x_j}\right) \left(\frac{\partial^2 z}{\partial x_j^2}\right) \bigg|_{x_j = E(x_j)} \right] m_3(x_j); \quad (127)$$

where $m_3(x_j)$ is the third central moment (about the mean) for the j^{th} variate. For the linear case, in which $z = \sum_j(x_j)$ and the x_j are uncorrelated, the first partials are again unity and the second partials are zero; hence, for this case

$$\text{Var}(z) = \sum_{j=1}^{m} \text{Var}(x_j) . \tag{128}$$

When the x_j are correlated, the system response variance is

$$\text{Var}(z) = \sum_{j=1}^{m} \text{Var}(x_j) + 2 \sum_{\substack{j=1 \\ j<k}}^{m-1} \sum_{k=2}^{m} E\,[x_j - E(x_j)][x_k - E(x_k)]$$

or simplifying:
$$\text{Var}(z) = \sum_{j=1}^{m} \sigma^2_{x_j} + 2 \sum_{\substack{j=1 \\ j<k}}^{m-1} \sum_{k=2}^{m} \sigma_{x_j}\sigma_{x_k} ; \tag{129}$$

where $\sigma^2_{x_j}$ is the variance and $\sigma_{x_j}\sigma_{x_k}$ is the covariance of the j^{th} component and the j^{th}-k^{th} components, respectively.

Using the Taylor series expansion in the same manner, expressions for the third and fourth system response moments can be obtained. For the case in which the component variables are linearly related (as in the capital budgeting problem), the third system moments, $m_3(z)$, are:

For Uncorrelated Component Variables:

$$m_3(z) = \sum_{j=1}^{m} m_3(x_j) \tag{130}$$

For Correlated Component Variables:

$$m_3(z) = \sum_{j=1}^{m} m_3(x_j) + 6 \sum_{\substack{j=1 \\ j<k<s}}^{m-2} \sum_{k=2}^{m-1} \sum_{s=3}^{m} \left(\frac{\partial z}{\partial x_j}\right)\left(\frac{\partial z}{\partial x_k}\right)\left(\frac{\partial z}{\partial x_s}\right)\Big|_{x_r} \tag{131}$$

where $r = j, k, s$.

For a linear combination of component variables, the fourth system response moments (as condensed from Hahn and Shapiro's results) are:

For Uncorrelated Component Variables:

$$m_4(z) = \sum_{j=1}^{m} m_4(x_j) + 6 \sum_{\substack{j=1 \\ j<k}}^{m-1} \sum_{k=2}^{m} \sigma_{x_j}^2 \sigma_{x_k}^2 \quad . \qquad (132)$$

For Correlated Component Variables:

$$\begin{aligned} m_4(z) = & \sum_{j=1}^{m} m_4(x_j) + 4 \sum_{j} \sum_{\substack{k \\ j \neq k}} m_3(x_j)\, \sigma_{x_k} \\ & + 6 \sum_{j} \sum_{\substack{k \\ j<k}} \sigma_{x_j}^2 \sigma_{x_k}^2 + 12 \sum_{j} \sum_{k} \sum_{\substack{s \\ j \neq k \neq s}} \sigma_{x_j}^2 \sigma_{x_k} \sigma_{x_s} \\ & + 24 \sum_{j} \sum_{k} \sum_{s} \sum_{\substack{t \\ j<k<s<t}} \left[\sigma_{x_j} \sigma_{x_k} \sigma_{x_s} \sigma_{x_t}\right] . \qquad (133) \end{aligned}$$

With these preliminary relationships now defined, the numerical example based on a quartic utility function can be presented, which is the subject of the next section.

Formulation of Numerical Example for Independent Projects

It is assumed, for illustrative purposes, that the decision-maker in this numerical example is R. F. Mellon (referred to in Chapter V), and that he possesses a quartic utility function determined by regression analysis to be:

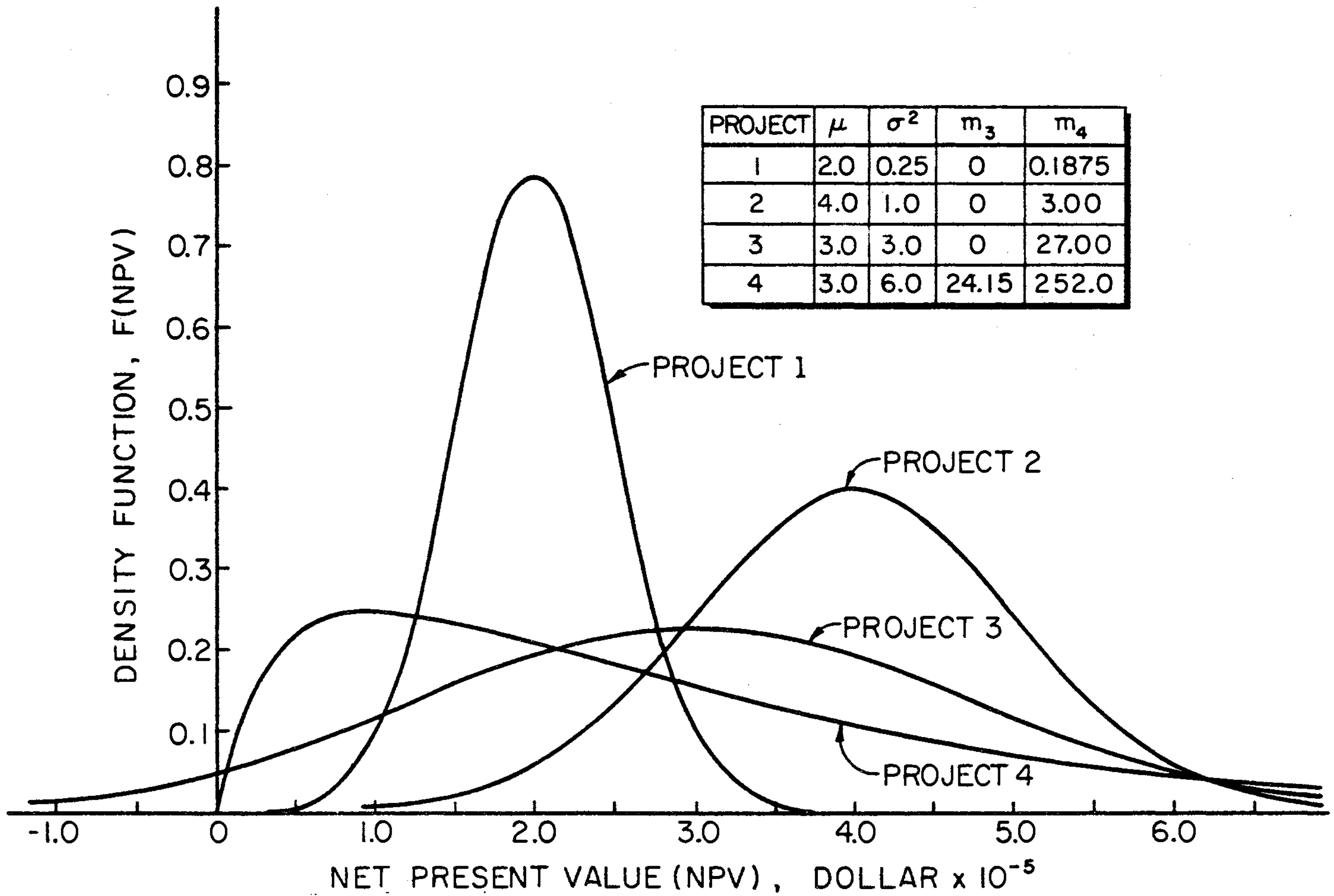
$$\hat{U} = -17.63 + 48.74X - 17.967X^2 + 2.721X^3 - 0.123X^4$$

where X is a distributed project net present value in Dollars x 10^{-5}. It is also assumed that Mellon, or another person with expert knowledge in whom Mellon has confidence, has evaluated four prospective <u>in</u>dependent projects and has estimated (subjectively, if necessary) each of the project net present values and their distributions, as tabulated in Table V and illustrated in Figure 17. (These distributions, obviously, result from assumed solutions to the <u>analysis</u> problem described in Chapter III, and such solutions could result from any of the types of formulations described there. For the quartic utility function it is required, of course, that all four moments be known for each project net present value).

TABLE V

PARAMETERS OF PROJECT NET PRESENT VALUES

Project Identification (j)	Mean NPV (μ)	V(NPV) (σ^2)	Third Moment About Mean (m_3)	Fourth Moment About Mean (m_4)
1	2.0	0.25	0	0.1875
2	4.0	1.00	0	3.00
3	3.0	3.00	0	27.00
4	3.0	6.00	24.15	252.00



PROJECT	μ	σ^2	m_3	m_4
1	2.0	0.25	0	0.1875
2	4.0	1.0	0	3.00
3	3.0	3.0	0	27.00
4	3.0	6.0	24.15	252.0

Figure 17. Density Functions of Project Net Present Values

When complete enumeration of all possible combinations of projects is the mode of solution, there are 2^m possible combinations of the net present value parameters. One combination is always that of "do nothing" or "invest in cash." Either term is acceptable, since either one implies retention of available funds without any return. The remaining $2^m - 1$ combinations encompass all of the remaining possible combinations of m projects, for each of which the mean and other moments of the system response function must be calculated. Thus, for the present case of $m = 4$, the mean, variance, third moment and fourth moment for each of the $2^4 - 1 = 15$ combinations of net present value must be calculated. For the independent case, equations (126), (128), (130) and (132) are used. As an example of a calculation, consider the project combination consisting of Projects 1, 2 and 3. For this combination, the mean of the resultant net present value distribution would be

$$E(z) = x_1 + x_2 + x_3 = 2.0 + 4.0 + 3.0 = 9.0$$

which follows directly from equation (126). The variance, from equation (128), is

$$V(z) = \sigma_z^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 0.25 + 1.0 + 3.0 = 4.25 \quad .$$

The third moment about the mean of z, from equation (130), is

$$m_3(z) = m_3(x_1) + m_3(x_2) + m_3(x_3) = 0 + 0 + 0 = 0 \; .$$

The fourth moment about the mean of z, from equation (132), is

$$\begin{aligned} m_4(z) &= m_4(x_1) + m_4(x_2) + m_4(x_3) + 6[\sigma_1^2(\sigma_2^2 + \sigma_3^2) + \sigma_2^2(\sigma_3^2)] \\ &= 0.1875 + 3.0 + 27.0 + 6[0.25(1.0 + 3.0) + 1.0(3.0)] \\ &= 54.1875 \; . \end{aligned}$$

In a similar fashion, the first four moments are calculated for the remaining 15 "bundles" of projects, the results of which are given in Table VI.

Now, by means of equations (112) and (113), the expected values $E(z)$, $E(z^2)$, $E(z^3)$ and $E(z^4)$ can be calculated from the moments, which, for the example project bundle (Projects 1, 2 and 3), are:

$$E(z) = 9.0$$

$$E(z^2) = \sigma_z^2 + [E(z)]^2 = 4.25 + (9.0)^2 = 85.25$$

$$E(z^3) = 3\sigma_z^2[E(z)] + [E(x)]^3 + m_3(z)$$

$$= 3(4.25)(9.0) + (9.0)^3 + (0) = 195.75$$

$$E(z^4) = [E(z)]^4 + 6\sigma_z^2[E(z)]^2 + 4\, m_3(z)\, E(z) + m_4(z)$$

$$= (9.0)^4 + 6(4.25)(9.0)^2 + 4(0)(9.0) + 54.1875$$

$$= 8,680.6875 .$$

Knowing the expected values of these four moments <u>about the origin</u>, the expected utility of the resultant net present value for the bundle, $E[U(z)]$, can be calculated by applying the expected value operator to the utility function itself, thus:

$$E[U(z)] = -17.63 + 48.74\, E(z) - 17.967\, E(z^2) + 2.721\, E(z^3) - 0.123\, E(z^4)$$

$$= -17.63 + 48.74(9.0) - 17.967(85.25) + 2.721(195.75) - 0.123(8680.6875)$$

$$= +117.46 \text{ utiles} .$$

TABLE VI

MOMENTS OF PROJECT BUNDLE NET PRESENT VALUE DISTRIBUTIONS

Project Bundle Number	Component Project Numbers	Mean Net Present Value	Variance of Net Pres. Val.	Third Moment of NPV	Fourth Moment of NPV
1	1	2.00	0.25	0	0.19
2	2	4.00	1.00	0	3.00
3	3	3.00	3.00	0	27.00
4	4	3.00	6.00	24.15	252.00
5	1,2	6.00	1.25	0	4.69
6	1,3	5.00	3.25	0	31.69
7	1,4	5.00	6.25	24.15	261.19
8	2,3	7.00	4.00	0	48.00
9	2,4	7.00	7.00	24.15	291.00
10	3,4	6.00	9.00	24.15	387.00
11	1,2,3	9.00	4.25	0	54.19
12	1,2,4	9.00	7.25	24.15	301.69
13	1,3,4	8.00	9.25	24.15	400.69
14	2,3,4	10.00	10.00	24.15	444.00
15	1,2,3,4	12.00	10.25	24.15	459.19

In a similar manner, the expected utility for each of the other combinations of projects can be calculated, the results of which are tabulated in Table VII.

TABLE VII

EXPECTED UTILITIES OF BUNDLES

Project Bundle Number	Component Project Numbers	Expected Utility, E[U(NPV)]
1	1	26.61
2	2	35.02
3	3	26.71
4	4	28.74
5	1,2	61.30
6	1,3	50.54
7	1,4	41.81
8	2,3	87.29
9	2,4	48.98
10	3,4	43.13
11	1,2,3	117.46
12	1,2,4	32.96
13	1,3,4	34.09
14	2,3,4	- 45.02
15	1,2,3,4	-271.39

Now, the question is which bundle represents the decision-maker's choice. It will be noted that one bundle, consisting of Projects 1, 2, 3, and 4, has a mean net present value of 12.0. For the particular utility function assumed here, namely, Mellon's quartic utility function, the range of validity is limited to values of mean net present value of 10.4988 or less (see Appendix C for the calculation of this result and for the validation of Mellon's utility function). That is, the intercept of the marginal utility function for Mellon's utility function is at $\mu_{NPV} = 10.4988$, and the project combinations whose mean net present values are greater than this value cannot be evaluated. Thus, regardless of whether the calculated utility of such a bundle is positive or negative, and regardless of whether or not such an expected utility is or may be the maximum of all bundles, all calculations for bundle net present values in excess of the validity limit must be ignored. This requirement follows from the discussion in Chapter V concerning the valid range of cubic and quartic utility functions. With reference to Table VII, then, project Bundle 15 (consisting of Projects 1, 2, 3 and 4) is eliminated from consideration. In a real situation, this would be an unfortunate occurrence. However, it simply means that the decision-maker's utility function was not determined initially over a wide enough range for the magnitude of the projects being considered. The remedy is to redetermine the utility function, extending the range of investigation.

Again referring to Table VII, in the absence of budget constraints, the project bundle with maximum expected utility would be chosen. That is, Bundle 11 (consisting of Projects 1, 2 and 3) gives maximum expected utility for R. F. Mellon, and if the criterion is maximization of expected utility, then only Projects 1, 2 and 3 would be executed. The effect of budget constraints can be illustrated by assuming some project "costs" for each project. Suppose, for example, that each project had an initial cost (in year zero) as follows:

Project 1	-	$ 50,000
Project 2	-	100,000
Project 3	-	40,000
Project 4	-	70,000

The cost of a project bundle is merely the summation of the costs of the component projects. To determine the optimal choice of a bundle, the project bundles are ranked by increasing bundle cost until the budget is reached (but not exceeded), and then the bundle with the maximum expected utility among feasible bundles is the optimum bundle.

Assuming that the budget is $175,000, the ranking procedure is illustrated in Table VIII, from which the optimum bundle is Bundle 8, consisting of Projects 2 and 3, with an expected utility of 87.29 utiles.

If more than one periodic budget were part of the problem, a second ranking by cost, a third, and so on, would be performed; and, the optimum bundle would be that one which would be common to all sets of rankings within the feasible region of each, which maximizes expected utility.

Technical constraints, such as a mutual exclusivity constraint, are handled by a notation system in the ranking procedure which rules out selection of a project bundle in which the constraint is operative. For example, if Projects 2 and 3 were mutually exclusive, then Bundle 8

TABLE VIII

PROJECT BUNDLES RANKED BY COST

Project Bundle Number	Component Project Numbers	Bundle Cost	Expected Utility, E[U(NPV)]
3	3	40,000	26.71
1	1	50,000	26.61
4	4	70,000	28.74
6	1,3	90,000	50.54
2	2	100,000	35.02
10	3,4	110,000	43.13
7	1,4	120,000	41.81
8	2,3	140,000	87.29
5	1,2	150,000	61.30
13	1,3,4	160,000	34.09
9	2,4	170,000	48.98
11	1,2,3	190,000	117.46
14	2,3,4	210,000	- 45.02
12	1,2,4	220,000	32.96

Note: project Bundles 11, 14 and 12 are infeasible because of budget limitation of $175,000.

would be lined out, thereby preventing the simultaneous selection of Projects 2 and 3. If this were the case, the optimum bundle would be Bundle 5, consisting of Projects 1 and 2, with a maximum expected utility of 61.30 utiles.

Probability Statements

When the mean and the three central moments (about the mean) of a distribution are known, as for any one of the project bundles in Table VI, the distribution can often be approximated by an empirical distribution such as the Johnson or Pearson distribution. If it is assumed, for example, that a Pearson distribution is a satisfactory spproximation to the unknown net present value distribution of a project bundle, then satisfactory approximations concerning probability statements about net present value of the project bundle can often be made. Probability approximations are obtained by reference to standardized Pearson or Johnson distribution tables, such as those reported in references (36) and (47) for the Pearson family of distributions.
A numerical example will suffice to demonstrate the method.

The required information consists of knowledge of the mean and the next three central moments of the unknown distribution. For example, refer to Table VI, and observe that the parameters for Bundle 12 are as follows:

$$E(NPV) = 9.0$$

$$\sigma^2_{NPV} = 7.25$$

$$m_3(NPV) = 24.15$$

$$m_4(NPV) = 301.69 .$$

Now, calculate the two Pearsonian shape parameters:[3]

$$\sqrt{\theta_1} = \frac{m_3}{(\sigma^2)^{1.5}} = \frac{24.15}{(7.25)^{1.5}} = 1.238$$

$$\theta_2 = \frac{m_4}{(\sigma^2)^2} = \frac{301.69}{(7.25)^2} = 5.74 .$$

These two shape parameters permit entry into the standardized Pearson tables for the determination of a standardized deviate

$$t_\alpha = \frac{X - E(NPV)}{\sigma}$$

corresponding to a given probability level, α, from which confidence intervals can be determined. Conversely, a given probability can be estimated from a knowledge of the actual deviate, X - E(NPV), by reversing the process, although interpolation in the tables is more difficult.

For the above example, assume that 90% confidence limits on NPV are desired. Referring to Table XIV (Appedix D), which is a reproduction of the appropriate section of the Pearson tables, for $\alpha = 0.05$ enter the table with $\sqrt{\theta_1} = 1.238$ and $\theta_2 = 5.74$, and read (by interpolation) the standardized deviate $t_{.05} = -1.265$. Similarly, for $\alpha = 0.95$, enter the appropriate table with $\sqrt{\theta_1}$ and θ_2, and interpolate the standard deviate $t_{.95} = 1.886$. The confidence limits corresponding to $\alpha = 0.05$ and $\alpha = 0.95$ are then found, thus:

$$X_{\alpha=.05} = E(NPV) - t_{.05}\,\sigma = 9.0 - 1.265\sqrt{7.25} = 5.59$$

$$X_{\alpha=.95} = E(NPV) + t_{.95}\,\sigma = 9.0 + 1.886\sqrt{7.25} = 14.08$$

Thus, $P[\$5.59(10^5) \leq NPV \leq \$14.08(10^5)] = 0.90$, and, if one is willing to accept a possible Type I error of 10%, the NPV will fall somewhere in the range between \$559,000 and \$1,408,000. Similarly, the probability of a net present value being less than \$559,000 is approximately 5%.

Now, assuming that one wanted to know what the probability of a loss is, the standard deviate for this assumption is

$$t_\alpha = \frac{0 - 9.0}{(7.25)^{\frac{1}{2}}} = -3.34 ;$$

and entering the table with $\sqrt{\theta_1}$ and θ_2, it is found that the largest negative standard deviate is approximately -2.3 for $\alpha = 0$. Thus, the probability of zero net present value or less is approximately zero, since $t_\alpha = -3.34 < t_0 = -2.3$.

This concludes the presentation of a numerical example to illustrate project selection by means of the expected utility criterion derived from a quartic utility function. By implication, the same procedure can be applied to the cubic utility function, with appropriate changes in the estimation of bundle moments and expected utilities.

FOOTNOTES

[1]See, for example, Winegartner (65). Traditionally, the problem has been solved by dynamic programming methods using a Lagrangian multiplier technique for merging the several constraints, as originally proposed by Bellman (5). However, as has been shown by Everett (18), and demonstrated emphatically by Winegartner (65), the Lagrangian may NOT exist in some cases, and therefore, the problem solved may not be (and often is not) the problem as originally stated for solution. Consequently, the Lagrangian method does not necessarily lead to a discrete optimum solution, as the problem requires when x_j is restricted to values of 0 or 1.

[2]The material in this section is based on Hahn and Shapiro (27), pages 228-236 and 252-257. Hahn and Shapiro call this method the "Generation of System Moments," which, as they point out, is equivalent to the terms "Statistical Error Propagation" or "Delta Method", used by other authors.

[3]The shape parameter $\sqrt{\theta_1}$ expresses the relative skewness of the distribution. When $\sqrt{\theta_1}$ is zero, the distribution is symmetrical about the mean, as a normal distribution is. When $\sqrt{\theta_1}$ is positive, the distribution is right-skewed (with the "long" tail pointed toward higher values of the variate), and when it is negative, the distribution is left-skewed (with the "long" tail pointed toward lower values of the variate).

The shape parameter θ_2 expresses the kurtosis, or relative peakedness of the distribution. The standard reference value is $\theta_2 = 3$ for the normal distribution. If $\theta_2 > 3$ the distribution is said to be leptokurtic, or more peaked than the normal density function. If $\theta_2 < 3$ the distribution is said to be platykurtic, or "flatter" than the normal density function.

CHAPTER VII

SUMMARY AND CONCLUSIONS

The principal problem approached in this research is a theoretical analysis of the probabilistic non-sequential capital budgeting problem from the standpoint of explaining, or predicting, the behavior of a non-risk-averse decision-maker. The objective is to provide a theoretical basis for a solution to the capital budgeting problem under conditions of uncertainty, especially where the decision-maker displays risk-seeking behavior for some combinations of return and uncertainty.

Secondary problems approached are (1) a rigorous statement of the probabilistic non-sequential capital budgeting model, (2) the development of a classification scheme by which prior research can be integrated into what is called the analysis problem, and (3) the correction of an interpretational error (in the Farrar model) which has affected prior studies of the capital budgeting problem under uncertainty.

The basis for this study lies in the assumption that not all decision-makers are risk-averse, at least for certain combinations of risk and return. This is contrary to the direction that most (in fact, nearly all) prior research has taken. The rationale for an assumption of non-risk-averse behavior originates in both observed behavior of decision-making in growth companies and in empirical evidence. While business decision-makers may be risk-averse in a gross sense in the long run, it is also a fact that they tend to generate "growth leverage by taking risks in order, ultimately, to reach the risk-aversion plateau of entrepreneurial maturity. Stated another way, the ultimate goal of the firm seems to be one of "satisifcing" decision behavior (as March and Simon call it) in which risk conversion--the control of the uncertainties in the environmental determinants of outcome--is an accomplished fact; but the intermediate growth process, on the way to the ultimate goal, is characterized more by the assumption of risks in order to generate the capacity for risk conversion. Furthermore, empirical evidence in the form of non-risk-avoiding utility functions exists in the published literature. Such utility functions imply, at least, that certain decision-makers are not always risk-averse over all ranges of return. For these reasons, the decision behavior of non-risk-avoiding decision-makers is of importance, and this research explores the theoretical implications of such behavior.

Many of the conclusions of this research are presented along with the development of the various problems appearing in each chapter. However, some of the major conclusions are as follows:

(1) The maximization of net present value, as a project selection criterion, is not adequate when the capital budgeting problem involves considerations of explicit uncertainty. Some other criterion, based at least upon the mean and the variance of the net present value distribution, is necessary in order for a realistic solution to the capital budgeting problem to be obtained.

(2) The capital budgeting problem consists of two separate phases: (a) the analysis problem, which is characterized by the mathematical formulations necessary to obtain a present value distribution for each project; and (b) the selection problem, by which specific

projects (or groups of projects, called "bundles") are selected from the set of all candidate projects under consideration. Selection of an optimal subset of projects is done in accordance with a selection criterion, which is a mathematical formulation derivable from utility functions that acts as the objective function in the selection model. The solution to the analysis problem becomes an input to the selection problem, and no inter-project comparisons can be made in the absence of a defined selection criterion.

(3) The non-sequential capital budgeting problem can be formulated in the form of a maximization of some function of Net Present Value to the firm, where maximum f(NPV) results from an optimal subset of projects. For the deterministic case, in which all net present values for all projects are assumed to be known with certainty, f(NPV) becomes simply equal to the net present value itself, and the objective function is then to maximize net present value. For probabilistic cases, maximization of expected utility (of net present value) is a reasonable criterion, derivable from the von Neumann-Morgenstern axioms of rational behavior. In these cases, the form of f(NPV) depends upon the mathematical form of the utility function assumed for the decision-maker. In all cases, the capital budgeting problem may be constrained by one or more technical constraints, but is always constrained by the requirement that projects be indivisible.

(4) Many of the probabilistic models appearing in the literature can be classified into various forms of the analysis problem, on the basis of the dependence-independence assumptions made in the model's cash flow stream. The classification scheme developed herein permits structural integration of nearly all of the published cash flow models.

(5) A derivation of the expected utility selection criterion is given in detail for the case where the assumed utility function is a quadratic polynomial. This derivation corrects a prior one by Farrar (19), who came to an incorrect conclusion in his derivation. This error was perpetuated by Watters (62), the first investigator (in 1967) to apply an assumption of a quadratic utility function to the 0/1 capital budgeting problem under uncertainty. Thus, the derivation stated herein also corrects Watters' work.

(6) The derivation of Freund's selection criterion from an assumed negative exponential utility function is also given in detail. While Freund (24) noted how this could be done, and used the result, he did not present the derivation step-by-step nor show the necessary transformation to obtain the result.

(7) The case for the existence of a complex utility function, in quartic polynomial form, is considerably strengthened by the use of a multiple regression technique. The analysis of variance of the regression for one decision-make, R. F. Mellon (reported by Grayson (25)), indicates that terms up to at least the quartic must be retained in the regression equation.

(8) The necessary and sufficient conditions for the existence of valid utility functions, from a theoretical standpoint, are rigorously derived for utility functions of cubic and quartic polynomial form. Such derivations have not been reported in the literature anywhere prior to this time.

(9) Finally, a numerical example of a project selection problem, based on a quartic utility function, is formulated and solved by complete enumeration (the only known method of solving such problems).

A method of making probability statements about the net present value of a project "bundle" is given, based on knowledge of the four principal moments of the bundle net present value distribution and an assumption that a Pearson distribution approximates the unknown net present value distribution satisfactorily.

Areas for Further Investigation

Von Neumann-Morgenstern Utility Function

One of the principal weaknesses in the use of selection criteria derived from utility functions is that an assumption must be made as to the mathematical form of the utility function itself, so that "best fit" of the decision-maker's response data (to the N-M standard lotteries) can be obtained. The manner in which the standard lotteries are presented to the respondent greatly affects the sampling, or experimental, error. Great care must be taken to insure consistent response data, and considerable "overlapping" of standard lotteries must be incorporated into the examination procedure in order to obtain a valid and independent measure of the sampling error. The sampling error magnitude is critical in the acceptance or rejection of hypotheses concerning the shape and form of the assumed utility function. More investigative work using multiple regression techniques for utility function validation needs to be done.

Moreover, the N-M utility function may not actually measure the decision-maker's "attitudes" toward risk taking or risk aversion. A better method of inferring such attitudes from other than the standard lotteries is needed, as, for example, overt decision behavior in actual viable organizational contexts.

A third point is that capital budgeting decisions in the context of the "firm" may not be--and, indeed, often are not--made by a single individual acting for the firm, but rather by a group of individuals who collectively comprise the actual decision-making "body." While at least one attempt has been made by Spetzler(58) to determine a "utility function" for the principal managing officers of an industrial firm, the results in this effort are far from conclusive. Indeed, some theoretical objections to a "group" utility function must be disposed of, or at least modified, in order for such a function to exist; and this requires investigation in depth into the decision-making processes of a group. While some investigations in the field of social psychology show promise in this direction, the problem is by no means solved.

Complex Negative Exponential Utility Function

One of the major disadvantages of the cubic and quartic forms of the assumed utility function is that the marginal utility (the first derivative of the utility function itself) becomes zero at some finite value of the return (net present value). These functions, therefore, are limited in their validity to ranges of net present value equal to or less than the point at which the marginal utility becomes zero. Possible relief from this limitation could be obtained if the utility function were of modified exponential form.

Consider, for example, the negative exponential utility function proposed by Freund, which is of the form

$$U(X) = 1 - e^{-AX} \qquad (134)$$

where A is a constant. This function is concave downward, and expresses a <u>general</u> aversion to risk over all ranges of X. Now, if this general aversion were to be "overcome" by the decision-maker in a certain range of X, so that within this range he would accept risks not offset by increased returns, then one would have the basis for complex decision behavior. Such an "overcoming" of general risk aversion, for a specific range of X, can be modeled mathematically by providing an additive term of negative Normal form; that is, by adding to equation (134) the following:

$$-\Delta U(X) = B\, e^{-C(X - D)^2} \tag{135}$$

where B, C and D are constants, not necessarily the same ones used in the quartic polynomial form of the utility function.

Equation (135) expresses a "negative" risk-aversion—that is, a positive risk-seeking attitude—over all values of X, but which becomes most pronounced and, in fact, overpowers the <u>general</u> tendency toward risk aversion in the vicinity where the deviate X - D becomes zero. Thus, the shape of the complex exponential utility function,

$$U(X) = 1 - e^{-AX} - B\, e^{-C(X - D)^2}, \tag{136}$$

would be similar to that of the quartic utility function, in that it would possess the necessary concave-upward curvature to indicate risk-seeking behavior over a portion of the range of X. Moreover, it would possess the Siegel "level of aspiration" inflection point, which seems to be a necessary ingredient in any description of complex risk behavior.

But the most obvious improvement that equation (136) implies, however, is a <u>positive</u> marginal utility over <u>all</u> ranges of X, up to and including positive infinity. Such a conclusion can be supported by taking the first derivative of equation (136) with respect to X:

$$U'(X) = \frac{dU(X)}{dX} = +Ae^{-AX} + 2BC(X - D)e^{-C(X-D)^2} \qquad (137)$$

which is the marginal utility of the complex exponential utility function. By proper choice of the coefficients A, B, C and D, equation (137) can be made positive over all values of X, and when the coefficients are so chosen, then the utility function in equation (136) becomes a valid utility function. The intercept at the origin is shifted somewhat by equation (136), but this needs only a minor linear adjustment to cause the function to pass through the origin, and this may be performed straightforward because of the assumption that a utility function is valid up to a linear transformation.

Obviously, a complex exponential utility function is freed from the restriction of a finite operating range, and can be used--even extrapolated somewhat beyond its range of investigation--more flexibly than can the quartic utility function. More investigation is needed about this form of utility function, particularly in the theoretical derivation of the necessary and sufficient conditions for its existence, and in the derivation of a selection criterion from it. This is a suggested area for further research.

These suggested areas for further research are not all of the possible ones, nor is this dissertation a complete solution to the capital budgeting problem. However, it is believed that the research reported here, particularly with reference to the derivations for the cubic and quartic utility functions and the specification of selection criteria from them, adds to the existing knowledge concerning decision-making behavior under conditions of uncertainty. Moreover, it is believed that, with continued investigation into the whole problem, there will result (one day) an understanding of the fundamental principles upon which organizational decisions are made, at least with respect to the risk-taking and risk-seeking propensities of the principal decision-makers in the organization. It is to this overall goal that this research has been directed.

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APPENDICES

APPENDIX A

NUMERICAL EXAMPLES

Problem 1

The cash flow stream for a project has been synthesized from relevant data, with the following estimates for the random variable cash flow increments:

End of Period	Mean Cash Flow Increment, $\bar{Y}_t$	Standard Deviation of Cash Flow, σ_t
0	$ -85,000	$ 6,000
1	+20,000	5,000
2	+30,000	2,500
3	+35,000	3,000
4	+38,000	3,500
5	+38,000	4,000

(a) Assuming the cash flow increments (each a random variable) are normally and independently distributed, and that the applicable discount rate is 15%, what is the probability that the project (if accepted) will have a negative net present value?

(b) Under the same assumptions, what is the probability that the net present value will be greater than $25,000?

(c) If no assumption can be made concerning the form of the cash flow distributions, what is the probability that a negative net present value might result?

Solution

The mean (expected value) of the project net present value is found from equation (21):[1]

$$E[NPV] = \overline{Y} = -85{,}000 + 20{,}000(PS_{15-1}) + 30{,}000(PS_{15-2}) + \dots + 38{,}000(PS_{15-5})$$

$$= -85{,}000 + 20{,}000(.8696) + 30{,}000(.7561) + 35{,}000(.6575) + 38{,}000(.5718) + 38{,}000(.4972)$$

$$= +18{,}750.$$

The variance of the project net present value is found from equation (23):

$$V[NPV] = (6000)^2 + (5000)^2(PS_{15-2}) + (2500)^2(PS_{15-4}) + \dots + (4000)^2(PS_{15-10})$$

$$= 10^6[36(1.0) + 25(.7561) + 6.25(.5718) + 9.0(.4323) + 12.25(.3269) + 16.0(.2472)]$$

$$= 70.42 \times 10^6.$$

Note, that even though the Period 0 cash flow is negative (a capital consumption, or "cost"), the variance of that increment is added in the variance summation.

Then, the standard deviation of the NPV is

$$\sigma_{PV} = \sqrt{V(NPV)} = 8{,}400.$$

(a) Since the cash flow increments are $NID(\overline{Y}_t, \sigma_t^2)$, then the Net Present Values is also $NID(\overline{Y}, \sigma_{PV}^2)$. The probability of a negative net present value is found from the standardized normal deviate in which Y = 0 is the upper bound:

$$Z_{\alpha} = \frac{Y - \bar{Y}}{\sigma_{PV}} = \frac{0 - 18750}{8400} = -2.23 .$$

From Normal tables,

$$\alpha = P[-\infty \leq Y \leq 0] = 0.0129 .$$

(b) Again, computing the standardized normal deviate for the upper bound Y = 25,000 and computing its complementary probability,

$$Z_{1-\alpha} = \frac{Y - \bar{Y}}{\sigma_{PV}} = \frac{25000 - 18750}{8400} = +0.745$$

$$1 - \alpha = P[-\infty \leq Y \leq 25000] = 0.772$$

from which $\alpha = P[Y > 25000] = 1 - 0.772 = 0.228 .$

(c) If the form of the cash flow increment distributions is not known, then Chebyshev's inequality is applicable. Compute the same standardized deviate as for the normal distribution. Then, if $t_c = Z_{\alpha}$, by Chebyshev's inequality

$$P[-\infty \leq Y \leq 0] \leq 1 - [1 - \frac{1}{t_c^2}] = \frac{1}{t_c^2}$$

$$\text{"} \qquad \leq \frac{1}{(-2.23)^2} = 0.201 .$$

If the form of the cash flow increment distributions is known to be unimodal, with the mode equal to the mean, then the Camp-Meidell inequality is applicable:

$$P[-\infty \leq Y \leq 0] \leq 1 - (1 - \frac{1}{2.25t_c^2}) = \frac{1}{2.25t_c^2}$$

$$" \qquad \leq \frac{1}{2.25(-2.23)^2} = 0.089 .$$

Problem 2

Using the cash flow stream and other data supplied in Problem 1, assume that the cash flow increments are also correlated among periods as indicated in Table IX.

TABLE IX

CORRELATION COEFFICIENTS ($\rho_{\delta\theta}$) OF CASH FLOW INCREMENTS AMONG PERIODS

Y_δ \ Y_θ	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5
Y_0	-	0	0	0	0	0
Y_1		-	0.100	0.050	0.020	0.010
Y_2			-	0.150	0.120	0.060
Y_3	(Symmetrical)			-	0.160	0.140
Y_4					-	-0.070
Y_5						-

Posing the same questions (a), (b), and (c) as in Problem 1, the only new information needed is the project NPV variance, which can be calculated from equation (34):

$$V[Y] = 70.42(10^6) + 2(10^6)[(.100)(25)(6.25)(PS_{15-3}) + $$
$$+ (0.05)(25)(9)(PS_{15-4}) + (0.02)(25)(12.25)(PS_{15-5})$$
$$+ (0.01)(25)(16)(PS_{15-6}) + (.15)(6.25)(9)(PS_{15-5})$$
$$+ \ldots + (-0.07)(12.25)(16)(PS_{15-9})],$$
$$= 149.8(10^6) \;;$$

from which $\sigma_{PV} = \sqrt{V(Y)} = 12{,}220.$

(Note that the negative correlation coefficient, ρ_{45}, decreases the variance V(Y) while the other positive coefficients increase it over the uncorrelated case).

The probability statements in (a), (b) and (c) for Problem 1 now become:

(a) $$Z_\alpha = \frac{0 - 18750}{12220} = -1.533$$

$\alpha = 0.0626$ (from Normal tables).

(b) $$t_c = \frac{0 - 18750}{12220} = -1.533$$

$$P[-\infty \leq Y \leq 0] \leq \frac{1}{t_c^2} = \frac{1}{(-1.533)^2} = 0.425$$

(c) $$P[-\infty \leq Y \leq 0] \leq \frac{1}{2.25t_c^2} = \frac{1}{2.25(-1.533)^2} = 0.189 \;.$$

Thus, while the expected Net Present Value remains unchanged, the effect of positively correlated cash flow increments is to increase the project variance (and therefore decrease the probability of "success," i.e., it increases the probability of lower net present values). Conversely, a negative correlational relationship will decrease project variance and increase the probability of a higher project NPV.

Problem 3

A simplified version of the Horowitz model for the Case IV problem can be formulated and solved as follows.

Let Q = plant output (demand), units;

P = market price, \$/unit;

e = price elasticity of demand (assumed positive), normally and independently distributed (μ_e, σ_e^2);

W = unit cost of production, \$/unit.

Then, according to Horowitz' formulation, the dependent plant output in any period t is a result of the independently distributed elasticity and the market price in that period, according to the relationship

$$Q = A\,P^{-e} \tag{A-1}$$

where A is a constant (assumed here to be 10^6).

Now, under the assumed economic condition of an elastic market (here, for convenience, it is assumed that $e > +1$), and with a goal of maximization of profit, the firm should adjust its price until marginal costs equal marginal revenue. At this operating point, it can be shown that the unit price to be charged is[2]

$$P = W\left[\frac{e}{e-1}\right] . \tag{A-2}$$

Now, if production costs (W) are assumed to be known and constant,[3] and if (for simplicity) fixed costs, depreciation and tax rate are ignored, then the "gross income" in any period t is PQ, and the cash flow increment is (P - W)Q. Substituting equations (A-1) and (A-2), the cash flow increment is

$$Y_t = (P - W)Q = \left[W(\frac{e}{e-1}) - W\right] A\ P^{-e}$$

$$= W\left[\frac{1}{e-1}\right] A \left[W \frac{e}{e-1}\right]^{-e}$$

$$= \frac{A}{e}\left[W \frac{e}{e-1}\right]^{1-e} \tag{A-3}$$

which is a <u>non</u>-linear relationship among two unlike functional constants (A, W) and a normally distributed variable, e. Hence, the cash flow increments (and, therefore, the project net present value) will be distributed non-normally. Summing over t, the project net present value is

$$NPV = Y_0 = \sum_{t=0}^{n} (1 + i)^t \left\{ A\ e^{-1} \left[W \frac{e}{e-1}\right]^{1-e} - C_t \right\} \tag{A-4}$$

where C_t = any non-variable "cost" (i.e., investment) in period t.

To obtain the <u>expected</u> net present value, it would be necessary to evaluate the expression

$$E[NPV] = \int_{e=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_e} \exp\left[-\frac{1}{2} \frac{(e - \mu_e)^2}{\sigma_e}\right] [NPV(e)]\ de \tag{A-5}$$

where [NPV(e)] is given by equation (A-4). As Horowitz says, "the potential for the evaluation of this integral would not seem to be very great." Accordingly, a numerical integration procedure is used to approximate the mean and variance of the net present value, which will be demonstrated.

Let the elasticity, e, be normally and independently distributed with mean $\mu_e = 1.5$ and variance $\sigma_e^2 = 0.04$. Let the constant $A = 10^6$. Let the unit cost of production W = \$10/unit. Now, choose standardized normal deviates, Z_e, ranging from approximately -2.5 to +3.0.

Calculate the corresponding values of elasticity, e, from the mean and standard deviates:

$$e = \mu_e + Z_e \sigma_e , \qquad \text{(A-6)}$$

and calculate the corresponding values of the cash flow increments from equation (A-3). These calculations are shown in Table X.

TABLE X

CASH FLOW INCREMENTS

$P[e' \leq e]$	Z_e	e (Eqn. (A-6))	$Y_t = f(A,W,e)$ (Eqn. (A-3))
0.0062	-2.50	1.001	990,000
0.010	-2.38	1.04	938,000
0.100	-1.28	1.24	321,000
0.200	-0.840	1.33	220,000
0.300	-0.524	1.39	175,000
0.400	-0.253	1.45	143,500
0.500	0	1.50	121,500
0.600	+0.253	1.55	102,500
0.700	+0.524	1.61	85,400
0.800	+0.840	1.67	69,600
0.900	+1.28	1.76	52,400
0.950	+1.64	1.83	41,500
0.999	+3.09	2.12	17,400

Now, the probability that e' is less than some value e is exactly equivalent to the probability that Y'_t is greater than some value Y_t which was computed from e (see Table X). Thus,

$$P[Y'_t < Y_t] = 1 - P[e' < e] \text{ , for all } Y_t, \qquad \text{(A-7)}$$

which allows the construction of Table XI from Table X.

TABLE XI

CUMULATIVE DISTRIBUTION OF CASH FLOWS

$P[Y'_t \leq Y_t]$	Y_t
0.9938	990,000
0.990	938,000
0.900	321,000
0.800	220,000
0.700	175,000
0.600	143,500
0.500	121,500
0.400	102,500
0.300	85,400
0.200	69,600
0.100	52,400
0.050	41,500
0.001	17,400

The objective then is to obtain a probability function from the cumulative cash flow distribution. This can be done as follows. Knowing that the probability, for example, of obtaining a cash flow of 321,000 or less is 0.900, and the probability of obtaining one of 220,000 or less is 0.800, then the cell mean cash flow that will lie in the p = 0.100 interval can be approximated by the geometric mean

$$(Y_t)_g = \sqrt{(10^{12})(.321)(.220)} = 266{,}000 ,$$

and the probability of getting $(Y_t)_g$ is 0.100, the difference between the upper and lower cell boundaries. In a similar fashion, the cell mean cash flows can be obtained for the other intervals in Table XI. The project mean net present value is then found by taking the mathematical expectation of the series, and the variance by taking moments about the project mean (grand mean). These calculations are given in Table XII.

The grand mean of the cash flow distribution is

$$E[\overline{Y}_t] = \overline{\overline{Y}}_t \doteq \frac{\text{Expectation}}{\Sigma p[\overline{Y}_t]} = \frac{16.68 \times 10^4}{0.9928} = 16.8 \times 10^4$$

and the deviations of $\overline{Y}_t$ in Table XII are calculated about this value. The standard deviation of the random cash flow variable is then estimated by

$$\hat{\sigma}_{Y_t} = \sqrt{\frac{\Sigma f_p[(Y_t)_g - \overline{\overline{Y}}]^2}{N - 1}} = \left[\frac{209.37 \times 10^8}{12 - 1}\right]^{\frac{1}{2}} = 4.36 \times 10^4.$$

Now, an assumption was tacitly made earlier that equation (A-3) was a general expression for the cash flow in any period t. This is true. Since equation (A-3) depends solely on the random variable e,

TABLE XII

MEAN AND VARIANCE OF CASH FLOW INCREMENTS

Cell Mean $(Y_t)_g$	$f_p = p[(Y_t)_g]$	Expectation	$(Y_t)_g - \bar{\bar{Y}}$	$f_p[(Y_t)_g - \bar{\bar{Y}}]^2$
96.3×10^4	0.0038	0.366×10^4	79.5×10^4	24.0×10^8
54.6	0.090	4.91	37.8	129.0
26.6	0.100	2.66	9.8	9.60
20.2	0.100	2.02	3.4	1.16
16.3	0.100	1.63	- 0.5	0.02
13.2	0.100	1.32	- 3.6	1.30
11.2	0.100	1.12	- 5.6	3.14
9.20	0.100	0.92	- 7.6	5.78
7.65	0.100	0.765	- 8.15	6.62
6.04	0.100	0.604	-10.76	11.60
4.66	0.050	0.233	-12.14	7.40
2.69	0.049	0.132	-14.11	9.75
TOTALS	0.9928	16.680×10^4	-----	209.37×10^8

then each of the cash flow distributions will have the <u>same</u> parent (population) distribution for each of the t periods, which causes the Y_t to be perfectly correlated among periods. Thus, this model is a special case of the Case IV models.

To complete the problem, assume a "known" project life of three years and a discount rate of 15%. Then, the project expected NPV is found from equation (26), with the right-hand member set to zero (there are no <u>in</u>dependent cash flows):

$$E[NPV] = \sum_{t=1}^{3} \frac{E[\bar{Y}t]}{(1+i)^t} = \frac{16.8x10^4}{(1.15)} + \frac{16.8x10^4}{(1.15)^2} + \frac{16.8x10^4}{(1.15)^3}$$

$$= (16.8 \times 10^4)(PR_{15-3}) = 384,000.$$

The variance of the project NPV is found from equation (30), with the first right-hand member set to zero:

$$V[NPV] = \sum_{t=1}^{3} \left[\frac{\sigma_{Y_t}}{(1 + i)^t}\right]^2$$

$$= (4.36 \times 10^4)^2(PS_{15-2} + PS_{15-4} + PS_{15-6})$$

$$= (19.0 \times 10^8)(1.7602) = 33.6 \times 10^8.$$

The distribution of the Net Present Value, of course, is non-normal. To get the NPV distribution, consider again the relative frequencies (cell probabilities) and cell means in Table XII. If we consider that <u>three</u> such "tables" of cash flows will exist for the three periods (one for each period), and if we randomly sample from these tables and discount to t = 0, then the present value expectation of a cell mean is:

$$f_0(PV)_{Y_0} = \frac{f_p\,(Y_1)_g}{(1+i)} + \frac{f_p\,(Y_2)_g}{(1+i)^2} + \frac{f_p\,(Y_3)_g}{(1+i)^3}$$

$$= \left[\frac{1}{1+i} + \frac{1}{1+i}^{2} + \frac{1}{1+i}^{3}\right] f_p(Y_t)_g$$

$$= (PR_{i-3})(Y_t)_g\, f_p\ .$$

Since numerically $(PV)_{Y_0} = (PR_{i-3})(Y_t)_g$, then by analogy $f_0 = f_p$. Thus, by analogy all other relative frequencies of cell mean present values will be identical to their undiscounted counterparts, and hence the <u>form</u> of the NPV discrete probability function is identical to the cash flow function in Table XII. This permits the calculation of the NPV probability function, and from it, the NPV cumulative distribution. The cumulative NPV distribution is given in Table XIII, which permits probability statements about the net present value of the project.

TABLE XIII

NET PRESENT VALUE CUMULATIVE DISTRIBUTION

$f_0 = p(Y_0' \leq Y_0)$	$Y_0 = PV(Y_t) = (Y_t)(PR_{15-3})$
0.9938	2160. $\times 10^3$
0.990	2140.
0.900	733.
0.800	503.
0.700	400.
0.600	326.
0.500	277.
0.400	234.
0.300	194.5
0.200	159.0
0.100	119.7
0.050	94.9
0.001	39.7

FOOTNOTES

[1]In the numerical solutions that follow, the notation PS_{i-t} indicates the single-sum present worth factor, from t years hence at i percent per annum compounded annually. Similarly, the notation PR_{i-t} indicates the uniform-series end-of-period present worth factor.

[2]See Bierman et al (6), pp. 385-387. In the price equation (A-2), our sign differs from Bierman's, which is the result of choosing e as a positive number rather than as a negative one, as Bierman does.

[3]This differs from Horowitz' assumption somewhat.

APPENDIX B

DERIVATION OF NECESSARY & SUFFICIENT CONDITIONS FOR A POSITIVE MARGINAL UTILITY FUNCTION

The discriminant of the reduced cubic polynomial

$$y^3 + py + q = 0 \tag{104}$$

is

$$\Delta_3 = 4p^3 + 27q^2 \tag{B-1}$$

where

$$p = \frac{B}{2D} - \frac{3C^2}{16D^2} \tag{B-2}$$

$$q = -\frac{A}{4D} + \frac{BC}{8D^2} - \frac{C^3}{32D^3} . \tag{B-3}$$

Substituting the literal values of p^3 and q^2 in (B-1), the discriminant becomes

$$\Delta_3 = \frac{27AC^3 - 9B^2C^2}{64D^4} - \frac{27ABC - 8B^3}{16D^3} + \frac{27A^2}{16D^2} . \tag{B-4}$$

If $\Delta_3 > 0$, as it is required to be for the reduced cubic to have one real root and two imaginary roots, then the right-hand side of (B-4) must be greater than zero. Setting (B-4) > 0, and multiplying by $64D^4$, we obtain

$$27AC^3 - 9\,B^2C^2 - 4(27\,ABC - 8\,B^3)D + 4(27)\,A^2D^2 > 0. \tag{B-5}$$

Now, the left-hand side of (B-5) is a quadratic polynomial in D, the roots of which are:

$$R_1 = \frac{(27ABC - 8B^3) + \sqrt{(27ABC - 8B^3)^2 - 27A^2(27AC^3 - 9B^2C^2)}}{2(27)\ A^2}; \quad \text{(B-6)}$$

$$R_2 = \frac{(27ABC - 8B^3) - \sqrt{(27ABC - 8B^3)^2 - 27A^2(27AC^3 - 9B^2C^2)}}{2(27)\ A^2}, \quad \text{(B-7)}$$

By the factor theorem, a polynomial can be expressed in terms of its roots. Thus, inequality (B-5) can be expressed as

$$(D - R_1)(D - R_2) > 0; \quad \text{(B-8)}$$

where R_1 and R_2 are given by (B-6) and (B-7). Inequality (B-8) is true when both R_1 and R_2 are real numbers and when either:

(a) $D - R_1 > 0$ **AND** $D - R_2 > 0$

OR

(b) $D - R_1 < 0$ **AND** $D - R_2 < 0$.

For case (a), $D - R_1 > 0$ when $D > R_1$. Note, however, that if $D > R_1$ then $D > R_2$ also, since $R_1 > R_2$ from (B-6) and (B-7). Thus, case (a) is satisfied merely if $D > R_1$. For case (b), $D - R_2 < 0$ if $D < R_2$. But note, that if $D < R_2$, then $D < R_1$ also, since $R_2 < R_1$. Thus, the second case (b) is satisfied if $D < R_2$. Hence, it can be said that $(D - R_1)(D - R_2) > 0$, and thus that the reduced cubic (104) will have only one real root if

(1) R_1 and R_2 are real numbers; and

(2) $D > R_1$ <u>or</u> $D < R_2$.

Taking these conditions in order, it can be shown that R_1 and R_2 are real numbers if the discriminant of the quadratic root

$$\Delta_2 = (27ABC - 8B^3)^2 - 27A^2(27AC^3 - 9B^2C^2) \qquad \text{(B-9)}$$

is zero or positive (assuming, of course, that the coefficients A, B, C and D a real numbers). The condition for $\Delta_2 \geq 0$ can be obtained by setting (B-9) ≥ 0, expanding and collecting terms. Thus:

$$\Delta_2 = (27)^2A^2B^2C^2 - 2(8)(27)AB^4C + 64B^6 - (27)^2A^3C^3 + (9)(27)A^2B^2C^2 \geq 0$$

or, restating numerical coefficients as powers of 2 and 3:

$$2^6B^6 - 2^43^3AB^4C + 2^23^5A^2B^2C^2 - (3^3)^2A^3C^3 \geq 0 \; ; \qquad \text{(B-10)}$$

the left-hand side of which can be factored (into a perfect cube root) by $4B^2 - 9AC$. Thus, (B-10) becomes

$$\Delta_2 = (4B^2 - 9AC)^3 \geq 0$$

or, extracting and simplifying,

$$B^2 \geq \frac{9}{4} AC \; . \qquad \text{(B-11)}$$

To summarize, if $\Delta_2 \geq 0$, then R_1 and R_2 are real roots; this condition is met by requiring $B^2 \geq 9AC/4$. If R_1 and R_2 are real roots and if $D > R_1$ or $D < R_2$, then $\Delta_3 = (D - R_1)(D - R_2) > 0$, which is the condition for the reduced cubic polynomial (104) to have a single real root. It follows, then, that if the reduced cubic has a single real root, y_1, then the marginal utility function (equation (102)) will also have a single real root, $X^* = y_1 + C/4D$. This final condition is met by requiring (1) that $B^2 \geq 9AC/4$, and (2) that $D > R_1$ or $D < R_2$.

APPENDIX C

VALIDATION OF QUARTIC UTILITY FUNCTION

FOR R. F. MELLON

The following data are derived from the coefficients of the regression equation of R. F. Mellon's utility function (Chapter V):

$$A = 48.74$$

$$B = 17.967$$

$$C = 2.7209$$

$$D = 0.1234$$

The following requirments must be met, with the above values, for equation (72) to be a valid utility function:

(a) $B^2 \geq 9AC/4$ (from equation (107));

(b) $D > R_1$ <u>or</u> $D < R_2$ (from inequality (108));

where $$R_1, R_2 = \frac{27ABC \pm \sqrt{(27ABC - 8B^3)^2 - 27A^2(27AC^3 - 9B^2C^2)}}{54\,A^2};$$

(c) $BD > 0$ (inequality (109));

(d) $C^2 \geq 8BD/3$ (inequality (110)).

Quite obviously, $BD = (17.967)(0.1234) > 0$; and condition (c) is already met. For condition (a):

$$B^2 = (17.967)^2 = 322.813089$$

and $$9AC/4 = 9(48.74)(2.7209)/4 = 298.3874985;$$

hence, $B^2 \geq 9AC/4$. For condition (d):

$$C^2 = (2.7209)^2 = 7.40329681$$

$$8BD/3 = 8(17.967)(0.1234)/3 = 5.9123408;$$

hence, $C^2 \geq 8BD/3$, and condition (d) is met. Now, for condition (b), let

$$X = 27ABC = 27(48.74)(17.967)(2.7209) = 64333.53823$$

$$Y = (27ABC - 8B^3)^2 = [64{,}333.53823 - 8(17.967)^3]^2$$

$$= 3.216167374 \times 10^8$$

$$Z = 27A^2(27AC^3 - 9B^2C^2) = 27(48.74)^2[27(48.74)(2.7209)^3 - 9(17.967)^2(2.7209)^2]$$

$$= 3.206840947 \times 10^8$$

$$M = 54A^2 = 54(48.74)^2 = 12.82817304 \times 10^4 .$$

Then:

$$R_2 = \frac{X - \sqrt{Y - Z}}{M} = 0.1322709145;$$

hence, $(D = 0.1234) < (R_2 = 0.13227)$, and condition (b) is met. Equation (72), therefore, is a valid quartic utility function, the four requirements having been met.

Now, the upper limit of the range of applicability of this utility function is found by solving for the single root of the marginal utility function. Let

$$p = \frac{B}{2D} - \frac{3C^2}{16D^2} = -18.35844034$$

$$q = -\frac{A}{4D} + \frac{BC}{8D^2} - \frac{C^3}{32D^3} = -32.44319461 .$$

Then $4p^3 = -24{,}749.55182$ and $27q^2 = +28{,}419.14365$.

The discriminant of the reduced cubic is

$$\Delta_3 = 4p^3 + 27q^2 = +3669.591830 > 0 ,$$

which implies a single, positive, real root.

Now, to find the root, define

$$\alpha = -\frac{q}{2} + \sqrt{\frac{\Delta_3}{108}} = +22.05063686$$

$$\beta = -\frac{q}{2} - \sqrt{\frac{\Delta_3}{108}} = +10.39255775$$

$$\sqrt[3]{\alpha} = (22.05063686)^{1/3} = 2.804187478$$

$$\sqrt[3]{\beta} = (10.39255775)^{1/3} = 2.182264974 .$$

Then the root, y_1, of the reduced cubic is

$$y_1 + \sqrt[3]{\alpha} + \sqrt[3]{\beta} = 4.986452452$$

and the real root of the marginal utility function, X*, is

$$X^* = y_1 + \frac{C}{4D} = 4.9865 + \frac{2.7209}{4(0.1234)} = +10.4988 \;;$$

which is the real root of the cubic <u>marginal</u> utility function. Therefore, since the marginal utility becomes zero at $X^* = 10.4988$, then this value is the upper limit of validity for the quartic utility function itself.

APPENDIX D

STANDARDIZED DEVIATES FOR PEARSON DISTRIBUTIONS

Appendix D consists of Table XIV, on the following pages. Table XIV is an abridgement of "Table A" in Reference (36), to which the reader is referred for the complete table of Pearson standardized deviates.

Table XIV assumes that all values of $\sqrt{\theta_1}$ (= $\sqrt{\beta_1}$ in the Table) are <u>positive</u>; that is, the tabular values are for right-skewed distributions. If deviates are required for a left-skewed distribution, in which $\sqrt{\theta_1}$ is <u>negative</u>, then the column headings become $(1 - \alpha)$ instead of α in percentage points. For example, if $\sqrt{\theta_1} = -1.2$, then for $\alpha = 0.05$ one would look in the $\sqrt{\beta_1} = 1.2$ table under the column headed $\alpha = 95.0\%$, to find the negative standardized deviate.

For most purposes, linear interpolation in the Table is satisfactory. However, under certain instances, a second difference interpolation may be required; for these conditions, see Reference (36).

TABLE XIV

STANDARDIZED DEVIATES FOR PEARSON DISTRIBUTIONS

α, Percent →	0·0	0·25	0·5	1·0	2·5	5·0	10·0	25·0	50·0
β_2					$\sqrt{\beta_1} = 1{\cdot}2$				
3·6	0·9312	0·9312	0·9311	0·9310	0·9303	0·9272	0·9140	0·8104	0·3786
·8	1·0000	·9999	·9997	·9991	·9960	·9873	·9593	·8077	·3356
4·0	1·0727	1·0721	1·0713	1·0692	1·0607	1·0424	0·9956	0·7994	0·3014
·2	1·1504	1·1480	1·1455	1·1400	1·1226	1·0913	1·0236	·7886	·2739
·4	1·2340	1·2274	1·2213	1·2103	1·1801	1·1336	1·0448	·7768	·2515
·6	1·3275	1·3092	1·2975	1·2784	1·2320	1·1696	1·0606	·7650	·2330
·8	1·4320	1·3921	1·3726	1·3433	1·2797	1·2000	1·0723	·7537	·2174
5·0	1·5520	1·4748	1·4453	1·4040	1·3215	1·2256	1·0809	0·7431	0·2043
·2	1·6982	1·5559	1·5147	1·4603	1·3586	1·2471	1·0872	·7332	·1930
·4	1·8832	1·6343	1·5802	1·5120	1·3913	1·2653	1·0917	·7242	·1833
·6	2·1430	1·7093	1·6415	1·5594	1·4201	1·2807	1·0949	·7158	·1749
·8	2·6198	1·7805	1·6986	1·6026	1·4456	1·2939	1·0972	·7081	·1674
6·0	∞	1·848	1·752	1·642	1·468	1·305	1·099	0·701	0·161
·2	—	1·911	1·801	1·678	1·488	1·315	1·100	·695	·155
·4	—	1·970	1·846	1·711	1·506	1·323	1·100	·689	·150
·6	—	2·025	1·888	1·741	1·522	1·330	1·100	·683	·145
·8	—	2·077	1·927	1·768	1·537	1·337	1·100	·678	·141
7·0	—	2·125	1·963	1·793	1·550	1·343	1·100	0·673	0·137
·2	—	2·170	1·996	1·816	1·562	1·348	1·100	·669	·133
·4	—	2·212	2·027	1·838	1·573	1·352	1·099	·665	·130
·6	—	2·252	2·056	1·858	1·583	1·356	1·099	·661	·127
·8	—	2·289	2·083	1·876	1·592	1·359	1·098	·658	·124
8·0	—	2·324	2·108	1·893	1·600	1·363	1·098	0·654	0·122
·2	—	2·356	2·132	1·909	1·608	1·365	1·097	·651	·119
·4	—	2·387	2·154	1·923	1·615	1·368	1·096	·648	·117
·6	—	2·416	2·175	1·937	1·621	1·370	1·096	·645	·115
·8	—	2·444	2·194	1·950	1·627	1·373	1·095	·643	·113
9·0	—	2·470	2·212	1·962	1·633	1·375	1·094	0·640	0·112
					$\sqrt{\beta_1} = 1{\cdot}3$				
4·0	0·9173	0·9172	0·9172	0·9171	0·9162	0·9128	0·8987	0·7947	0·3788
·2	·9795	·9794	·9792	·9786	·9754	·9668	·9395	·7934	·3411
·4	1·0449	1·0444	1·0436	1·0416	1·0338	1·0168	·9729	·7875	·3104
·6	1·1142	1·1123	1·1101	1·1054	1·0899	1·0617	·9996	·7792	·2850
·8	1·1887	1·1830	1·1781	1·1688	1·1427	1·1013	1·0205	·7697	·2639
5·0	1·2697	1·2560	1·2466	1·2307	1·1914	1·1357	1·0368	0·7600	0·2462
·2	1·3595	1·3302	1·3145	1·2902	1·2357	1·1653	1·0493	·7504	·2311
·4	1·4610	1·4046	1·3809	1·3466	1·2758	1·1907	1·0589	·7412	·2181
·6	1·5791	1·4783	1·4449	1·3994	1·3117	1·2125	1·0663	·7326	·2069
·8	1·7222	1·5502	1·5059	1·4486	1·3438	1·2312	1·0720	·7244	·1971
6·0	1·9070	1·6197	1·5637	1·4941	1·3725	1·2473	1·0763	0·7169	0·1886
·2	2·1764	1·6863	1·6180	1·5360	1·3981	1·2612	1·0795	·7099	·1809
·4	2·7530	1·7497	1·6689	1·5746	1·4211	1·2733	1·0820	·7034	·1742
·6	∞	1·810	1·716	1·610	1·442	1·284	1·084	·697	·168
·8	—	1·867	1·761	1·643	1·460	1·293	1·085	·692	·163
7·0	—	1·920	1·802	1·673	1·477	1·301	1·086	0·687	0·158
·2	—	1·970	1·840	1·700	1·492	1·308	1·087	·682	·153
·4	—	2·018	1·876	1·726	1·506	1·315	1·087	·677	·149
·6	—	2·062	1·910	1·750	1·519	1·320	1·087	·673	·145
·8	—	2·104	1·941	1·772	1·530	1·326	1·087	·669	·142
8·0	—	2·144	1·970	1·792	1·541	1·330	1·087	0·665	0·139
·2	—	2·181	1·998	1·811	1·551	1·334	1·087	·662	·136
·4	—	2·216	2·023	1·829	1·559	1·338	1·087	·659	·133
·6	—	2·249	2·047	1·845	1·568	1·341	1·087	·656	·131
·8	—	2·280	2·070	1·860	1·575	1·344	1·086	·653	·128
9·0	—	2·310	2·091	1·875	1·583	1·347	1·086	0·650	0·126
·2	—	2·337	2·112	1·888	1·589	1·350	1·085	·647	·124
·4	—	2·364	2·131	1·901	1·595	1·352	1·085	·645	·122

TABLE XIV (Continued)

α, Percent → / β_2	75.0	90.0	95.0	97.5	99.0	99.5	99.75	100.0
				$\sqrt{\beta_1} = 1{\cdot}2$				
3·6	0·5366	1·5755	2·1465	2·5572	2·9178	3·0944	3·2152	3·4696
·8	·5355	1·5219	2·1033	2·5580	3·0027	3·2478	3·4337	4·0000
4·0	0·5341	1·4793	2·0638	2·5482	3·0589	3·3647	3·6142	4·6934
·2	·5324	1·4446	2·0287	2·5334	3·0956	3·4531	3·7609	5·6504
·4	·5307	1·4159	1·9979	2·5167	3·1192	3·5201	3·8796	7·0767
·6	·5290	1·3917	1·9706	2·4996	3·1341	3·5713	3·9759	9·4703
·8	·5273	1·3710	1·9466	2·4829	3·1429	3·6106	4·0545	14·4320
5·0	0·5256	1·3531	1·9253	2·4669	3·1477	3·6412	4·1191	31·5520
·2	·5240	1·3375	1·9063	2·4519	3·1497	3·6651	4·1727	∞
·4	·5225	1·3237	1·8892	2·4378	3·1497	3·6839	4·2174	
·6	·5211	1·3115	1·8739	2·4246	3·1484	3·6988	4·2552	
·8	·5197	1·3005	1·8600	2·4124	3·1461	3·7106	4·2872	
6·0	0·518	1·291	1·847	2·401	3·143	3·720	4·314	
·2	·517	1·282	1·836	2·390	3·140	3·727	4·337	
·4	·516	1·274	1·825	2·380	3·136	3·733	4·358	
·6	·515	1·266	1·816	2·371	3·133	3·738	4·375	
·8	·514	1·259	1·807	2·362	3·129	3·742	4·391	
7·0	0·513	1·253	1·798	2·354	3·125	3·745	4·404	
·2	·512	1·247	1·791	2·347	3·121	3·748	4·416	
·4	·511	1·242	1·783	2·340	3·117	3·749	4·427	
·6	·510	1·237	1·777	2·333	3·114	3·751	4·436	
·8	·510	1·232	1·771	2·327	3·110	3·752	4·445	
8·0	0·509	1·228	1·765	2·321	3·107	3·753	4·452	
·2	·508	1·224	1·759	2·315	3·103	3·753	4·459	
·4	·508	1·220	1·754	2·310	3·100	3·753	4·465	
·6	·507	1·217	1·749	2·305	3·097	3·754	4·471	
·8	·506	1·213	1·745	2·300	3·094	3·754	4·476	
9·0	0·506	1·210	1·740	2·295	3·091	3·753	4·480	
				$\sqrt{\beta_1} = 1{\cdot}3$				
4·0	0·5024	1·5457	2·1550	2·6192	3·0555	3·2851	3·4520	3·8814
·2	·5057	1·4996	2·1111	2·6082	3·1171	3·4112	3·6435	4·4851
·4	·5077	1·4625	2·0722	2·5915	3·1568	3·5065	3·7999	5·2828
·6	·5087	1·4318	2·0380	2·5726	3·1820	3·5785	3·9267	6·3976
·8	·5092	1·4062	2·0080	2·5532	3·1973	3·6331	4·0294	8·0866
5·0	0·5093	1·3843	1·9816	2·5343	3·2060	3·6748	4·1130	10·9893
·2	·5091	1·3654	1·9583	2·5163	3·2102	3·7067	4·1815	17·2699
·4	·5088	1·3490	1·9375	2·4994	3·2114	3·7315	4·2380	41·9054
·6	·5083	1·3345	1·9189	2·4835	3·2106	3·7507	4·2850	∞
·8	·5078	1·3217	1·9022	2·4688	3·2083	3·7657	4·3244	
6·0	0·5072	1·3102	1·8871	2·4551	3·2050	3·7774	4·3576	
·2	·5065	1·2999	1·8733	2·4423	3·2011	3·7866	4·3858	
·4	·5059	1·2906	1·8608	2·4304	3·1969	3·7938	4·4100	
·6	·505	1·282	1·849	2·419	3·192	3·799	4·430	
·8	·505	1·274	1·839	2·409	3·188	3·803	4·448	
7·0	0·504	1·267	1·829	2·399	3·183	3·807	4·464	
·2	·503	1·261	1·820	2·390	3·178	3·809	4·477	
·4	·503	1·255	1·812	2·382	3·173	3·811	4·489	
·6	·502	1·249	1·804	2·374	3·169	3·812	4·499	
·8	·502	1·244	1·797	2·367	3·164	3·813	4·509	
8·0	0·501	1·239	1·790	2·359	3·160	3·814	4·517	
·2	·501	1·234	1·784	2·353	3·156	3·814	4·524	
·4	·500	1·230	1·778	2·347	3·152	3·814	4·531	
·6	·500	1·226	1·772	2·341	3·148	3·813	4·536	
·8	·499	1·222	1·767	2·335	3·144	3·813	4·542	
9·0	0·499	1·219	1·762	2·330	3·140	3·813	4·546	
·2	·498	1·216	1·758	2·325	3·137	3·812	4·551	
·4	·498	1·213	1·753	2·320	3·133	3·811	4·554	

VITA

Lynn Edward Bussey

Candidate for the Degree of

Doctor of Philosophy

Thesis: CAPITAL BUDGETING PROJECT ANALYSIS AND SELECTION WITH COMPLEX UTILITY FUNCTIONS

Major Field: Engineering (Industrial and Management)

Biographical:

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