NONLINEAR DYNAMIC RESPONSE OF THIN RECTANGULAR PLATES SUBJECTED TO PULSE-TYPE LOADS

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LIST OF SYMBOLS

A	Coefficient Matrix of Equation (2-45)
a_{ij}	Elements of A-Matrix
a, b	Plate Dimensions
B_{mn}	Defined by Equation (3-10)
B_i, BB_i, C_i	Defined by Equations (2-51), (2-52), and (2-47)
$C1$, $C2$, and $C3$	Defined by Equations (2-48), (2-49), and (2-50)
C_x, C_y	Defined by Equations (2-33) and (2-32)
c_i	Elements of Equation (2-47)
D	Flexural Rigidity of the Plate
DC	Vector Defined on Page 32
E	Young's Modulus
F	Stress Function
h	Plate Thickness
M, N	Number of Grid-Points in x- and y-Direction
m_i	Defined by Equation (2-54)
P_j, p, q	Load on Plate
t	Time
u, v	Inplane Displacements of the Plate
w	Transverse Displacement of the Plate
x, y	Rectangular Coordinates
γ	Specific Weight
ν	Poisson's Ratio
ρ	Mass Density

σ_x, σ_y, τ_{xy}	Membrane Stresses
ω_{mn}	Defined by Equation (3-11)
Δx	Grid-Spacing
Δt	Time Increment
∇^4	Biharmonic Operator, $\nabla^4 \equiv \partial^4/\partial x^4 + 2\partial^4/\partial x^2 \partial y^2 + \partial^4/\partial y^4$
$w_{,x}$, $w_{,t}$	Partial Derivatives, $\partial w/\partial x$, $\partial w/\partial t$, ...
$w_{i,j}$, $F_{i,j}$, ...	Subscript i Denotes Grid-Point Location and j Denotes Time-Step

CHAPTER I

INTRODUCTION

Origin and Objective of this Study

During recent studies at Oklahoma State University on structural response from sonic booms, the necessity of further study of nonlinear plate vibrations became apparent. Specifically, the need arose for a lumped parameter model of a large plate-glass window valid for large dynamic deflections. Such a model was found in the literature, but the accuracy and the applicability to the specific problem at hand could not be determined without further study.

Large deflections of plates are of basic interest in structural dynamics. A research study in this area should have a broad application. A variety of important problems in modern design cannot be adequately analyzed on the basis of classical linear theory. Nonlinear plate theories have been developed to describe the response of physical systems, but an exact solution of the governing differential equations is not available. The application of the nonlinear theory is confined to only very special cases.

The only solutions known are fundamental mode approximations which result in a lumped parameter model of the problem. A more general method of solving the differential equations is needed, as well as evaluation and perhaps improvement of the lumped parameter model.

With the availability of larger, faster digital computers, a numerical analysis of the problem is now feasible and practical. It appears at this time that an approximate numerical method may be the best possible approach for this problem. The finite-difference method has very broad applications and is suitable here. This method has been used successfully for the analysis of linear plate vibrations and a great variety of other problems. A proper application of the method to the problem at hand may be expected to provide solutions sufficiently accurate to evaluate the lumped-parameter model, as well as to permit solutions for boundary conditions that would otherwise defy analysis.

The objective of this study is to develop a numerical method to determine the large amplitude dynamic response of a thin elastic plate subjected to a pulse-type load, and to investigate the stability, convergence, accuracy, and application of the method.

Historical Background

The following brief history of the early development of plate theory was taken from references (6) and (13). The governing differential equation for the static deflection of plates by linear theory, ($\nabla^4 w = q/D$), was obtained by Lagrange in 1811. The first to consider equilibrium of plates with large deflections appears to be Clebsch in 1862. Kirchoff, in 1883, was apparently the first to analyze motions of plates with large deflections. A set of membrane plate equations was obtained by A. Föppl in 1907. Similar equations for static deflection by nonlinear theory were obtained by von Kármán in 1910. A numerical solution of the membrane equations by finite-differences was discussed by H. Hencky in 1921.

There have been two theories developed that probably represent the most significant efforts for the formulation of a general plate theory. In 1955, Herrmann (6) used a variational technique to derive a large-deflection plate theory of motion, starting with the general equations of the three-dimensional nonlinear theory of elasticity. The theory is valid for an isotropic material obeying Hooke's law, and for the case of small elongations and shears with moderately large rotations. In 1960, Tadjbakhsh and Saibel (12) considered the problem from an equilibrium point of view to develop a more general theory. This system of differential equations contains the equations derived by Herrmann as a special case. The theories have not been investigated fully or applied to practical problems because an exact solution of the differential equations is not known. However, it is significant to note that each set of equations may be reduced to the well-known static von Kármán equations, as presented by Timoshenko and Woinowsky-Krieger (13), by certain simplifying assumptions.

Several authors have used a simplified plate theory of motion to investigate the influence of large amplitudes on the vibrations of plates. The theory corresponds to a first-order approximation of the theory developed by Herrmann (6), which may be identified as the dynamic von Kármán theory. In 1956, Chu and Herrmann (3) studied the problem of free vibrations of rectangular plates. They solved the differential equations, with boundary conditions for hinged immovable edges, by a perturbation method. They also obtained identical results by the principle of conservation of energy. They were able to show the influence of large amplitudes on the period of vibration, the maximum membrane stress, and the maximum total stress. In 1961, Yamaki (15) extended the work of Chu and Herrmann by considering free and forced vibrations for both rectangular and circular plates with various boundary conditions. He used a different approximate method to solve the differential equations, but the results compare favorably with those previously obtained. In both of the analysis referred to above the approximate methods were essentially a lumped-parameter representation of the problem. For each set of boundary conditions this lumped-parameter model took the form of a mass on a cubic hardening spring. In 1968, Bauer (1) used the models previously developed and presented a method for solving the problem for various types of pulse loads. There are a limited number of related articles in the literature, but it appears that the lumped-parameter representation is the most accurate solution available at this time.

Whitehouse (14) and Seshadri (11), in the study of structural response to sonic booms, used a lumped-parameter model of glass windows based on linear plate theory. They demonstrated the validity and application of the model in analyzing systems with mechanical and acoustical coupling, and Seshadri established the necessity of using a nonlinear model. Bowles and Sugarman (2) observed, from experimental investigations of glass panels under uniform pressure, a definite flattening of the panel at the center as the deformation increased into the nonlinear range. Freynik (5) concluded that the maximum principal tensile stress in a simply supported square window migrates along a diagonal away from the center of the panel as the load increases. This was attributed to the effect of the membrane stress, but it has not been fully explained. The nonlinear models that are available do not consider this flattening at the center. Although the models may adequately represent some physical systems, their accuracy and range of validity are not known. This is one example of the need for a more accurate analysis of large amplitude vibrations of plates.

The finite-difference method of solving partial differential equations is adequately described in most textbooks on numerical methods. Unlike ordinary differential equations, partial differential equations cannot be solved by general type computer programs. The development of specific programs for each set of equations is required. The method has been used successfully on many varied problems and numerous articles are available that provide useful suggestions. A numerical solution to the problem at hand could not be found.

Statement of Problem

The basic problem is to determine the dynamic response of an elastic plate. Since several mathematical theories are available that describe the behavior of plates in terms of their physical properties, the first consideration is to determine which theory is applicable to the problem at hand. After a certain theory has been selected the problem takes the form of developing a numerical solution for a system of differential equations with their associated initial and boundary conditions. The problem is simplified by assumptions and limitations that restrict the solution to a certain class of plates.

Plate Theory

A. Differential Equations

The governing differential equations selected as the most suitable for this study are:

$$\nabla^4 F = E\left[w_{,xy}^2 - w_{,xx}\, w_{,yy}\right], \tag{1-1}$$

$$D\,\nabla^4 w + \rho h\, w_{,tt} = p(t) + h\left[F_{,yy}\, w_{,xx} + F_{,xx}\, w_{,yy} - 2F_{,xy}\, w_{,xy}\right]. \tag{1-2}$$

The subscripts following a comma stand for differentiation. The x and y are Cartesian coordinates, t represents time, w is the transverse deflection, h the plate thickness, ρ the mass density, p the load, and $D \equiv E\,h^3/12\,(1 - \nu^2)$ denotes the bending stiffness, where E is Young's modulus of elasticity and ν is Poisson's ratio. The operator $\nabla^4 \equiv \partial^4/\partial x^4 + 2\partial^4/\partial x^2\,\partial y^2 + \partial^4/\partial y^4$. F is Airy's stress function defined by $F_{,xx} = \sigma_y$, $F_{,yy} = \sigma_x$, and $-F_{,xy} = \tau_{xy}$, where σ_x, σ_y, and τ_{xy} are membrane stresses.

These equations were used by Yamaki (15), Bauer (1), and others in recent investigations. They have been described as the dynamic analogue of the von Kármán large-deflection plate theory, which is valid for moderately large amplitudes. The equations may be derived simply by adding the inertia term to the static von Kármán equations presented by Timoshenko and Woinowsky-Krieger (13). They correspond to the first order approximation to the plate equations derived by Hermann (6). The equations derived by Tadjbakhsh and Saibel (12) may be reduced to these equations by discarding certain terms. This theory should adequately describe the physical system for deflections within a certain limit. Sufficient studies have not been made to determine this limit. The limit for the corresponding linear theory is generally considered to be $w/h < 1/2$.

B. Assumptions

The derivation of the plate equations will not be presented here, but a brief discussion of the basic assumptions and limitations of the theory is necessary.

The theory may be developed in various ways, and consequently the assumptions are made or developed in various ways. The relationships between stresses, strains, and displacements of nonlinear theory of elasticity are valid, with the simplifications of plate theory. The following list provides a description of the physical system for which the theory is valid:

1. The material is isotropic and obeys Hooke's law.

2. The planform dimensions are much greater than the thickness, $(a, b > 10 h)$.

3. Normals to the middle plane of the plate remain normal to the middle surface after deformation.

4. The normal stresses in the direction transverse to the plate can be disregarded.

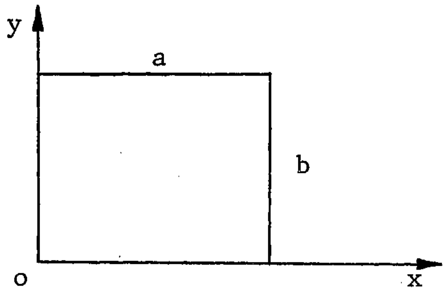
5. Effects of both longitudinal and rotary inertia are negligible.

6. Nonlinearities are introduced only geometrically.

7. The maximum deflection is moderate, (limit is unknown).

<u>Boundary Conditions</u>

The dimensions and coordinates of the plate are as shown:



Four different sets of boundary conditions are considered. The conditions for the displacement w are designated displacement conditions, and those for the stress function F as stress conditions. The four cases are:

Displacement Conditions	I.	All edges simply supported.
	II.	All edges clamped.
Stress Conditions	(a)	All edges free of membrane stress.
	(b)	All edges immovably constrained.

This nomenclature will hereafter identify the boundary conditions, such as case I (a) for simply supported, stress-free edges.

The boundary conditions may be expressed as follows:

	$\underline{x = 0, a}$	$\underline{y = 0, b}$
I:	$w = w,_{xx} + \nu w,_{yy} = 0$	$w = w,_{yy} + \nu w,_{xx} = 0$
II:	$w = w,_{x} = 0$	$w = w,_{y} = 0$
(a):	$F,_{yy} = F,_{xy} = 0$	$F,_{xx} = F,_{xy} = 0$
(b):	$u = F,_{xy} = 0$	$v = F,_{xy} = 0$,

(1-3)

where u and v are midplane displacements in the x and y directions, respectively. They may be expressed as

$$u = \int_o^x \left\{ \frac{1}{E} \left[F,_{yy} - \nu F,_{xx} \right] - \tfrac{1}{2} w,_x^2 \right\} dx \ , \qquad (1\text{-}4)$$

and

$$v = \int_o^y \left\{ \frac{1}{E} \left[F,_{xx} - \nu F,_{yy} \right] - \tfrac{1}{2} w,_y^2 \right\} dy \ . \qquad (1\text{-}5)$$

Symmetry Conditions

The only type of loading considered in this study is that of uniform pressure over the surface of the plate. No restrictions are placed on the variation of the load with time. Many practical problems fall into this category.

For the symmetrical boundary conditions and the uniform pressure loads, a physical analysis of the problem results in the following symmetry conditions:

$$\begin{array}{cc} \underline{x = a/2} & \underline{y = b/2} \\ w,_{x} = 0 & w,_{y} = 0 \\ F,_{x} = 0 & F,_{y} = 0 \\ u = 0 & v = 0 \end{array} \quad , \qquad (1\text{-}6)$$

which are valid for each set of boundary conditions.

The restriction of uniform pressure loads is not necessary for the numerical method, but it permits a much more efficient and accurate solution by considering only one quarter of the plate.

Initial Conditions

The initial conditions considered are

$$w = 0 \text{ and } w,_{t} = 0 \text{ at } t = 0 \text{ .} \qquad (1\text{-}7)$$

These conditions are the most common for this type of problem. However, the numerical solution may easily be adapted to other types of initial conditions.

CHAPTER II

NUMERICAL SOLUTION

Finite-Difference Equations

An approximate solution of Equations (1-1) and (1-2) may be obtained by replacing each derivative by its finite-difference approximation and solving the resulting algebraic equations. This method is explained adequately in most books on numerical analysis, (e.g. (4), (9), and (10)).

Only the final forms of the finite-difference equations are given. Each approximation used is a centered-difference formula with an error of $(\Delta x)^2$. The formula for each derivative can easily be determined by examination of the final equations. Only a square grid, ($\Delta x = \Delta y$), is considered. The subscript i denotes the grid-point location, with M grid-points in the x-direction, (see Figure 1). The subscript j denotes the time increment.

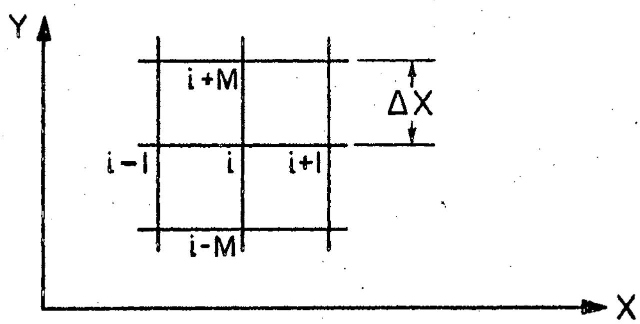


Figure 1. Numbering of Grid-Points

The finite-difference equations for Equations (1-1) and (1-2) are, respectively:

$$\begin{aligned}
&\Big\{F_{i-2} - 8\,F_{i-1} + 20\,F_{i} - 8\,F_{i+1} + F_{i+2} \\
&\quad + 2\,F_{i+M-1} - 8\,F_{i+M} + 2\,F_{i+M+1} + F_{i+2M} \\
&\quad + 2\,F_{i-M-1} - 8\,F_{i-M} + 2\,F_{i-M+1} + F_{i-2M}\Big\}_j \qquad (2\text{-}1) \\
&= E\Big\{\frac{1}{16}\Big[w_{i-M-1} - w_{i+M-1} - w_{i-M+1} + w_{i+M+1}\Big]^2 \\
&\quad - \Big[w_{i-1} - 2\,w_{i} + w_{i+1}\Big] \cdot \Big[w_{i-M} - 2\,w_{i} + w_{i+M}\Big]\Big\}_j
\end{aligned}$$

$$\begin{aligned}
&\frac{D}{(\Delta x)^4}\Big\{w_{i-2} - 8\,w_{i-1} + 20\,w_{i} - 8\,w_{i+1} + w_{i+2} \\
&\quad + 2\,w_{i+M-1} - 8\,w_{i+M} + 2\,w_{i+M+1} + w_{i+2M} \\
&\quad + 2\,w_{i-M-1} - 8\,w_{i-M} + 2\,w_{i-M+1} + w_{i-2M}\Big\}_j \\
&+ \frac{\rho h}{(\Delta t)^2}\Big\{w_{i,j-1} - 2\,w_{i,j} + w_{i,j+1}\Big\} = \Big\{p\,(t)\Big\}_{i,j} \qquad (2\text{-}2) \\
&+ \frac{h}{(\Delta x)^4}\Big\{\Big[F_{i-M} - 2\,F_{i} + F_{i+M}\Big] \cdot \Big[w_{i-1} - 2\,w_{i} + w_{i+1}\Big] \\
&\quad + \Big[F_{i-1} - 2\,F_{i} + F_{i+1}\Big] \cdot \Big[w_{i-M} - 2\,w_{i} + w_{i+M}\Big] \\
&\quad - \frac{1}{8}\Big[F_{i-M-1} - F_{i+M-1} - F_{i-M+1} + F_{i+M+1}\Big] \cdot \\
&\qquad \Big[w_{i-M-1} - w_{i+M-1} - w_{i-M+1} + w_{i+M+1}\Big]\Big\}_j
\end{aligned}$$

These equations are not the only finite-difference formulas which may be employed for this problem. Several alternatives may be developed by using higher-order approximations, implicit formulas, or predictor-corrector techniques. A comparison of different formulas is a study in itself. Other possibilities were considered, but Equations (2-1) and (2-2) appeared to be the most satisfactory for this study, primarily because of their simplicity. The results of this study may lead to improvements, (if necessary), in accuracy and efficiency.

The method applied to Equation (2-1) is basically the same as using the finite-difference method to determine the static displacement of a plate by linear theory. This is a common method that works very well, (e.g. see (9)). When Equation (2-1) is applied to each grid-point at any given time-step, a system of linear algebraic equations is developed. This system of equations can be solved for $F_{i,j}$ at each time-step, (assuming $w_{i,j}$ and sufficient boundary conditions are known).

If $F_{i,j}$, $w_{i,j}$, and $w_{i,j-1}$ are known, the only unknown term in Equation (2-2) is $w_{i,j+1}$. Assuming sufficient boundary and initial conditions are known, Equation (2-2) can be solved explicitly for $w_{i,j+1}$.

With these two equations a complete displacement-time, and stress function-time, history can be generated for each grid-point. From these a stress history can be determined through the use of the stress resultant-displacement equations.

Error Analysis

A major concern in using any finite-difference method is the error in the solution. A rigorous analysis was not attempted, but only a sufficient study to insure successful use of the method.

There are at least two types of errors associated with these equations. The truncation error, (inherent in the finite-difference approximations), and the round-off error, (due to using finite arithmetic in the calculations). The truncation error is of order $(\Delta x)^2$. This error, (and generally the total error), will decrease as

Δx is reduced, until some point where the round-off error becomes more significant than the truncation error. At this point the total error will increase with a decrease of Δx. This was investigated empirically for the complete solution.

The stress function will have an error resulting from the approximate displacements that will be used in the right-hand side of Equation (2-1). This error was not considered, except in the empirical study of the complete solution. The truncation and round-off error for Equation (2-1) was investigated empirically by replacing the right-hand side with a function for which there was an exact solution. This problem was then solved by the finite-difference method and compared with the exact solution. The results were not expected to be exactly the same as for Equation (2-1), if there were an exact solution, but they should be very similar. This checked the program and gave a good indication of the grid-size necessary for convergence to a specific accuracy.

Initial value problems are complicated by numerical stability requirements which result from the round-off error. This error at one time-step may propagate with increasing magnitude through the remainder of the calculation. The circumstances were investigated under which the error does not grow with time, but instead dies out and thus provides an acceptable solution.

Leech (8) successfully used von Neumann's method of stability analysis for the linear plate vibration problem. An identical procedure is applied here to Equation (2-2). This equation is the same as the formula used by Leech, except for the additional nonlinear terms. It is necessary to assume that the membrane stresses are known exactly. This can only be justified empirically.

A double subscript notation is used here to denote grid-points. k is used for the grid-point location in the x-direction and ℓ for the y-direction. Subscript j is used for the time-step.

It may be shown that the round-off error $\delta(x,y,t)$ must satisfy an equation similar to Equation (2-2).

$$\begin{aligned}
&\frac{D}{(\Delta x)^4}\Big\{\delta_{k-2,\ell} - 8\,\delta_{k-1,\ell} + 20\,\delta_{k,\ell} - 8\,\delta_{k+1,\ell} + w_{k+2,\ell} \\
&\qquad + 2\,\delta_{k-1,\ell+1} - 8\,\delta_{k,\ell+1} + 2\,\delta_{k+1,\ell+1} + \delta_{k,\ell+2} \\
&\qquad + 2\,\delta_{k-1,\ell-1} - 8\,\delta_{k,\ell-1} + 2\,\delta_{k+1,\ell-1} + \delta_{k,\ell-2}\Big\}_j \\
&+ \frac{\rho h}{(\Delta t)^2}\Big\{\delta_{j-1} - 2\,\delta_j + \delta_{j+1}\Big\}_{k,\ell} \\
&- \frac{h}{(\Delta x)^2}\Big\{F,_{yy}\left(\delta_{k-1,\ell} - 2\,\delta_{k,\ell} + \delta_{k+1,\ell}\right) \\
&\qquad + F,_{xx}\left(\delta_{k,\ell-1} - 2\,\delta_{k,\ell} + \delta_{k,\ell+1}\right) \\
&\qquad - \tfrac{1}{4}\,F,_{xy}\left(\delta_{k-1,\ell-1} - \delta_{k-1,\ell+1} - \delta_{k+1,\ell-1} + \delta_{k+1,\ell+1}\right)\Big\}_j = 0
\end{aligned} \tag{2-3}$$

Assume that the general term for the error may be expressed in the form

$$\delta(x,y,t) = e^{\alpha t}\, e^{i\beta x}\, e^{i\gamma y} , \tag{2-4}$$

or equivalently

$$\delta_{k,\ell,j} = e^{\alpha j \Delta t}\, e^{i\beta k \Delta x}\, e^{i\gamma \ell \Delta y} . \tag{2-5}$$

If one lets $\xi = e^{\alpha \Delta t}$, Equations (2-3) and (2-5) may be combined to give

$$\xi^2 - 2A\xi + 1 = 0 , \tag{2-6}$$

where

$$A = 1 - \frac{8\ D(\Delta t)^2}{\rho h\ (\Delta x)^4}\left[\sin^2\frac{\beta\Delta x}{2} + \sin^2\frac{\gamma\Delta y}{2}\right]^2$$

$$- \frac{(\Delta t)^2}{\rho(\Delta x)^2}\left[F,_{yy}\ (1 - \cos\beta\Delta x) + F,_{xx}\ (1 - \cos\gamma\Delta y) + F,_{xy}\ (\tfrac{1}{4} - \sin\beta\Delta x \sin\gamma\Delta y)\right] . \qquad (2\text{-}7)$$

It can be shown that the error will not grow with increasing time as long as the following necessary and sufficient condition is applied:

$$|\xi| \leq 1 \qquad (2\text{-}8)$$

In terms of A, this requirement becomes

$$-1 \leq A \leq +1 . \qquad (2\text{-}9)$$

If the maximum displacement is less than one-half the thickness, the membrane stresses may be neglected and the stability criterion is identical to the linear analysis by Leech.

$$\frac{(\Delta t)^2}{(\Delta x)^4} \leq \frac{\rho h}{16\ D} \qquad (2\text{-}10)$$

For displacements greater than one-half the thickness, the membrane stresses may have a significant influence on the stability of the method. Since the membrane stresses are always positive, the worst condition for stability will probably be when $\beta\Delta x = \gamma\Delta y = \pi$. Applying this condition to Equations (2-7) and (2-9), the limiting stability requirement becomes

$$-2 \leq -\frac{8\ D(\Delta t)^2}{\rho h\ (\Delta x)^4}\ (4) - \frac{(\Delta t)^2}{\rho(\Delta x)^2}\left(2\ F,_{yy} + 2\ F,_{xx} + \tfrac{1}{4}\ F,_{xy}\right) \leq 0 . \qquad (2\text{-}11)$$

The right-hand inequality will always be satisfied. The left-hand inequality imposes the following restriction on Δt:

$$(\Delta t)^2 \leq 2 \div \left[\frac{32\ D}{\rho h(\Delta x)^4} + \frac{1}{\rho(\Delta x)^2}\left(2\ F,_{yy} + 2\ F,_{xx} + \tfrac{1}{4}\ F,_{xy}\right)\right] \qquad (2\text{-}12)$$

By satisfying this condition, a stable solution should be assured. The magnitude of the membrane stresses must be known before this expression can be simplified.

The results of an empirical study of the total error, convergence, and stability are presented in Chapter III.

Grid Numbering System

Only one-quarter of the plate need be considered for the finite-difference solution because of the symmetry resulting from the uniform pressure loads and the symmetric boundary conditions. Figures 2 and 3 illustrate the numbering system for the two stress conditions. Both case I and II are used with each of the stress conditions. The time-grid may be considered the third dimension, and the space-grids illustrated are for any given time-step. The external, (fictitious), grid-points are not numbered but will be referred to in a general sense by the system of Figures 1 and 4.

The advantages of the system for stress condition (a) are obvious, but unfortunately the other system is necessary for stress condition (b). The reader is cautioned to be aware of the two systems. For stress condition (a)

$$\Delta x = \frac{a}{2 \cdot M} = \frac{b}{2 \cdot N} \tag{2-13}$$

For stress condition (b)

$$\Delta x = \frac{a}{2(M-1)} = \frac{b}{2(N-1)} \tag{2-14}$$

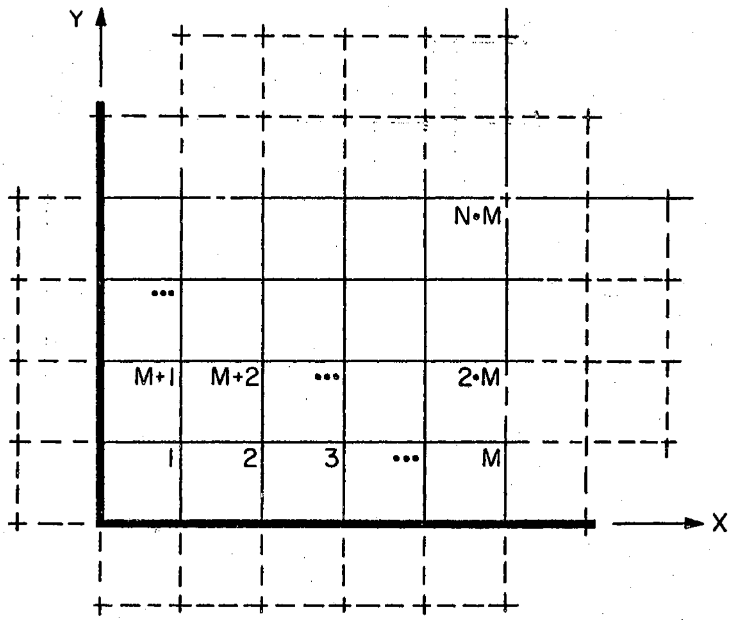


Figure 2. Grid Numbering System for Stress Condition (a)

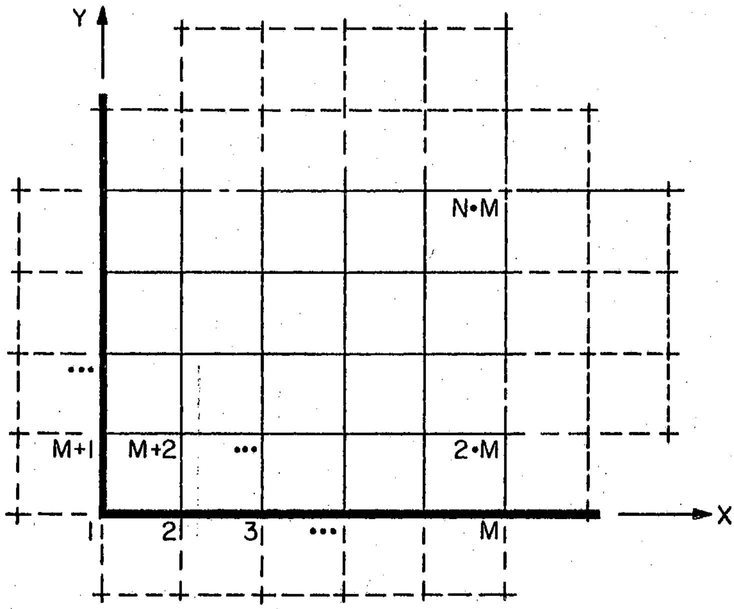


Figure 3. Grid Numbering System for Stress Condition (b)

Figure 4 illustrates the system of identifying the fictitious and boundary grid-points. The subscript k simply identifies the unnumbered points in relation to the numbered points, and no grid-point number should be associated with it.

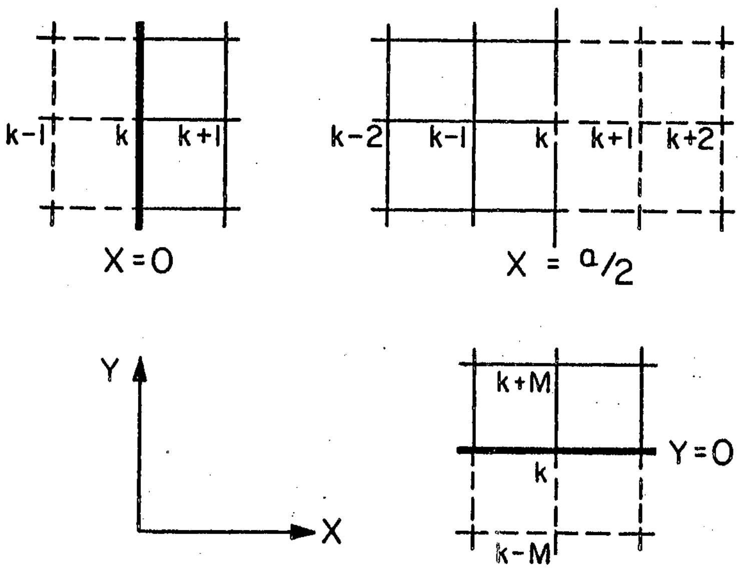


Figure 4. Identifying Fictitious Grid-Points

Boundary Conditions

In carrying out the solution of Equations (2-1) and (2-2), each equation is applied to every interior grid-point at every time-step. In order to do this some knowledge of the displacements and stress functions at the fictitious and boundary grid-points must be known. This knowledge comes from the boundary and symmetry conditions. The values at the fictitious points are identified as functions of the interior points. Basically, the problem is to develop a system of equations corresponding to Equation (2-1) with the same number of equations as unknown $F_{i,j}$'s. Equation (2-2) is solved explicitly for displacements at time-step (j+1).

The following finite-difference approximations are used in satisfying the boundary and symmetry conditions:

$$w,_x \approx \frac{1}{2(\Delta x)}\left(- w_{k-1} + w_{k+1}\right) + 0(\Delta x)^2 \qquad (2\text{-}15)$$

$$w,_y \approx \frac{1}{2(\Delta x)}\left(- w_{k-M} + w_{k+M}\right) + 0(\Delta x)^2 \qquad (2\text{-}16)$$

$$w,_{xx} \approx \frac{1}{(\Delta x)^2}\left(w_{k-1} - 2\, w_k + w_{k+1}\right) + 0(\Delta x)^2 \qquad (2\text{-}17)$$

$$w,_{yy} \approx \frac{1}{(\Delta x)^2}\left(w_{k-M} - 2\, w_k + w_{k+M}\right) + 0(\Delta x)^2 \qquad (2\text{-}18)$$

$$w,_{xy} \approx \frac{1}{4(\Delta x)^2}\left(w_{k-M-1} - w_{k+M-1} + w_{k+M+1} - w_{k-M+1}\right) + 0(\Delta x)^2 \qquad (2\text{-}19)$$

The same approximations are used for the stress function F. One additional formula that is used is the familiar Trapezoidal Rule.

$$\int_{f_1}^{f_N} f(x)\, dx \approx \frac{\Delta x}{2}\left(f_1 + 2f_2 + 2f_3 + \cdots\cdots + 2f_{N-1} + f_N\right) + 0(\Delta x)^2 \qquad (2\text{-}20)$$

The combination of these equations and the boundary conditions yield a relationship between the fictitious points and the numbered grid-points. The following examples for certain conditions illustrate the methods. The results for the remaining conditions should be obvious. The complete results are tabulated in Table I.

<u>$w,_x = 0$ at $x = 0$.</u>

From Equation (2-15), $\frac{1}{2(\Delta x)}\left(- w_{k-1} + w_{k+1}\right) \approx 0$,

or

$$w_{k-1} \approx w_{k+1} \text{ at } x = 0 . \qquad (2\text{-}20)$$

<u>$w,_y = 0$ at $y = 0$.</u>

From Equation (2-16), $\frac{1}{2(\Delta x)}\left(- w_{k-M} + w_{k+M}\right) \approx 0$,

or

$$w_{k-M} \approx w_{k+M} \text{ at } y = 0 . \qquad (2\text{-}21)$$

$w,_{xx} + \nu\, w,_{yy} = 0$ and $w = 0$ at $x = 0$.

From Equations (2-17) and (2-18),

$$\frac{1}{(\Delta x)^2}\left(w_{k-1} - 0 + w_{k+1}\right) + \nu \frac{1}{(\Delta x)^2}(0) \approx 0 ,$$

or

$$w_{k-1} \approx - w_{k+1} \text{ at } x = 0 . \qquad (2\text{-}22)$$

$F,_x = 0$ at $x = a/2$.

From Equation (2-15), $\frac{1}{2(\Delta x)}\left(- F_{k-1} + F_{k+1}\right) \approx 0$,

or

$$F_{k+1} \approx F_{k-1} \text{ at } x = a/2 . \qquad (2\text{-}23)$$

By a similar formula with double the grid size,

$$F_{k+2} \approx F_{k-2} \text{ at } x = a/2 \qquad (2\text{-}24)$$

$F,_{xx} = 0$ at $y = 0$.

From Equation (2-17),

$$F_{k-1} - 2\, F_k + F_{k+1} \approx 0 \text{ at } y = 0 . \qquad (2\text{-}25)$$

This formula may be applied to each boundary grid-point along $y = 0$ to obtain a system of algebraic equations. One additional equation is obtained from the symmetry condition at $x = a/2$ and $y = 0$, (Equation (2-23)). The number of unknowns is one more than the number of equations, but fortunately the equations may be reduced to

$$F_{k-1} = F_k = F_{k+1} = F_{k+2} = \cdots\cdots \text{ at } y = 0 . \qquad (2\text{-}26)$$

A similar result is obtained for $F,_{yy} = 0$ at $x = 0$. Therefore, at any given time-step, these boundary conditions indicate the value of F on the boundary is a constant. Since only derivatives of F appear in the differential equations, this constant may be set equal to zero without any loss of generality.

<u>$F,_{xy} = 0$ at $y = 0$.</u>

From Equation (2-19),

$$F_{k-M-1} - F_{k+M-1} + F_{k+M+1} - F_{k-M+1} \approx 0 \text{ at } y = 0 . \qquad (2\text{-}27)$$

This formula may also be applied to each boundary grid-point along $y = 0$ to obtain a system of equations.

At $y = 0$, $F,_y$ is unknown, but it can only be a function of x. Therefore, let

$$F,_y = f(x) \text{ at } y = 0 . \qquad (2\text{-}28)$$

From Equations (2-16) and (2-28),

$$- F_{k-M} + F_{k+M} \approx 2(\Delta x)\ f(x) \text{ at } y = 0 . \qquad (2\text{-}29)$$

This formula may also be applied to each grid-point along $y = 0$. Combining these equations with the system of equations from Equation (2-27) results in

$$f_{k-2} = f_k = f_{k+2} = f_{k+4} = \cdots\cdots \text{ at } y = 0 , \qquad (2\text{-}30)$$

and

$$f_{k-1} = f_{k+1} = f_{k+3} = f_{k+5} = \cdots\cdots \text{ at } y = 0 . \qquad (2\text{-}31)$$

From symmetry conditions and an examination of the problem from a physical nature, Equations (2-30) and (2-31) must also be equal. Therefore, f(x) has a constant value at each boundary grid-point along $y = 0$. Let

$$- F_{k-M} + F_{k+M} \approx C_y \text{ at } y = 0 . \qquad (2\text{-}32)$$

By the same procedure for $F,_{xy} = 0$ at $x = 0$,

$$- F_{k-1} + F_{k+1} \approx C_x \text{ at } x = 0 . \qquad (2\text{-}33)$$

<u>$F,_{xy} = 0$ at $x = 0$ and $F,_{xx} = 0$ at $y = 0$.</u>

A combination of Equations (2-33) and (2-26) results in $C_x = 0$. Therefore,

$$F_{k-1} \approx F_{k+1} \text{ at } x = 0 \text{ .} \tag{2-34}$$

The boundary conditions for stress condition (a) may now be recognized as identical, (for the approximations used), to the displacement boundary conditions for a clamped plate.

<u>u = 0 at x = 0, a/2</u> .

Equation (1-4) is repeated here as

$$u,_x = \frac{1}{E}\left(F,_{yy} - \nu F,_{xx}\right) - \tfrac{1}{2}\, w,_x{}^2 \tag{2-35}$$

By using the boundary conditions the value of the following integral is

$$\int_{x=0}^{x=a/2} u,_x \, dx = \Big[u\Big]_0^{a/2} = 0, \text{ for } y = \text{constant.} \tag{2-36}$$

This result may now be applied to the right-hand side of Equation (2-35),

$$\int_{x=0}^{x=a/2} \left\{\frac{1}{E}\left(F,_{yy} - \nu F,_{xx}\right) - \tfrac{1}{2}\, w,_x{}^2\right\} dx = 0, \text{ for } y = \text{constant.} \tag{2-37}$$

This integral may be converted, by the Trapezoidal Rule and the finite-difference approximations, to an algebraic equation in F_i and w_i. After simplification the result is

$$\begin{aligned} &\left[\left(F_{k-M} - 2\,F_k + F_{k+M}\right)_{k=i} + 2\left(F_{k-M} - 2\,F_k + F_{k+M}\right)_{k=i+1} + \cdots\cdots\right] \\ &-\nu\left[\left(F_{k-1} - 2\,F_k + F_{k+1}\right)_{k=i} + 2\left(F_{k-1} - 2\,F_k + F_{k+1}\right)_{k=i+1} + \cdots\cdots\right] \\ &= \frac{E}{8}\left[\left(- w_{k-1} + w_{k+1}\right)^2_{k=i} + \left(- w_{k-1} + w_{k+1}\right)^2_{k=i+1} + \cdots\cdots\right] \end{aligned} \tag{2-38}$$

This expression may be simplified further by using Equation (2-33) for $F,_{xy} = 0$ at $x = 0$ and Equation (2-23) for $F,_x = 0$ at $x = a/2$.

Equation (2-38) now becomes

$$\left[\left(F_{k-M} - 2\,F_k + F_{k+M}\right)_{k=i} + 2\left(F_{k-M} - 2\,F_k + F_{k+M}\right)_{k=i+1} + \cdots\right]$$

$$+ \nu\,C_x = \frac{E}{8}\left[\left(-\,w_{k-1} + w_{k+1}\right)^2_{k=i} + \left(-\,w_{k-1} + w_{k+1}\right)^2_{k=i+1} + \cdots\right] \quad (2\text{-}39)$$

This equation may be applied to each line of grid-points where y is constant to obtain a set of N algebraic equations in F_i, w_i, C_x, and C_y. For example, at y = 0, (using Equation (2-32) and w = 0), Equation (2-39) becomes

$$- 2\,F_1 - 4\,F_2 - 4\,F_3 - \cdots\cdots - 4\,F_{M-1} - 2\,F_M \quad (2\text{-}40)$$

$$+ 2F_{M+1} + 4F_{M+2} + 4F_{M+3} + \cdots\cdots + 4F_{2m-1} + 2F_{2m} - 2(M-1)\,C_y + \nu\,C_x = 0$$

<u>v = 0 at y = 0, b/2</u> .

By the same procedure as above, the following similar equation may be developed:

$$\left[\left(F_{k-1} - 2F_k + F_{k+1}\right)_{k=i} + 2\left(F_{k-1} - 2\,F_k + F_{k+1}\right)_{k=i+M} + \cdots\cdots\right]$$

$$+ \nu\,C_y = \frac{E}{8}\left[\left(-\,w_{k-M} + w_{k+M}\right)^2_{k=i} + 2\left(-\,w_{k-M} + w_{k+M}\right)^2_{k=i+M} + \cdots\cdots\right] \quad (2\text{-}41)$$

This equation is valid for any line of grid-points where x is constant, and may be used to obtain a set of M algebraic equations in F_i, w_i, C_x and C_y.

For stress condition (b) it is convenient to let $C_y = F_{MN+1}$ and $C_x = F_{MN+2}$. It is also convenient to set $F_1 = 0$, which can be done without any loss of generality since only derivatives of F appear in the differential equations.

It is now possible to set up a system of linear algebraic equations in F_i for both stress conditions (a) and (b) that may be solved for F_i at any time-step j, (assuming $w_{i,j}$ are known).

TABLE I

FICTITIOUS GRID-POINTS IDENTIFIED AS FUNCTIONS OF THE INTERIOR GRID-POINTS

Boundary Conditions	$x = 0$	$y = 0$
I	$w = 0$ $w_{k-1} \approx - w_{k+1}$	$w = 0$ $w_{k-m} \approx - w_{k+M}$
II	$w = 0$ $w_{k-1} \approx w_{k+1}$	$w = 0$ $w_{k-M} \approx w_{k+M}$
(a)	$F = 0$ $F_{k-1} \approx F_{k+1}$	$F = 0$ $F_{k-M} \approx F_{k+M}$
(b)	$F_{k-1} \approx F_{k+1} + C_x$ Equation (39)	$F_{k-M} \approx F_{k+M} + C_y$ Equation (41)

Symmetry Conditions	$x = a/2$	$y = b/2$
For Uniform Pressure Loads and Symmetric Boundary Conditions	$w_{k+1} \approx w_{k-1}$ $w_{k+2} \approx w_{k-2}$ $F_{k+1} \approx F_{k-1}$ $F_{k+2} \approx F_{k-2}$	$w_{k+M} \approx w_{k-M}$ $w_{k+2m} \approx w_{k-2M}$ $F_{K+M} \approx F_{k-M}$ $F_{k+2M} \approx F_{k-2M}$

Initial Conditions

The values of $w_{i,j}$ and $w_{i,j-1}$ must be known before $w_{i,j+1}$ can be calculated from Equation (2-2). These values for starting the solution may be determined from the initial conditions.

When the finite-difference approximation

$$w,_t \approx \frac{1}{2(\Delta t)} \left(- w_{j-1} + w_{j+1} \right) + 0(\Delta t)^2 \tag{2-42}$$

is applied to $w,_t = 0$ at $t = 0$, the result is

$$w_{j-1} \approx w_{j+1} \text{ at } t = 0 . \tag{2-43}$$

With equation (2-43), and $w = 0$ at $t = 0$, Equation (2-2) becomes

$$w_{i,j+1} \approx \tfrac{1}{2} \frac{(\Delta t)^2}{\rho h} \left\{ p(t) \right\}_j , \tag{2-44}$$

which is valid for the first time-step only.

Equation (2-44) may be used to start the solution, after which the solution may be continued in a marching sequence with Equations (2-1) and (2-2).

Computer Programming

It is convenient to consider Equations (2-1) and (2-2) in the form

$$\left[A \right] \left\{ F_{i,j} \right\} = \left\{ C_i \right\}_j \tag{2-45}$$

and

$$w_{i,j+1} = 2\, w_{i,j} - w_{i,j-1} - C1 \cdot B_i + C2 \cdot BB_i + C3 \cdot P_j , \tag{2-46}$$

where A is the matrix of the coefficients of the unknown stress function, $F_{i,j}$,

$$C_i = E \left\{ \frac{1}{16} \left[w_{i-M-1} - w_{i+M-1} - w_{i-M+1} + w_{i+M+1} \right]^2 - \left[w_{i-1} - 2\, w_i + w_{i+1} \right] \cdot \left[w_{i-M} - 2\, w_i + w_{i+M} \right] \right\}_j , \tag{2-47}$$

$$C1 = \frac{(\Delta t)^2}{\rho h} \frac{D}{(\Delta x)^4} , \tag{2-48}$$

$$C2 = \frac{(\Delta t)^2}{\rho h} \frac{h}{(\Delta x)^4} , \tag{2-49}$$

$$C3 = \frac{(\Delta t)^2}{\rho h} , \tag{2-50}$$

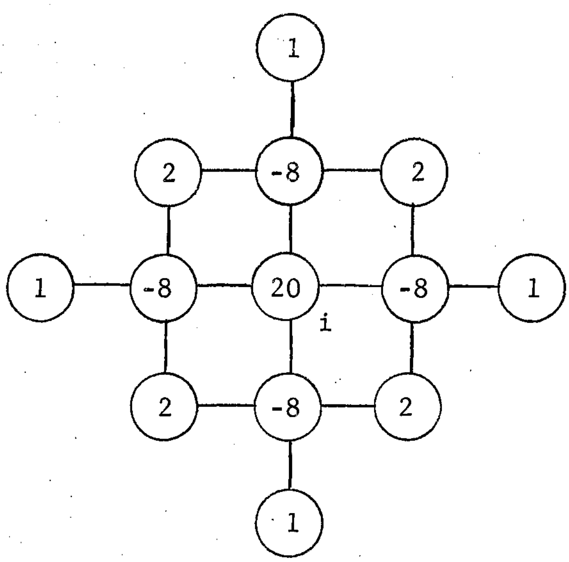
$$\begin{aligned} B_i = \Big\{ & w_{i-2} - 8\, w_{i-1} + 20\, w_i - 8\, w_{i+1} + w_{i+2} \\ & + 2\, w_{i+M-1} - 8\, w_{i+M} + 2\, w_{i+M+1} + w_{i+2M} \\ & + 2\, w_{i-M-1} - 8\, w_{i-M} + 2\, w_{i-M+1} + w_{i-2M} \Big\}_j , \end{aligned} \tag{2-51}$$

$$\begin{aligned} BB_i = \Big\{ & \left[F_{i-M} - 2\, F_i + F_{i+M}\right] \cdot \left[w_{i-1} - 2\, w_i + w_{i+1}\right] \\ & + \left[F_{i-1} - 2\, F_i + F_{i+1}\right] \cdot \left[w_{i-M} - 2\, w_i + w_{i+M}\right] \\ & - \frac{1}{8} \left[F_{i-M-1} - F_{i+M-1} - F_{i-M+1} + F_{i+M+1}\right] \\ & \cdot \left[w_{i-M-1} - w_{i+M-1} - w_{i-M+1} + w_{i+M+1}\right] \Big\}_j , \end{aligned} \tag{2-52}$$

and P_j is the load at time-step j.

A stencil for C_i, B_i, or BB_i may easily be made and is very useful in programming. For example, the stencil for B_i, or $\nabla^4 F$, is

$\nabla^4_i \approx \frac{1}{(\Delta x)^4} \cdot$



$+ \; 0(\Delta x)^2$.

A, C_i, B_i, and BB_i are all dependent on the boundary conditions. For case I (a) there are nine different grid-points, or groups of grid-points, where the equations for C_i and BB_i have a different form because of the boundary conditions. For B_i there are 25 different equations after the substitutions for boundary conditions have been made. For example, at the first grid-point Equation (2-47) becomes

$$C_1 = E \left\{ \frac{1}{16} \left[w_{1+M+1} \right]^2 - \left[- 2\ w_1 + w_2 \right] \cdot \left[- 2\ w_1 + w_{1+M} \right] \right\}_j \quad .(2\text{-}53)$$

The results for the other cases are similar. Of course the fictitious grid-points could be numbered and only one equation for each term would be necessary, but the boundary conditions would have to be included in the program and the storage requirements for the computer would be increased considerably.

Figures 5 and 6 illustrate the system of equations from Equation (2-45) for the two stress conditions. For stress condition (a) each equation results from Equation (2-1) with boundary conditions incorporated in each equation. For stress condition (b) Equation (2-1) is also used at each interior grid-point. The boundary conditions of Equations (2-40) and (2-41), along with $F_1 = 0$, make up the additional equations required to determine the unknown values of F along the boundaries and the two additional unknowns of C_x and C_y, (see page 21). These equations may be included at any position in the matrix, but they are arranged to facilitate the solution. The coefficients for either matrix may be set up for any value of M and N by a subroutine.

The complete solution is illustrated in the block diagram of Figure 7. The programs basically follow this diagram but with several variations to improve efficiency. Separate programs must be prepared for each set of boundary conditions.

$$
\left[\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrr}
22 & -8 & 1 & & & -8 & 2 & & & & 1 & & & & & & & & & & & & & & \\
-8 & 21 & -8 & 1 & & 2 & -8 & 2 & & & & 1 & & & & & & & & & & & & & \\
1 & -8 & 21 & -8 & 1 & & 2 & -8 & 2 & & & & 1 & & & & & & & & & & & & \\
 & 1 & -8 & 22 & -8 & & & 2 & -8 & 2 & & & & 1 & & & & & & & & & & & \\
 & & 2 & -16 & 21 & & & & 4 & -8 & & & & & 1 & & & & & & & & & & \\
-8 & 2 & & & & 21 & -8 & 1 & & & -8 & 2 & & & & 1 & & & & & & & & & \\
2 & -8 & 2 & & & -8 & 20 & -8 & 1 & & 2 & -8 & 2 & & & & 1 & & & & & & & & \\
 & 2 & -8 & 2 & & 1 & -8 & 20 & -8 & 1 & & 2 & -8 & 2 & & & & 1 & & & & & & & \\
 & & 2 & -8 & 2 & & 1 & -8 & 21 & -8 & & & 2 & -8 & 2 & & & & 1 & & & & & & \\
 & & & 4 & -8 & & & 2 & -16 & 20 & & & & 4 & -8 & & & & & 1 & & & & & \\
1 & & & & & -8 & 2 & & & & 21 & -8 & 1 & & & -8 & 2 & & & & 1 & & & & \\
 & 1 & & & & 2 & -8 & 2 & & & -8 & 20 & -8 & 1 & & 2 & -8 & 2 & & & & 1 & & & \\
 & & 1 & & & & 2 & -8 & 2 & & 1 & -8 & 20 & -8 & 1 & & 2 & -8 & 2 & & & & 1 & & \\
 & & & 1 & & & & 2 & -8 & 2 & & 1 & -8 & 21 & -8 & & & 2 & -8 & 2 & & & & 1 & \\
 & & & & 1 & & & & 4 & -8 & & & 2 & -16 & 20 & & & & 4 & -8 & & & & & 1 \\
 & & & & & 1 & & & & & -8 & 2 & & & & 22 & -8 & 1 & & & -8 & 2 & & & \\
 & & & & & & 1 & & & & 2 & -8 & 2 & & & -8 & 21 & -8 & 1 & & 2 & -8 & 2 & & \\
 & & & & & & & 1 & & & & 2 & -8 & 2 & & 1 & -8 & 21 & -8 & 1 & & 2 & -8 & 2 & \\
 & & & & & & & & 1 & & & & 2 & -8 & 2 & & 1 & -8 & 22 & -8 & & & 2 & -8 & 2 \\
 & & & & & & & & & 1 & & & & 4 & -8 & & & 2 & -16 & 21 & & & & 4 & -8 \\
 & & & & & & & & & & 2 & & & & & -16 & 4 & & & & 21 & -8 & 1 & & \\
 & & & & & & & & & & & 2 & & & & 4 & -16 & 4 & & & -8 & 20 & -8 & 1 & \\
 & & & & & & & & & & & & 2 & & & & 4 & -16 & 4 & & 1 & -8 & 20 & -8 & 1 \\
 & & & & & & & & & & & & & 2 & & & & 4 & -16 & 4 & & 1 & -8 & 21 & -8 \\
 & & & & & & & & & & & & & & 2 & & & & 8 & -16 & & & 2 & -16 & 20
\end{array}\right]
\left[\begin{array}{c}
F(1) \\ F(2) \\ F(3) \\ F(4) \\ F(5) \\ F(6) \\ F(7) \\ F(8) \\ F(9) \\ F(10) \\ F(11) \\ F(12) \\ F(13) \\ F(14) \\ F(15) \\ F(16) \\ F(17) \\ F(18) \\ F(19) \\ F(20) \\ F(21) \\ F(22) \\ F(23) \\ F(24) \\ F(25)
\end{array}\right]_j
=
\left[\begin{array}{c}
C(1) \\ C(2) \\ C(3) \\ C(4) \\ C(5) \\ C(6) \\ C(7) \\ C(8) \\ C(9) \\ C(10) \\ C(11) \\ C(12) \\ C(13) \\ C(14) \\ C(15) \\ C(16) \\ C(17) \\ C(18) \\ C(19) \\ C(20) \\ C(21) \\ C(22) \\ C(23) \\ C(24) \\ C(25)
\end{array}\right]_j
$$

Figure 5. System of Equations Developed from Equation (2-1) and Boundary Conditions for Stress Condition (a), (M = N = 5)

$$
\left[\begin{array}{ccccccccccccccccccccccccccc}
1&&&& & &&&& & &&&& & &&&& & &&&& & & \\
-2&2&&& & -4&4&&& & -4&4&&& & -4&4&&& & -2&2&&& & \nu & -8 \\
1&-2&1&& & 2&-4&2&& & 2&-4&2&& & 2&-4&2&& & 1&-2&1&& & \nu & \\
&1&-2&1& & &2&-4&2& & &2&-4&2& & &2&-4&2& & &1&-2&1& & \nu & \\
&&1&-2&1 & &&2&-4&2 & &&2&-4&2 & &&2&-4&2 & &&1&-2&1 & \nu & \\
&&&2&-2 & &&&4&-4 & &&&4&-4 & &&&4&-4 & &&&2&-2 & \nu & \\
2&-8&2&& & -8&22&-8&1& & 2&-8&2&& & &1&&& & &&&& & -1 & -1 \\
&2&-8&2& & 1&-8&21&-8&1 & &2&-8&2& & &&1&& & &&&& & -1 & \\
&&2&-8&2 & &1&-8&22&-8 & &&2&-8&2 & &&&1& & &&&& & -1 & \\
&&&4&-8 & &&2&-16&21 & &&&4&-8 & &&&&1 & &&&& & -1 & \\
-2&-4&-4&-4&-2 & 2&4&4&4&2 & &&&& & &&&& & &&&& & -8 & \nu \\
&1&&& & 2&-8&2&& & -8&21&-8&1& & 2&-8&2&& & &1&&& & & -1 \\
&&1&& & &2&-8&2& & 1&-8&20&-8&1 & &2&-8&2& & &&1&& & & \\
&&&1& & &&2&-8&2 & &1&-8&21&-8 & &&2&-8&2 & &&&1& & & \\
&&&&1 & &&&4&-8 & &&2&-16&20 & &&&4&-8 & &&&&1 & & \\
1&2&2&2&1 & -2&-4&-4&-4&-2 & 1&2&2&2&1 & &&&& & &&&& & & \nu \\
&&&& & &1&&& & 2&-8&2&& & -8&22&-8&1& & 2&-8&2&& & & -1 \\
&&&& & &&1&& & &2&-8&2& & 1&-8&21&-8&1 & &2&-8&2& & & \\
&&&& & &&&1& & &&2&-8&2 & &1&-8&22&-8 & &&2&-8&2 & & \\
&&&& & &&&&1 & &&&4&-8 & &&2&-16&21 & &&&4&-8 & & \\
&&&& & 1&2&2&2&1 & -2&-4&-4&-4&-2 & 1&2&2&2&1 & &&&& & & \nu \\
&&&& & &&&& & &2&&& & 4&-16&4&& & -8&21&-8&1& & & -1 \\
&&&& & &&&& & &&2&& & &4&-16&4& & 1&-8&20&-8&1 & & \\
&&&& & &&&& & &&&2& & &&4&-16&4 & &1&-8&21&-8 & & \\
&&&& & &&&& & &&&&2 & &&&8&-16 & &&2&-16&20 & & \\
&&&& & &&&& & 1&2&2&2&1 & -2&-4&-4&-4&-2 & 1&2&2&2&1 & & \nu \\
&&&& & &&&& & &&&& & 2&4&4&4&2 & -2&-4&-4&-4&-2 & & \nu
\end{array}\right]
\left[\begin{array}{c}
F(1)\\ F(2)\\ F(3)\\ F(4)\\ F(5)\\ F(6)\\ F(7)\\ F(8)\\ F(9)\\ F(10)\\ F(11)\\ F(12)\\ F(13)\\ F(14)\\ F(15)\\ F(16)\\ F(17)\\ F(18)\\ F(19)\\ F(20)\\ F(21)\\ F(22)\\ F(23)\\ F(24)\\ F(25)\\ F(26)\\ F(27)
\end{array}\right]_j
=
\left[\begin{array}{c}
0\\ C(2)\\ C(3)\\ C(4)\\ C(5)\\ C(6)\\ C(7)\\ C(8)\\ C(9)\\ C(10)\\ C(11)\\ C(12)\\ C(13)\\ C(14)\\ C(15)\\ C(16)\\ C(17)\\ C(18)\\ C(19)\\ C(20)\\ C(21)\\ C(22)\\ C(23)\\ C(24)\\ C(25)\\ C(26)\\ C(27)
\end{array}\right]_j
$$

Figure 6. System of Equations Developed from Equation (2-1) and Boundary Conditions for Stress Condition (b), (M=N=5).

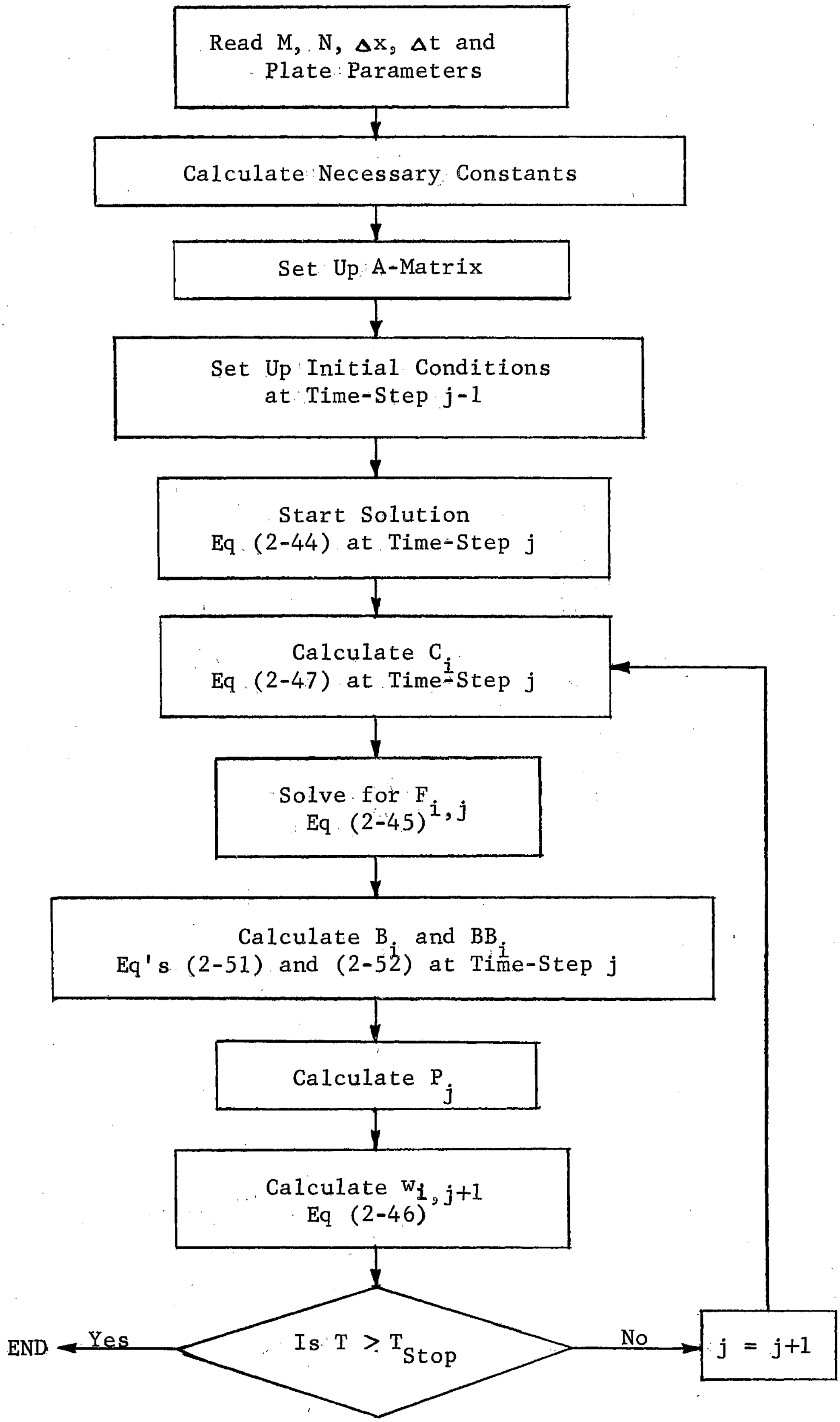


Figure 7. Block Diagram for Solution of Equations (2-1) and (2-2)

Solution of Simultaneous Equations

Several things should be considered in the solution of Equation (2-45). First, the system of equations must be solved at each time-step so an efficient method is essential. Next, the system may be quite large and an accurate solution could be a difficult problem. Fortunately, the A-matrix may be set up so that both an accurate and an efficient solution may be obtained. The method is a modified form of "Gaussian elimination with back substitution." Several other schemes were considered, but none were successful.

The A-matrix for stress condition (a) may be arranged into a narrow centered band structure as shown in Figure 8. This matrix is well-conditioned and is not a function of time. The Gaussian elimination method will not be described here, except to explain the particular modifications made. The Gaussian elimination algorithm is

$$
\begin{aligned}
m_i^{(k-1)} &= a_{ik}^{(k-1)} / a_{kk}^{(k-1)} \\
a_{ij}^{(k)} &= a_{ij}^{(k-1)} - m_i^{(k-1)} a_{kj}^{(k-1)} \\
c_i^{(k)} &= c_i^{(k-1)} - m_i^{(k-1)} c_k^{(k-1)}
\end{aligned}
\qquad (2\text{-}54)
$$

The triangular block of zero elements above and below the band will not be changed by the Gaussian elimination, except by round-off errors. Therefore, the algorithm is only applied to the band elements. This results in a considerable savings in computer time. Also, there is a very significant reduction in round-off error, simply because of the reduced number of arithmetic operations. Since the A-matrix does not change with time, the elimination scheme on the A-matrix is only made one time for each problem. However, the C-vector varies with time and the elimination scheme must be made on the new vector at each time-step.

In order to do this, the values for $m_i^{(k-1)}$, (Equation (2-54)), must be retained for repeated use. They are stored in the order they are used as the vector DC. In the back substitution process only the diagonal elements and the band elements above the diagonal are used. The zero elements outside the band are never used in the calculation. This is a considerable waste of computer storage, but no satisfactory alternative could be found. Figure 10 is a block diagram of the elimination scheme on the A-matrix. This part of the solution is developed as a separate subroutine. Row-interchange has no value for this particular matrix. Figure 11 is a block diagram of the elimination scheme on the C-vector and the back substitution. This part of the complete solution is all that is needed at each time-step and is included in the main subroutine.

The structure of the A-matrix for stress condition (b) is shown in Figure 9. All elements above the main diagonal must be used. This increases the computer time, but has very little effect on the accuracy. The large triangular block of zeroes in the lower left corner still permits an accurate solution for large systems. A similar method as for stress condition (a) is used here, but row-interchange is necessary.

The significance of the band structure of the A-matrix increases with the number of grid-points. To take full advantage of this M should be less than or equal to N. The results of an empirical check on accuracy and convergence are included in Chapter 3.

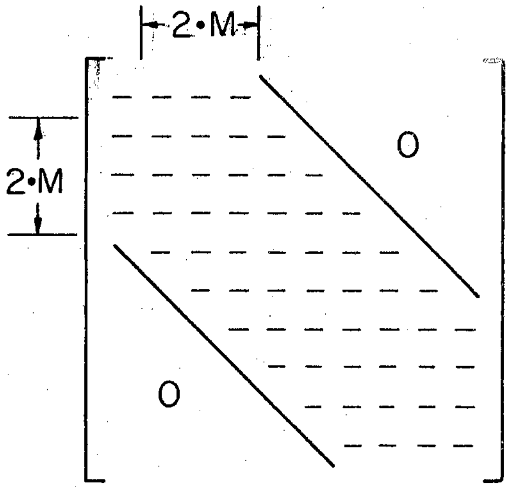


Figure 8. Band Structure of A-matrix for Stress Condition (a)

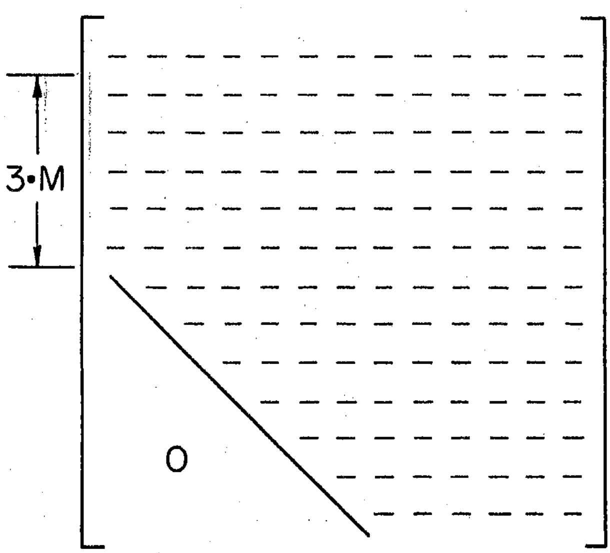


Figure 9. Structure of A-matrix for Stress Condition (b)

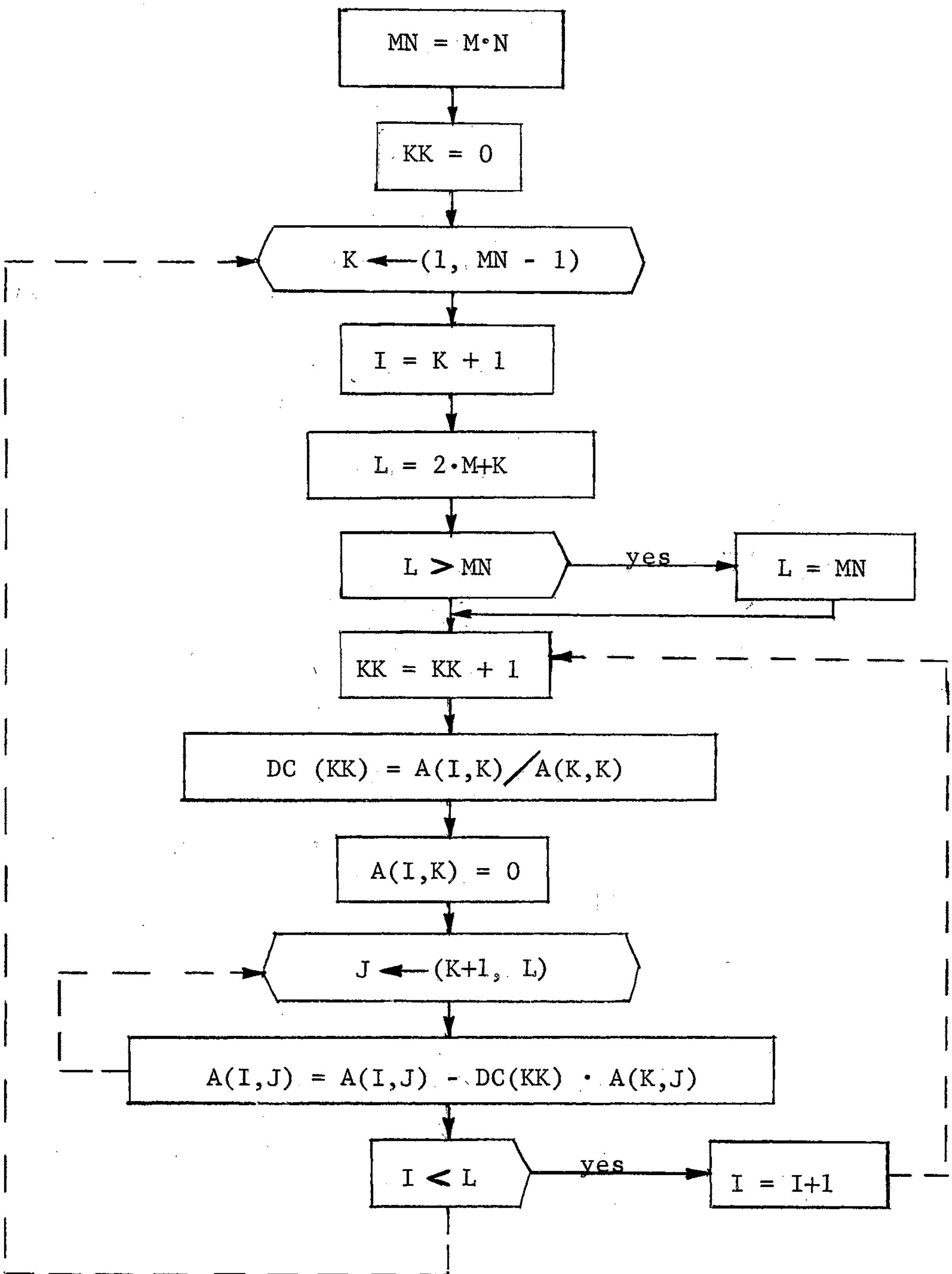


Figure 10. Gaussian Elimination on A-matrix for Stress Condition (a)

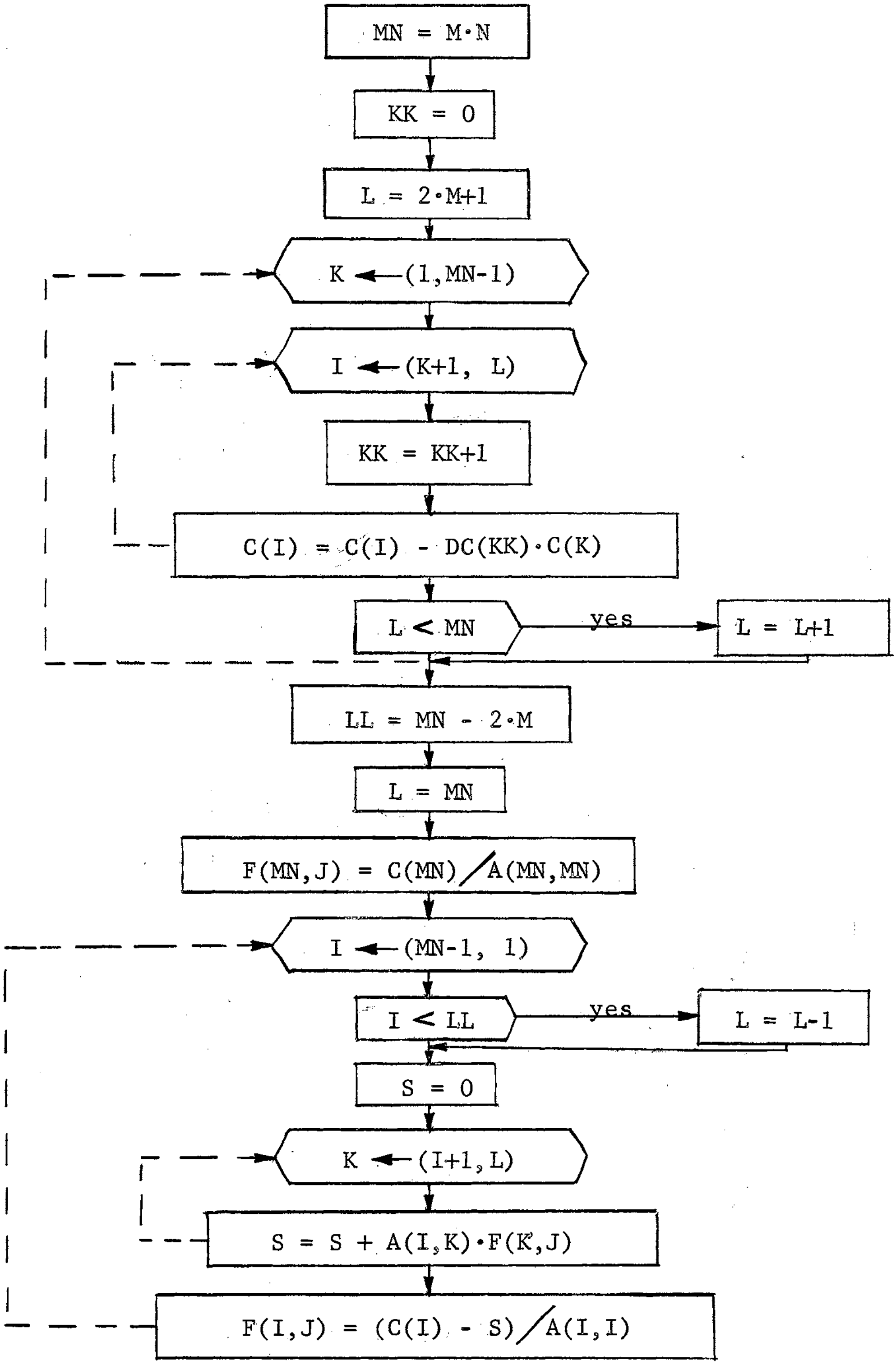


Figure 11. Gaussian Elimination on the C-vector and Back Substitution for Stress Condition (a)

CHAPTER III

NUMERICAL RESULTS

Numerical Error Analysis

The most practical method of checking a numerical method and the program to execute it is to select a particular problem for which an exact solution is known and compare the results. For the problems at hand an exact solution is not known. However, special problems may be devised to check components of the programs. The complete solution may be checked for stability and convergence and compared with other approximate solutions.

The finite-difference technique and the numerical methods used have been proven by numerous applications and the methods as such are not under investigation here. The purpose of this error analysis is to verify the particular applications of the methods and to check the programs.

Special Problems

One method of checking the numerical solution of Equation (1-1) is to change the right-hand side of the equation to some function of x and y for which an exact solution is known. The simplest procedure for doing this is to assume a solution that satisfies exactly the boundary conditions and then determine the differential equation for which the solution is valid. The boundary conditions are those stated for stress conditions (a) and (b) in Equation (1-3). The symmetry conditions of Equation (1-6) are also valid. It is convenient to let a = b = 2.

For stress condition (a) the assumed solution is

$$F = \sin^2 \frac{\pi x}{2} \sin^2 \frac{\pi y}{2} , \tag{3-1}$$

and the differential equation becomes

$$\nabla^4 F = - \frac{\pi^4}{2} \left(\cos \pi x \sin^2 \frac{\pi y}{2} - \cos \pi x \cos \pi y + \sin^2 \frac{\pi x}{2} \cos \pi y \right) . \tag{3-2}$$

For stress condition (b) the assumed solutions are

$$u = \frac{\pi(1+\nu)}{E} \sin \pi x \sin^2 \frac{\pi y}{2} , \tag{3-3}$$

and

$$v = \frac{\pi(1+\nu)}{E} \sin^2 \frac{\pi x}{2} \sin \pi y . \tag{3-4}$$

The relationship between u, v, and F is taken as

$$F,_{xy} = \frac{-E}{2(1+\nu)} \left(u,_y + v,_x \right) . \tag{3-5}$$

This expression is similar to the expression relating displacements and the membrane shear stress of classical nonlinear plate theory. The stress function may now be determined as

$$F = \tfrac{1}{2} (1 - \cos \pi x \cos \pi y) , \tag{3-6}$$

and the differential equation becomes

$$\nabla^4 F = - 2 \pi^4 \cos \pi x \cos \pi y . \tag{3-7}$$

For both problems stated above the solutions exactly satisfy the differential equations and the boundary conditions; therefore, by the uniqueness theorem the solutions are complete. The problems may easily be solved by the finite-difference method by using appropriate components of the programs developed for the solution of Equation (2-1).

The results are shown in Figure 12. The maximum value of F is plotted, which occurs at point (1, 1) for stress condition (a) and at points (1, 0) and (0, 1) for stress condition (b). The exact value at each point is 1. The percent error may be read directly from the plot. The results for each problem are almost identical for the same value of Δx, (see Equations (2-13) and (2-14)). Also, the results for each grid-point were similar.

For stress condition (b) the system of equations contains two unknowns in addition to the unknown values of F at each grid-point, (see page 21). These unknowns are $F,_x$ at x = 0 and $F,_y$ at y = 0. They were calculated by the finite-difference method to be zero, which agrees with the exact solution.

Another check was made on the solution of Equation (1-1). A fundamental mode shape was assumed for w and used to evaluate the right-hand side of Equation (2-1). The results have not been shown but they appear to be almost identical, from a percentage viewpoint, to the results shown in Figure 12. Of course, the exact solutions were not known.

Part of the numerical solution of Equation (1-2) may be checked by neglecting the nonlinear terms. For linear plate theory, Equation (1-2) becomes

$$\rho h\ w,_{tt} + D\ \nabla^4 w = p(t)\ , \qquad (3\text{-}8)$$

and Equation (1-1) is not applicable. An exact series solution of this equation, for a simply supported square plate subjected to a step pressure load, may be obtained by classical methods to be

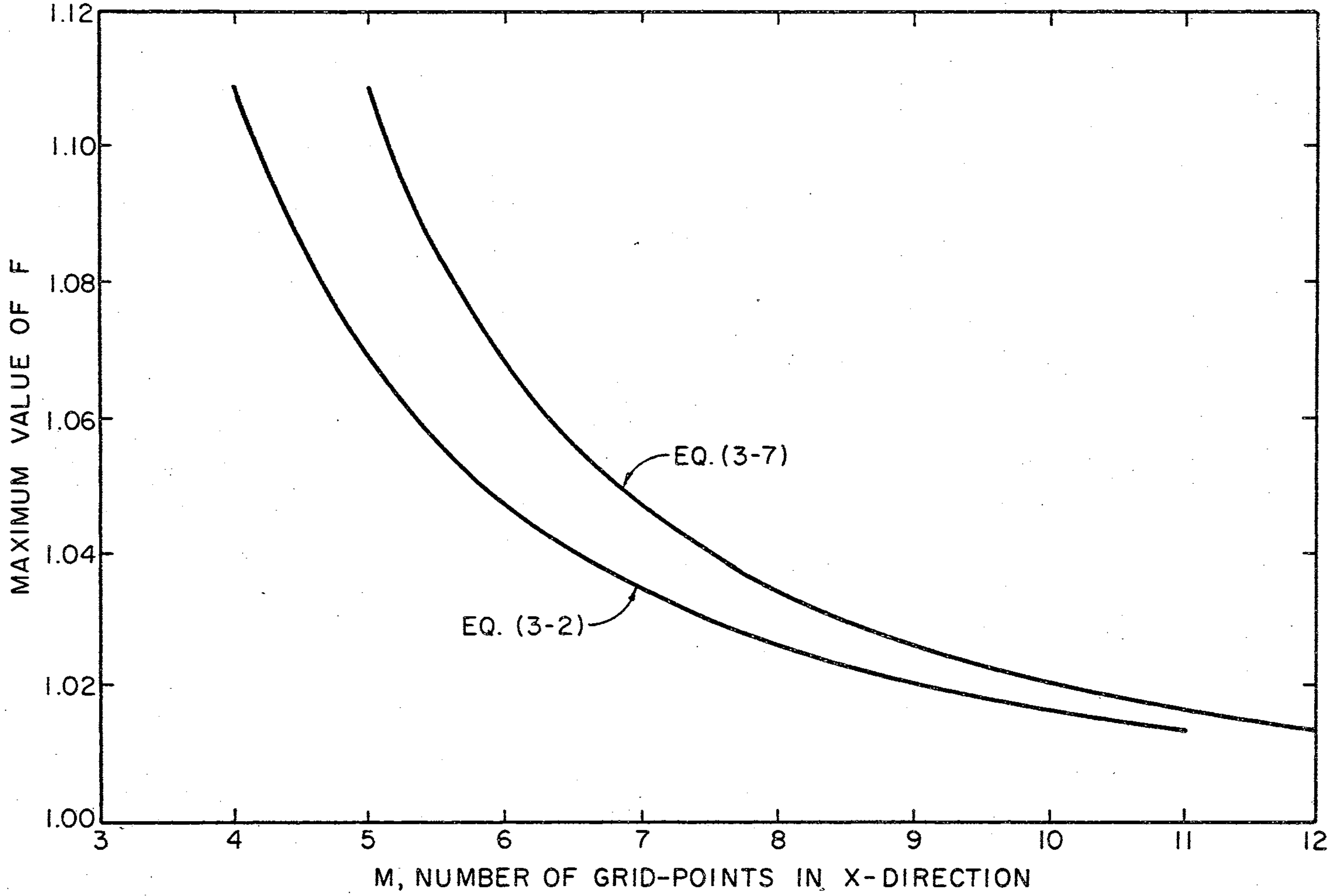


Figure 12. Convergence Check for Equations (3-2) and (3-7)

$$w(x,y,t)=\frac{16p_o a^4}{\pi^6 D} \sum_{m=1,3,\cdots}^{\infty} \sum_{n=1,3,\cdots}^{\infty} B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \left(1-\cos \omega_{mn} t\right), \tag{3-9}$$

where

$$B_{mn} = \frac{1}{mn(m^2+n^2)^2}, \tag{3-10}$$

and

$$\omega_{mn} = \frac{\pi^2}{a^2} \sqrt{\frac{D}{\rho h}} \, (m^2 + n^2) \, . \tag{3-11}$$

A numerical solution of Equation (3-8) may be obtained by using appropriate components of the programs developed for Equation (2-2). Instead of Equation (2-46), the solution takes the form

$$w_{i,j+1} = 2\, w_{i,j} - w_{i,j-1} - C1 \cdot B_i + C3 \cdot P_j \, . \tag{3-12}$$

The initial conditions and the starting formula do not change. The boundary conditions may be simply supported or clamped.

The specific problem to be considered is a glass window with $a = b = 8$ feet, $h = .25$ inches, $\gamma = 157.5$ pounds/foot^3, $\nu = .23$, and $E = 10 \times 10^6$ psi. The load is a 1 psf step-function.

The exact solution of this problem for simply supported edges is shown in Figure 13, along with the finite-difference solution for two values of Δx and Δt. Eight terms of the series of Equation (3-9) were used. The finite-difference solution may be improved with a reduction of the time-step. The magnitude of the displacement indicates that linear theory is not valid for this problem, but this does not affect the comparison from a mathematical standpoint. The convergence of the finite-difference solution is shown in Figure 14. The plot is of the maximum center displacement. The time-step is different for each grid-size and there is a slight variation in the time of the maximum displacement. The irregularity of the data is probably due to this

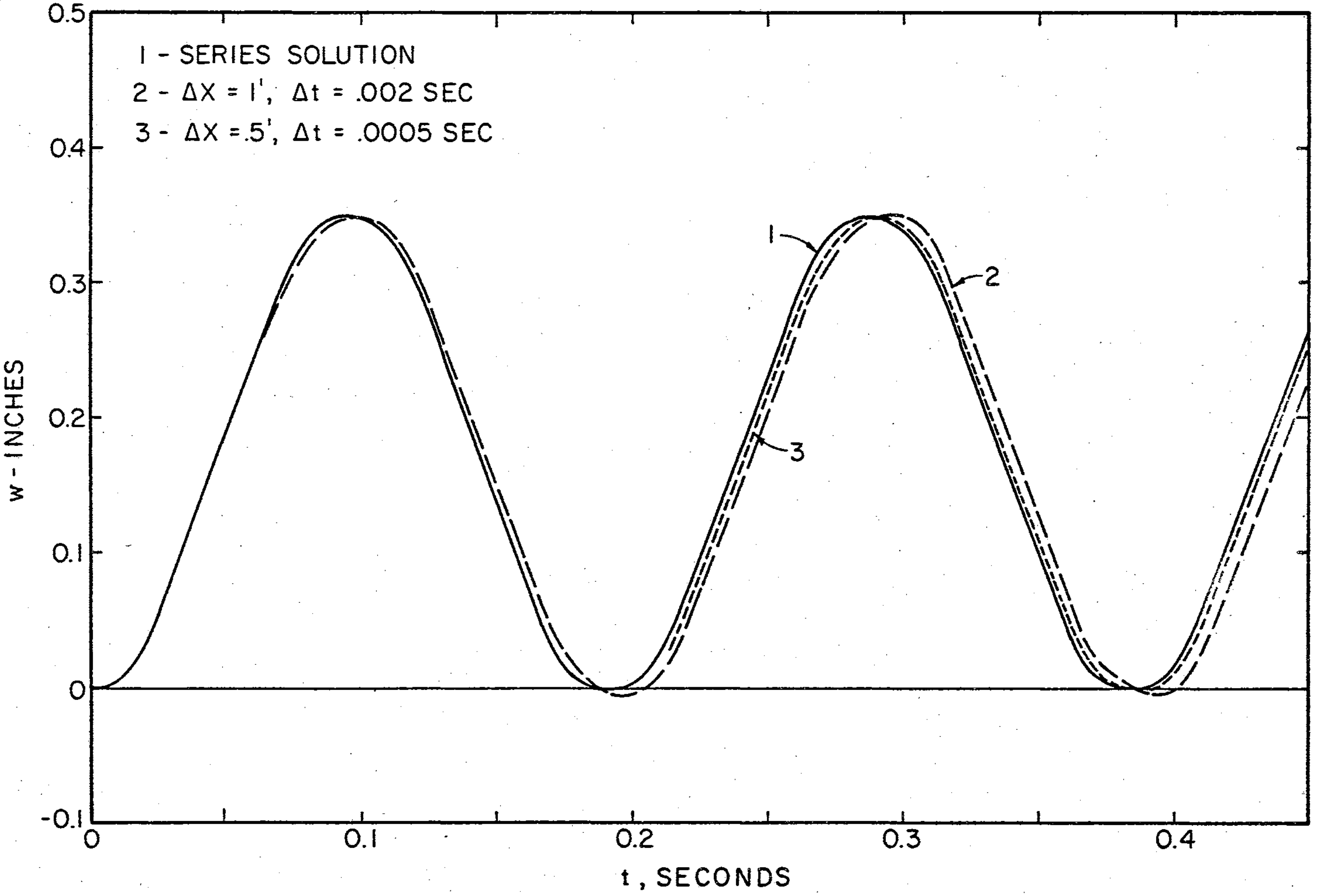


Figure 13. Plate Center Displacement vs Time for a Step Pressure Load

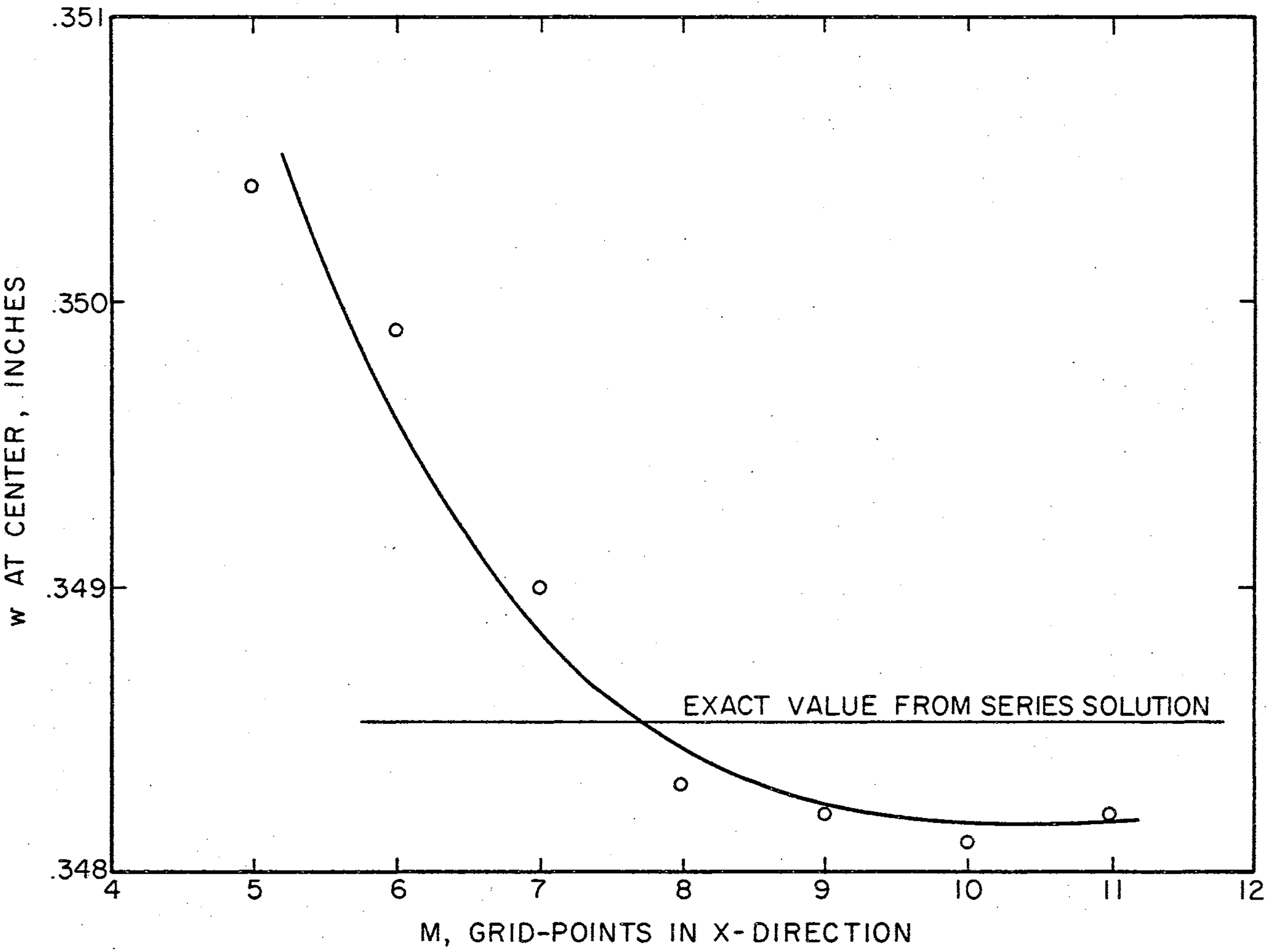


Figure 14. Convergence of Finite-Difference Solution of Equation (3-8) with Simply Supported Boundary Conditions

time factor. The maximum variation of the amplitude determined by using values of M from 7 through 11 is less than one percent.

The convergence of the finite-difference solution for clamped edges is shown in Figure 15. For values of M from 8 through 12, the variation of the maximum amplitude is again less than one percent. An exact solution of this problem is not known.

Stability

The necessary conditions for a stable solution of Equation (2-2) can be determined from Equation (2-12). This equation is plotted in Figure 16 for a particular material and for constant values of the membrane stresses, (assuming $F,_{xx} = F,_{yy}$). The area below the curve is the stable region.

This theory may be checked by attempting a solution for various time-steps. The results for case I (a) are shown in Figure 17. The physical properties of the plate are the same as in the previous section. The load is a 1 psf step-function and the maximum membrane stress is less than 100 psi. For a Δt in the unstable region the magnitude of w and F became obviously unrealistic. Similar checks were made at various points for the other boundary conditions. Although the results were not identical, there was no significant deviation.

Similar stability checks were also made for higher values of the membrane stresses by increasing the pressure of the step-function load. Since the maximum stresses only occur periodically, the error resulting from an unstable condition might require several cycles before it would grow to a point it could be recognized as such. The

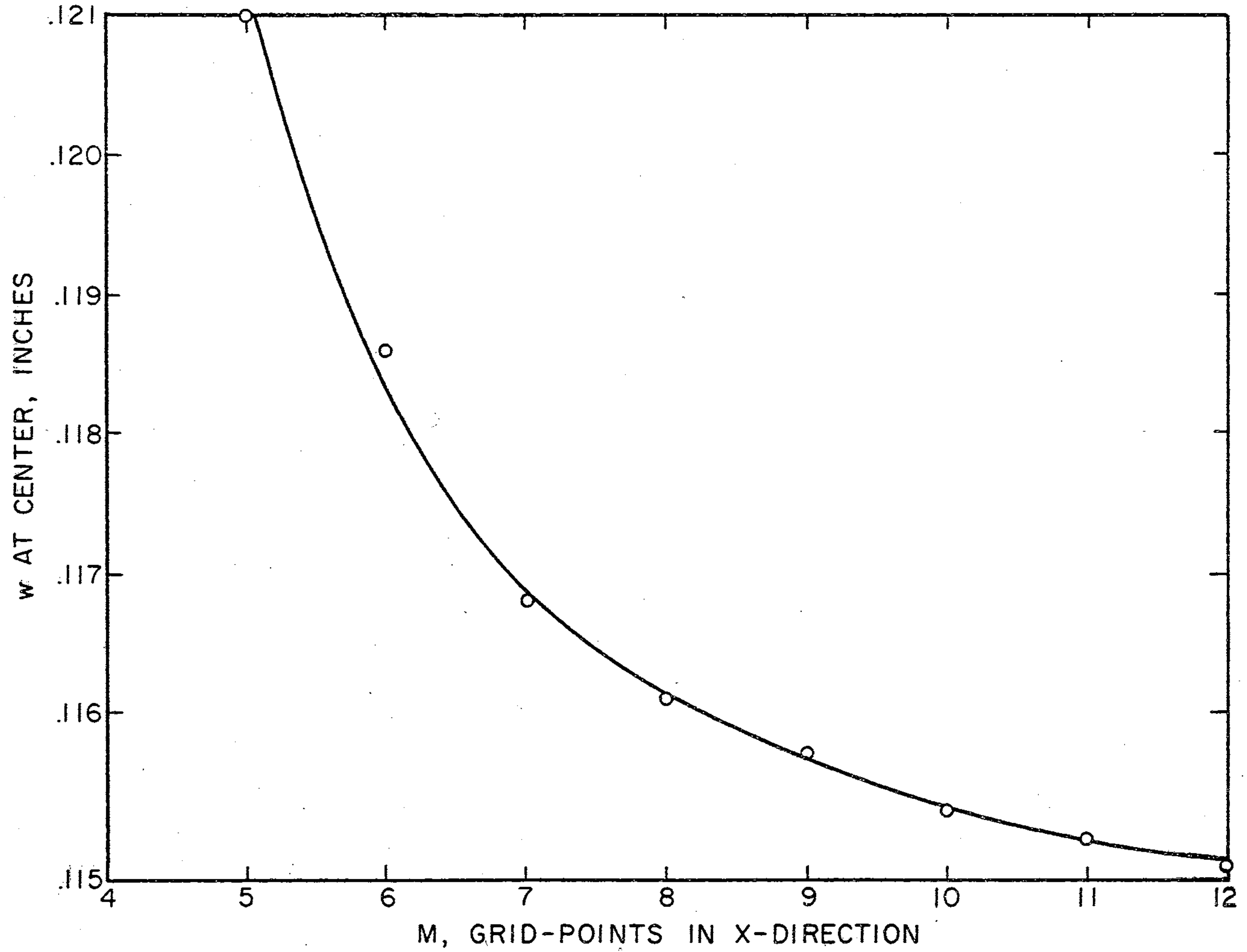


Figure 15. Convergence of Finite-Difference Solution of Equation (3-8) with Clamped Boundary Conditions

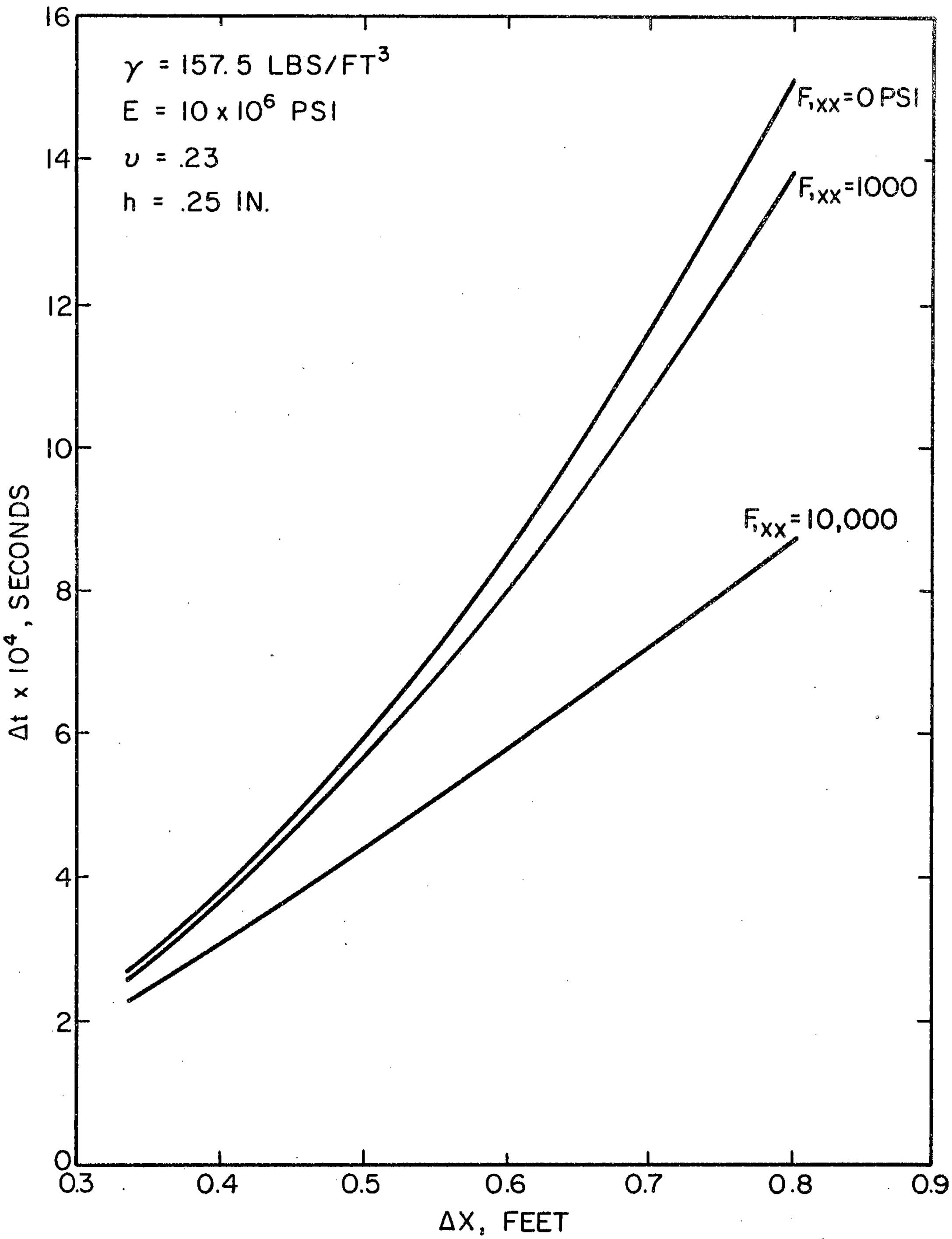


Figure 16. Nonlinear Stability Criterion, (Equation 2-12)

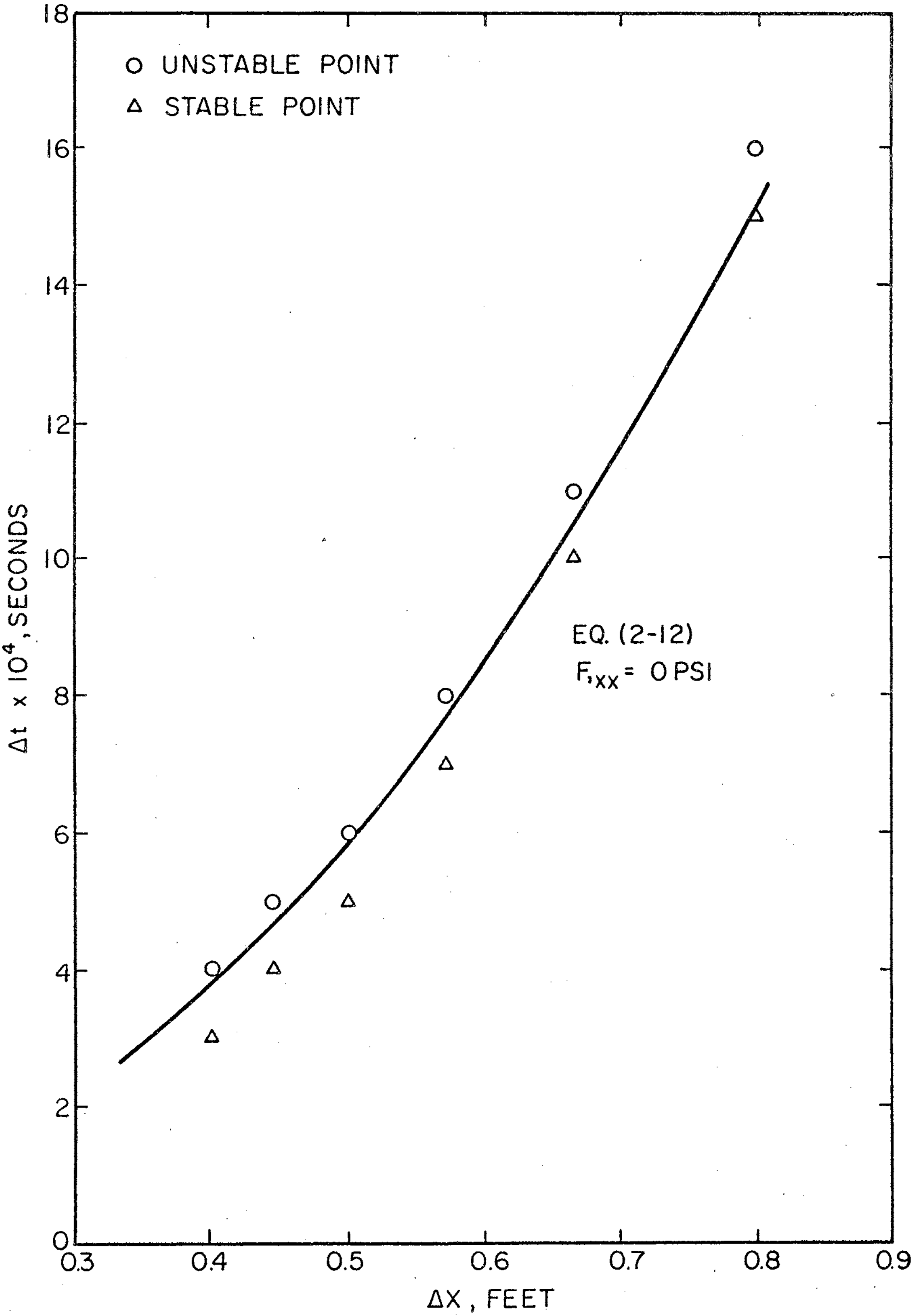


Figure 17. Empirical Stability Check

solutions appeared to be stable, (for a few cycles), for values of Δt larger than those predicted necessary by the theory. A complete check at a higher stress-level would require excessive computer time and was not attempted. However, sufficient checks were made to determine that the value of Δt necessary for a stable solution always decreased as the membrane stresses increased. Although the validity of Equation (2-12) has not been proven, sufficient evidence has been presented to have confidence and to recommend its use.

Example Problems

Several particular problems have been solved to illustrate to a certain extent what can be expected from this method. A comparison is made with other approximate solutions to the same problems, and also with some experimental observations that have not previously been accounted for theoretically. A complete parameter study is beyond the scope of this study and is not intended. The results for various types of loading, magnitude of response, and plate parameters should be similar to the results presented here for specific problems. However, a careless application of the programs presented could lead to significant errors.

An approximate solution of Equations (1-1) and (1-2) was obtained for each set of boundary conditions by Yamaki (15). The method is essentially a lumped-parameter model of the plate. A fundamental mode shape for displacement is assumed and the problem is reduced to one of solving an ordinary differential equation. This equation, with any type of forcing function, may easily be solved by various numerical integration methods. The one used in this study was the Hamming predictor-corrector method, which is available as subroutine DHPCG from the IBM Scientific Subroutine Package, (7).

Some specific problems have been solved by using the lumped-parameter models, and also by the finite-difference method. The results are shown in Figures 18 through 31. The plate size and physical properties are the same as in the previous section. The load is indicated on the figure. The N-Wave loads represent typical overpressures, or sonic booms, that are produced by supersonic aircraft.

For some cases the agreement between the two solutions is excellent. For others the higher frequencies in the response are evident in the finite-difference solutions and result in a significant deviation between the two solutions. As expected, the effect of the higher frequencies becomes more significant as the amplitude of deflection increases. There are also displacement shapes associated with these higher frequencies. Since the lumped parameter model was developed by assuming a fundamental mode shape, the method is limited to some range of deflection where the effect of the higher frequencies is negligible. As mentioned before, the limit of this range has not been established. The influence of the higher frequencies on the deflected shape of the plate is shown in Figures 27 and 28. Notice that the time when the deflected surface has the greatest deviation from a fundamental mode shape corresponds to the time on the response curves, (Figures 22 and 25), when the higher frequencies are evident. The deviation from the fundamental mode shape appears to be a flattening of the center section of the plate. A flattening was observed experimentally by Bowles and Sugarman (2) and by Freynik (5). Also, the point of maximum stress was found to migrate along a diagonal away from the center as the load increased. A comparison with their experimental data has not been made, but it is believed that the finite-difference solution is the only one available that will predict such a behavior of the plate. The shape of the deflected surface may be the most serious drawback of the lumped-parameter method. Even when the center deflection and frequency determined by this method are reasonably accurate, there may still be a significant error in determining the stresses. For the examples cited the time responses by the finite-difference method always have a negative deflection at the end of the first cycle. Since the total error in the finite-difference method always grows with time, it is natural to suspect this may be an error in the method. This was actually the case for the linear problem and the negative deflection converged to zero as the grid-size and time-step were reduced, (see Figure 13). For the nonlinear analysis this was not generally the case. Only the center section of the plate deflects into the negative range, which indicates that the negative deflections are a result of the higher frequencies.

The convergence of the finite-difference solutions for the same example problems is illustrated in Figures 26, 30, and 31 and also in Tables II and III. The values in the tables are for the first peak of the response and at the center of the plate. The maximum deviation of any two corresponding values in the tables is less than five percent. Although the figures show the response for only two values of grid-spacing, the problems were solved for several values. The general pattern of convergence was excellent. As the grid-spacing was reduced, the deviation between the results for two successive grid-sizes became less. For a given grid-size, a reduction of the time-step to a value lower than those used did not produce a significant difference in the response. In Figure 29, the deviation in the response determined by reduced grid-sizes was not sufficient to show on the curve. Similar convergence checks were made for each set of boundary conditions and the results were similar. Convergence of the solution is necessary, but not sufficient, to insure an accurate solution.

The effect of the boundary conditions on some of the example problems is clearly shown by Figures 32 and 33.

The computer used for all calculations was an IBM 360 Model 50 with 256 K main core and 2361 K large core storage under OSMFT Release 15/16. The high speed main core with a Fortran-G compiler was used to obtain the computer times. Exact computer times were not available and those presented are only rough approximations that are normally intended for accounting purposes. The results are shown in Figure 34. The particular problem used to obtain this data was a square plate with a step-function load. The times are the same for displacement conditions I or II. The execution times for Subroutines COEF and AGE range from two seconds for M=5 to twenty seconds for M=10. The curves may be used to estimate the computer time required for a particular problem.

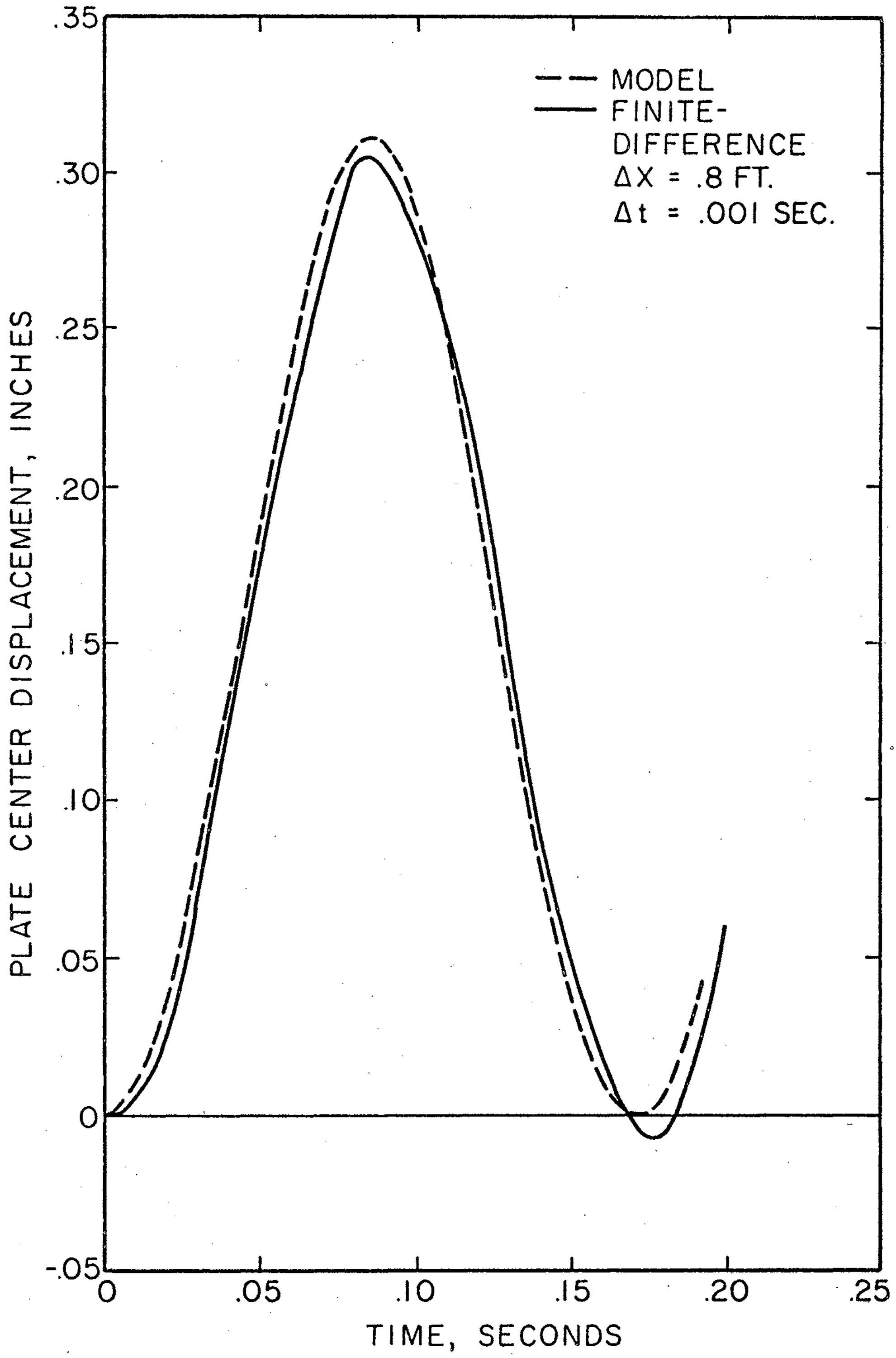


Figure 18. Displacement-Time Response for Case I(a) and a 1 psf Step Load

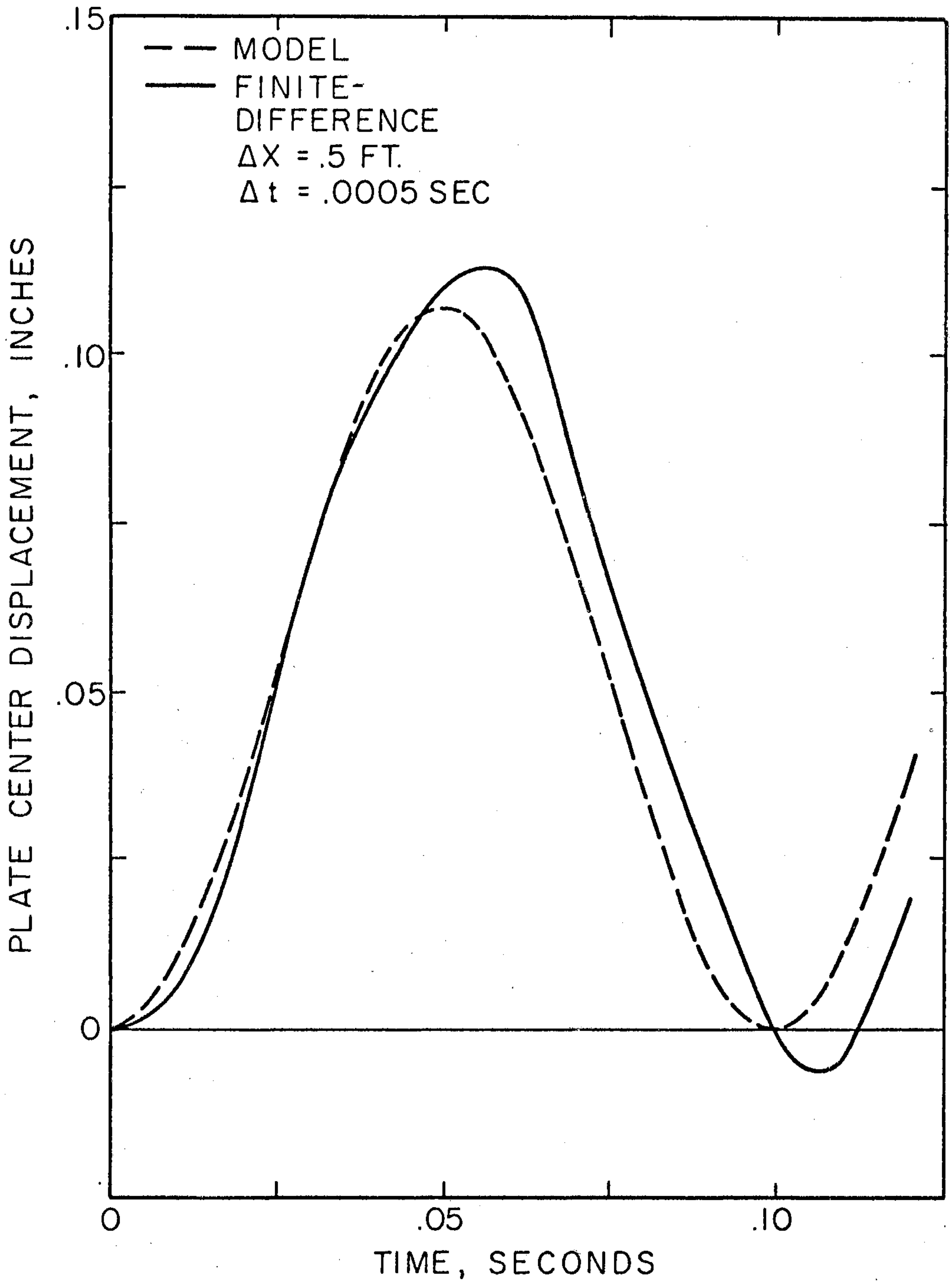


Figure 19. Displacement-Time Response for
Case II(a) and a 1 psf
Step Load

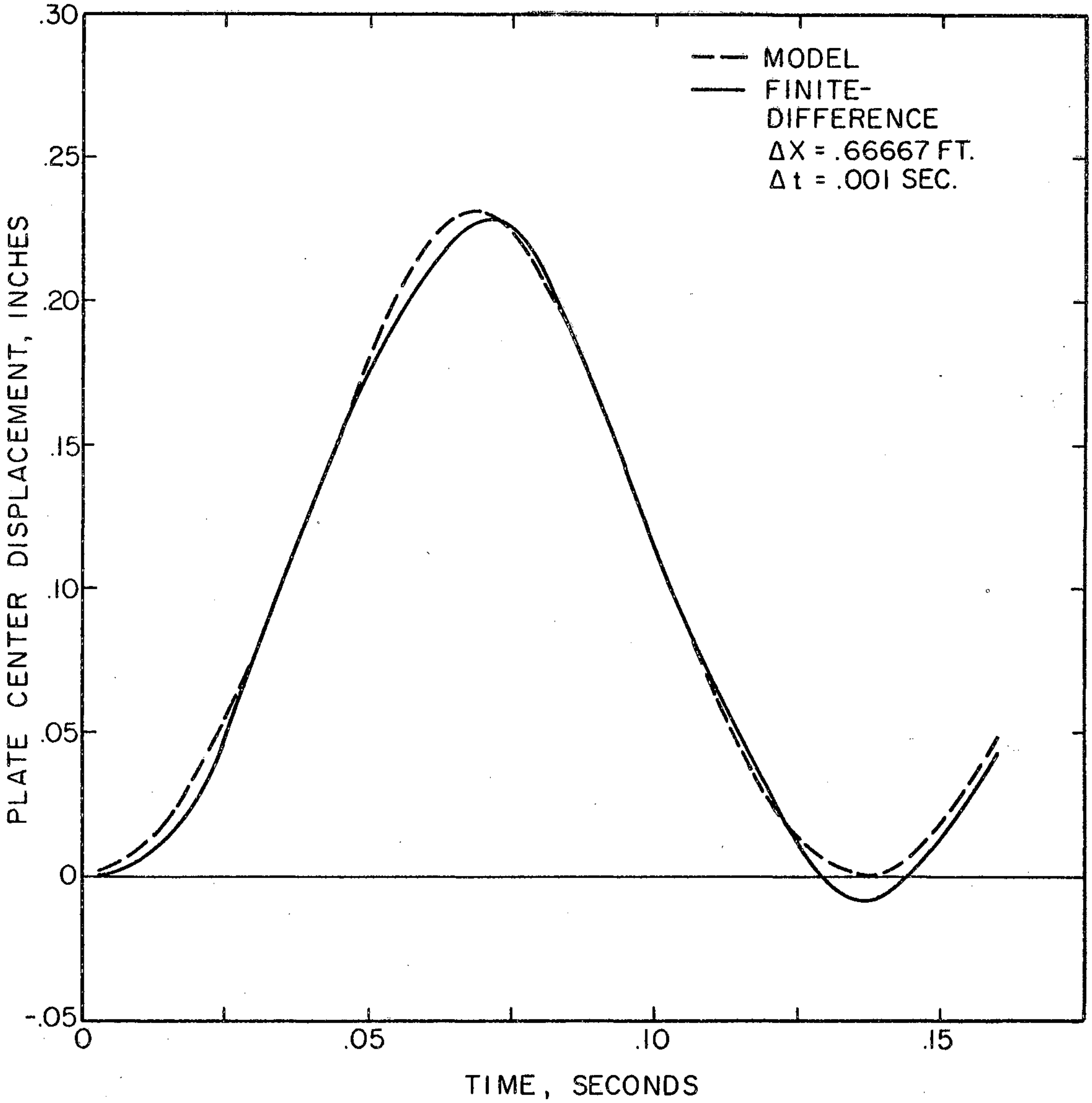


Figure 20. Displacement-Time Response for
Case I(b) and a 1 psf
Step Load

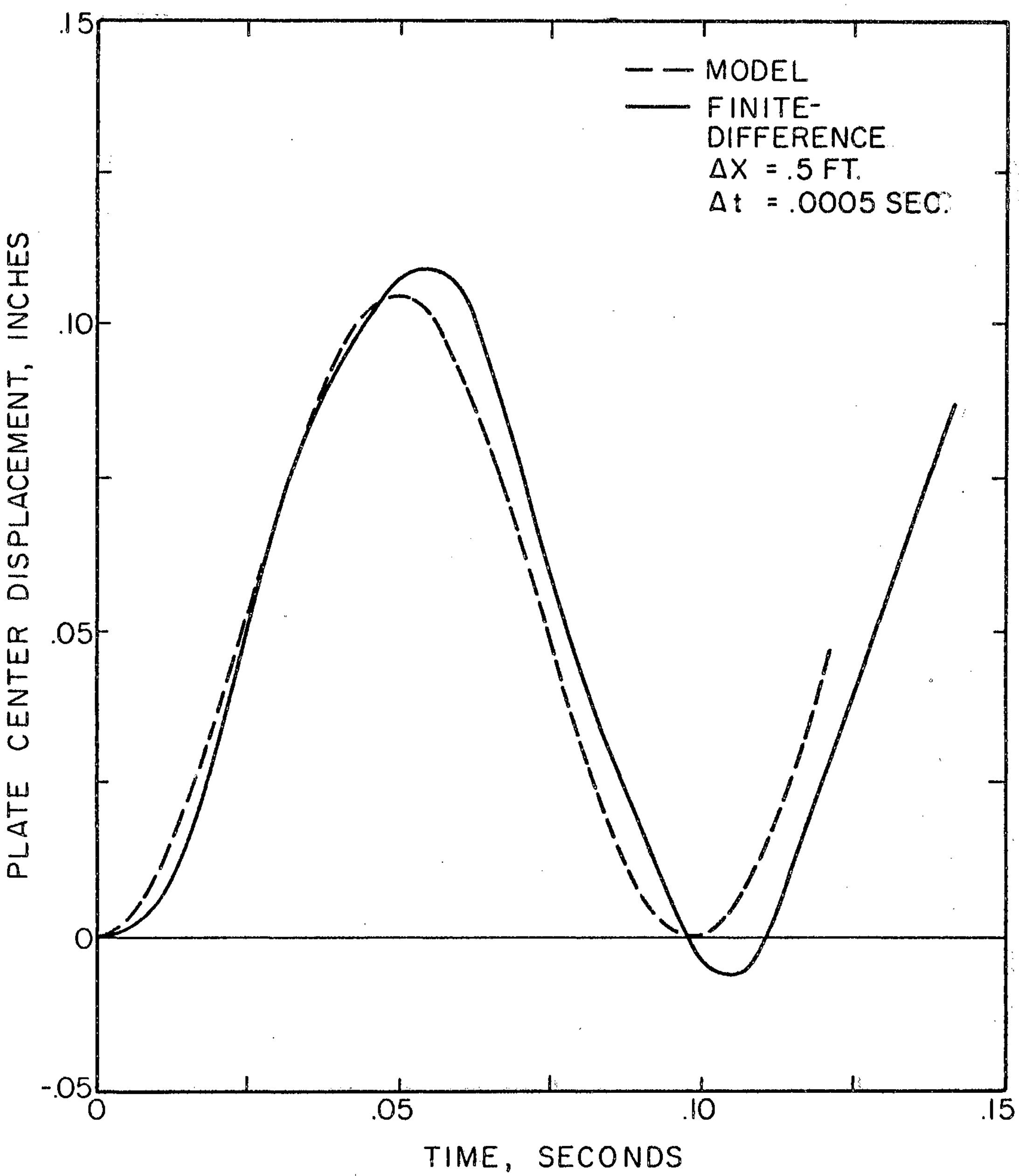


Figure 21. Displacement-Time Response for Case II(b) and a 1 psf Step Load

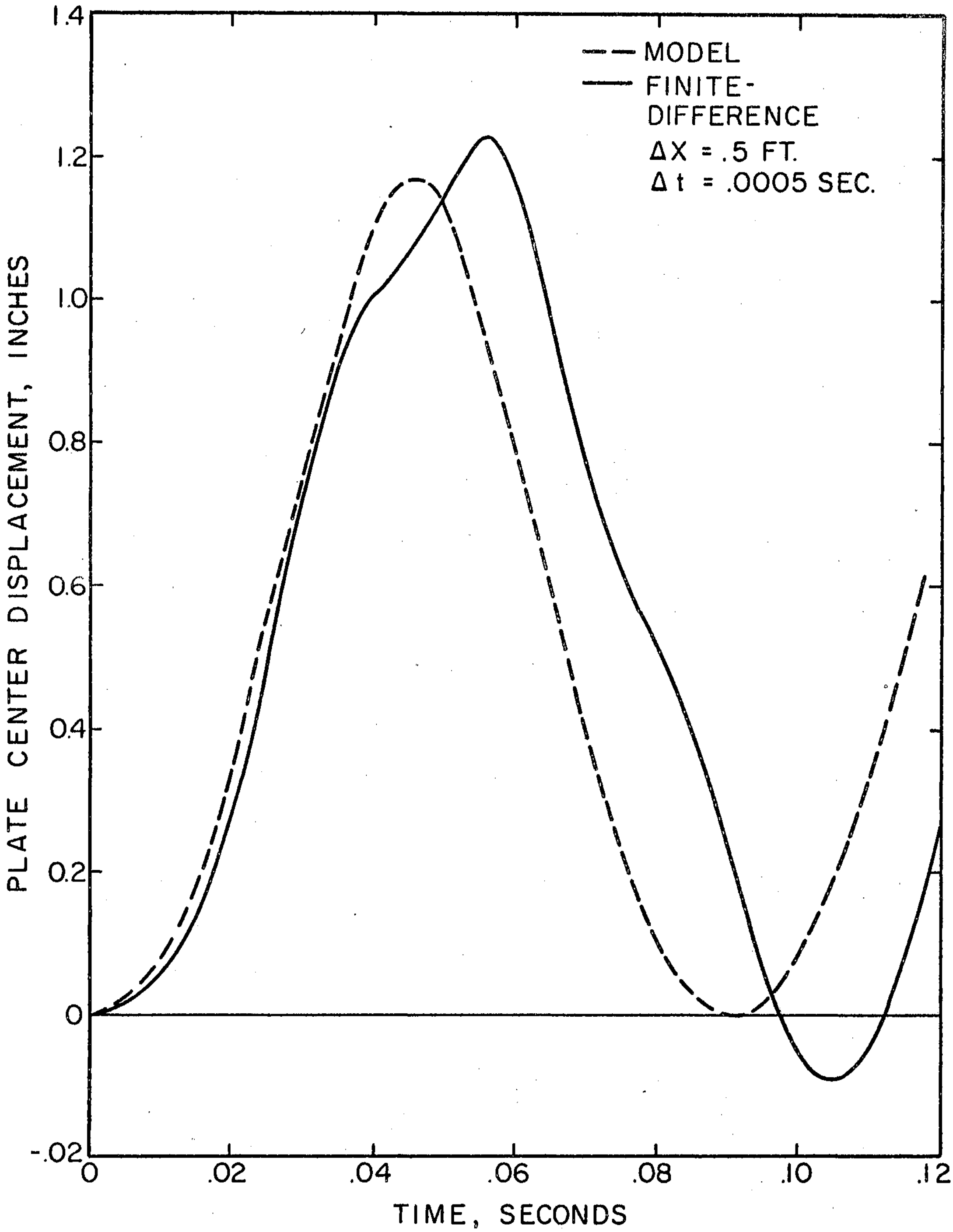


Figure 22. Displacement-Time Response for Case I(a) and a 10 psf Step Load

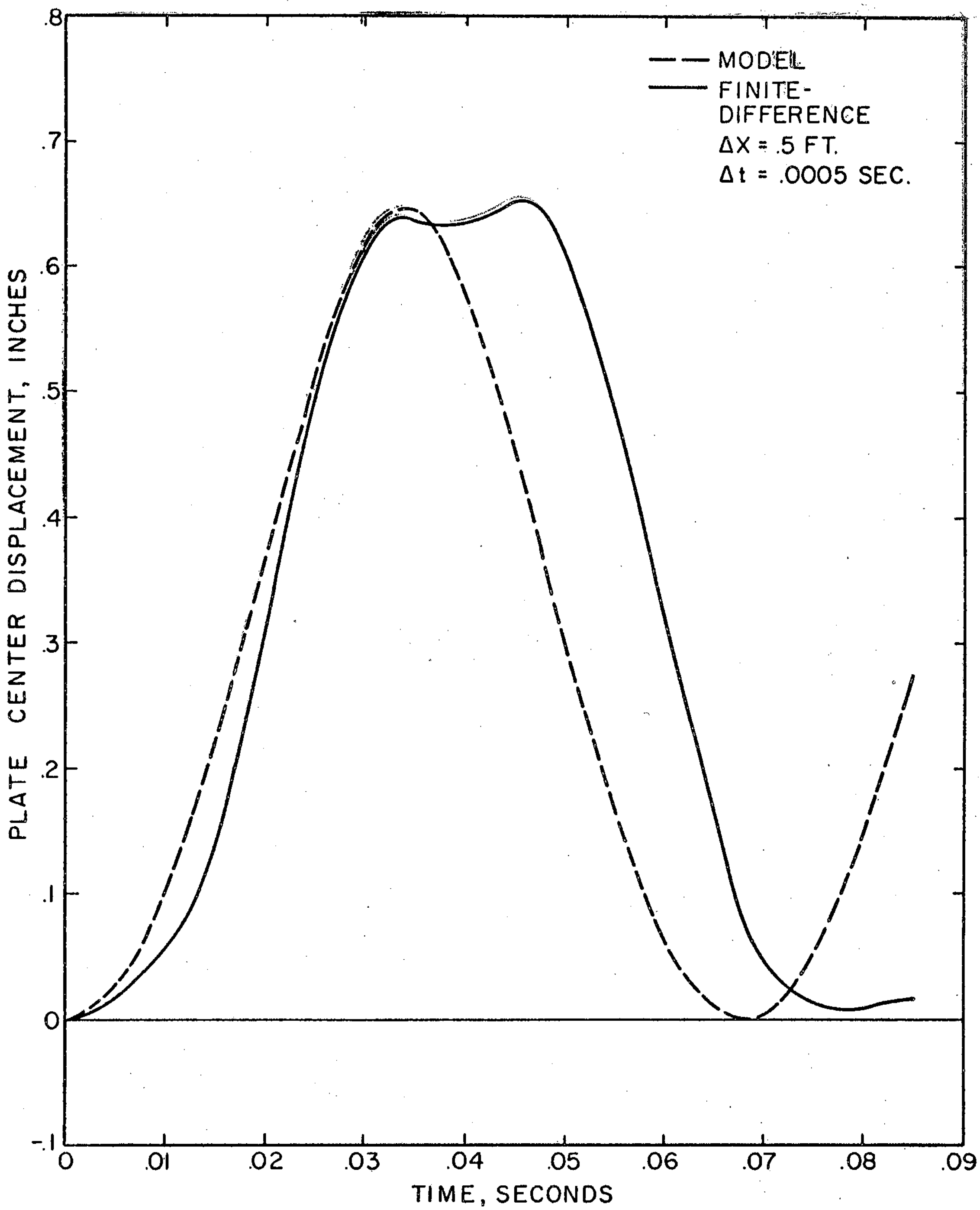


Figure 23. Displacement-Time Response for Case II(a) and a 10 psf Step Load

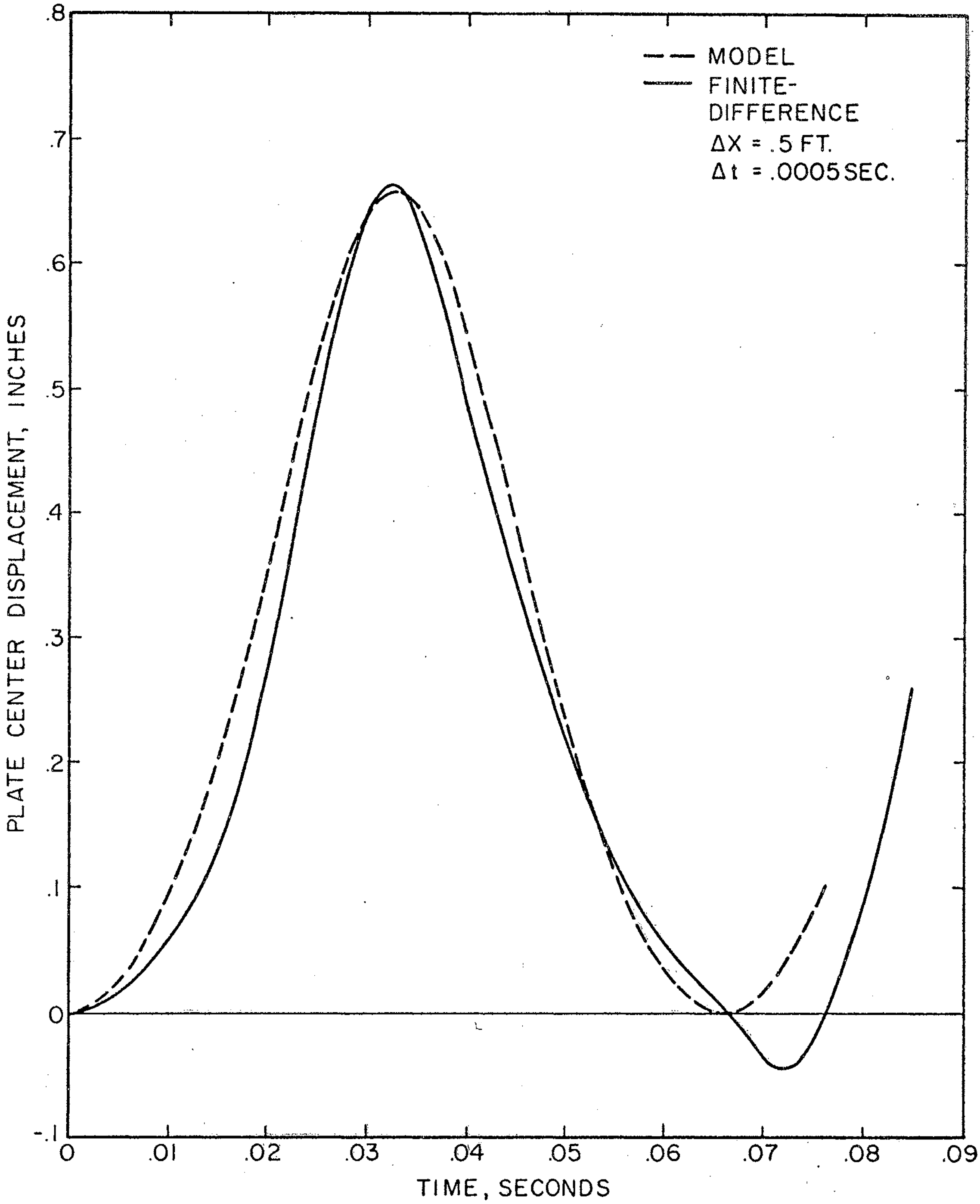


Figure 24. Displacement-Time Response for Case I(b) and a 10 psf Step Load

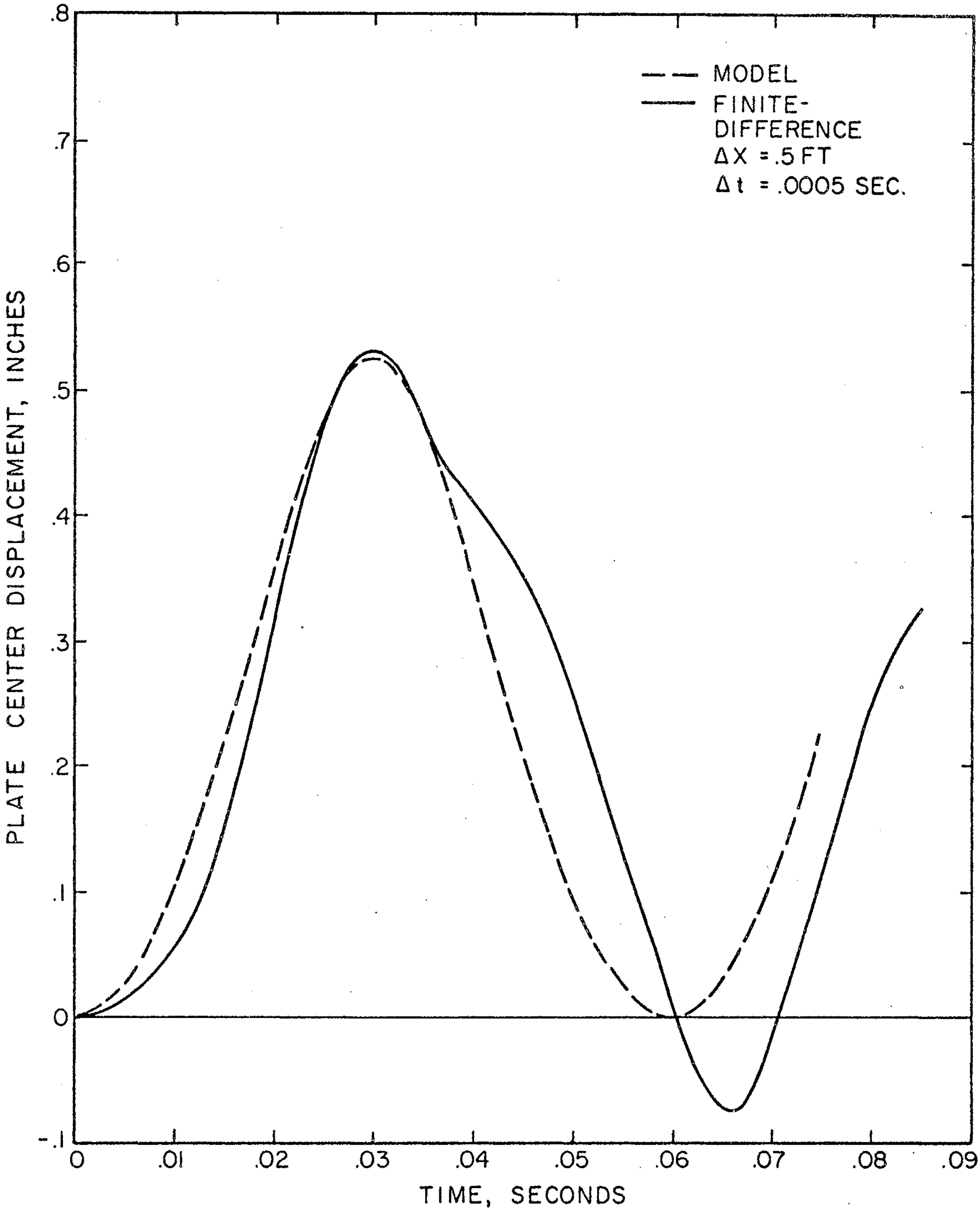


Figure 25. Displacement-Time Response for Case II(b) and a 10 psf Step Load

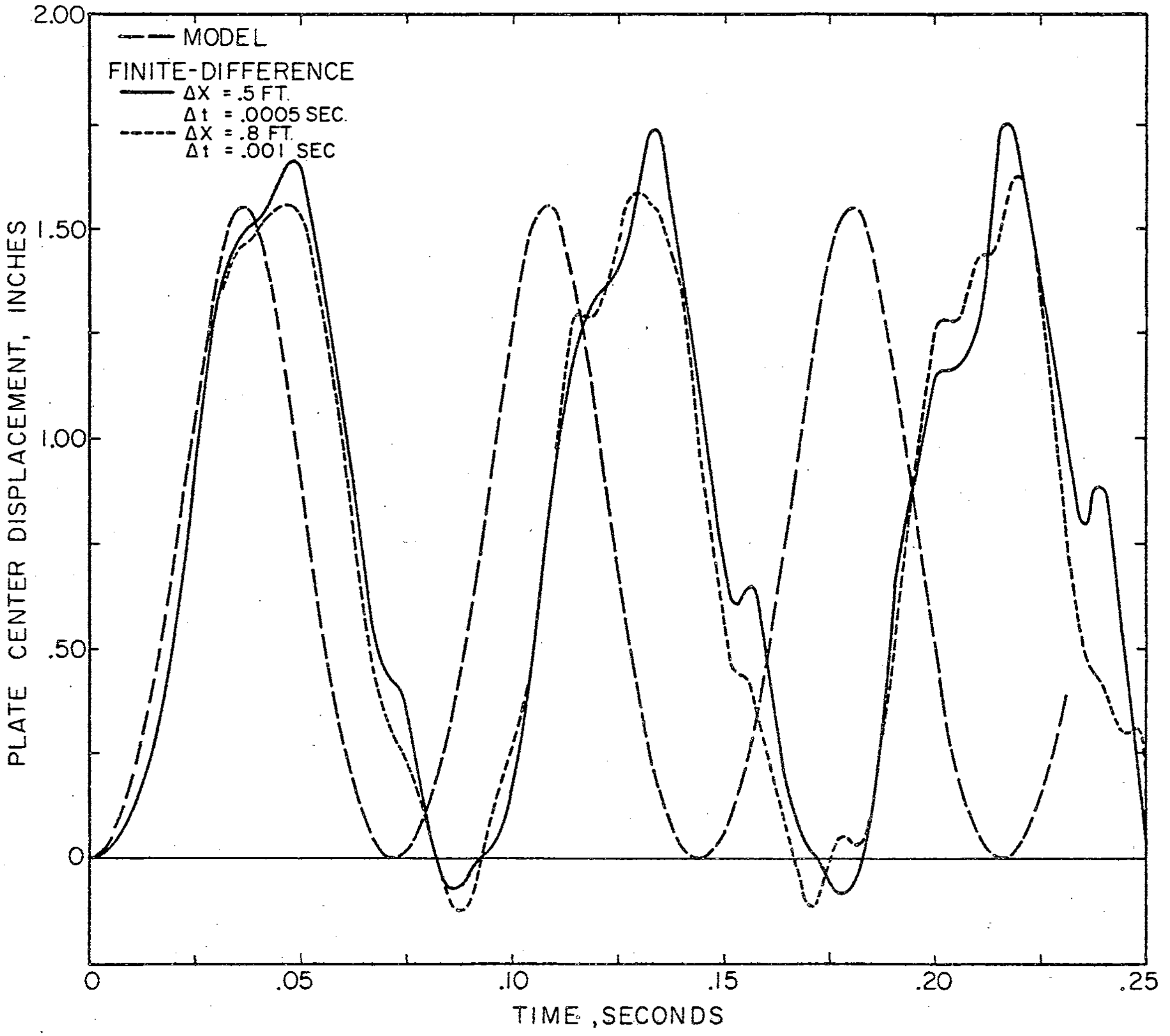


Figure 26. Displacement-Time Response for Case I(a) and a 20 psf Step Load

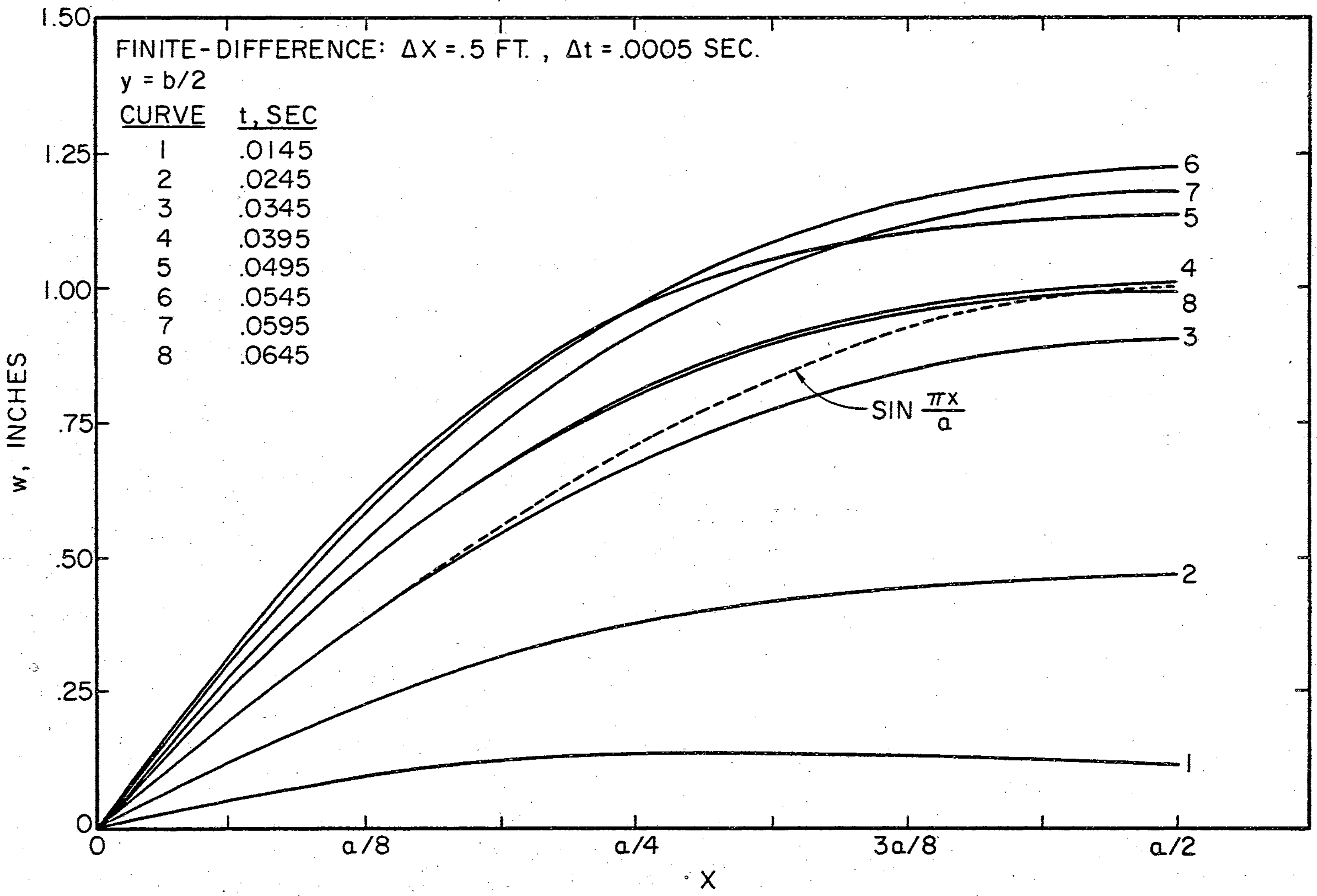


Figure 27. Deflected Shape as a Function of Time for Case I(a) and a 10 psf Step Load

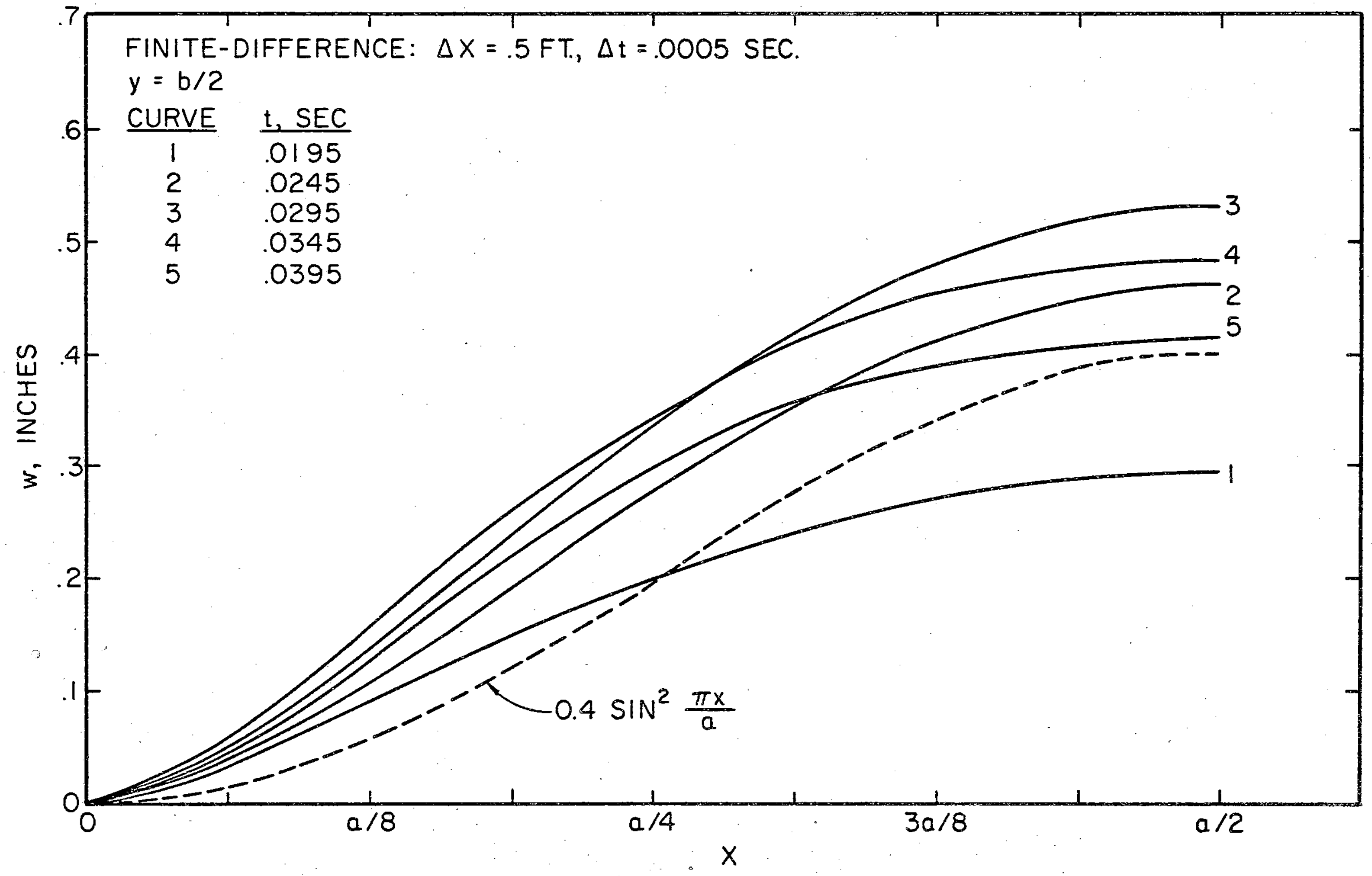


Figure 28. Deflected Shape as a Function of Time for Case II(b) and a 10 psf Step Load

TABLE II

CONVERGENCE OF FINITE-DIFFERENCE SOLUTION
FOR CASE I (a)

p = 1 psf step-function load				
Δx Feet	Δt Seconds	t Seconds	w_{Max} Inches	F_{Max} Pounds
.80000	.0014	.0840	.30402	-58,618
.80000	.0010	.0830	.30421	-58,670
.66667	.0010	.0830	.30357	-57,447
.66667	.0007	.0833	.30364	-57,478
.57143	.0007	.0840	.30394	-56,755
.57143	.0005	.0840	.30396	-56,762
.50000	.0005	.0835	.30404	-56,299
.50000	.0004	.0836	.30406	-56,302
.44444	.0004	.0836	.30429	-55,966
.44444	.0003	.0837	.30427	-55,968

TABLE III

CONVERGENCE OF FINITE-DIFFERENCE SOLUTION FOR CASE II (b)

p = 10 psf step-function load				
Δx Feet	Δt Seconds	t Seconds	w_{Max} Inches	F_{Max} Pounds
.80000	.0014	.0308	.55504	-1,278,300
.80000	.0010	.0300	.55503	-1,280,200
.66667	.0010	.0300	.54120	-1,267,300
.66667	.0007	.0301	.54111	-1,269,100
.57143	.0007	.0301	.53330	-1,263,200
.57143	.0005	.0300	.53330	-1,263,300
.50000	.0005	.0300	.53055	-1,262,400
.50000	.0004	.0300	.53031	-1,262,500
.44444	.0004	.0296	.52769	-1,262,800
.44444	.0003	.0296	.52778	-1,262,900

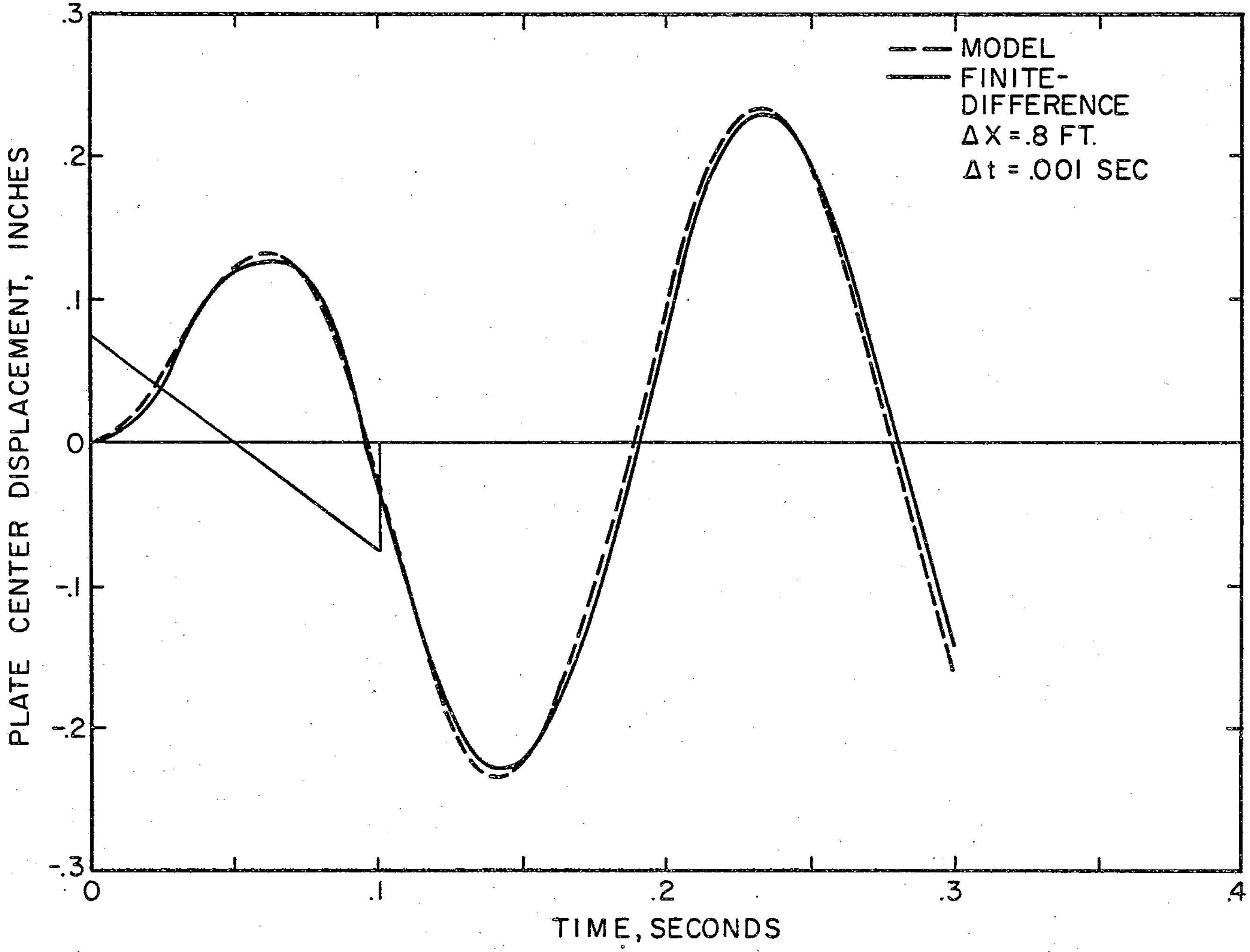


Figure 29. Displacement-Time Response for Case I(a) and a 1 psf by 0.1 Second N-Wave Load

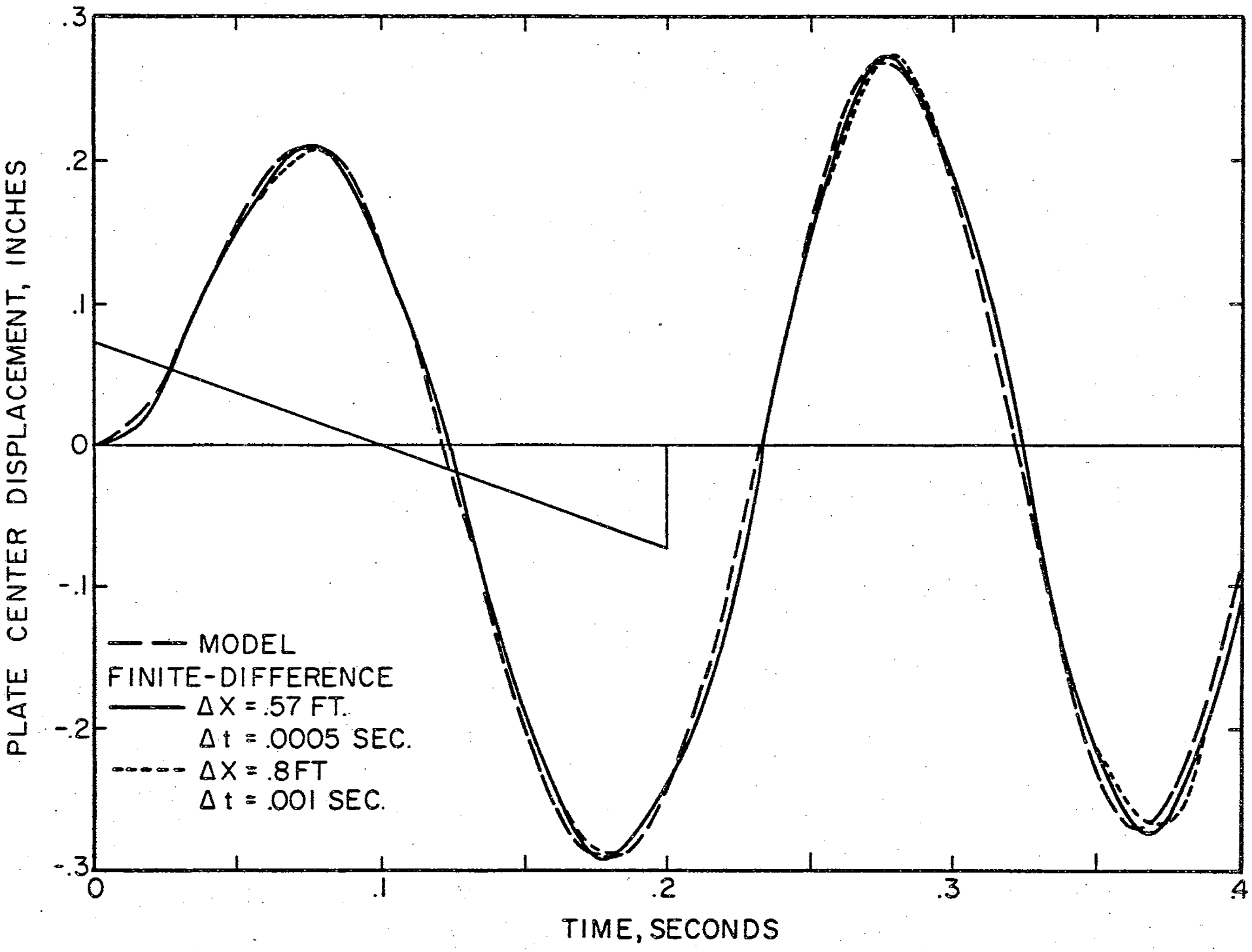


Figure 30. Displacement-Time Response for Case I(a) and a 1 psf by 0.2 Second N-Wave Load

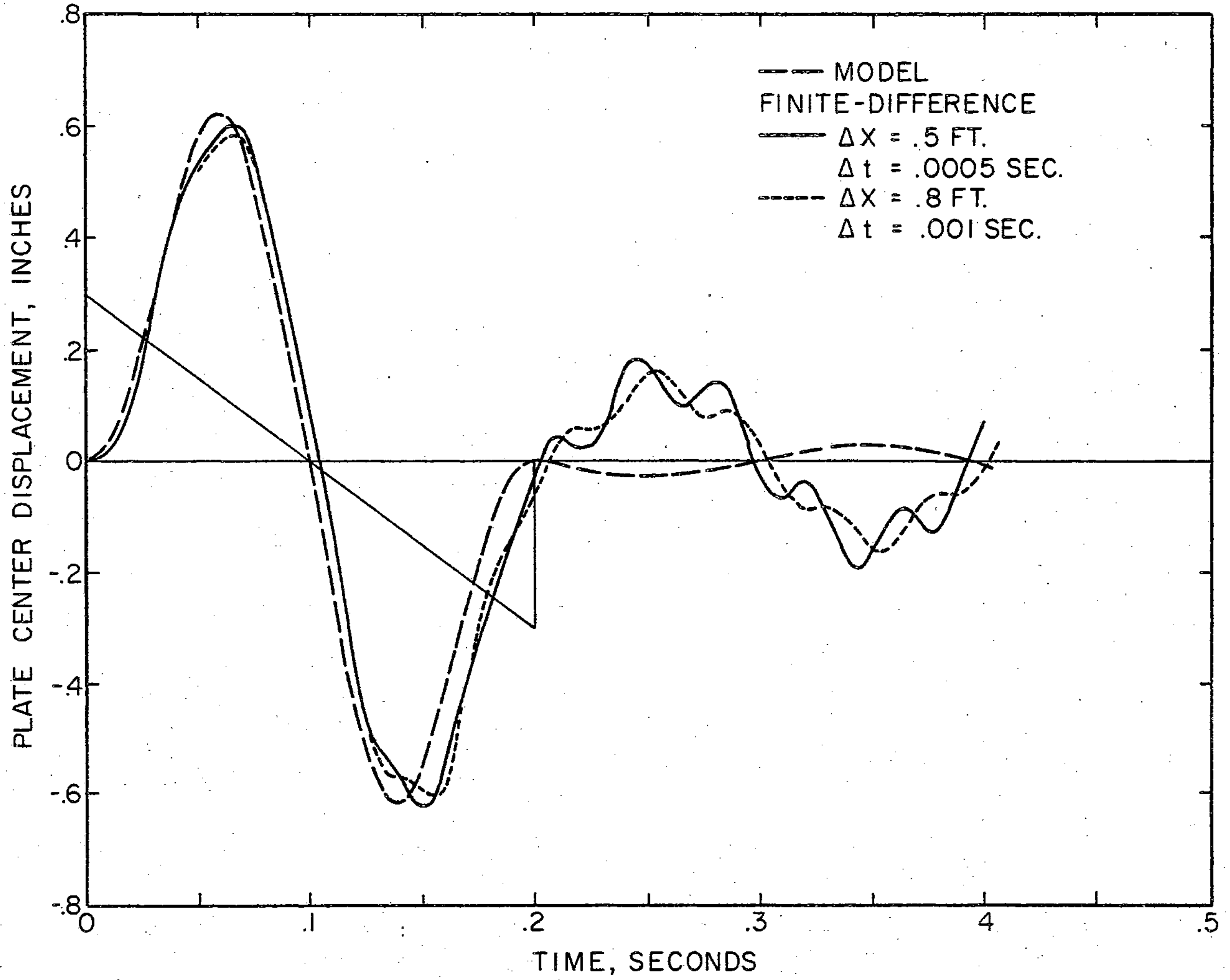


Figure 31. Displacement-Time Response for Case I(a) and a 4 psf by 0.2 Second N-Wave Load

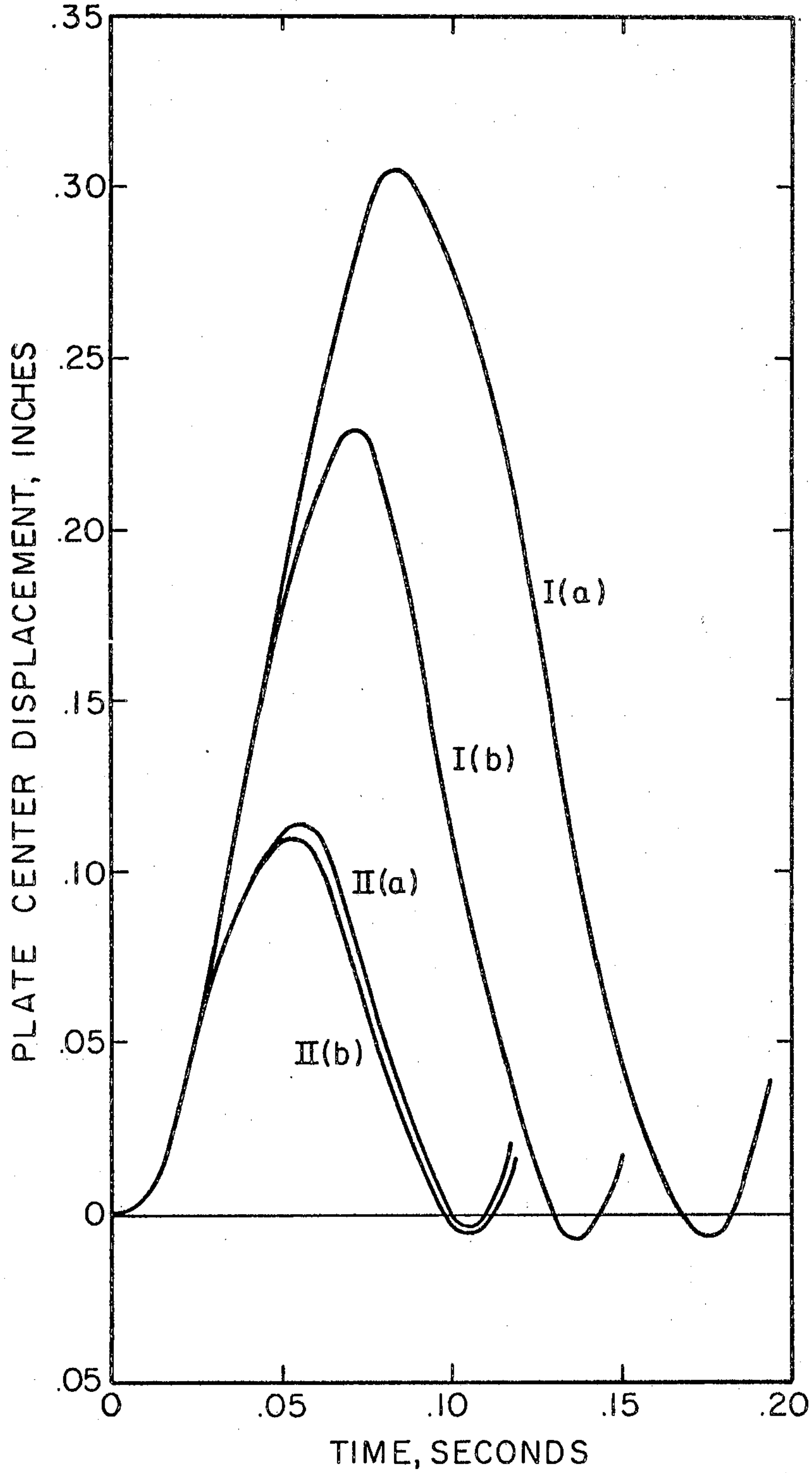


Figure 32. Finite-Difference Solutions for a
1 psf Step Load

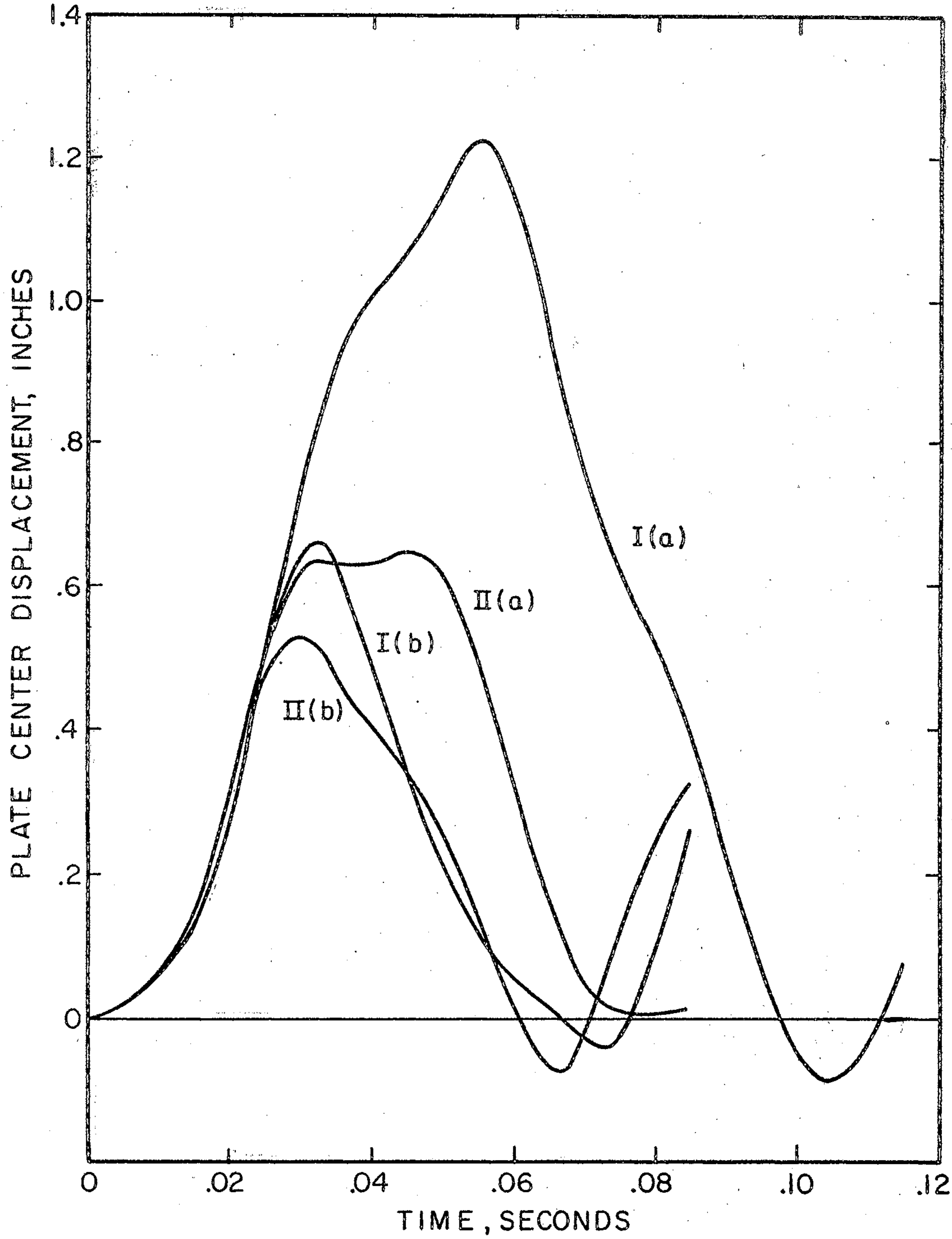


Figure 33. Finite-Difference Solutions for a 10 psf Step Load

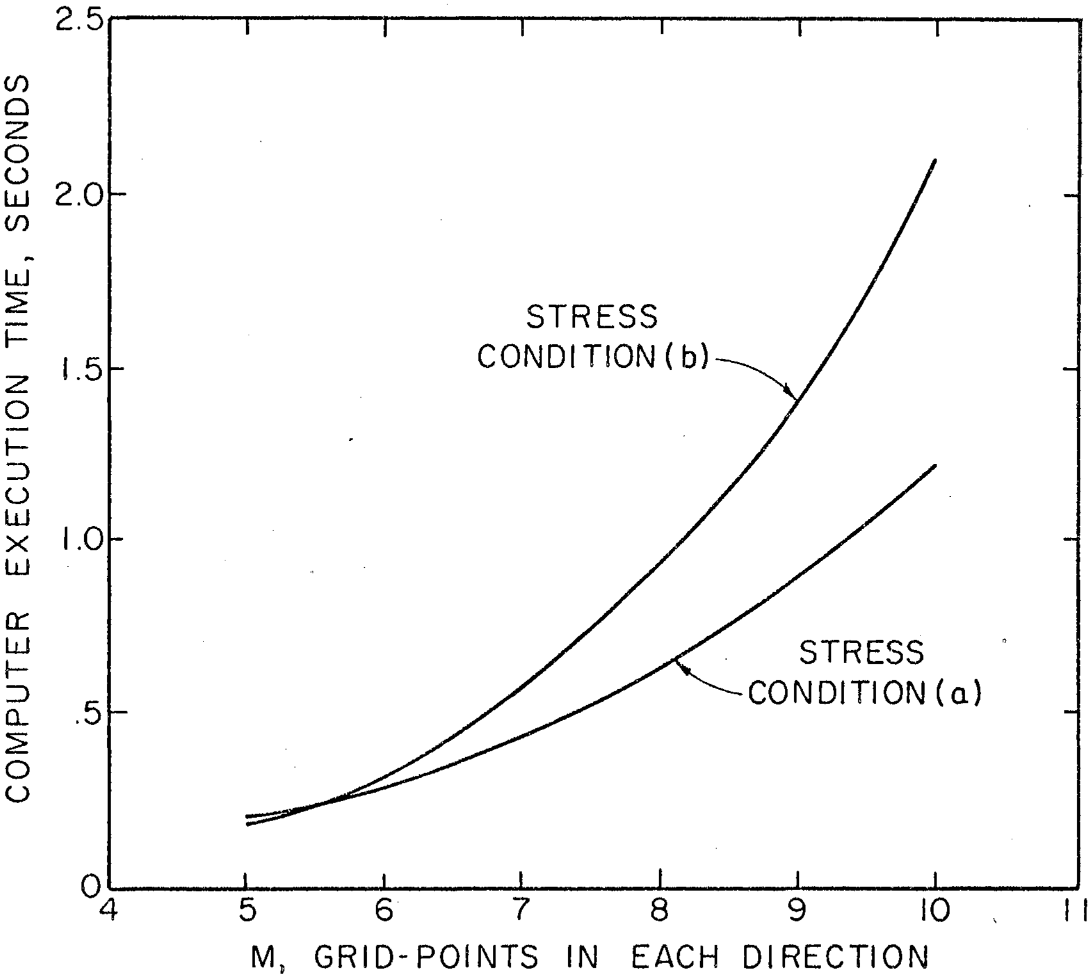


Figure 34. Computer Time Required to Execute One Time-Step

CHAPTER IV

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

A numerical method was developed for determining the nonlinear dynamic responses of thin elastic rectangular plates subjected to pulse-type uniform pressure loads. The nonlinear plate theory used may be identified as the dynamic von Kármánn theory. The numerical method was based on finite-difference approximations of the differential equations using central-difference formulas. A special form of Gaussian elimination was used to solve the system of algebraic equations resulting from the finite-difference method. A stability criterion for the method was derived and checked numerically, and the convergence of the method was demonstrated numerically.

Four sets of boundary conditions were considered. Fortran computer programs were written and are included in the appendix. The use of the method was demonstrated by specific example problems, and the results compared with other approximate solutions.

The following conclusions are made from this study:

1. The numerical method presented provides an accurate and efficient approximate solution to the problem. The programs should be useful for design and for future research studies.

2. The approximate solutions obtained by this method are the most accurate approximations available. They should be useful as a check on other approximate solutions.

3. The stability criterion developed is adequate.

4. The necessary grid-sizes and time-steps depend on the particular problem. For many cases the method converges rapidly and rather large grid-sizes and time-steps are adequate.

5. The numerical solutions reveal that the accuracy of the other approximate solutions is dependent on the problem and/or the amplitude of the response.

6. This method may be extended to other boundary conditions and/or plate theories. This will involve program changes, but the current programs should provide a very useful groundwork to build from.

The following recommendations are made for further study:

1. The finite-difference method may be improved by:

(a) Using a predictor-corrector technique to permit a larger time-step.

(b) Using higher order difference approximations to permit a larger grid-spacing.

Each of these are feasible and should not be too difficult. The boundary conditions and programming could be handled in much the same way as in this study. It is possible that both accuracy and efficiency could be improved.

2. Finite-difference methods, in conjunction with the programs developed here, may be developed to determine the stress and strain conditions in the plate. The accuracy will be less than for the displacement, but it may still be the most accurate method available.

3. The finite-difference method should provide an accurate solution to the differential equations for any reasonable magnitude of deflection, but the theory is only valid for moderately large

deflections. Further research to determine the limit of the theory would be valuable.

4. It appears at this time that the development of plate theories is more advanced than the mathematics necessary to carry out solutions. The finite-difference method may be used to solve the differential equations of more sophisticated plate theories. It may be possible to compare and evaluate theories and assumptions by this method when no other method is possible.

5. The ultimate test of both the plate theory and the solution is the comparison with actual tests of the physical system. The finite-difference solutions should be checked with experimental data. It appears that additional experimental work is necessary for this purpose.

6. The approximate solution resulting in a lumped-parameter model of the plate is a significant simplification of the problem. An improvement of these models is needed. The finite-difference solutions should be useful in developing and evaluating improved models.

BIBLIOGRAPHY

(1) Bauer, H. F. "Nonlinear Response of Elastic Plates to Pulse Excitations." Journal of Applied Mechanics, (March, 1968), pp. 47-52.

(2) Bowles, R. and B. Sugarman. "The Strength and Deflection Characteristics of Large Rectangular Glass Panels Under Uniform Pressure." Glass Technology, Vol. 3 (1962), pp. 156-170.

(3) Chu, H. N. and G. Herrmann. "Influence of Large Amplitudes on Free Flexural Vibrations of Rectangular Elastic Plates." Journal of Applied Mechanics, Vol. 78 (1956), pp. 532-540.

(4) Crandall, S. H. Engineering Analysis. McGraw Hill Book Company, Inc., 1956.

(5) Freynik, H. S. "The Nonlinear Response of Windows to Random Noise." NASA Technical Note, No. D-2025 (1963).

(6) Herrmann, G. "Influence of Large Amplitudes on Flexural Motions of Elastic Plates." NACA Technical Note, No. 3578 (1955).

(7) IBM Application Program H20-0205-2, "System/360 Scientific Subroutine Package (360A-CM-03X) Version II Programmer's Manual," International Business Machines Corp., White Plains, New York, 1966.

(8) Leech, J. W. "Stability of Finite-Difference Equation for the Transient Response of a Plate." AIAA Journal, Technical Note, Vol. 3, No. 9 (1965), pp. 1772-1773.

(9) McCormick, J. M. and M. G. Salvadori. Numerical Methods in Fortran. Prentice-Hall, Inc., 1964.

(10) McCracken, D. D. and W. S. Dorn. Numerical Methods and Fortran Programming. John Wiley and Sons, Inc., 1964.

(11) Seshadri, T. V. "Transient Response of Mechano-Acoustical Networks." (Ph.D. Thesis, Oklahoma State University, May, 1968.)

(12) Tadjbakhsh, I. and E. Saibel. "On the Large Elastic Deflections of Plates." ZAMM, Vol. 40 (1960), pp. 259-268.

(13) Timoshenko, S. and S. Woinowsky - Kreiger. Theory of Plates and Shells. McGraw Hill Book Company, Inc., 1959.

(14) Whitehouse, G. D. "Coupled and Uncoupled Panel Response to Sonic Boom Type Inputs." (Ph.D. Thesis, Oklahoma State University, May, 1967.)

(15) Yamaki, N. "Influence of Large Amplitudes on Flexural Vibrations of Elastic Plates." ZAMM, Vol 41 (1961), pp. 501-510.

APPENDIX

FORTRAN PROGRAMS

This appendix contains sufficient Fortran programs for a solution to Equations (1-1) and (1-2), for each of the boundary, symmetry, and initial conditions given by Equations (1-3), (1-6), and (1-7). The development of the numerical methods and the flow charts for programming are included in the main body of this thesis.

A study of the thesis should be made before attempting to use these programs. The programming was made as simple and straightforward as possible, and everything necessary for successful use of the programs should be obvious once the reader has become familar with the method and with the additional information given here.

The main program should generally be tailored to a specific problem. The main programs included here are only intended as samples that may be a useful guide. Basically, the main program should set up the parameters of the problem, set up the initial conditions, calculate the starting values, calculate the load at each time-step, calculate the time, re-identify the time variables after each ten time-steps, call the subroutines, and write out the desired data. The subroutines may be used without change for any problem with the boundary and symmetry conditions specified.

As far as possible the variable names in the programs correspond to the variable names in the text of the thesis. The exceptions are identified in Table IV, along with the units for all the parameters.

The subroutines are set up for particular boundary conditions. Subroutine COEFA sets up the coefficient matrix of Equation (2-45) for stress condition (a), and Subroutine COEFB is for stress condition (b). Subroutine AGEA performs the Gaussian Elimination scheme of Figure 10 on the coefficient matrix for stress condition (a), and Subroutine AGEB is for stress condition (b). These subroutines are needed only one time for a given problem. The Gaussian Elimination on the constant vector of Equation (2-45), and the back substitution, are included in the following subroutines. These subroutines all have FD as the first two characters of their name, and the remaining characters indicate the boundary conditions for which the subroutine is valid, such as FDIA for case I (a). These FD subroutines carry out the solutions of Equations (2-45) and (2-46) for ten time-steps, and then return to the main program.

The dimension statement in the main program for an M by N grid-system, ($M \leq N$), must include:

W(L, 12), F(L, 12), A(L, L), B(L), C(L) ,

BB(L), DC(LC), DI(LI), P(12) .

The subroutines have variable dimension statements that are set up by the parameter KR in the main program, (KR = L). For stress condition (a),

$$M \geq 5$$

$$L = M \cdot N$$

$$LC = 2 \cdot M \cdot M \cdot (N-1) - M$$

LI - (DI is not used)

For stress condition (b),

$$M \geq 4$$

$$L = M \cdot N + 2$$

$$LC = 3 \cdot (M \cdot M \cdot (2 \cdot N - 3) + 3 \cdot M) \div 2$$

$$LI = M \cdot N + 1$$

All of the programs are in double precision. No error messages are included.

TABLE IV

DIMENSIONAL UNITS FOR PROGRAM VARIABLES

Variable	Name	Units
AX	Plate Dimension in x-direction	feet
BY	Plate Dimension in y-direction	feet
H	Plate Thickness	inches
E	Modulus of Elasticity	lbs/sq. inch
PR	Poisson's Ratio	-
SW	Specific Weight	lbs/cu. foot
DX	Grid-Spacing (DX = $\Delta x = \Delta y$)	feet
DT	Time-Step	seconds
TS	Stop-Time	seconds
P	Load	lbs/sq. foot
T	Time	seconds
W	Transverse Displacement	inches
F	Stress-Function	lbs.

```
C   MAIN PROGRAM FOR CASE I(A) AND AN N-WAVE LOAD
       IMPLICIT REAL*8 (A-H,O-Z)
       DIMENSION W(100,12),F(100,12),A(100,100),B(100),C(100),BB(100)
      1  ,DC(1800),P(12)
       KR=100
  100 FORMAT (6D10.3)
  101 FORMAT (2I5,2D15.6)
  110 FORMAT (1X,F7.5)
  111 FORMAT (1X,10D13.5)
  115 FORMAT (1H1,20X,'CASE I(A),  SIMPLY SUPPORTED, STRESS FREE EDGES')
  120 FORMAT      (/,5X,'AX = ',F6.2,' FT',5X,'BY = ',F6.2,' FT',5X,
      1  'H = ',F7.4,' IN',5X,'E = ',D10.3,' PSI',5X,'PR = ',F5.3,5X,
      2  'SW = ',F7.2,' PCF')
  121 FORMAT  (/,5X,'N-WAVE LOAD',5X,'PL = ',F7.2,' PSF'
      1  5X,'TAU = ',F7.3,' SEC')
  122 FORMAT  (//,5X,'M = ',I2,5X,'N = ',I2,5X,'DX = ',F10.6,' FT',5X,
      1  'DT = ',F10.6,' SEC')
       WRITE (6,115)
       READ  (5,100)  AX,BY,H,E,PR,SW
       WRITE(6,120)   AX,BY,H,E,PR,SW
       READ (5,100) TS,PL,TAU
       WRITE(6,121) PL,TAU
       RO=SW/32.2D0
       D=E*H**3/(12.D0*(1.D0-PR**2))
       MC=1
     1 READ (5,101)  M,N,DX,DT
       WRITE(6,122)  M,N,DX,DT
       T=0.D0
       MN=M*N
       C1=DT**2*D/(RO*H*DX**4)
       C2=DT**2/(RO*DX**4)
       C3=DT**2*144.D0/(RO*H)
       IF(MC.EQ.M) GO TO 2
       MC=M
C   SET UP (A) MATRIX
       CALL        COEFA (A,M,N,KR)
       CALL        AGEA (A,DC,M,N,KR)
     2 CONTINUE
C   SET UP INITIAL CONDITIONS
       DO3I=1,MN
       F(I,1)=0.D0
     3 W(I,1)=0.D0
C   CALCULATE LOAD AT FIRST TIME STEP
C   N-WAVE
       P(1)=PL
C   STARTING FORMULA
       DO5I=1,MN
     5 W(I,2)=.5D0*C3*P(1)
     6 CONTINUE
C   CALCULATE LOAD FOR NEXT 10 TIME STEPS
C   N-WAVE
       IF(T.GT.TAU) GO TO 8
       DO7J=2,11
       T=T+DT
```

```
      P(J)=PL*(1.D0-2.D0*T/TAU)
      IF(T.LE.TAU) GO TO 7
      P(J)=0. D0
    7 CONTINUE
      GO TO 10
    8 DO9J=2,11
      T=T+DT
    9 P(J)=0. D0
   10 CONTINUE
C  CALCULATE W & F FOR 10 TIME STEPS
      CALL FDIA  (M,N,MN,W,F,A,B,C,BB,DC,P,C1,C2,C3,E,KR)
      TT=T-10. D0*DT
      WRITE(6,110)  TT
      WRITE(6,111) (W(MN,J),J=1,10)
      WRITE(6,111) (F(MN,J),J=1,10)
      WRITE(6,111) (W(I,10),I=M,MN,M)
      WRITE(6,111) (F(I,10),I=M,MN,M)
      DO71I=1,MN
      F(I,1)=F(I,11)
      W(I,1)=W(I,11)
   71 W(I,2)=W(I,12)
      IF(TT.LT.TS) GO TO 6
      GO TO 1
   90 STOP
      END
```

```
C  MAIN PROGRAM FOR CASE II(B) AND A STEP LOAD

      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 DABS
      DIMENSION W(102,12),F(102,12),A(102,102),B(100),C(102),BB(100),
     1  DC(2610),DI(101),P(12)
      KR=102
  100 FORMAT (6D10.3)
  101 FORMAT (2I5,2D15.6)
  110 FORMAT (1X,F7.5)
  111 FORMAT (1X,10D13.5)
  115 FORMAT(1H1,20X,'CASE II(B), CLAMPED, IMMOVABLY CONSTRAINED EDGES')
  120 FORMAT      (/,5X,'AX = ',F6.2,' FT',5X,'BY = ',F6.2,' FT',5X,
     1  'H = ',F7.4,' IN',5X,'E = ',D10.3,' PSI',5X,'PR = ',F5.3,5X,
     2  'SW = ',F7.2,' PCF')
  121 FORMAT  (/,5X,'STEP FUNCTION LOAD',5X,'PL = ',F7.2,' PSF')
  122 FORMAT  (//,5X,'M = ',I2,5X,'N = ',I2,5X,'DX = ',F10.6,' FT',5X,
     1  'DT = ',F10.6,' SEC')
      WRITE (6,115)
      READ  (5,100)  AX,BY,H,E,PR,SW
      WRITE(6,120)   AX,BY,H,E,PR,SW
      READ (5,100)  TS,PL
      WRITE(6,121) PL
      RO=SW/32.2 D0
```

```
      D=E*H**3/(12.D0*(1.D0-PR**2))
      MC=1
    1 READ (5,101)  M,N,DX,DT
      WRITE(6,122)  M,N,DX,DT
      T=0.D0
      MN=M*N
      MN2=MN+2
      C1=DT**2*D/(RO*H*DX**4)
      C2=DT**2/(RO*DX**4)
      C3=DT**2*144.D0/(RO*H)
      IF(MC.EQ.M) GO TO 2
      MC=M
C   SET UP  (A)  MATRIX
      CALL        COEFB (A,M,N,PR,KR)
      CALL        AGEB (A,DC,DI,M,N,KR)
    2 CONTINUE
C   SET UP INITIAL AND BOUNDARY CONDITIONS
      DO3I=1,MN
      F(I,1)=0. D0
      DO3J=1,12
    3 W(I,J)=0. D0
C   CALCULATE LOAD AT FIRST TIME STEP
C   STEP LOAD
      P(1)=PL
C   FORMULA FOR STARTING SOLUTION
      N1=N-1
      DO5K=1,N1
      KM=K*M
      DO5L=2,M
      I=KM+L
    5 W(I,2)=.5D0*C3*P(1)
C   CALCULATE LOAD FOR NEXT 10 TIME STEPS
C   STEP LOAD
      DO6J=2,11
    6 P(J)=PL
   10 CONTINUE
C   CALCULATE W & F FOR 10 TIME STEPS
      CALL   FDIIB      (M,N,MN,W,F,A,B,C,BB,DC,DI,P,C1,C2,C3,E,KR)
      WRITE(6,110)  T
      WRITE(6,111) (W(MN,J),J=1,10)
      WRITE(6,111) (F(MN,J),J=1,10)
      WRITE(6,111) (W(I,10),I=M,MN,M)
      WRITE(6,111) (F(I,10),I=M,MN,M)
      DO71I=1,MN
      F(I,1)=F(I,11)
      W(I,1)=W(I,11)
   71 W(I,2)=W(I,12)
      T=T+10. D0*DT
      IF(T.LT.TS) GO TO 10
      GO TO 1
   90 STOP
      END
```

```
C   SUBROUTINE COEFA, SETS UP A-MATRIX FOR STRESS FREE EDGES
      SUBROUTINE COEFA (A,M,N,KR)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(KR,1)
      MN=M*N
      DO1I=1,MN
      DO1J=1,MN
    1 A(I,J)=0.D0
      DO2K=1,MN
    2 A(K,K)=20.D0
      A(1,1)=22.D0
      L=M-2
      DO3K=2,L
    3 A(K,K)=21.D0
      A(M-1,M-1)=22.D0
      A(M,M)=21.D0
      L=MN-3*M
      DO4K=M,L,M
      A(K+1,K+1)=21.D0
      KK=K+M-1
    4 A(KK,KK)=21.D0
      L1=(N-2)*M+2
      A(L1-1,L1-1)=22.D0
      L=L1+M-1
      DO5K=L1,L
    5 A(K,K)=21.D0
      A(MN-M-1,MN-M-1)=22.D0
      A(MN-1,MN-1)=21.D0
      DO6K=2,MN
      A(K,K-1)=-8.D0
    6 A(K-1,K)=-8.D0
      MNM=MN-M
      DO7K=M,MNM,M
      A(K+1,K)=0.D0
    7 A(K,K+1)=0.D0
      DO8K=3,MN
      A(K,K-2)=1.D0
    8 A(K-2,K)=1.D0
      DO9K=M,MNM,M
      A(K+1,K-1)=0.D0
      A(K+2,K)=0.D0
      A(K-1,K+1)=0.D0
    9 A(K,K+2)=0.D0
      DO10K=M,MN,M
      A(K,K-1)=-16.D0
   10 A(K,K-2)=2.D0
      M1=M+1
      DO11K=M1,MN
      KM=K-M
      A(K,KM)=-8.D0
      A(KM,K)=-8.D0
      A(K-1,KM)=2.D0
   11 A(KM,K-1)=2.D0
      MN1=MN-1
```

```
      DO12K=M1,MN1
      A(K+1,K-M)=2.D0
   12 A(K-M,K+1)=2.D0
      MN1=MN+1
      DO13K=M1,MN1,M
      KM=K-M
      A(K-1,KM)=0.D0
   13 A(KM,K-1)=0.D0
      M2=2*M
      DO14K=M2,MNM,M
      KM=K-M
      A(K+1,KM)=0.D0
   14 A(KM,K+1)=0.D0
      M21=M2+1
      DO15K=M21,MN
      KM2=K-M2
      A(K,KM2)=1.D0
   15 A(KM2,K)=1.D0
      DO16K=M2,MN,M
      A(K,K-M-1)=4.D0
   16 A(K-M,K-1)=4.D0
      L=MN-M+1
      DO17K=L,MN
      KM=K-M
      A(K,KM)=-16.D0
      A(K,KM+1)=4.D0
      A(K,KM-1)=4.D0
   17 A(K,KM-M)=2.D0
      A(L,MN-M2)=0.D0
      A(MN,MN-M-1)=8.D0
      A(MN,L)=0.D0
      RETURN
      END


C   SUBROUTINE COEFB, SETS UP A-MATRIX FOR IMMOVABLY CONSTRAINED EDGES

      SUBROUTINE COEFB (A,M,N,PR,KR)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(KR,1)
      MN=M*N
      MN2=MN+2
      M2=2*M
      M3=3*M
      M1=M+1
      MP2=M+2
      MP3=M+3
      MP4=M+4
      M21=M2+1
      M22=M2+2
      M31=M3+1
```

```
      M32=M3+2
      MNM=MN-M
      DO1I=1,MN2
      DO1J=1,MN2
    1 A(I,J)=0.D0
      DO2K=MP2,MN
    2 A(K,K)=20.D0
      A(MP2,MP2)=22.D0
      DO3K=MP3,M2
    3 A(K,K)=21.D0
      A(M2-1,M2-1)=22.D0
      L=MN-M3
      DO4K=M2,L,M
      A(K+2,K+2)=21.D0
      KK=K+M-1
    4 A(KK,KK)=21.D0
      L1=(N-2)*M+2
      A(L1,L1)=22.D0
      L=L1+M-3
      DO5K=L1,L
    5 A(K+1,K+1)=21.D0
      A(L,L)=22.D0
      A(L+3,L+3)=21.D0
      A(MN-1,MN-1)=21.D0
      A(MP2,M1)=-8.D0
      DO6K=MP3,MN
      A(K,K-1)=-8.D0
    6 A(K-1,K)=-8.D0
      DO7K=M2,MNM,M
    7 A(K,K+1)=0.D0
      A(MP3,M+1)=1.D0
      DO8K=MP4,MN
      A(K,K-2)=1.D0
    8 A(K-2,K)=1.D0
      DO9K=M2,MNM,M
      A(K+2,K)=0.D0
      A(K-1,K+1)=0.D0
    9 A(K,K+2)=0.D0
      DO10K=M2,MN,M
      A(K,K-1)=-16.D0
   10 A(K,K-2)=2.D0
      DO11K=2,M
      KM=K+M
      A(KM,K)=-8.D0
      A(KM,K+1)=2.D0
   11 A(KM,K-1)=2.D0
      DO12K=M22,MN
      KM=K-M
      A(K,KM)=-8.D0
      A(KM,K)=-8.D0
      A(K-1,KM)=2.D0
   12 A(KM,K-1)=2.D0
      A(M22,M1)=2.D0
      MN1=MN-1
      DO13K=M22,MN1
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```
      A(K+1,K-M)=2.D0
   13 A(K-M,K+1)=2.D0
      DO14K=M2,MNM,M
      A(K,K+M+1)=0.D0
      A(K,K-M+1)=0.D0
      A(K,K+M-1)=4.D0
   14 A(K,K-M-1)=4.D0
      DO15K=2,M
   15 A(M2+K,K)=1.D0
      DO16K=M32,MN
      KM2=K-M2
      A(K,KM2)=1.D0
   16 A(KM2,K)=1.D0
      L=MN-M+2
      DO17K=L,MN
      KM=K-M
      A(K,KM)=-16.D0
      A(K,KM+1)=4.D0
      A(K,KM-1)=4.D0
   17 A(K,KM-M)=2.D0
      A(MN,MN-M-1)=8.D0
      A(MN,MN-M+1)=0.D0
      DO20I=M2,MNM,M
      DO20J=1,MN
   20 A(I+1,J)=0.D0
      A(1,1)=1.D0
      L=N-2
      A(2,1)=-2.D0
      A(2,2)=2.D0
      DO21J=1,L
      K=J*M
      A(2,K+1)=-4.D0
   21 A(2,K+2)=4.D0
      A(2,MNM+1)=-2.D0
      A(2,MNM+2)=2.D0
      DO22I=3,M
      K=MNM+I
      A(I,I-2)=1.D0
      A(I,K-2)=1.D0
      A(I,I-1)=-2.D0
      A(I,K-1)=-2.D0
      A(I,I)=1.D0
   22 A(I,K)=1.D0
      DO23I=3,M
      DO23J=1,L
      K=J*M+I
      A(I,K-2)=2.D0
      A(I,K-1)=-4.D0
   23 A(I,K)=2.D0
      A(M1,M-1)=2.D0
      A(M1,M)=-2.D0
      DO24K=M2,MNM,M
      A(M1,K-1)=4.D0
   24 A(M1,K)=-4.D0
      A(M1,MN-1)=2.D0
```

```
      A(M1,MN)=-2.D0
      L=M-2
      J=1
      A(M21,J)=-2.D0
      J=J+1
      DO25K=1,L
      A(M21,J)=-4.D0
   25 J=J+1
      A(M21,J)=-2.D0
      J=J+1
      A(M21,J)=2.D0
      J=J+1
      DO26K=1,L
      A(M21,J)=4.D0
   26 J=J+1
      A(M21,J)=2.D0
      KK=1
      MN1=MN+1
      DO29I=M31,MN1,M
      J=KK
      A(I,J)=1.D0
      A(I,J+M2)=1.D0
      J=J+1
      DO27K=1,L
      A(I,J)=2.D0
      A(I,J+M2)=2.D0
   27 J=J+1
      A(I,J)=1.D0
      A(I,J+M2)=1.D0
      J=J+1
      A(I,J)=-2.D0
      J=J+1
      DO28K=1,L
      A(I,J)=-4.D0
   28 J=J+1
      A(I,J)=-2.D0
   29 KK=KK+M
      J=KK
      A(MN2,J)=2.D0
      J=J+1
      DO30K=1,L
      A(MN2,J)=4.D0
   30 J=J+1
      A(MN2,J)=2.D0
      J=J+1
      A(MN2,J)=-2.D0
      J=J+1
      DO31K=1,L
      A(MN2,J)=-4.D0
   31 J=J+1
      A(MN2,J)=-2.D0
      DO32I=2,M1
   32 A(I,MN1)=PR
      A(2,MN2)=-2.D0*(N-1)
      DO33I=MP2,M2
```

```
   33 A(I,MN1)=-1.D0
      A(MP2,MN2)=-1.D0
      A(M21,MN1)=-2.D0*(M-1)
      DO34I=M21,MN1,M
      A(I,MN2)=PR
   34 A(I+1,MN2)=-1.D0
      A(MN2,MN2)=PR
      RETURN
      END


C   SUBROUTINE AGEA, PERFORMS GAUSSIAN ELIMINATION ON A-MATRIX AND SETS
C   UP DC-VECTOR FOR USE ON C-VECTOR, STRESS FREE EDGES

      SUBROUTINE AGEA (A,DC,M,N,KR)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(KR,1),DC(1)
      MN=M*N
      K=1
      KK=0
    1 I=K+1
      L=2*M+K
      IF(L.GT.MN)L=MN
    2 KK=KK+1
      DC(KK)=A(I,K)/A(K,K)
      A(I,K)=0.
      J=K+1
    3 A(I,J)=A(I,J)-DC(KK)*A(K,J)
      IF(J-L)4,5,30
    4 J=J+1
      GOTO3
    5 IF(I-L)6,7,30
    6 I=I+1
      GOTO2
    7 IF(K-MN+1)8,30,30
    8 K=K+1
      GOTO1
   30 RETURN
      END


C   SUBROUTINE AGEB, PERFORMS GAUSSIAN ELIMINATION ON A-MATRIX AND SETS
C   UP DC- & DI-VECTORS FOR USE ON C-VECTOR, IMMOVABLY CONSTRAINED EDGES

      SUBROUTINE AGEB (A,DC,DI,M,N,KR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 DABS
      DIMENSION A(KR,1),DC(1),DI(1)
      MN1=M*N+1
```

```
      MN2=M*N+2
      LL=3*M
      KK=0
      DO3K=1,MN1
      K1=K+1
      IF(LL.LT.MN2) LL=LL+1
      L=K
      DO11KI=K1,LL
   11 IF(DABS(A(KI,K)).GT.DABS(A(L,K))) L=KI
      DI(K)=L
      IF(L.EQ.K) GO TO 2
      DO12J=K,MN2
      D=A(L,J)
      A(L,J)=A(K,J)
   12 A(K,J)=D
    2 DO3I=K1,LL
      KK=KK+1
      DC(KK)=A(I,K)/A(K,K)
      DO3J=K,MN2
    3 A(I,J)=A(I,J)-DC(KK)*A(K,J)
      RETURN
      END



C   SUBROUTINE FDIA, CALCULATES W AND F FOR TEN TIME STEPS

      SUBROUTINE FDIA  (M,N,MN,W,F,A,B,C,BB,DC,P,C1,C2,C3,E,KR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 DABS
      DIMENSION W(KR,1),F(KR,1),A(KR,1),B(1),C(1),BB(1),DC(1),P(1)
C   SET UP CONSTANTS ONE TIME ONLY
    1 IF(W(MN,1).NE.0.) GO TO 2
      M1=M+1
      M2=2*M
      M3=3*M
      M4=4*M
      MNM=MN-M
      LT=(N-2)*M+1
      LN=N-2
      LM=M-1
      LLN=N-3
      LLM=M-2
      LLT=LT+2
      LLS=MNM-2
      M21=M2+1
      LS=(N-3)*M+1
      LST=(N-2)*M
    2 CONTINUE
      DO70J=2,11
C
C   USE LINEAR TERMS ONLY FOR VERY SMALL  W
```

```
C
      IF(DABS(W(MN,J)).GT.0.01D0) GO TO 10
      DO3I=1,MN
      F(I,J)=0.D0
    3 BB(I)=0.D0
      GO TO 50
   10 CONTINUE
C
C        CALCULATE CONSTANT VECTOR FOR  A F=C  (SS OR C)
C
      C(1)=(W(M+2,J)**2/16.D0-(-2.D0*W(1,J)+W(2,J))*(-2.D0*W(1,J)+
     1  W(M+1,J)))*E
      C(M)=(-(2.D0*W(M-1,J)-2.D0*W(M,J))*(W(M2,J)-2.D0*W(M,J)))*E
      C(MN)=(-(2.D0*W(MN-1,J)-2.D0*W(MN,J))*(2.D0*W(MNM,J)-2.D0*W(MN,J)
     1  ))*E
      K=MNM+1
      C(K)=(-(-2.D0*W(K,J)+W(K+1,J))*(2.D0*W(K-M,J)-2.D0*W(K,J)))*E
      DO11I=2,LM
      K=I+M
      C(I)=((-W(K-1,J)+W(K+1,J))**2/16.D0-(W(I-1,J)-2.D0*W(I,J)+W(I+1,J)
     1  )*(-2.D0*W(I,J)+W(K,J)))*E
      K=MNM+I
   11 C(K)=(-(W(K-1,J)-2.D0*W(K,J)+W(K+1,J))*(2.D0*W(K-M,J)-2.D0*W(K,J)
     1  ))*E
      DO12I=M1,LT,M
      IM=I+M
      IL=I-M
   12 C(I)=((W(IM+1,J)-W(IL+1,J))**2/16.D0-(-2.D0*W(I,J)+W(I+1,J))*
     1  (W(IL,J)-2.D0*W(I,J)+W(IM,J)))*E
      DO13I=M2,MNM,M
      IM=I+M
      IL=I-M
   13 C(I)=(-(2.D0*W(I-1,J)-2.D0*W(I,J))*(W(IL,J)-2.D0*W(I,J)+W(IM,J))
     1  )*E
      DO14K=1,LN
      KM=K*M
      DO14L=2,LM
      I=KM+L
      IM=I+M
      IL=I-M
   14 C(I)=((W(IL-1,J)-W(IM-1,J)+W(IM+1,J)-W(IL+1,J))**2/16.D0-(W(I-1,J)
     1  -2.D0*W(I,J)+W(I+1,J))*(W(IL,J)-2.D0*W(I,J)+W(IM,J)))*E
C
C        PERFORM GAUSS ELIMINATION ON C(I)
C
   21 KK=0
      L=2*M+1
      K=1
   22 I=K+1
   23 KK=KK+1
      C(I)=C(I)-DC(KK)*C(K)
      IF(I-L)24,25,40
   24 I=I+1
      GOTO23
   25 IF(L.LT.MN)L=L+1
```

```
   26 IF(K-MN+1)27,31,40
   27 K=K+1
      GOTO22
C
C        PERFORM BACK SUBSTITUTION FOR F(I)
C
   31 LL=MN-M2
      L=MN
      F(L,J)=C(L)/A(L,L)
      I=MN-1
   32 IF(I.LT.LL)L=L-1
      K=I+1
      S=0.D0
   33 S=S+A(I,K)*F(K,J)
      IF(K-L)34,35,40
   34 K=K+1
      GOTO33
   35 F(I,J)=(C(I)-S)/A(I,I)
      IF(I-1)40,40,36
   36 I=I-1
      GOTO32
   40 CONTINUE
C
C        CALCULATE NONLINEAR TERMS FOR SS OR C
C
      BB(1)=(-2.D0*F(1,J)+F(M+1,J))*(-2.D0*W(1,J)+W(2,J))+(-2.D0*F(1,J)+
     1  F(2,J))*(-2.D0*W(1,J)+W(M+1,J))-(F(M+2,J)*W(M+2,J))/8.D0
      BB(M)=(-2.D0*F(M,J)+F(M2,J))*(2.D0*W(M-1,J)-2.D0*W(M,J))+(2.D0*
     1  F(M-1,J)-2.D0*F(M,J))*(-2.D0*W(M,J)+W(M2,J))
      BB(MN)=(2.D0*F(MNM,J)-2.D0*F(MN,J))*(2.D0*W(MN-1,J)-2.D0*W(MN,J))+
     1  (2.D0*F(MN-1,J)-2.D0*F(MN,J))*(2.D0*W(MNM,J)-2.D0*W(MN,J))
      K=MNM+1
      BB(K)=(2.D0*F(K-M,J)-2.D0*F(K,J))*(-2.D0*W(K,J)+W(K+1,J))+
     1  (-2.D0*F(K,J)+F(K+1,J))*(2.D0*W(K-M,J)-2.D0*W(K,J))
      DO41I=2,LM
      K=I+M
      BB(I)=(-2.D0*F(I,J)+F(K,J))*(W(I-1,J)-2.D0*W(I,J)+W(I+1,J))+
     1  (F(I-1,J)-2.D0*F(I,J)+F(I+1,J))*(-2.D0*W(I,J)+W(K,J))-
     2  (-F(K-1,J)+F(K+1,J))*(-W(K-1,J)+W(K+1,J))/8.D0
      K=MNM+I
   41 BB(K)=(2.D0*F(K-M,J)-2.D0*F(K,J))*(W(K-1,J)-2.D0*W(K,J)+W(K+1,J))+
     1  (F(K-1,J)-2.D0*F(K,J)+F(K+1,J))*(2.D0*W(K-M,J)-2.D0*W(K,J))
      DO42I=M1,LT,M
      IM=I+M
      IL=I-M
   42 BB(I)=(F(IL,J)-2.D0*F(I,J)+F(IM,J))*(-2.D0*W(I,J)+W(I+1,J))+
     1  (-2.D0*F(I,J)+F(I+1,J))*(W(IL,J)-2.D0*W(I,J)+W(IM,J))-
     2  (F(IM+1,J)-F(IL+1,J))*(W(IM+1,J)-W(IL+1,J))/8.D0
      DO43I=M2,MNM,M
      IM=I+M
      IL=I-M
   43 BB(I)=(F(IL,J)-2.D0*F(I,J)+F(IM,J))*(2.D0*W(I-1,J)-2.D0*W(I,J))+
     1  (2.D0*F(I-1,J)-2.D0*F(I,J))*(W(IL,J)-2.D0*W(I,J)+W(IM,J))
      DO44K=1,LN
      KM=K*M
```

```
      DO44L=2,LM
      I=KM+L
      IM=I+M
      IL=I-M
   44 BB(I)=(F(IL,J)-2.DO*F(I,J)+F(IM,J))*(W(I-1,J)-2.DO*W(I,J)+W(I+1,J)
     1 )+(F(I-1,J)-2.DO*F(I,J)+F(I+1,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J))-
     2 (F(IL-1,J)-F(IM-1,J)+F(IM+1,J)-F(IL+1,J))*
     3 (W(IL-1,J)-W(IM-1,J)+W(IM+1,J)-W(IL+1,J))/8.DO
   50 CONTINUE
C
C        CALCULATE DEL FOURTH W FOR SIMPLY SUPPORTED
C
      B(1)=18.DO*W(1,J)-8.DO*(W(2,J)+W(M1,J))+2.DO*W(M1+1,J)+W(3,J)+
     1 W(M2+1,J)
      B(2)=19.DO*W(2,J)-8.DO*(W(1,J)+W(3,J)+W(M+2,J))+2.DO*(W(M1,J)+
     1 W(M+3,J))+W(4,J)+W(M2+2,J)
      B(M1)=19.DO*W(M1,J)-8.DO*(W(M+2,J)+W(M2+1,J)+W(1,J))+2.DO*
     1 (W(M2+2,J)+W(2,J))+W(M+3,J)+W(M3+1,J)
      B(M+2)=20.DO*W(M+2,J)-8.DO*(W(M1,J)+W(M+3,J)+W(M2+2,J)+W(2,J))+
     1 2.DO*(W(M2+1,J)+W(M2+3,J)+W(1,J)+W(3,J))+W(M+4,J)+W(M3+2,J)
      B(M-1)=20.DO*W(M-1,J)-8.DO*(W(M-2,J)+W(M,J)+W(M2-1,J))+2.DO*
     1 (W(M2-2,J)+W(M2,J))+W(M-3,J)+W(M3-1,J)
      B(M)=19.DO*W(M,J)-8.DO*(2.DO*W(M-1,J)+W(M2,J))+4.DO*W(M2-1,J)+
     1 2.DO*W(M-2,J)+W(M3,J)
      B(M2-1)=21.DO*W(M2-1,J)-8.DO*(W(M2-2,J)+W(M2,J)+W(M3-1,J)+W(M-1,J)
     1 )+2.DO*(W(M3-2,J)+W(M3,J)+W(M-2,J)+W(M,J))+W(M2-3,J)+W(M4-1,J)
      B(M2)=20.DO*W(M2,J)-8.DO*(2.DO*W(M2-1,J)+W(M3,J)+W(M,J))+4.DO*
     1 (W(M3-1,J)+W(M-1,J))+2.DO*W(M2-2,J)+W(M4,J)
      K=LT
      B(K)=20.DO*W(K,J)-8.DO*(W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
     1 (W(K+M1,J)+W(K-LM,J))+W(K+2,J)+W(K-M2,J)
      K=K+1
      B(K)=21.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
     1 (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K+2,J)+W(K-M2,J)
      K=MNM+1
      B(K)=19.DO*W(K,J)-8.DO*(W(K+1,J)+2.DO*W(K-M,J))+4.DO*W(K-LM,J)+
     1 W(K+2,J)+2.DO*W(K-M2,J)
      K=K+1
      B(K)=20.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+2.DO*W(K-M,J))+4.DO*
     1 (W(K-M1,J)+W(K-LM,J))+W(K+2,J)+2.DO*W(K-M2,J)
      K=MNM-1
      B(K)=22.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
     1 (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K-2,J)+W(K-M2,J)
      K=K+1
      B(K)=21.DO*W(K,J)-8.DO*(2.DO*W(K-1,J)+W(K+M,J)+W(K-M,J))+4.DO*
     1 (W(K+LM,J)+W(K-M1,J))+2.DO*W(K-2,J)+W(K-M2,J)
      K=MN-1
      B(K)=21.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+2.DO*W(K-M,J))+4.DO*
     1 (W(K-M1,J)+W(K-LM,J))+W(K-2,J)+2.DO*W(K-M2,J)
      B(MN)=20.DO*W(MN,J)-16.DO*(W(K,J)+W(MNM,J))+8.DO*W(MN-M1,J)+
     1 2.DO*(W(MN-2,J)+W(MN-M2,J))
      DO51I=3,LLM
      B(I)=19.DO*W(I,J)-8.DO*(W(I-1,J)+W(I+1,J)+W(I+M,J))+2.DO*
     1 (W(I+LM,J)+W(I+M1,J))+W(I-2,J)+W(I+2,J)+W(I+M2,J)
      K=I+M
```

```
51 B(K)=20.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
  1  (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K-2,J)+W(K+2,J)+
  2  W(K+M2,J)
   DO52I=LLT,LLS
   B(I)=21.DO*W(I,J)-8.DO*(W(I-1,J)+W(I+1,J)+W(I+M,J)+W(I-M,J))+2.DO*
  1  (W(I+LM,J)+W(I+M1,J)+W(I-M1,J)+W(I-LM,J))+W(I-2,J)+W(I+2,J)+
  2  W(I-M2,J)
   K=I+M
52 B(K)=20.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+2.DO*W(K-M,J))+4.DO*
  1  (W(K-M1,J)+W(K-LM,J))+W(K-2,J)+W(K+2,J)+2.DO*W(K-M2,J)
   DO53I=M21,LS,M
   B(I)=19.DO*W(I,J)-8.DO*(W(I+1,J)+W(I+M,J)+W(I-M,J))+2.DO*
  1  (W(I+M1,J)+W(I-LM,J))+W(I+2,J)+W(I+M2,J)+W(I-M2,J)
   K=I+1
53 B(K)=20.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
  1  (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K+2,J)+W(K+M2,J)+
  2  W(K-M2,J)
   DO54I=M3,LST,M
   B(I)=20.DO*W(I,J)-8.DO*(2.DO*W(I-1,J)+W(I+M,J)+W(I-M,J))+4.DO*
  1  (W(I+LM,J)+W(I-M1,J))+2.DO*W(I-2,J)+W(I+M2,J)+W(I-M2,J)
   K=I-1
54 B(K)=21.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+W(K-M,J)+W(K+M,J))+2.DO*
  1  (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K-2,J)+W(K+M2,J)+
  2  W(K-M2,J)
   DO55K=2,LLN
   KM=K*M
   DO55L=3,LLM
   I=KM+L
55 B(I)=20.DO*W(I,J)-8.DO*(W(I-1,J)+W(I+1,J)+W(I+M,J)+W(I-M,J))+2.DO*
  1  (W(I+LM,J)+W(I+M1,J)+W(I-M1,J)+W(I-LM,J))+W(I-2,J)+W(I+2,J)+
  2  W(I+M2,J)+W(I-M2,J)
C
C       CALCULATE W(I,J+1)
C
   DO60I=1,MN
60 W(I,J+1)=2.DO*W(I,J)-W(I,J-1)-C1*B(I)+C2*BB(I)+C3*P(J)
70 CONTINUE
   RETURN
   END


C   SUBROUTINE FDIIA, CALCULATES W AND F FOR TEN TIME STEPS

   SUBROUTINE FDIIA (M,N,MN,W,F,A,B,C,BB,DC,P,C1,C2,C3,E,KR)
   IMPLICIT REAL*8 (A-H,O-Z)
   DIMENSION W(KR,1),F(KR,1),A(KR,1),B(1),C(1),BB(1),DC(1),P(1)
C   SET UP CONSTANTS ONE TIME ONLY
 1 IF(W(MN,1).NE.0.) GO TO 2
   M1=M+1
   M2=2*M
   M3=3*M
```

```
      M4=4*M
      MNM=MN-M
      LT=(N-2)*M+1
      LN=N-2
      LM=M-1
      LLN=N-3
      LLM=M-2
      LLT=LT+2
      LLS=MNM-2
      M21=M2+1
      LS=(N-3)*M+1
      LST=(N-2)*M
    2 CONTINUE
      DO70J=2,11
C
C  USE LINEAR TERMS ONLY FOR VERY SMALL  W
C
      IF(W(MN,J).GT.0.01) GO TO 10
      DO3I=1,MN
      F(I,J)=0.D0
    3 BB(I)=0.D0
      GO TO 50
   10 CONTINUE
C
C        CALCULATE CONSTANT VECTOR FOR  A F=C  (SS OR C)
C
      C(1)=(W(M+2,J)**2/16.D0-(-2.D0*W(1,J)+W(2,J))*(-2.D0*W(1,J)+
     1  W(M+1,J)))*E
      C(M)=(-(2.D0*W(M-1,J)-2.D0*W(M,J))*(W(M2,J)-2.D0*W(M,J)))*E
      C(MN)=(-(2.D0*W(MN-1,J)-2.D0*W(MN,J))*(2.D0*W(MNM,J)-2.D0*W(MN,J)
     1  ))*E
      K=MNM+1
      C(K)=(-(-2.D0*W(K,J)+W(K+1,J))*(2.D0*W(K-M,J)-2.D0*W(K,J)))*E
      DO11I=2,LM
      K=I+M
      C(I)=((-W(K-1,J)+W(K+1,J))**2/16.D0-(W(I-1,J)-2.D0*W(I,J)+W(I+1,J)
     1  )*(-2.D0*W(I,J)+W(K,J)))*E
      K=MNM+I
   11 C(K)=(-(W(K-1,J)-2.D0*W(K,J)+W(K+1,J))*(2.D0*W(K-M,J)-2.D0*W(K,J)
     1  ))*E
      DO12I=M1,LT,M
      IM=I+M
      IL=I-M
   12 C(I)=((W(IM+1,J)-W(IL+1,J))**2/16.D0-(-2.D0*W(I,J)+W(I+1,J))*
     1  (W(IL,J)-2.D0*W(I,J)+W(IM,J)))*E
      DO13I=M2,MNM,M
      IM=I+M
      IL=I-M
   13 C(I)=(-(2.D0*W(I-1,J)-2.D0*W(I,J))*(W(IL,J)-2.D0*W(I,J)+W(IM,J))
     1  )*E
      DO14K=1,LN
      KM=K*M
      DO14L=2,LM
      I=KM+L
      IM=I+M
```

```
      IL=I-M
   14 C(I)=((W(IL-1,J)-W(IM-1,J)+W(IM+1,J)-W(IL+1,J))**2/16.D0-(W(I-1,J)
     1  -2.D0*W(I,J)+W(I+1,J))*(W(IL,J)-2.D0*W(I,J)+W(IM,J)))*E
C
C        PERFORM GAUSS ELIMINATION ON C(I)
C
   21 KK=0
      L=2*M+1
      K=1
   22 I=K+1
   23 KK=KK+1
      C(I)=C(I)-DC(KK)*C(K)
      IF(I-L)24,25,40
   24 I=I+1
      GOTO23
   25 IF(L.LT.MN)L=L+1
   26 IF(K-MN+1)27,31,40
   27 K=K+1
      GOTO22
C
C        PERFORM BACK SUBSTITUTION FOR F(I)
C
   31 LL=MN-M2
      L=MN
      F(L,J)=C(L)/A(L,L)
      I=MN-1
   32 IF(I.LT.LL)L=L-1
      K=I+1
      S=0
   33 S=S+A(I,K)*F(K,J)
      IF(K-L)34,35,40
   34 K=K+1
      GOTO33
   35 F(I,J)=(C(I)-S)/A(I,I)
      IF(I-1)40,40,36
   36 I=I-1
      GOTO32
   40 CONTINUE
C
C        CALCULATE NONLINEAR TERMS FOR SS OR C
C
      BB(1)=(-2.D0*F(1,J)+F(M+1,J))*(-2.D0*W(1,J)+W(2,J))+(-2.D0*F(1,J)+
     1  F(2,J))*(-2.D0*W(1,J)+W(M+1,J))-(F(M+2,J)*W(M+2,J))/8.D0
      BB(M)=(-2.D0*F(M,J)+F(M2,J))*(2.D0*W(M-1,J)-2.D0*W(M,J))+(2.D0*
     1  F(M-1,J)-2.D0*F(M,J))*(-2.D0*W(M,J)+W(M2,J))
      BB(MN)=(2.D0*F(MNM,J)-2.D0*F(MN,J))*(2.D0*W(MN-1,J)-2.D0*W(MN,J))+
     1  (2.D0*F(MN-1,J)-2.D0*F(MN,J))*(2.D0*W(MNM,J)-2.D0*W(MN,J))
      K=MNM+1
      BB(K)=(2.D0*F(K-M,J)-2.D0*F(K,J))*(-2.D0*W(K,J)+W(K+1,J))+
     1  (-2.D0*F(K,J)+F(K+1,J))*(2.D0*W(K-M,J)-2.D0*W(K,J))
      DO41I=2,LM
      K=I+M
      BB(I)=(-2.D0*F(I,J)+F(K,J))*(W(I-1,J)-2.D0*W(I,J)+W(I+1,J))+
     1  (F(I-1,J)-2.D0*F(I,J)+F(I+1,J))*(-2.D0*W(I,J)+W(K,J))-
     2  (-F(K-1,J)+F(K+1,J))*(-W(K-1,J)+W(K+1,J))/8.D0
```

```
      K=MNM+I
   41 BB(K)=(2.DO*F(K-M,J)-2.DO*F(K,J))*(W(K-1,J)-2.DO*W(K,J)+W(K+1,J))+
     1  (F(K-1,J)-2.DO*F(K,J)+F(K+1,J))*(2.DO*W(K-M,J)-2.DO*W(K,J))
      DO42I=M1,LT,M
      IM=I+M
      IL=I-M
   42 BB(I)=(F(IL,J)-2.DO*F(I,J)+F(IM,J))*(-2.DO*W(I,J)+W(I+1,J))+
     1  (-2.DO*F(I,J)+F(I+1,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J))-
     2  (F(IM+1,J)-F(IL+1,J))*(W(IM+1,J)-W(IL+1,J))/8.DO
      DO43I=M2,MNM,M
      IM=I+M
      IL=I-M
   43 BB(I)=(F(IL,J)-2.DO*F(I,J)+F(IM,J))*(2.DO*W(I-1,J)-2.DO*W(I,J))+
     1  (2.DO*F(I-1,J)-2.DO*F(I,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J))
      DO44K=1,LN
      KM=K*M
      DO44L=2,LM
      I=KM+L
      IM=I+M
      IL=I-M
   44 BB(I)=(F(IL,J)-2.DO*F(I,J)+F(IM,J))*(W(I-1,J)-2.DO*W(I,J)+W(I+1,J)
     1  )+(F(I-1,J)-2.DO*F(I,J)+F(I+1,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J))-
     2  (F(IL-1,J)-F(IM-1,J)+F(IM+1,J)-F(IL+1,J))*
     3  (W(IL-1,J)-W(IM-1,J)+W(IM+1,J)-W(IL+1,J))/8.DO
   50 CONTINUE
C
C        CALCULATE DEL FOURTH W FOR CLAMPED
C
      B(1)=22.DO*W(1,J)-8.DO*(W(2,J)+W(M1,J))+2.DO*W(M1+1,J)+W(3,J)+
     1  W(M2+1,J)
      B(2)=21.DO*W(2,J)-8.DO*(W(1,J)+W(3,J)+W(M+2,J))+2.DO*(W(M1,J)+
     1  W(M+3,J))+W(4,J)+W(M2+2,J)
      B(M1)=21.DO*W(M1,J)-8.DO*(W(M+2,J)+W(M2+1,J)+W(1,J))+2.DO*
     1  (W(M2+2,J)+W(2,J))+W(M+3,J)+W(M3+1,J)
      B(M+2)=20.DO*W(M+2,J)-8.DO*(W(M1,J)+W(M+3,J)+W(M2+2,J)+W(2,J))+
     1  2.DO*(W(M2+1,J)+W(M2+3,J)+W(1,J)+W(3,J))+W(M+4,J)+W(M3+2,J)
      B(M-1)=22.DO*W(M-1,J)-8.DO*(W(M-2,J)+W(M,J)+W(M2-1,J))+2.DO*
     1  (W(M2-2,J)+W(M2,J))+W(M-3,J)+W(M3-1,J)
      B(M)=21.DO*W(M,J)-8.DO*(2.DO*W(M-1,J)+W(M2,J))+4.DO*W(M2-1,J)+
     1  2.DO*W(M-2,J)+W(M3,J)
      B(M2-1)=21.DO*W(M2-1,J)-8.DO*(W(M2-2,J)+W(M2,J)+W(M3-1,J)+W(M-1,J)
     1  )+2.DO*(W(M3-2,J)+W(M3,J)+W(M-2,J)+W(M,J))+W(M2-3,J)+W(M4-1,J)
      B(M2)=20.DO*W(M2,J)-8.DO*(2.DO*W(M2-1,J)+W(M3,J)+W(M,J))+4.DO*
     1  (W(M3-1,J)+W(M-1,J))+2.DO*W(M2-2,J)+W(M4,J)
      K=LT
      B(K)=22.DO*W(K,J)-8.DO*(W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
     1  (W(K+M1,J)+W(K-LM,J))+W(K+2,J)+W(K-M2,J)
      K=K+1
      B(K)=21.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
     1  (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K+2,J)+W(K-M2,J)
      K=MNM+1
      B(K)=21.DO*W(K,J)-8.DO*(W(K+1,J)+2.DO*W(K-M,J))+4.DO*W(K-LM,J)+
     1  W(K+2,J)+2.DO*W(K-M2,J)
      K=K+1
      B(K)=20.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+2.DO*W(K-M,J))+4.DO*
```

```
     1 (W(K-M1,J)+W(K-LM,J))+W(K+2,J)+2.D0*W(K-M2,J)
      K=MNM-1
      B(K)=22.D0*W(K,J)-8.D0*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.D0*
     1 (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K-2,J)+W(K-M2,J)
      K=K+1
      B(K)=21.D0*W(K,J)-8.D0*(2.D0*W(K-1,J)+W(K+M,J)+W(K-M,J))+4.D0*
     1 (W(K+LM,J)+W(K-M1,J))+2.D0*W(K-2,J)+W(K-M2,J)
      K=MN-1
      B(K)=21.D0*W(K,J)-8.D0*(W(K-1,J)+W(K+1,J)+2.D0*W(K-M,J))+4.D0*
     1 (W(K-M1,J)+W(K-LM,J))+W(K-2,J)+2.D0*W(K-M2,J)
      B(MN)=20.D0*W(MN,J)-16.D0*(W(K,J)+W(MNM,J))+8.D0*W(MN-M1,J)+
     1 2.D0*(W(MN-2,J)+W(MN-M2,J))
      DO51I=3,LLM
      B(I)=21.D0*W(I,J)-8.D0*(W(I-1,J)+W(I+1,J)+W(I+M,J))+2.D0*
     1 (W(I+LM,J)+W(I+M1,J))+W(I-2,J)+W(I+2,J)+W(I+M2,J)
      K=I+M
   51 B(K)=20.D0*W(K,J)-8.D0*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.D0*
     1 (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K-2,J)+W(K+2,J)+
     2 W(K+M2,J)
      DO52I=LLT,LLS
      B(I)=21.D0*W(I,J)-8.D0*(W(I-1,J)+W(I+1,J)+W(I+M,J)+W(I-M,J))+2.D0*
     1 (W(I+LM,J)+W(I+M1,J)+W(I-M1,J)+W(I-LM,J))+W(I-2,J)+W(I+2,J)+
     2 W(I-M2,J)
      K=I+M
   52 B(K)=20.D0*W(K,J)-8.D0*(W(K-1,J)+W(K+1,J)+2.D0*W(K-M,J))+4.D0*
     1 (W(K-M1,J)+W(K-LM,J))+W(K-2,J)+W(K+2,J)+2.D0*W(K-M2,J)
      DO53I=M21,LS,M
      B(I)=21.D0*W(I,J)-8.D0*(W(I+1,J)+W(I+M,J)+W(I-M,J))+2.D0*
     1 (W(I+M1,J)+W(I-LM,J))+W(I+2,J)+W(I+M2,J)+W(I-M2,J)
      K=I+1
   53 B(K)=20.D0*W(K,J)-8.D0*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.D0*
     1 (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K+2,J)+W(K+M2,J)+
     2 W(K-M2,J)
      DO54I=M3,LST,M
      B(I)=20.D0*W(I,J)-8.D0*(2.D0*W(I-1,J)+W(I+M,J)+W(I-M,J))+4.D0*
     1 (W(I+LM,J)+W(I-M1,J))+2.D0*W(I-2,J)+W(I+M2,J)+W(I-M2,J)
      K=I-1
   54 B(K)=21.D0*W(K,J)-8.D0*(W(K-1,J)+W(K+1,J)+W(K-M,J)+W(K+M,J))+2.D0*
     1 (W(K+LM,J)+W(K+M1,J)+W(K-M1,J)+W(K-LM,J))+W(K-2,J)+W(K+M2,J)+
     2 W(K-M2,J)
      DO55K=2,LLN
      KM=K*M
      DO55L=3,LLM
      I=KM+L
   55 B(I)=20.D0*W(I,J)-8.D0*(W(I-1,J)+W(I+1,J)+W(I+M,J)+W(I-M,J))+2.D0*
     1 (W(I+LM,J)+W(I+M1,J)+W(I-M1,J)+W(I-LM,J))+W(I-2,J)+W(I+2,J)+
     2 W(I+M2,J)+W(I-M2,J)
C
C        CALCULATE W(I,J+1)
C
      DO60I=1,MN
   60 W(I,J+1)=2.D0*W(I,J)-W(I,J-1)-C1*B(I)+C2*BB(I)+C3*P(J)
   70 CONTINUE
      RETURN
      END
```

```
C   SUBROUTINE FDIB, CALCULATES W AND F FOR TEN TIME STEPS
      SUBROUTINE FDIB (M,N,MN,W,F,A,B,C,BB,DC,DI,P,C1,C2,C3,E,KR)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 DABS
      DIMENSION W(KR,1),F(KR,1),A(KR,1),B(1),C(1),BB(1),DC(1),DI(1),P(1)
C   SET UP CONSTANTS ONE TIME ONLY
    1 IF(W(MN,1).NE.0.) GO TO 2
      MN1=MN+1
      MN2=MN+2
      M1=M+1
      M2=2*M
      M3=3*M
      MP2=M+2
      MP3=M+3
      M22=M2+2
      MNM=MN-M
      MNM2=MNM+2
      MM1=M-1
      MM2=M-2
      M2M1=M2-1
      M2M2=M2-2
      MNM1=MN-1
      N1=N-1
      N2=N-2
      N3=N-3
      LT=(N-2)*M+2
      LT1=LT+1
      LT2=MNM-2
      LS=LT-M
      LST=LT-2
    2 CONTINUE
      DO70J=2,11
C
C   USE LINEAR TERMS ONLY FOR VERY SMALL  W
C
      IF(DABS(W(MN,J)).GT.0.01) GO TO 10
      DO3I=1,MN
      F(I,J)=0. DO
    3 BB(I)=0. DO
      GO TO 50
   10 CONTINUE
C
C   CALCULATE CONSTANT VECTOR FOR  (A)F=C     I(B)
C
      C(1)=0. DO
      C(2)=0. DO
      DO12I=2,M
      K=I
      S=2.DO*W(I+M,J)**2
      DO11L=1,N2
      K=K+M
   11 S=S+(W(K+M,J)-W(K-M,J))**2
   12 C(I+1)=E*S/4.DO
      C(M2+1)=0. DO
```

```
      K=2
      DO14I=M3,MN,M
      K=K+M
      S=2.D0*W(K,J)**2
      KK=K
      DO13L=1,MM2
      KK=KK+1
   13 S=S+(W(KK,J)-W(KK-2,J))**2
   14 C(I+1)=E*S/4.D0
      KK=K+M
      S=2.D0*W(KK,J)**2
      DO15L=1,MM2
      KK=KK+1
   15 S=S+(W(KK,J)-W(KK-2,J))**2
      C(MN2)=E*S/4.D0
      DO16K=1,N2
      KM=K*M
      DO16L=2,MM1
      I=KM+L
      IM=I+M
      IL=I-M
   16 C(I)=E*((W(IL-1,J)-W(IM-1,J)+W(IM+1,J)-W(IL+1,J))**2/16.D0-
     1 (W(I-1,J)-2.D0*W(I,J)+W(I+1,J))*(W(IL,J)-2.D0*W(I,J)+W(IM,J)))
      DO17I=M2,MNM,M
      IM=I+M
      IL=I-M
   17 C(I)= -E*(2.D0*W(I-1,J)-2.D0*W(I,J))*(W(IL,J)-2.D0*W(I,J)+W(IM,J))
      DO18I=MNM2,MNM1
      IL=I-M
   18 C(I)=E*(-(W(I-1,J)-2.D0*W(I,J)+W(I+1,J))*(2.D0*W(IL,J)-2.D0*
     1  W(I,J)))
      C(MN)=E*(-(2.D0*W(MN-1,J)-2.D0*W(MN,J))*(2.D0*W(MNM,J)-2.D0*
     1  W(MN,J)))
   20 CONTINUE
C
C       PERFORM GAUSS ELIMINATION ON C(I)
C
   21 KK=0
      LL=3*M
      DO23K=1,MN1
      K1=K+1
      IF(LL.LT.MN2) LL=LL+1
      L=DI(K)
      IF(L.EQ.K) GO TO 22
      D=C(L)
      C(L)=C(K)
      C(K)=D
   22 DO23I=K1,LL
      KK=KK+1
   23 C(I)=C(I)-DC(KK)*C(K)
C
C       PERFORM BACK SUBSTITUTION FOR F(I,J)
C
   31 F(MN2,J)=C(MN2)/A(MN2,MN2)
      I=MN2-1
```

```
   32 I1=I+1
      S=0. DO
      DO33K=I1,MN2
   33 S=S+A(I,K)*F(K,J)
      F(I,J)=(C(I)-S)/A(I,I)
      IF(I.EQ.2) GO TO 34
      I=I-1
      GO TO 32
   34 F(1,J)=0.
C
C        CALCULATE NONLINEAR TERMS FOR I OR II(B)
C
      DO41 K=1,N2
      KM=K*M
      DO41L=2,MM1
      I=KM+L
      IM=I+M
      IL=I-M
   41 BB(I)=(F(IL,J)-2.DO*F(I,J)+F(IM,J))*(W(I-1,J)-2.DO*W(I,J)+W(I+1,J)
     1 )+(F(I-1,J)-2.DO*F(I,J)+F(I+1,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J))
     2 -(F(IL-1,J)-F(IM-1,J)+F(IM+1,J)-F(IL+1,J))
     3 *(W(IL-1,J)-W(IM-1,J)+W(IM+1,J)-W(IL+1,J))/8.DO
      DO42I=M2,MNM,M
      IM=I+M
      IL=I-M
   42 BB(I)=(F(IL,J)-2.DO*F(I,J)+F(IM,J))*(2.DO*W(I-1,J)-2.DO*W(I,J))+
     1  (2.DO*F(I-1,J)-2.DO*F(I,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J))
      DO43I=MNM2,MNM1
      IL=I-M
   43 BB(I)=(2.DO*F(IL,J)-2.DO*F(I,J))*(W(I-1,J)-2.DO*W(I,J)+W(I+1,J))+
     1  (F(I-1,J)-2.DO*F(I,J)+F(I+1,J))*(2.DO*W(IL,J)-2.DO*W(I,J))
      BB(MN)=(2.DO*F(MNM,J)-2.DO*F(MN,J))*(2.DO*W(MNM1,J)-2.DO*W(MN,J))+
     1  (2.DO*F(MNM1,J)-2.DO*F(MN,J))*(2.DO*W(MNM,J)-2.DO*W(MN,J))
   50 CONTINUE
C
C        CALCULATE DEL FOURTH W FOR SIMPLY SUPPORTED
C
      B(MP2)=18.DO*W(MP2,J)-8.DO*(W(MP3,J)+W(MP2+M,J))+2.DO*W(MP2+M1,J)+
     1  W(MP2+2,J)+W(M3+2,J)
      B(M2M1)=20.DO*W(M2M1,J)-8.DO*(W(M2-2,J)+W(M2,J)+W(M3-1,J))+
     1  2.DO*(W(M3-2,J)+W(M3,J))+W(M2-3,J)+W(M2M1+M2,J)
      B(M2)=19.DO*W(M2,J)-8.DO*(2.DO*W(M2-1,J)+W(M3,J))+4.DO*W(M3-1,J)+
     1  2.DO*W(M2-2,J)+W(M3+M,J)
      K=LT
      B(K)=20.DO*W(K,J)-8.DO*(W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
     1  (W(K+M1,J)+W(K-MM1,J))+W(K+2,J)+W(K-M2,J)
      K=K+M
      B(K)=19.DO*W(K,J)-8.DO*(W(K+1,J)+2.DO*W(K-M,J))+4.DO*W(K-MM1,J)+
     1  W(K+2,J)+2.DO*W(K-M2,J)
      K=MNM-1
      B(K)=22.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
     1 (W(K+MM1,J)+W(K+M1,J)+W(K-M1,J)+W(K-MM1,J))+W(K-2,J)+W(K-M2,J)
      K=K+1
      B(K)=21.DO*W(K,J)-8.DO*(2.DO*W(K-1,J)+W(K+M,J)+W(K-M,J))+4.DO*
     1  (W(K+MM1,J)+W(K-M1,J))+2.DO*W(K-2,J)+W(K-M2,J)
```

```
      K=MN-1
      B(K)=21.D0*W(K,J)-8.D0*(W(K-1,J)+W(K+1,J)+2.D0*W(K-M,J))+4.D0*
     1  (W(K-M1,J)+W(K-MM1,J))+W(K-2,J)+2.D0*W(K-M2,J)
      B(MN)=20.D0*W(MN,J)-16.D0*(W(K,J)+W(MNM,J))+8.D0*W(MN-M1,J)+
     1  2.D0*(W(MN-2,J)+W(MN-M2,J))
      DO51I=MP3,M2M2
   51 B(I)=19.D0*W(I,J)-8.D0*(W(I-1,J)+W(I+1,J)+W(I+M,J))+2.D0*
     1  (W(I+MM1,J)+W(I+M1,J))+W(I-2,J)+W(I+2,J)+W(I+M2,J)
      DO52I=LT1,LT2
      B(I)=21.D0*W(I,J)-8.D0*(W(I-1,J)+W(I+1,J)+W(I+M,J)+W(I-M,J))+2.D0*
     1  (W(I+MM1,J)+W(I+M1,J)+W(I-M1,J)+W(I-MM1,J))+W(I-2,J)+W(I+2,J)+
     2  W(I-M2,J)
      K=I+M
   52 B(K)=20.D0*W(K,J)-8.D0*(W(K-1,J)+W(K+1,J)+2.D0*W(K-M,J))+4.D0*
     1  (W(K-M1,J)+W(K-MM1,J))+W(K-2,J)+W(K+2,J)+2.D0*W(K-M2,J)
      DO53I=M22,LS,M
   53 B(I)=19.D0*W(I,J)-8.D0*(W(I+1,J)+W(I+M,J)+W(I-M,J))+2.D0*
     1  (W(I+M1,J)+W(I-MM1,J))+W(I+2,J)+W(I+M2,J)+W(I-M2,J)
      DO54I=M3,LST,M
      B(I)=20.D0*W(I,J)-8.D0*(2.D0*W(I-1,J)+W(I+M,J)+W(I-M,J))+4.D0*
     1  (W(I+MM1,J)+W(I-M1,J))+2.D0*W(I-2,J)+W(I+M2,J)+W(I-M2,J)
      K=I-1
   54 B(K)=21.D0*W(K,J)-8.D0*(W(K-1,J)+W(K+1,J)+W(K-M,J)+W(K+M,J))+
     1  2.D0*(W(K+MM1,J)+W(K+M1,J)+W(K-M1,J)+W(K-MM1,J))+W(K-2,J)+
     2  W(K+M2,J)+W(K-M2,J)
      DO55K=2,N3
      KM=K*M
      DO55L=3,MM2
      I=KM+L
   55 B(I)=20.D0*W(I,J)-8.D0*(W(I-1,J)+W(I+1,J)+W(I+M,J)+W(I-M,J))+
     1  2.D0*(W(I+MM1,J)+W(I+M1,J)+W(I-M1,J)+W(I-MM1,J))+
     2  W(I-2,J)+W(I+2,J)+W(I+M2,J)+W(I-M2,J)
C
C        CALCULATE W(I,J+1)
C
      DO60K=1,N1
      KM=K*M
      DO60L=2,M
      I=KM+L
   60 W(I,J+1)=2.D0*W(I,J)-W(I,J-1)-C1*B(I)+C2*BB(I)+C3*P(J)
   70 CONTINUE
      RETURN
      END
```

```
C   SUBROUTINE FDIIB, CALCULATES W AND F FOR TEN TIME STEPS

       SUBROUTINE FDIIB (M,N,MN,W,F,A,B,C,BB,DC,DI,P,C1,C2,C3,E,KR)
       IMPLICIT REAL*8 (A-H,O-Z)
       REAL*8 DABS
       DIMENSION W(KR,1),F(KR,1),A(KR,1),B(1),C(1),BB(1),DC(1),DI(1),P(1)
C   SET UP CONSTANTS ONE TIME ONLY
     1 IF(W(MN,1).NE.0.) GO TO 2
       MN1=MN+1
       MN2=MN+2
       M1=M+1
       M2=2*M
       M3=3*M
       MP2=M+2
       MP3=M+3
       M22=M2+2
       MNM=MN-M
       MNM2=MNM+2
       MM1=M-1
       MM2=M-2
       M2M1=M2-1
       M2M2=M2-2
       MNM1=MN-1
       N1=N-1
       N2=N-2
       N3=N-3
       LT=(N-2)*M+2
       LT1=LT+1
       LT2=MNM-2
       LS=LT-M
       LST=LT-2
     2 CONTINUE
       DO70J=2,11
C
C   USE LINEAR TERMS ONLY FOR VERY SMALL  W
C
       IF(DABS(W(MN,J)).GT.0.01) GO TO 10
       DO3I=1,MN
       F(I,J)=0. DO
     3 BB(I)=0. DO
       GO TO 50
    10 CONTINUE
C
C   CALCULATE CONSTANT VECTOR FOR  (A)F=C    II(B)
C
       C(1)=0.DO
       C(2)=0.DO
       DO12I=2,M
       S=0.DO
       K=I
       DO11L=1,N2
       K=K+M
    11 S=S+(W(K+M,J)-W(K-M,J))**2
    12 C(I+1)=E*S/4.DO
       C(M2+1)=0.DO
```

```
      K=2
      DO14I=M3,MN,M
      K=K+M
      S=0.DO
      KK=K
      DO13L=1,MM2
      KK=KK+1
   13 S=S+(W(KK,J)-W(KK-2,J))**2
   14 C(I+1)=E*S/4.DO
      KK=K+M
      S=0.DO
      DO15L=1,MM2
      KK=KK+1
   15 S=S+(W(KK,J)-W(KK-2,J))**2
      C(MN2)=E*S/4.DO
      DO16K=1,N2
      KM=K*M
      DO16L=2,MM1
      I=KM+L
      IM=I+M
      IL=I-M
   16 C(I)=E*((W(IL-1,J)-W(IM-1,J)+W(IM+1,J)-W(IL+1,J))**2/16.DO-
     1 (W(I-1,J)-2.DO*W(I,J)+W(I+1,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J)))
      DO17I=M2,MNM,M
      IM=I+M
      IL=I-M
   17 C(I)= -E*(2.DO*W(I-1,J)-2.DO*W(I,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J))
      DO18I=MNM2,MNM1
      IL=I-M
   18 C(I)=E*(-(W(I-1,J)-2.DO*W(I,J)+W(I+1,J))*(2.DO*W(IL,J)-2.DO*
     1  W(I,J)))
      C(MN)=E*(-(2.DO*W(MN-1,J)-2.DO*W(MN,J))*(2.DO*W(MNM,J)-2.DO*
     1  W(MN,J)))
   20 CONTINUE
C
C       PERFORM GAUSS ELIMINATION ON C(I)
C
   21 KK=0
      LL=3*M
      DO23K=1,MN1
      K1=K+1
      IF(LL.LT.MN2) LL=LL+1
      L=DI(K)
      IF(L.EQ.K) GO TO 22
      D=C(L)
      C(L)=C(K)
      C(K)=D
   22 DO23I=K1,LL
      KK=KK+1
   23 C(I)=C(I)-DC(KK)*C(K)
C
C       PERFORM BACK SUBSTITUTION FOR F(I,J)
C
   31 F(MN2,J)=C(MN2)/A(MN2,MN2)
      I=MN2-1
```

```
   32 I1=I+1
      S=0. DO
      DO33K=I1,MN2
   33 S=S+A(I,K)*F(K,J)
      F(I,J)=(C(I)-S)/A(I,I)
      IF(I.EQ.2) GO TO 34
      I=I-1
      GO TO 32
   34 F(1,J)=0.
C
C        CALCULATE NONLINEAR TERMS FOR I OR II(B)
C
      DO41 K=1,N2
      KM=K*M
      DO41L=2,MM1
      I=KM+L
      IM=I+M
      IL=I-M
   41 BB(I)=(F(IL,J)-2.DO*F(I,J)+F(IM,J))*(W(I-1,J)-2.DO*W(I,J)+W(I+1,J)
     1  )+(F(I-1,J)-2.DO*F(I,J)+F(I+1,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J))
     2  -(F(IL-1,J)-F(IM-1,J)+F(IM+1,J)-F(IL+1,J))
     3  *(W(IL-1,J)-W(IM-1,J)+W(IM+1,J)-W(IL+1,J))/8.DO
      DO42I=M2,MNM,M
      IM=I+M
      IL=I-M
   42 BB(I)=(F(IL,J)-2.DO*F(I,J)+F(IM,J))*(2.DO*W(I-1,J)-2.DO*W(I,J))+
     1  (2.DO*F(I-1,J)-2.DO*F(I,J))*(W(IL,J)-2.DO*W(I,J)+W(IM,J))
      DO43I=MNM2,MNM1
      IL=I-M
   43 BB(I)=(2.DO*F(IL,J)-2.DO*F(I,J))*(W(I-1,J)-2.DO*W(I,J)+W(I+1,J))+
     1  (F(I-1,J)-2.DO*F(I,J)+F(I+1,J))*(2.DO*W(IL,J)-2.DO*W(I,J))
      BB(MN)=(2.DO*F(MNM,J)-2.DO*F(MN,J))*(2.DO*W(MNM1,J)-2.DO*W(MN,J))+
     1  (2.DO*F(MNM1,J)-2.DO*F(MN,J))*(2.DO*W(MNM,J)-2.DO*W(MN,J))
   50 CONTINUE
C
C       CALCULATE DEL FOURTH W FOR CLAMPED
C
      B(MP2)=22.DO*W(MP2,J)-8.DO*(W(MP3,J)+W(MP2+M,J))+2.DO*W(MP2+M1,J)+
     1  W(MP2+2,J)+W(M3+2,J)
      B(M2M1)=22.DO*W(M2M1,J)-8.DO*(W(M2-2,J)+W(M2,J)+W(M3-1,J))+
     1  2.DO*(W(M3-2,J)+W(M3,J))+W(M2-3,J)+W(M2M1+M2,J)
      B(M2)=21.DO*W(M2,J)-8.DO*(2.DO*W(M2-1,J)+W(M3,J))+4.DO*W(M3-1,J)+
     1  2.DO*W(M2-2,J)+W(M3+M,J)
      K=LT
      B(K)=22.DO*W(K,J)-8.DO*(W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
     1  (W(K+M1,J)+W(K-MM1,J))+W(K+2,J)+W(K-M2,J)
      K=K+M
      B(K)=21.DO*W(K,J)-8.DO*(W(K+1,J)+2.DO*W(K-M,J))+4.DO*W(K-MM1,J)+
     1  W(K+2,J)+2.DO*W(K-M2,J)
      K=MNM-1
      B(K)=22.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+W(K+M,J)+W(K-M,J))+2.DO*
     1 (W(K+MM1,J)+W(K+M1,J)+W(K-M1,J)+W(K-MM1,J))+W(K-2,J)+W(K-M2,J)
      K=K+1
      B(K)=21.DO*W(K,J)-8.DO*(2.DO*W(K-1,J)+W(K+M,J)+W(K-M,J))+4.DO*
     1  (W(K+MM1,J)+W(K-M1,J))+2.DO*W(K-2,J)+W(K-M2,J)
```

```
      K=MN-1
      B(K)=21.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+2.DO*W(K-M,J))+4.DO*
     1 (W(K-M1,J)+W(K-MM1,J))+W(K-2,J)+2.DO*W(K-M2,J)
      B(MN)=20.DO*W(MN,J)-16.DO*(W(K,J)+W(MNM,J))+8.DO*W(MN-M1,J)+
     1 2.DO*(W(MN-2,J)+W(MN-M2,J))
      DO51I=MP3,M2M2
   51 B(I)=21.DO*W(I,J)-8.DO*(W(I-1,J)+W(I+1,J)+W(I+M,J))+2.DO*
     1 (W(I+MM1,J)+W(I+M1,J))+W(I-2,J)+W(I+2,J)+W(I+M2,J)
      DO52I=LT1,LT2
      B(I)=21.DO*W(I,J)-8.DO*(W(I-1,J)+W(I+1,J)+W(I+M,J)+W(I-M,J))+2.DO*
     1 (W(I+MM1,J)+W(I+M1,J)+W(I-M1,J)+W(I-MM1,J))+W(I-2,J)+W(I+2,J)+
     2 W(I-M2,J)
      K=I+M
   52 B(K)=20.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+2.DO*W(K-M,J))+4.DO*
     1 (W(K-M1,J)+W(K-MM1,J))+W(K-2,J)+W(K+2,J)+2.DO*W(K-M2,J)
      DO53I=M22,LS,M
   53 B(I)=21.DO*W(I,J)-8.DO*(W(I+1,J)+W(I+M,J)+W(I-M,J))+2.DO*
     1 (W(I+M1,J)+W(I-MM1,J))+W(I+2,J)+W(I+M2,J)+W(I-M2,J)
      DO54I=M3,LST,M
      B(I)=20.DO*W(I,J)-8.DO*(2.DO*W(I-1,J)+W(I+M,J)+W(I-M,J))+4.DO*
     1 (W(I+MM1,J)+W(I-M1,J))+2.DO*W(I-2,J)+W(I+M2,J)+W(I-M2,J)
      K=I-1
   54 B(K)=21.DO*W(K,J)-8.DO*(W(K-1,J)+W(K+1,J)+W(K-M,J)+W(K+M,J))+
     1 2.DO*(W(K+MM1,J)+W(K+M1,J)+W(K-M1,J)+W(K-MM1,J))+W(K-2,J)+
     2 W(K+M2,J)+W(K-M2,J)
      DO55K=2,N3
      KM=K*M
      DO55L=3,MM2
      I=KM+L
   55 B(I)=20.DO*W(I,J)-8.DO*(W(I-1,J)+W(I+1,J)+W(I+M,J)+W(I-M,J))+
     1 2.DO*(W(I+MM1,J)+W(I+M1,J)+W(I-M1,J)+W(I-MM1,J))+
     2 W(I-2,J)+W(I+2,J)+W(I+M2,J)+W(I-M2,J)
C
C        CALCULATE W(I,J+1)
C
      DO60K=1,N1
      KM=K*M
      DO60L=2,M
      I=KM+L
   60 W(I,J+1)=2.DO*W(I,J)-W(I,J-1)-C1*B(I)+C2*BB(I)+C3*P(J)
   70 CONTINUE
      RETURN
      END
```

VITA

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