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GENERAL ALGEBRAS AND THEIR MANIPULATIVE SYNTAXES

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GENERAL ALGEBRAS AND THEIR MANIPULATIVE SYNTAXES

INTRODUCTION

It is part of the folklore of algebra that certain systems — groups in particular — have differing formulations which are essentially equivalent; the equivalence being shown by informal comparison of properties. More formally, Post, Sierpinski, Webb, and Los ([10,11,13,7], 1921-1950) investigated the reduction of operations over abstract sets to compositions of binary operators, while G. Birkhoff ([2], 1935) propounded the notions of general algebras and their species. E. Marczewski appears to have been the first ([8], 1958) to have explicitly used these notions to identify algebras of different species. In this dissertation, the above ideas are combined rather differently and for quite different purposes from those of Marczewski by wedding them with an idea of A. S. Davis — that of a Total Transformation Algebra ([5], 1966). The latter — in the form of a morphology — has been coupled by Davis and Chance to the mainstream of formal language theory ([6,3], 1969). In Chapter 0 following, an extention of a particular kind of Total Transformation Algebra — a <u>Function System</u> — is described in detail. These morphology-like structures provide the formal framework for the algebraic development in chapters I - III.

In Chapter I, there is associated with each algebra a formal Syntax (<u>Manipulative Syntax</u>) in a Function System. Each such Syntax leads to an equivalence class of algebras such that congruences are invariant among members of the class; i.e. a congruence on <u>any</u> algebra in a class is a congruence on <u>every</u> member of the class (Theorem 1.16). It follows that all factors of a given class by a (particular) congruence are equivalent (Theorem 1.22). Again, any congruence on an algebra induces a set-theoretically defined congruence on the Syntax of the algebra (Theorem 1.25) and, in the finitary case (at least), the converse is also true (Theorem 1.28). The question of the unrestricted converse is open, as is the question of whether the 1-1 mapping of congruences defined in 1.26 is onto the congruences of the Syntax.

Chapter II is devoted to the notion of structure-preserving mappings and the related concept of homomorphism. A fundamental result (Theorem 2.13.2) is that (non-trivial) homomorphisms of Syntaxes generate mappings (<u>linguistic morphisms</u>) of their associated finitary algebras. Linguistic morphisms, while not necessarily homomorphisms, can be thought of as structure preserving due to their association with their Syntax morphisms. (This idea seems completely new.) Furthermore (Theorem 2.13.1), given a linguistic morphism φ between algebras $\mathfrak{A}$ and $\mathfrak{B}$, there exists an algebra $\mathfrak{C}$ such that φ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$ and $\mathfrak{C}$ is subsumed by $\mathfrak{B}$ (that is, the Syntax of $\mathfrak{C}$ is contained in the Syntax of $\mathfrak{B}$). Another primary result in Chapter II (Theorem 2.20.2) is that isomorphic algebras have isomorphic Syntaxes in their isomorphic Function Systems. From this it follows in particular that epimorphisms are linguistic morphisms (Theorem 2.22). A question for future research which is posed by the results of Chapter II is whether the carriers of Syntaxes constitute a language-theoretically characterizable class of distinguished languages.

Chapter III is a view of Universal Algebra as the study of a category of certain equivalence classes of algebras. Intuitively, elements in a class generate the same set of equations through composition and concatenation of their operations. Morphisms in the category amount to identification of certain sets of equations. The classes are developed by using the equivalence of Chapter I (Functional Equivalence) in combination with the isomorphism equivalence in the class of algebras. It is shown that these relations commute, and the resultant category is distinguished in that the only isomorphisms that exist are identity maps. Then, the analogue of the fundamental homomorphism theorem is reduced to triviality. This suggests that the notions of Isomorphism and Functional Equivalence, taken together, provide as much identification in the class of algebras as is consistent with maintaining the lattice structure of algebras and their factor algebras.

It is with apologies that a rather primitive notation is used. The reasons for its use are that it shows explicitly the duality of finitary expressions expressed informally with those expressed in the Syntax of an algebra and it also remains the most easily read of the possibilities encountered. Terms being defined in the text are underlined. If f is a function, its value at a point x is denoted f(x), and its restriction to a set B is denoted f B. Due to the great need for parentheses in the function-value constructs, symbolic-logic statements use squared parentheses [and] rather than the usual. Triangular parentheses ⟨ and ⟩ denote either algebras or n-tuples. The end of a proof is denoted by //.

CHAPTER 0

PRELIMINARIES

In [5], A. S. Davis introduced the notion of a <u>Total Transformation Algebra</u> over a set in order to obtain an axiomatic treatment of the common notions of "composition" and "concatenation" of functions. In obtaining his axioms, Davis briefly describes the construction of a particular algebra over a given set, then characterizes this algebra. This characterization has been adapted by Davis [6] to the study of formal languages, and that work has been extended by Chance in her dissertation [3] directed by Davis. In the work of this paper, the <u>particular</u> construction of [5], in the slightly generalized form of a partial algebra, plays a central role. Because of this fact, a development of the ideas involved is included here.

The following hypotheses and notational conventions are assumed to hold throughout the paper.

0.1 Let A be a non-empty set.

0.2 Let N be the set of natural numbers (including zero).

Let $N^+ = N - \{0\}$.

0.3 Let $\mathcal{N}$ be the set of ordinals from $\emptyset$ up to and including the first infinite ordinal. The latter is denoted by ω. Let $\mathcal{N}^+ = \mathcal{N} - \{\emptyset\}$.

0.4 Let I be the set of all integers.

0.5 Let $m \in \mathcal{N}^+$, $n \in \mathcal{N}$. The following notation will be used.

i) $g : A^0 \longrightarrow A^m$ iff $g \in A^m$.

ii) $g : A^n \longrightarrow A^m$ iff g is a function with domain A^n and range in A^m.

0.6 Let $A^* = \bigsqcup_{i \in N^+} A^i$, $A^{**} = A^* \cup A^\omega$. Thus A^{**} is the set of all sequences (finite and infinite) to A.

0.7 If $n \in \mathcal{N}^+$, denote $x \in A^n$ by $\langle x_1, \ldots \rangle_n$. Occasionally intermediate terms will be included, e.g. $\langle x_1, \ldots, x_k, x_{k+1}, \ldots \rangle_n$. If $n \in N^+$, $\langle x_1, \ldots, x_n \rangle$ will also be used.

0.8 DEFINITION. For each $n \in \mathcal{N}^+$, let $\underline{n} : A^{**} \longrightarrow A^{**}$ be the mapping such that, for each $m \in \mathcal{N}^+$, for each $x \in A^m$, if $n \leq m$, $\underline{n}(x) = \langle x_1, \ldots \rangle_n$ and if $n > m$, $\underline{n}(x) = \langle z_1, \ldots \rangle_n$, where $z_j = x_k$ when $j - k \equiv 0 \pmod{m}$, $1 \leq j \leq n$, and $1 \leq k \leq m$.

Definition 0.8 provides a countable infinity of operators for converting sequences of one length into sequences of other lengths. Heuristically, given a finite m-tuple $\langle x_1, \ldots, x_m \rangle$ and finite n, $\underline{n}(\langle x_1, \ldots, x_m \rangle)$ can be thought of as being formed mechanically by writing $\langle x_1, \ldots, x_m, x_1, \ldots, x_m, \ldots \rangle$ to form an m·n-tuple and then restricting this m·n-tuple to the first n components. This is a wasteful procedure but involves no decision processes, hence can be carried out by a simple machine. Note that if $m < \omega$, then $\underline{\omega}(\langle x_1, \ldots, x_m \rangle)$ is the infinite sequence $\langle x_1, \ldots, x_m, x_1, \ldots, x_m, x_1, \ldots \rangle$, and that $\underline{\omega} | A^\omega$ is the identity mapping on A^ω, and similarly for any $n \in N^+$.

0.9 DEFINITION. Let $n \in \mathcal{N}$, $m \in \mathcal{N}^+$. Let $f : A^n \longrightarrow A^m$. Then

i) If $n \neq 0$, $\overline{f} : A^{**} \longrightarrow A^{**}$ is the mapping $f \circ \underline{n}$.

ii) If $n = 0$ (so $f \in A^m$), $\overline{f} : A^{**} \longrightarrow A^{**}$ is the constant map with range $\{f\}$ (i.e. $\overline{f}(x) = f$ for each $x \in A^{**}$). $\overline{f}$ is the <u>d-extention</u> of f to A**.

The significance of this construction of $\overline{f}$ is that $\overline{f}$ accepts any sequence as a domain-point; in particular, two d-extentions $\overline{f}$ and $\overline{g}$ always compose. For each $n \in \mathcal{N}^+$, for each $x \in A^n$, the d-extention of ii) is called n-constant.

0.10 EXAMPLE. Let $f : A^3 \longrightarrow A$. Then $\underline{3}\langle a_1\rangle = \langle a_1,a_1,a_1\rangle$, $\underline{3}\langle a_1,a_2\rangle = \langle a_1,a_2,a_1\rangle$, $\underline{3}\langle a_1,a_2,a_3\rangle = \langle a_1,a_2,a_3\rangle$, and $\underline{3}\langle a_1,a_2,a_3,a_4\rangle = \langle a_1,a_2,a_3\rangle$; whence $\overline{f}\langle a_1\rangle = f\langle a_1,a_1,a_1\rangle$, $\overline{f}\langle a_1,a_2\rangle = f\langle a_1,a_2,a_1\rangle$, $\overline{f}\langle a_1,a_2,a_3\rangle = f\langle a_1,a_2,a_3\rangle$, and $\overline{f}\langle a_1,a_2,a_3,a_4\rangle = f\langle a_1,a_2,a_3\rangle$. As usual, $f\langle a_1,a_2,a_3\rangle$ is written for $f(\langle a_1,a_2,a_3\rangle)$.

0.11 REMARK. Note in particular (ignoring the 1-tuple — element distinction) that for $n \neq 0$, if $f : A^n \longrightarrow A^m$ then $\overline{f}|A^n = f$; thus given an extention and the index n of the domain of f, f can be recovered from $\overline{f}$. (Caution! Note that $\overline{\overline{f}|A^k} \neq \overline{f}$ in general. Let $f : \{a,b\} \times \{a,b\} \longrightarrow \{a,b\}$ be such that $f\langle a,a\rangle = f\langle b,a\rangle = a$, $f\langle a,b\rangle = f\langle b,b\rangle = b$. Then $\overline{\overline{f}|A^1}\langle a,b\rangle = \overline{f}|A(\underline{1}\langle a,b\rangle) = \overline{f}\langle a\rangle = a$ whereas $\overline{f}\langle a,b\rangle = f\langle a,b\rangle = b$.)

0.12 LEMMA. Let $k,n \in \mathcal{N}$, $m \in \mathcal{N}^+$. Then $f : A^n \longrightarrow A^m$, $g : A^k \longrightarrow A^m$, and $\overline{f} = \overline{g}$ implies

i) $n = k$ implies $f = g$

ii) $0 = n < k$ implies g is m-constant with range $\{f\}$

iii) $0 < n < k$ implies $f\langle x_1,\ldots\rangle_n = g\langle x_1,\ldots,x_n,x_{n+1},\ldots\rangle_k$ for all $x_1,\ldots,x_n,x_{n+1},\ldots$ in A.

Proof: If $n = k$, then $f = \overline{f}|A^n = \overline{g}|A^k = g$. If $0 = n < k$, then $f\langle x_1,\ldots,x_n\rangle = \overline{f}\langle x_1,\ldots\rangle_k = \overline{g}\langle x_1,\ldots\rangle_k = g\langle x_1,\ldots\rangle_k$ for any $x_1,\ldots$ in A. If $0 < n < k$, then $f\langle x_1,\ldots,x_n\rangle = \overline{f}\langle x_1,\ldots\rangle_n$. But $\langle x_1,\ldots\rangle_n = \underline{n}\langle x_1,\ldots,x_n,x_{n+1},\ldots\rangle_k$ implies $f\langle x_1,\ldots\rangle_n = \overline{f}\langle x_1,\ldots,x_{n+1},\ldots\rangle_k = \overline{g}\langle x_1,\ldots\rangle_k = g\langle x_1,\ldots\rangle_k$. //

The significant import of Lemma 0.12 is that when two functions $f : A^n \longrightarrow A^m$ and $g : A^k \longrightarrow A^m$ have the same d-extention and $n < k$, the last $k - n$ components of a k-tuple x play no role in determining the value of $g(x)$.

There is need for the usual notion of projections from a product A^n and the d-extentions of them.

0.13 DEFINITION. For each $n \in \mathcal{N}^+$ and $j \in N^+$ with $j < n$, let ${}_n\pi_j$ be the mapping of A^n into A such that ${}_n\pi_j\langle x_1, \ldots \rangle_n = x_j$.

Note explicitly that for all $n, m \in \mathcal{N}^+$ and $j \in N$ with $j < n, m$; ${}_n\bar{\pi}_j = {}_m\bar{\pi}_j$.

0.14 DEFINITION. Let $\pi_j = {}_n\bar{\pi}_j$ (for any $n \in \mathcal{N}^+$ with $j < n$).

Observe that for $\langle x_1, \ldots \rangle_k \in A^{**}$, $\pi_j\langle x_1, \ldots \rangle_k = x_t$ where $t = j$ if $k = \omega$ and $t \equiv j \pmod{k}$ if k is finite.

A Function System over A can now be constructed. The first step is specification of the set $\underline{A}$, over which operations will be defined.

0.15 DEFINITION.

i) For each $m, n \in \mathcal{N}^+$, let $A_n^{\ m}$ be the set of all functions on A^n into A^m.

ii) For each $m \in \mathcal{N}^+$, let $A_0^{\ m} = A^m$.

iii) Let $T = \bigsqcup_{m \in N^+} \bigsqcup_{n \in N} A_n^{\ m} \cup \bigsqcup_{m \in N^+} A_\omega^{\ m}$; $\underline{T} = T \cup \bigsqcup_{n \in N} A_n \cup A_\omega^\omega$.

v) Let $\hat{A} = \{\alpha \subseteq A^{**} \times A^{**} : \alpha \text{ is a function and } [\exists g \in T][\alpha = \bar{g}]\}$.

vi) Let $\underline{A} = \{\alpha \subseteq A^{**} \times A^{**} : \alpha \text{ is a function and } [\exists g \in \underline{T}][\alpha = \bar{g}]\}$.

Specification of the binary operations $\circ$ (composition) and $+$ (concatenation) in $\underline{A} \times \underline{A}$, the unary operation ' (shift) on $\underline{A}$, and the distinguished element π will complete the formulation.

0.16 DEFINITION of <u>composition</u>. $\circ : \underline{A} \times \underline{A} \longrightarrow \underline{A}$ is the mapping such that, for $\alpha, \beta \in \underline{A}$, $\alpha \circ \beta : A^{**} \longrightarrow A^{**}$ where for each $x \in A^{**}$, $(\alpha \circ \beta)(x) = \alpha(\beta(x))$. Hereafter, juxtaposition will denote this operation.

0.17 DEFINITION of Concatenation. $+ : \hat{A} \times \underline{A} \longrightarrow \underline{A}$ is the mapping such that for each $\alpha \in \hat{A}$, $\beta \in \underline{A}$, $\alpha + \beta : A^{**} \longrightarrow A^{**}$ is the mapping defined by $(\alpha+\beta)(x) = \langle z_1, \ldots \rangle_{k+j}$ where $\alpha(x) = \langle u_1, \ldots, u_k \rangle$, $\beta(x) = \langle v_1, \ldots \rangle_j$, $z_i = u_i$ if $1 < i < k$, and $z_i = v_{i-k}$ for all i involved which exceed k. (The convention is hereby established that sub or superscript arithmetic is cardinal arithmetic, thus $k = \omega$ implies $j+k = \omega$.)

Where convenient, $(\alpha+\beta)(x)$ will be denoted by $\langle\langle \alpha(x), \beta(x) \rangle\rangle$.

A lemma is needed to support the definition of Shift.

0.18 LEMMA. Let $\alpha \in \underline{A}$. Then i) $[\exists m \in \mathcal{N}^+][\text{ range}(\alpha) \subseteq A^m]$ and ii) $\text{range}(\alpha) \subseteq A^k$ implies $[\exists n \in \mathcal{N}][$ n is the least ordinal such that $[\exists f \in A_n{}^m][\alpha = \bar{f}\,]]$.

Proof of i): By 0.15.vi, $\alpha \in \underline{A}$ implies $[\exists g \in \underline{T}][\alpha = \bar{g}\,]$. But $g \in \underline{T}$ implies $[\exists n \in \mathcal{N}][\exists m \in \mathcal{N}^+][g \in A_n{}^m]$. Suppose $g \in A_s{}^t$. Then $s > 0$ implies g is a function and $\text{range}(g) \subseteq A^t$, whence range $(\alpha) =$ $\text{range}(\bar{g}) \subseteq A^t$. On the other hand, $s = 0$ implies $g \in A^t$, whence $\text{range}(\bar{g}) =$ $\{g\} \subseteq A^t$. In either case, then, i) follows.

Proof of ii). Again $\alpha \in \underline{A}$ implies $[\exists g \in \underline{T}][\alpha = \bar{g}\,]$, and $[\exists n \in \mathcal{N}]$ $[\exists m \in \mathcal{N}^+][g \in A_n{}^m]$, whence for some $t \in \mathcal{N}^+$, $\text{range}(g) \subseteq A^t$. But then $\text{range}(\alpha) \subseteq A^t$ also, i.e. $\text{range}(\alpha) \subseteq A^t \cap A^k$ and this can occur only if $\text{range}(\alpha) = \emptyset$ or $t = k$. By construction of $\underline{A}$ $\text{range}(\alpha) \neq \emptyset$, whence $g \in A_s{}^k$ for some $s \in \mathcal{N}^+$. Let $B = \{n \in \mathcal{N}^+ : [\exists f \in A_n{}^k][\alpha = \bar{f}\,]\}$. We have shown $B \neq \emptyset$. But then, as a non-empty set of ordinals, B has a least element which serves as the n of ii.

0.19 DEFINITION of Shift. $' : \underline{A} \longrightarrow \underline{A}$ is the mapping such that, for $\alpha \in \underline{A}$, $\alpha' : A^{**} \longrightarrow A^{**}$ where; if $\text{range}(\alpha) \subseteq A^m$ and n is the least ordinal such that $[\exists g \in A_n{}^m][\alpha = \bar{g}\,]$, then i) $0 = n$ implies $\alpha' = \alpha$ whereas ii) $0 < n <$ implies $[\forall x \in A^{**}][\alpha'(x) = g\langle z_2, \ldots \rangle_{n+1}]$ where $\langle z_1, \ldots \rangle_{n+1} =$

$\underline{n+1}(x)$.

0.20 REMARK. 1/ By 0.9.i there is no ambiguity in 0.19 relative to the function g.

2/ It is clear that the operations $\circ$ and $'$, and the partial operation $+$ are closed (into $\underline{A}$), for given $\overline{f}, \overline{g} \in \underline{A}$, $\overline{h} \in A$, with $f \in A_n{}^m$, $g \in A_r{}^s$, and $h \in A_u{}^v$, it follows that $\overline{f} \circ \overline{g} = f \circ \underline{n} \circ g \circ \underline{r} = \overline{f \circ \underline{n} \circ g}$ while $f \circ \underline{n} \circ g \in A_r{}^m$. Again, $\overline{h} + \overline{f} = \overline{\gamma}$ where $\gamma = C \circ \langle f \circ \underline{n}, g \circ \underline{r} \rangle \,|\, A^{\max(r,n)}$. ($\langle f \circ \underline{n}, g \circ \underline{r} \rangle$ is the function on $A^{\max(r,n)}$ into $A^m \times A^s$ such that $\langle f \circ \underline{n}, g \circ \underline{r} \rangle(x) = \langle f(\underline{n}(x)), g(\underline{r}(x)) \rangle$ and $C : A^m \times A^s \longrightarrow A^{m+s}$ is the canonical map of such 2-tuples to m+s-tuples.) This gives $\gamma \in A_{\max(r,n)}{}^{m+s}$ so $\overline{h} + \overline{f} \in \underline{A}$. Finally, $\overline{f}' = \overline{h}$ where $h \in A_n{}^m$ is defined by $h\langle x_1, \ldots, x_n \rangle = f\langle x_2, \ldots, x_n, x_1 \rangle$ for n finite, or $h\langle x_1, \ldots \rangle_\omega = f\langle x_2, \ldots \rangle_\omega$ if $m = \omega$, as is shown by the following:

0.21 LEMMA. Let $f \in A_n{}^m$. Let $x \in A^{**}$. Then $\overline{f}'(x) = f\langle z_2, \ldots \rangle_{n+1}$ where $\langle z_1, \ldots \rangle = \underline{n+1}(x)$. In particular, n finite and $x \in A^n$ implies $\overline{f}'(x) = f\langle z_2, \ldots, z_n, z_1 \rangle$.

Proof: Let $g \in A_k{}^m$ with $\overline{f} = \overline{g}$ and k minimal (as in 0.18). If $k = 0$, $\overline{g}$ is m-constant and $\overline{f}' = \overline{g}' = \overline{g}$. This implies $[\forall y \in A^{**}][\ \overline{f}'(y) = \overline{g}(y) = g = \overline{f}(y)\ ]$ by 0.12.ii. In particular $f\langle z_2, \ldots \rangle_{n+1} = g = f'(x)$ for any $x \in A^{**}$. If $0 < k < \omega$, then 0.12.iii implies $g\langle z_2, \ldots \rangle_{k+1} = f\langle z_2, \ldots, z_{k+1}, z_{k+2}, \ldots \rangle_{n+1}$ for any $z_{k+2}, \ldots \in A$. Then $g\langle z_2, \ldots \rangle_{k+1} = f\langle z_2, \ldots \rangle_{n+1}$ where $\underline{n+1}(x) = \langle z_1, \ldots \rangle_{n+1}$. It follows that $\overline{f}'(x) = g\langle z_2, \ldots \rangle_{k+1}$. This gives the conclusion also. Finally, $k = \omega$ implies $n = \omega$ whence 0.12.i implies $f = g$ and the definition 0.19.ii provides equality. The conclusion has been shown to follow for any n. For the second part, note that for n finite, $\underline{n+1}\langle x_1, \ldots, x_n \rangle = \langle x_1, \ldots, x_n, x_1 \rangle$. //

0.22 DEFINITION. i) $\pi = \pi_1 = {}_n\overline{\pi}_1$ for every $n \in \mathcal{N}^+$ (uniformly).

ii) $\pi_m = {}_n\bar{\pi}_m$ for all $m,n \in \mathcal{N}^+$ with $m < n$.

0.23 DEFINITION. The partial algebra $\langle \underline{A}, \pi, \circ, +, ' \rangle$ is the Function System over A.

0.24 REMARK. The above formulation provides several conveniences.

Non-trivial compositions of functions obtain whenever domains and codomains, whether they overlap or not, are products of the same set. The advantage is evident — there are no fussy domain-range questions in compositions. In addition, ordinary functions and composition can be thought of as a sort of "cross-section" (as a plane is a cross-section of E^n) of this much more general class of functions with their special form of composition.

In concatenation, there is a formalization of oft-used statements like: "given functions $f_1, \ldots, f_m$, on E^n to the real numbers, let g be the function on E^n to E^m such that $g(x) = \langle f_1(x), \ldots, f_m(x) \rangle$". In a Function System, this is captured in letting $\bar{g} = \bar{f}_1 + \ldots + \bar{f}_m$.

Finally, there is a convenient way to introduce constants into statements (formulas) — especially to restrict functions to constants in various "slots" in their domains; e.g. suppose f is a function of five "variables" with domain(f) $= A^5$. If $x,y \in A$, then the function $\bar{f}(\bar{x} + \pi_2 + \bar{y} + \pi_4 + \pi_5)$ has values determined by 3 "slots" in any n-tuple to which it is applied. In particular, for any $a_1, a_2, \ldots, a_5 \in A$, $\bar{f}(\bar{x} + \pi_2 + \bar{y} + \pi_4 + \pi_5)\langle a_1, \ldots, a_5 \rangle = f\langle x, a_2, y, a_4, a_5 \rangle$.

It is noted in passing that it may be productive to re-do this Function System to allow for functions which are not on entire cartesian products; i.e. for partial functions, but for the initial work that refinement has not been attempted.

In closing this introductory chapter, some results will be established

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Now suppose A contains at least 2 elements. Then $[\forall k \in N^+][\pi_k = {}_k\bar{\pi}_k]$. Furthermore, k is the least integer j such that $[\exists g \in A_j{}^1][\pi_k = \bar{g}\ ]$. This follows since (given two distinct elements) π_j is not constant (whence $j \neq 0$) and further, if $f \in A_{k-1}^{\ 1}$, $\pi_k = \bar{f}$, and $u, v \in A$ with $u \neq v$ then a contradiction follows from observing that $u = \pi_k\langle x_1, \ldots, x_{k-1}, u\rangle = \bar{f}\langle x_1, \ldots, x_{k-1}, u\rangle = f\langle x_1, \ldots, x_{k-1}\rangle = \bar{f}\langle x_1, \ldots, x_{k-1}, v\rangle = \pi_k\langle x_1, \ldots, x_{k-1}, v\rangle = v$. It is emphasized; k is minimal.

The theorem can now be completed by induction on n. Let $\langle x_1, \ldots\rangle_j \in A^{**}$. For $k \in N$, let $\langle z_1, \ldots\rangle_{k+1} = \underline{k+1}\langle x_1, \ldots\rangle_j$.

For $n = 1$: $\pi'\langle x_1, \ldots\rangle_j = \pi_1{}'\langle x_1, \ldots\rangle_j = {}_1\pi_1\langle z_2\rangle$ where $\langle z_1, z_2\rangle = \underline{2}\langle x_1, \ldots\rangle_j$. But ${}_1\pi_1\langle z_2\rangle = z_2 = {}_2\pi_2\langle z_1, z_2\rangle = \pi_2\langle x_1, \ldots\rangle_j$. We therefore have $\pi' = \pi_2$.

Suppose now, $\forall k < n$ that $\pi^{(k)} = \pi_{k+1}$. Then $\pi^{(n+1)}\langle x_1, \ldots\rangle_j = (\pi^{(n)})'\langle x_1, \ldots\rangle_j = (\pi_{n+1})'\langle x_1, \ldots\rangle_j = {}_{n+1}\pi_{n+1}\langle z_2, \ldots, z_{n+2}\rangle = z_{n+2} = {}_{n+2}\pi_{n+2}(\underline{n+2}\langle x_1, \ldots\rangle_j) = \pi_{n+2}\langle x_1, \ldots\rangle_j$. As previously, $\langle z_1, \ldots, z_{n+2}\rangle$ is $\underline{n+2}\langle x_1, \ldots\rangle_j$. //

0.28 LEMMA. Let $g \in A_n{}^m$ and let $j \in N^+$ and let $a_1, \ldots, a_j \in A$. Then: i) $\bar{g}(\bar{a}_1 + \ldots + \bar{a}_j) = \overline{\bar{g}\langle a_1, \ldots, a_j\rangle}$.

ii) If $n = j$ (so j is finite and $g : A^j \longrightarrow A^m$), then also $\bar{g}(\bar{a}_1 + \ldots + \bar{a}_j) = \overline{g\langle a_1, \ldots, a_j\rangle}$.

Proof: Let $x \in A^{**}$. Then $(\bar{g}(\bar{a}_1 + \ldots + \bar{a}_j))(x) = \bar{g}\langle \bar{a}_1(x), \ldots, \bar{a}_j(x)\rangle = \bar{g}\langle a_1, \ldots, a_j\rangle = (\overline{\bar{g}\langle a_1, \ldots, a_j\rangle})(x)$. For part ii), simply note $\bar{g}\langle a_1, \ldots, a_j\rangle = g\langle a_1, \ldots, a_j\rangle$. //

The next result (Theorem 0.30, due to Davis) shows that the shift operation on d-extentions of finitary functions can be duplicated by appropriate composition with a concatenation of projections. A lemma is needed which is essentially a specialization of theorem 3 of Davis [5].

0.29 LEMMA. $[\forall n, m \in N][\ (\pi^{(n)} + \pi^{(m)})' = \pi^{(n+1)} + \pi^{(m+1)}\ ]$

Proof: Pointwise.

0.30 THEOREM. Let $\alpha \in \underline{A}$ with range$(\alpha) \subseteq A^m$. Let $n \in \mathcal{N}$ be the least ordinal such that $[\exists g \in A_n{}^m][\ \alpha = \bar{g}\ ]$. Then, n is finite implies $[\forall k \in N^+]$ $[\ \alpha^{(k)} = \alpha(\pi^{(k)} + \ldots + \pi^{(k+n-1)})\ ]$. (By fiat, $\pi^{(k)} + \ldots + \pi^{(k+0-1)}$ is defined to be π.)

Proof: Suppose n is finite. If $n = 0$, then $\alpha^{(k)} = \alpha = \alpha\pi$ since α is m-constant, and the conclusion is obtained. The proof proceeds by induction on k with $n \neq 0$. Note that for $x \in A^{**}$, $(\pi^{(1)} + \ldots + \pi^{(n)})(x) =$ $\langle\ {}_2\pi_2(\underline{2}(x)), \ldots, {}_{n+1}\pi_{n+1}(\underline{n+1}(x))\ \rangle = \langle z_2, \ldots, z_{n+1} \rangle$ where $\langle z_1, \ldots, z_{n+1} \rangle =$ $\underline{n+1}(x)$. (This follows by definition of + and the minimality of k giving ${}_k\bar{\pi}_k = \pi_k$ per the proof of 0.27 and the fact that for $j < n+1$, $\underline{j}(\underline{n+1}(x))$ is $\underline{n+1}(x)$ restricted to the first j slots, i.e. $\underline{j}(x)$.

Suppose $k = 1$. Then $x \in A^{**}$ implies $\alpha'(x) = g\langle z_2, \ldots, z_{n+1} \rangle =$ $\bar{g}\langle z_2, \ldots, z_{n+1} \rangle = \alpha((\pi^{(1)} + \ldots + \pi^{(n)})(x)) = \alpha(\pi^{(1)} + \ldots + \pi^{(n)})(x)$, whence $\alpha' = \alpha(\pi^{(1)} + \ldots + \pi^{(1+n-1)})$ and the statement follows for $k = 1$. Suppose now $\alpha^{(j)} = \alpha(\pi^{(j)} + \ldots + \pi^{(j+n-1)})$ for all $j < k$. Then $\alpha^{(k)} = (\alpha^{(k-1)})' =$ $(\alpha(\pi^{(k-1)} + \ldots + \pi^{(k+n-2)})'$ by hypothesis. Therefore $\alpha^{(k)} =$ $\alpha(\pi^{(k-1)} + \ldots + \pi^{(k+n-2)})' = \alpha(\pi^{(k)} + \ldots + \pi^{(k+n-1)})$ by 0.25.iv and 0.29. //

The preliminary development is now complete, and the machinery prepared for the integration of algebraic concepts with Function Subsystems in Chapter I.

CHAPTER I

MANIPULATIVE SYNTAXES AND CONGRUENCES

Representations and Manipulative Syntaxes. Definitions 1.1 and 1.2 are essentially those of Pierce [9]. They provide reference and help establish notation and terminology.

1.1 DEFINITION. Let U be a set and $\alpha : U \longrightarrow \eta$ be a mapping. Then $\langle A, \tau \rangle$ is a U,α representation of an Algebraic System iff τ is a mapping on U such that $[\forall u \in U][\ \alpha(u) = n$ implies $\tau(u) \in A^{A^n}]$.

A is the carrier of $\langle A, \tau \rangle$.

The above will be abbreviated to "representation" or, when necessary, "U,α representation". Carriers are always assumed to be non-empty. "Representation" is used here in place of the customary "algebra" to emphasize the equivalence-class nature of algebraic systems (Definitions 1.8,3.1, and especially 3.6).

1.2 DEFINITION. Let $\langle A, \tau \rangle$ and $\langle B, \gamma \rangle$ be U,α representations. Let $f : A \longrightarrow B$. Then

i) f is a homomorphism of $\langle A, \tau \rangle$ into $\langle B, \gamma \rangle$ iff $[\forall u \in U]$ $[\forall x = \langle x_1, \ldots \rangle_{\alpha(u)} \in A^{\alpha(u)}][\ f((\tau(u))(x)) = \gamma(u) \langle f(x_1), \ldots \rangle_{\alpha(u)}\ ]$.

ii) f is an isomorphism of $\langle A, \tau \rangle$ onto $\langle B, \gamma \rangle$ iff f is a homomorphism and f is 1-1 on A onto B.

1.3 REMARK. 1.1 and 1.2, while standard, are not altogether satisfactory due to their restrictiveness. For example, if $\vee$ and $\wedge$ are operations which make $\langle A, \{\langle 1,\vee\rangle, \langle 2,\wedge\rangle\}\rangle$ a lattice, it seems unfortunate to insist that $\langle A, \{\langle a,\vee\rangle, \langle b,\wedge\rangle\}\rangle$ is a "different" lattice and allow only a gross intuitive comparison of these (under the guise that their relationship is "evident"). Indeed, by 1.2 and now standard convention, one cannot even call the identity map on A a homomorphism between the two, as homomorphisms exist only between systems indexed by the same set. After a few preliminary steps, there will be set forth a formal criterion (Definition 1.10) for calling two representations equivalent when not only the indexing sets may be different, but also the cardinalities of the sets of indexed operations. In particular, the two lattices mentioned will be equivalent. Some natural results on congruences will then follow for the equivalence classes of representations. The question of "structure preserving" maps will be taken up later (Chapter II).

1.4 DEFINITION. Let $\langle A, \mathcal{T}\rangle$ be a U,α representation. Let $\mathfrak{A} = \langle \underline{A}, \pi, \circ, +, ' \rangle$ be the Function System over A. Then $\mathcal{X}$ is the <u>Manipulative Syntax of $\langle A, \mathcal{T}\rangle$ in $\mathfrak{A}$</u> iff $\mathcal{X}$ is the subsystem of $\mathfrak{A}$ generated by $\{\bar{f} : f \in A \cup \mathcal{T}[U]\}$.

The set of $\mathcal{X}$ will be denoted by $[\bar{A} \cup \bar{\mathcal{T}}]$.

1.5 REMARK. As usual, by "$\mathcal{X}$ is the subsystem of $\mathfrak{A}$ generated by set $\bar{G}$", is meant $\mathcal{X}$ is the intersection of subsystems of $\mathfrak{A}$ whose carriers contain $\bar{G}$. Also; $\mathcal{T}[U] = \{\mathcal{T}(u) : u \in U\}$. $\bar{f}$ indicates the d-extention of f. For the purposes of this definition, it is convenient to consider the operations in the Funotion System to be naturally indexed by $\{1,2,3,4\}$.

1.6 DEFINITION. Let $\langle A, \mathcal{T}\rangle$ be a U,α representation. Let $\langle A, \gamma\rangle$ be a V,β representation. Let $\mathfrak{A}$ be the Function System over A. Then $\langle A, \mathcal{T}\rangle$

is <u>functionally</u> <u>subsumed</u> in $\langle A, \gamma \rangle$ iff the Manipulative Syntax of $\langle A, \tau \rangle$ in $\mathfrak{A}$ is a subsystem of the Manipulative Syntax of $\langle A, \gamma \rangle$ in $\mathfrak{A}$. Notation: $\langle A, \tau \rangle \prec \langle A, \gamma \rangle$.

1.7 REMARK. Observe that the class of all representations (algebras) over A is partially ordered by $\prec$, that a maximal representation in this class is given by $\langle A, I_F \rangle$ where I_F is the identity function on the set of all operations with carrier A, and a (unique) minimum representation is given by $\langle A, \emptyset \rangle$. An upper bound for $\langle A, \tau \rangle$ and $\langle A, \gamma \rangle$ is found in $\langle A, I_U \rangle$ where I_U is the identity map on $\tau[U] \cup \gamma[V]$. A lower bound is given by $\langle A, I_\cap \rangle$ where $I_\cap$ is the identity map on $\tau[U] \cap \gamma[V]$.

1.8 DEFINITION. Let $\langle A, \tau \rangle$ be a U, α representation. Let $\langle A, \gamma \rangle$ be a V, β representation. Then $\langle A, \tau \rangle$ is <u>A-Functionally</u> <u>Equivalent</u> with $\langle A, \gamma \rangle$ iff $\langle A, \tau \rangle \prec \langle A, \gamma \rangle$ and $\langle A, \gamma \rangle \prec \langle A, \tau \rangle$.

1.9 REMARK. i) It is clear that A-Functional Equivalence relative to a set A is an equivalence relation on the class of representations with carrier A. Denote this equivalence by A-FE. ii) Each A-FE class consists of those representations which — through d-extentions — form generating sets for their (unique) Manipulative Syntax. In particular, the lattice representations of 1.3 are in the same class. iii) Given a ring $\langle R, +, \cdot \rangle$ with identity, there is now formal meaning to the statement that the group $\langle R, + \rangle$ and the semi-group $\langle R, \cdot \rangle$ are included ("subsumed") in the ring. Furthermore, distinguishing the identity to give $\langle R, e, +, \cdot \rangle$ simply provides a representation R-FE to the given one. Similar statements apply to the other familiar systems. iv) If $A = \{a\}$, then $\Pi = \bar{a} = \bar{f}$ for <u>any</u> operation $f \neq \emptyset$ on a product of A to A; thus <u>all</u> representations on $\{a\}$ are A-Functionally Equivalent.

The next theorem is standard. It is included for ease of reference and to establish notation.

1.10 THEOREM. Let $\mathfrak{A} = \langle \underline{A}, \Pi, \circ, +, ' \rangle$ be the Function System over A. Let $M \subseteq \underline{A}$ and let $\langle [M], \Pi, \circ, +, ' \rangle$ be the subsystem of $\mathfrak{A}$ generated by M. Let $M_0 = M$. For each $k \in N$, let $M_{k+1} = \{ f \in \underline{A} : f \in M_k$ or $[\exists g, h \in M_k][f = gh$ or $f = g + h]$ or $[\exists g \in M_k][f = g'] \}$. Then $[M] = \bigsqcup_{k \in N} M_k$.

Proof: Let $\bigsqcup_{k \in N} M_k = X$. $M_0 \subseteq [M]$, and by the definition of M_{k+1}, $M_k \subseteq [M]$ implies $M_{k+1} \subseteq [M]$ so that $X \subseteq [M]$. X also is carrier for a subsystem of $\mathfrak{A}$ since $[\forall m_1, m_2 \in X][\exists r, s \in N][\ m_1 \in M_r$ and $m_2 \in M_s$ implies $m_1 m_2 \in M_{\max(r,s)+1}$; if $m_1 + m_2$ is defined then $m_1 + m_2 \in M_{\max(r,s)+1}$; and $m_1' \in M_{r+1}$ while $m_2' \in M_{s+1}$]. But then the X subsystem contains the [M] subsystem. //

In the sequel, M will be a set of d-extentions of operations of a representation $\langle A, \mathcal{T} \rangle$. Where necessary for clarity, this notation will be expanded to $M(\mathcal{T})$. Each M_k for $\langle A, \mathcal{T} \rangle$ will then be denoted by $M_k(\mathcal{T})$.

<u>Congruences and Factor Representations</u>. At this point, attention is focused on a sequence of definitions and lemmas culminating in theorem 1.16. That theorem can be stated roughly as "congruences are invariant under A-Functional Equivalence". A consequence of this is that A-FE representations have $^{A}/_{\equiv}$-FE factor representations (theorem 1.22). Throughout this sequence, it will be assumed that $\langle A, \mathcal{T} \rangle$ is a U,α representation and that $\equiv$ is a congruence on $\langle A, \mathcal{T} \rangle$.

1.11 DEFINITION. Let $\underline{\equiv} \subseteq A^{**} \times A^{**}$ such that $\langle a_1, \ldots \rangle_n \underline{\equiv} \langle b_1, \ldots \rangle_m$ iff $n = m$ and for all i involved $a_i \equiv b_i$; that is, $a_i \equiv b_i$ for all i where $1 \leq i \leq n$ if $n \neq \omega$ or $1 \leq i < n$ if $n = \omega$.

1.12 LEMMA. Let $f, g \in \mathcal{T}[U]$. Let $\langle a_1, \ldots \rangle_k, \langle b_1, \ldots \rangle_k \in A^{**}$ with

$\langle a_1,\ldots\rangle_k \equiv \langle b_1,\ldots\rangle_k$. Then:

i) $\bar{f}\langle a_1,\ldots\rangle_k \equiv \bar{f}\langle b_1,\ldots\rangle_k$

ii) $\bar{f}'\langle a_1,\ldots\rangle_k \equiv \bar{f}'\langle b_1,\ldots\rangle_k$

iii) $(\bar{f}+\bar{g})\langle a_1,\ldots\rangle_k \equiv (\bar{f}+\bar{g})\langle b_1,\ldots\rangle_k$

iv) $(\bar{f}\bar{g})\langle a_1,\ldots\rangle_k \equiv (\bar{f}\bar{g})\langle b_1,\ldots\rangle_k$

Proof: Suppose $f \in A_r^{\;1}$ and $g \in A_s^{\;1}$ with $r \neq 0 \neq s$. (If r or s is zero, touching up of the following gives a proof.) Let $\underline{r}\langle a_1,\ldots\rangle_k = \langle u_1,\ldots\rangle_r$ and $\underline{s}\langle b_1,\ldots\rangle_k = \langle v_1,\ldots\rangle_s$. Also let $\underline{s}\langle a_1,\ldots\rangle_k = \langle u_1,\ldots\rangle_s$ and $\underline{r}\langle b_1,\ldots\rangle_k = \langle v_1,\ldots\rangle_r$. There is no notational ambiguity: u_j represents the same object wherever it appears; similarly for v_j. Furthermore, $u_i \equiv v_i$ for all i involved ($1 \leq i \leq \max(r,s)$ if r,s finite; $1 \leq i < \omega$ otherwise). In i): $\bar{f}\langle a_1,\ldots\rangle_k = f\circ\underline{r}\langle a_1,\ldots\rangle_k = f\langle u_1,\ldots\rangle_r \equiv f\langle v_1,\ldots\rangle_r = f\circ\underline{r}\langle b_1,\ldots\rangle_k = \bar{f}\langle b_1,\ldots\rangle_k$. In ii): by 0.21, $\bar{f}'\langle a_1,\ldots\rangle_k = f\langle z_1,\ldots\rangle_{r+1}$ where $\langle z_1,\ldots\rangle_{r+1} = \underline{r+1}\langle a_1,\ldots\rangle_k$. But $\underline{r+1}\langle a_1,\ldots\rangle_k \equiv \underline{r+1}\langle b_1,\ldots\rangle_k = \langle y_1,\ldots\rangle_{r+1}$ implies $\langle z_2,\ldots\rangle_{r+1} \equiv \langle y_2,\ldots\rangle_{r+1}$ whence $f\langle z_2,\ldots\rangle_{r+1} \equiv f\langle y_2,\ldots\rangle_{r+1} = \bar{f}'\langle b_1,\ldots\rangle_k$. In iii) merely note that $(\bar{f}+\bar{g})\langle a_1,\ldots\rangle_k = \langle\langle \bar{f}\langle a_1,\ldots\rangle_k, \bar{g}\langle a_1,\ldots\rangle_k \rangle\rangle = \langle f\circ\underline{r}\langle a_1,\ldots\rangle_k, g\circ\underline{s}\langle a_1,\ldots\rangle_k \rangle \in A^2$ and similarly $(\bar{f}+\bar{g})\langle b_1,\ldots\rangle_k = \langle f\circ\underline{r}\langle b_1,\ldots\rangle_k, g\circ\underline{s}\langle b_1,\ldots\rangle_k \rangle \in A^2$. Inspection shows pointwise $\equiv$ congruence, hence the 2-tuples are $\equiv$ equivalent. Finally in iv) $g\circ\underline{n}\langle a_1,\ldots\rangle_k \equiv g\circ\underline{n}\langle b_1,\ldots\rangle_k$ implies $\bar{f}\bar{g}\langle a_1,\ldots\rangle_k = \bar{f}(\bar{g}\langle a_1,\ldots\rangle_k) = \bar{f}(g\circ\underline{n}\langle a_1,\ldots\rangle_k) = f\circ\underline{m}(g\circ\underline{n}\langle a_1,\ldots\rangle_k) = f(\underline{m}(g\circ\underline{n}\langle a_1,\ldots\rangle_k)) = f(\underline{m}(g\circ\underline{n}\langle b_1,\ldots\rangle_k)) = \bar{f}\bar{g}\langle b_1,\ldots\rangle_k$. //

1.13 NOTATION. The Manipulative Syntax of $\langle A, \tau\rangle$ will hereafter be denoted M.S.$\langle A, \tau\rangle$.

1.14 LEMMA. For each $\bar{f} \in$ M.S.$\langle A,\tau\rangle$, $\langle a_1,\ldots\rangle_k \equiv \langle b_1,\ldots\rangle_k$ implies $\bar{f}\langle a_1,\ldots\rangle_k \equiv \bar{f}\langle b_1,\ldots\rangle_k$.

Proof: In the notation of theorem 1.10; let $M_0(\tau) = \bar{A} \cup \overline{\tau[U]} \cup \{\bar{\pi}\}$ and for each $k \in N^+$, let $M_{k+1}(\tau) = \{\bar{g} \in \underline{A} : \bar{g} \in M_k(\tau)$ or $[\exists \bar{h}, \bar{l} \in M_k(\tau)][\ \bar{g} = \bar{h} + \bar{l}$

or $\bar{g} = \bar{h}\bar{1}$] or $[\exists\bar{h} \in M_k(\tau)][\ \bar{g} = \bar{h}'\ ]\ \}$. The lemma follows by induction on the statement $[\forall n \in N][\ \bar{f} \in M_n(\tau)$ implies $\bar{f}\langle a_1, \ldots\rangle_k \equiv \bar{f}\langle b_1, \ldots\rangle_k\ ]$. Let $n = 0$. Then $\bar{f}$ is π, 1-constant, or the d-extention of an operation in $\tau[U]$. If π or 1-constant, the conclusion follows trivially. If a d-extention of an operation, the conclusion follows by 1.12.i. Suppose the hypothesis for $n < m$. Then $\bar{f} \in M_m(\tau)$ implies $\bar{f} \in M_{m-1}(\tau)$, $[\exists\bar{g}, \bar{h} \in M_{m-1}(\tau)]$ $[\ \bar{f} = \bar{g} + \bar{h}$ or $\bar{f} = \bar{g}\bar{h}\ ]$, or $[\exists\bar{g} \in M_{m-1}(\tau)][\ \bar{f} = \bar{g}'\ ]$. We may suppose $\bar{f} \in M_m(\tau) - M_{m-1}(\tau)$ by the induction hypothesis. If $\bar{g}, \bar{h} \in M_{m-1}(\tau)$ and $\bar{f} = \bar{g} + \bar{h}$, then $\bar{f}\langle a_1, \ldots\rangle_k = \langle\langle\ \bar{g}\langle a_1, \ldots\rangle_k, \bar{h}\langle a_1, \ldots\rangle_k\ \rangle\rangle$ and $\bar{f}\langle b_1, \ldots\rangle_k =$ $\langle\langle\ \bar{g}\langle b_1, \ldots\rangle_k, \bar{h}\langle b_1, \ldots\rangle_k\ \rangle\rangle$. These are $\equiv$ equivalent by the induction hypothesis and pointwise comparison. If $\bar{g}, \bar{h} \in M_{m-1}(\tau)$ and $\bar{f} = \bar{g}\bar{h}$, then $\bar{h}\langle a_1, \ldots\rangle_k \equiv \bar{h}\langle b_1, \ldots\rangle_k$ by the induction hypothesis and therefore also $\bar{g}\bar{h}\langle a_1, \ldots\rangle_k \equiv \bar{g}\bar{h}\langle b_1, \ldots\rangle_k$. Finally, suppose $\bar{h} \in M_{m-1}(\tau)$ and $\bar{f} = \bar{h}'$. It may be supposed that $h \in A_r^{\ s}$ for some $r, s \in \mathcal{N}^+$. Then $\underline{r+1}\langle a_1, \ldots\rangle_k =$ $\langle z_1, \ldots\rangle_{r+1}$, $\underline{r+1}\langle b_1, \ldots\rangle_k = \langle y_1, \ldots\rangle_{r+1}$, and $z_i \equiv y_i$ for all i involved implies $\langle z_2, \ldots\rangle_{r+1} \equiv \langle y_2, \ldots\rangle_{r+1}$. But then $\bar{f}\langle a_1, \ldots\rangle_k = \bar{h}\langle z_2, \ldots\rangle_{r+1} \equiv$ $\bar{h}\langle y_2, \ldots\rangle_{r+1} = \bar{f}\langle b_1, \ldots\rangle_k$. //

1.15 COROLLARY. If $n \in \mathcal{N}^+$, $f \in A_n^{\ m}$, and $\bar{f} \in$ M.S.$\langle A, \tau\rangle$, then $[\forall\langle a_1, \ldots\rangle_n, \langle b_1, \ldots\rangle_n \in A^n][\ \langle a_1, \ldots\rangle_n \equiv \langle b_1, \ldots\rangle_n$ implies $f\langle a_1, \ldots\rangle_n \equiv$ $f\langle b_1, \ldots\rangle_n\ ]$.

Proof: $\langle a_1, \ldots\rangle_n \equiv \langle b_1, \ldots\rangle_n$ implies $f\langle a_1, \ldots\rangle_n = \bar{f}\langle a_1, \ldots\rangle_n \equiv$ $\bar{f}\langle b_1, \ldots\rangle_n = f\langle b_1, \ldots\rangle_n$. //

1.16 THEOREM. Let $\langle A, \tau\rangle$ and $\langle A, \gamma\rangle$ be A-Functionally Equivalent representations. Let $\equiv$ be a congruence on $\langle A, \tau\rangle$. Then $\equiv$ is also a congruence on $\langle A, \gamma\rangle$.

Proof. $\bar{f} \in [\bar{A} \cup \bar{\tau}]$ iff $\bar{f} \in [\bar{A} \cup \bar{\gamma}]$ and corollary 1.15 where $\bar{f} = \bar{a}$ for some $a \in A$ or else for some $n \in \mathcal{N}^+$, $f \in A_n^{\ 1}$. //

1.17 NOTATION. Given U,α representation $\langle A, \mathcal{T}\rangle$ and a congruence $\equiv$ on it, the quotient set of A by $\equiv$ is denoted by ${}^{A}/_{\equiv}$; for each $a \in A$, the equivalence class of a is denoted by ${}^{a}/_{\equiv}$; and $\mathcal{T}/_{\equiv}$ is the function on U into operations on ${}^{A}/_{\equiv}$ such that $\mathcal{T}/_{\equiv}(u)$ is the operation on the quotient representation corresponding to $\mathcal{T}(u)$ on $\langle A, \mathcal{T}\rangle$. The quotient or factor representation of $\langle A, \mathcal{T}\rangle$ by $\equiv$ will thus be denoted by $\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv}\rangle$.

1.18 DEFINITION. Let $\bar{f} \in M.S.\langle A, \mathcal{T}\rangle$. Then

i) ${}^{\bar{f}}/_{\equiv}$ is the quotient of $\bar{f}$ (by $\equiv$) iff ${}^{\bar{f}}/_{\equiv} : {}^{A}/_{\equiv}{}^{**} \longrightarrow {}^{A}/_{\equiv}{}^{**}$ is the mapping such that for each $\langle Q_1, \ldots\rangle_n \in {}^{A}/_{\equiv}{}^{**}$, ${}^{\bar{f}}/_{\equiv}\langle Q_1, \ldots\rangle_n = \langle {}^{r_1}/_{\equiv}, \ldots\rangle_m$ where $\bar{f}\langle q_1, \ldots\rangle_n = \langle r_1, \ldots\rangle_m$ for some $\langle q_1, \ldots\rangle_m \in A^{**}$ with $q_i \in Q_i$ for every i involved.

ii) For $f \in A_n^{\,m}$ with $\bar{f} \in M.S.\langle A, \mathcal{T}\rangle$, define the quotient ${}^{f}/_{\equiv}$ similarly.

1.19 REMARK. By 1.14 and 1.15, the quotients ${}^{\bar{f}}/_{\equiv}$ and ${}^{f}/_{\equiv}$ of 1.18 are well-defined functions. It is clear that if π is the distinguished projection of the Function System over A, then ${}^{\pi}/_{\equiv}$ is the distinguished projection of the Function System over ${}^{A}/_{\equiv}$. It is also clear that ${}^{\mathcal{T}(u)}/_{\equiv}$ is the same as $\mathcal{T}/_{\equiv}(u)$ and that if range$(\bar{f}) \subseteq A$, then ${}^{\bar{f}}/_{\equiv}\langle Q_1, \ldots\rangle_n = \bar{f}\langle q_1, \ldots\rangle_n/_{\equiv}$ for all $n \in \mathcal{N}^+$, given that $q_i \in Q_i$ for all i involved.

1.20 LEMMA. Let $\bar{f}, \bar{g} \in M.S.\langle A, \mathcal{T}\rangle$. Then

i) ${}^{\bar{g}\bar{f}}/_{\equiv} = {}^{\bar{g}}/_{\equiv}{}^{\bar{f}}/_{\equiv}$

ii) ${}^{\bar{g}+\bar{f}}/_{\equiv} = {}^{\bar{g}}/_{\equiv} + {}^{\bar{f}}/_{\equiv}$

iii) ${}^{\bar{g}'}/_{\equiv} = ({}^{\bar{g}}/_{\equiv})'$

i.e. the mapping $\bar{f} \longrightarrow {}^{\bar{f}}/_{\equiv}$ is a homomorphism of $M.S.\langle A, \mathcal{T}\rangle$ into $M.S.\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv}\rangle$ where the Syntax operations are taken as indexed by

$\{1,2,3,4\}$ as shown.

In 1.23 it will be shown that this mapping is onto M.S.$\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle$.

Proof of 1.20: These results are all obtained pointwise. Let $\langle Q_1,\ldots\rangle_n \in {}^{A}/_{\equiv}{}^{**}$ and $\langle p_1,\ldots\rangle_n, \langle q_1,\ldots\rangle_n \in A^n$ with $p_i, q_i \in Q_i$ for each i involved. Then ${}^{\overline{g}\overline{f}}/_{\equiv}\langle Q_1,\ldots\rangle_n = \langle {}^{r_1}/_{\equiv},\ldots\rangle_m$ where $\langle r_1,\ldots\rangle_m = \overline{g}\overline{f}\langle q_1,\ldots\rangle_n = \overline{g}(\overline{f}\langle q_1,\ldots\rangle_n)$ so $r_i = (\overline{g}(\overline{f}\langle q_1,\ldots\rangle_n))_i$ for all i involved. Now, ${}^{\overline{g}}/_{\equiv}{}^{\overline{f}}/_{\equiv}\langle Q_1,\ldots\rangle_n = {}^{\overline{g}}/_{\equiv}({}^{\overline{f}}/_{\equiv}\langle Q_1,\ldots\rangle_n) = {}^{\overline{g}}/_{\equiv}\langle {}^{s_1}/_{\equiv},\ldots\rangle_k$ where $\overline{f}\langle p_1,\ldots\rangle_n = \langle s_1,\ldots\rangle_k$; whence ${}^{\overline{g}}/_{\equiv}\langle {}^{s_1}/_{\equiv},\ldots\rangle_k = \langle {}^{t_1}/_{\equiv},\ldots\rangle_m$ where $\langle t_1,\ldots\rangle_k = \overline{g}\langle s_1,\ldots\rangle_k = \overline{g}(\overline{f}\langle p_1,\ldots\rangle_n)$ and therefore $t_i = (\overline{g}(\overline{f}\langle p_1,\ldots\rangle_n))_i$ for all i involved. But then, using 1.14, $\overline{f},\overline{g} \in$ M.S.$\langle A, \mathcal{T} \rangle$ and $\langle p_1,\ldots\rangle_n \equiv \langle q_1,\ldots\rangle_n$ implies $\overline{f}\langle p_1,\ldots\rangle_n \equiv \overline{f}\langle q_1,\ldots\rangle_n$, which gives $\overline{g}\overline{f}\langle p_1,\ldots\rangle_n \equiv \overline{g}\overline{f}\langle q_1,\ldots\rangle_n$. From the latter it follows that $r_i \equiv t_i$ for all i involved, whence $\langle {}^{r_1}/_{\equiv},\ldots\rangle_m = \langle {}^{t_1}/_{\equiv},\ldots\rangle_m$. The remaining parts are proved similarly. //

1.21 LEMMA. Let $f \in A_n{}^m$ with $\overline{f} \in$ M.S.$\langle A, \mathcal{T} \rangle$. Then $\overline{{}^{f}/_{\equiv}} = {}^{\overline{f}}/_{\equiv}$.

Proof: Pointwise as in 1.20. //

1.22 THEOREM. Let $\langle A, \mathcal{T} \rangle$ and $\langle A, \mathcal{Y} \rangle$ be A-Functionally Equivalent representations. Let $\equiv$ be a congruence on either (hence both) of them. Then the two factor representations $\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle$ and $\langle {}^{A}/_{\equiv}, {}^{\mathcal{Y}}/_{\equiv} \rangle$ are ${}^{A}/_{\equiv}$-Functionally Equivalent.

The proof of 1.22 requires the following lemma. The hypotheses of 1.20 are assumed. Notation is that of the proof of 1.14 and 1.20.

1.23 LEMMA. For each $k \in N$, let ${}^{M_k(\mathcal{T})}/_{\equiv} = \{\mathfrak{f} : [\exists \overline{g} \in M_n(\mathcal{T})][\mathfrak{f} = {}^{\overline{g}}/_{\equiv}]\}$. Then $[\forall n \in N][M_n({}^{\mathcal{T}}/_{\equiv}) = {}^{M_n(\mathcal{T})}/_{\equiv}]$.

Proof of the lemma: Proceed by induction. Suppose $\mathfrak{f} \in M_0({}^{\mathcal{T}}/_{\equiv})$. Then $\mathfrak{f} \in \overline{{}^{A}/_{\equiv}} \cup \overline{{}^{\mathcal{T}}/_{\equiv}[U]}$ or $\mathfrak{f} = {}^{\Pi}/_{\equiv}$. If $\mathfrak{f} = {}^{\Pi}/_{\equiv}$, then it is in both 0-indexed sets by definition and Remark 1.19. If $\mathfrak{f} \in \overline{A/_{\equiv}}$ there exists $Q \in \overline{A/_{\equiv}}$ such that $\{Q\} = \text{range}(\mathfrak{f})$ and it follows that $[\forall q \in Q][\overline{q}/_{\equiv} = \mathfrak{f}]$. But then 1.21 implies $[\exists q \in Q][\overline{q}/_{\equiv} = \mathfrak{f}]$ so $\mathfrak{f} \in {}^{M_0(\mathcal{T})}/_{\equiv}$. If $\mathfrak{f} \in \overline{\mathcal{T}/_{\equiv}[U]}$, then $\mathfrak{f} = \mathcal{T}/_{\equiv}(u)$ for some u and by 1.19 $\overline{\mathcal{T}/_{\equiv}(u)} = \overline{\mathcal{T}(u)}/_{\equiv}$. But $\overline{\mathcal{T}(u)/_{\equiv}} = \overline{\mathcal{T}(u)}/_{\equiv}$ gives $\mathfrak{f} \in {}^{M_0}/_{\equiv}$. It follows that $M_0(\mathcal{T}/_{\equiv}) \subseteq {}^{M_0(\mathcal{T})}/_{\equiv}$. The opposite inclusion is clear, so the induction basis is established.

Suppose the statement for all $n < m$; i.e. $[\forall n < m][M_n(\mathcal{T}/_{\equiv}) = {}^{M_n(\mathcal{T})}/_{\equiv}]$. Suppose $\mathfrak{f} \in {}^{M_m(\mathcal{T})}/_{\equiv}$. Then there is a $\overline{g} \in M_m(\mathcal{T})$ with $\mathfrak{f} = \overline{g}/_{\equiv}$. It may be supposed that $\overline{g} \in M_m(\mathcal{T}) - M_{m-1}(\mathcal{T})$ by the induction hypothesis. Now, for some $\overline{f}, \overline{g} \in M_{m-1}(\mathcal{T})$, $\overline{g}$ is one of $\overline{f}\,\overline{h}$, $\overline{f} + \overline{h}$, or $\overline{f}'$. If $\overline{g} = \overline{f}\,\overline{h}$, then $\overline{f}/_{\equiv}, \overline{h}/_{\equiv} \in {}^{M_{m-1}(\mathcal{T})}/_{\equiv} = M_{m-1}(\mathcal{T}/_{\equiv})$ together with $\overline{f}\,\overline{h}/_{\equiv} = \overline{f}/_{\equiv}\overline{h}/_{\equiv}$ implies $\overline{g}/_{\equiv} \in M_m(\mathcal{T}/_{\equiv})$. If $\overline{g} = \overline{f} + \overline{h}$ the argument follows similarly. If $\overline{g} = \overline{f}'$ for $\overline{f} \in M_{m-1}(\mathcal{T})$, then $\overline{g}/_{\equiv} = \overline{f}'/_{\equiv} = (\overline{f}/_{\equiv})'$ for $\overline{f}/_{\equiv} \in {}^{M_{m-1}(\mathcal{T})}/_{\equiv} = M_{m-1}(\mathcal{T}/_{\equiv})$, i.e. $\overline{g}/_{\equiv} \in M_m(\mathcal{T}/_{\equiv})$. From these it follows that ${}^{M_m(\mathcal{T})}/_{\equiv} \subseteq M_m(\mathcal{T}/_{\equiv})$. Suppose $\mathfrak{f} \in M_m(\mathcal{T}/_{\equiv}) - M_{m-1}(\mathcal{T}/_{\equiv})$. Then for some $\mu, \nu \in M_{m-1}(\mathcal{T}/_{\equiv})$; $\mathfrak{f} = \mu\nu$ or $\mathfrak{f} = \mu + \nu$ or $\mathfrak{f} = \mu'$. If $\mathfrak{f} = \mu\nu$ or $\mathfrak{f} = \mu + \nu$, the induction hypothesis implies there exist $\overline{g}, \overline{h} \in M_{m-1}(\mathcal{T})$ such that $\mu = \overline{g}/_{\equiv}$, $\nu = \overline{h}/_{\equiv}$, and $\mathfrak{f} = \overline{g}/_{\equiv}\overline{h}/_{\equiv} = \overline{g}\,\overline{h}/_{\equiv}$ ($\mathfrak{f} = \overline{g}/_{\equiv} + \overline{h}/_{\equiv} = \overline{g} + \overline{h}/_{\equiv}$, respectively) where $\overline{g}\,\overline{h}$ ($\overline{g} + \overline{h}$) $\in M_m(\mathcal{T})$. It follows that $\mathfrak{f} \in {}^{M_m(\mathcal{T})}/_{\equiv}$. The case $\mathfrak{f} = \mu'$ follows similarly. //

Proof of Theorem 1.24: Let $\mathfrak{f} \in$ M.S.$\langle A/_{\equiv}, \mathcal{T}/_{\equiv}\rangle$. Then for some $n \in N$, $\mathfrak{f} \in M_n(\mathcal{T}/_{\equiv}) = {}^{M_n(\mathcal{T})}/_{\equiv}$, whence there exists $\overline{f} \in M_n(\mathcal{T})$ such that $\mathfrak{f} = \overline{f}/_{\equiv}$. By A-FE, then $\overline{f} \in$ M.S.$\langle A, \gamma\rangle$ implies there is an $m \in N$ such that $\overline{f} \in M_m(\gamma)$. It follows that $\mathfrak{f} = \overline{f}/_{\equiv} \in {}^{M_m(\gamma)}/_{\equiv} = M_m(\gamma/_{\equiv})$. This shows $\langle A/_{\equiv}, \mathcal{T}/_{\equiv}\rangle < \langle A/_{\equiv}, \gamma/_{\equiv}\rangle$. The opposite inequality follows by the symmetry of the argument; thus the two factor representations are $^{A}/_{\equiv}$-Functionally Equivalent. //

1.24 REMARK. If $\underline{G}$ is the epimorphism of M.S.$\langle A, \mathcal{T} \rangle$ into M.S.$\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle$ such that $\underline{G}(\overline{f}) = {}^{\overline{f}}/_{\equiv}$, then $\underline{G}$ has a kernel, say $(\equiv)$, which (since a Manipulative Syntax is an "algebra: in the sense of Universal Algebra) is a congruence. But then $^{\text{M.S.}\langle A, \mathcal{T} \rangle}/_{(\equiv)}$ is isomorphic with M.S.$\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle$ and the following diagram is commutative.

$$
\begin{tikzcd}
& \text{M.S.}\langle A, \mathcal{T} \rangle \arrow[r, "\underline{G}"] \arrow[dl, "\text{Nat}(\equiv)"'] & \text{M.S.}\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle \\
{}^{\text{M.S.}\langle A, \mathcal{T} \rangle}/_{(\equiv)} \arrow[urr, "\text{Isomorphism}"'] & &
\end{tikzcd}
$$

Theorem 1.22 and Remark 1.24 show that congruences on representations (or "algebras") are intimately associated with certain congruences on the corresponding Manipulative Syntaxes. Several questions then naturally arise:

1/ Is there a precise relationship between congruences on representations and congruences on their Syntaxes?

2/ Given a congruence $\equiv$ on $\langle A, \mathcal{T} \rangle$, is there an explicit characterization of $(\equiv)$ on M.S.$\langle A, \mathcal{T} \rangle$?

3/ Since congruences are associated with homomorphisms, are relationships between representations with different carriers reflected by relationships of their Syntaxes and/or conversely? In particular, is there any relationship between hom$(\langle A, \mathcal{T} \rangle, \langle B, \gamma \rangle)$ and hom(M.S.$\langle A, \mathcal{T} \rangle$, M.S.$\langle B, \gamma \rangle$) ?

The remainder of this chapter deals with the first two of these questions. Chapter II will deal with the third.

The next three theorems provide a class of set-theoretically specified congruences on a Syntax which are in 1-1 correspondence with the congruences on the algebras generating that Syntax. In particular, 1.25 answers question 2/ above and the first part of 1.27 shows that the characterization given in 1.25 is indeed what it is claimed to be.

1.25 THEOREM. Let $\langle A, \mathcal{T} \rangle$ be a representation. Let Γ be a congruence on $\langle A, \mathcal{T} \rangle$. Let $\underline{\Gamma} = \{ \langle x,y \rangle : x,y \in A^{**}$ and $[\exists n \in \mathcal{N}^+][\, x,y \in A^n$ and $x_i \,\Gamma\, y_i$ for all i involved $] \}$. Let $\Xi \subseteq M.S.\langle A, \mathcal{T} \rangle \times M.S.\langle A, \mathcal{T} \rangle$ such that $\langle \bar{f}, \bar{g} \rangle \in \Xi$ iff $[\forall x,y \in A^{**}][\, x \,\underline{\Gamma}\, y$ implies $\bar{f}(x) \,\underline{\Gamma}\, \bar{g}(y) \,]$. Then Ξ is a congruence on $M.S.\langle A, \mathcal{T} \rangle$.

Proof: It is clear that $\underline{\Gamma}$ is an equivalence on A^{**}. Ξ is reflexive by 1.12 and it is evidently symmetric and transitive, thus is an equivalence. Let $\mu, \nu, \hat{\mu}, \hat{\nu} \in M.S.\langle A, \mathcal{T} \rangle$ with $\langle \mu, \hat{\mu} \rangle, \langle \nu, \hat{\nu} \rangle \in \Xi$. Then $x \,\underline{\Gamma}\, y$ implies: i) $\mu + \nu(x) = \langle\langle \mu(x), \nu(x) \rangle\rangle$ and $\langle\langle \mu(x), \nu(x) \rangle\rangle \,\underline{\Gamma}\, \langle\langle \hat{\mu}(y), \hat{\nu}(y) \rangle\rangle$ whence $\mu + \nu(x) \,\underline{\Gamma}\, \hat{\mu} + \hat{\nu}(y)$; ii) $\mu\nu(x) = \mu(\nu(x)) \,\underline{\Gamma}\, \hat{\mu}(\nu(x)) \,\underline{\Gamma}\, \hat{\mu}(\hat{\nu}(y)) = \hat{\mu}\hat{\nu}(y)$; and $\mu'(x) = \mu(u) \,\underline{\Gamma}\, \hat{\mu}(v) = \hat{\mu}'(y)$ where if $x = \langle x_1, \ldots \rangle_n$ then $u = \langle x_2, \ldots \rangle_{n+1}$ and if $y = \langle y_1, \ldots \rangle_n$ then $v = \langle y_2, \ldots \rangle_{n+1}$ (so that $x \,\underline{\Gamma}\, y$ implies $u \,\underline{\Gamma}\, v$). //

1.26 THEOREM. In the hypotheses of 1.25, let ψ be the mapping of congruences on $\langle A, \mathcal{T} \rangle$ into congruences on $M.S.\langle A, \mathcal{T} \rangle$ as defined in 1.25. Then ψ is 1-1 .

Proof: Suppose $\equiv$ and Γ are congruences on $\langle A, \mathcal{T} \rangle$ with $\psi(\equiv) = \psi(\Gamma)$. Then for all $a,b \in A$; $\langle a,b \rangle \in \Gamma$ implies $\langle a \rangle \,\underline{\Gamma}\, \langle b \rangle$, whence $\langle \bar{a}, \bar{b} \rangle \in \psi(\Gamma)$ and is also in $\psi(\equiv)$. But then $[\forall x,y \in A^{**}][\, x \,\underline{\equiv}\, y$ implies $\langle a \rangle = \bar{a}(x) \,\underline{\equiv}\, \bar{b}(y) = \langle b \rangle$, i.e. $a \equiv b$], thus $\Gamma \subseteq \equiv$. By symmetry, $\equiv \subseteq \Gamma$ and therefore $\equiv = \Gamma$. //

1.27 THEOREM. In hypotheses of theorems 1.25 and 1.26, let $\underline{G}: M.S.\langle A, \mathcal{T}\rangle \longrightarrow M.S.\langle A/_{\equiv}, \mathcal{T}/_{\equiv}\rangle$ such that $\underline{G} = \bar{f}/_{\equiv}$. Then kernel($\underline{G}$) = $\psi(\equiv)$, whence $M.S.\langle A, \mathcal{T}\rangle/_{\psi(\equiv)}$ is isomorphic with $M.S.\langle A/_{\equiv}, \mathcal{T}/_{\equiv}\rangle$.

Proof: By 1.20 and 1.23, the mapping $\underline{G}$ is an epimorphism. Let $\langle \bar{f}, \bar{g}\rangle \in \psi(\equiv)$. Then $x, y \in A^{**}$ with $x \underline{\equiv} y$ implies $\bar{f}(x) \underline{\equiv} \bar{g}(y)$. In particular then, $x = y$ implies $\bar{f}(x) = \bar{g}(x)$. But then Definition 1.18 applied straightforward gives $\bar{f}/_{\equiv} = \bar{g}/_{\equiv}$, whence $\langle \bar{f}, \bar{g}\rangle \in$ kernel($\underline{G}$) and $\psi(\equiv) \subseteq$ kernel($\underline{G}$). Let $\langle \bar{f}, \bar{g}\rangle \in$ kernel($\underline{G}$) so $\bar{f}/_{\equiv} = \bar{g}/_{\equiv}$. Then for each $x, y \in A^{**}$ with $x \underline{\equiv} y$, if $x/_{\equiv} = \langle x_1/_{\equiv}, \ldots\rangle_n$ and $y/_{\equiv} = \langle y_1/_{\equiv}, \ldots\rangle_n$ so $x/_{\equiv} = y/_{\equiv}$, then $\bar{f}/_{\equiv}(x/_{\equiv}) = \bar{g}/_{\equiv}(y/_{\equiv})$. But this is so (Def. 1.18) only if $\bar{f}(x) \underline{\equiv} \bar{g}(y)$, i.e. only if $\langle \bar{f}, \bar{g}\rangle \in \psi(\equiv)$. It follows that kernel($\underline{G}$) $\subseteq \psi(\equiv)$. //

It is natural at this point to wonder if each congruence on a Syntax similarly induces a congruence on the Syntax-generating representations and if the mapping ψ of 1.26 is thereby a 1-1 and onto order isomorphism. ψ is evidently inclusion preserving, for $\equiv \subseteq \Gamma$ implies $[\forall x, y \in A^{**}][\, x \underline{\equiv} y$ implies $x \underline{\Gamma} y\,]$ so for all $\bar{f}, \bar{g}$; if $\bar{f}(x) \underline{\equiv} \bar{g}(y)$, then $\bar{f}(x)\, \underline{\Gamma}\, \bar{g}(y)$. For finitary representations, the first part of the question is answered affimatively in 1.28. The second part of the question remains unanswered at the time of this writing. Theorem 1.31 embodies essentially all that is known.

1.28 THEOREM. Let $\langle A, \mathcal{T}\rangle$ be a finitary representation. Let Ξ be a congruence on $M.S.\langle A, \mathcal{T}\rangle$. Let $\equiv \subseteq A \times A$ such that $a \equiv b$ iff $\bar{a} \,\Xi\, \bar{b}$. Then $\equiv$ is a congruence on $\langle A, \mathcal{T}\rangle$.

Proof: $\equiv$ is clearly an equivalence on A. Let $\mathcal{T}_u$ be an operation of $\langle A, \mathcal{T}\rangle$. Define the equivalence $\underline{\equiv}$ on A^{**} as in 1.25. Suppose $x, y \in$ domain($\mathcal{T}_u$) and $x \underline{\equiv} y$. By hypothesis, for some finite m, $\mathcal{T}_u \in A_m^{\ 1}$.

Then $\mathcal{T}_u(x) \in A$ implies $\overline{\mathcal{T}_u(x)} = \overline{\mathcal{T}_u}(\bar{x}_1 + \ldots + \bar{x}_m)$ is 1-constant. Now $x \equiv y$ implies $\bar{x}_i \mathrel{\overline{\equiv}} \bar{y}_i$ for each i involved. Since $\overline{\equiv}$ is a congruence it follows that $(\bar{x}_1 + \ldots + \bar{x}_m) \mathrel{\overline{\equiv}} (\bar{y}_1 + \ldots + \bar{y}_m)$ and therefore $\overline{\mathcal{T}_u}(\bar{x}_1 + \ldots + \bar{x}_m) \mathrel{\overline{\equiv}} \overline{\mathcal{T}_u}(\bar{y}_1 + \ldots + \bar{y}_m)$, i.e. $\overline{\mathcal{T}_u(x)} \mathrel{\overline{\equiv}} \overline{\mathcal{T}_u(y)}$. But then $\mathcal{T}_u(x) \equiv \mathcal{T}_u(y)$. Since $\mathcal{T}_u$ was arbitrary, $\equiv$ is a congruence relative to all operations of $\langle A, \mathcal{T} \rangle$.
//

Discourse at this point will be considerably eased by some definitions and elementary results. In 1.29 and 1.30, let $\mathcal{X} = \langle X, \pi, \circ, +, ' \rangle$ be a partial algebra satisfying the conditions of theorem 0.25 and such that $\pi \neq \pi'$. Also let $x \in X$.

1.29 DEFINITION. i) $\pi^{(0)} = \pi$ and for each $n \in N$, let $\pi_{k+1} = \pi^{(k)}$.
ii) $\dim(x)$ is the least $m \in N^+$ such that $(\pi_1 + \ldots + \pi_m)x = x$, or else ω.
iii) $\deg(x)$ is the least $n \in N^+$ such that $x(\pi_1 + \ldots + \pi_n) = x$, or else ω.

1.30 LEMMA.

i) $[\forall n \in \mathcal{N}^+][\ \dim(x) = n$ and $m \neq m$ implies $(\pi_1 + \ldots + \pi_m)x \neq x\ ]$.

If φ is a homomorphism of $\mathcal{X}$ into a partial algebra $\mathcal{Y} = \langle Y, \pi, \circ, +, ' \rangle$, then: ii) $\deg(x) = n$ implies $\deg(\varphi(x)) \leq n$.

iii) $\dim(x) = m$ implies $\dim(\varphi(x)) = m$.

Proof of i): Suppose $\dim(x) = n \in N^+$ (otherwise the conclusion follows immediately), $m \in N^+$, $n \neq m$, and $(\pi_1 + \ldots + \pi_n)x = x = (\pi_1 + \ldots + \pi_m)x$. Then $\pi_1 x + \ldots + \pi_n x = \pi_1 x + \ldots + \pi_m x$. $m < n$ contradicts minimality of n, so suppose $m > n$. Since $\pi \neq \pi'$, $\pi_m \neq \pi_n$; whence if $\pi_{n+1}x \neq \pi_m$, let $z = \pi_m$ whereas if $\pi_{n+1}x = \pi_m$ let $z = \pi_n$. Then $\pi_1 x + \ldots + \pi_n x + z = \pi_1 x + \ldots + \pi_m x + z$ and it follows that $z = \pi_{n+1}(\pi_1 x + \ldots + \pi_n x + z) = \pi_{n+1}(\pi_1 x + \ldots + \pi_m x + z) = \pi_{n+1}x$, a contradiction of the definition of z.

Proof of ii): $x(\pi_1 + \ldots + \pi_n) = x$ implies $\varphi(x)(\pi_1 + \ldots + \pi_n) =$

$\varphi(X)$ if $n \in N^+$. $n = \omega$ is evident.

Proof of iii): $(\pi_1 + \ldots + \pi_m)x = x$ implies $(\pi_1 + \ldots + \pi_m)\varphi(x) = \varphi(x)$ and part i) implies this is true for no other $n \in \mathcal{N}^+$.

The next result, for finitary $\langle A, \mathcal{T} \rangle$, closes this chapter. It reduces the question of whether ψ is an onto mapping to the question of whether the minimal congruence in M.S.$\langle A, \mathcal{T} \rangle$ generated by $\{ \langle \bar{a}, \bar{b} \rangle : \langle a,b \rangle \in \equiv \}$ coincides with $\psi(\equiv)$, for the latter is shown to be maximal relative to that condition.

1.31 THEOREM. Let $\langle A, \mathcal{T} \rangle$ be finitary. Let Γ be a congruence on M.S.$\langle A, \mathcal{T} \rangle$. Let $\equiv \subseteq A \times A$ such that $a \equiv b$ iff $\bar{a}\,\Gamma\,\bar{b}$. Let $\underline{\equiv} = \{ \langle x,y \rangle : [\exists n \in \mathcal{N}^+][\ x,y \in A^n$ and for all i involved, $x_i \equiv y_i\ ] \}$. Let ψ be as in 1.28. Then $\equiv$ is a congruence on $\langle A, \mathcal{T} \rangle$ and $\Gamma \subseteq \psi(\equiv)$; i.e. $\psi(\equiv)$ is the <u>unique</u> congruence $\mathcal{C}$ on M.S.$\langle A, \mathcal{T} \rangle$ which is maximal with respect to inclusion and the condition that $a \equiv b$ iff $\bar{a}\,\mathcal{C}\,\bar{b}$.

Proof: $\equiv$ is a congruence by 1.28, so $\underline{\equiv}$ is an equivalence. Furthermore, for each $x,y \in A^*$, $x \underline{\equiv} y$ iff $\bar{x}\,\Gamma\,\bar{y}$. (That $x \underline{\equiv} y$ implies $\bar{x}\,\Gamma\,\bar{y}$ is clear. Conversely, $\bar{x}\,\Gamma\,\bar{y}$ implies that for some homomorphism φ of M.S.$\langle A, \mathcal{T} \rangle$, $\varphi(\bar{x}) = \varphi(\bar{y})$, so there exists $m \in N^+$ such that $m = \dim(\bar{x}) = \dim(\bar{y})$ by 1.30. But then for all $i < m$; $\bar{x}_i\,\Gamma\,\bar{y}_i$ so $x_i \equiv y_i$ and $x \underline{\equiv} y$.) Suppose $\langle \bar{f}, \bar{g} \rangle \in \Gamma$ and $x,y \in A^{**}$ with $x \underline{\equiv} y$. Then, as above, $\dim(\bar{f}) = \dim(\bar{g}) = m$ for some $m \in N^+$. In particular, $\bar{f}(x), \bar{g}(y) \in A^m$. But if $x,y \in A^n$, then for all i involved, $\bar{x}_i\,\Gamma\,\bar{y}_i$ so that $\bar{f}(\bar{x}_1 + \ldots + \bar{x}_j)\ \Gamma\ \bar{g}(\bar{y}_1 + \ldots + \bar{y}_j)$ for all finite $j < n$. If n is finite, this gives $\bar{f}(x) = \bar{f}(\bar{x}_1 + \ldots + \bar{x}_n)\ \Gamma\ \bar{g}(\bar{y}_1 + \ldots + \bar{y}_n) = \bar{g}(y)$ and by the first part of the proof $\bar{f}(x) \underline{\equiv} \bar{g}(y)$. If n is infinite and p is the maximum of $\deg(\bar{f})$ and $\deg(\bar{g})$; then $\bar{f} \circ \underline{p}(x) = \bar{f}(\bar{x}_1 + \ldots + \bar{x}_p)\ \Gamma\ \bar{g}(\bar{y}_1 + \ldots + \bar{y}_p) = \bar{g} \circ \underline{p}(y)$. It again follows that

$\bar{f}(x) = \bar{f}\ \underline{p}(x) \equiv \bar{g}\ \underline{p}(y) = \bar{g}(y)$. //

Further results on congruences are deferred until Chapter III, where Functional Equivalence is combined with Isomorphism to provide congruence transfer between representations with different carriers.

CHAPTER II

STRUCTURE PRESERVING MAPPINGS

The central problem of this chapter can be posed as follows: What is it about a homomorphism that makes one consider it as structure preserving? The underlying premise of what is to follow is that the answer to this question is that the homomorphism is — in some way — related to a transformation of languages which turns equations in one language into equations in another in some consistent fashion. Indeed, it is held herein that the language mapping is the crucial component of the notion. In particular, any A-FE class generates a unique Manipulative Syntax which in turn can be thought of as associated with a mass of equations; that is, pairings of strings of concatenations, compositions, and shifts of generating elements where each string defines the same function on A**. In the light of these considerations, it seems reasonable to make the following definitions.

2.1 DEFINITION. Let $\langle A, \tau \rangle$ and $\langle B, \gamma \rangle$ be representations. A function φ on the Manipulative Syntax of $\langle A, \tau \rangle$ into that of $\langle B, \gamma \rangle$ is a

1/ <u>Comparitor</u> of $\langle A, \tau \rangle$ and $\langle B, \gamma \rangle$ iff i) φ is a homomorphism of M.S.$\langle A, \tau \rangle$ into M.S.$\langle B, \gamma \rangle$ and ii) $\varphi[\bar{A} \cup \{\pi_A\}] \subseteq \bar{B} \cup \{\pi_B\}$.

2/ <u>Replicator</u> of $\langle A, \tau \rangle$ in $\langle B, \gamma \rangle$ iff i) φ is a Comparitor of $\langle A, \tau \rangle$ and $\langle B, \gamma \rangle$ and ii) φ is 1-1.

3/ <u>Simulator</u> of $\langle A, \mathcal{T} \rangle$ with $\langle B, \gamma \rangle$ iff i) φ is a Comparitor of $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$, ii) φ is onto $[\bar{B} \cup \bar{\gamma}]$, and iii) $\varphi[\bar{A} \cup \{\pi_A\}] = \bar{B} \cup \{\pi_B\}$.

4/ <u>Complete Identification</u> of $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$ iff i) φ is a Replicator of $\langle A, \mathcal{T} \rangle$ in $\langle B, \gamma \rangle$ and ii) φ is a Simulator of $\langle A, \mathcal{T} \rangle$ with $\langle B, \gamma \rangle$.

Corresponding to the above will be 1/ $\langle A, \mathcal{T} \rangle$ <u>compares</u> with $\langle B, \gamma \rangle$, etc.

To simplify notation, $\mathcal{T}_u$ will hereafter be used in place of $\mathcal{T}(u)$.

2.2 REMARK. It is clear that the identity mapping on the Manipulative Syntax of two A-FE representations is a Complete Identification of them, so A-Functionally Equivalent representations are completely identified. ii) Again, if $\langle A, \mathcal{T} \rangle \prec \langle A, \gamma \rangle$, then $\langle A, \mathcal{T} \rangle$ is replicated in $\langle A, \gamma \rangle$, the Replicator being the injection mapping of Syntaxes.

All algebraic representations with a given one-element carrier have been completely identified (Remark 1.9.iv). In addition, their intrinsic algebraic character — being very meager — should be reflected in the algebraic syntax of all other representations. This remark is formalized in

2.3 THEOREM. Let $\langle A, \mathcal{T} \rangle$ be a representation with $A = \{a\}$. Then, if $\langle B, \gamma \rangle$ is a representation, there exists a Replicator of $\langle A, \mathcal{T} \rangle$ in $\langle B, \gamma \rangle$.

Proof: In earlier notation, observe that $M_0(\mathcal{T}) = \{\pi_A\}$, and for $n \in N$, $M_{n+1}(\mathcal{T}) = \{ m\pi_A : 1 \leq m \leq n+2 \}$ (where $1\pi = \pi$, $(k+1)\pi = k\pi + \pi$). From this it follows that the Syntax of $\langle A, \mathcal{T} \rangle$ consists precisely of the functions $n\pi_A$ where $n \in N$. But then the function φ defined by

$\varphi(m\pi_A) = m\pi_B$ is clearly the required Replicator. //

The proof of 2.3 anticipates the following more general result.

2.4 LEMMA. Let $\langle A, \tau \rangle$ and $\langle B, \gamma \rangle$ be representations. Let φ be a Comparitor of $\langle A, \tau\rangle$ and $\langle B, \gamma \rangle$. Then $[\exists a \in A][\ \varphi(\overline{a}) = \pi_B]$ implies $[\forall a \in A][\ \varphi(\overline{a}) = \pi_B\ ]$.

Proof: Suppose $a_1, a_2 \in A$, $\varphi(\overline{a_1}) = \pi_B$, and $\varphi(\overline{a_2}) \neq \pi_B$. Then for some $b \in B$, $\varphi(\overline{a_2}) = \overline{b}$ and $\overline{b} \neq \pi_B$ by 2.1.1.ii and hypothesis. But then it follows that $\pi_B = \varphi(\overline{a_1}) = \varphi(\overline{a_1}\ \overline{a_2}) = \varphi(\overline{a_1})\varphi(\overline{a_2}) = \pi_B\varphi(\overline{a_2}) = \pi_B\overline{b} = \overline{b}$, a contradiction. //

Similar to 2.3, if one has a representation $\langle B, \gamma \rangle$, identifies all elements, and thereby refuses to distinguish different functions in any class B_n^m, then he is ignoring all algebraic distinctions. What is then being considered should essentially be mimicked completely by a one-element representation. This is formalized in:

2.5 THEOREM. In hypothesis of 2.3, there exists a Simulator of $\langle B, \gamma\rangle$ with $\langle A, \tau \rangle$.

Proof: Define $\varphi(\overline{f}) = n\pi_A$ if $\overline{f} \in [\ \overline{B} \cup \overline{\gamma}\ ]$ and $\dim(\overline{f}) = n$. Evidently then, $\varphi[\ \overline{B} \cup \{\pi_B\}\ ] = \overline{A} = \{\pi_A\}$ and φ is onto $[\ \overline{A} \cup \overline{\tau}\ ]$. φ is a homomorphism, for if $\overline{f}, \overline{g} \in [\ \overline{B} \cup \overline{\gamma}\ ]$, $\dim(\overline{f}) = n$, and $\dim(\overline{g}) = m$, then $\dim(\overline{f}\ \overline{g}) = n$ (whence $\varphi(\overline{f})\varphi(\overline{g}) = n\pi_A m\pi_A = n\pi_A = \varphi(\overline{f}\ \overline{g})$), $\dim(\overline{f}') = n$ (whence $\varphi(\overline{f}') = n\pi_A = (n\pi_A)' = \varphi(\overline{g})'$), and since $\dim(\overline{f} + \overline{g}) = n+m$, etc. //

2.6 LEMMA. Let $n \in N$. Let $y \in A^{**}$. Then $(\pi_1 + \ldots + \pi_n)(y) = (\pi_1 + \ldots + \pi_n)(\underline{n}(y)) = \underline{n}(y)$.

Proof: Suppose $n \in N$ and let $y = \langle y_1, \ldots \rangle_m$, $\underline{n}(y) = \langle z_1, \ldots \rangle_n$. The proof proceeds by induction on n. Suppose $n = 1$. Then $\pi_1(y) = \pi_1(\underline{1}(y)) = \pi_1\langle y_1 \rangle = y_1 = \underline{1}(y)$. Suppose the statement for all $j < m$. Then

$(\pi_1 + \dots + \pi_{n-1} + \pi_n)(y) = \langle\langle (\pi_1 + \dots + \pi_{n-1})(y), \pi_n(y) \rangle\rangle =$ $\langle\langle (\pi_1 + \dots + \pi_{n-1}((\underline{n-1}(y)), \pi_n(y) \rangle\rangle$. But pointwise comparison shows this is $\langle\langle (\pi_1 + \dots + \pi_{n-1})(\underline{n}(y)), \pi_n(\underline{n}(y)) \rangle\rangle$, which is $(\pi_1 + \dots + \pi_n)(\underline{n}(y))$. The left equality is therefore established for $n \in N$. The right hand equality is now also evident. //

2.7 LEMMA. Let φ be a homomorphism in the Function System of B. Suppose $\bar{f} \in \mathrm{domain}(\varphi)$. Then for each $n \in N^+$, $\bar{f}(\pi_1 + \dots + \pi_n) = \bar{f}$ implies $\overline{\varphi(\bar{f})|B^n} = \varphi(\bar{f}) = \varphi(\bar{f})(\pi_1 + \dots + \pi_n)$.

Proof: $\varphi(\bar{f}) = \underline{\varphi(\bar{f})(\pi_1 + \dots + \pi_n)}$ since φ is a homomorphism. But then $y \in B^{**}$ implies $\overline{\varphi(\bar{f})|B^n}(y) = \overline{\varphi(\bar{f})(\pi_1 + \dots + \pi_n)|B^n}(y) =$ $\varphi(\bar{f})(\pi_1 + \dots + \pi_n)(\underline{n}(y)) = \varphi(\bar{f})(\pi_1 + \dots + \pi_n)(y)$ by 2.6. The latter is just $\varphi(\bar{f})(y)$. //

2.8 REMARK. It is evident that if $\bar{f}$ is in the Function System over A, then $\mathrm{range}(\bar{f}) \subseteq A^m$ iff $\dim(\bar{f}) = m$; and if m is finite, $(\pi_1 + \dots + \pi_m)(\bar{f}) = \bar{f}$. This fact will hereafter be used without specific mention. Note explicitly that $\bar{f} \in M.S.\langle A, \tau \rangle$ implies $\dim(\bar{f})$ is finite.

2.9 THEOREM. Let φ be a homomorphism in the Function System of A into the Function System of B. Then for $m, n \in N^+$, $1 < m$ and $f \in A_m{}^n$ implies $[\exists g \in B_m{}^n][\varphi(\bar{f}) = \bar{g}]$.

Proof: $f \in A_m{}^n$ implies $\bar{f}(\pi_1 + \dots + \pi_m) = \bar{f}$, whence $\overline{\varphi(\bar{f})|B^m} = \varphi(\bar{f})$ by 2.8. Furthermore, $\mathrm{range}(\bar{f}) \subseteq A^n$ implies $(\pi_1 + \dots + \pi_n)\varphi(\bar{f}) = \varphi(\bar{f})$ so $\mathrm{range}(\varphi(\bar{f})) \subseteq B^n$. Thus $\varphi(\bar{f})|B^m \in B_m{}^n$ suffices for the required g. //

2.10 REMARK. It is evident at this point that the mapping φ of 2.5 is unique, for the only functions with extentions in $[\bar{A} \cup \bar{\tau}]$ are restrictions of concatenations of π_A, whence the only available image in $[\bar{B} \cup \bar{\gamma}]$ with range a subst of B^n is $n\pi_B$ by 2.9.

2.11 DEFINITION. Let $\langle A, \tau \rangle$ and $\langle B, \gamma \rangle$ be representations. Let

Φ be a Comparitor of $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$. Then Φ is <u>trivial</u> iff $\Phi[\bar{A}] = \{ \pi_B \}$.

2.12 COROLLARY. Let $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$ be representations. Then there always exists a trivial Comparitor of $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$.

Proof: Define Φ as in 2.5. //

The next result sharpens the relationship between Comparitors and homomorphisms.

2.13 THEOREM. Let $\langle A, \mathcal{T} \rangle$ be a <u>finitary</u> U,α representation (i.e. $[\forall u \in U][\, \mathcal{T}_u \in A_m{}^1$ implies m is finite]). Let Φ be a Comparitor of $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$. Then:

1/ $[\exists \langle B, \delta \rangle \prec \langle B, \gamma \rangle][\exists \varphi : A \longrightarrow B][\, \varphi$ is a homomorphism of $\langle A, \mathcal{T} \rangle$ into $\langle B, \delta \rangle$.]

2/ If Φ is non-trivial, then φ and δ can be chosen such that for each $a \in A$, $\Phi(\bar{a}) = \overline{\varphi(a)}$ and for each $u \in U$, $\Phi(\overline{\mathcal{T}_u}) = \overline{\delta(u)}$.

3/ If Φ is a Replicator, then φ constructed for 1/ and 2/ is 1-1.

4/ If Φ is a Simulator, then φ constructed for 1/ and 2/ is onto.

5/ If Φ is a Complete Identification, then φ constructed for 1/ and 2/ is an isomorphism.

2.14 REMARK. From the proof to follow, observe that if Φ is non-trivial in 2.13, $x \in A$, and $u \in U$, then $\varphi(a) = \Phi(\bar{a})(x)$ and $\delta(u) = \Phi(\overline{\mathcal{T}_u}) \big| B^{\alpha(u)}$.

Proof of 1/ and 2/. Suppose Φ is trivial and let $b \in B$ be arbitrary (but fixed). Define $\varphi(a) = b$ for every $a \in A$. Define δ on U by $\delta(u) = \alpha(u)^{\pi}{}_1$ (on B) so $\overline{\delta(u)} = \pi_B$ for each $u \in U$. Then $\langle B, \delta \rangle$ is a U,α representation and $\langle B, \delta \rangle \prec \langle B, \gamma \rangle$. Furthermore, letting $m = \alpha(u)$ gives $\varphi(\tau_u \langle a_1, \ldots \rangle_m) = b = {}_m\pi_1 \langle b, \ldots \rangle_m = \delta(u) \langle \varphi(a_1), \ldots \rangle_m$, therefore φ is a homomorphism.

Suppose now that $\underline{\varphi}$ is non-trivial. This implies $|A| \geqslant 2$ and $|B| \geqslant 2$ (for $|A| = 1$ implies $\bar{a} = \pi_A$ and $|B| = 1$ implies $\underline{\varphi}(\bar{a}) = \bar{b} = \pi_B$). Let $u \in U$ with $\alpha(u) = m$. Then $\tau_u \in A_m{}^1$ and by 2.9 there is $g \in B_m{}^1$ such that $\underline{\varphi}(\overline{\tau_u}) = \bar{g}$, so $g = \bar{g} | B^m = \underline{\varphi}(\overline{\tau_u}) | B^m$. In addition, this g is unique in $B_m{}^1$ by 0.12.i. Define $\delta(u) = g$. Then $\langle B, \delta \rangle$ is a U, α representation (each $\delta(u)$ has domain $B^{\alpha(u)}$). $\langle B, \delta \rangle \prec \langle B, \gamma \rangle$ since $\bar{g} = \underline{\varphi}(\overline{\tau_u}) \in [\bar{B} \cup \bar{\gamma}]$. Since $\underline{\varphi}$ is non-trivial, given $a \in A$, there exists $b \in B$ with $\underline{\varphi}(\bar{a}) = \bar{b}$ and b is unique. Define $\varphi(a) = b$. It remains to show φ to be a homomorphism. Again, let $u \in U$, $\alpha(u) = m$, and recall m is finite by hypothesis. Then $\varphi(\tau_u \langle a_1, \ldots, a_m \rangle) = b$ where $\underline{\varphi}(\overline{\tau_u \langle a_1, \ldots, a_m \rangle}) = \bar{b}$. But $\overline{\tau_u \langle a_1, \ldots, a_m \rangle} = \overline{\tau_u}(\overline{a_1} + \ldots + \overline{a_m})$, hence $\bar{b} = \underline{\varphi}(\overline{\tau_u \langle a_1, \ldots, a_m \rangle}) = \underline{\varphi}(\overline{\tau_u})(\underline{\varphi}(\overline{a_1}) + \ldots + \underline{\varphi}(\overline{a_m})) = \overline{\delta(u)}(\underline{\varphi}(\overline{a_1}) + \ldots + \underline{\varphi}(\overline{a_m})) = \overline{\delta(u) \langle \varphi(a_1), \ldots, \varphi(a_m) \rangle}$, i.e. $b = \delta_u \langle \varphi(a_1), \ldots, \varphi(a_m) \rangle$.

Proof of 3/. If $\underline{\varphi}$ is 1-1 and trivial, then $|A| = 1$ and $\underline{\varphi}(\bar{a}) = \pi_B = \underline{\varphi}(\pi_A)$ gives $\bar{a} = \pi_A$. But then φ contains one pair exactly and thereby is 1-1. If $\underline{\varphi}$ is 1-1 and not trivial, $x, y \in A$ and $\varphi(x) = \varphi(y)$ implies $\underline{\varphi}(\bar{x}) = \underline{\varphi}(\bar{y})$ whence $\bar{x} = \bar{y}$ and $x = y$.

Proof of 4/. If $b \in B$, $\bar{b} \in [\bar{B} \cup \bar{\gamma}]$ and therefore there is an $\bar{a} \in \bar{A}$ with $\underline{\varphi}(\bar{a}) = \bar{b}$ by 2.1.1.ii. If $\underline{\varphi}$ is trivial, $a \in A$ implies $\underline{\varphi}(\bar{a}) = \pi_B$, whence $\pi_B = \bar{b}$ and $|B| = 1$. This gives φ onto B. If $\underline{\varphi}$ is not trivial, $b \in B$ implies there is $\bar{a} \in \bar{A}$ with $\underline{\varphi}(\bar{a}) = \bar{b} = \overline{\varphi(a)}$ and $\varphi(a) = b$.

5/ is immediate from 3/ and 4/. //

While the above shows that Comparitors induce homomorphisms, it is not clear at this time whether every homomorphism can be thought of as induced by a suitable Comparitor. Even so, it is natural to consider the interrelationship between Comparitors and homomorphisms which <u>are</u> related and to question whether homomorphisms — in their turn — induce Comparitors. The remainder of this chapter is concerned with these questions.

2.15 DEFINITION. Let $\langle A, \mathcal{T} \rangle$ be a U,α representation and $\langle B, \gamma \rangle$ be a representation. Let $|A| \geqslant 2$. Let φ be a Comparitor of $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$. Let $\varphi \subseteq A \times B$ such that for each $a \in A$, $\varphi[a] = \varphi(\bar{a})[B]$. Then φ is <u>Descriptive</u> iff φ is a homomorphism of $\langle A, \mathcal{T} \rangle$ into $\langle B, \delta \rangle$ where $[\forall u \in U][\ \delta(u) = \varphi(\overline{\mathcal{T}_u}) | B^{\alpha(u)}\ ]$.

φ will be called the <u>linguistic morphism</u> or l-morphism of φ.

2.16 REMARK. i) It is clear that the relation φ is a function whenever $\varphi(\bar{a})$ is constant on B. In particular this is true if φ is non-trivial or if $|B| = 1$. ii) $\langle B, \delta \rangle$, where δ is given in 2.15, is a U,α representation and $\langle B, \delta \rangle \prec \langle B, \gamma \rangle$.

2.17 LEMMA. Let $\bar{g}$ be in the Function System over A. Let $k = \deg(\bar{g})$ and let $k < m$. Then $\overline{\bar{g}|A^m} = \bar{g}$.

Proof: $\overline{\bar{g}|A^m} = \bar{g}|A^m \circ \underline{m}$, thus for any $y \in A^{**}$, $\overline{\bar{g}|A^m}(y) = \bar{g}|A^m \langle z_1, \ldots \rangle_m$ where $\underline{m}(y) = \langle z_1, \ldots \rangle_m$. But $\langle z_1, \ldots \rangle_m \in A^m$ then gives $\overline{\bar{g}|A^m}(y) = \bar{g}\langle z_1, \ldots \rangle_m$. But if $k < m$, it is clear that for m finite that $(\pi_1 + \ldots + \pi_k)\langle z_1, \ldots \rangle_m = (\pi_1 + \ldots + \pi_k)(y)$, so $\overline{\bar{g}|A^m}(y) = \bar{g}(\pi_1 + \ldots + \pi_k)(y)$ and since k is $\deg(\bar{g})$ the conclusion follows. If $k = \omega = m$, then the conclusion follows by the fact that $\bar{g} \circ \underline{\omega} = \bar{g}$. //

2.18 COROLLARY. i) If φ is a Descriptive Comparitor of $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$, then for each $a \in A$, $\overline{\varphi(a)} = \varphi(\bar{a})$ and $\overline{\delta_u} = \varphi(\overline{\mathcal{T}_u})$. ii) If $\langle A, \mathcal{T} \rangle$ is finitary, then any Comparitor on $\langle A, \mathcal{T} \rangle$ is Descriptive.

Proof: 2.13 and 2.17. //

2.19 THEOREM. Let $\underset{\sim}{\varphi}$ be a Descriptive Comparitor of $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$. Let φ be the linguistic morphism of $\underset{\sim}{\varphi}$. Then for each $a \in A$, $\underset{\sim}{\varphi}(\mathcal{L}_A(a)) = \mathcal{L}_B(\varphi(a))$ and for each $u \in U$, $\underset{\sim}{\varphi}(\mathcal{L}_A(\mathcal{T}_u)) = \mathcal{L}_B(\delta_u)$ where δ is as in 2.15 and $\mathcal{L}$ is the appropriate mapping $x \longrightarrow \overline{x}$; i.e. the following diagram is commutative.

$$\begin{array}{ccccc} M.S.\langle A, \mathcal{T} \rangle & \xrightarrow{\underset{\sim}{\varphi}} & M.S.\langle B, \delta \rangle & \subseteq & M.S.\langle B, \gamma \rangle \\ \uparrow{\scriptstyle \mathcal{L}_A} & & \uparrow{\scriptstyle \mathcal{L}_B} & & \\ \langle A, \mathcal{T} \rangle & \xrightarrow{\varphi} & \langle B, \delta \rangle & \prec & \langle B, \gamma \rangle \end{array}$$

Proof: Let $a \in A$. Then $\underset{\sim}{\varphi}(\mathcal{L}_A(a)) = \underset{\sim}{\varphi}(\overline{a})$ and $\mathcal{L}_B(\varphi(a)) = \overline{\varphi(a)}$. But then for $y \in B^{**}$, $\underset{\sim}{\varphi}(\overline{a})(y) = \varphi(a) = \overline{\varphi(a)}(y)$ and the first conclusion is established. Let $u \in U$. Then $\mathcal{L}_A(\mathcal{T}_u) = \overline{\mathcal{T}_u}$ and $\delta_u = \underset{\sim}{\varphi}(\overline{\mathcal{T}_u}) \big| B^{\alpha(u)}$. Let $k = \deg(\underset{\sim}{\varphi}(\overline{\mathcal{T}_u}))$ and $m = \alpha(u)$. Then $\deg(\overline{\mathcal{T}_u}) \prec m$ implies $k \prec m$, so $\underset{\sim}{\varphi}(\overline{\mathcal{T}_u}) \big| B^m = \underset{\sim}{\varphi}(\overline{\mathcal{T}_u}) = \overline{\delta_u}$. //

2.20 THEOREM. Let $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$ be U, α representations. Let $\varphi : A \longrightarrow B$ be an isomorphism of $\langle A, \mathcal{T} \rangle$ onto $\langle B, \gamma \rangle$. Let $\hat{\varphi} : A^{**} \longrightarrow B^{**}$ be defined by $\hat{\varphi}\langle a_1, \ldots \rangle_n = \langle \varphi(a_1), \ldots \rangle_n$ for all $n \in \mathcal{N}^+$ and $a_i \in A$. Then:

1/ $\underset{\sim}{\varphi} : \underline{A} \longrightarrow \underline{B}$ defined by $\underset{\sim}{\varphi}(\overline{f}) = \hat{\varphi} \circ \overline{f} \circ \hat{\varphi}^{-1}$ is an isomorphism of the Function System of A onto the Function System of B.

2/ $\underset{\sim}{\varphi} \big| M.S.\langle A, \mathcal{T} \rangle$ is an isomorphism of $M.S.\langle A, \mathcal{T} \rangle$ onto $M.S.\langle B, \gamma \rangle$ (thus $\underset{\sim}{\varphi} \big| M.S.\langle A, \mathcal{T} \rangle$ is a Complete Identification of $\langle A, \mathcal{T} \rangle$ and $\langle B, \gamma \rangle$).

i.e. Isomorphic representations generate isomorphic Syntaxes in their isomorphic Function Systems; thus isomorphic representations are completely identified.

Proof: Suppose $|A| \geq 2$. The degenerate case only requires specialization of what follows. Let $\varphi : A \longrightarrow B$ be an isomorphism and let

$\hat{\varphi} : A^{**} \longrightarrow B^{**}$ be as above. It is evident that $\hat{\varphi}$ is 1–1 and onto, since φ is. Let $\Phi(\bar{f}) = \hat{\varphi}\cdot\bar{f}\cdot\hat{\varphi}^{-1}$ for every $\bar{f} \in \underline{A}$. Since $\hat{\varphi}$ is 1–1, Φ is a function on $\underline{A}$ onto $\underline{B}$ ($\bar{g} \in \underline{B}$ implies $\hat{\varphi}^{-1}\cdot\bar{g}\cdot\hat{\varphi}$ is mapped onto $\bar{g}$). Now, Φ is 1–1, for $\Phi(\bar{f}) = \Phi(\bar{g})$ iff $\hat{\varphi}\cdot\bar{f}\cdot\hat{\varphi}^{-1} = \hat{\varphi}\cdot\bar{g}\cdot\hat{\varphi}^{-1}$ and therefore only if $\bar{f} = \bar{g}$. It remains to show Φ is a homomorphism. Let $\bar{f}, \bar{g} \in \underline{A}$ and $x = \langle x_1, \ldots \rangle_n$ be in A^{**}, and suppose $y = \hat{\varphi}(x)$, $\bar{f}(x) = \langle u_1, \ldots \rangle_m$, and $\bar{g}(x) = \langle v_1, \ldots \rangle_k$. Then $\bar{f} + \bar{g}$ is defined implies $\Phi(\bar{f} + \bar{g}) = \hat{\varphi}\cdot(\bar{f} + \bar{g})\cdot\hat{\varphi}^{-1}$ so $\Phi(\bar{f} + \bar{g})(y) =$ $\hat{\varphi}\cdot(\bar{f} + \bar{g})\cdot\hat{\varphi}^{-1}(y) = \hat{\varphi}\cdot(\bar{f} + \bar{g})(x) = \hat{\varphi}\langle\langle \bar{f}(x), \bar{g}(x) \rangle\rangle = \hat{\varphi}\langle u_1, \ldots, u_m, v_1, \ldots \rangle_{m+k} =$ $\langle \varphi(u_1), \ldots, \varphi(u_m), \varphi(v_1), \ldots \rangle_{m+k}$. On the other hand, $((\hat{\varphi}\cdot\bar{f}\cdot\hat{\varphi}^{-1}) + (\hat{\varphi}\cdot\bar{g}\cdot\hat{\varphi}^{-1}))(y) = \langle\langle \hat{\varphi}\cdot\bar{f}\cdot\hat{\varphi}^{-1}(y), \hat{\varphi}\cdot\bar{g}\cdot\hat{\varphi}^{-1}(y) \rangle\rangle =$ $\langle\langle \hat{\varphi}(\bar{f}(x)), \hat{\varphi}(\bar{g}(x)) \rangle\rangle = \langle \varphi(u_1), \ldots, \varphi(u_m), \varphi(v_1), \ldots \rangle_{m+k}$, thus $\Phi(\bar{f} + \bar{g}) =$ $\Phi(\bar{f}) + \Phi(\bar{g})$. The composition case is straightforward. Finally, consider $\Phi(\bar{f}')$. For notational convenience, suppose n is finite. (For $n = \omega$ the proof is essentially unchanged, but auxiliary variables — e.g. z_i, $i = 1, 2, \ldots$ such that $\underline{\omega}(x) = \langle z_1, \ldots \rangle_\omega$ — must be introduced.) Then

$\hat{\varphi}\cdot\bar{f}'\cdot\hat{\varphi}^{-1}\langle y_1, \ldots, y_n \rangle = \hat{\varphi}(\bar{f}'\langle x_1, \ldots, x_n \rangle) = \hat{\varphi}(\bar{f}\langle x_2, \ldots, x_n, x_1 \rangle) =$
$\hat{\varphi}(\bar{f}(\hat{\varphi}^{-1}\langle y_2, \ldots, y_n, y_1 \rangle)) = \hat{\varphi}\cdot\bar{f}\cdot\hat{\varphi}^{-1}\langle y_2, \ldots, y_n, y_1 \rangle = (\hat{\varphi}\cdot\bar{f}\cdot\hat{\varphi}^{-1})'(y)$.

Since $\hat{\varphi}\cdot\pi_A\cdot\hat{\varphi}^{-1}(y) = \hat{\varphi}\cdot\pi_A\langle x_1, \ldots \rangle_n = \hat{\varphi}\langle x_1 \rangle = \varphi(x_1) = y_1$, $\Phi(\pi_A)$ is π_B and Φ is a homomorphism as claimed.

As before, let $M_0(\tau) = \bar{A} \cup \overline{\tau[U]} \cup \{\pi_A\}$, and let $M_{k+1}(\tau) = \{\bar{f} :$ $\bar{f} \in M_k(\tau)$ or $[\exists \bar{g}, \bar{h} \in M_k(\tau)][\bar{f} = \bar{g} + \bar{h}$ or $\bar{f} = \bar{g}\,\bar{h}]$ or $[\exists \bar{g} \in M_k(\tau)][\bar{f} = \bar{g}']\}$. Define $M_k(\gamma)$ similarly. Then M.S.$\langle A, \tau \rangle$ has carrier $\bigsqcup_{i \in N} M_k(\tau)$ and similarly for M.S.$\langle B, \gamma \rangle$. By construction, the M_k form ascending chains (ordered by inclusion). It will be shown that $\psi = \Phi|$M.S.$\langle A, \tau \rangle$ is into and onto M.S.$\langle B, \gamma \rangle$ by showing that for each $m \in N$, $\psi|M_m(\tau)$ is into and onto $M_m(\gamma)$. As a restriction of a monomorphism, ψ will then be an isomorphism. Projections may be ignored since it is already known that they map properly. Let $\overline{f} \in M_0(\mathcal{T})$ with $\overline{f} \neq \pi_A$ and $f \in {A_n}^1$. If $f = a \in {A_0}^1$, then $\psi(\overline{a}) = \hat{\varphi}\cdot\overline{a}\cdot\hat{\varphi}^{-1} = \overline{\varphi(a)} \in \overline{B}$. If $n > 0$, then observe that $\hat{\varphi}\cdot\overline{f}\cdot\hat{\varphi}^{-1}|B^n = \hat{\varphi}\cdot f\cdot\hat{\varphi}^{-1}|B^n$ and that $\hat{\varphi}\cdot\overline{f}\cdot\hat{\varphi}^{-1} = \overline{\hat{\varphi}\cdot f\cdot\hat{\varphi}^{-1}|B^n}$. But $f = \mathcal{T}_u$ for some $u \in U$ implies $\hat{\varphi}\cdot f\cdot\hat{\varphi}^{-1}|B^n = \gamma_u$ by definition of homomorphism. But then $\psi(\overline{f}) = \overline{\hat{\varphi}\cdot f\cdot\hat{\varphi}^{-1}|B^n} = \overline{\gamma}_u \in M_0(\gamma)$ shows $\psi[M_0(\mathcal{T})] \subseteq M_0(\gamma)$. Equality evidently obtains since each $\overline{\mathcal{T}_u}$ maps onto $\overline{\gamma_u}$. Now suppose the induction statement for each $k < m$. Let $\overline{f} \in M_m(\mathcal{T})$. If $\overline{f} \in M_{m-1}(\mathcal{T})$, $\psi(\overline{f}) \in M_m(\gamma)$ by hypothesis, so suppose $\overline{f} \in M_m(\mathcal{T}) - M_{m-1}(\mathcal{T})$. Then $[\exists \overline{g}, \overline{h} \in M_{m-1}(\mathcal{T})][\ \overline{f} = \overline{g} + \overline{h}$ or $\overline{f} = \overline{g}\,\overline{h}$ or $\overline{f} = \overline{g}'\ ]$. Since ψ is a homomorphism and by the induction hypothesis $\psi(\overline{g}), \psi(\overline{h}) \in M_{m-1}(\gamma)$, $\psi(\overline{f})$ has a unique image in $M_m(\gamma)$, and it follows that $\psi[M_m(\mathcal{T})] \subseteq M_m(\gamma)$. Since any $\overline{f} \in M_m(\gamma)$ is in $M_{m-1}(\gamma)$ or is formed from elements in $M_{m-1}(\gamma)$, the mapping $\psi|M_m(\mathcal{T})$ is onto $M_m(\gamma)$. //

2.21 COROLLARY. In the hypothesis of 2.20, $|A| \geqslant 2$ implies φ is the 1-morphism of ψ.

Proof: ψ is a Comparitor, φ is a homomorphism, and $\psi(\overline{a}) = \overline{\varphi(a)}$. It remains to show that $[\forall u \in U][\ \gamma_u = \psi(\overline{\mathcal{T}_u})|B^{\alpha(u)}\ ]$. Let $u \in U$. Then $\psi(\overline{\mathcal{T}_u}) = \hat{\varphi}\circ\overline{\mathcal{T}_u}\circ\hat{\varphi}^{-1}$. Furthermore, $\deg(\overline{\mathcal{T}_u}) \leqslant \alpha(u)$ implies $\overline{\psi(\overline{\mathcal{T}_u})|B^{\alpha(u)}} = \psi(\overline{\mathcal{T}_u}) = \overline{\hat{\varphi}\cdot\overline{\mathcal{T}_u}\circ\hat{\varphi}^{-1}|B^{\alpha(u)}}$. Now φ is an isomorphism implies $[\forall x \in A^{\alpha(u)}]$ $[\ \varphi(\mathcal{T}_u(x)) = \gamma_u(\hat{\varphi}(x))\ ]$ whence $[\forall y \in B^{\alpha(u)}][\ \varphi(\mathcal{T}_u(\hat{\varphi}^{-1}(y))) = \gamma_u(y)\ ]$ i.e. $\varphi\cdot\mathcal{T}_u\cdot\hat{\varphi}^{-1}|B^{\alpha(u)} = \gamma_u$. Observe that $\psi(\overline{\mathcal{T}_u})|B^{\alpha(u)} = \hat{\varphi}\cdot\overline{\mathcal{T}_u}\cdot\hat{\varphi}^{-1}|B^{\alpha(u)}$. Let $\xi = \hat{\varphi}\cdot\overline{\mathcal{T}_u}\cdot\hat{\varphi}^{-1}|B^{\alpha(u)}$. Let $y \in B^{\alpha(u)}$ and let $x = \hat{\varphi}^{-1}(y)$, so $x \in A^{\alpha(u)}$. Then $\gamma_u(y) = \varphi\cdot\mathcal{T}_u(x) = \varphi(\mathcal{T}_u(x))$ whereas $\xi(y) = \hat{\varphi}\circ\overline{\mathcal{T}_u}\circ\hat{\varphi}^{-1}|B^{\alpha(u)}(y) = \hat{\varphi}\cdot\overline{\mathcal{T}_u}(x)$. But $\overline{\mathcal{T}_u}(x) = \mathcal{T}_u\circ\underline{\alpha(u)}(x) = \mathcal{T}_u(x)$ and $\mathcal{T}_u(x) \in A$ implies $\xi(y) = \varphi(\mathcal{T}_u(x))$, whence $\psi(\overline{\mathcal{T}_u})|B^{\alpha(u)} = \xi = \gamma_u$. //

2.22 THEOREM. Let $\langle A, \mathcal{T}\rangle$ and $\langle B, \gamma\rangle$ be U, α representations. Let

$|A| \geq 2$. Let $\varphi : A \longrightarrow B$ be an epimorphism of $\langle A, \tau \rangle$ onto $\langle B, \gamma \rangle$. Then there exists a unique Simulator $\underline{\varphi}$ of $\langle A, \tau \rangle$ with $\langle B, \gamma \rangle$ such that φ is the 1-morphism of $\underline{\varphi}$.

Proof: If two Simulators have φ as 1-morphism, then they coincide on $M_0(\tau)$ and it is clear that if they coincide on $M_k(\tau)$ they also coincide on $M_{k+1}(\tau)$ by the homomorphism property, hence are the same function.

To construct $\underline{\varphi}$; let $\equiv$ be kernel(φ) and consider the following diagram:

$$\begin{tikzcd}
\mathrm{M.S.}\langle A, \tau \rangle \arrow[dr, "\underline{G}"] & & \\
& \mathrm{M.S.}\langle A/_{\equiv}, \tau/_{\equiv} \rangle \arrow[r, "\underline{I}"] & \mathrm{M.S.}\langle B, \gamma \rangle \\
\langle A, \tau \rangle \arrow[uu, "\mathcal{L}_A"] \arrow[dr, "\mathrm{Nat}(\equiv)"'] & & \\
& \langle A/_{\equiv}, \tau/_{\equiv} \rangle \arrow[uu, "\mathcal{L}(A/_{\equiv})"] \arrow[r, "I"] & \langle B, \gamma \rangle \arrow[uu, "\mathcal{L}_B"]
\end{tikzcd}$$

where $\underline{G}(\bar{f}) = \bar{f}/_{\equiv}$.

By the fundamental homomorphism theorem, $\langle B, \gamma \rangle$ is isomorphic with $\langle A/_{\equiv}, \tau/_{\equiv} \rangle$. But then $\mathrm{M.S.}\langle B, \gamma \rangle$ is isomorphic with $\mathrm{M.S.}\langle A/_{\equiv}, \tau/_{\equiv} \rangle$. The rectangle commutes by 2.21. Let $\underline{\varphi} = \underline{I} \circ \underline{G}$. Then $\underline{\varphi}(\bar{f}) = \underline{I}(\bar{f}/_{\equiv}) = \hat{I} \circ \bar{f}/_{\equiv} \circ \hat{I}^{-1}$. Since $\underline{G}$ and $\underline{I}$ are epimorphisms, $\underline{\varphi}$ is also. The trapezoid commutes by 1.21, so in particular $\overline{\varphi(a)} = \underline{\varphi}(\bar{a})$ and constants map into constants. This shows $\underline{\varphi}$ is a Simulator. For each $u \in U$, $\underline{\varphi}(\overline{\tau_u}) = \hat{I} \cdot \overline{\tau_u}/_{\equiv} \circ \hat{I}^{-1}$, and since I is the 1-morphism of $\underline{I}$ by 2.21, $\gamma_u = \hat{I} \cdot \overline{\tau_u}/_{\equiv} \cdot \hat{I}^{-1} | B^{\alpha(u)}$. From the proof of 2.21, $\overline{\gamma_u} = \hat{I} \circ \overline{\tau_u}/_{\equiv} \cdot \hat{I}^{-1}$, thus $\underline{\varphi}(\overline{\tau_u}) = \overline{\gamma_u}$. But then by remark 0.11, $\underline{\varphi}(\overline{\tau_u}) | B^{\alpha(u)} = \gamma_u$ and φ is the 1-morphism of $\underline{\varphi}$. //

While 2.22 shows that a large class of homomorphisms are linguistic maps, it happens that not all homomorphisms are.

2.23 EXAMPLE. Let A be any set with $|A| \geqslant 2$. Suppose $x, y \notin A$ with $x \neq y$. Let $B = A \cup \{x, y\}$. Let I_1 be the identity mapping on A and let $f : B \longrightarrow B$ be such that $f(a) = a$ for each $a \in A$ while $f(x) = y$ and $f(y) = x$. Then $\langle A, \{\langle 1, I_1 \rangle\}\rangle$ and $\langle B, \{\langle 1, f \rangle\}\rangle$ are $\{1\}, \{\langle 1,1 \rangle\}$ representations. The injection φ of A into B is a homomorphism. Now $\overline{I}_1 = \pi_A$ and $\overline{f} \neq \pi_B$. For any Comparitor $\underline{\varphi}$, $\pi_B = \underline{\varphi}(\overline{I}_1) \neq \overline{f}$, so $\underline{\varphi}$ cannot be Descriptive with 1-morphism φ.

CHAPTER III

FUNCTIONAL ISOMORPHISM

This chapter is devoted to further investigation of the interplay of the notions of Functional Equivalence and Isomorphism.[1]

3.1 DEFINITION. Let $\langle A, \tau \rangle$ and $\langle B, \gamma \rangle$ be representations. Then $\langle A, \tau \rangle$ is <u>Functionally Equivalent</u> with $\langle B, \gamma \rangle$ iff $A = B$ and $\langle A, \tau \rangle$ is A-Functionally Equivalent with $\langle B, \gamma \rangle$.

3.2 REMARK AND NOTATION. i) It is clear that "Functionally Equivalent" is an equivalence relation in the class of all representations ("algebras"). Denote this relation by $\mathcal{F}$; i.e. $\mathcal{F} = \Big\{ \langle \langle A, \tau \rangle, \langle B, \gamma \rangle \rangle :$ for some set C, $A = C = B$ and $\langle A, \tau \rangle$ and $\langle B, \gamma \rangle$ are C-Functionally Equivalent $\Big\}$. ii) In the same vein, it is clear that "isomorphic with" is also an equivalence relation in the class of all representations. Denote the latter by $\mathcal{I}$.

3.3 DEFINITION. Let $\langle A, \tau \rangle$ and $\langle B, \gamma \rangle$ be representations. Then $\langle A, \tau \rangle$ is <u>Functionally Isomorphic</u> with $\langle B, \gamma \rangle$ iff

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[1] The definitions and theorems could be stated in the usual Gödel-Bernays set context by using "sequences of representations, each of which is isomorphic or Functionally Equivalent (relative to some set) with the one just preceeding it". This will not be done. The foundations viewpoint taken herein is as in Quine [12], wherein sets can be "virtual". In that context, if a virtual set (that is, a symbolic specifier for a non-existent set) is used in a statement, then the offending statement can be reduced to logically valid quantified statements about things which are assumed to exist. The particular use to which this system will be put is that the term "relation" will be used in what is more usually

$$\langle\langle A,\mathcal{T}\rangle,\langle B,\gamma\rangle\rangle \in \bigcup_{j=-\infty}^{\infty}(\mathcal{F}\cdot\mathcal{J})^{j}$$ where $(\mathcal{F}\cdot\mathcal{J})^{0}$ is the identity relation; and $n \in N$ implies $(\mathcal{F}\cdot\mathcal{J})^{-n}$ is $(\mathcal{F}\cdot\mathcal{J})^{n}$ converse.

Denote the Functionally Isomorphic relation by $\mathcal{F}/\mathcal{J}$.

The following lemma is essentially the proof of theorem 3.5. It is included here for notational convenience. If R is a relation, R^{-1} is R converse.

3.4 LEMMA. Let R and T be quivalence relations over the same set X. Let $V = \bigcup_{n=-\infty}^{\infty}(R\circ T)^{n}$ where $(R\circ T)^{0} = \Delta$ (identity) and $(R\circ T)^{-n} = [(R\circ T)^{n}]^{-1}$. Then V is an equivalence relation over X.

Proof: Let $\Delta = (R\circ T)^{0}$. $\Delta \subseteq V$ by hypothesis and V is symmetric from the definitions of $(R\circ T)^{-n}$ and V. Transitivity remains to be shown. $\langle u,v\rangle, \langle v,w\rangle \in V$ implies for some integers m and n, $\langle u,v\rangle \in (R\circ T)^{m}$ and $\langle v,w\rangle \in (R\circ T)^{n}$. It will therefore suffice to show

$$(R\circ T)^{m}\circ(R\circ T)^{n} \subseteq (R\circ T)^{m+n}. \tag{1}$$

(1) is clearly true if m or n is zero. Furthermore, if m and n are both non-positive or both non-negative, (1) follows from the definitions and an easy induction. (1) will be shown for $m > 0$ and $n < 0$ by induction on $-n$. The case $m < 0$ and $n > 0$ is similar. Let $-i = n$, so $i \in N$. Then $i = 1$ implies $(R\circ T)^{m}\circ(R\circ T)^{-1} = [(R\circ T)^{m-1}\circ(R\circ T)]\circ(T^{-1}\circ R^{-1})$. But $R = R^{-1}$, $T = T^{-1}$, and associativity of composition gives equality with $(R\circ T)^{m-1}\circ[(R\circ T)\circ(T\circ R)]$. But this is equal to $(R\circ T)^{m-1}\circ[(R\circ T)\circ(T\circ R\circ\Delta)]$ and since $\Delta \subseteq T$, this latter is contained in $(R\circ T)^{m-1}\circ[(R\circ T)\circ(T\circ R\circ T)]$.

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a "class — but not set" context and these relations will be treated — as per Quine — in a "do what comes naturally" fashion. The advantage to exposition of what follows will be evident. To those unwilling to accept the proffered approach we extend apologies together with the hope that the ideas and constructions offered are yet sufficient for them to follow the train of thought.

But $(R\circ T)\circ(T\circ R\circ T) = (R\circ T\circ T)\circ(R\circ T) = (R\circ T)\circ(R\circ T) = (R\circ T)^2$ implies $(R\circ T)^m\circ(R\circ T)^{-1} \subseteq (R\circ T)^{m-1}\circ(R\circ T)^2 = (R\circ T)^{m+1}$, and the induction basis is established. Suppose (1) holds for all $i<k$. Then $(R\circ T)^m\circ(R\circ T)^{-k} =$ $(R\circ T)^m\circ[(R\circ T)^{-(k-1)}\circ(R\circ T)^{-1}]$ (since k is at least 2) $=$ $[(R\circ T)^m\circ(R\circ T)^{-(k-1)}]\circ(R\circ T)^{-1} \subseteq (R\circ T)^{m-(k-1)}\circ(R\circ T)^{-1} = (R\circ T)^{m-k+1}\circ(R\circ T)^1$. The latter is $(R\circ T)^{m-k}$ by the basis of the induction. //

3.5 THEOREM. $\mathscr{T}/\mathscr{I}$ is an equivalence over the class of all representations.

Proof: $\mathscr{T}$ and $\mathscr{I}$ are such equivalences and 3.4. //

At this point, the use of "representation" instead of "algebra" becomes clear. Implicit is the idea that an algebra is "something", and that one chooses some notationally convenient means to represent "it".

3.6 DEFINITION. An Algebraically Representable System or Algebra is an $\mathscr{T}/\mathscr{I}$ equivalence class.

The following example will illustrate the conceptual power of the above.

3.7 EXAMPLE. In what follows, $\langle A, \mathcal{T}\rangle$ will be denoted by listing in order the operations chosen by $\mathcal{T}$. Let $\mathscr{G}_1 = \langle A, e, \cdot, ^{-1}\rangle$ be a group. In defining a group in this manner one postulates that the operations satisfy the conditions i) $e\cdot a = a\cdot e = a$, ii) $a\cdot(b\cdot c) = (a\cdot b)\cdot c$, and iii) $a^{-1}\cdot a = a\cdot a^{-1} = e$ (which is essentially a composition rule) for each $a,b,c \in A$. Another formulation of the group concept is given by $\mathscr{G}_2 =$ $\langle B, /\rangle$ with postulates i) $a/a = b/b$, ii) $a/(b/b) = a$, and iii) $(a/c)/(b/c) = a\circ b$ for all $a,b,c \in B$ (cf. Barnes [1]). Other forms of definition can use $\mathscr{G}_3 = \langle C, +, -\rangle$ with postulates including existence of a universal (re. C) identity, and $\mathscr{G}_4 = \langle D, *\rangle$ with postulates including identity and pointwise inverse existence (the classical definition).

For $i = 1, 2, 3, 4$; consider the following.

Since $\langle \mathcal{G}_i, \mathcal{G}_i \rangle \in \mathcal{F}$, anything isomorphic with $\mathcal{G}_i$ is Functionally Isomorphic with it, hence represents the same algebra. This fits nicely with the common viewpoint that isomorphic representations essentially indicate only a notational change. (That this _is_ a common viewpoint is supported by the many times it is suggested that one "identify" isomorphic systems.)

Similarly, since $\langle \mathcal{G}_i, \mathcal{G}_i \rangle \in \mathcal{E}$, anything Functionally Equivalent with $\mathcal{G}_i$ is Functionally Isomorphic with it and represents the same algebra.

1/ Starting with $\mathcal{G}_1$, define the representation $\langle A, /\!\!/ \rangle$ by letting $/\!\!/ = /\!\!\!/\!\!/ \big| A \times A$ where $/\!\!\!/\!\!/$ is the function $\bar{\cdot}(\pi_1 + (\overline{^{-1}} \cdot \pi_2))$. Then $\langle A, /\!\!/ \rangle < \langle A, e, \cdot, ^{-1} \rangle$. Conversely, let $a \in A$ be arbitrary (but fixed) and then notice $\bar{e} = /\!\!\!/\!\!/(a + a)$, $\bar{\cdot} = /\!\!\!/\!\!/(\pi_1 + (\overline{^{-1}} \cdot \pi_2))$, and $\overline{^{-1}} = /\!\!\!/\!\!/((/\!\!\!/\!\!/(\bar{a} + \bar{a})) + \pi_1)$ so $\langle A, /\!\!/ \rangle$ F.E. $\mathcal{G}_1$. Direct computation shows that $a /\!\!/ b = a \cdot b^{-1}$ and $/\!\!/$ is seen to be the division operator associated with $\mathcal{G}_1$. But then if $\mathcal{G}_2 \mathcal{E} \langle A, /\!\!/ \rangle$, it follows that $\mathcal{G}_1 \mathcal{F}/\mathcal{E}\, \mathcal{G}_2$ so $\mathcal{G}_1$ and $\mathcal{G}_2$ represent the same thing.

2/ $\langle A, \cdot, ^{-1} \rangle < \langle A, e, \cdot, ^{-1} \rangle$. Also, $e \in A$ implies $\bar{e} \in \bar{A}$, so $\langle A, \cdot, ^{-1} \rangle$ F.E. $\mathcal{G}_1$. If $\mathcal{G}_3 \mathcal{E} \langle A, \cdot, ^{-1} \rangle$ then $\mathcal{G}_1 \mathcal{F}/\mathcal{E}\, \mathcal{G}_3$.

Given 1/ and 2/, then also $\mathcal{G}_2 \mathcal{F}/\mathcal{E}\, \mathcal{G}_3$. Unfortunately, the case of the classically defined $\mathcal{G}_4$ is less clear. While it is manifest that $\langle A, \cdot \rangle < \langle A, \cdot, ^{-1} \rangle$, there is no apparent way to construct $\overline{^{-1}}$ from $\bar{\cdot}$. It is unsettled whether a "group" as in $\mathcal{G}_4$ can be shown to be $\mathcal{F}/\mathcal{E}$ with $\mathcal{G}_1$ or not.

3.8 REMARK. In 2.20 it was observed that isomorphic representations generate isomorphic Syntaxes of their isomorphic Function Systems. The counterpart of this theorem for Functionally Equivalent representations is evident: Functionally Equivalent representations generate Functionally Equivalent Syntaxes of (in the finitary case) their Functionally Equivalent Function Systems, for the respective representations are identical.

The next three results show the elegant relationship between $\not{f}$ and $\mathscr{I}$, and that — where convenient — the definition of $\not{f}/\mathscr{I}$ can be simplified to $\not{f}\cdot\mathscr{I}$ or $\mathscr{I}\cdot\not{f}$.

3.9 THEOREM. Let $\langle B, \mu\rangle$ be a representation. Let $\langle A, \tau\rangle$ and $\langle A, \gamma\rangle$ be Functionally Equivalent and let $\varphi: A \twoheadrightarrow B$ be an isomorphism of $\langle A, \gamma\rangle$ onto $\langle B, \mu\rangle$. Then $[\exists\langle B, \delta\rangle][\ \varphi: A \twoheadrightarrow B$ is an isomorphism of $\langle A, \tau\rangle$ onto $\langle B, \delta\rangle$ and $\langle B, \delta\rangle$ and $\langle B, \mu\rangle$ are Functionally Equivalent.]

Diagrammatically:

$$\text{Given}\quad \begin{tikzcd} \langle A, \tau\rangle \arrow[r, "\not{f}"] & \langle A, \gamma\rangle \arrow[d, "\varphi"] \\ & \langle B, \mu\rangle \end{tikzcd} \quad, \exists\langle B, \delta\rangle \text{ such that } \begin{tikzcd} \langle A, \tau\rangle \arrow[d, "\varphi"'] & \\ \langle B, \delta\rangle \arrow[r, "\not{f}"] & \langle B, \mu\rangle . \end{tikzcd}$$

$$\textit{Same } \varphi$$

Proof: Suppose $\langle A, \tau\rangle$ is a U,α representation. For each $u \in U$, let $\delta_u = \hat{\varphi}\cdot\bar{\tau}_u\cdot\hat{\varphi}^{-1}\Big|B^{\alpha(u)}$ as in 2.20. Then $\langle B, \delta\rangle$ is a U,α representation and by construction φ is a homomorphism (hence isomorphism) of $\langle A, \tau\rangle$ and $\langle B, \delta\rangle$. It remains to show that $\bar{f} \in [\bar{B}U\bar{\delta}]$ iff $\bar{f} \in [\bar{B}U\bar{\mu}]$.

Suppose $\bar{f} \in [\bar{B}U\bar{\mu}]$. Then $\hat{\varphi}^{-1}\cdot\bar{f}\cdot\hat{\varphi} = \underset{\sim}{\varphi}^{-1}(\bar{f}) \in [\bar{A}U\bar{\gamma}]$ where $\underset{\sim}{\varphi}$ is the isomorphism of M.S.$\langle A, \gamma\rangle$ onto M.S.$\langle B, \mu\rangle$ as in 2.20. But M.S.$\langle A, \tau\rangle =$ M.S.$\langle A, \gamma\rangle$, so $\underset{\sim}{\varphi}$ is also an isomorphism of M.S.$\langle A, \tau\rangle$ onto M.S.$\langle B, \delta\rangle$. This gives $\bar{f} \in [\bar{B}U\bar{\delta}]$. By symmetry, the opposite inclusion follows also, hence $\langle B, \mu\rangle \not{f} \langle B, \delta\rangle$. //

3.10 COROLLARY. Let $\langle A, \tau\rangle$ be a representation. Let $\langle B, \mu\rangle$ and

$\langle B, \delta \rangle$ be Functionally Equivalent and let $\varphi: A \longrightarrow B$ be an isomorphism of $\langle A, \tau \rangle$ onto $\langle B, \delta \rangle$. Then $[\exists \langle A, \gamma \rangle][\ \varphi: A \longrightarrow B$ is an isomorphism of $\langle A, \gamma \rangle$ onto $\langle B, \mu \rangle$ and $\langle A, \tau \rangle$ and $\langle A, \gamma \rangle$ are Functionally Equivalent].

Diagrammatically:

Given
$$\begin{tikzcd} \langle A, \tau \rangle \arrow[d, "\varphi"'] & \\ \langle B, \delta \rangle \arrow[r, dash, "\mathcal{F}"] & \langle B, \mu \rangle \end{tikzcd}$$
, $\exists \langle A, \gamma \rangle$ such that
$$\begin{tikzcd} \langle A, \tau \rangle \arrow[r, dash, "\mathcal{F}"] & \langle A, \gamma \rangle \arrow[d, "\varphi"] \\ & \langle B, \mu \rangle \end{tikzcd}.$$

Proof: In the hypotheses of 3.9, interchange $\langle A, \tau \rangle$ with $\langle B, \mu \rangle$, $\langle A, \gamma \rangle$ with $\langle B, \delta \rangle$, and φ with φ^{-1}. Obtain $\langle A, \gamma \rangle$ as was $\langle B, \delta \rangle$ in 3.9. //

3.11 COROLLARY. $\mathcal{F} \cdot \mathcal{I} = \mathcal{I} \circ \mathcal{F}$. In particular, $\mathcal{F}/\mathcal{I} = \mathcal{F} \cdot \mathcal{I}$.

Proof: $\langle A, \tau \rangle \ \mathcal{F}/\mathcal{I} \ \langle B, \mu \rangle$ provides the hypotheses of 3.9, which gives $\langle A, \tau \rangle \ \mathcal{I} \cdot \mathcal{F} \ \langle B, \mu \rangle$. 3.10 provides the alternative. //

The following analogue of 1.22 shows that Functional Isomorphism is a sufficient condition for a congruence to transfer from one representation to another.

3.12 THEOREM. Let $\langle A, \tau \rangle \ \mathcal{F}/\mathcal{I} \ \langle B, \gamma \rangle$. Let $\equiv$ be a congruence on $\langle A, \tau \rangle$. Then there exists a congruence Γ on $\langle B, \gamma \rangle$ such that $\langle A/_{\equiv}, \tau/_{\equiv} \rangle \ \mathcal{F}/\mathcal{I} \ \langle B/_{\Gamma}, \gamma/_{\Gamma} \rangle$.

Proof: Suppose $\langle A, \tau \rangle \ \mathcal{F} \cdot \mathcal{I} \ \langle B, \gamma \rangle$. Then for some $\langle A, \mu \rangle$, $\langle A, \tau \rangle \ \mathcal{F} \ \langle A, \mu \rangle$ and there exists $\varphi: A \longrightarrow B$ which is an isomorphism of $\langle A, \mu \rangle$ onto $\langle B, \gamma \rangle$. Now, $\equiv$ is a congruence on $\langle A, \mu \rangle$ by 1.16, and $\Gamma = \left\{ \langle \varphi(a), \varphi(a') \rangle : \langle a, a' \rangle \in \equiv \right\}$ is a congruence on $\langle B, \gamma \rangle$. (If $\langle b_1, \ldots \rangle_m, \langle b_1', \ldots \rangle_m \in B^{\alpha(u)}$ with $b_i = \varphi(a_i)$ and $b_i' = \varphi(a_i')$ for all i involved, then $b_i \Gamma b_i'$ for all i involved implies $\gamma_u \langle b_1, \ldots \rangle_m = \hat{\varphi} \circ \mu_u \cdot \hat{\varphi}^{-1} \langle b_1, \ldots \rangle_m = \varphi(\mu_u \langle a_1, \ldots \rangle_m) = \varphi(\mu_u \langle a_1', \ldots \rangle_m) = \gamma_u \langle b_1', \ldots \rangle_m$). It is readily seen that

$\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle \not\mathscr{F}/\mathscr{I} \langle {}^{B}/_{\Gamma}, {}^{\gamma}/_{\Gamma} \rangle$; for $\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle \not\mathscr{F} \langle {}^{A}/_{\equiv}, {}^{\mu}/_{\equiv} \rangle$ by 1.22 while $\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle \mathscr{I} \langle {}^{B}/_{\Gamma}, {}^{\gamma}/_{\Gamma} \rangle$ by standard arguments. //

At this very abstract level, it is possible to define a Factor Algebra and thereby a category of Algebras. Doing so leads to reduction of the fundamental "homomorphism" theorem to triviality, suggesting that $\not\mathscr{F}/\mathscr{I}$ is as large an equivalence in the category of representations as offers non-trivial results.

3.13 DEFINITION. Let $\mathfrak{C}$ and $\mathfrak{G}$ be algebraically representable systems. Then $\mathfrak{G}$ is a <u>factor system</u> of $\mathfrak{C}$ iff $[\exists \langle A, \mathcal{T} \rangle \in \mathfrak{C}]$ $[\exists \equiv \subseteq A \times A][\equiv$ is a congruence on $\langle A, \mathcal{T} \rangle$ and $\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle \in \mathfrak{G}]$.

3.14 REMARK. 1.16 guarantees there is no ambiguity in 3.13 due to choice of representations, for in passing to another representation $\langle B, \gamma \rangle \in \mathfrak{C}$, the congruence $\equiv$ is carried into a congruence $\Gamma \subseteq B \times B$ under some isomorphism of systems with A and B as carriers. 3.12 then gives $\langle {}^{B}/_{\Gamma}, {}^{\gamma}/_{\Gamma} \rangle \in \mathfrak{G}$.

3.15 DEFINITION. Let $\mathscr{A}$ be the class of algebraically representable systems. Let $\mathscr{F}$ be the class of pairs $\langle \mathfrak{C}, \mathfrak{G} \rangle$ of Algebras such that $\mathfrak{G}$ is a factor system of $\mathfrak{C}$. For $\langle \mathfrak{C}, \mathfrak{G} \rangle$ and $\langle \mathfrak{D}, \mathfrak{E} \rangle$ in $\mathscr{F}$, define composition by $\langle \mathfrak{C}, \mathfrak{G} \rangle \langle \mathfrak{D}, \mathfrak{E} \rangle = \langle \mathfrak{C}, \mathfrak{E} \rangle$ iff $\mathfrak{G} = \mathfrak{D}$.

Since it is clear that $\mathfrak{A} = \langle \mathscr{A}, \mathscr{F}, \circ \rangle$ is a category, it is hereby named the "<u>Category of Algebraic Systems and Abstract Morphisms</u>."

3.16 THEOREM. Let $\langle \mathfrak{G}, \mathfrak{C} \rangle$ be a category isomorphism of $\mathfrak{G}$ and $\mathfrak{C}$. Then $\mathfrak{G} = \mathfrak{C}$. That is, the only isomorphisms in this category are the identity morphisms.

Proof: $\langle \mathfrak{G}, \mathfrak{C} \rangle$ is an isomorphism in $\mathfrak{A}$ implies $\langle \mathfrak{C}, \mathfrak{G} \rangle \in \mathfrak{A}$ also, since the identity on $\mathfrak{G}$ is $\langle \mathfrak{G}, \mathfrak{G} \rangle$. Then: $[\exists \langle A, \mathcal{T} \rangle \in \mathfrak{G}][\exists \equiv \subseteq A \times A]$ $[\equiv$ is a congruence on $\langle A, \mathcal{T} \rangle$ and $\langle {}^{A}/_{\equiv}, {}^{\mathcal{T}}/_{\equiv} \rangle \in \mathfrak{C}]$ and similarly

$[\exists\langle B,\gamma\rangle\in\mathfrak{C}][\exists\Gamma\subseteq B\times B][\ \Gamma$ is a congruence on $\langle B,\gamma\rangle$ and $\langle B/_\Gamma,\ \gamma/_\Gamma\rangle\in\mathfrak{G}\ ]$. This gives the diagram

$$\langle A,\tau\rangle \xrightarrow{\text{nat}} \langle A/_\equiv,\ \tau/_\equiv\rangle \xrightarrow{\mathcal{F}/\mathcal{L}} \langle B,\gamma\rangle \xrightarrow{\text{nat}} \langle B/_\Gamma,\ \gamma/_\Gamma\rangle$$

with $\langle A,\tau\rangle\ \mathcal{F}/\mathcal{L}\ \langle B/_\Gamma,\ \gamma/_\Gamma\rangle$. Due to the structure of $\mathcal{F}/\mathcal{L}$, the diagram may be refined to

$$\begin{tikzcd}
 & \mathfrak{G}\ni\langle A,\tau\rangle \arrow[dl, dash, "\mathcal{F}"'] \arrow[dr, leftrightarrow, "I_2"] \arrow[rrr, "\text{nat}_1"] & & & \langle A/_\equiv,\ \tau/_\equiv\rangle\in\mathfrak{C} \arrow[dl, dash, "\mathcal{F}"'] \arrow[dr, leftrightarrow, "I_1"] & \\
\mathfrak{G}\ni\langle A,\delta\rangle \arrow[dr, leftrightarrow, "I_2"'] & & \langle B/_\Gamma,\ \mu/_\Gamma\rangle \arrow[dl, dash, "\mathcal{F}"] & \langle A/_\equiv,\ \xi/_\equiv\rangle\in\mathfrak{C} \arrow[dr, leftrightarrow, "I_1"'] & & \langle B,\nu\rangle\in\mathfrak{C} \arrow[dl, dash, "\mathcal{F}"] \\
 & \mathfrak{G}\ni\langle B/_\Gamma,\ \gamma/_\Gamma\rangle \arrow[rrr, "\text{nat}_2"] & & & \langle B,\gamma\rangle\in\mathfrak{C} &
\end{tikzcd}$$

where $\langle A,\delta\rangle$, $\langle A/_\equiv,\ \xi/_\equiv\rangle$, $\langle B,\nu\rangle$, and $\langle B/_\Gamma,\ \mu/_\Gamma\rangle$ are suitable representations whose existence is assumed in the definition. I_1 and I_2 are isomorphisms. But then in particular one has

$$\begin{tikzcd}
 & \langle A/_\equiv,\ \xi/_\equiv\rangle \\
\langle B,\gamma\rangle \arrow[ur, "I_1"] \arrow[dr, "\text{nat}_2"'] & \\
 & \langle B/_\Gamma,\ \gamma/_\Gamma\rangle
\end{tikzcd} \quad .$$

Since kernel(I_1) = Identity on $B\subseteq\Gamma$, by a standard result (cf. Cohn [4], p. 60, Corollary 3.8) this completes uniquely to a commutative diagram. Let φ be the completion map. Then $\varphi\circ\text{nat}_2 = I_1$, whence $I_1^{-1}\circ(\varphi\circ\text{nat}_2)$ is the identity on B. But then nat_2 is a monomorphism, since its inverse $I_1^{-1}\circ\varphi$, exists. But then $\langle B/_\Gamma,\ \gamma/_\Gamma\rangle\ \mathcal{F}/\mathcal{L}\ \langle B,\gamma\rangle$; i.e.

$\langle {}^{B}/_{\Gamma}, {}^{\gamma}/_{\Gamma} \rangle \in \mathfrak{C} \cap \mathfrak{G}$, from which it follows that $\mathfrak{C}$ and $\mathfrak{G}$ must be the same equivalence class. //

3.17 REMARK. In this context, one obtains the following analogue of the fundamental homomorphism theorem: Given an abstract morphism $\langle \mathfrak{C}, \mathfrak{G} \rangle$, there exists a commutative diagram

$$\begin{tikzcd} \mathfrak{C} \arrow[rr, "\langle \mathfrak{C}, \mathfrak{G} \rangle"] \arrow[dr, "\langle \mathfrak{C}, \mathfrak{D} \rangle"'] & & \mathfrak{G} \\ & \mathfrak{D} \arrow[ur, "\langle \mathfrak{D}, \mathfrak{G} \rangle"'] & \end{tikzcd}$$

if and only if $\mathfrak{G}$ is a factor of $\mathfrak{D}$. If $\langle \mathfrak{D}, \mathfrak{G} \rangle$ is an isomorphism, then $\mathfrak{D} = \mathfrak{G}$.

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