A STUDY OF A REINFORCED CONCRETE BEAM-COLUMN JUNCTION HAVING BEAMS AT DIFFERENT ELEVATIONS

By

JOHN ALAN KELLY

Bachelor of Architectural Engineering

Oklahoma State University

Stillwater, Oklahoma

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Thesis Approved:

Louis Bass
Thesis Adviser

W. H. Chamberlain

D. D. Durham
Dean of the Graduate College

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CHAPTER I

INTRODUCTION

The purpose of this study is to determine the nature of structural action in a reinforced concrete beam-column joint in which two beams of slightly different elevations frame into the column in the same plane. An unusual situation in most buildings, this junction is common in ramp-type parking garage structures. In one particular structure the diagonal cracks which occurred in the column at these joints required an independent structural steel shoring system to prevent failure due to the cracks.

While large quantities of literature have been published regarding the structural action and failure of individual beam and column elements, very little research has been done on the failure of the monolithic joints of these elements. As far as the author could determine, no studies have been made of the particular joint conditions which occur in the ramp-type parking structures.

For this investigation preliminary experimental tests of several typical joints were made using reinforced plaster models, and several joint variations were then computer analyzed for stresses and displacements using the finite element method.

The plaster model studies, though simple in nature and qualitative in approach, yielded cracking patterns which corresponded very closely to those of the particular structure mentioned above. This correspondence is especially significant in view of the different properties of reinforced concrete and the model materials. Also, it was found that the joint which had similar beam elevations was capable of supporting greater loads than were those joints in which the beam elevations were significantly different.

One of the main parameters indicated by the model tests was the relative elevations of the beams framing into the column. The effects of this parameter on the stresses and displacements in the joint was investigated by theoretical analyses based on the finite element approach. The results of these analyses show that much higher stresses result in joints with a significant difference in the beam elevations. In addition, high principal tensile stresses were found in the regions of the column cracks within the models and in the directions consistent with these cracking patterns.

CHAPTER II

PRELIMINARY REINFORCED PLASTER MODEL INVESTIGATION

In beginning this project, a simple form of preliminary experimental analysis was desired for determining first, the feasibility and usefulness of a more detailed study and second, the parameters to be used in case a more quantitative investigation were to be carried out. Furthermore, it was felt that an experimental study would be helpful in obtaining a physical understanding of the structural action of the joints. With these criteria in mind, a reinforced plaster model study was decided upon for this preliminary phase because of the minimum amount of time, equipment and expense involved in working with these materials.

Figure 1 shows the crack pattern of a typical joint in a parking structure in Atlanta, Georgia. The cracks have been grouted in, but the pattern is still visible, and the added steel supports of Figure 2 testify to the extent of the damage. The proportions of this joint were used in determining the model dimensions and the dimensions used in the theoretical analysis described later. The cracking pattern of this junction is typical of others found in this structure and was used as a basis of comparison for the cracking configurations of the model joints.

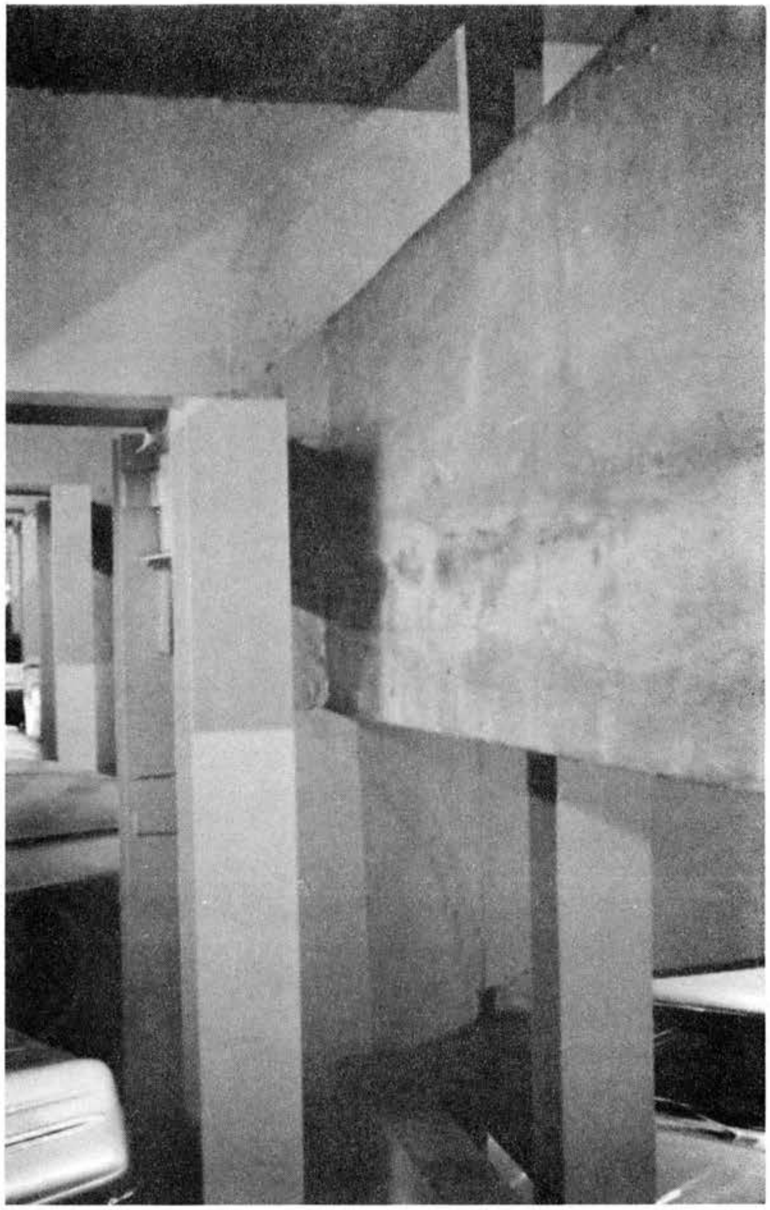


Figure 1. Crack Pattern of Typical Joint in an Atlanta, Georgia, Parking Structure

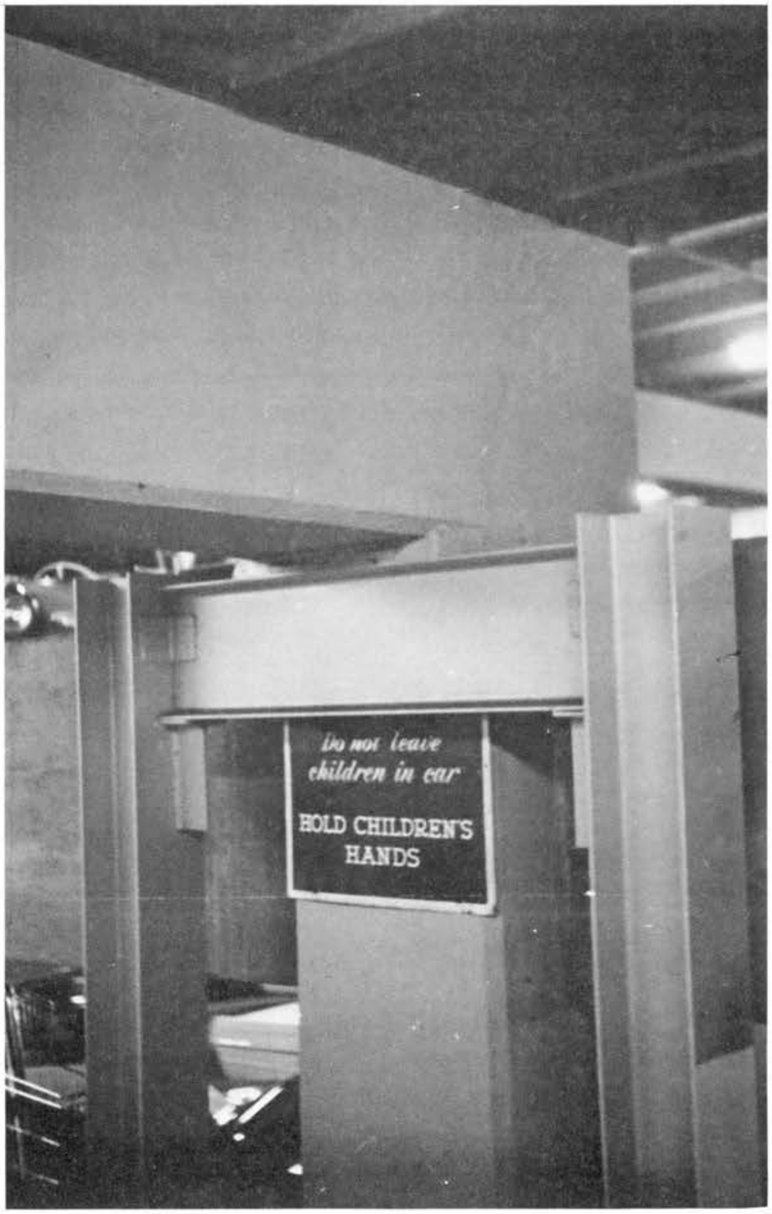


Figure 2. Steel Shoring System Necessitated by Joint Crack

Isolation of the Joint

The simplest joint isolation which the author could devise was employed for the model configuration. Figure 5 shows the general configuration of the model. The cantilevered beams permitted direct loading, and their greater depth resulted from the necessity for cracking in the column to be given the chance to occur before failure in the beam. A more complete structural framework such as a complete bent would have entailed greater care and more work in fabrication as well as a more elaborate loading system.

Selection of Materials

Both James (6) and Kornegay (7) suggest the use of a 0.9 water to plaster ratio because of its facility for filling the mold, completely covering the reinforcing and leaving no air pockets. The same ratio was used in this study for all models and found to be very satisfactory. An ordinary gypsum plaster was used because of its availability, although for exacting studies a more refined material such as a dental casting plaster may be desired.

After considering several possibilities for reinforcing, including threaded steel rods and annealed wire, hardware cloth was decided upon based upon the favorable recommendation of Kornegay (7) concerning the material's bonding properties, strength and ductility. This material can be obtained at any hardware store. It is an easily worked material since it can be cut with ordinary metal shears and readily formed into reinforcing cages.

Fabrication of Reinforcement

The hardware cloth was cut into strips of a width equal to the perimeter of the desired cage size. The cages were then bent and tapped around a wood form of dimensions slightly less than the finished cage size. After tack soldering in a few spots to hold the cage together, the wood form was slipped out and the soldering completed. Each cage element was fabricated in this manner, the finished size being small enough to allow about 1/4" plaster cover over all reinforcing. The transverse strands of the hardware cloth served as evenly spaced "stirrups" and "ties" for the beams and column. Figure 3 shows the assembled cage elements for the reinforcing of a typical model. The attachment of the beam cage elements to the continuous column cage was accomplished by soldering the beam reinforcing strands to the column reinforcing.

Fabrication of Molds

Molds to receive the plaster were made of plywood. One-quarter inch plywood vertical strips were nailed to a thicker plywood base to prevent movement and warping. A minimum of exact fitting was required by lapping the vertical strips where feasible. The mold with reinforcing in place ready to be filled with plaster is shown in Figure 4.

Mixing and Pouring the Plaster

A coating of machine oil was applied to the insides of the forms to facilitate removal, and after carefully positioning the reinforcing cage

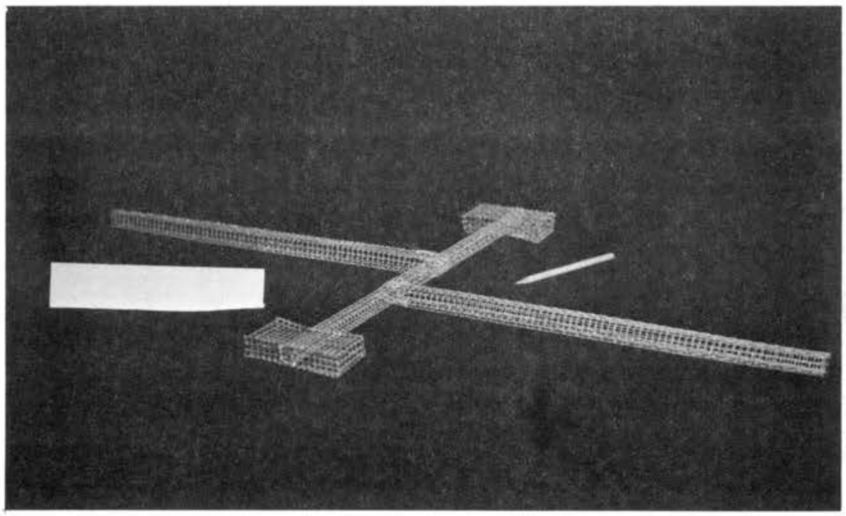


Figure 3. Assembled Cage for a Typical Model
Joint

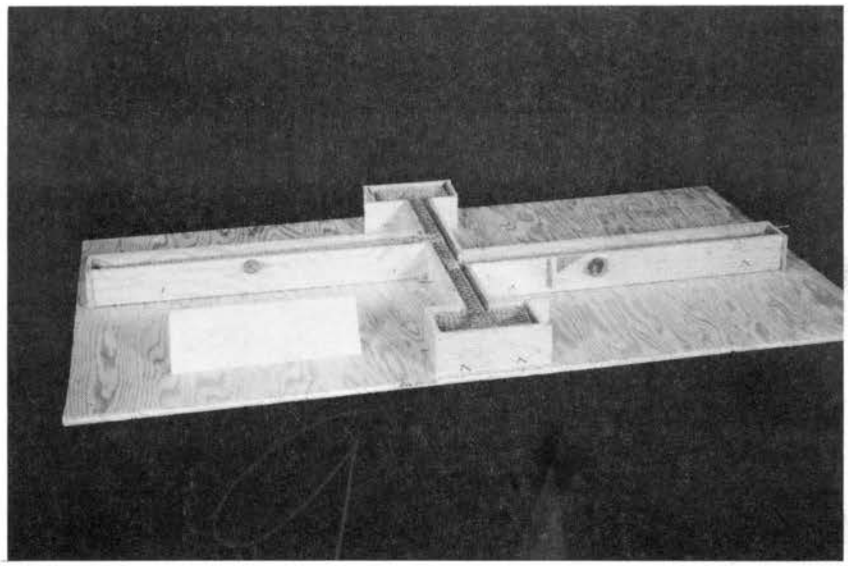


Figure 4. Plywood Mold With Cage Element in Place Prior to Pouring of the Plaster

in the mold to allow adequate cover on all sides of the reinforcement, the plaster and water were mixed. The convenient procedure outlined in reference (6) was used during the mixing operation. Pouring was accomplished as quickly as possible so that the plaster was in its mold before it began setting up. No vibration was required. The thin plaster mixture covered the reinforcing well, filling the smallest of spaces between reinforcing without the formation of air pockets. The plaster was poured until it reached above the top edges of the mold forming a meniscus. This was done to allow for the gathering of any free water at the top surface. The excess was scraped off with a wooden screed after the plaster had achieved its initial set.

The wood forms were removed after about ten to twelve hours. Removing the nails proved difficult, and the use of screws would have facilitated the form removal. The models were kept in a room of constant temperature for curing. Some investigators have recommended using as short a curing time as one hour, but it was felt that a longer time would assure a more distinct cracking pattern. Thus a curing time of four days was planned; however, circumstances prevented testing of the models until ten days had elapsed. As the results will indicate this discrepancy seems to have mattered little in the final cracking pattern.

Loading the Models

One of the advantages of the plaster model is the relatively small loads required to produce failure. This advantage coupled with the informal nature of the tests permitted the use of the very simple loading apparatus shown in Figure 5. The loading frame was designed to permit direct vertical loading by use of one pound weight increments hung from the extended cantilevers of the model. The one pound loads were applied simultaneously to each cantilever. However, in some of the models at higher loads, an unbalanced load condition was necessitated by the imminent failure of one of the cantilevers. This difference in the cantilever loadings was significant only for joint D, as can be observed from Table I. The lengths of steel reinforcing bars used as weights proved cumbersome and at times difficult to apply, especially when several weights were already in place. Also the tendency for the weights to swing into a slight pendulum motion no doubt initiated some dynamic effects which have been ignored. In spite of these drawbacks, the device served its purpose well. The apparatus was designed to provide lateral support to the cantilevers, fixity at the top and bottom of the column and a clear view of the joint itself during the loading procedure.

The six joints tested are shown in Figure 6. Joint A and joint A-1 are of the same dimensions, the only difference being that joint A was the first model poured and tested. Joint A-1 was poured and tested at the same time as the other models, B through E. The crack sequence for joint A-1 is shown in Figure 7.

Results

The final cracking patterns for the various joints tested are remark-

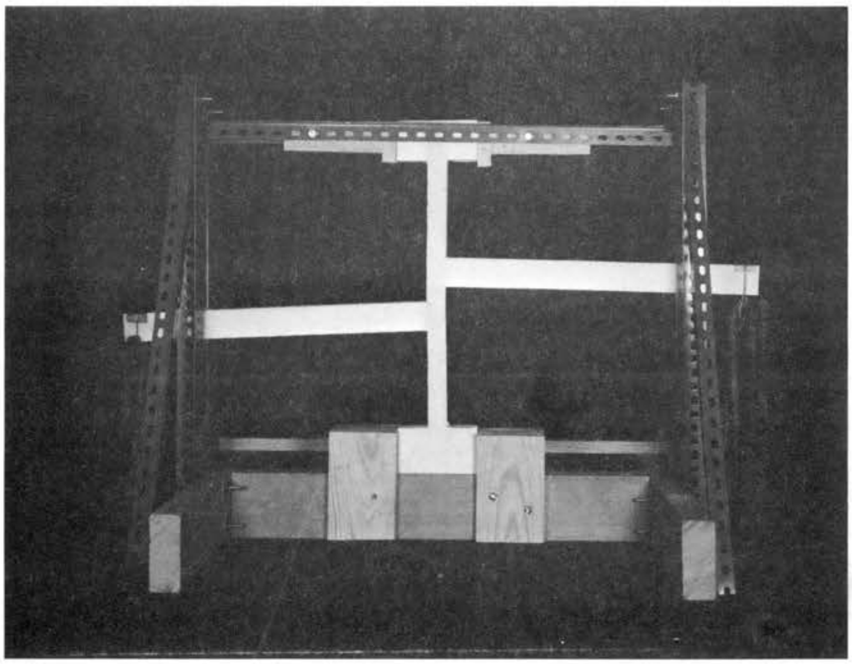


Figure 5. Model Loading Apparatus Showing Model Joint in Place With Hung Weights Applied at the Ends of the Cantilevers

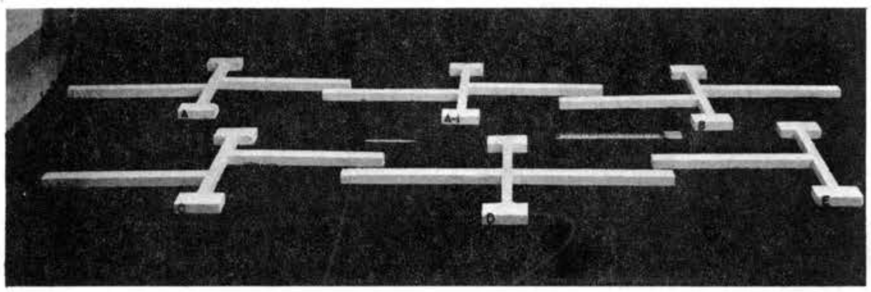
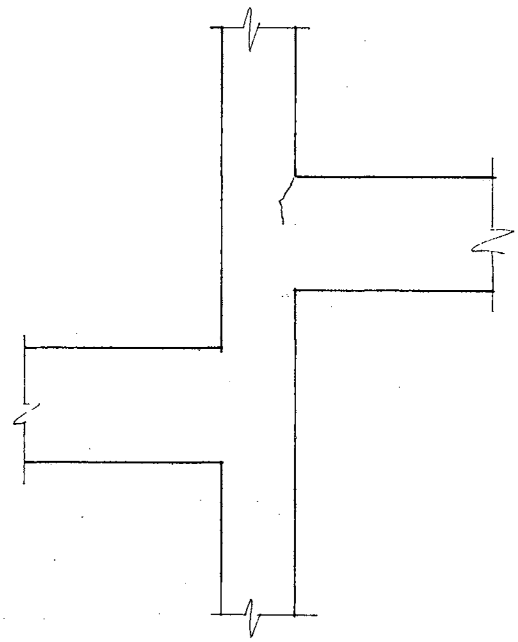
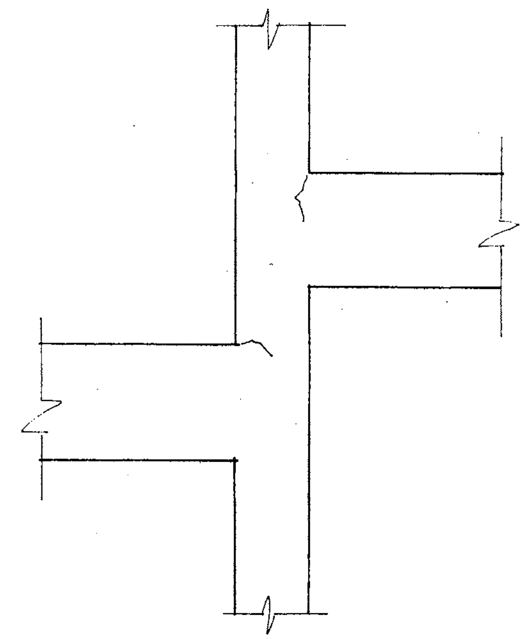


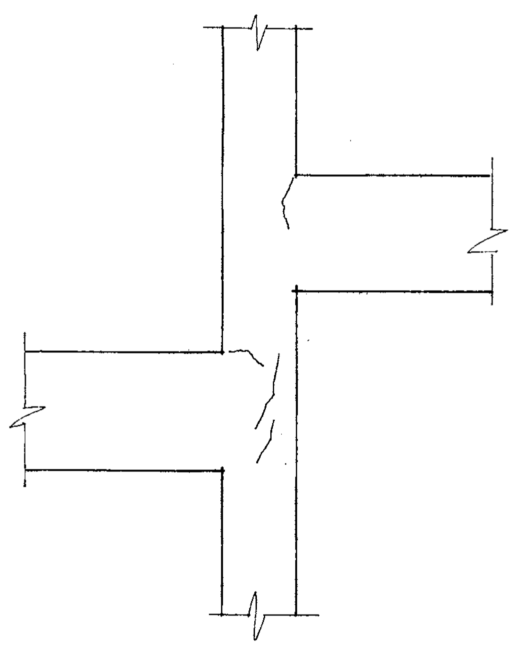
Figure 6. Model Joints Tested in Preliminary Investigation



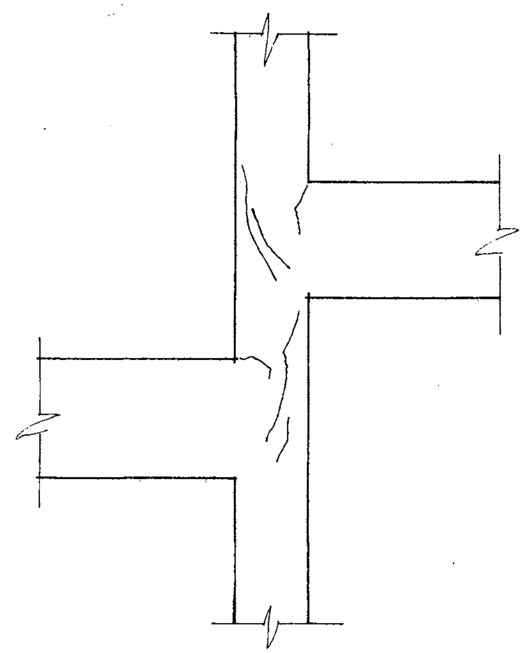
A. 5 lb. on each cantilever



B. 7 lb. on each cantilever



C. 8 lb. on each cantilever



D. 9 lb. on each cantilever

Figure 7. Crack Sequence of Joint A-1

ably similar as can be seen in Figures 8 through 13. Test joints A and A-1 showed very close failure patterns even though they were of different plaster batches and tested at different times. The repetition of this joint configuration was done to be certain the cracking pattern of joint A was not an accident. The resulting cracking pattern of both joints is in close agreement to that of the parking garage failure as can be seen by a comparison of Figures 8 and 9 with Figure 1. Joints B and C were tested to see if there could be any correlation between the difference in elevation of the beams and the type of crack resulting or the carrying capacity of the joint prior to cracking. The crack patterns vary so slightly that such a correlation cannot be established from them; however, it is the author's feeling that such a correlation might appear in a series of tests designed with longer column lengths.

Perhaps the clearest result of the tests can be seen in a comparison of the cracking pattern and loads of the joint D with the other joints tested. The reinforcing in this joint was kept as continuous as possible by threading the column cage through the middle of the continuous beam reinforcement instead of breaking it off and tying into the column steel as was necessary in all other joints tested. The magnitude of the total loads at failure for the "continuous" joint D was roughly one and one-half that of the other joints. The crack pattern in this joint also differs from the others as can be seen in Figure 12.

Joint D exhibited no diagonal cracking in the column as did all the other models. Joint E showed a very similar pattern of cracking to that

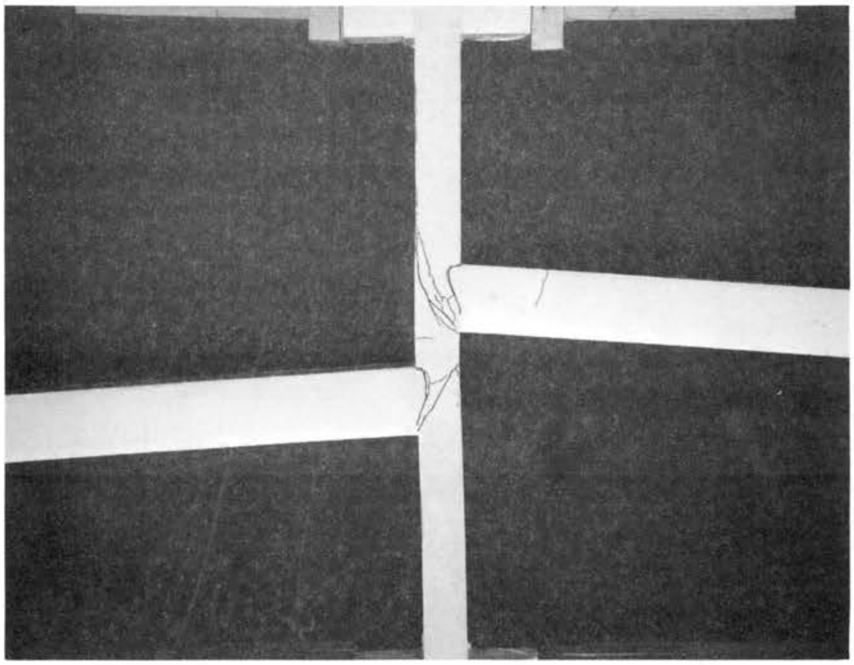


Figure 8. Final Cracking Pattern of Joint A

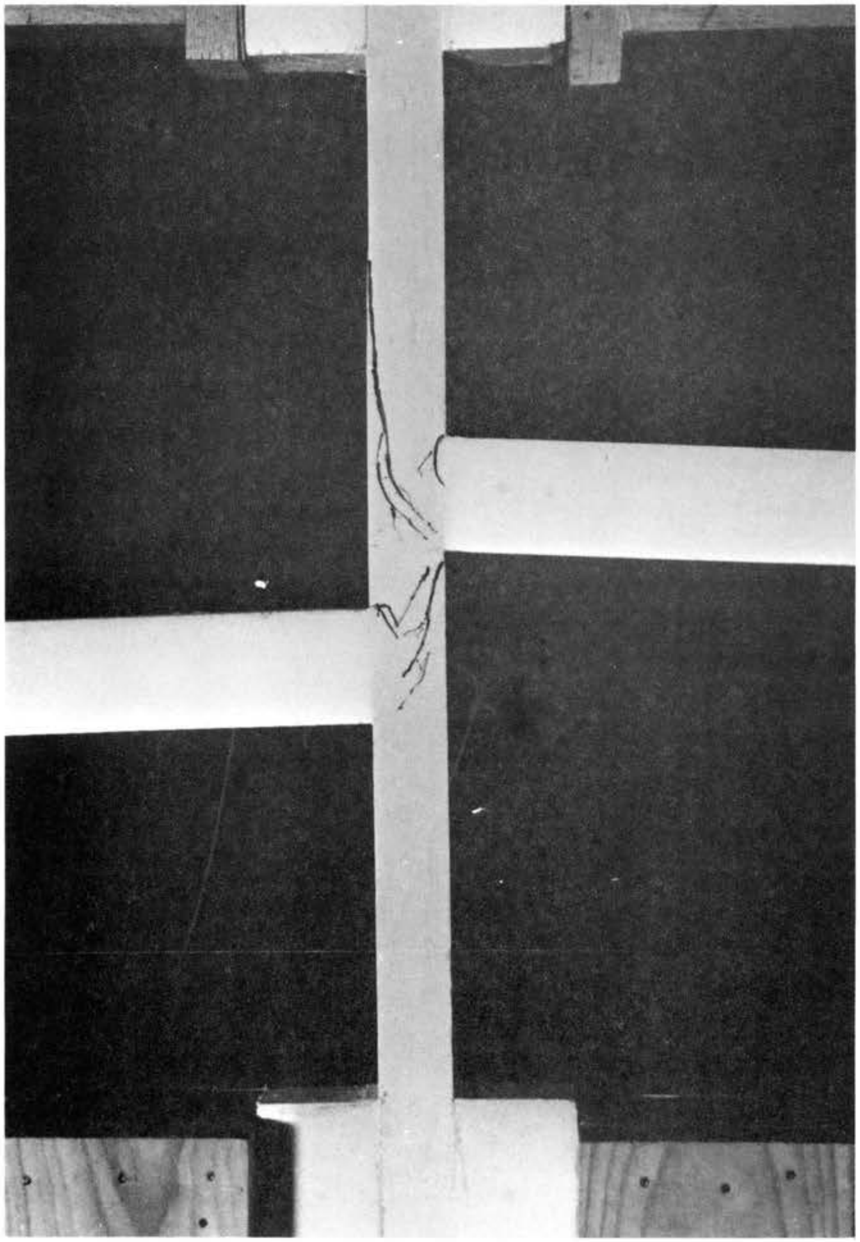


Figure 9. Final Cracking Pattern of Joint A-1

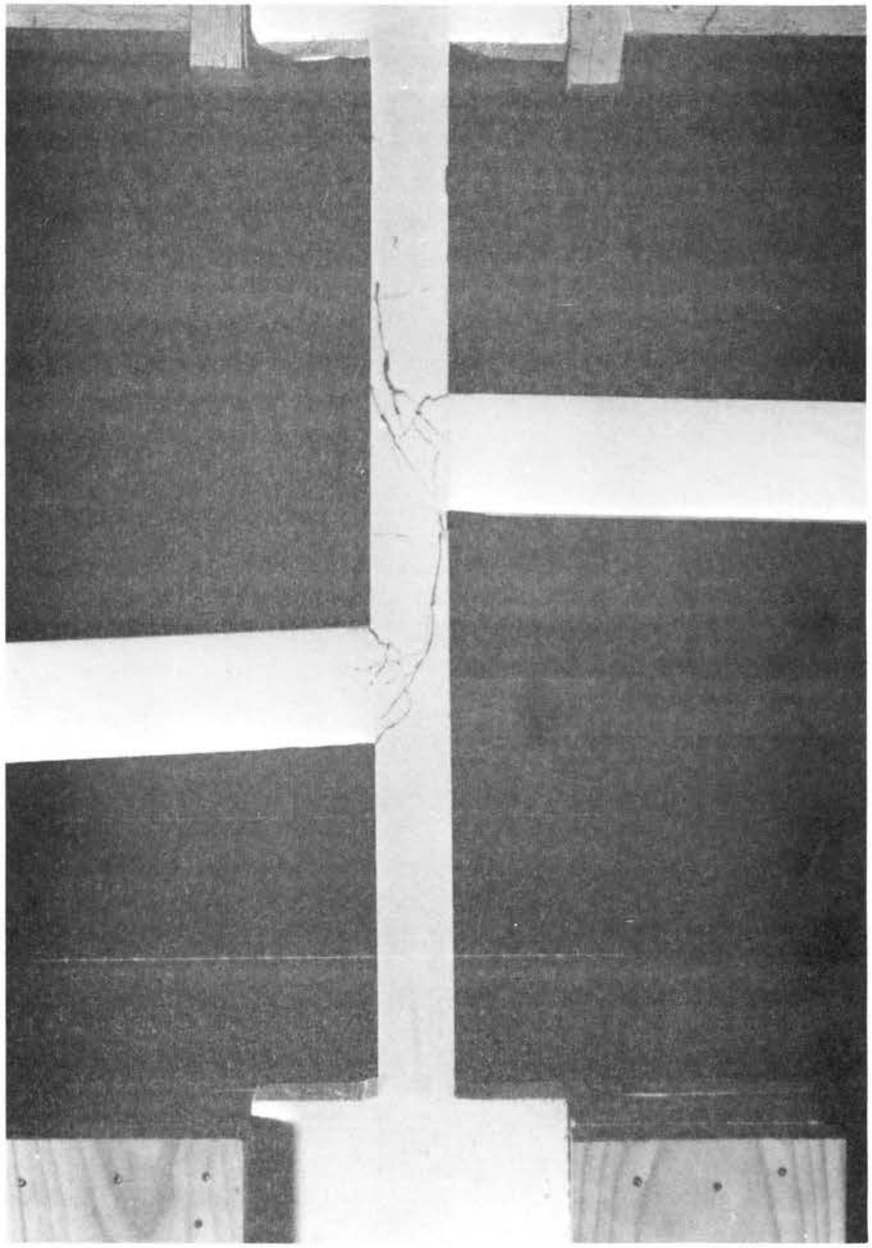


Figure 10. Final Cracking Pattern of Joint B

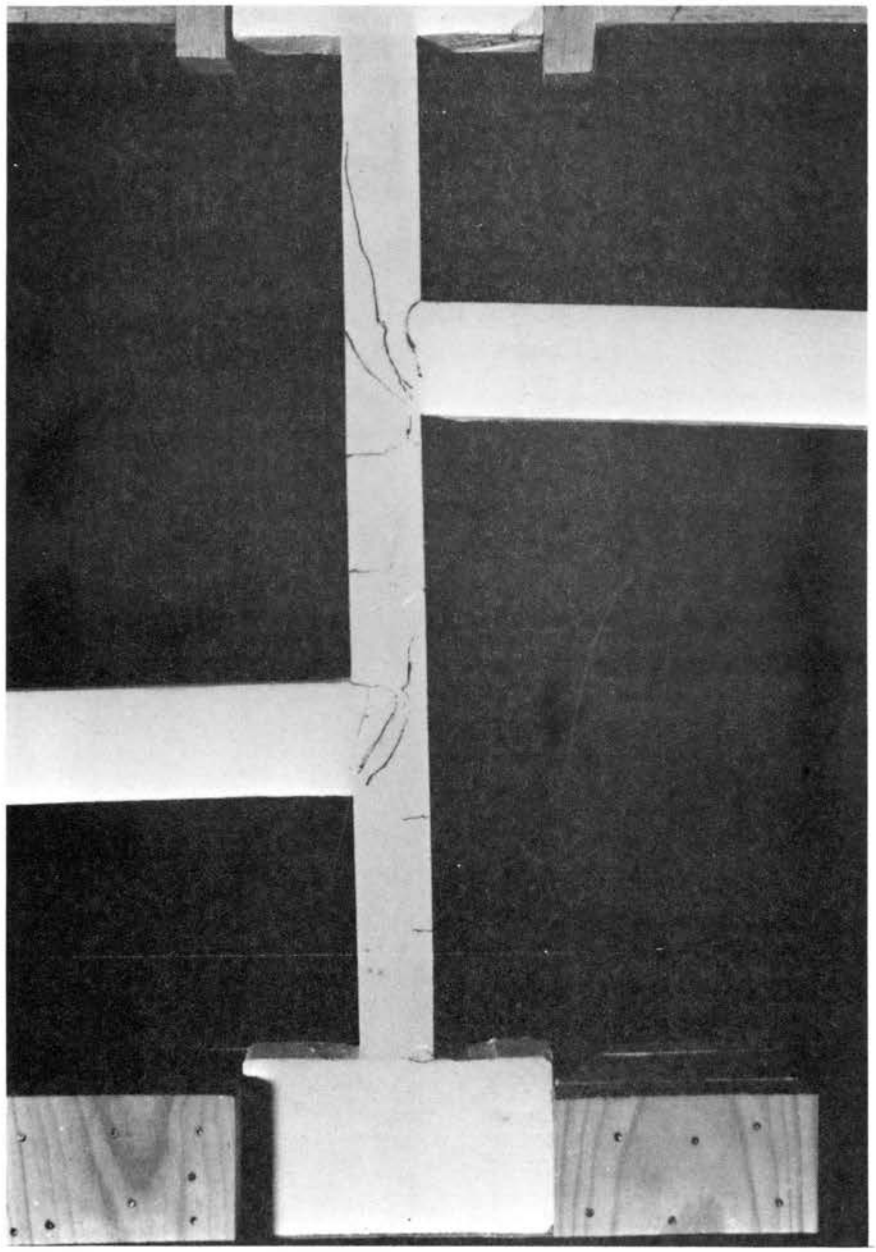


Figure 11. Final Cracking Pattern of Joint C

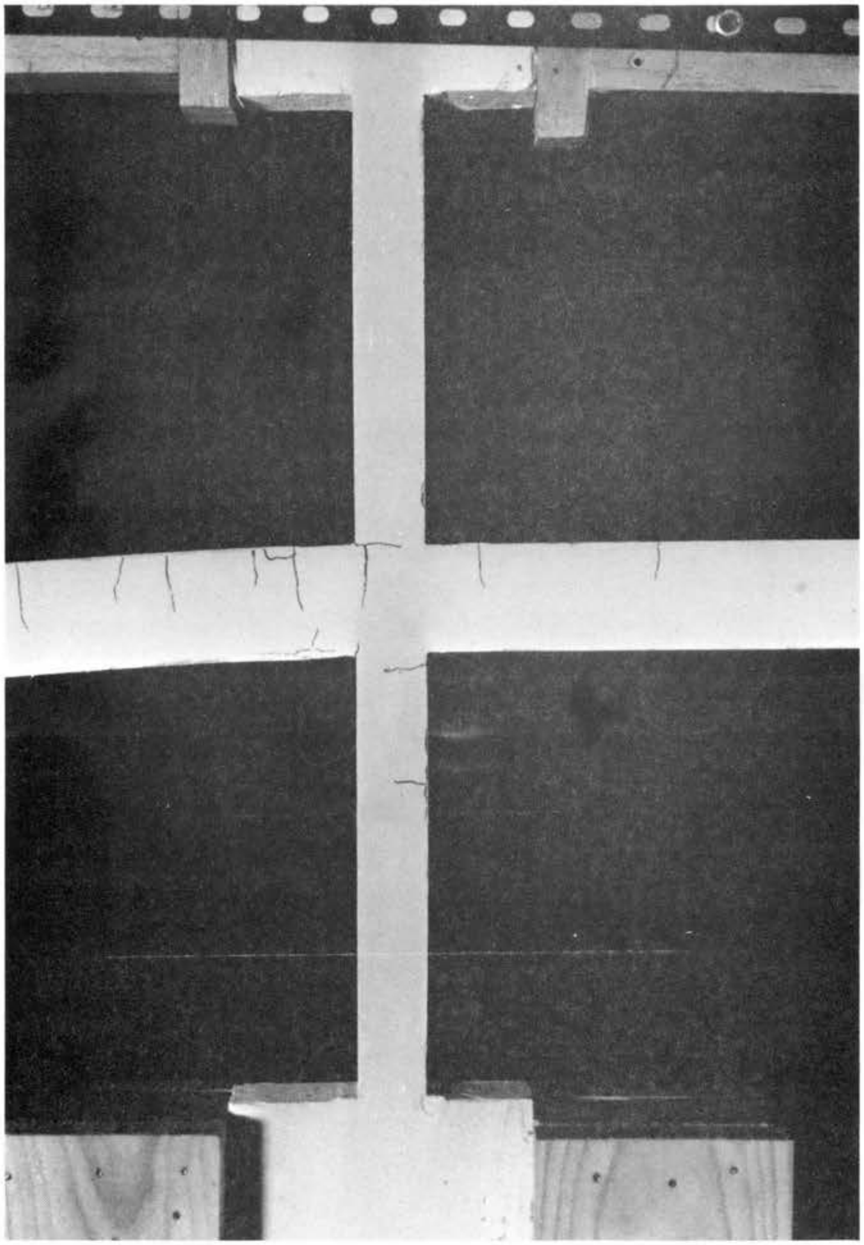


Figure 12. Final Cracking Pattern of Joint D

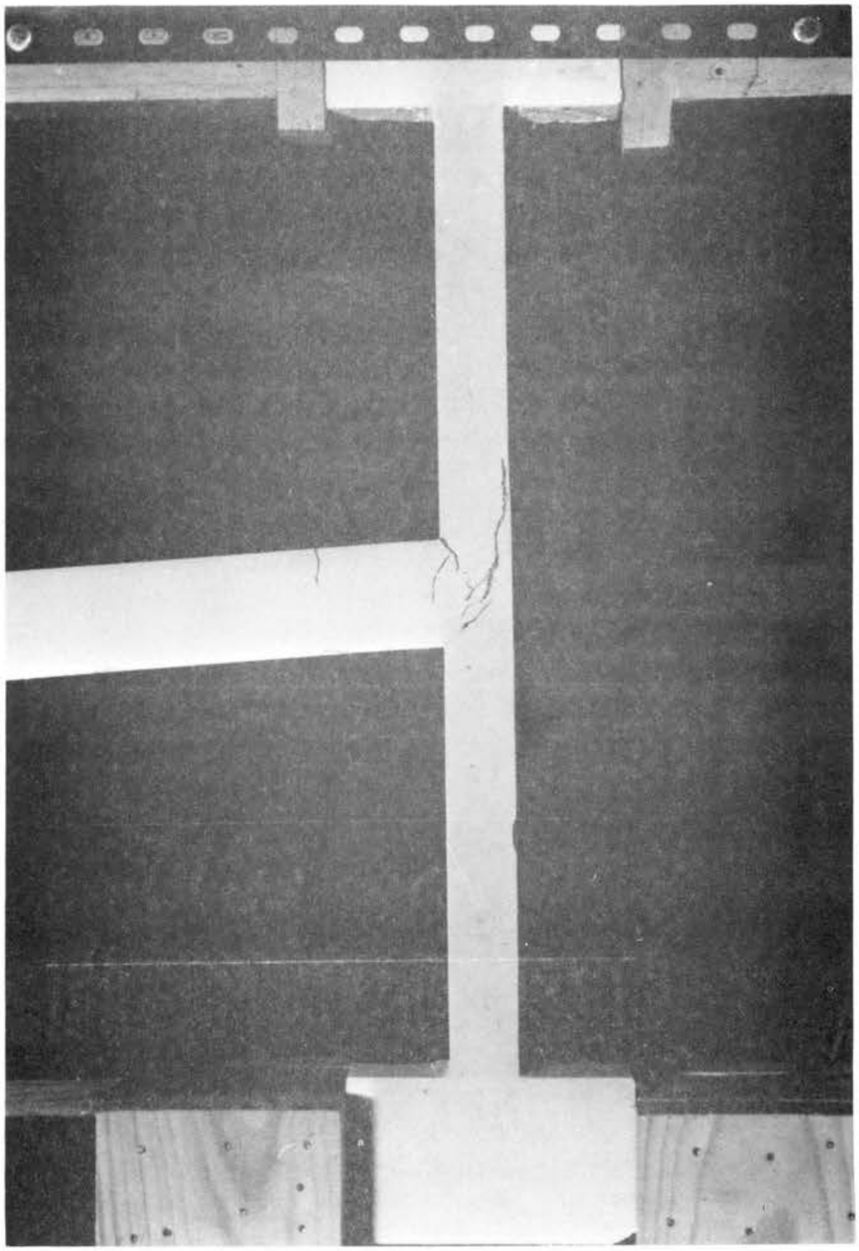


Figure 13. Final Cracking Pattern of Joint E

of the joints A, A-1, B and C in spite of the fact that joint E had only one beam framing into the column instead of two.

TABLE I

LOAD VALUES ON THE JOINTS AT CRACK FAILURE WITHIN THE COLUMN

Joint	Load Right (lb)	Load Left (lb)	Total load (lb)
A	9	7	16
A-1	9	9	18
B	10	10	20
C	10	11	21
D	12	23	35
E	—	10	10

Model Test Conclusions

The qualitative nature and limited number of tests made on the joints prevent any rigid conclusions regarding the beam-column joint under investigation. However, several qualified conclusions may be put forth.

First, the similarity of the cracks in the model tests and in the actual concrete joints in the parking structure indicate that a unique condition exists where a joint has beams or girders framing into the column at different elevations. This is a situation which could easily be overlooked in the design of such a structure.

Secondly, the unconventional cracks which occurred in both model and prototype indicate a stress distribution which is largely dependent on the relative position of the beams and the manner in which the reinforcing is detailed. The model tests indicated that differences exist in load capacities of a joint in which the reinforcement of the beams was truly continuous (i.e. carried from one beam into the other) from one in which the reinforcement was terminated and tied into the column. The joint having beams at the same elevation carried one and one-half times as great a load before failing as that carried by the other joints tested.

The positive results of the preliminary phase of this paper also demonstrate the value of the plaster model investigations as a tool for preliminary qualitative studies of various reinforced concrete situations. The major advantages of a study of this type are the ease with which the materials can be handled, their ready availability, and the relatively short time expended upon the preparation and fabrication of the models as well as the minimum requirements for testing apparatus. In addition, the information gained from such a study, though highly qualitative in nature, allows a rapid means of observing the structural phenomena in question. This information is of value in planning and launching further investigations, whether they be more refined experimental tests or mathematical analyses. A preliminary study such as this one can help the investigator formulate propositions and theories which may then be examined by more sophisticated means.

CHAPTER III

THEORETICAL ANALYSIS

There are numerous factors which might enter into the cracking of the beam-column joint shown in Figure 1. Some of the more important ones include 1) the total structural action of the monolithic reinforced concrete frame, 2) the nature of the moving loads, 3) the possibility of faulty engineering and/or construction, 4) the placement of reinforcement and 5) the relative location of the beams. The positive results of the model tests described in Chapter II on the isolated fixed column with loaded cantilevers seems to indicate the relative unimportance of 1), 2), and 3) when compared with 4) and 5). Due to the complexity of the problem, it was decided to limit the theoretical investigation mainly to the effects of the location of the beams. The basic set up for this analysis is very similar to the model loading conditions. A column of constant length and cross section is assumed fixed at each end and the beam positions are taken as the main parameters. The beam depths are assumed constant. The desired analytical results for each beam position are as follows:

1) deflected configuration of the joint, and
2) determination of the distribution and relative values of the principal stresses in the joint.

These results will allow comparisons of the effects of various beam positions as well as comparison of the regions of high tensile stress with the regions of cracking in the model tests.

Finite Element Method of Analysis

The analysis of framed structures of two and three dimensions has been greatly simplified by the advent of electronic computers and the formulation of the well-known methods of matrix structural analysis. The finite element method of analysis is based on the ordinary structural methods and their assumptions, i.e. that the structural system is an assemblage of distinct structural elements and that the forces and displacements of the structural assembly can be determined once the characteristics of the individual members are known. In the application to framed structures the elements are often entities in themselves, and their properties are assumed to be functions of a single variable, the distance along the axis of the member.

The finite element procedure, however, extends the basic methods to the analysis of continuum structures in which the continuum is replaced by a finite number of two dimensional idealized plate elements joined only at their corners, or nodes, each element having the same material property as the continuum. The resulting idealized structure can be treated as any other structure to be analyzed by matrix methods, once the stiffness characteristics of the individual elements have been determined. Naturally both the accuracy and the degree of complexity increase with the number of elements, for the greater the number of elements, the more closely the idealized structure approximates the real one.

The first attempt at idealizing an elastic continuum as an assemblage of structural elements was carried out by A. Hrennikoff (5). This was followed shortly by a similar approach, the "lattice analogy," developed by McHenry (9). Later improvements in the form of the finite element idealization were initiated by aeronautical engineers, led chiefly by Argyris (1,2). R.W. Clough (3) has been mainly responsible for the application of this method to non-aeroengineering structures.

In this thesis the beam-column joint is idealized by replacing the beams by boundary forces. The resulting rectangular shaped joint is assumed to be an elastic continuum idealized as a series of plate elements connected at the corners or nodes of the adjacent elements. The stiffnesses of the individual plate elements are computed and the displacement method of analysis is applied to evaluate the stresses and deflections of the joint. The investigation is thus a two dimensional stress analysis to determine the effects of the location of the beams on the stress distribution at the joint.

The finite element method is proving to be a very powerful analytical tool as indicated by the increasing amount of literature appearing about the method and its many uses. Because thorough treatises dealing with the theoretical development of the method are available, only a brief discussion of the method as applied in this investigation will be included here (1), (11).

Since the basic idea of the method is that of assuming the structural configuration to be a series of nodal connected elements, the choice of element geometry and the development of element structural properties become of prime importance. Any number of element shapes might be used, but the rectangle and triangle are most commonly employed. In this investigation the rectangular shape of the joint with its regular boundaries permitted the use of rectangular elements which tend to yield better approximations of stresses and deflections for a given nodal pattern than do triangular elements, although triangular elements offer many advantages for irregular boundary situations (2).

Having selected the element shape the next phase is the determination of the stiffness of the element. Various approaches to the determination of the stiffness matrix for a rectangular element have been made by Clough (3) and Martin (8), but the most immediately applicable derivation is made by Argyris (1) on page 251. The results of the derivation are used in this thesis for the stiffness matrix of a typical rectangular element. The basis of the derivation is as follows: each of the corners or nodes of the element is assumed to have two degrees of freedom, one in the horizontal direction and one in the vertical direction; the eight degrees of freedom of the element are represented by element nodal coordinates which are used to refer to forces and displacements at the node; these coordinates are numbered in sequence as shown in Figure 14.

The assumptions mentioned above of the element boundaries deforming as straight lines must of course be employed here. The determination of the stiffness coefficients is similar to that of a one-dimensional element, the main difference being the degree of complexity involved in the calculations. A unit displacement is applied at each coordinate with the other coordinate deformations held to zero and the coordinate forces required to create this deformation form one column of the stiffness matrix. The displacement used for forming the third column of the element stiffness matrix can be seen in Figure 15. Since there are eight coordinates, the procedure must be repeated eight times, resulting in an eight by eight matrix.

Virtual work concepts are employed to calculate the coefficients (11). Referring to figure 15, the assumption of linear element boundary deformation means that the displacement of an arbitrary point x,y within the element varies from zero at the top boundary, $y = 0$, to α at the bottom boundary, $y = L$; α in turn varies linearly from $\alpha = 1$ at $x = 0$, to $\alpha = 0$ at $x = D$.

Then $$\frac{\alpha}{1} = \frac{D - x}{D} = 1 - \frac{x}{D} ,$$

and $$\frac{w(x,y)}{\alpha} = \frac{y}{L} .$$

Thus $$w(x,y) = \frac{y}{L}\alpha = \frac{y}{L}\left(1 - \frac{x}{D}\right) ,$$

and $$u(x,y) = 0 .$$

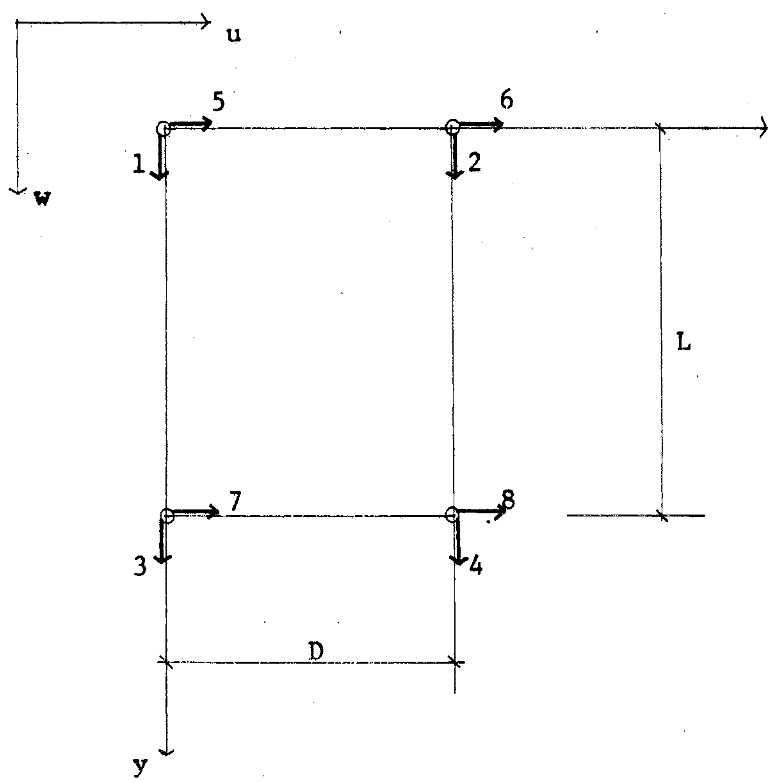


Figure 14. Typical Rectangular Element Coordinate System

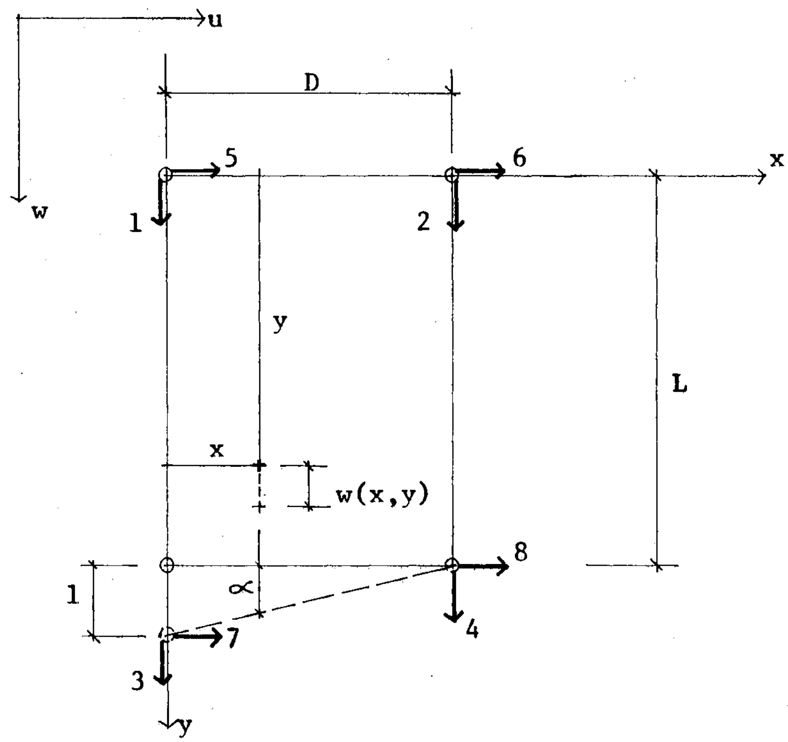


Figure 15. Unit Deformation at Coordinate 3 Showing Corresponding Displacement of Point (x,y)

The strain energy formula for any coefficient of the stiffness matrix is

$$k_{jh} = \int_0^D \int_0^L \int_0^T \sigma_j \varepsilon_h \, dx\,dy\,dt \ .$$

Since $\varepsilon_{xx} = \dfrac{\partial u(x,y)}{\partial x}$,

$$\varepsilon_{yy} = \frac{\partial w(x,y)}{\partial y} \ ,$$

and $\varepsilon_{xy} = \dfrac{\partial u(x,y)}{\partial y} + \dfrac{\partial w(x,y)}{\partial x}$,

the element strains due to a unit displacement at coordinate 3 are:

$$\varepsilon_{yy_3} = \frac{\partial}{\partial y}\left(\frac{y}{L}\left(1 - \frac{x}{D}\right)\right) = \frac{1}{L}\left(1 - \frac{x}{D}\right) \ ,$$

$$\varepsilon_{xx_3} = 0 \ ,$$

$$\varepsilon_{xy_3} = \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial x}\left(\frac{y}{L}\left(1 - \frac{x}{D}\right)\right) = -\frac{y}{LD} \ .$$

The stresses corresponding to these strains are found from the elastic relationships:

$$\sigma_x = (\varepsilon_{xx} + V\varepsilon_{yy})E' \quad ;$$

$$\sigma_y = (\varepsilon_{yy} + V\varepsilon_{xx})E' \quad ;$$

$$\sigma_{xy} = G\varepsilon_{xy} \quad ;$$

where E = Modulus of Elasticity,

V = Poisson's Ratio,

$$E' = \frac{E}{(1 - V^2)} \quad ,$$

and $$G = \frac{E}{2(1 + V)} \quad .$$

Thus $$\sigma_{yy_3} = \frac{E'}{L}(1 - \frac{x}{D}) \quad ,$$

$$\sigma_{xx_3} = \frac{VE'}{L}(1 - \frac{x}{D}) \quad ,$$

and $$\sigma_{xy_3} = \frac{-Gy}{LD} \quad .$$

By applying unit displacements at the remaining coordinates similar formulas can be obtained for each of these conditions. The element coefficients may then be found by application of the strain energy formula. For example, applying a unit displacement at element coordinate 7 yields the following:

$$\frac{\alpha}{1} = \frac{y}{L}$$

and $$\frac{\alpha}{D} = \frac{u(x,y)}{D - x} ,$$

thus $$u(x,y) = \alpha\left(\frac{D - x}{D}\right) = \frac{y}{L}\left(1 - \frac{x}{D}\right) ,$$

and $$w(x,y) = 0 .$$

Then $$\varepsilon_{xx} = \frac{-y}{LD} ,$$

$$\varepsilon_{yy} = 0 ,$$

$$\varepsilon_{xy} = \frac{1}{L}\left(1 - \frac{x}{D}\right) ,$$

and $$\sigma_{xx} = \frac{-E'y}{LD} ,$$

$$\sigma_{yy} = \frac{-VE'y}{LD} ,$$

$$\sigma_{xy} = \frac{Gy}{L}\left(1 - \frac{x}{D}\right) .$$

Applying the strain energy formula for coefficient 3,7 and assuming a constant element thickness, T,

$$KE_{3,7} = T\int_0^D \int_0^L \sigma_3 \varepsilon_7 dxdy = T\int_0^D \int_0^L$$

$$= T\int_0^D \int_0^L \frac{VE'}{L}\left(1 - \frac{x}{D}\right).\left(\frac{-y}{LD}\right) \, dxdy$$

$$+T \int_0^D \int_0^L \frac{-Gy}{LD} . \frac{1}{L}\left(1 - \frac{x}{D}\right)dxdy$$

$$\frac{-VE'T}{4} - \frac{GT}{4} .$$

Other element coefficients of the stiffness matrix are obtained in the same manner. For convenience in writing, the final element stiffness matrix is separated into the contributions due to shear strains and direct strains:

$$[KE] = [KS] + [KD] \text{ , where}$$

$$KS = GT \begin{bmatrix} \frac{L}{3D} & & & & & & & \\ -\frac{L}{3D} & \frac{L}{3D} & & & & & \text{(symmetrical)} & \\ \frac{L}{6D} & -\frac{L}{6D} & \frac{L}{3D} & & & & & \\ -\frac{L}{6D} & \frac{L}{6D} & -\frac{L}{3D} & \frac{L}{3D} & & & & \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{D}{3L} & & & \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{D}{6L} & \frac{D}{3L} & & \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{D}{3L} & -\frac{D}{6L} & \frac{D}{3L} & \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{D}{6L} & -\frac{D}{3L} & \frac{D}{6L} & \frac{D}{3L} \end{bmatrix},$$

$$KD=E'T\begin{bmatrix} \frac{D}{3L} & & & & & & & \\ \frac{D}{6L} & \frac{D}{3L} & & & & \text{(symmetrical)} & & \\ -\frac{D}{3L} & -\frac{D}{6L} & \frac{D}{3L} & & & & & \\ -\frac{D}{6L} & -\frac{D}{3L} & \frac{D}{6L} & \frac{D}{3L} & & & & \\ \frac{V}{4} & \frac{V}{4} & -\frac{V}{4} & -\frac{V}{4} & \frac{L}{3D} & & & \\ -\frac{V}{4} & -\frac{V}{4} & \frac{V}{4} & \frac{V}{4} & -\frac{L}{3D} & \frac{L}{3D} & & \\ \frac{V}{4} & \frac{V}{4} & -\frac{V}{4} & -\frac{V}{4} & \frac{L}{6D} & -\frac{L}{6D} & \frac{L}{3D} & \\ -\frac{V}{4} & -\frac{V}{4} & \frac{V}{4} & \frac{V}{4} & -\frac{L}{6D} & \frac{L}{6D} & -\frac{L}{3D} & \frac{L}{3D} \end{bmatrix}$$

Synthesis of the System Stiffness

The system stiffness matrix establishes the relationship between the forces acting on the system and the displacements due to these forces. The determination of the overall system stiffness matrix requires an independent set of coordinates at each nodal point of the structure which must be related in some manner to the coordinates of the individual elements. Figure 16 shows the coordinates for a coarse mesh system of six elements. The numbering of the system nodal points is carried out across the shortest dimension of the structure, a procedure which encourages the formation of a well-conditioned system stiffness matrix.

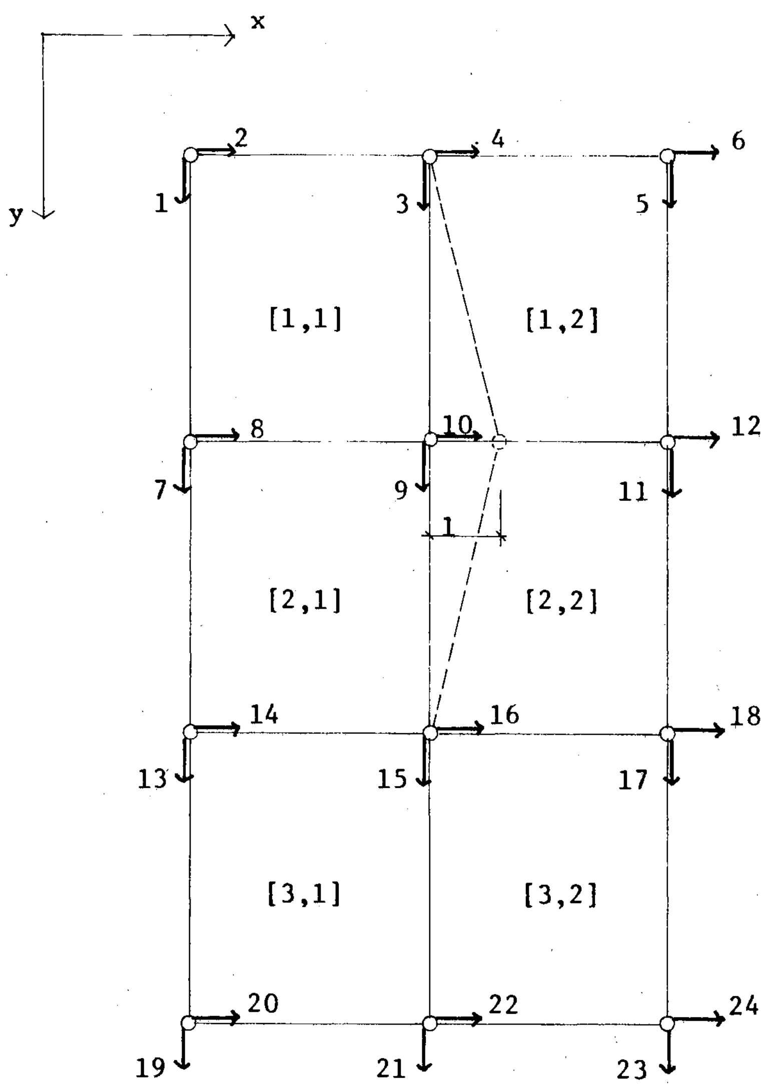


Figure 16. System Coordinates for a Coarse Mesh

The construction of the stiffness matrix for the system proceeds similarly to that of the individual element stiffness matrix. A unit displacement is made at the jth coordinate. The coordinate forces necessary to maintain the unit displacement of the node form the jth column of the stiffness matrix. The resulting coordinate forces can be determined from the stiffness matrices of the individual elements connected to the displaced node. For example, in Figure 16 a unit displacement at coordinate 10 (with all other nodal displacements held to zero) will require unit displacements at coordinate 8 in element [1,1], at coordinate 7 in element [1,2], at coordinate 6 in element [2,1], and at coordinate 5 in element [2,2]. The forces resulting from these displacements have already been tabulated in the element stiffness matrix and it is a simple matter to transfer the appropriate value of the element coefficients to their proper places in the system stiffness matrix. Of course this procedure must be repeated for each coordinate of the system in order to form the complete system stiffness matrix (11).

System Equations

In this investigation the shears and moments of the beams on each side of the column are replaced by equivalent forces acting at appropriate nodal points of the isolated structural system. The relationship between these applied forces and the resulting nodal displacements is expressed by a set of simultaneous equations which in matrix form become

[FORCE] = [STF] [UNODE]

where

[FORCE] is the system of forces acting at the coordinates,

[STF] is the system stiffness matrix,

[UNODE] is the column of nodal displacements at the coordinates of the structure.

Solution of the System Equations

Since the forces acting on the system are known and the stiffness matrix can be determined, the linear simultaneous equations can be solved for the displacements of the system. Several methods are available for this purpose including matrix inversion, iteration, relaxation, and substitution. In this paper, the method of substitution will be used as developed for banded symmetric equations by E.L. Wilson (15). This method utilizes the tendency for stiffness matrices to have their non-zero elements located in a band near the main diagonal of the matrix. This quality and the symmetrical nature of the stiffness matrix are exploited to achieve a significant saving in calculation time and required computer storage.

Determination of Element Stresses

With the determination of the nodal displacements a back substitution is made into previously established relationships to find the desired element stresses. This is accomplished as follows: the element nodal displacements are easily determined from the system nodal displacements by a coordinate transformation. The relationships between the element displacements and element strains outlined earlier have been developed into a more convenient matrix form for triangular elements by Rubenstein (11) and Clough (3). Here these relationships are adapted for use with rectangular elements.

The displacement functions defining the linear boundary assumption for the deformed rectangular element are as modified for the coordinate system used in this paper,

$$w(x,y) = a_1 + a_2 x + a_3 y + a_4 xy$$

$$u(x,y) = a_5 + a_6 x + a_7 y + a_8 xy$$

where, referring to Figure 15, "w" and "u" refer to the displacement of a point (x,y) within the element in the "y" and "x" directions respectively. These functions may be expressed in matrix form as

$$\delta(x,y) = \begin{Bmatrix} w(x,y) \\ u(x,y) \end{Bmatrix} = [A] \quad \{a\}$$

in which

$$[A] = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix}$$

and

$$a = \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_8 \end{Bmatrix}$$

The strains within the element can be found from the appropriate partial derivatives of the displacement functions:

$$\varepsilon_{xx} = a_6 + a_8 y$$

$$\varepsilon_{yy} = a_3 + a_4 x$$

$$\varepsilon_{xy} = a_2 + a_4 y + a_7 + a_8 x \quad .$$

The matrix form of these equations is

$$\{\varepsilon\} = [B] \{a\}$$

in which

$$[B] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & y \\ 0 & 0 & 1 & x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & y & 0 & 0 & 1 & x \end{bmatrix} \quad .$$

The displacements in the "w" and "u" directions at the element nodal points can be expressed in terms of $\{a\}$ by the equation

$$\{\delta_i\} = [C] \{a\} \quad .$$

In this equation $\{\delta_i\}$ represents the nodal displacement vector

$$\begin{Bmatrix} \delta_1 \\ \vdots \\ \delta_8 \end{Bmatrix}$$

which has been previously determined from the system displacements. The matrix [C] is the result of substituting the element coordinate locations into the proper displacement function, or more directly by substitution into [A] :

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & D & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & L & 0 & 0 & 0 & 0 & 0 \\ 1 & D & L & DL & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 1 & D & L & DL \end{bmatrix} .$$

Since $\{\delta\}$ and [C] are known, it is convenient to solve for $\{a\}$ by matrix inversion. Thus

$$\{a\} = [C]^{-1}\{\delta\}$$

which may be substituted into the matrix strain equation to yield

$$\{\varepsilon\} = [B] \quad \{a\} = [B] \quad [C]^{-1}\{\delta\} \qquad .$$

Further, the relationship of stress to strain is expressed by Wang (13) in matrix formulation as

$$\{\sigma\} = [SS]\{\varepsilon\} = \frac{E}{1-V^2} \begin{bmatrix} 1 & V & 0 \\ V & 1 & 0 \\ 0 & 0 & \frac{1-V}{2} \end{bmatrix} \{\varepsilon\} \quad ,$$

and therefore

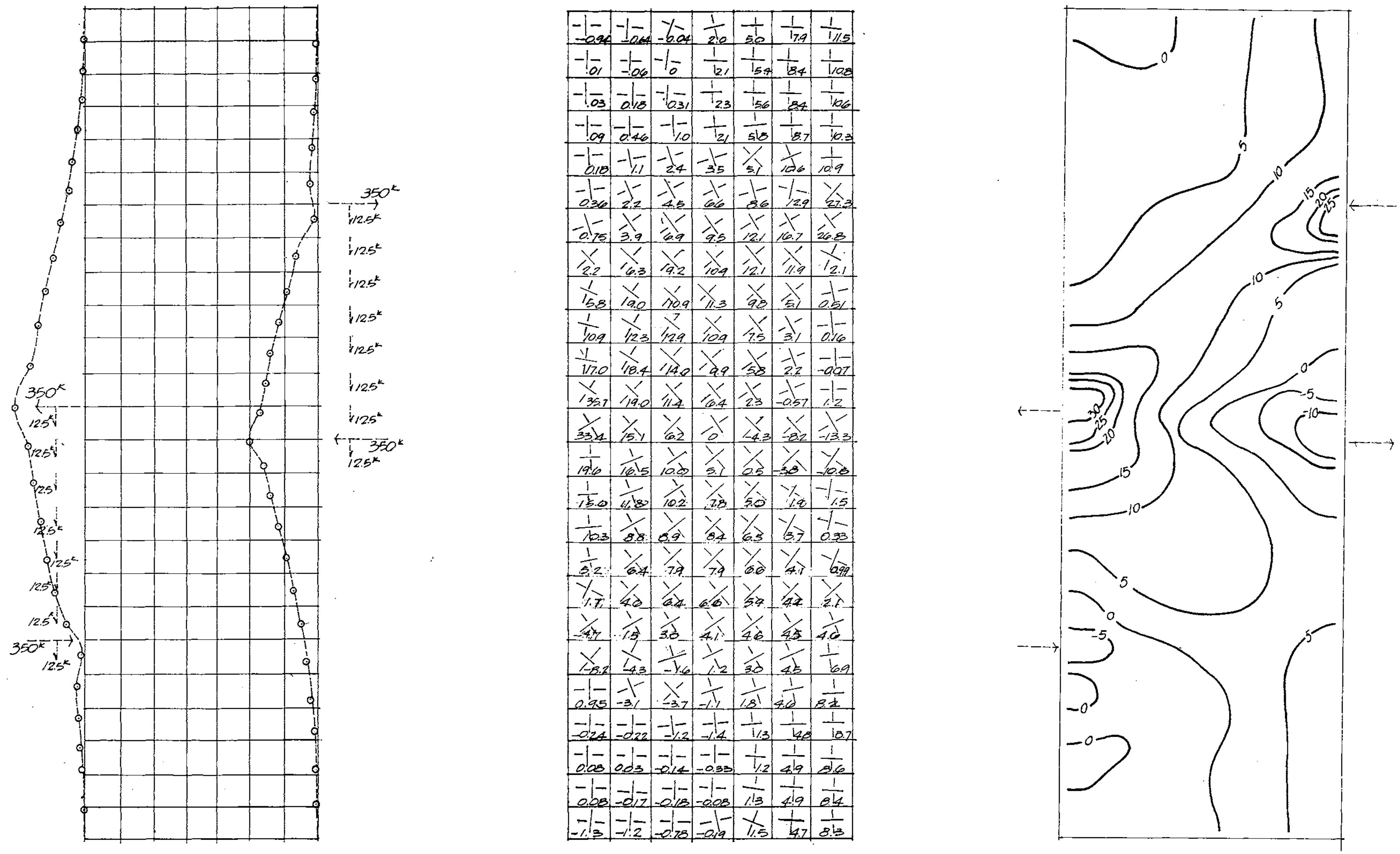
$$\{\sigma\} = [SS] \quad [B] \quad [C]^{-1} \quad \{\varepsilon\} \ .$$

Thus the desired stresses can be determined by a series of matrix operations once the element displacements have been found.

Analytical Idealization of the Joint

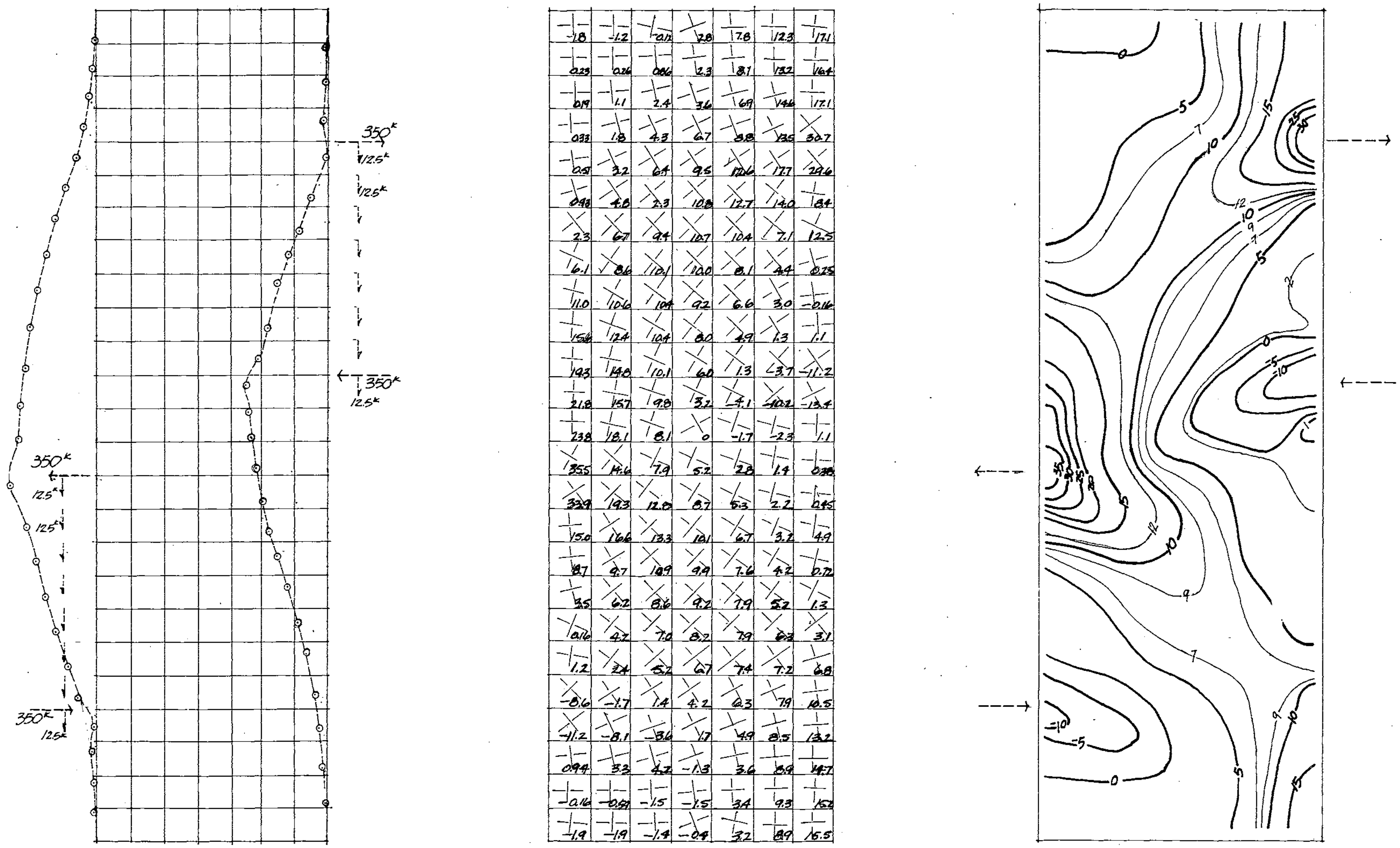
The complex nature of the reinforced concrete material and the configuration of the joint require certain assumptions in the analytical representation of the structure. As discussed in Chapter III, the structure is assumed to be a fixed-end column, the beams having been replaced with boundary forces of shear and internal force-couples; further, the loading is assumed to be in-plane so that the analysis is two-dimensional. The numerous variables which affect the actual non-linear stress-strain relationship of concrete have been ignored and the material is assumed to be elastic. Because of the extreme complications surrounding the composite action of the steel and concrete, the analytical material is assumed to be homogeneous concrete with a Modulus of Elasticity of 3,000,000 pounds per square inch. It is thought that the resulting displacements and stresses based on this assumption will give a general picture of the action of the joint though the specific values may be somewhat in error.

The idealization for each joint condition analyzed may be seen in part A of Figures 17 through 21. In each case, the column has a cross section 24 inches by 24 inches and is 100 inches long. The location of the boundary forces is the only variable in the various joint configurations. Each beam of the joint is assumed to be replaced by shear forces based on a linear shear stress distribution and moments created by simple tension-compression force-couples. The values of these forces were approximated from the loading conditions which might exist on a



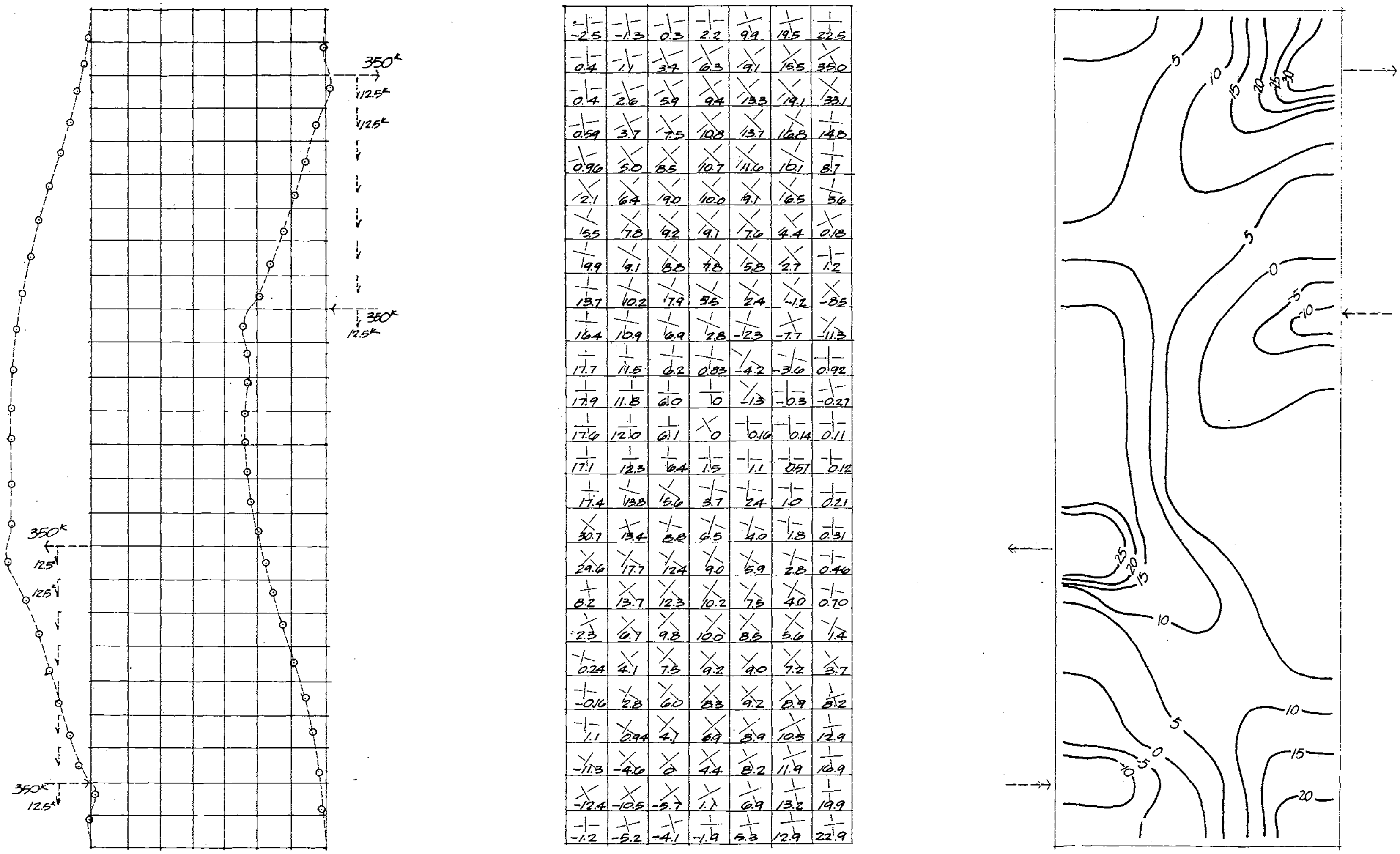
A. Joint Displacement B. Principal Tensile Stresses C. Tensile Stress Contours

Figure 17. Results of Finite Element Analysis for Joint Condition I



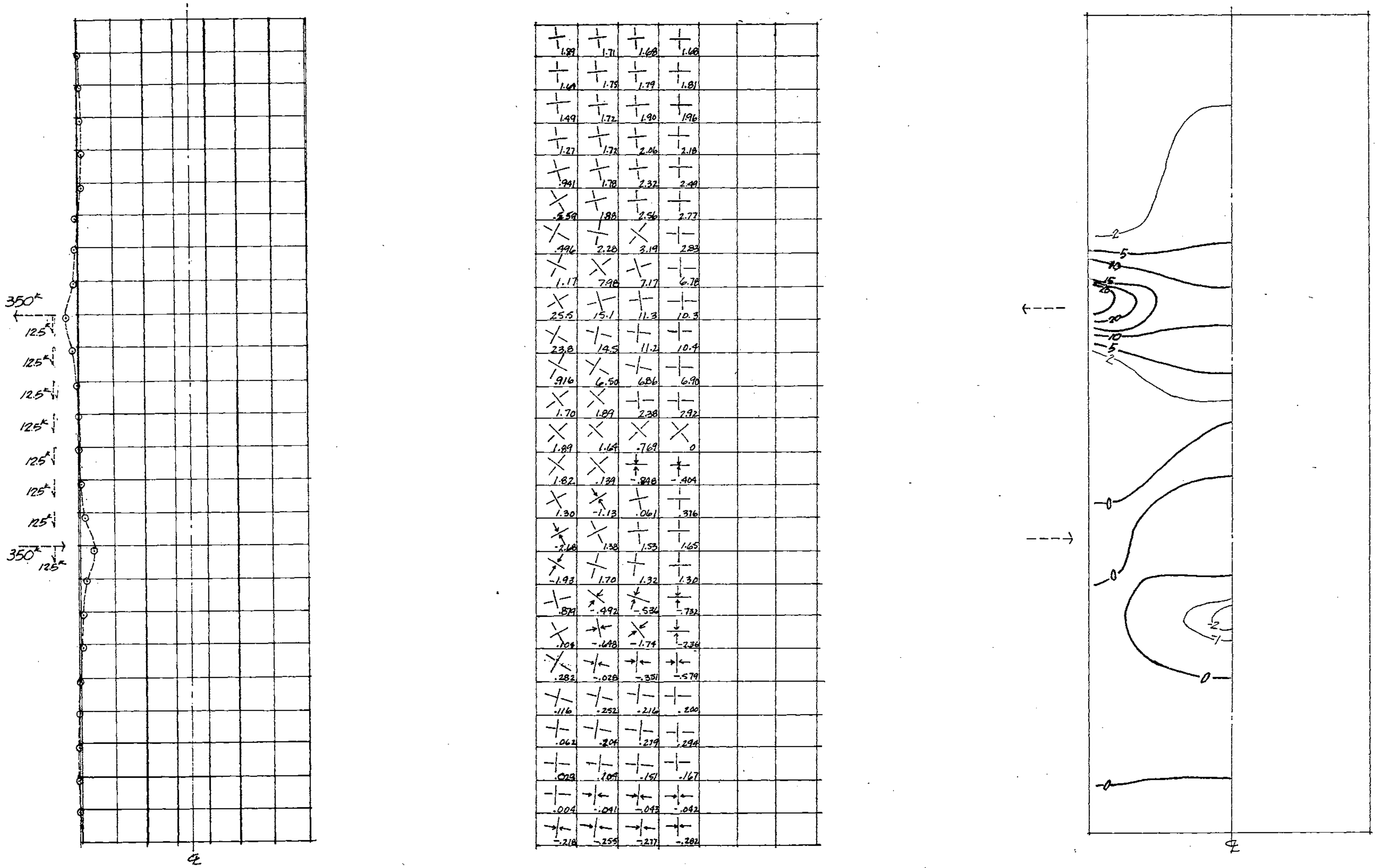
A. Joint Displacement B. Principal Tensile Stresses C. Tensile Stress Contours

Figure 18. Results of Finite Element Analysis for Joint Condition II



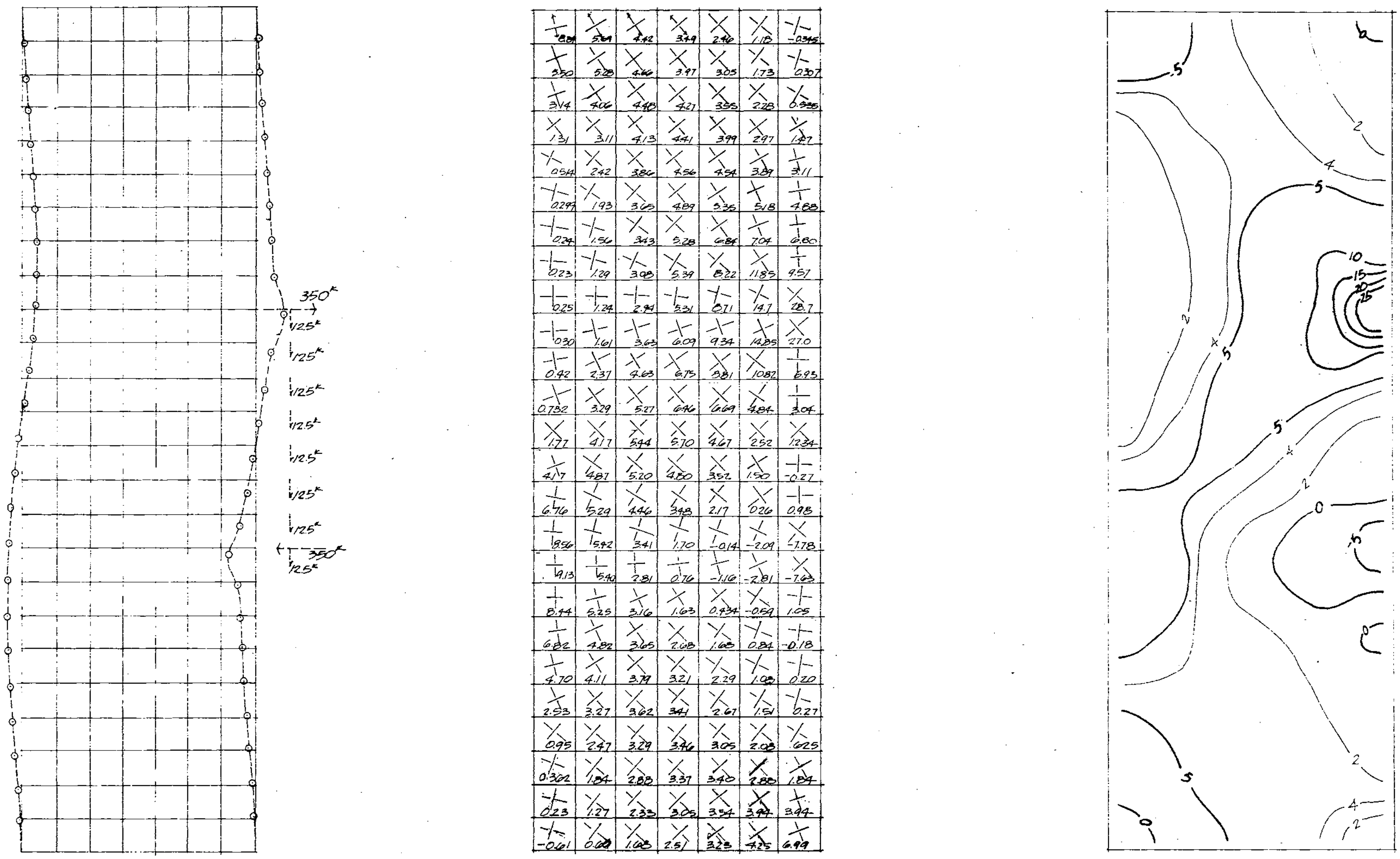
A. Joint Displacement   B. Principal Tensile Stresses   C. Tensile Stress Contours

Figure 19. Results of Finite Element Analysis for Joint Condition III



A. Joint Displacement B. Principal Tensile Stresses C. Tensile Stress Contours

Figure 20. Results of Finite Element Analysis for Joint Condition IV



A. Joint Displacement B. Principal Tensile Stresses C. Tensile Stress Contours

Figure 21. Results of Finite Element Analysis for Joint Condition V

parking garage beam with a span of 50 feet.

A 25 by 7 element mesh was used creating a total of 416 system coordinates. Subtracting from this total the 16 coordinates at each end of the column which do not enter into the system equations due to the assumption of fixed column ends gives 384 system equations which must be solved.

CHAPTER IV

COMPUTER PROGRAMS

Due to the large number of elements required for acceptable accuracy and the subsequently large number of simultaneous equations which must be solved when the finite element method is employed, the operations described in Chapter III have been programmed for electronic computation. The computer used, a Burroughs B-5500 located on the campus of the Georgia Institute of Technology, employs an ALGOL language as does the program of Table II.

The program is limited to the use of a single element size and a homogeneous material in the column. The maximum mesh size which can be employed is a 40 by 15, although in this thesis a more coarse 25 by 7 mesh proved adequate.

Program Sequence for the Finite Element Analysis

The general program sequence is as follows: The required input of joint geometry, material properties, and nodal forces is read into storage. The system nodal points are numbered and the system coordinates labeled. The element stiffness matrix [KE] is then computed and stored. Next, the system coordinates are assigned for each element in the same sequence as the rectangular element coordinates. This step permits the

TABLE II

COMPUTER PROGRAM FOR FINITE ELEMENT ANALYSIS

```
        BEGIN

COMMENT         DETERMINATION OF STRESSES AND DISPLACEMENTS IN RECT- -
                ANGULAR ELASTIC CONTINUUM BY THE FINITE ELEMENT METHOD;
FILE IN         CDJK (2,10);
FILE OUT        LINE  6(2,15);
FILE OUT        PUNCH 0(2,10) ; %
INTEGER         I,J,K,H,NVERTDIV,NHORZDIV,SUM,NUMBELEMENTS,Z;
INTEGER         HZ ,II   ; %
INTEGER      BW,KK,N,M,R,S,NFORCEVECT,NCARDS ; %
REAL            SUMSTRESS,DIFSTRESS,SN1,SN2,SN,STRESSMAX,ANGMAX,STRESSMIN
                ,ANGMIN,PHI2  ;
REAL            JTLGTH,JTWIDTH,L,D;
REAL            G,T,V,E,EPRIME;
REAL ARRAY      STFBD[0:800,0:25] ; %
REAL ARRAY      FORCE[0:800] ; %
REAL ARRAY      KELEMENT,KD,KS[0:8,0:8] ; %
REAL ARRAY                         B,BB,PR[0:3,0:8],C[0:8,0:8],SS[0:3,0:3],
                UELEMENT[0:8,0:1],STR[0:3,0:1]  ; %
INTEGER ARRAY   NODE[0:40,0:15],COORD[0:40,0:15,0:8],SYSTCOORDX[0:40,0:15
                ],SYSTCOORDY[0:40,0:15];              %
LABEL           POT,SEED        ; %
LABEL           EQUSTRESS               ;
LABEL           L9 ;
LIST            LIST1(NVERTDIV,NHORZDIV,JTLGTH,JTWIDTH,NCARDS,NFORCEVECT,
                T,E,G,V) ;
LIST           LIST2(NVERTDIV,NHORZDIV,JTLGTH,JTWIDTH,NFORCEVECT,T,E,G,V);
LIST          LIST3(FOR R ← 1 STEP 1 UNTIL 8 DO FOR S ← 1 STEP 1 UNTIL 8
                DO KELEMENT[R,S]) ; %
LIST            L7(I,J,FOR R ← 1 STEP 1 UNTIL 3 DO STR[R,1],
                STRESSMAX,ANGMAX,STRESSMIN,ANGMIN) ;             %
FORMAT OUT      FMT1(8E12.5/) ; %

FORMAT OUT      FOUT2("L= ",F5.3/,"D= ",F5.3//) ; %

FORMAT OUT      FOUT3(I3,X1,",",I3,3(X3,E12.5),X7,2(E18.11,X1,F7.3,X3)) ;

FORMAT OUT      FOUT4(I4,",",X2,E16.9) ; %


FORMAT OUT      FOUT5(X48,"ELEMENT STIFFNESS MATRIX"//) ; %


FORMAT OUT      FOUT6("NODAL FORCE MATRIX"//) ; %


FORMAT OUT      FOUT7("NODAL DISPLACEMENTS"//) ; %


FORMAT OUT      FOUT8("ELEMENT   STRESS X-DIR,  STRESS Y-DIR,  SHEARSTRES

S XY       MAXIMUM STRESS      ANGMAX    MINIMUM STRESS      ANGMIN"//) ;

FORMAT OUT      FOUT9(I4,",",X2,E18.11) ; %


FORMAT OUT      FOUT14(X54,"PROBLEM DATA"//) ; %
FORMAT OUT      FOUT15   ("NO.JOINT DIVISIONS VERTICALLY =",I3/,"NO. JOIN

T DIVISIONS HORIZONTALLY =",I3/,"JOINT LENGTH = ",F9.5/,"JOINT WIDTH = "
,F9.5/,"NO. FORCE SYSTEMS = ",I3/,"MATL THICKNESS = ",F5.3,"INCHES"/,"MO
DULUS OF ELASTICITY = ",F12.2,"PSI"/,"SHEAR MODULUS = ",F12.2,"PSI"/,"PO
ISSONS RATIO = ",F4.2) ;
```

TABLE II (Continued)

```
FORMAT OUT      FOUT19(//12E10.3//) ; %

PROCEDURE       BANDSOLTN(A,B,NN,MM) ; %
% THIS PROCEDURE SOLVES SYMMETRIC LINEAR EQUATIONS OF BANDED TYPE BY THE
% METHOD OF ELIMINATION.  THE EQUATIONS MUST BE IN THE REDUCED MATRIX
% FORM AX = B.  A IS THE COEFICIENT MATRIX (NN×NN), NN IS THE NUMBER OF
% EQUATIONS, MM IS THE BAND WIDTH.  THE UNKNOWNS ARE FOUND AND STORED
% BACK IN B.       %
VALUE           NN,MM ; %
INTEGER         NN,MM ; %
REAL ARRAY      A[0,0],B[0] ; %
BEGIN %
INTEGER         I,J,K,L,N ; %

REAL ARRAY      C[0:MM] ; %
LABEL           CONTINUE,REPEAT,AGAIN,EXIT,CARYON ; %
                N ← 0 ; %
REPEAT:         N ← N + 1 ; %
%   REDUCE N TH EQUATION
%         1. DIVIDE RIGHT SIDE BY DIAGONAL ELEMENT
                B[N] ← B[N] / A[N,1] ; %
%         2. CHECK FOR LAST EQUATION
                IF N - NN = 0 THEN GO TO AGAIN ; %
%         3. DIVIDE N TH EQUATION BY DIAGONAL ELEMENT
                FOR K ← 2 STEP 1 UNTIL MM DO %
          BEGIN %
                C[K] ← A[N,K] ; %
                A[N,K] ← A[N,K] / A[N,1] ; %
          END ; %
%         4. REDUCE REMAINING EQUATIONS
                FOR L ← 2 STEP 1 UNTIL MM DO %
          BEGIN %
                I ← N + L - 1 ; %
                IF NN - I < 0  THEN GO TO CONTINUE ; %
                J ← 0 ; %
                FOR K ← L STEP 1 UNTIL MM DO %
          BEGIN %
                J ← J + 1 ; %
                A[I,J] ← A[I,J] - C[L] × A[N,K] ; %
          END ; %
                B[I] ← B[I] - C[L] × B[N] ; %
CONTINUE:END ; GO TO REPEAT ; %
%   BACK SUBSTITUTION ; %
AGAIN:          N ← N - 1 ; %
%               1. CHECK FOR FIRST EQUATION
                IF N = 0   THEN GO TO EXIT ; %
%         2. CALCULATE UNKNOWN B[N] ; %
                FOR K ← 2 STEP 1 UNTIL MM DO %
          BEGIN %
                L ← N + K - 1 ; %
                IF NN - L < 0 THEN GO TO CARYON ; %
                B[N] ← B[N] - A[N,K] × B[L] ; %
CARYON:   END;  GO TO AGAIN ; %
EXIT:  %
          END BANDSOLTN; %
                WRITE(LINE[NO]);
                READ (CDJK,/,LIST2);%
                HZ ← 2 × (NHORZDIV + 1) ; %
                Z ← HZ × (NVERTDIV + 1 ) ; %
                KK ← Z -2×HZ ; %
                BW  ← (NHORZDIV + 1) × 2 + 4 ; %
WHILE TRUE DO READ(CDJK,/,I,FORCE[I-HZ]) [L9] ; L9:
CLOSE(CDJK,RELEASE) ;
                WRITE(LINE,FOUT14) ; %
                WRITE(LINE,FOUT15,LIST2) ; %
                WRITE(LINE,FOUT6) ; %
```

TABLE II (Continued)

```
                FOR I ← 1 STEP 1 UNTIL Z DO
                IF FORCE[I] ≠ 0 THEN
                WRITE(LINE,FOUT4,I+HZ,FORCE[I] ) ; %
COMMENT         COMPUTE ELEMENT DIMENSIONS;
                L ← JTLGTH/NVERTDIV;
                D ← JTWIDTH/NHORZDIV;
                WRITE(LINE,FOUT2,L,D) ; %
COMMENT         LABEL SYSTEM NODAL POINTS;
                SUM ← 0;
                FOR I←1 STEP 1 UNTIL NVERTDIV + 1 DO
                FOR J←1 STEP 1 UNTIL NHORZDIV + 1 DO
          BEGIN
          BEGIN
                NODE[I,J] ← SUM + 1;
                SUM ← SUM + 1;
          END  ;
          END  ;
COMMENT         LABEL SYSTEM COORDINATES, TWO ARRAYS; ONE FOR X AND Y;
                FOR I ← 1 STEP 1 UNTIL NVERTDIV + 1 DO
          BEGIN
                FOR J ← 1 STEP 1 UNTIL NHORZDIV + 1 DO
         BEGIN
                SYSTCOORDX[I,J] ← 2 × NODE[I,J];
                SYSTCOORDY[I,J] ← SYSTCOORDX[I,J] - 1;
          END  ;
          END  ;
%   COMPUTE TYPICAL RECTANGULAR ELEMENT STIFFNESS, WHERE G= SHEAR MODUL-
%   US, T= ELEMENT THICKNESS, V = POISSONS RATIO, E = MODULUS OF ELAS-
%   TICITY, KS = STIFFNESS DUE TO SHEAR STRAIN, KD= STIFFNESS DUE TO
%   DIRECT STRAINS.
                FOR I←1 STEP 1 UNTIL 4 DO
                FOR J←1 STEP 1 UNTIL 4 DO
                KS[I,J] ← (G × T × L)/(3 × D);
                KS[2,1] ← KS[4,3] ← KS[1,1] ×(-1);
                KS[3,1] ← KS[4,2] ← KS[1,1] × .5;
                KS[3,2] ← KS[4,1] ← KS[1,1] × (-.5);
                FOR I←5 STEP 1 UNTIL 8 DO
                FOR J←1 STEP 1 UNTIL 4 DO
                KS[I,J] ← G × T × .25;
                KS[5,2] ← KS[5,4] ← KS[6,2] ← KS[6,4] ← KS[7,1] ← KS[7,3]
                ← KS[8,1] ← KS[8,3] ← KS[5,1] × (-1) ; %
                FOR I ← 5 STEP 1 UNTIL 8 DO
                FOR J ← 5 STEP 1 UNTIL 8 DO
                KS[I,J] ← (G × T × D)/(3 × L);
                KS[7,5] ← KS[8,6] ← KS[5,5] × (-1);
                KS[6,5] ← KS[8,7] ← KS[5,5] × .5;
                KS[7,6] ← KS[8,5] ← KS[5,5] × (-.5);
                FOR I ← 2 STEP 1 UNTIL 8 DO
          BEGIN
                H ← I - 1;
                FOR J ← 1 STEP 1 UNTIL H DO
                KS[J,I] ← KS[I,J];
          END  ;
                EPRIME ← E/(1-V*2);
                FOR I ← 1 STEP 1 UNTIL 4 DO
                FOR J ← 1 STEP 1 UNTIL 4 DO
                KD[I,J] ← (EPRIME × T × D)/(L × 3);
                KD[2,1] ← KD[4,3] ← KD[1,1] × .5;
                KD[3,1] ← KD[4,2] ← KD[1,1] × (-1);
                KD[4,1] ← KD[3,2] ← KD[1,1] × (-.5);
                FOR I ← 5 STEP 1 UNTIL 8 DO
                FOR J ← 1 STEP 1 UNTIL 4 DO
```

TABLE II (Continued)

```
            KD[I,J] ← EPRIME × T × V × .25;
            KD[5,3] ← KD[5,4] ← KD[6,1] ← KD[7,3] ← KD[7,4] ← KD[6,2]
            ← KD[8,1] ← KD[8,2] ← (-1) × KD[5,1];
            FOR I ← 5 STEP 1 UNTIL 8 DO
            FOR J ← 5 STEP 1 UNTIL 8 DO
            KD[I,J] ← (EPRIME × L × T)/(3 × D);
            KD[6,5] ← KD[8,7] ← KD[5,5] × (-1);
            KD[7,5] ← KD[8,6] ← KD[5,5] × .5;
            KD[7,6] ← KD[8,5] ← KD[5,5] × (-.5);
            FOR I ← 2 STEP 1 UNTIL 8 DO
       BEGIN
            H ← I-1;
            FOR J ← 1 STEP 1 UNTIL H DO
            KD[J,I] ← KD[I,J];
       END  ;
            FOR I ← 1 STEP 1 UNTIL 8 DO
            FOR J ← 1 STEP 1 UNTIL 8 DO
            KELEMENT[I,J] ← KS[I,J] + KD[I,J];
            WRITE(LINE,FOUT5) ; %
            WRITE (LINE,FMT1,LIST3) ; %
%   ASSIGN SYSTEM COORDINATES FOR EACH ELEMENT IN SAME SEQUENDE AS
%   TYPICAL ELEMENT COORDINATES.
            FOR I ← 1  STEP 1 UNTIL NVERTDIV DO
       BEGIN
            FOR J ← 1  STEP 1 UNTIL NHORZDIV DO
       BEGIN
            COORD[I,J,1] ← SYSTCOORDY[I,J];
            COORD[I,J,2] ← SYSTCOORDY[I,J+1];
            COORD[I,J,3] ← SYSTCOORDY[I+1,J];
            COORD[I,J,4] ← SYSTCOORDY[I+1,J+1];
            COORD[I,J,5] ← SYSTCOORDX[I,J];
            COORD[I,J,6] ← SYSTCOORDX[I,J+1];
            COORD[I,J,7] ← SYSTCOORDX[I+1,J];
            COORD[I,J,8] ← SYSTCOORDX[I+1,J+1];
       END  ;
       END  ;
%   DETERMINE THE SYSTEM STIFFNESS MATRIX BY ADDING THE CONTRIBUTION
%   FROM EACH RECTANGULAR ELEMENT.
            FOR I ← 1 STEP 1 UNTIL NVERTDIV DO %
            FOR J ← 1 STEP 1 UNTIL NHORZDIV DO %
   BEGIN %
            FOR K ← 1 STEP 1 UNTIL 8 DO %
     BEGIN %
            R ← COORD[I,J,K] ; %
            IF R≤ HZ OR R > Z-HZ THEN GO TO SEED ; %
            FOR N ← 1 STEP 1 UNTIL 8 DO %
       BEGIN %
            S ← COORD[I,J,N] ; %
            IF S < R THEN GO TO POT %
       ELSE IF S = R THEN
      BEGIN %
      H ← R-HZ ; %
            STFBD[H,1] ← STFBD[H,1] + KELEMENT[K,N] ; %
      END %
       ELSE IF S > R THEN
      BEGIN %
            H ← R-HZ ; %
            STFBD[H,S-R+1] ← STFBD[H,S-R+1] + KELEMENT[K,N] ; %
      END; %
    POT:
   END ;
    SEED:
```

TABLE II (Continued)

```
            END ;
          END ;
                 BANDSOLTN(STFBD,FORCE,KK,BW) ; %
                 WRITE(LINE,FOUT7) ; %
                 FOR I ← 1 STEP 1 UNTIL KK DO %
           BEGIN %
                 K  ← I + HZ ; %
                 WRITE(LINE,FOUT9,K,FORCE[I])  ; %
           END ; %
%   DETERMINE THE ELEMENT STRESS VECTORS FOR EACH ELEMENT BY MATRIX
%   MULTIPLICATION: ELSTRESS = PR × UELEMENT, WHERE PR IS THE MATRIX
%   PRODUCT SS × B × C(INVERTED), UELEMENT IS THE DISPLACEMENT VECTOR
%   FOR EACH ELEMENT, LBETA × UNODE, SS IS THE STRESS-STRAIN MATRIX FOR
%   THE MATERIAL, C IS THE MATRIX WHICH RELATES NODAL DISPLACEMENTS OF
%   THE ELEMENTS TO THE GENERALIZED COORDINATES, B IS THE MATRIX WHICH
%   RELATES THE ELEMENT STRAINS TO THE GENERALIZED COORDINATES.
                 FOR I ← 1 STEP 1 UNTIL 4 DO
                 C[I,1] ← C[I+4,5] ← 1 ;
                 C[2,2] ← C[4,2] ← C[6,6] ← C[8,6] ← D ;
                 C[3,3] ← C[4,3] ← C[7,7] ← C[8,7] ← L ;
                 C[4,4] ← C[8,8] ← D × L ;
                 INVERT(8,C,SN1,LINE) ;
                 B[1,6] ← B[2,3] ← B[3,2] ← B[3,7] ← 1 ; %
                 B[1,8] ← B[3,4] ← L/2 ; %
                 B[2,4] ← B[3,8] ← D/2 ; %
                 SS[1,1] ← SS[2,2] ← EPRIME ;
                 SS[2,1] ← SS[1,2] ← V × EPRIME ;
                 SS[3,3] ←((1-V)/2)× (EPRIME) ;
                 MATPROD(3,3,8,SS,B,BB) ;
                 MATPROD(3,8,8,BB,C,PR) ;
                 WRITE(LINE,FOUT8) ;
                 FOR I ← 1 STEP 1 UNTIL NVERTDIV DO
BEGIN
                 FOR J ← 1 STEP 1 UNTIL NHORZDIV DO
            BEGIN
                 FOR K ← 1 STEP 1 UNTIL 8 DO
            BEGIN
                 II ← COORD[I,J,K] ; %
                 IF (II≤HZ) OR (II>Z-HZ) THEN UELEMENT[K,1] ← 0 %
            ELSE %
UELEMENT[K,1] ← FORCE[II-HZ] ; %
            END  ;
                 MATPROD(3,8,1,PR,UELEMENT,STR) ; %
%   CALCULATE THE PRINCIPAL STRESSES AND THE DIRECTIONS OF THE PLANES ON
%   WHICH THEY ACT.
                 SUMSTRESS ←(STR[1,1] + STR[2,1])/2 ; %
                 DIFSTRESS ←(STR[1,1] - STR[2,1])/2 ; %
                 IF DIFSTRESS = 0 THEN GO TO EQUSTRESS  ;
                 SN1 ← SUMSTRESS + SQRT(DIFSTRESS*2 + STR[3,1]*2) ; %
                 SN2 ← SUMSTRESS - SQRT(DIFSTRESS*2 + STR[3,1]*2) ; %
                 STRESSMAX ← SN1; STRESSMIN ← SN2 ;
                 PHI2 ← ARCTAN(((-2) × STR[3,1])/(DIFSTRESS×2)) ; %
                 IF STR[1,1] < STR[2,1] AND STR[3,1] < 0  THEN
                 PHI2 ← PHI2 + 22/7  ; %
                 IF STR[1,1] < STR[2,1] AND STR[3,1] > 0 THEN
                      PHI2 ← PHI2 - 22/7  ; %
                 ANGMAX ←(PHI2/2)× 57.2950 ; ANGMIN ← ANGMAX + 90 ;
EQUSTRESS:       IF DIFSTRESS = 0 THEN BEGIN STRESSMAX ← STRESSMIN
                 ← STR[1,1]          ;  ANGMAX ← 0 ; ANGMIN ← 90 END ;
                 WRITE(LINE,FOUT3,L7) ;
      END;
END;
           END .
```

building of the system stiffness matrix by adding the contribution from each of the rectangular elements. Since the resulting matrix is symmetrical and of a band form, only the main diagonal and the non-zero elements to the right are retained as a condensed matrix [STFBD] . The substitution procedure is then used to solve the system equations

$$[\text{FORCE}] = [\text{STFBD}] \; [\text{NODE DISPLACEMENTS}] \quad ,$$

for the node displacements. Rather than take up more storage space with another variable, the values of the displacements are stored in the variable formerly occupied by the forces [FORCE] .

The stress strain matrix [SS] is next formed, after which [B] and [C] are constructed as discussed earlier in Chapter III. The product of these three matrices [PR] is found by a standard matrix multiplication subroutine and stored:

$$[\text{PR}] = [\text{SS}] \; [\text{B}] \; [\text{C}] \quad .$$

The displacements of the system nodal coordinates are transferred to the corresponding element coordinates resulting in an element displacement vector [UELEMENT] for each of the elements.

The stress vector [STR] for the element is then determined by the matrix multiplication

$$[\text{STR}] = [\text{PR}] \; [\text{UELEMENT}] \quad .$$

These stresses are then employed in Mohr's equations of plane stress to determine the principal stresses and their planes of action (12).

The output of the program includes a printout of the input geometry and loading, the element stiffness matrix, the nodal displacements of the structure, and the stress vector and principal stress values for each element.

Results of the Finite Element Analyses

A portion of the printed output for the analysis of joint condition I can be seen in Tables III, IV, and V. Table III shows the geometry, material properties and applied boundary forces. With the exception of the coordinates at the fixed ends of the column where the displacement is zero, the displacements for each coordinate in the system are listed by number as in Table IV. The stress values for the elements are listed horizontally one row per element as indicated by the column headings in Table V.

The results of the analyses can be seen more graphically in Figures 17 through 21 where for each joint loading condition, the following are presented: a plot of the boundary displacement of the structure to an exaggerated scale, the value of the principal tensile stress and its direction for each element and principal tensile stress contours in the joint.

The result most evident from these figures is that joint IV, in which the beams are at the same elevation, has the least critical deformation and stress condition. The higher stresses within this joint would

TABLE III

PROGRAM DATA PRINTOUT

```
        PROBLEM DATA
NO.JOINT DIVISIONS VERTICALLY = 25
NO. JOINT DIVISIONS HORIZONTALLY =  7
JOINT LENGTH = 100.00000
JOINT WIDTH =  24.00000
NO. FORCE SYSTEMS =   1
MATL THICKNESS = *****INCHES
MODULUS OF ELASTICITY =    3000000.00PSI
SHEAR MODULUS =    1200000.00PSI
POISSONS RATIO = 0.25
 NODAL FORCE MATRIX
 111,    1.250000000e+04
 112,    3.500000000e+05
 127,    1.250000000e+04
 143,    1.250000000e+04
 159,    1.250000000e+04
 175,    1.250000000e+04
 191,    1.250000000e+04
 193,    1.250000000e+04
 194,   -3.500000000e+05
 207,    1.250000000e+04
 209,    1.250000000e+04
 223,    1.250000000e+04
 224,   -3.500000000e+05
 225,    1.250000000e+04
 241,    1.250000000e+04
 257,    1.250000000e+04
 273,    1.250000000e+04
 289,    1.250000000e+04
 305,    1.250000000e+04
 306,    3.500000000e+05
L= 4.000
D= 3.429
```

TABLE IV

PORTION OF DISPLACEMENT OUTPUT

NODAL DISPLACEMENTS

17,	-1.31149731456@-03
18,	-4.71796762940@-04
19,	-8.24961125180@-04
20,	-2.17177558085@-04
21,	-3.80282813648@-04
22,	-1.02685725866@-04
23,	4.79452001069@-05
24,	-7.92045877420@-05
25,	4.51793414410@-04
26,	-1.24852254447@-04
27,	8.34034797250@-04
28,	-2.32744589902@-04
29,	1.21506719445@-03
30,	-4.15199516586@-04
31,	1.74311178588@-03
32,	-7.48185424960@-04
33,	-2.65113379035@-03
34,	-1.22025882913@-03
35,	-1.72431421674@-03
36,	-9.77194943600@-04
37,	-7.85606115120@-04
38,	-8.26988661510@-04
39,	1.11344961876@-04
40,	-7.81538841250@-04
41,	9.56239244020@-04
42,	-8.38951262140@-04
43,	1.74707586673@-03
44,	-9.94900776600@-04
45,	2.52094565824@-03
46,	-1.24888065587@-03
47,	3.33460838064@-03
48,	-1.55361806139@-03
49,	-4.06078518700@-03
50,	-2.56305514601@-03
51,	-2.61787620070@-03
52,	-2.30666427729@-03
53,	-1.19615643052@-03
54,	-2.13910033244@-03
55,	1.73882892344@-04
56,	-2.04434134347@-03
57,	1.48679486431@-03
58,	-2.04199849139@-03
59,	2.70264017598@-03
60,	-2.17517041707@-03
61,	3.80480721443@-03
62,	-2.40631590830@-03
63,	4.86934623876@-03
64,	-2.72240305287@-03
65,	-5.50035889270@-03
66,	-4.58211381556@-03
67,	-3.49927084077@-03
68,	-4.32102029523@-03
69,	-1.61249625063@-03
70,	-4.11333905572@-03
71,	1.84817823250@-04
72,	-3.91043066992@-03
73,	1.94484874964@-03
74,	-3.71667185832@-03
75,	3.68640319866@-03
76,	-3.62220483564@-03
77,	5.12717025245@-03
78,	-3.83815728051@-03
79,	6.28511204590@-03

TABLE V

PORTION OF ELEMENT STRESS OUTPUT

ELEMENT	STRESS X-DIR.	STRESS Y-DIR.	SHEARSTRESS XY	MAXIMUM STRESS	ANGMAX	MINIMUM STRESS
1 , 1	-9,48235@+01	-8,24878@+02	-1,82023@+01	-9,43699950140@+01	1,427	-8,25331355360@+02
1 , 2	-6,70949@+01	-4,68740@+02	2,98392@+01	-6,48901465120@+01	-4,226	-4,70944920774@+02
1 , 3	-2,22759@+01	-1,30196@+02	4,76564@+01	-4,24412282743@+00	-20,725	-1,48227353372@+02
1 , 4	2,86716@+01	1,94570@+02	4,00649@+01	2,03738913621@+02	-77,145	1,95025880597@+01
1 , 5	7,82331@+01	5,01744@+02	1,32527@+01	5,02158150956@+02	-88,244	7,78187591960@+01
1 , 6	1,19765@+02	7,98354@+02	-3,05109@+01	7,99723468150@+02	87,466	1,18395487129@+02
1 , 7	1,40424@+02	1,14442@+03	-8,20999@+01	1,15109248826@+03	85,391	1,33755222031@+02
2 , 1	8,35315@+00	-8,37533@+02	2,10653@+01	8,87742340010@+00	-1,426	-8,38057071040@+02
2 , 2	-6,94185@+00	-4,90989@+02	1,94446@+01	-6,16200459930@+00	-2,297	-4,91768958580@+02
2 , 3	-2,02457@+00	-1,28727@+02	1,79108@+01	4,58630383015@-01	-7,893	-1,31210674507@+02
2 , 4	8,68985@+00	2,15115@+02	6,06495@+00	2,15292599842@+02	-88,354	8,51181141010@+00
2 , 5	1,86225@+01	5,36213@+02	-1,61496@+01	5,36716614164@+02	88,250	1,81190896026@+01
2 , 6	1,82224@+01	8,36650@+02	-3,72677@+01	8,38343927580@+02	87,433	1,65288689258@+01
2 , 7	-7,86671@+00	1,08455@+03	-1,10683@+01	1,08466110182@+03	89,455	-7,97883896530@+00
3 , 1	2,75755@+00	-8,63016@+02	1,38627@+01	2,97945737466@+00	-0,917	-8,63237541040@+02
3 , 2	1,78815@+01	-4,84572@+02	1,68377@+01	1,84451599847@+01	-1,917	-4,85135343952@+02
3 , 3	3,06295@+01	-1,22847@+02	1,04862@+01	3,13426872370@+01	-3,891	-1,23560407649@+02
3 , 4	3,36102@+01	2,30813@+02	7,73863@+00	2,31115851257@+02	-87,791	3,33070089770@+01
3 , 5	1,36887@+01	5,60717@+02	-6,32819@+00	5,60790333790@+02	89,372	1,36154575655@+01
3 , 6	-2,44925@+00	8,39172@+02	-2,23493@+01	8,39765454420@+02	88,515	-3,04232114181@+00
3 , 7	-7,85818@+00	1,05501@+03	-2,02477@+01	1,05539581596@+03	88,944	-8,24376302210@+00
4 , 1	9,39588@+00	-8,68014@+02	-2,31272@+00	9,40197624920@+00	0,151	-8,68020255540@+02
4 , 2	4,53410@+01	-4,75315@+02	-1,93027@+01	4,60556166712@+01	2,120	-4,76029822548@+02
4 , 3	9,83710@+01	-1,27434@+02	-2,17624@+01	1,00449211248@+02	5,455	-1,29512353896@+02
4 , 4	1,38413@+02	2,10474@+02	6,65060@+00	2,11082705324@+02	-84,806	1,37804339938@+02
4 , 5	1,26119@+02	5,72211@+02	4,92888@+01	5,77592212570@+02	-83,805	1,20738381334@+02
4 , 6	2,19669@+01	8,70289@+02	1,31821@+01	8,70493785890@+02	-89,145	2,17621150659@+01
4 , 7	2,50746@+01	1,03307@+03	-2,57437@+01	1,03372402472@+03	88,573	2,44175529154@+01
5 , 1	1,71737@+01	-8,20024@+02	-3,77204@+01	1,88697919044@+01	2,575	-8,21720005210@+02
5 , 2	8,64262@+01	-4,41072@+02	-1,03580@+02	1,06036131819@+02	10,720	-4,60682322423@+02
5 , 3	1,99310@+02	-1,43487@+02	-1,24226@+02	2,39594391937@+02	17,967	-1,83771588947@+02
5 , 4	3,30620@+02	1,30117@+02	-7,25496@+01	3,54117343313@+02	17,946	1,06619255425@+02
5 , 5	4,38478@+02	4,44954@+02	7,00032@+01	5,11793838389@+02	-46,360	3,71637683989@+02
5 , 6	4,07632@+02	9,58127@+02	2,58655@+02	1,06058809091@+03	-68,425	3,05171256239@+02
5 , 7	-9,46693@+01	1,08666@+03	9,41772@+00	1,08673907089@+03	-89,578	-9,47443786710@+01
6 , 1	2,33290@+01	-6,76174@+02	-9,41021@+01	3,57671283706@+01	7,529	-6,88612105090@+02
6 , 2	1,17577@+02	-3,48114@+02	-2,40479@+02	2,19466425512@+02	22,962	-4,50002619367@+02
6 , 3	2,80897@+02	-1,31454@+02	-3,17321@+02	4,53140598052@+02	28,493	-3,03698216953@+02
6 , 4	5,06598@+02	3,30182@+01	-3,06631@+02	6,57224798420@+02	26,161	-1,17608933656@+02
6 , 5	8,11774@+02	2,05644@+02	-1,77360@+02	8,59857103920@+02	15,168	1,57561117817@+02

be carried by the major beam or column reinforcement. The other joints show high tensile stresses too far away from the normal locations of major reinforcement to be carried by it. These stresses are of sufficient magnitude to create tensile cracks in concrete and are in directions consistent with the formation of the cracking patterns of the model tests.

The greatest deformation and the highest general stress pattern occur in joint III, followed in order by joints II, I, V and IV. In each joint the regions of greatest tensile stress occur near the location of the tensile boundary forces. The exaggerated displacements of the nodes at which the large tension and compression forces are applied indicate that large and unrealistic stress concentrations have occurred in these regions.

CHAPTER V

SUMMARY AND CONCLUSIONS

An investigation of a reinforced concrete beam-column joint, precipitated by observance of severe cracks in an existing structure, has been made in two phases. The first phase involved experimental reinforced plaster model tests; the second, a theoretical analysis for displacements and stresses by the finite element method. In each phase the primary parameter was the location of the beams framing into the column.

The plaster model tests indicated, and the theoretical analysis confirmed, that in reinforced concrete beam-column joints in which the beams frame into the column at different elevations a much more critical condition exists than that in a joint where the beams are at the same elevation. These results indicate that special attention must be given by the designer in detailing the reinforcement for such joints if severe cracks are to be avoided.

Suggestions for Further Study

The plaster model technique was found to be a valuable form of preliminary study since it provided insight into the nature of the various joint conditions with a minimum cost in time, materials, and equipment. Various reinforcement details for these joint conditions could be investigated by this method.

Since the finite element method is such an extremely powerful tool for continuum stress analysis, it should be extended to more sophisticated analyses of these joint conditions when a computer system of sufficient speed and storage size is available. In particular, the effects of steel reinforcement on the stress distributions should be studied. A study of this type would require the development of a computer program capable of handling the composite nature of reinforced concrete.

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VITA

John Alan Kelly

Candidate for the Degree of

Master of Architectural Engineering

Thesis: A STUDY OF A REINFORCED CONCRETE BEAM-COLUMN JUNCTION HAVING BEAMS AT DIFFERENT ELEVATIONS

Major Field: Architectural Engineering

Biographical:

Personal Data: Born in Tulsa, Oklahoma, January 10, 1940, the son of Curtis Hodge and Anita Mae Kelly.

Education: Attended grade school in Tulsa, Oklahoma; graduated from Daniel Webster High School, Tulsa, Oklahoma, in 1958; attended Oklahoma Christian College, Oklahoma City; received the Bachelor of Architectural Engineering degree from the Oklahoma State University, with a major in Architectural Engineering, in May, 1963; completed requirements for the Master of Architectural Engineering degree in July, 1967.

Professional Experience: Student Assistant in the School of Architecture, Oklahoma State University, 1961-62, and 1962-63; Graduate Assistant, 1963-64; worked as structural engineer for Bill J. Blair and Associates, Architects, Oklahoma City, summers of 1963 and 1964; traveled and studied in Western Europe on the Lloyd Warren Fellowship in Architecture from August 1964 to August 1965; has been, since September 1965, Assistant Professor of Architecture, Georgia Institute of Technology, Atlanta, Georgia; had part-time experience as architectural designer with Stevens and Wilkinson, Architects, Atlanta, Georgia, Spring, 1966. Member of Sigma Delta Tau at Georgia Tech; American Institute of Architects, Association of Collegiate Schools of Architecture, and Chi Epsilon.