69

2198

 69-21,987

REESE, Charles D., 1937-
BUCKLING OF ORTHOTROPIC CONICAL
SANDWICH SHELLS UNDER VARIOUS
LOADINGS.

The University of Oklahoma, Ph.D., 1969
Engineering Mechanics

University Microfilms, Inc., Ann Arbor, Michigan

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

# BUCKLING OF ORTHOTROPIC CONICAL SANDWICH SHELLS UNDER VARIOUS LOADINGS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

CHARLES D. REESE

Norman, Oklahoma

1969

BUCKLING OF ORTHOTROPIC CONICAL SANDWICH

SHELLS UNDER VARIOUS LOADINGS

APPROVED BY

Charles W. Bert

L. A. Comp

William N. Huff

Franklin J. Appl

D. M. Egle

DISSERTATION COMMITTEE

TABLE OF CONTENTS

## ACKNOWLEDGEMENT

The author is grateful to have had the opportunity to work under the guidance of Dr. Charles W. Bert for both his Master's and Doctor's degrees. Dr. Bert's encouragement and enthusiasm has gone far beyond what could normally be expected of an advisor and teacher.

Drs. Davis M. Egle and Franklin J. Appl have both provided valuable suggestions concerning the analysis and computer program.

The staff of the Merrick Computer Center of the University of Oklahoma has provided considerable assistance in both the programming and operational stages of the computation. The author is particularly indebted to Mr. Ken Bray and Mr. Stan Tyler.

The financial support of the National Science Foundation, in the form of a Graduate Traineeship, is gratefully acknowledged.

Finally, this work is dedicated to my wife Kay and family. Their constant encouragement has been extraordinary.

## LIST OF TABLES

Page

## SYMBOLS

| | |
|---|---|
| $A_{ij}$, $A^k_{ij}$, $A'^k_{ij}$ | Arbitrary dimensionless coefficients of assumed modal functions, Eqs. (2-1) and (2-7) |
| A, B | $\Delta/L$, $\Delta'/L$ defined in Eq. (C-9a) |
| $B_i$ | Stiffness coefficients defined in Appendix B |
| $C_A$, $C_B$ | $\equiv \cos AR_o$, $\cos BR_o$ defined in Eq. (C-9a) |
| $D^k_{ij}$ | Arbitrary dimensionless coefficients for simple-support assumed modal functions |
| $D_x$, $D_y$, $D_{xy}$ | Flexural rigidities for an orthotropic cylindrical sandwich shell, defined in Eqs. (F-3) (in.lb.) |
| $D_{qx}$, $D_{qy}$ | Transverse shear rigidities of orthotropic core material, defined in Eqs. (F-4) (lb./inch) |
| $E_x$, $E_\theta$ | Facing elastic moduli in the x and $\theta$ directions, respectively (psi) |
| $\bar{E}_x$, $\bar{E}_\theta$ | $\equiv E_x/(1-\nu_{\theta x}\nu_{x\theta})$, $E_\theta/(1-\nu_{\theta x}\nu_{x\theta})$ (psi) |
| $e_{ij}$ | Ordinary engineering strains defined in Eq. (A-13) |
| $\tilde{e}_{ij}$ | Membrane facing strains |
| $G_{x\theta}$ | Facing shear modulus in the x-$\theta$ plane (psi) |
| $G_{zx}$, $G_{\theta z}$ | Core shear moduli in the z-x and $\theta$-z planes, respectively (psi) |
| $g_{ij}$, $g^{ij}$ | Components of Euclidean metric tensor and associated metric tensor, respectively |
| h | Core half-thickness (inches) |
| I___(i,j) | Integral forms defined in Appendix B |
| i2, j2 | Array positions associated with the $m_i + 2$ and $m_j + 2$ integers. |

| | |
|---|---|
| $K_x$, $K_\theta$ | Core transverse shear coefficients in the z-x and θ-z planes, respectively (dimensionless) |
| M, M' | Bending moment and bending load factor, respectively |
| $m_k$ | Number of longitudinal half-waves in the assumed modal term |
| $\tilde{m}$ | $\equiv m_i \pi/L$ |
| N | $\equiv - \tilde{N}_x$, Eq. (F-1) |
| $\tilde{N}_i$, $\tilde{N}_{ij}$ | Membrane stress resultants (lb./in.) |
| n, $n_\ell$ | Number of circumferential full-waves |
| $\tilde{n}$ | $\equiv n_i/R$ |
| K | Buckling parameter defined in Eq. (F-6) |
| L | Shell slant length (inches); load matrix |
| P, P' | Axial load and axial load factor respectively |
| R | $\equiv R_o + x \sin \alpha$ |
| $R_o$ | Radius of middle surface at small end of cone (inches) (see Fig.A.1) |
| R' | $\equiv R_o + x \sin \alpha + z \cos \alpha$ |
| S | Stiffness matrix |
| $S_A$, $S_B$ | $\equiv \sin AR_o$, $\sin BR_o$ defined in Eq. (C-7a) |
| $S_x$, $S_y$, $S_{xy}$ | Stretching rigidities for an orthotropic cylindrical sandwich shell, defined in Eqs. (F-3) (lb./inch) |
| T, T' | Torque and torsional load factor, respectively |
| t | Facing half-thickness (inches) |
| u | Middle surface displacement in the x-direction (inches); dummy variable used in Appendix C |
| $u_i$ | Covariant components of displacement vector |
| V | Strain energy (in.-lb) |

| | |
|---|---|
| $\bar{V}_i$ | Load coefficients defined in Appendix B |
| $v$ | Middle surface displacement in $\theta$-direction (inches) |
| $W_1$, $W_2$ | $\equiv$ Au, Bu dummy variables used in Appendix C |
| $w$ | Middle surface displacement in the z-direction (inches) |
| $x^i$ | General curvilinear coordinates |
| $y$ | Circumferential coordinate on the cylinder middle surface, Appendix F |
| $y_i$ | Generalized coordinate, Eq. (2-1) |
| $\alpha$ | Cone semi-vertex angle (see Fig. A.1) |
| $\beta_x$, $\beta_\theta$ | Angle of rotation of the normal to the middle surface for the facings in the meridional and circumferential direction, respectively (radians) |
| $\Gamma^k_{ij}$ | Euclidean Christoffel symbols |
| $\Delta$, $\Delta'$ | $\equiv (m_j - m_k)\pi/L$, $(m_j + m_k)\pi/L$ defined in Eq. (C-6) |
| $\varepsilon_{ij}$ | Tensorial strain components |
| $\varepsilon'_{ij}$ | Physical components of strain defined in Eq. (A-10) |
| $\zeta$ | $\equiv R^{-1}$ |
| $\lambda$ | Eigenvalues |
| $\nu_{x\theta}$, $\nu_{\theta x}$ | Poisson's ratios defined in Eqs. (A-27) |
| $\sigma$ | Stress, psi |
| $\phi$ | Factor defined in Eq. (F-13) |
| $\phi_j$ | Assumed modal functions |
| $\Psi_x$, $\Psi_\theta$ | Angle of rotation of the normal to the shell middle surface for the core in the meridional and circumferential directions, respectively (radians) |
| $\xi_i$ | Physical components of displacement defined in Eq. (A-10) |

**Subscripts**

| | |
|---|---|
| b | Bending |
| c | Compression |
| cr | Critical |
| e | Elastic |
| rc | Rigid core |
| s | Secant |
| T | Total |
| t | Tangent |
| $x, \theta, z$ | Coordinates |

**Superscripts**

| | |
|---|---|
| c | Refers to core |
| f | Refers to facings |
| i | Refers to inner facing |
| o | Refers to outer facing |
| $u, v, w, x, \theta$ | Refer to displacements |

## BUCKLING OF ORTHOTROPIC CONICAL SANDWICH SHELLS UNDER VARIOUS LOADINGS

# CHAPTER I

# INTRODUCTION

### 1.1 Survey of Sandwich Shell Buckling

Although shell structures have been used extensively in aircraft and space vehicles for several years, it has been only in the past few years that sandwich-type construction has come into prominent usage. Due to its high strength-to-weight and stiffness-to-weight capabilities, sandwich-type construction is now being considered for many primary structural applications. Thus, it becomes of increasing importance to be able to predict the performance of various types of sandwich structures without costly experimental studies.

The theoretical and experimental buckling studies for thin-walled cylindrical shell structures are far too numerous to be listed individually; however, a few of the reports with extensive bibliographies of their own will be given. For the case of cylindrical shells in axial compression, Hoff [1] provided an extensive review of the subject. For torsional loading, one of the earliest analyses is that by Donnell [2]. More recently, Chehill and Cheng [3] presented a torsional analysis for composite cylindrical shells.

Batdorf, et al [4-7] extended the work of Donnell to other types of loading and boundary conditions. For pure bending loading, Seide and Weingarten [8] applied the Galerkin method to Batdorf's modified Donnell equation. The pure bending of laminated, long, composite cylindrical shells was treated by Ugural and Cheng [9].

For thin-walled conical shells, numerous analyses have been conducted also. To name but a few, in axial compression, there are the isotropic analyses by Kobayashi [10] and Baruch, et al [11]; in torsion, there is the analysis by Seide [12]; and in bending, there is the analysis by Seide [13]. For orthotropic conical shells, there is the analysis by Leyko [14] for axial compression. Singer [15] presented a set of Donnell-type equations for bending and buckling of orthotropic conical shells.

For sandwich-type construction, there are not many analyses available even for the buckling of isotropic shells. Plantema [16] has provided the most complete survey for this type of construction.

The most complete analysis for buckling of isotropic sandwich shells under axial compression is that of Bartelds and Mayers [17,18]. This is a unified theory in that it applies to the face-wrinkling (short wavelength) type of buckling in addition to general instability. Sylvester [19] has provided a nonlinear analysis for an axially compressed sandwich cylinder. Wang, et al [20], using a Donnell-type analysis and a Galerkin solution, treated the combined loading case for a long sandwich cylinder.

For an axially compressed sandwich cylinder with orthotropic facings and core, three basic analyses exist. The analysis by March and Kuenzi [21] is a large-deflection analysis that has recently been shown to be physically invalid (see Ref. [22] for a discussion). Design curves based on this large-deflection theory were given by Norris and Zahn [23]. More recently, Bert et al [22] presented a general instability analysis based on the small-deflection, Donnell-type, theory of Stein and Mayers [24]. Reese and Bert [25] investigated this linear analysis [22] using a digital computer and then presented a set of approximate design equations.* Later Peterson [26] also presented an orthotropic analysis based on the sandwich shell theory of Stein and Mayers. With one exception, this analysis is identical to the Ref. [22] analysis. The Poisson's ratios for bending and extension are assumed equal in Ref. [22]. In practice the effect of this assumption is not restrictive.

The only existing analysis for torsional buckling of orthotropic sandwich cylinders is the small-deflection analysis by March and Kuenzi [27].

All other orthotropic linear analyses are for special cases: isotropic facings and unidirectional core [28,29], isotropic facings and orthotropic core [30,31], and orthotropic facings and rigid core [32].

Most of the previous cylindrical sandwich analyses consider what was termed by Hoff [1] as classical simple-support boundary conditions. The rest considered the cylinder to be long, thereby

---

*This work was undertaken as a preliminary investigation for the dissertation. The results of this preliminary work are abstracted in Appendix F.

making the boundary conditions unimportant.

The author knows of no buckling analysis for orthotropic sandwich conical shells.

## 1.2 Research Objectives

The purpose of this research is to develop a general instability analysis for conical sandwich shells with orthotropic facings and core under various loadings based on Love's first-approximation shell theory. The loadings considered are axial compression, pure torsion, and pure bending. Cylindrical sandwich shells are considered as a special case (zero cone angle).

The particular cases treated are as follows:

1. Classical Simple-Support Boundary Condition with Axial Compression Loading
2. Clamped Boundary Condition
    a. Axial Compression Loading
    b. Pure Torsional Loading
    c. Combined Torsion and Axial Compression
    d. Pure Bending Loading
    e. Combined Bending and Axial Compression

# CHAPTER II

# FORMULATION OF THE THEORY

## 2.1 Method of Analysis

The potential energy for a truncated conical sandwich shell of symmetric construction is presented in Appendix A. The Rayleigh-Ritz approximate method of solution is applied to the total energy to obtain a standard eigenvalue problem in which the eigenvalue represents the critical load or combination of loadings. The final eigenvalue problem is solved with the aid of a high-speed digital computer.

## 2.2 Hypotheses

The analysis is based on the following assumptions:

1. The facings are capable of developing extensional in-surface strain energy and bending strain energy, but not transverse shear strain energy or strain energy from extension in the thickness direction.
2. The facings are linearly elastic and can be orthotropic. This requirement will still allow symmetric angle orientations providing that the facings are sufficiently thin, such that the bending-stretching coupling effect can be ignored.

3. The facings are of identical construction.
4. The core is capable of developing transverse shear strain energy only.
5. The core is linearly elastic and can be orthotropic with respect to its transverse shear properties.
6. The deflections are assumed to be sufficiently small to allow linearization of the strain-displacement relations (second-order extensional strain terms are included where needed to maintain a consistent order within the energy function of Appendix A).
7. All thermal and dynamic effects are neglected, including damping.
8. The shell thickness is small compared to the minimum radius of curvature of the shell.
9. For the core, lines which are straight and normal to the middle surface before deformation remain straight during deformation but not necessarily normal to the middle surface.
10. For the facings, lines which are straight and normal to the facing-core interface before deformation remain so after deformation.
11. The Kirchchoff-Love approximations, with the exception of transverse shear deformation in the core, are assumed applicable. (Reissner's version of Love's first-approximation as given by Kraus [33] is used in the formulation of Appendix A.)

## 2.3 Rayleigh-Ritz Method

The Rayleigh-Ritz approximate assumed-mode method of solution is based on a minimum principle and thus represents an upper-bound solution. Although the Galerkin (assumed-mode) approximate method appears to hold an equal position of prominence for buckling analyses, the Rayleigh-Ritz method has certain advantages which appear to make it the most practical for problems which involve extensive computer programs. The Rayleigh-Ritz method always gives rise to a symmetric stiffness matrix while the Galerkin method may not. This symmetry can be an aid in debugging a complicated computer program. The other principal advantage of the Rayleigh-Ritz method results from the boundary conditions. For the Galerkin method, the assumed modal functions must satisfy all of the boundary conditions exactly. For the Rayleigh-Ritz method it is necessary to satisfy exactly the geometric boundary conditions only. However, since it is impossible to utilize an infinite number of terms in the assumed modal functions, the resulting convergence will be dependent on how nearly all of the boundary conditions are satisfied, particularly the geometric boundary conditions.

The Rayleigh-Ritz method consists of assuming a series of deflection modal functions of the form

$$y_i = \sum_{j=1}^{\infty} A_{ij}\phi_j \tag{2-1}$$

where $y_i$ is the generalized coordinate, $A_{ij}$ are the undetermined modal coefficients, and $\phi_j$ are the assumed modal functions. If Eqs. (2-1) are truncated to a given number of terms and then substituted into the total energy expression, Eq. (A-43), the resulting equation is second order with respect to the $A_{ij}$,

$$S(A_{ij}A_{kl}) = \lambda L(A_{ij}A_{kl}) \tag{2-2}$$

Then Eq. (2-2) can be minimized with respect to each of the $A_{ij}$, thus resulting in i + j equations which are linear in the modal coefficient, $A_{ij}$. These i + j equations can be written in matrix form as

$$([S] - \lambda[L])\{A_{ij}\} = 0 \tag{2-3}$$

Eq. (2-3) is in standard eigenvalue form. The eigenvalue represents the critical load, torsion, moment or a particular combined state of loading.

## 2.4 Simple-Support Boundary Conditions

The conditions generally assumed to represent simple supports for a beam or shell are $w = w,_{xx} = 0$. As pointed out by Hoff [1], for a thin-walled cylindrical shell with displacements u,v, and w, there are four possible simple-support cases:

$$\begin{array}{llll} w = 0 & w,_{xx} = 0 & \sigma_x = 0 & \sigma_{x\theta} = 0 \\ w = 0 & w,_{xx} = 0 & u = 0 & \sigma_{x\theta} = 0 \\ w = 0 & w,_{xx} = 0 & \sigma_x = 0 & v = 0 \\ w = 0 & w,_{xx} = 0 & u = 0 & v = 0 \end{array} \qquad \text{(2-4a through d)}$$

Eq. (2-4c) represents what is generally termed as classical simple-support boundary conditions. For a sandwich shell with displacements $w,u,v,\Psi_x$, and $\Psi_\theta$, the possible combinations are further expanded.

In this analysis, displacement functions which satisfy the classical condition on u,v, and w are utilized. The same functional forms are used for the core rotations, $\Psi_x$ and $\Psi_\theta$, as are used for the displacements, u and v respectively. The functions used are as follows:

$$
\begin{aligned}
w(x,\theta) &= \sum_{\ell,k} D^{w}_{\ell k} \cos n_\ell\theta \sin(m_k \pi x/L) \\
u(x,\theta) &= \sum_{\ell,k} D^{u}_{\ell k} \cos n_\ell\theta \cos(m_k \pi x/L) \\
v(x,\theta) &= \sum_{\ell,k} D^{v}_{\ell k} \sin n_\ell\theta \sin(m_k \pi x/L) \qquad (2\text{-}5) \\
\Psi_x(x,\theta) &= \sum_{\ell,k} D^{x}_{\ell k} \cos n_\ell\theta \cos(m_k \pi x/L) \\
\Psi_\theta(x,\theta) &= \sum_{\ell,k} D^{\theta}_{\ell k} \sin n_\ell\theta \sin(m_k \pi x/L)
\end{aligned}
$$

where the D's are the undetermined coefficients. This functional form is satisfactory for solving the axial compression loading case; however, the other load terms vanish for these functions. These are the same functions used in the analysis of Ref. [22]. Since the simple-support case was undertaken primarily as a check case for the energy expression, Eq. (A-43), it was desirable to use the same functions.

Substituting Eqs. (2-5) into the total energy expression, Eq. (A-43), it is seen that all of the θ-terms are uncoupled as a result of the θ integration. Thus, it is necessary to include the nth θ-term only. Since the final order of the stiffness matrix is 5(No. of n terms)(No. of m terms), this reduces the final order of the stiffness matrix. It is still necessary to use the particular value of n which results in a minimum value of critical stress. Since it is generally not known in advance which n is critical, it becomes necessary to perform the computation for several values of n in order to establish the minimum.

As a result of the $\theta$ integration, a factor of $\pi$ appears in each term and can thus be divided out of the equation. It is advantageous to minimize the energy expression with respect to the undetermined coefficients. All of these resulting equations are linear in the undetermined coefficients and can be arranged in the form of Eq. (2-3). The x-integrals identified in Appendix B can be substituted into these equations. Figures 2.1 and 2.2 show the form of the resulting stiffness and load matrices respectively. The cross-hatched partitions are populated whereas the blank partitions are completely filled with zeros. The generating terms for each of the populated partitions are catalogued in Appendix D. A flow chart for the algorithm used to solve the resulting eigenvalue problem is given in Appendix G.

## 2.5 Clamped Boundary Conditions

The conditions generally associated with clamped edges of a beam or shell are $w = w,_x = 0$. Then for a thin-walled shell with displacements u, v, and w, the possible clamped conditions are

$$
\begin{array}{llll}
w = 0 & w,_x = 0 & \sigma_x = 0 & \sigma_{x\theta} = 0 \\
w = 0 & w,_x = 0 & u = 0 & \sigma_{x\theta} = 0 \\
w = 0 & w,_x = 0 & \sigma_x = 0 & v = 0 \\
w = 0 & w,_x = 0 & u = 0 & v = 0
\end{array}
\qquad \text{(2-6 a through d)}
$$

As in the simple-support case, Eq. (2-6c) would apply for axial com-

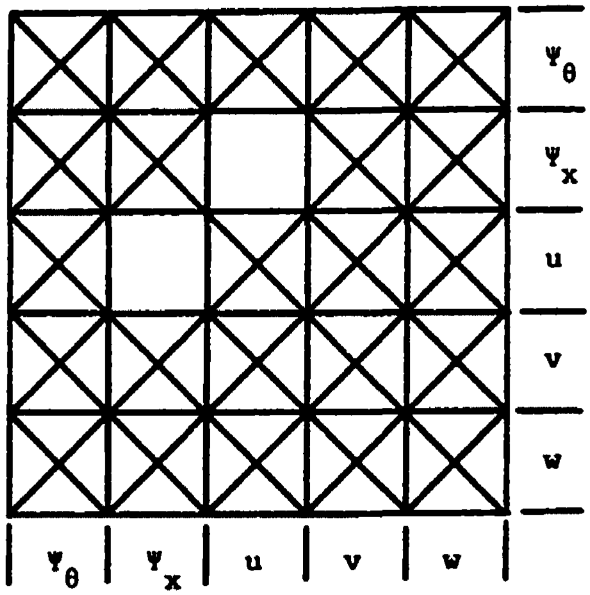


**Figure 2.1 - Simple-Support Stiffness Matrix**

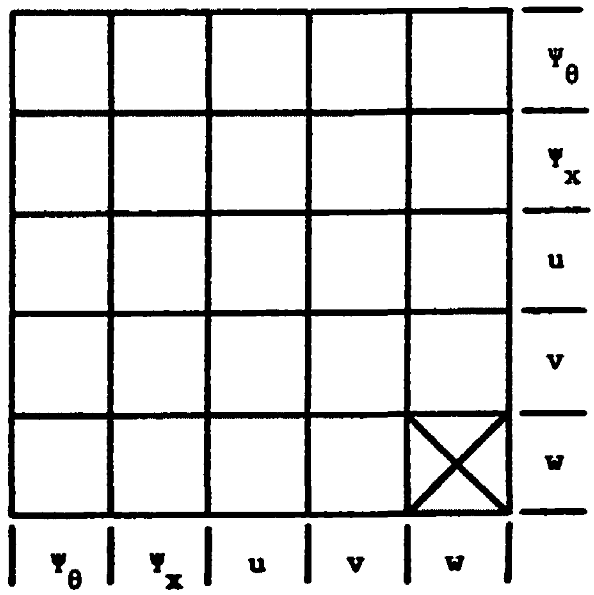


**Figure 2.2 - Simple-Support Load Matrix**

pression or pure bending. For pure torsion, Eq. (2-6b) is the most reasonable set of conditions. The first set of conditions would apply to the combined loading case, axial compression and torsion or bending and torsion.

To the author's knowledge, the only reported displacement function for the clamped case is the one given by Batdorf, et al [4-7] for w. A generalized form of this function will be used here. For u, v, $\Psi_x$ and $\Psi_\theta$, the same general functional forms used for the simple-support case can be utilized; however, it is necessary to fit these functions to the load terms which appear in Eq. (A-43). The general set of functions used are as follows:

$$\begin{aligned}
w = \sum_k [&A^w_{1k} \sin (n-1)\theta + A^w_{2k} \sin n\theta + A^w_{3k} \sin (n+1)\theta \\
&+ A^w_{4k} \cos (n-1)\theta + A^w_{5k} \cos n\theta + A^w_{6k} \cos (n+1)\theta]\cdot \\
&\{\cos (m_k \pi x/L) - \cos[(m_k + 2)\pi x/L]\}
\end{aligned} \tag{2-7}$$

$$\begin{aligned}
u = \sum_k \{ [&A^u_{1k} \sin (n-1)\theta + A^u_{2k} \sin n\theta + A^u_{3k} \sin (n+1)\theta \\
&+ A^u_{4k} \cos (n-1)\theta + A^u_{5k} \cos n\theta + A^u_{6k} \cos (n+1)\theta] \cos (m_k \pi x/L) \\
&+ [A'^u_{1k} \sin (n-1)\theta + A'^u_{2k} \sin n\theta + A'^u_{3k} \sin (n+1)\theta + \\
&A'^u_{4k} \cos (n-1)\theta + A'^u_{5k} \cos n\theta + A'^u_{6k} \cos (n+1)\theta] \sin (m_k \pi x/L)\}
\end{aligned}$$

$$\begin{aligned}
v = \sum_k \{ [&A^v_{1k} \sin (n-1)\theta + A^v_{2k} \sin n\theta + A^v_{3k} \sin (n+1)\theta + \\
&A^v_{4k} \cos (n-1)\theta + A^v_{5k} \cos n\theta + A^v_{6k} \cos (n+1)\theta] \sin (m_k \pi x/L) \\
&+ [A'^v_{1k} \sin (n-1)\theta + A'^v_{2k} \sin n\theta + A'^v_{3k} \sin (n+1)\theta + \\
&A'^v_{4k} \cos (n-1)\theta + A'^v_{5k} \cos n\theta + A'^v_{6k} \cos (n+1)\theta] \cos (m_k \pi x/L)\}
\end{aligned}$$

$$\begin{aligned}
\Psi_x = \sum_k \{ [&A^x_{1k} \sin (n-1)\theta + A^x_{2k} \sin n\theta + A^x_{3k} \sin (n+1)\theta + \\
&A^x_{4k} \cos (n-1)\theta + A^x_{5k} \cos n\theta + A^x_{6k} \cos (n+1)\theta] \cos (m_k \pi x/L) \\
&+ [A'^x_{1k} \sin (n-1)\theta + A'^x_{2k} \sin n\theta + A'^x_{3k} \sin (n+1)\theta +
\end{aligned}$$

$$A'^{x}_{4k} \cos (n-1)\theta + A'^{x}_{5k} \cos n\theta + A'^{x}_{6k} \cos (n+1)\theta] \sin (m_k\pi x/L)\}$$

$$\Psi_\theta = \sum_k \{[A^{\theta}_{1k} \sin (n-1)\theta + A^{\theta}_{2k} \sin n\theta + A^{\theta}_{3k} \sin (n+1)\theta +$$

$$A^{\theta}_{4k} \cos (n-1)\theta + A^{\theta}_{5k} \cos n\theta + A^{\theta}_{6k} \cos (n+1)\theta] \sin (m_k\pi x/L)$$

$$+ [A'^{\theta}_{1k} \sin (n-1)\theta + A'^{\theta}_{2k} \sin n\theta + A'^{\theta}_{3k} \sin (n+1)\theta +$$

$$A'^{\theta}_{4k} \cos (n-1)\theta + A'^{\theta}_{5k} \cos n\theta + A'^{\theta}_{6k} \cos (n+1)\theta] \cos (m_k\pi x/L)\}$$

The additional cos $\theta$ term which appears in the bending term of Eq. (A-43), $\bar{V}_2$ term, results in a $\theta$ coupling of the nth term with the n-1 and n+1 terms. It must be noted that additional coupling is also present. For example, the n+1 term is coupled with the n+2 term in addition to the nth term, and this secondary coupling has been neglected in the set of functions of Eq. (2-7). This simplification was necessitated due to the resultant size of the matrices.

In a similar manner, the $w,_x w,_\theta$ torsion term makes it necessary to retain both sine and cosine terms for $\theta$.

The unprimed set of functions was selected primarily for bending, Eq. (2-6c). The w functions plus the primed u, v, $\Psi_x$, and $\Psi_\theta$ set of functions were chosen primarily for the torsional case, Eq. (2-6b).

For the combined loading case, torsion, axial compression, and bending, it is necessary to retain all of these terms. The stiffness matrix dimension would be 54 times the number of x-terms used. Due to the overall size involved, it was not considered feasible to undertake this combined loading case until a greater understanding of the lesser combined cases is obtained. The subcases selected for this analysis are combined torsion and axial compression and combined bending and axial compression. The individual loading cases are included as special cases of the combined loading cases. Since the axisymmetric case is not realistic for bending or torsional loading, it was treated as a separate case. The series retained for each of these special cases are as follows: torsion and axial compression- $A^{w}_{2k}$, $A^{w}_{5k}$, $A^{u}_{2k}$, $A^{u}_{5k}$, $A'^{u}_{2k}$, $A'^{u}_{5k}$, $A^{v}_{2k}$, $A^{v}_{5k}$, $A'^{v}_{2k}$, $A'^{v}_{5k}$, $A^{x}_{2k}$, $A^{x}_{5k}$, $A'^{x}_{2k}$, $A'^{x}_{5k}$, $A^{\theta}_{2k}$, $A^{\theta}_{5k}$, $A'^{\theta}_{2k}$, and $A'^{\theta}_{5k}$; bending and axial compression- $A^{w}_{4k}$, $A^{w}_{5k}$, $A^{w}_{6k}$, $A^{u}_{4k}$, $A^{u}_{5k}$, $A^{u}_{6k}$, $A^{v}_{1k}$, $A^{v}_{2k}$, $A^{v}_{3k}$, $A^{x}_{4k}$, $A^{x}_{5k}$, $A^{x}_{6k}$, $A^{\theta}_{1k}$, $A^{\theta}_{2k}$, and $A^{\theta}_{3k}$; and axial compression (axisymmetric buckling)- $A^{w}_{5k}$, $A^{u}_{5k}$, and $A^{x}_{5k}$.

Substitution of these functions into Eq. (A-43) and reduction to the standard eigenvalue form is the same as outlined in Section 2.4. The forms of the resulting stiffness and load matrices are shown in Figs. 2.3 - 2.8 and the generating terms are catalogued in Appendix E. In the matrices, the rows represent $(\partial\ /\partial A^{k}_{ij})$ and the columns represent the coefficient of $A^{n}_{\ell m}$.

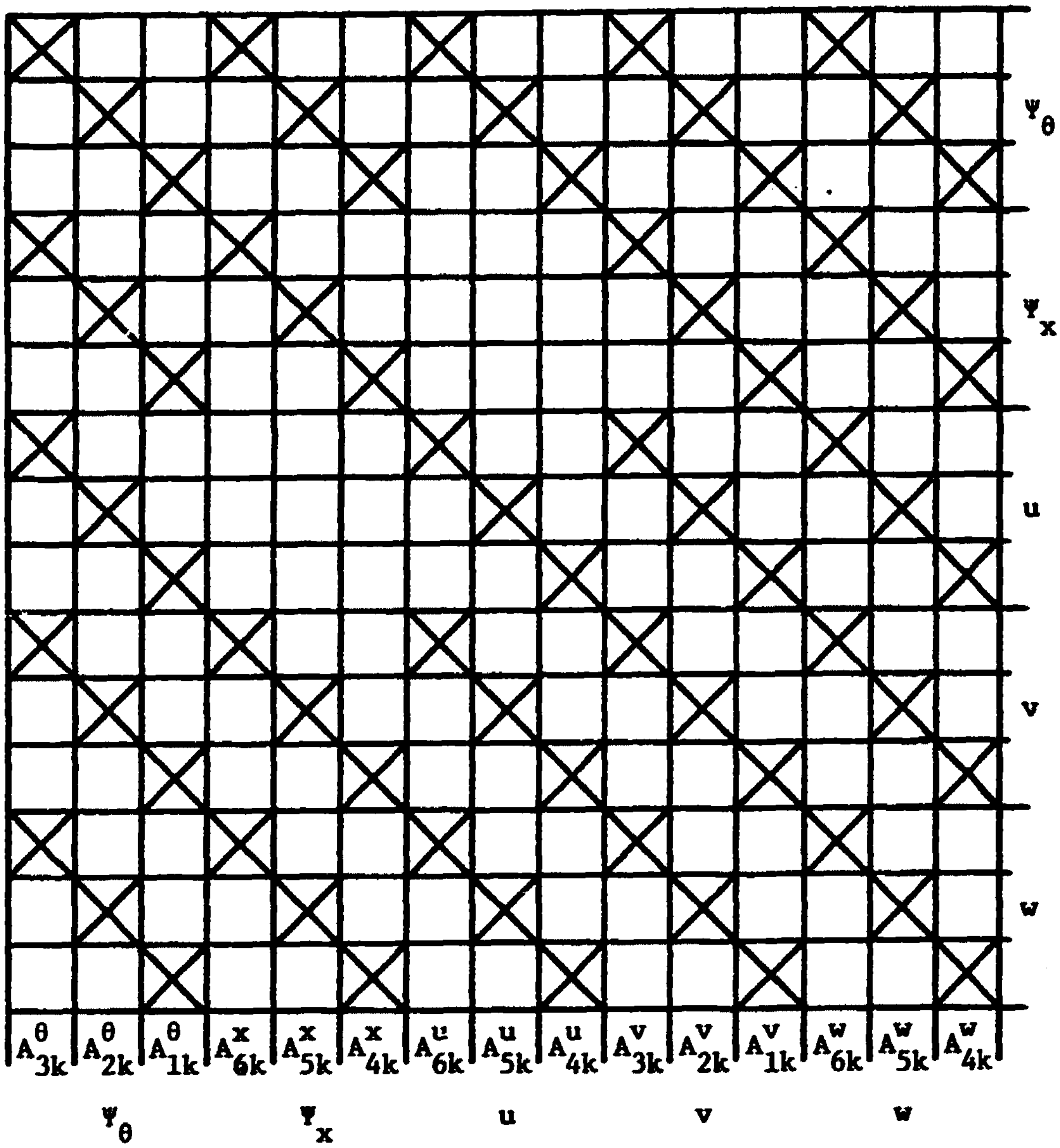


**Figure 2.3 - Clamped-Support Stiffness Matrix Axial Compression and Bending Loading**

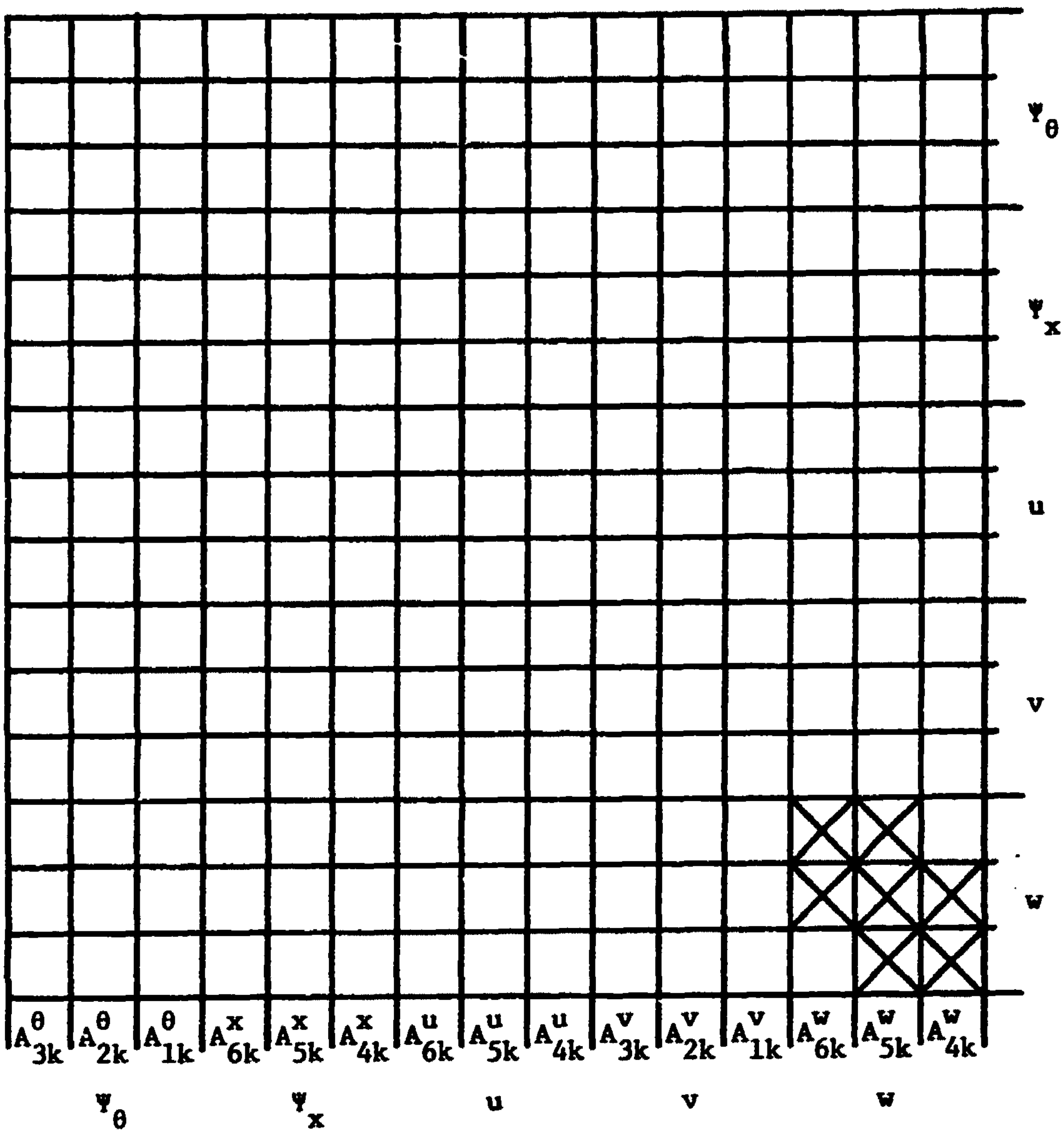


**Figure 2.4 - Clamped-Support Load Matrix Axial Compression and Bending Loading**

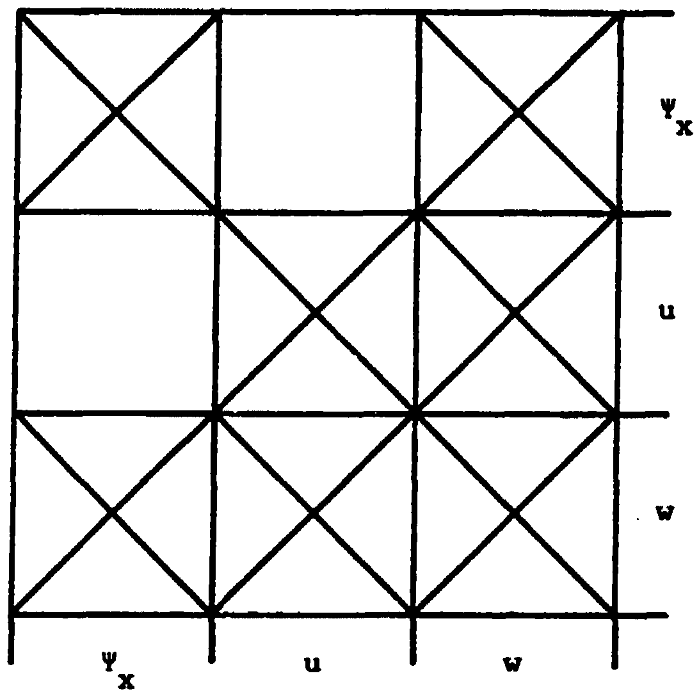


Figure 2.5 - Clamped-Support Stiffness Matrix
Axial Compression Loading (Axisymmetric Buckling)

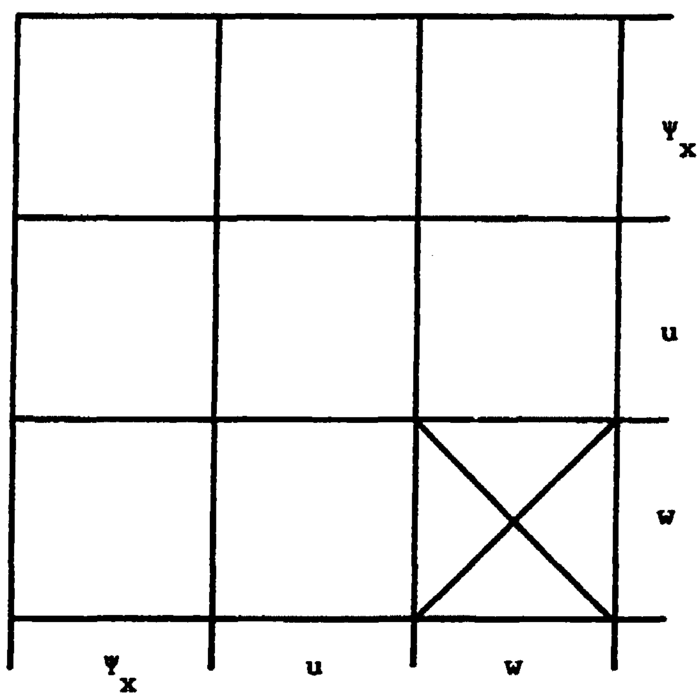


Figure 2.6 - Clamped-Support Load Matrix
Axial Compression Loading (Axisymmetric Buckling)

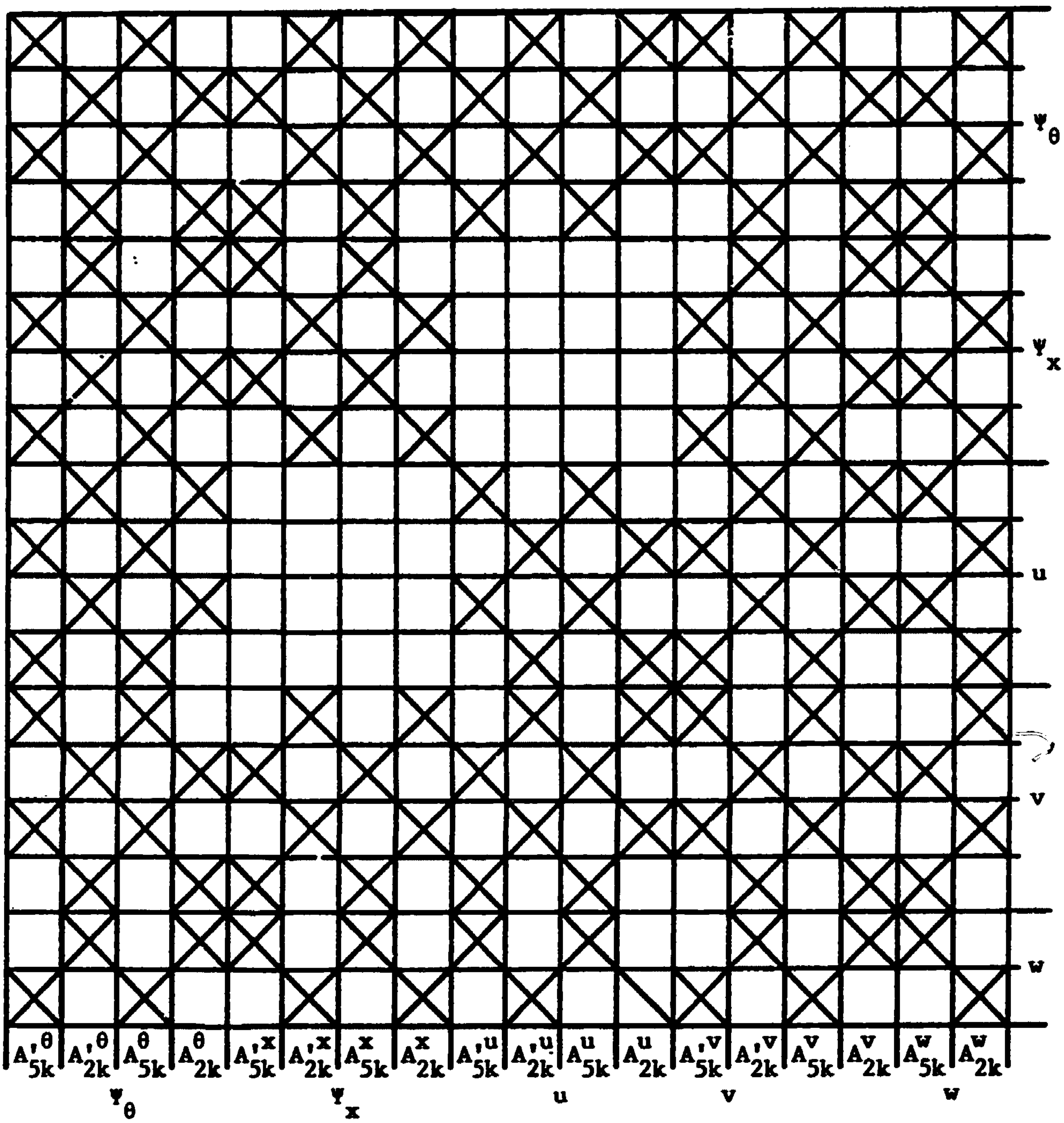


Figure 2.7 - Clamped-Support Stiffness Matrix
Axial Compression and Torsion Loading

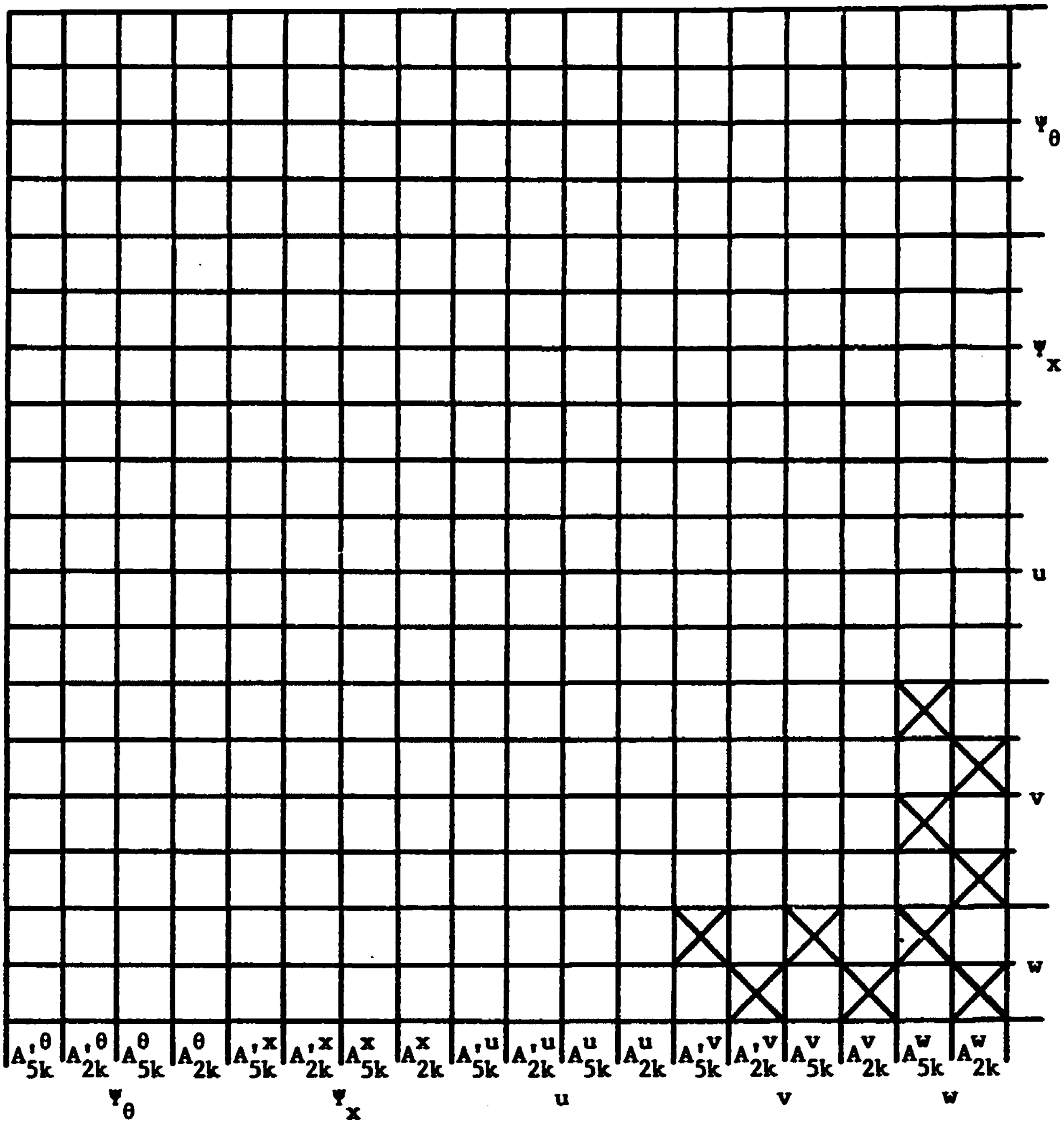


**Figure 2.8 - Clamped-Support Load Matrix Axial Compression and Torsion Loading**

# CHAPTER III

# EVALUATION OF THE THEORY

## 3.1 Simple-Support Boundary Conditions

Although most previous analyses have been for the simple-support case, there are no experimental data available since this type of boundary condition is impractical. Thus, the theory must be evaluated by comparing it with existing analyses.

### 3.1.1 Circular Cylindrical Shells

For the cylindrical case, the analysis can be compared with the Donnell-type axial-compression analysis presented in Refs. [22,25] which is abstracted in Appendix F. For the cylindrical specimens of Ref. [22], (See Appendix F for the material and geometrical properties.) this analysis predicts a critical stress of 24,869 psi. The analysis of Ref. [22] and Appendix F predicts a value of 25,285 psi. Thus, less than 2 percent difference occurred as a result of the more accurate Love's first-approximation analysis. It should be pointed out that the Ref. [22] analysis was based on the assumption that the longitudinal modes were uncoupled and could be evaluated separately. This analysis did not make that assumption; however, the results verified that the longitudinal modes were uncoupled. Both analyses predicted that the critical stress occurs for a

circumferential full-wave number of 5 and a longitudinal half-wa
number of 5.

A comparison of the Donnell-type shear flexible core an
of Appendix F with the rigid core Donnell-type analysis of Ref.
is made in Appendix F.

### 3.1.2 Truncated Conical Shell

There is no existing analysis with which to compare the
conical analysis. For a 5° semi-vertex angle cone and the mate
and geometrical properties of the previous case, Section 3.1.1,
analysis predicts a critical stress of 25,100 psi. In obtainin
value, eight longitudinal terms were used; however, the eigenve
indicated that the 4th, 5th, and 6th terms were predominant. T
critical stress occurred for a circumferential full-wave number
In recomputing the critical stress using the three predominant
a critical value of 25,160 psi was obtained. Again, the critic
stress was found to occur for a circumferential full-wave numbe
however, the variation of critical stress with circumferential
number was found to be much more pronounced.

## 3.2 Clamped Boundary Conditions

For the clamped boundary condition case, the theory wa
evaluated for axial compression loading, pure bending loading,
combined axial compression and bending loading. Even though cl
boundary conditions are the most feasible from a test standpoin
are very few experimental data available. This is due primaril
to the difficulty and cost involved in fabricating the large

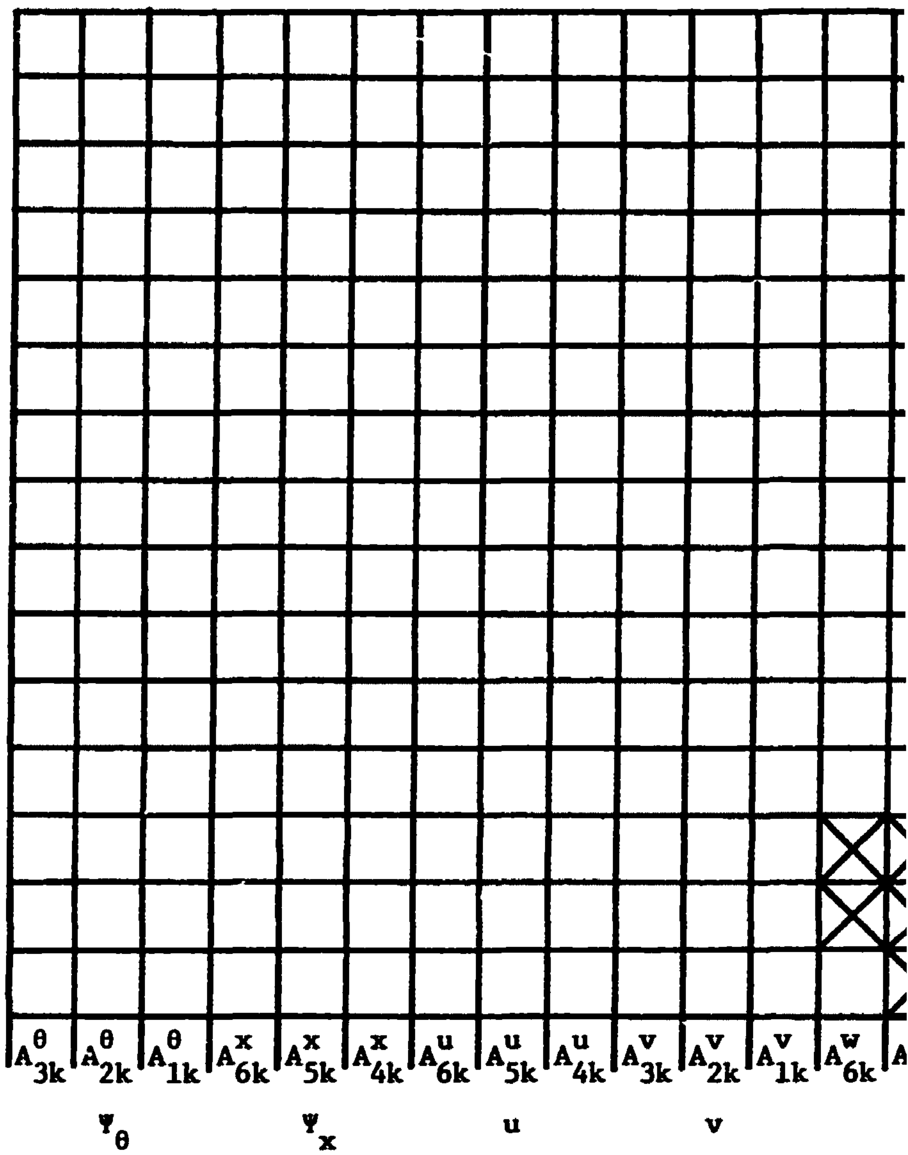


Figure 2.4 - Clamped-Support Load Matrix
Axial Compression and Bending Loading

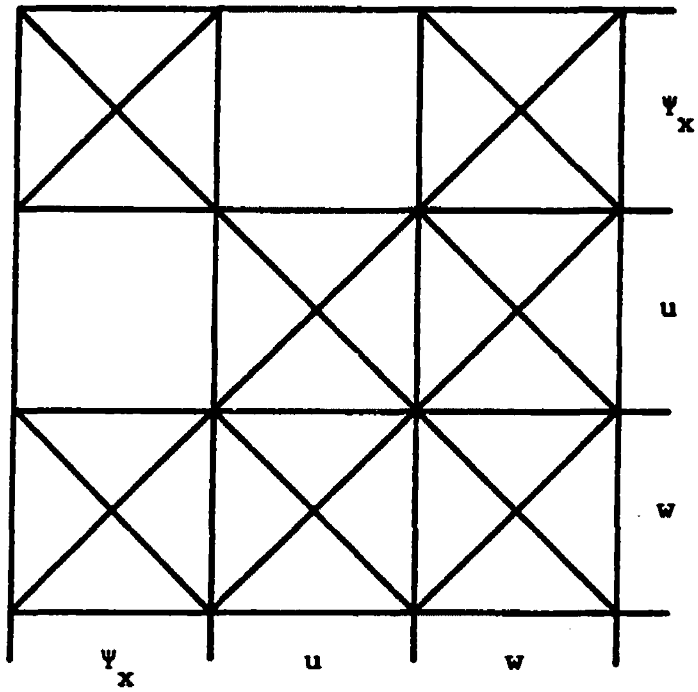


Figure 2.5 - Clamped-Support Stiffness Matrix
Axial Compression Loading (Axisymmetric Buckling)

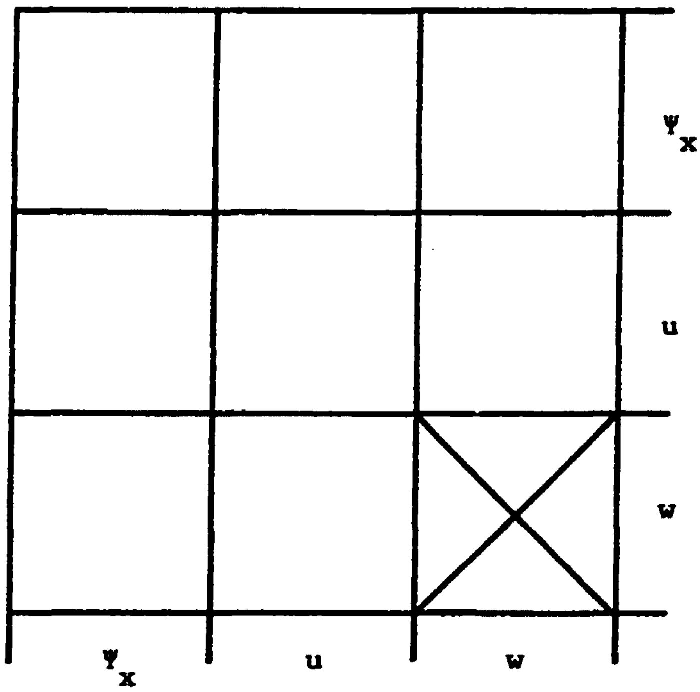


Figure 2.6 - Clamped-Support Load Matrix
Axial Compression Loading (Axisymmetric Buckling)

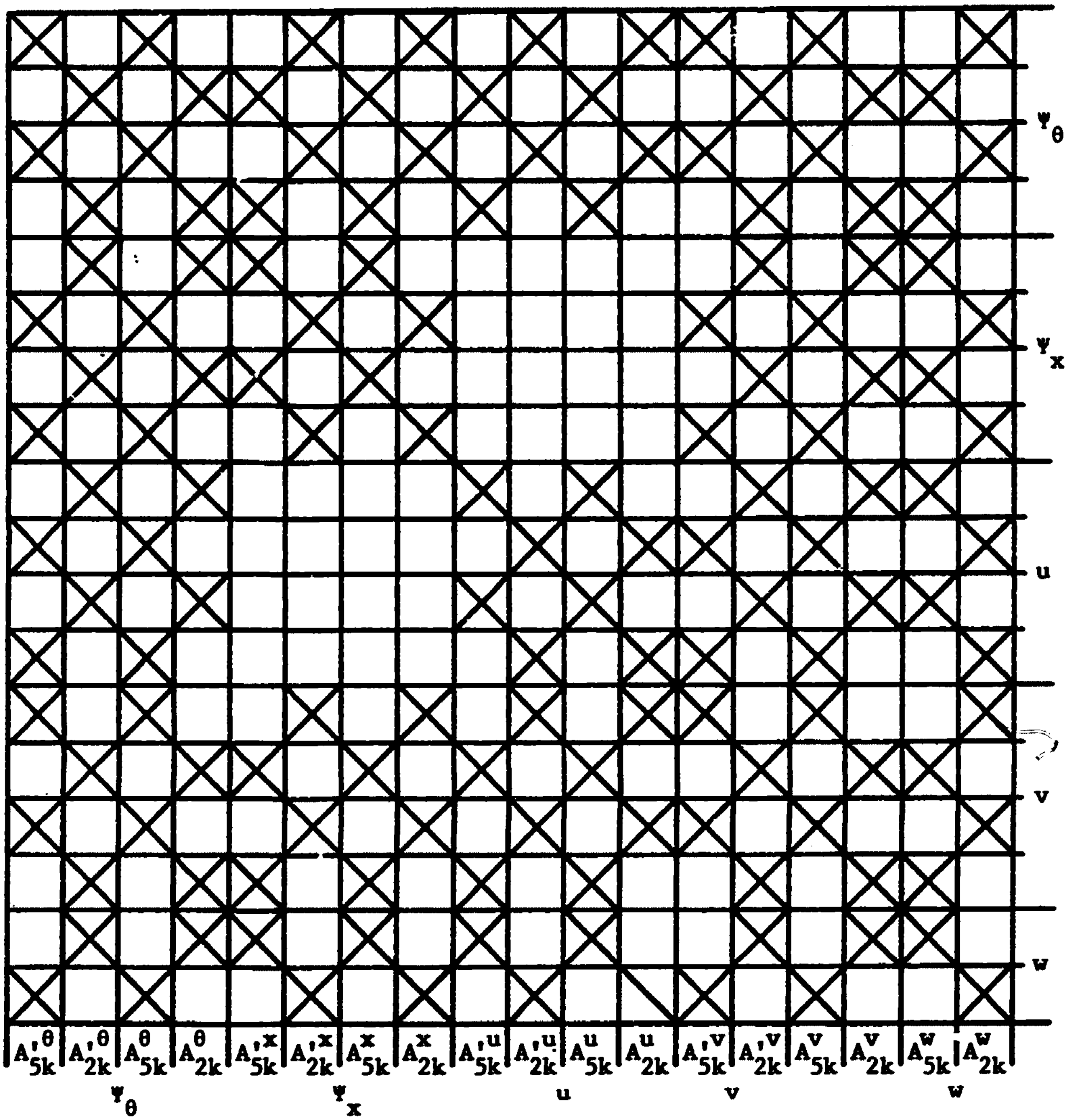


Figure 2.7 - Clamped-Support Stiffness Matrix
Axial Compression and Torsion Loading

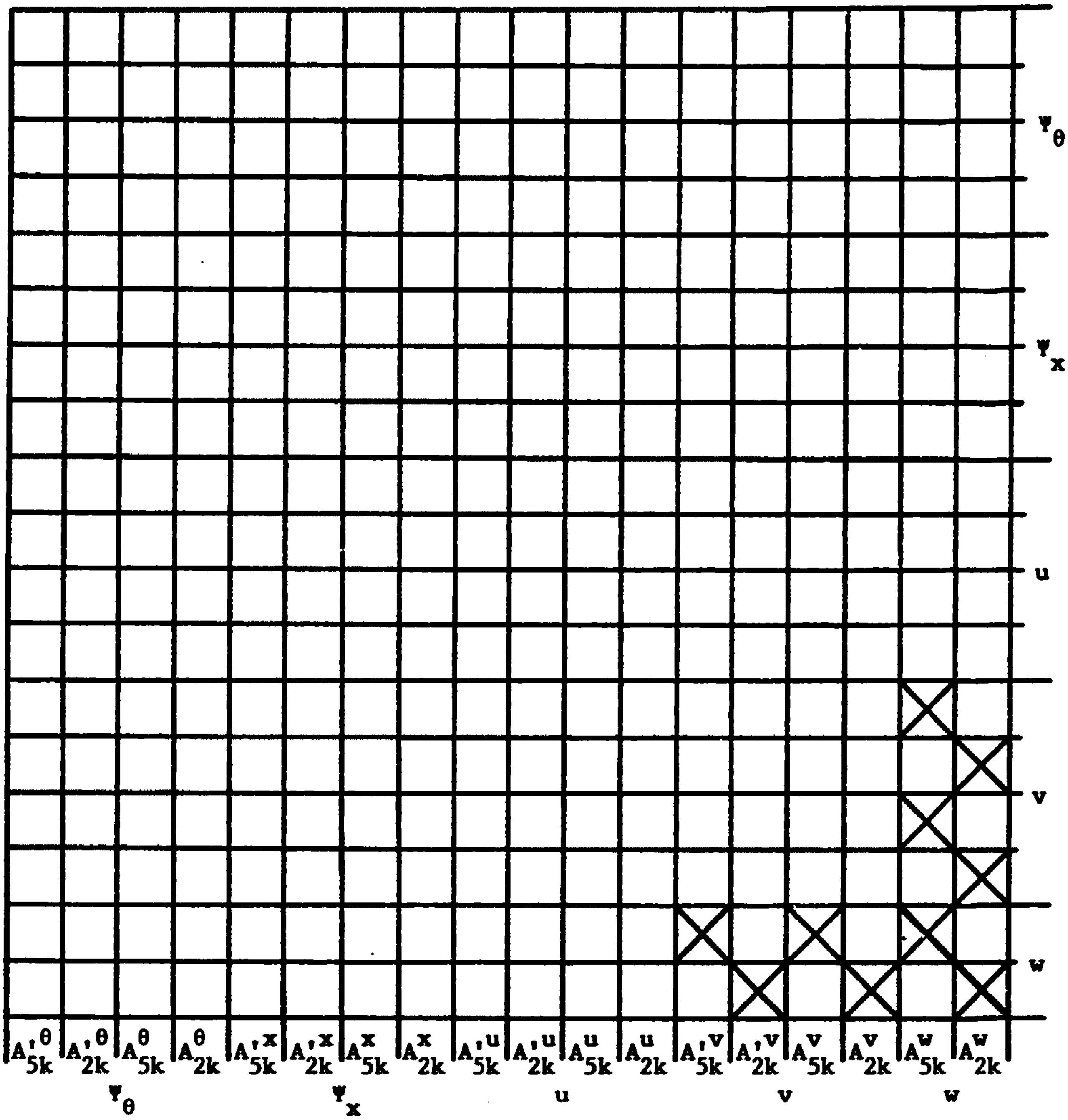


**Figure 2.8 - Clamped-Support Load Matrix**
**Axial Compression and Torsion Loading**

# CHAPTER III

# EVALUATION OF THE THEORY

## 3.1 Simple-Support Boundary Conditions

Although most previous analyses have been for the simple-support case, there are no experimental data available since this type of boundary condition is impractical. Thus, the theory must be evaluated by comparing it with existing analyses.

### 3.1.1 Circular Cylindrical Shells

For the cylindrical case, the analysis can be compared with the Donnell-type axial-compression analysis presented in Refs. [22,25] which is abstracted in Appendix F. For the cylindrical specimens of Ref. [22], (See Appendix F for the material and geometrical properties.) this analysis predicts a critical stress of 24,869 psi. The analysis of Ref. [22] and Appendix F predicts a value of 25,285 psi. Thus, less than 2 percent difference occurred as a result of the more accurate Love's first-approximation analysis. It should be pointed out that the Ref. [22] analysis was based on the assumption that the longitudinal modes were uncoupled and could be evaluated separately. This analysis did not make that assumption; however, the results verified that the longitudinal modes were uncoupled. Both analyses predicted that the critical stress occurs for a circumferential full-wave number of 5 and a longitudinal half-wave number of 5.

A comparison of the Donnell-type shear flexible core analysis of Appendix F with the rigid core Donnell-type analysis of Ref. [32] is made in Appendix F.

### 3.1.2 Truncated Conical Shell

There is no existing analysis with which to compare the conical analysis. For a 5° semi-vertex angle cone and the material and geometrical properties of the previous case, Section 3.1.1, the analysis predicts a critical stress of 25,100 psi. In obtaining this value, eight longitudinal terms were used; however, the eigenvector indicated that the 4th, 5th, and 6th terms were predominant. The critical stress occurred for a circumferential full-wave number of 6. In recomputing the critical stress using the three predominant terms, a critical value of 25,160 psi was obtained. Again, the critical stress was found to occur for a circumferential full-wave number of 6; however, the variation of critical stress with circumferential wave number was found to be much more pronounced.

## 3.2 Clamped Boundary Conditions

For the clamped boundary condition case, the theory was evaluated for axial compression loading, pure bending loading, and combined axial compression and bending loading. Even though clamped boundary conditions are the most feasible from a test standpoint, there are very few experimental data available. This is due primarily to the difficulty and cost involved in fabricating the large specimens needed for sandwich-type construction. The analysis was compared with the experimental results reported in Ref. [22] for orthotropic sandwich cylinders and cones loaded in pure bending and the experimental results of Ref. [34] for axially compressed sandwich cylinders and cones with isotropic facings. The effects of plasticity were accounted for in comparing with the Ref. [34] data by substituting $(E_s E_t)^{1/2}$ for the longitudinal elastic modulus. This is the same procedure suggested in Ref. [34] and was obtained from Ref. [35]. The variation in Poisson's ratio was accounted for by using the relationship proposed by Nádai [36]:

$$\nu = (1/2) - [(1/2) - \nu_e](E_s/E) \tag{3-1}$$

It is realized that more rigorous methods exist for taking the plasticity effect into account; however, the complicated form of the solution makes it advantageous to use a simple procedure such as outlined above. This simple procedure can result in several iterations since it is necessary to obtain the secant and tangent moduli at the critical stress.

The axisymmetric computer program, BOSS-AA, will not reduce automatically to the axisymmetric core shear buckling mode of failure. This mode can be obtained by using Eq. (F-10). This mode of failure is characterized by many longitudinal waves and is thus independent of the boundary conditions.

### 3.2.1 Axial Compression Loading

In Ref. [34] axial compression tests were conducted on sandwich cylinders and 10-degree half-angle cones with 7178-T6 bare aluminum-alloy facings and Hexcel (1/8) - 5052-0.0015P aluminum-alloy honeycomb core. The cylinders had an overall length of 60.0 inches and an inside diameter of 55.0 inches. The inside diameter of the cone was 39.4 inches at the small end and 66.8 at the large end; the slant length was 53.0 inches. The test results are shown in Table 3.1.

Since the facings were isotropic, the axial compression axisymmetric computer algorithm, BOSS-AA, was used for calculating the theoretical stress. The theoretical values are also shown in Table 3.1. No other analyses exist for clamped cylinders or cones. The cylindrical simple-support analysis of Refs. [22,25], Eq. (F-7), provides a slightly lower estimate as would be expected.

| Cone half-angle, degrees | Facing sheet thickness, inches | Nominal core depth, inches | Failure stress, ksi | Theoretical buckling stress, ksi |
|---|---|---|---|---|
| 0 | 0.0143 | 0.200 | 56.7 | 69.8 |
| 0 | 0.0149 | 0.200 | 48.7 | 69.8 |
| 0 | 0.0150 | 0.200 | 48.3 | 69.8 |
| 10 | 0.0144 | 0.200 | 56 | 68.8 |
| 10 | 0.0148 | 0.200 | 61 | 68.8 |

Table 3.1 Ref. [34] Test Results, 7178-T6 Aluminum Alloy Facings

### 3.2.2 Pure Bending Loading

In Ref. [22] pure bending tests were conducted on an orthotropic sandwich cylinder and 5-degree half-angle cone. The material and geometric properties are given in Section F.2 with the following additions:

$$G_{x\theta} = 416{,}000 \text{ psi}$$

$$G_{\theta z} = 18{,}300 \text{ psi}$$

$$G_{xz} = 32{,}000 \text{ psi}$$

For the cylinder, the computer program BOSS-AB gave a theoretical value for the critical stress of 37,550 psi. The experimental value was 14,100 psi. For the cone, the theoretical value was 43,250 psi as compared with an experimental value of 26,000 psi. Both of the theoretical values were obtained using four terms in the assumed longitudinal modal functions. Figure 3.1 shows the convergence with increasing number of assumed terms for the Ref. [22] cylindrical specimens for both axial compression and bending. It should be mentioned that in the past, pure bending loading has been predicted by the assumption that failure occurs when the maximum bending stress reaches the critical value for axial compression buckling. As seen in Fig. 3.1, theoretically, this is a very conservative estimate.

For the cylinder, the experimental value was 38 percent of the value predicted by theory. For the cone, failure occurred at 60 percent of the theoretical value. The conical shell was fabricated after the cylindrical shell and was probably of a better quality. An improvement in shell quality would account for the better agreement in the conical test.

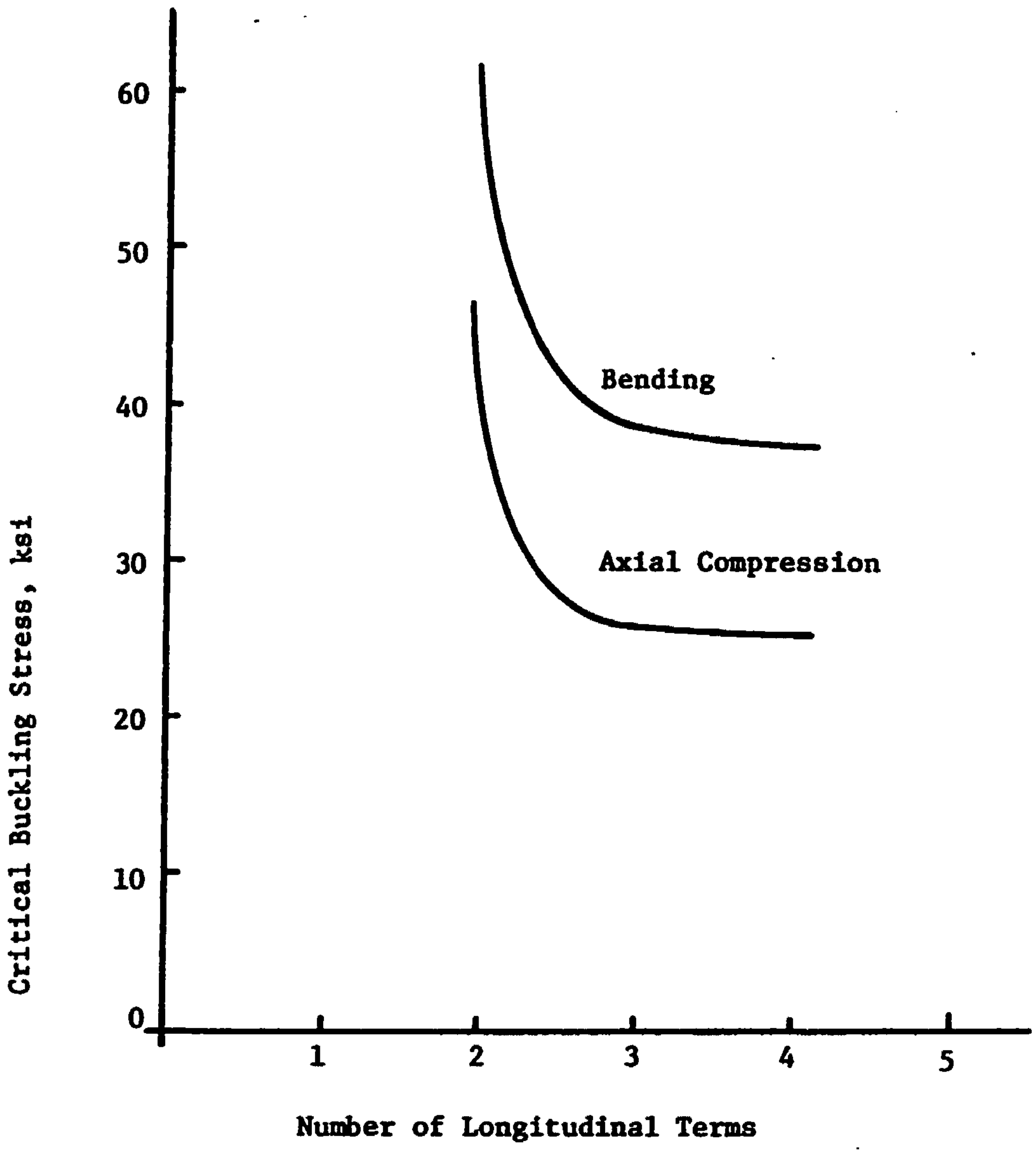


Figure 3.1 - Convergence for Cylinders of Ref. [22]

There are no existing analyses available for pure bending of sandwich cylinders or cones even in the isotropic case. However, somewhat of a comparison can be made with the isotropic thin-walled analysis of Seide and Weingarten [8] and to the anisotropic analysis of Ugural and Cheng [9]. The comparison was made using an effective wall thickness for the sandwich construction of $2\sqrt{3}$ (h+t). This gives an effective thickness of 0.5545 inches and a (R/thickness) ratio of 40 for the cylinder of Ref. [22]. An ($m\pi R/L$) ratio of 5 was found for these same cylinders. For the isotropic case, Ref. [8] found that the ratio of critical stress in bending to critical stress in axial compression, $\sigma_{b_o}/\sigma_{c_o}$, for a wave-length parameter of 5 and (R/thickness) ratio of 100 was approximately 1.20. Ref. [9] gave a value of approximately 1.35 for this ratio. For the Ref. [22] cylinders, this analysis gives a value of 1.48 for this ratio. The analyses of Refs. [8,9] were for simply supported cylinders.

### 3.2.3 Combined Axial Compression and Bending Loading

There are no experimental data available for this combined loading case; however, the interaction curve for the cylinders of Ref. [22], see Fig. 3.2, is of the same general appearance as the isotropic cylinder interaction curves reported in Ref. [8]. The Ref. [8] interaction curve for a (R/thickness) ratio of 100 is given in Fig. 3.3 for three different values of the wave length parameter. It is noted that both interaction curves, Figs. 3.2 and 3.3, are of the same general form. The set of interaction curves given in

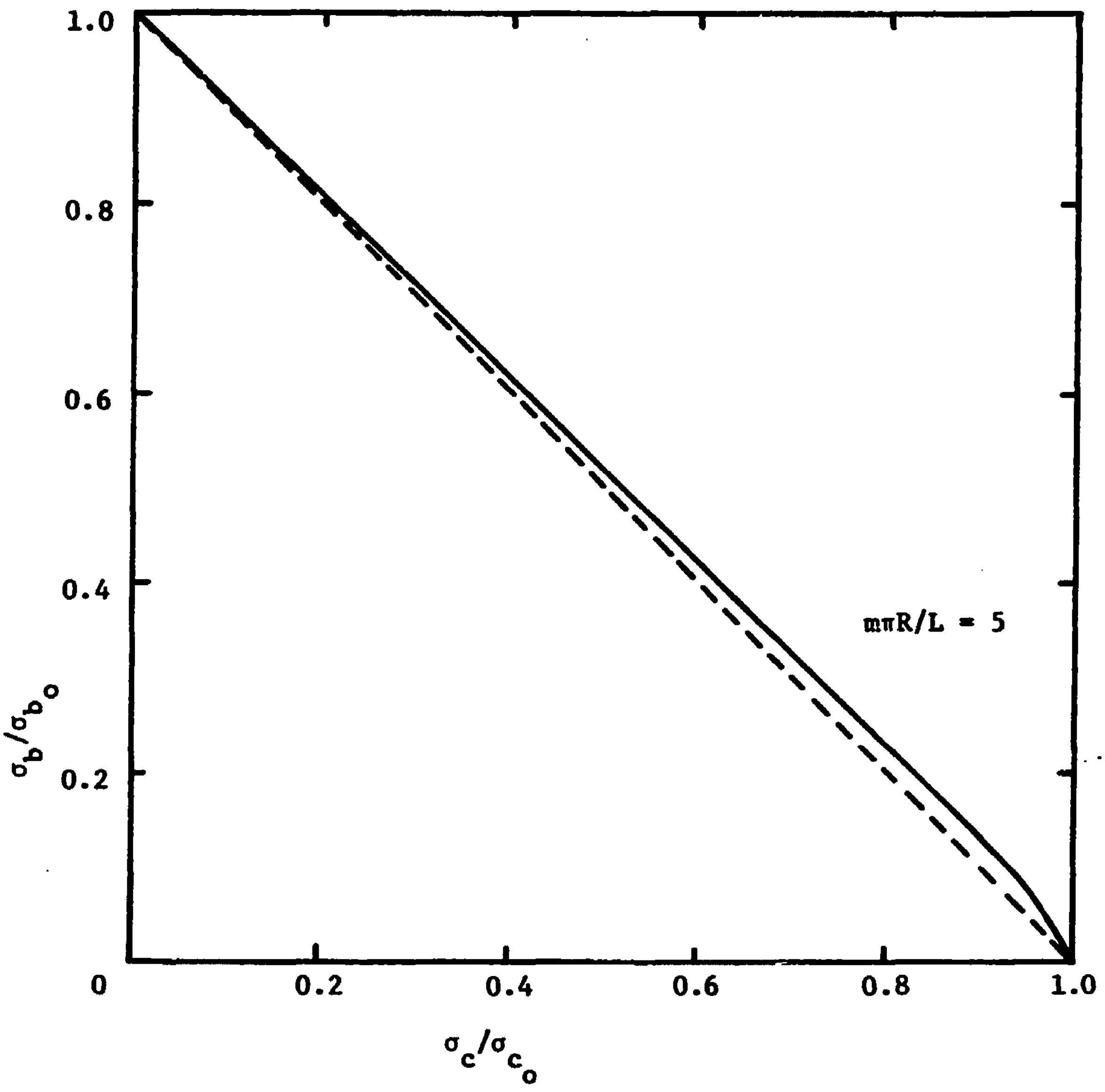


**Figure 3.2 - Axial Compression and Bending Interaction Curve Ref. [22] Cylinders**

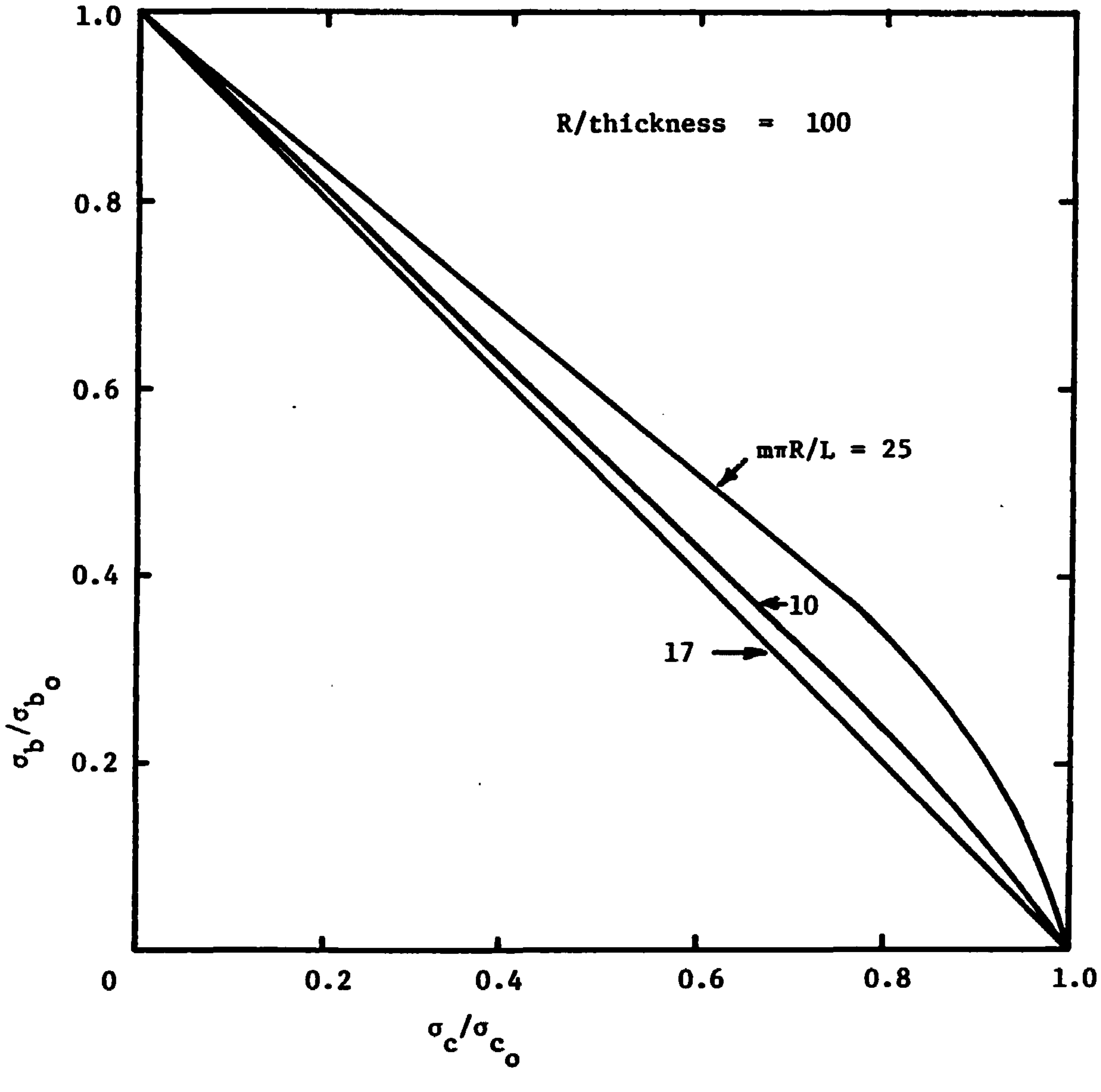


**Figure 3.3 - Typical Axial Compression and Bending Interaction Curves (Ref. [8]) for an Isotropic Cylinder**

Fig. 3.3 is typical of the ones presented in Ref. [8] for different (radius/thickness) ratios.

For a (radius/thickness) ratio of 100, the ratio of $(\sigma_{b_o}/\sigma_{c_o})$ reaches a minimum for a wave length parameter of 17. This accounts for the apparent shift in the curves shown in Fig. 3.3.

# CHAPTER IV

## CLOSURE

The general theory presented here for general instability buckling under axial compression, bending, and torsional loads can be very easily applied to cases involving different assumed modal shapes. The primary difficulty occurs in programming the numerical results.

The evaluation of the present theory for axial compression loading and both simple supports and clamped supports is in very good agreement with what was expected. For bending, the results are considerably higher than originally expected; however, the results are very similar to those predicted in Refs. [8,9] for the isotropic case. There does exist a considerable difference between theory and experiments.

There appears to be two areas where considerable additional work is needed. First, the experimental work needed to verify the various cases is extremely scarce. Second, the present analysis involves many different material and geometric parameters which make it extremely difficult to appreciate fully their interaction. Thus, various parametric studies are needed to understand better the present theory. Appendix F provides a first step in this direction. Once these interactions are better understood, reduction of the complicated analysis for design purposes can be achieved more easily. This analysis will be of maximum benefit for design purposes only after it is reduced in a similar manner to that employed in Appendix F.

## REFERENCES


1. Hoff, N.J., "The Perplexing Behavior of Thin Cylindrical Shells in Axial Compression", Israel J. Technol., Vol. 4, 1966, pp. 1-28.

2. Donnell, L. H., "Stability of Thin-Walled Tubes under Torsion", NACA Rep. No. 479, 1933.

3. Chehill, D.S. and Cheng, S., "Elastic Buckling of Composite Cylindrical Shells under Torsion", J. Spacecraft, Vol. 5, No. 8, 1968, pp. 973-978.

4. Batdorf, S.B., "A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells I - Donnell's Equation", NACA TN 1341, June 1947.

5. Batdorf, S.B., "A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells II - Modified Equilibrium Equation", NACA TN 1342, June 1947.

6. Batdorf, S.B., Stein, M., and Schildcrout, M., "Critical Stress of Thin-Walled Cylinders in Axial Compression", NACA TN 1343, June 1947.

7. Batdorf, S.B., Stein, M., and Schildcrout, M., "Critical Stress of Thin-Walled Cylinders in Torsion," NACA TN 1344, June 1947.

8. Seide, P. and Weingarten, V.I., "On the Buckling of Circular Cylindrical Shells under Pure Bending", J. Appl. Mech., Vol. 28, 1961, pp. 112-116.

9. Ugural, A.C. and Cheng, S., "Buckling of Composite Cylindrical Shells under Pure Bending", AIAA J., Vol. 6, No. 2, 1968, pp. 349-354.

10. Kobayashi, S., "The Influence of Prebuckling Deformation on the Buckling Load of Truncated Conical Shells under Axial Compression", NASA CR-707, January 1967.

11. Baruch, M., Harari, O., and Singer, J., "Low Buckling Loads of Axially Compressed Conical Shells", Technion-Israel Institute of Technology, TAE Report No. 76, January 1968.

12. Seide, P., "On the Buckling of Truncated Conical Shells in Torsion", J. Appl. Mech., Vol. 29, 1962, pp. 321-328.

13. Seide, P., "A Donnell-Type Theory for Asymmetrical Bending and Buckling of Thin Conical Shells", J. Appl. Mech., Vol. 24, 1957, pp. 547-552.

14. Leyko, J., "Stability of an Orthotropic Conical-Sector-Shaped Shell Compressed Along the Generator", Archiwum Budowy Maszyn, Vol. 8, 1961, pp. 447-460.

15. Singer, J., "Donnell-Type Equations for Bending and Buckling of Orthotropic Conical Shells", J. Appl. Mech., Vol. 30, 1963, pp. 303-305.

16. Plantema, F.J., Sandwich Construction, John Wiley and Sons, Inc., New York, 1966, 171-231.

17. Bartelds, G. and Mayers, J., "Unified Theory for the Bending and Buckling of Sandwich Shells - Application to Axially Compressed Circular Cylindtrical Shells", SUDAER No. 287, Nov. 1966, Stanford University, Dept. of Aeronautics and Astronautics, Stanford, Calif.

18. Bartelds, G. and Mayers, J., "Unified Theory for the Bending and Buckling of Sandwich Shells-Application to Axially Compressed Circular Cylindrical Shells", AIAA/ASME 8th Structures, Structural Dynamics and Materials Conference, Palm Springs, Calif., March 29-31, 1967.

19. Sylvester, R.J., "Buckling of Sandwich Cylinders under Axial Load", J. Aerospace Sci., Vol. 29, No. 7, 1962, pp. 863-872.

20. Wang, C.T., Vaccaro, R.J., and Desanto, D.F., "Buckling of Sandwich Cylinders under Combined Compression, Torsion, and Bending Loads", J. Appl. Mech., Vol. 22, 1955, pp. 324-328.

21. March, H.W. and Kuenzi, E.W., "Buckling of Cylinders of Sandwich Construction in Axial Compression", Report 1830, June 1952, revised Dec. 1957, Forest Products Laboratory, Madison, Wis.

22. Bert, C.W., Crisman, W.C., and Nordby,G.M., "Buckling of Cylindrical and Conical Sandwich Shells with Orthotropic Facings", AIAA/ASME 9th Structures, Structural Dynamics and Materials Conference, Palm Springs, Calif., April 1968, AIAA Paper No. 68-294; AIAA J., Vol. 7, Feb. 1969, pp. 250-257.

23. Norris, C.B. and Zahn, J.J., "Compressive Buckling Curves for Sandwich Cylinders Having Orthotropic Facings", Report 1876, July 1960, Forest Products Laboratory, Madison, Wis.

24. Stein, M. and Mayers, J., "A Small-Deflection Theory for Curved Sandwich Plates", NACA Report 1008, 1951.

25. Reese, C.D. and Bert, C.W., "Simplified Design Equations for Buckling of Axially Compressed Sandwich Cylinders with Orthotropic Facings and Core", Presented at the Meeting on Fiber Composites, Case Western Reserve University, Cleveland, Ohio, Oct. 8-9, 1968; to appear in J. Aircraft, Vol. 6, 1969.

26. Peterson, J.P., "Plastic Buckling of Plates and Shells under Biaxial Loading", NASA TN D-4706, August 1968, Langley Research Center, Hampton, Va.

27. March, H.W., and Kuenzi, E.W., "Buckling of Sandwich Cylinders in Torsion", Report 1840, June 1953, revised Jan. 1958, Forest Products Laboratory, Madison, Wis.

28. Stein, M. and Mayers. J., "Compressive Buckling of Simply Supported Curved Plates and Cylinders of Sandwich Construction", NACA TN 2601, Jan. 1952, Langley Aeronautical Laboratory, Langley Field, Va.

29. Baker, E.H., "Stability of Circumferentially Corrugated Sandwich Cylinders under Combined Loads", AIAA J., Vol. 2, No. 12, Dec. 1964, pp. 2142-2149.

30. Almroth, B.O., "Buckling of Axially Compressed Sandwich Cylinders", TR 6-62-64-9, July 1964, Lockheed Missiles and Space Co., Sunnyvale, Calif.

31. Maki, A.C., "Elastic Stability of Cylindrical Sandwich Shells under Axial and Lateral Load", FPL-0173, October 1967, Forest Products Laboratory, Madison, Wis.

32. Dow, N.F., and Rosen, B.W., "Structural Efficiency of Orthotropic Cylindrical Shells Subjected to Axial Compression", AIAA J., Vol. 4, No. 3, March 1966, pp. 481-485.

33. Kraus, H., Thin Elastic Shells, John Wiley and Sons, Inc., New York, 1967, pp. 24-58.

34. Baker, E. H., "Experimental Investigation of Sandwich Cylinders and Cones Subjected to Axial Compression", AIAA J., Vol. 6, No. 9, 1968, pp. 1769-1770.

35. Peterson, J.P. and Anderson, J.K., "Structural Behavior and Buckling Strength of Honeycomb Sandwich Cylinders Subjected to Bending", NASA TN D-2926, August 1965, Langley Research Center, Langley Station, Hampton, Va.

36. Nádai, A., Theory of Flow and Fracture of Solids, Vol. I, second edition, McGraw-Hill Book Co., Inc., New York, N.Y., 1950, pp. 379-387.

37. Fung, Y.C., Foundations of Solid Mechanics, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965, pp. 31-53.

38. Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity, Dover Publications, Inc., New York, 4th Ed., 1944.

39. Wilkins, D.J., Jr., "Free Vibrations of Orthotropic Conical Sandwich Shells with Various Boundary Conditions", unpublished Ph.D. dissertation, University of Oklahoma, Norman, Oklahoma, 1969.

40. Siu, C.C., "Free Vibrations of Sandwich Conical Shells with Free Edges", unpublished Master of Engineering thesis, University of Oklahoma, Norman, Oklahoma, 1969.

41. Timoshenko, S.P. and Gere, J.M., Theory of Elastic Stability, McGraw-Hill Book Co., Inc., New York, second edition, 1961, p. 337.

42. Sechler, E.E., Elasticity in Engineering, John Wiley and Sons, Inc., New York, 1952, p. 41.

43. Bodner, S.R., "General Instability of a Ring-Stiffened, Circular Cylindrical Shell under Hydrostatic Pressure", J. Appl. Mech., Vol. 24, 1957, pp. 269-277.

44. Timoshenko and Gere, loc. cit., pp. 337-340, 350.

45. System/360 Scientific Subroutine Package Version III Programmer's Manual, Form H20-0205, International Business Machines Corporation, 1968.

46. Cunningham, J.H. and Jacobson, M.J., "Design and Testing of Honeycomb Sandwich Cylinders under Axial Compression", pp. 341-352 in "Collected Papers on Instability of Shell Structures", TN D-1510, Dec. 1962, NASA, Langley Research Center, Hampton, Va.

47. Structural Sandwich Composites, Chapter 12, MIL-HDBK-23A, Dec. 30, 1968, Dept. of Defense, Washington, D.C.

## APPENDIX A

## DERIVATION OF POTENTIAL ENERGY EXPRESSION FOR AN ORTHOTROPIC SANDWICH SHELL

### A.1 Strain-Displacement Equations

In general curvilinear coordinates, the infinitesimal tensorial strain components are given by Fung [37] as

$$\varepsilon_{ij} = (1/2)[\, u_i|_j + u_j|_i \,] \tag{A-1}$$

where the $u_i$, $i = 1,2$, and 3, are the covariant components of the displacement vector and $|_i$ denotes covariant differentiation. Then

$$\varepsilon_{ij} = (1/2)[\, u_{i},_{j} + u_{j},_{i} \,] - \Gamma^{\sigma}_{ij}u_{\sigma} \tag{A-2)*}$$

where the $\Gamma^{\sigma}_{ij}$ are the Euclidean Christoffel symbols:

$$\Gamma^{i}_{\alpha\beta}(x^1,x^2,x^3) = (1/2)g^{i\sigma}(g_{\sigma\beta},_{\alpha} + g_{\alpha\sigma},_{\beta} - g_{\alpha\beta},_{\sigma}) \tag{A-3}$$

In the above equation, the $x^i$ are the general curvilinear coordinates, the $g_{ij}$ are the components of the Euclidean metric tensor and the $g^{ij}$ are the components of the associated metric tensor.

For the right-hand, orthogonal, truncated-conical coordinate system shown in Figs. A.1 and A.2, the components of the metric tensor

---

*The repetition of an index in a term denotes summation with respect to that index over its range, i.e. the Einstein convention is used. The notation $,_i$ indicates partial differentiation with respect to the coordinate which follows the comma. This notation is used throughout this publication.

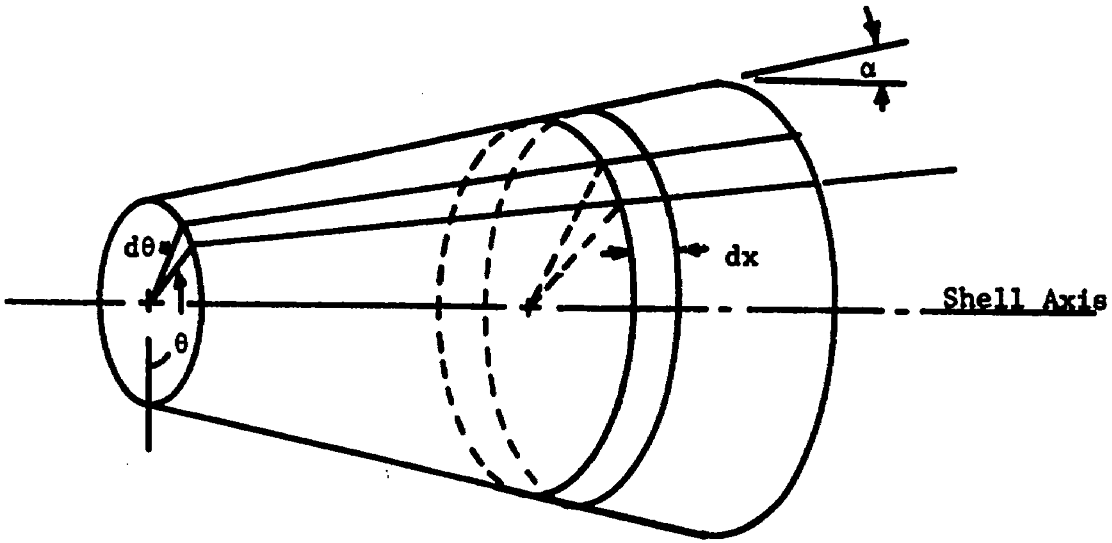


Figure A.1 - Shell Middle Surface

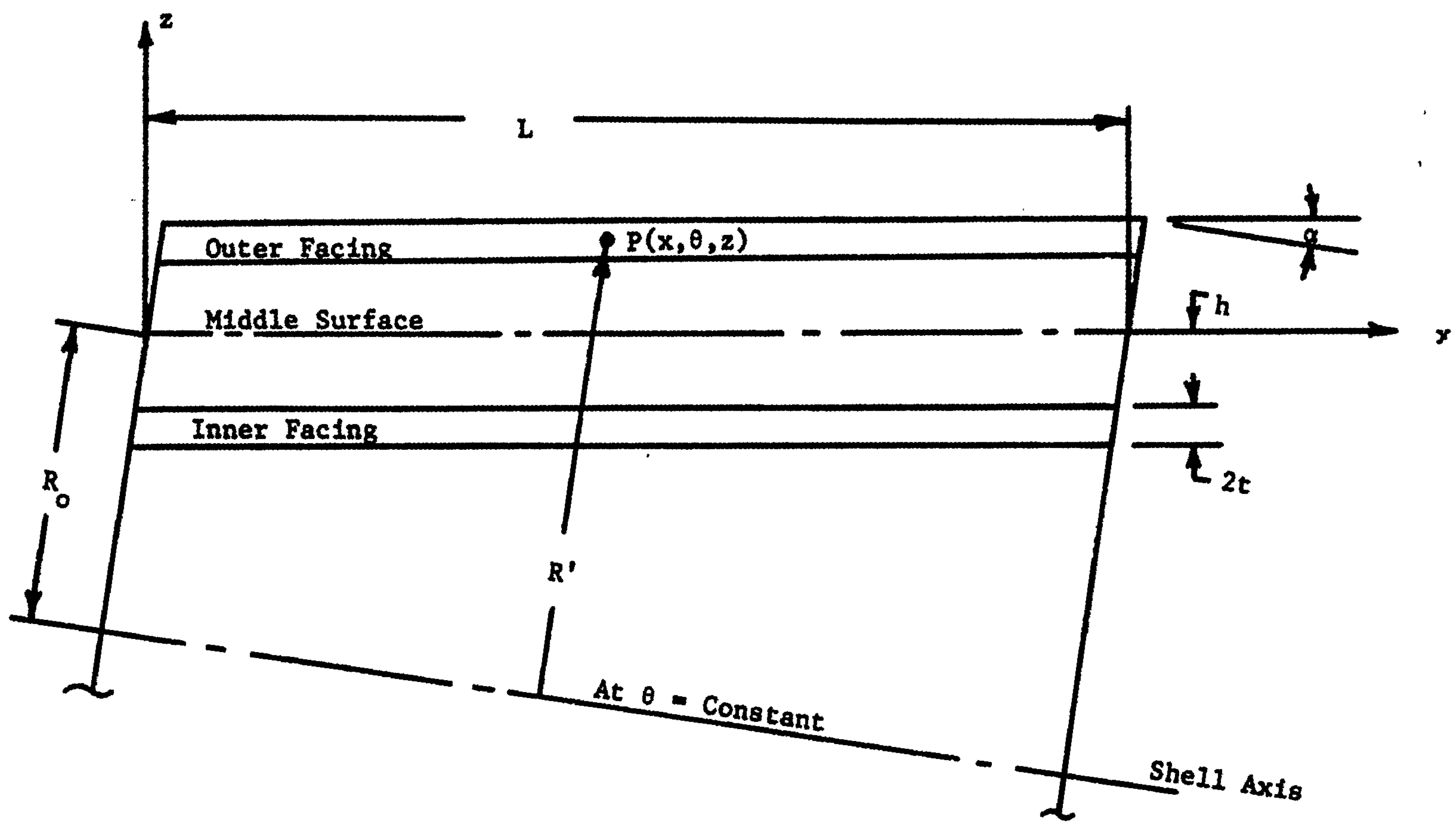


Figure A.2 - Shell Geometry

can be found from the square of a general, differential line element:

$$(ds)^2 = (dx)^2 + (R'd\theta)^2 + (dz)^2 \qquad \text{(A-4)}$$

where

$$R' = R_o + x \sin \alpha + z \cos \alpha \qquad \text{(A-5)}$$

Then,

$$\begin{aligned} g_{11} &= 1 \\ g_{22} &= (R_o + x \sin \alpha + z \cos \alpha)^2 = (R')^2 \\ g_{33} &= 1 \\ g_{ij} &= 0 \quad \text{for} \quad i \neq j \end{aligned} \qquad \text{(A-6)}$$

Since the coordinate system is orthogonal,

$$\begin{aligned} g^{11} &= 1/g_{11} = 1 \\ g^{22} &= 1/g_{22} = (R')^{-2} \\ g^{33} &= 1/g_{33} = 1 \\ g^{ij} &= 0 \quad \text{for} \quad i \neq j \end{aligned} \qquad \text{(A-7)}$$

From Eq. (A-3), the nonzero Christoffel symbols are

$$\begin{aligned} \Gamma^1_{22} &= -R' \sin \alpha \\ \Gamma^2_{12} &= \Gamma^2_{21} = (R')^{-1} \sin \alpha \\ \Gamma^2_{23} &= \Gamma^2_{32} = (R')^{-1} \cos \alpha \\ \Gamma^3_{22} &= -R' \cos \alpha \end{aligned} \qquad \text{(A-8)}$$

Substituting the Eqs. (A-8) into Eqs. (A-2), the tensorial strain components are found to be:

$$\begin{aligned} \varepsilon_{xx} &= u_1{,}_x \\ \varepsilon_{\theta\theta} &= u_2{,}_\theta + R'[(\sin \alpha)u_1 + (\cos \alpha)u_3] \\ \varepsilon_{zz} &= u_3{,}_z \\ \varepsilon_{x\theta} &= (1/2)[u_1{,}_\theta + u_2{,}_x] - (\sin \alpha)\, u_2(R')^{-1} \\ \varepsilon_{\theta z} &= (1/2)[u_2{,}_z + u_3{,}_\theta] - (\cos \alpha)\, u_2(R')^{-1} \\ \varepsilon_{zx} &= (1/2)[u_3{,}_x + u_1{,}_z] \end{aligned} \qquad \text{(A-9)}$$

The $u_i$ and $\varepsilon_{ij}$ above are tensorial components. Their corresponding physical components are given respectively as:

$$\begin{aligned} \xi_i &= \sqrt{g^{ii}}\ u_i \\ \xi'_{ij} &= \sqrt{g^{ii}g^{jj}}\ \varepsilon_{ij} \end{aligned} \tag{A-10}$$

Substituting Eqs. (A-10) into Eqs. (A-9)

$$\begin{aligned} \varepsilon'_{xx} &= \xi_x,_x \\ \varepsilon'_{\theta\theta} &= (R')^{-1}[\ \xi_\theta,_\theta + (\sin\alpha)\ \xi_x + (\cos\alpha)\ \xi_z] \\ \varepsilon'_{zz} &= \xi_z,_z \\ \varepsilon'_{x\theta} &= (2R')^{-1}[\xi_x,_\theta + R'\ \xi_\theta,_x - \xi_\theta \sin\alpha] \\ \varepsilon'_{\theta z} &= (2R')^{-1}[R'\ \xi_\theta,_z - \xi_\theta \cos\alpha + \xi_z,_\theta\ ] \\ \varepsilon'_{zx} &= (1/2)[\ \xi_z,_x + \xi_x,_z\ ] \end{aligned} \tag{A-11}$$

It is noted that the ordinary strains used in engineering are obtained from the tensorial strains as follows:

$$\begin{aligned} e_{ij} &= \varepsilon'_{ij} \qquad \text{if}\quad i = j \\ e_{ij} &= 2\varepsilon'_{ij} \qquad \text{if}\quad i \neq j \end{aligned} \tag{A-12}$$

Thus, the ordinary strains become:

$$\begin{aligned} e_{xx} &= \xi_x,_x \\ e_{\theta\theta} &= [\ \xi_\theta,_\theta + (\sin\alpha)\ \xi_x + (\cos\alpha)\ \xi_z]/R' \\ e_{zz} &= \xi_z,_z \\ e_{x\theta} &= [\ \xi_x,_\theta + R'\ \xi_\theta,_x - \xi_\theta \sin\alpha]/R' \\ e_{\theta z} &= [R'\xi_\theta,_z - \xi_\theta \cos\alpha + \xi_z,_\theta\ ]/R' \\ e_{zx} &= \xi_z,_x + \xi_x,_z \end{aligned} \tag{A-13}$$

It should be noted that Eqs. (A-13) can also be found by particularizing Love's [38] general curvilinear coordinates to the coordinate system shown in Figs. A.1 and A.2. This procedure was used by Wilkins [39] and Siu [40] to obtain these same strain equations.

Since the shell thickness was assumed to be small in comparison with the smallest radius of curvature, it is justifiable for the mathematical simplicity gained to ignore the term $z \cos \alpha$ in $R'$. Then

$$R = R_o + x \sin \alpha \approx R' \tag{A-14}$$

For later convenience, the following notation is introduced:

$$\zeta = R^{-1} \tag{A-15}$$

In view of the Hypotheses of Chapter 2, the displacements can be written in terms of the middle-surface displacements (u,v, and w) and the rotations of the normals to the shell middle surface.

For the core, these angles are denoted by $\Psi_x$ and $\Psi_\theta$ in the meridional and circumferential directions, respectively. Thus,

$$\begin{aligned} \xi_x^c &= u(x,\theta) + z\Psi_x(x,\theta) \\ \xi_\theta^c &= v(x,\theta) + z\Psi_\theta(x,\theta) \\ \xi_z^c &= w(x,\theta) \end{aligned} \tag{A-16}$$

For the facings, the angles of rotation are denoted as $\beta_x^o$ and $\beta_\theta^o$ for the outer facing and as $\beta_x^i$ and $\beta_\theta^i$ for the inner facing. For the outer facing,

$$\begin{aligned} \xi_x^o &= u(x,\theta) + h\Psi_x(x,\theta) + (z - h)\beta_x^o(x,\theta) \\ \xi_\theta^o &= v(x,\theta) + h\Psi_\theta(x,\theta) + (z - h)\beta_\theta^o(x,\theta) \\ \xi_z^o &= w(x,\theta) \end{aligned} \tag{A-17}$$

For the inner facing,

$$\begin{aligned} \xi_x^i &= u(x,\theta) - h\Psi_x(x,\theta) + (z + h)\beta_x^i(x,\theta) \\ \xi_\theta^i &= v(x,\theta) - h\Psi_\theta(x,\theta) + (z + h)\beta_\theta^i(x,\theta) \\ \xi_z^i &= w(x,\theta) \end{aligned} \tag{A-18}$$

Substituting Eqs. (A-15) and (A-16) into Eqs. (A-13), the following equations for the nonzero core strains are obtained:

$$e^c_{zx} = w,_x + \Psi_x \qquad \text{(A-19)}$$
$$e^c_{\theta z} = \zeta[\,- v\cos\alpha - z\Psi_\theta \cos\alpha + w,_\theta] + \Psi_\theta$$

Likewise for the outer facing,

$$e^o_{xx} = \underline{u,_x} + h\Psi_{x},_{x} + (z - h)\beta^o_{x},_{x}$$
$$e^o_{\theta\theta} = \zeta[\underline{v,_\theta} + h\Psi_\theta,_\theta + (z - h)\beta^o_\theta,_\theta + \underline{u \sin\alpha} \qquad \text{(A-20)}$$
$$+ h\Psi_x \sin\alpha + (z - h)\beta^o_x \sin\alpha + \underline{w\cos\alpha}]$$
$$e^o_{x\theta} = \zeta[\underline{u,_\theta} + h\Psi_x,_\theta + (z - h)\beta^o_x,_\theta - \underline{v\sin\alpha} - h\Psi_\theta \cdot$$
$$\sin\alpha - (z - h)\beta^o_\theta \sin\alpha] + \underline{v,_x} + h\Psi_\theta,_x +$$
$$(z - h)\beta^o_\theta,_x$$

In the above equations, the membrane terms are underscored for later convenience. Since it was assumed in the Hypotheses of Chapter 2 that $e^o_{\theta z} = e^o_{zx} = 0$ and following the procedure of Ref. [33], the rotations $\beta^o_x$ and $\beta^o_\theta$ can be found from the last two of Eqs. (A-13).

$$\beta^o_x = -\,w,_x \qquad \text{(A-21)}$$
$$\beta^o_\theta = [w,_\theta - v\cos\alpha - h\Psi_\theta \cos\alpha]/[(z - h)\cos\alpha - R']$$

In a similar manner, for the inner facing:

$$e^i_{xx} = \underline{u,_x} - h\Psi_x,_x + (z + h)\beta^i_x,_x$$
$$e^i_{\theta\theta} = \zeta[\underline{v,_\theta} - h\Psi_\theta,_\theta + (z + h)\beta^i_\theta,_\theta + \underline{u\sin\alpha} - h\Psi_x \sin\alpha \qquad \text{(A-22)}$$
$$+ (z + h)\beta^i_x \sin\alpha + \underline{w\cos\alpha}]$$
$$e^i_{x\theta} = \zeta[\underline{u,_\theta} - h\Psi_x,_\theta + (z + h)\beta^i_x,_\theta - \underline{v\sin\alpha} + h\Psi_\theta \sin\alpha$$
$$- (z + h)\beta^i_\theta \sin\alpha] + \underline{v,_x} - h\Psi_\theta,_x + (z + h)\beta^i_\theta,_x$$

Since $e^i_{\theta z} = e^i_{zx} = 0$ ,

$$\beta^i_x = -\,w,_x \qquad \text{(A-23)}$$
$$\beta^i_\theta = [w,_\theta - v\cos\alpha + h\Psi_\theta \cos\alpha]/[(z + h)\cos\alpha - R']$$

Based on the same argument as used in obtaining Eq. (A-14), the denominator of the relationships for $\beta^o_\theta$ and $\beta^i_\theta$ can be replaced by R.

In addition to the membrane strain components (underscored terms) given in Eqs. (A-20) and (A-22), there is an additional stretching elongation due to displacement in the thickness direction. It should be noted that if it is assumed that no stretching of the middle surface occurs during buckling, the thickness direction displacements would result in additional movement of the boundary forces in the x - θ plane. Thus, they would still enter into the analysis. The membrane strain contributions due to deflection in the thickness direction are then [41,42]

$$\begin{aligned} e_{xx} &= (1/2)\beta_x^2 \\ e_{\theta\theta} &= (1/2)\beta_\theta^2 \\ e_{x\theta} &= \beta_x\beta_\theta \end{aligned} \qquad \text{(A-24)}$$

Combining these components with the linear membrane components of Eqs. (A-20) and (A-22) gives the following membrane strain equations:

$$\begin{aligned} \tilde{e}^o_{xx} &= u,_x + (1/2)(\beta^o_x)^2 \\ \tilde{e}^o_{\theta\theta} &= \zeta[v,_\theta + u \sin\alpha + w\cos\alpha] + (\beta^o_\theta)^2/2 \\ \tilde{e}^o_{x\theta} &= \zeta[u,_\theta - v\sin\alpha] + v,_x + \beta^o_x\beta^o_\theta \\ \tilde{e}^i_{xx} &= u,_x + (1/2)(\beta^i_x)^2 \\ \tilde{e}^i_{\theta\theta} &= \zeta[v,_\theta + u\sin\alpha + w\cos\alpha] + (\beta^i_\theta)^2/2 \\ \tilde{e}^i_{x\theta} &= \zeta[u,_\theta - v\sin\alpha] + v,_x + \beta^i_x\beta^i_\theta \end{aligned} \qquad \text{(A-25)}$$

## A.2 Constitutive Equations

Both the core and facing materials are assumed to be linearly elastic.

In view of Hypotheses (4) and (5) of Chapter 2, the generalized Hooke's law for the core is as follows:

$$\sigma^c_{zx} = e^c_{zx}\, G_{zx}$$
$$\sigma^c_{\theta z} = e^c_{\theta z}\, G_{\theta z} \qquad \text{(A-26)}$$

A generalized state of plane stress is assumed to exist in the facings. The generalized Hooke's law for the facings then becomes:

$$\begin{aligned}
\sigma^o_{xx} &= \bar{E}_x(e^o_{xx} + \nu_{\theta x}e^o_{\theta\theta})\\
\sigma^o_{\theta\theta} &= \bar{E}_\theta(e^o_{\theta\theta} + \nu_{x\theta}e^o_{xx})\\
\sigma^o_{x\theta} &= G_{x\theta}\, e^o_{x\theta}\\
\sigma^i_{xx} &= \bar{E}_x(e^i_{xx} + \nu_{\theta x}e^i_{\theta\theta})\\
\sigma^i_{\theta\theta} &= \bar{E}_\theta(e^i_{\theta\theta} + \nu_{x\theta}e^i_{xx})\\
\sigma^i_{x\theta} &= G_{x\theta}\, e^i_{x\theta}
\end{aligned} \qquad \text{(A-27)}$$

where $\bar{E}_x = E_x/(1 - \nu_{x\theta}\nu_{\theta x})$ and $\bar{E}_\theta = E_\theta/(1 - \nu_{x\theta}\nu_{\theta x})$. It should also be noted that the following relationship holds as a consequence of the symmetry of the stiffness coefficient matrix for an arbitrary elastic material:

$$E_x\nu_{\theta x} \equiv E_\theta\nu_{x\theta} \qquad \text{(A-28)}$$

## A.3 Core Strain Energy

The strain energy for the core is due to transverse shear strain only. Therefore,

$$V^c = (1/2)\int_x\int_\theta\int_z(\sigma^c_{\theta z}e^c_{\theta z} + \sigma^c_{zx}e^c_{zx})dz\ Rd\theta dx \qquad \text{(A-29)}$$

Substituting Eqs. (A-26) and inserting the limits for z,

$$V^c = (1/2)\int_x\int_\theta\int_{-h}^{h}[G_{\theta z}(e^c_{\theta z})^2 + G_{zx}(e^c_{zx})^2]dz\ Rd\theta dx \qquad \text{(A-30)}$$

Substituting Eqs. (A-19) and integrating over z,

$$V^c = \int_x\int_\theta\{hK_xG_{zx}R\ [w,^2_x + 2\ w,_x\Psi_x + \Psi^2_x\ ] + hK_\theta G_{\theta z}[\zeta w,^2_\theta +$$

$$([h^2\zeta\cos^2\alpha]/3 + R)\Psi_\theta^2 + \zeta v^2\cos^2\alpha - 2\zeta v w,_\theta \cos\alpha$$

$$- 2\Psi_\theta v\cos\alpha + 2\Psi_\theta w,_\theta]\}\, d\theta dx \qquad \text{(A-31)}$$

It should be noted that $(h^2\zeta\cos^2\alpha)/3 << R$ and thus can be neglected.

## A.4 Facing Strain Energy

The strain energy for the facings consists of two parts. In addition to the usual part, there is a contribution due to the membrane load condition that exists at the onset of buckling. This membrane state of stress is assumed to remain essentially constant during buckling. Then

$$V_1^f = \int_x\int_\theta\int_z [\bar{\sigma}_{xx}^o\bar{e}_{xx}^o + \bar{\sigma}_{x\theta}^o\bar{e}_{x\theta}^o + \bar{\sigma}_{xx}^i\bar{e}_{xx}^i + \bar{\sigma}_{x\theta}^i\bar{e}_{x\theta}^i]dzRd\theta dx$$

$$V_2^f = (1/2)\int_x\int_\theta\int_z [\sigma_{xx}^o e_{xx}^o + \sigma_{\theta\theta}^o e_{\theta\theta}^o + \sigma_{x\theta}^o e_{x\theta}^o + \sigma_{xx}^i e_{xx}^i + \sigma_{\theta\theta}^i e_{\theta\theta}^i + \sigma_{x\theta}^i e_{x\theta}^i]dzRd\theta dx \qquad \text{(A-32)}$$

It can be seen at this point that, since the membrane stresses are assumed to remain constant during buckling, it was necessary to include the second-order membrane-strain contribution for deflections in the thickness direction. As will be seen in the next section, the linear membrane-strain terms will cancel identical terms in the potential energy of the external load system. Thus, the instability occurs when the usual strain energy expression is balanced by the second-order membrane-strain energy resulting from deflections in the thickness direction. The same mathematical formulation would result if the middle surface were assumed to be inextensible, thus causing movement at the boundaries to obtain the deflected position. In that case, the second-order terms would apply to the potential energy of the external load system. Since it appears more feasible to have stretching of the middle surface, the first model was used in this analysis. Additional discussion concerning this formulation can be found in Refs. [43] and [44].

It is convenient to rewrite $V_1^f$ separating the linear and nonlinear portions of the strain. If $V_1^f$ is also integrated with respect to z and written in terms of stress resultants, ($\tilde{N}_x^o, \tilde{N}_{x\theta}^o, \tilde{N}_x^i$ and $\tilde{N}_{x\theta}^i$), it becomes:

$$V_1^f = \int_x \int_\theta [u,_x(\tilde{N}_x^o + \tilde{N}_x^i) + \{\zeta(u,_\theta - v \sin \alpha) + v,_x\}(\tilde{N}_{x\theta}^o + \tilde{N}_{x\theta}^i)] \, Rd\theta dx + \int_x \int_\theta \{(1/2)[\tilde{N}_x^o(\beta_x^o)^2 + \tilde{N}_x^i(\beta_x^i)^2] + \tilde{N}_{x\theta}^o \beta_x^o \beta_\theta^o + \tilde{N}_{x\theta}^i \beta_x^i \beta_\theta^i\} \, Rd\theta dx \qquad \text{(A-33)}$$

The initial prebuckling loads are assumed to be equally divided between the two facings; therefore, $\tilde{N}_x^o = \tilde{N}_x^i$ and $\tilde{N}_{x\theta}^o = \tilde{N}_{x\theta}^i$, and Eq. (A-33) can be rewritten

$$V_1^f = \int_x \int_\theta 2[u,_x \tilde{N}_x^o + \{\zeta(u,_\theta - v \sin \alpha) + v,_x\}\tilde{N}_{x\theta}^o] \, Rd\theta dx + \int_x \int_\theta \{\tilde{N}_x^o w,_x^2 + 2\tilde{N}_{x\theta}^o [w,_x w,_\theta - w,_x \, v \cos \alpha]\zeta\} Rd\theta dx \qquad \text{(A-34)}$$

It must be remembered that the values for $\tilde{N}_x^o$ and $\tilde{N}_{x\theta}^o$ appearing in Eq. (A-34) are the stress resultants for one facing only.

Substituting Eqs. (A-27) and (A-28) into the relationship for $V_2^f$ gives the following result:

$$V_2^f = (1/2)\int_x \int_\theta \int_z \{\bar{E}_x[(e_{xx}^o)^2 + (e_{xx}^i)^2] + \bar{E}_\theta[(e_{\theta\theta}^o)^2 + (e_{\theta\theta}^i)^2]$$

$$+ 2\,\bar{E}_x \nu_{\theta x}(e^o_{\theta\theta} e^o_{xx} + e^i_{\theta\theta} e^i_{xx})$$

$$+ G_{x\theta}[(e^o_{x\theta})^2 + (e^i_{x\theta})^2]\}\ dzRd\theta dx \qquad \text{(A-35)}$$

Integrating over z with the aid of the z-integrals listed in Appendix B, $V_2^f$ becomes

$$V_2^f = \int_x \int_\theta \bar{E}_x[2tu,_x^2 + 2\ th^2\Psi_{x,x}^2 + (8t^3w,_{xx}^2)/3$$

$$- 4ht^2\Psi_{x,x}w,_{xx}]\ Rd\theta dx + \int_x \int_\theta \bar{E}_\theta \zeta\{\ 2tv,_\theta^2 +$$

$$(8/3)t^3v,_\theta^2\zeta^2 \cos^2\alpha + 2th^2\Psi_{\theta,\theta}^2 + (8/3)t^3h^2\zeta^2\Psi_{\theta,\theta}^2 \cos^2\alpha + (8/3)t^3\zeta^2w,_{\theta\theta}^2$$

$$+ 2tu^2 \sin^2\alpha + 2h^2t\Psi_x^2 \sin^2\alpha + (8/3)t^3w,_x^2 \sin^2\alpha + 2tw^2 \cos^2\alpha +$$

$$8ht^2v,_\theta\Psi_{\theta,\theta}\zeta \cos\alpha + 4tuv,_\theta \sin\alpha + 4tv,_\theta w \cos\alpha - 4ht^2\zeta\Psi_{\theta,\theta}w,_{\theta\theta} +$$

$$4h^2t\Psi_x\Psi_{\theta,\theta} \sin\alpha - 4ht^2w,_x\Psi_{\theta,\theta} \sin\alpha - (16/3)v,_\theta w,_{\theta\theta}\zeta^2 \cos\alpha -$$

$$4ht^2\zeta\Psi_x w,_{\theta\theta} \sin\alpha + (16/3)t^3\zeta w,_x w,_{\theta\theta} \sin\alpha + 4ht^2\zeta v,_\theta\Psi_x \sin\alpha \cos\alpha$$

$$- (16/3)t^3\zeta v,_\theta w,_x \sin\alpha \cos\alpha + 4ht^2\zeta u\Psi_{\theta,\theta} \sin\alpha \cos\alpha + 4ht^2\zeta\Psi_{\theta,\theta}w \cos^2\alpha$$

$$+ 4tuw \sin\alpha \cos\alpha - 4ht^2\Psi_x w,_x \sin^2\alpha\ \}\ d\theta dx\ + \int_x \int_\theta 2\bar{E}_x\nu_{\theta x}\{2tu,_x v,_\theta +$$

$$2ht^2\zeta u,_x\Psi_{\theta,\theta} \cos\alpha + 2tuu,_x \sin\alpha + 2tu,_x w \cos\alpha + 2h^2t\Psi_{x,x}\Psi_{\theta,\theta} -$$

$$2ht^2\zeta\Psi_{x,x}w,_{\theta\theta} + 2ht^2\zeta v,_\theta\Psi_{x,x} \cos\alpha + 2h^2t\Psi_x\Psi_{x,x} \sin\alpha - 2ht^2\Psi_{x,x}w,_x \sin\alpha$$

$$- 2ht\ w,_{xx}\Psi_{\theta,\theta} + (8/3)t^3\zeta w,_{xx}w,_{\theta\theta} - (8/3)t^3\zeta w,_{xx}v,_\theta \cos\alpha -$$

$$2ht^2w,_{xx}\Psi_x \sin\alpha + (8/3)t^3w,_{xx}w,_x \sin\alpha\ \}\ d\theta dx + \int_x \int_\theta G_{x\theta}\zeta\ \{2tu,_\theta^2 +$$

$$2th^2\Psi_{x,\theta}^2 + 2tv^2 \sin^2\alpha + (32/3)t^3w,_{x\theta}^2 + 2th^2\Psi_\theta^2 \sin^2\alpha +$$

$$(32/3)t^3\zeta^2w,_\theta^2 \sin^2\alpha + 2tR^2v,_x^2 + 2th^2R^2\Psi_{\theta,x}^2 - 4tvu,_\theta \sin\alpha + 4tRu,_\theta v,_x -$$

$$4h^2t\Psi_{x,\theta}\Psi_\theta \sin\alpha + 4h^2tR\Psi_{x,\theta}\Psi_{\theta,x} - 4tRvv,_x \sin\alpha - 4h^2tR\Psi_\theta\Psi_{\theta,x} \sin\alpha -$$

$$8ht^2\Psi_{x,\theta}w,_{x\theta} + 8ht^2\zeta\Psi_{x,\theta}w,_\theta \sin\alpha + 8ht^2\Psi_\theta w,_{x\theta} \sin\alpha -$$

$$(64/3)t^3\zeta w,_\theta w,_{x\theta} \sin\alpha - 8ht^2R\Psi_{\theta,x}w,_{x\theta} - 8ht^2\zeta\Psi_\theta w,_\theta \sin^2\alpha +$$

$$8ht^2w,_\theta\Psi_{\theta,x} \sin\alpha + (32/3)t^3\zeta^2v^2 \sin^2\alpha \cos^2\alpha + (32/3)t^3h^2\zeta^2\Psi_\theta^2 \sin^2\alpha \cos^2\alpha$$

$$+ (8/3)t^3v,_x^2 \cos^2\alpha + (8/3)t^3h^2\Psi_{\theta,x}^2 \cos^2\alpha - 8ht^2\zeta\Psi_\theta u,_\theta \sin\alpha \cos\alpha +$$

$4ht^2 u,_\theta \Psi_{\theta},_x \cos\alpha - 8ht^2 \zeta v \Psi_x,_\theta \sin\alpha\cos\alpha + 4ht^2 v,_x \Psi_x,_\theta \cos\alpha +$

$(64/3)t^3 \zeta v w,_{x\theta} \sin\alpha\cos\alpha - (32/3)t^3 v,_x w,_{x\theta} \cos\alpha + 16ht^2 \zeta v \Psi_\theta \sin^2\alpha\cos\alpha$

$- 12ht^2 v \Psi_{\theta},_x \sin\alpha\cos\alpha - 12ht^2 v,_x \Psi_\theta \sin\alpha\cos\alpha - (64/3)t^3 \zeta^2 v w,_\theta \sin^2\alpha\cos\alpha$

$+ (32/3)t^3 \zeta w,_\theta v,_x \sin\alpha\cos\alpha - (32/3)t^3 \zeta v v,_x \cos^2\alpha\sin\alpha -$

$(32/3)h^2 t^3 \zeta \Psi_\theta \Psi_{\theta},_x \sin\alpha\cos^2\alpha + 8ht^2 R v,_x \Psi_{\theta},_x \cos\alpha \}d\theta dx \qquad \text{(A-36)}$

## A.5 Potential Energy of External Forces

For convenience in obtaining the potential energy of the external forces ($\tilde{N}_x$ and $\tilde{N}_{x\theta}$), it is assumed that the small end does not move. The total displacement for u is then,

$$u_T = -\int_0^L u,_x \, dx \qquad \text{(A-37)}$$

Therefore, the potential energy due to $\tilde{N}_x$ is

$$U_1 = -\int_x \int_\theta \tilde{N}_x u,_x dxRd\theta \qquad \text{(A-38)}$$

In a similar manner for $\tilde{N}_{x\theta}$,

$$U_2 = -\int_x \int_\theta \{\zeta[u,_\theta - v\sin\alpha] + v,_x\}\tilde{N}_{x\theta} dxRd\theta \qquad \text{(A-39)}$$

Since $\tilde{N}_x = \tilde{N}_x^o + \tilde{N}_x^i$ and $\tilde{N}_{x\theta} = \tilde{N}_{x\theta}^o + \tilde{N}_{x\theta}^i$, these last two expressions cancel the linear membrane strain energy terms given in Section A.4.

## A.6 Stress Resultants

For the loading shown in Fig. A.3, the stress resultants prior to buckling are assumed to be given by the membrane relations* [8,12,20]

*Under the membrane stress state which exists prior to buckling, the conical shell is assumed to possess the geometry shown in Figs. A.1 and A.2. For a discussion of the influence of prebuckling deformation on the buckling load, see Ref. [10].

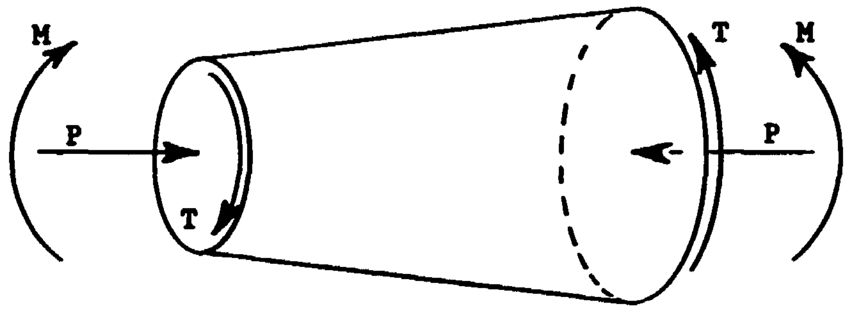


Figure A.3 - Loads Considered in the Analysis

$$\tilde{N}_x = [-P/2 + M(\cos\theta)/R]/(\pi R \cos\alpha)$$
$$\tilde{N}_{x\theta} = T/(2\pi R^2) \qquad \text{(A-40)}$$

The single-facing values which appear in Eq. (A-34) are then

$$\tilde{N}_x^o = \tilde{N}_x/2$$
$$\tilde{N}_{x\theta}^o = \tilde{N}_{x\theta}/2 \qquad \text{(A-41)}$$

For the combined loading case, it is necessary to assume that P, M, and T can be represented by a common factor $\lambda$, such that

$$P = \lambda P' \quad ; \quad M = \lambda M' \quad ; \text{ and } \quad T = \lambda T' \qquad \text{(A-42)}$$

This common factor represents the eigenvalue for the combined loading case.

## A.7 Total Energy

Combining Eqs. (A-31),(A-34),(A-36),(A-38),(A-39),(A-41), and (A-42), and substituting the coefficients listed in Appendix B, the total energy becomes as follows:

$$
\begin{aligned}
\int_x \int_\theta [ & B_1 \zeta w^2 + B_2 R w_{,x}^2 + B_3 \zeta w_{,x}^2 + B_4 \zeta w_{,\theta}^2 + B_5 \zeta^3 w_{,\theta}^2 + B_6 R w_{,xx}^2 + B_7 \zeta^3 w_{,\theta\theta}^2 + \\
& B_8 \zeta w_{,x\theta}^2 + B_9 \zeta u^2 + B_{10} R u_{,x}^2 + B_{11} \zeta u_{,\theta}^2 + B_{12} \zeta v^2 + B_{13} \zeta^3 v^2 + B_{14} R v_{,x}^2 + \\
& B_{15} \zeta v_{,x}^2 + B_{16} \zeta v_{,\theta}^2 + B_{17} \zeta^3 v_{,\theta}^2 + B_{18} R \Psi_x^2 + B_{19} \zeta \Psi_x^2 + B_{20} R \Psi_{x,x}^2 + B_{21} \zeta \Psi_{x,\theta}^2 \\
& + B_{22} R \Psi_\theta^2 + B_{23} \zeta \Psi_\theta^2 + B_{24} \zeta^3 \Psi_\theta^2 + B_{25} R \Psi_{\theta,x}^2 + B_{26} \zeta \Psi_{\theta,x}^2 + B_{27} \zeta \Psi_{\theta,\theta}^2 + \\
& B_{28} \zeta^3 \Psi_{\theta,\theta}^2 + B_{29} \zeta w u + B_{30} w u_{,x} + B_{31} \zeta w v_{,\theta} + B_{32} \zeta^2 w \Psi_{\theta,\theta} + B_{33} w_{,x} w_{,xx} \\
& + B_{34} \zeta^2 w_{,x} w_{,\theta\theta} + B_{35} \zeta^2 w_{,x} v_{,\theta} + B_{36} R w_{,x} \Psi_x + B_{37} \zeta w_{,x} \Psi_x + B_{38} w_{,x} \Psi_{x,x} + \\
& B_{39} \zeta w_{,x} \Psi_{\theta,\theta} + B_{40} \zeta^2 w_{,\theta} w_{,x\theta} + B_{41} \zeta w_{,\theta} v + B_{42} \zeta^3 w_{,\theta} v + B_{43} \zeta^2 w_{,\theta} v_{,x} + \\
& B_{44} \zeta^2 w_{,\theta} \Psi_{x,\theta} + B_{45} w_{,\theta} \Psi_\theta + B_{46} \zeta^2 w_{,\theta} \Psi_\theta + B_{47} \zeta w_{,\theta} \Psi_{\theta,x} + B_{48} \zeta w_{,xx} w_{,\theta\theta} + \\
& B_{49} \zeta w_{,xx} v_{,\theta} + B_{50} w_{,xx} \Psi_x + B_{51} R w_{,xx} \Psi_{x,x} + B_{52} w_{,xx} \Psi_{\theta,\theta} + B_{53} \zeta^3 w_{,\theta\theta} v_{,\theta} \\
& + B_{54} \zeta^2 w_{,\theta\theta} \Psi_x + B_{55} \zeta w_{,\theta\theta} \Psi_{x,x} + B_{56} \zeta^2 w_{,\theta\theta} \Psi_{\theta,\theta} + B_{57} \zeta^2 w_{,x\theta} v + \\
& B_{58} \zeta w_{,x\theta} v_{,x} + B_{59} \zeta w_{,x\theta} \Psi_{x,\theta} + B_{60} \zeta w_{,x\theta} \Psi_\theta + B_{61} w_{,x\theta} \Psi_{\theta,x} + B_{62} u u_{,x} + \\
& B_{63} \zeta u v_{,\theta} + B_{64} \zeta^2 u \Psi_{\theta,\theta} + B_{65} u_{,x} v_{,\theta} + B_{66} \zeta u_{,x} \Psi_{\theta,\theta} + B_{67} \zeta u_{,\theta} v + \\
& B_{68} u_{,\theta} v_{,x} + B_{69} \zeta^2 u_{,\theta} \Psi_\theta + B_{70} \zeta u_{,\theta} \Psi_{\theta,x} + B_{71} v v_{,x} + B_{72} \zeta^2 v v_{,x} + \\
& B_{73} \zeta^2 v \Psi_{x,\theta} + B_{74} v \Psi_\theta + B_{75} \zeta^2 v \Psi_\theta + B_{76} \zeta v \Psi_{\theta,x} + B_{77} \zeta v_{,x} \Psi_{x,\theta} + B_{78} \zeta v_{,x} \Psi_\theta \\
& + B_{79} v_{,x} \Psi_{\theta,x} + B_{80} \zeta^2 v_{,\theta} \Psi_x + B_{81} \zeta v_{,\theta} \Psi_{x,x} + B_{82} \zeta^2 v_{,\theta} \Psi_{\theta,\theta} + B_{83} \Psi_x \Psi_{x,x} \\
& + B_{84} \zeta \Psi_x \Psi_{\theta,\theta} + B_{85} \Psi_{x,x} \Psi_{\theta,\theta} + B_{86} \zeta \Psi_{x,\theta} \Psi_\theta + B_{87} \Psi_{x,\theta} \Psi_{\theta,x} + B_{88} \Psi_\theta \Psi_{\theta,x} \\
& + B_{89} \zeta^2 \Psi_\theta \Psi_{\theta,x} ] \, d\theta dx = \lambda \int_x \int_\theta [\bar{V}_1 w_{,x}^2 + \bar{V}_2 \zeta w_{,x}^2 \cos\theta + \bar{V}_3 \zeta^2 w_{,x} w_{,\theta} + \\
& \bar{V}_4 \zeta^2 w_{,x} v] \, d\theta dx \qquad \text{(A-43)}
\end{aligned}
$$

APPENDIX B

IDENTIFICATION OF COEFFICIENTS AND INTEGRAL FORMS

## B.1 Coefficients

The following coefficients appear in Eq. (A-43):

$$B_1 = 2t\bar{E}_\theta \cos^2\alpha$$

$$B_2 = hK_x G_{zx}$$

$$B_3 = (8/3)t^3\bar{E}_\theta \sin^2\alpha$$

$$B_4 = hK_\theta G_{\theta z}$$

$$B_5 = (32/3)t^3 G_{x\theta} \sin^2\alpha$$

$$B_6 = (8/3)t^3\bar{E}_x$$

$$B_7 = (8/3)t^3\bar{E}_\theta$$

$$B_8 = (32/3)t^3 G_{x\theta}$$

$$B_9 = 2t\bar{E}_\theta \sin^2\alpha$$

$$B_{10} = 2t\bar{E}_x$$

$$B_{11} = 2tG_{x\theta}$$

$$B_{12} = hK_\theta G_{\theta z} \cos^2\alpha + 2tG_{x\theta} \sin^2\alpha$$

$$B_{13} = (32/3)t^3 G_{x\theta} \sin^2\alpha \cos^2\alpha$$

$$B_{14} = 2tG_{x\theta}$$

$$B_{15} = (8/3)t^3 G_{x\theta} \cos^2\alpha$$

$$B_{16} = 2t\bar{E}_\theta$$

$$B_{17} = (8/3)t^3\bar{E}_\theta \cos^2\alpha$$

$$B_{18} = hK_xG_{zx}$$

$$B_{19} = 2h^2t\bar{E}_\theta \sin^2\alpha$$

$$B_{20} = 2th^2\bar{E}_x$$

$$B_{21} = 2th^2G_{x\theta}$$

$$B_{22} = hK_\theta G_{\theta z}$$

$$B_{23} = 2th^2G_{x\theta} \sin^2\alpha$$

$$B_{24} = (32/3)t^3h^2G_{x\theta} \sin^2\alpha \cos^2\alpha$$

$$B_{25} = 2th^2G_{x\theta}$$

$$B_{26} = (8/3)t^3h^2G_{x\theta} \cos^2\alpha$$

$$B_{27} = 2th^2\bar{E}_\theta$$

$$B_{28} = (8/3)t^3h^2\bar{E}_\theta \cos^2\alpha$$

$$B_{29} = 4t\bar{E}_\theta \sin\alpha \cos\alpha$$

$$B_{30} = 4t\bar{E}_x\nu_{\theta x} \cos\alpha$$

$$B_{31} = 4t\bar{E}_\theta \cos\alpha$$

$$B_{32} = 4ht^2\bar{E}_\theta \cos^2\alpha$$

$$B_{33} = (16/3)t^3\bar{E}_x\nu_{\theta x} \sin\alpha$$

$$B_{34} = (16/3)t^3\bar{E}_\theta \sin\alpha$$

$$B_{35} = -(16/3)t^3\bar{E}_\theta \sin\alpha \cos\alpha$$

$$B_{36} = 2hK_xG_{zx}$$

$$B_{37} = -4ht^2\bar{E}_\theta \sin^2\alpha$$

$$B_{38} = -4ht^2\bar{E}_x\nu_{\theta x} \sin\alpha$$

$$B_{39} = -4ht^2\bar{E}_\theta \sin\alpha$$

$$B_{40} = -(64/3)t^3G_{x\theta} \sin\alpha$$

$$B_{41} = -2hK_\theta G_{\theta z} \cos\alpha$$

$$B_{42} = -(64/3)t^3G_{x\theta} \sin^2\alpha \cos\alpha$$

$B_{43} = (32/3)t^3 G_{x\theta} \sin\alpha \cos\alpha$

$B_{44} = 8ht^2 G_{x\theta} \sin\alpha$

$B_{45} = 2hK_\theta G_{\theta z}$

$B_{46} = -8ht^2 G_{x\theta} \sin^2\alpha$

$B_{47} = 8ht^2 G_{x\theta} \sin\alpha$

$B_{48} = (16/3)t^3 \bar{E}_x \nu_{\theta x}$

$B_{49} = -(16/3)t^3 \bar{E}_x \nu_{\theta x} \cos\alpha$

$B_{50} = -4ht^2 \bar{E}_x \nu_{\theta x} \sin\alpha$

$B_{51} = -4ht^2 \bar{E}_x$

$B_{52} = -4ht^2 \bar{E}_x \nu_{\theta x}$

$B_{53} = -(16/3)t^3 \bar{E}_\theta \cos\alpha$

$B_{54} = -4ht^2 \bar{E}_\theta \sin\alpha$

$B_{55} = -4ht^2 \bar{E}_x \nu_{\theta x}$

$B_{56} = -4ht^2 \bar{E}_\theta$

$B_{57} = (64/3)t^3 G_{x\theta} \sin\alpha \cos\alpha$

$B_{58} = -(32/3)t^3 G_{x\theta} \cos\alpha$

$B_{59} = -8ht^2 G_{x\theta}$

$B_{60} = 8ht^2 G_{x\theta} \sin\alpha$

$B_{61} = -8ht^2 G_{x\theta}$

$B_{62} = 4t\bar{E}_x \nu_{\theta x} \sin\alpha$

$B_{63} = 4t\bar{E}_\theta \sin\alpha$

$B_{64} = 4ht^2 \bar{E}_\theta \sin\alpha \cos\alpha$

$B_{65} = 4t\bar{E}_x \nu_{\theta x}$

$B_{66} = 4ht^2 \bar{E}_x \nu_{\theta x} \cos\alpha$

$B_{67} = -4tG_{x\theta} \sin\alpha$

$B_{68} = 4tG_{x\theta}$

$B_{69} = -8ht^2G_{x\theta} \sin\alpha \cos\alpha$

$B_{70} = 4ht^2G_{x\theta} \cos\alpha$

$B_{71} = -4tG_{x\theta} \sin\alpha$

$B_{72} = -(32/3)t^3G_{x\theta} \sin\alpha \cos^2\alpha$

$B_{73} = -8ht^2G_{x\theta} \sin\alpha \cos\alpha$

$B_{74} = -2hK_\theta G_{\theta z} \cos\alpha$

$B_{75} = 16ht^2G_{x\theta} \sin^2\alpha \cos\alpha$

$B_{76} = -12ht^2G_{x\theta} \sin\alpha \cos\alpha$

$B_{77} = 4ht^2G_{x\theta} \cos\alpha$

$B_{78} = -12ht^2G_{x\theta} \sin\alpha \cos\alpha$

$B_{79} = 8ht^2G_{x\theta} \cos\alpha$

$B_{80} = 4ht^2\bar{E}_\theta \sin\alpha \cos\alpha$

$B_{81} = 4ht^2\bar{E}_x\nu_{\theta x} \cos\alpha$

$B_{82} = 8ht^2\bar{E}_\theta \cos\alpha$

$B_{83} = 4h^2t\bar{E}_x\nu_{\theta x} \sin\alpha$

$B_{84} = 4h^2t\bar{E}_\theta \sin\alpha$

$B_{85} = 4h^2t\bar{E}_x\nu_{\theta x}$

$B_{86} = -4h^2tG_{x\theta} \sin\alpha$

$B_{87} = 4h^2tG_{x\theta}$

$B_{88} = -4h^2tG_{x\theta} \sin\alpha$

$B_{89} = -(32/3)h^2t^3G_{x\theta} \sin\alpha \cos^2\alpha$

$\bar{V}_1 = P'/(4\pi \cos\alpha)$

$\bar{V}_2 = -M'/(2\pi \cos\alpha)$

$\bar{V}_3 = -T'/(2\pi)$

$\bar{V}_4 = T' \cos\alpha /(2\pi)$

## B.2 z-Integrals

The following integrals are used repeatedly in obtaining Eq. (A-36):

$$\int_{h}^{h+2t} dz = \int_{-h-2t}^{-h} dz = 2t \tag{B-1a}$$

$$\int_{h}^{h+2t} z dz = -\int_{-h-2t}^{-h} z dz = 2t(h + t) \tag{B-1b}$$

$$\int_{h}^{h+2t} z^2 dz = \int_{-h-2t}^{-h} z^2 dz = t(8t^2 + 12ht + 6h^2)/3 \tag{B-1c}$$

## B.3 θ-Integrals

The θ-integrals which are used throughout this analysis are tabulated here. The $n_i$ which appear in these integrals represent the number of circumferential waves. They are always integers. Most of these integrals are standard integrals which appear in most tables of integrals and are repeated here for convenience only. The evaluation of Eq. (B-2d) is given in Appendix C. The integrals are as follows:

$$\begin{aligned}\int_0^{2\pi} \cos n_i\theta \cos n_j\theta \, d\theta &= 0 \quad \text{for } n_i \neq n_j \\ &= \pi \quad \text{for } n_i = n_j\end{aligned} \tag{B-2a}$$

$$\begin{aligned}\int_0^{2\pi} \sin n_i\theta \sin n_j\theta \, d\theta &= 0 \quad \text{for } n_i \neq n_j \\ &= \pi \quad \text{for } n_i = n_j\end{aligned} \tag{B-2b}$$

$$\int_0^{2\pi} \sin n_i\theta \cos n_j\theta \, d\theta = 0 \tag{B-2c}$$

$$\begin{aligned}\int_0^{2\pi} \cos\theta \cos n_i\theta \cos n_j\theta \, d\theta &= \pi/2 \quad \text{for } n_i - n_j = \pm 1 \\ &= 0 \quad \text{otherwise}\end{aligned} \tag{B-2d}$$

## B.4 x-Integrals

There are fifteen integral forms for x which are used here repeatedly. Due to the x dependence of R, several of these integrals cannot be integrated in closed form; however, they can be transformed into a form for which there is a readily available computer algorithm. The evaluation or transformation for all of the x-integrals is given in Appendix C. Only the forms are listed here.

Since these integrals are needed repeatedly for a particular set of axial half-wave numbers, it is convenient to use an array type identification. Since there are two wave numbers appearing in each of these integrals, a two-dimensional array is needed for storage. The first "I" in the array title signifies "integral". The final number which appears in the title represents the trigonometric form: 1 - sine, sine; 2 - cosine, cosine; and 3 - sine, cosine. The central letters (R, IR, IRR, and IRRR) denote how R appears. For example, IRR denotes "inverse R squared". The integral forms are as follows:

$$I1(j,k) = \int_0^L \sin(m_j \pi x/L)\ \sin(m_k \pi x/L)\ dx$$

$$I2(j,k) = \int_0^L \cos(m_j \pi x/L)\ \cos(m_k \pi x/L)\ dx$$

$$I3(j,k) = \int_0^L \sin(m_j \pi x/L)\ \cos(m_k \pi x/L)\ dx$$

$$IR1(j,k) = \int_0^L R\ \sin(m_j \pi x/L)\ \sin(m_k \pi x/L)\ dx$$

$$IR2(j,k) = \int_0^L R\ \cos(m_j \pi x/L)\ \cos(m_k \pi x/L)\ dx$$

$$IR3(j,k) = \int_0^L R \sin(m_j \pi x/L) \cos(m_k \pi x/L)\, dx$$

$$IIR1(j,k) = \int_0^L \zeta \sin(m_j \pi x/L) \sin(m_k \pi x/L)\, dx$$

$$IIR2(j,k) = \int_0^L \zeta \cos(m_j \pi x/L) \cos(m_k \pi x/L)\, dx$$

$$IIR3(j,k) = \int_0^L \zeta \sin(m_j \pi x/L) \cos(m_k \pi x/L)\, dx$$

$$IIRR1(j,k) = \int_0^L \zeta^2 \sin(m_j \pi x/L) \sin(m_k \pi x/L)\, dx$$

$$IIRR2(j,k) = \int_0^L \zeta^2 \cos(m_j \pi x/L) \cos(m_k \pi x/L)\, dx$$

$$IIRR3(j,k) = \int_0^L \zeta^2 \sin(m_j \pi x/L) \cos(m_k \pi x/L)\, dx$$

$$IIRRR1(j,k) = \int_0^L \zeta^3 \sin(m_j \pi x/L) \sin(m_k \pi x/L)\, dx$$

$$IIRRR2(j,k) = \int_0^L \zeta^3 \cos(m_j \pi x/L) \cos(m_k \pi x/L)\, dx$$

$$IIRRR3(j,k) = \int_0^L \zeta^3 \sin(m_j \pi x/L) \cos(m_k \pi x/L)\, dx$$

## APPENDIX C

## EVALUATION OF INTEGRALS

Of the θ-integrals, only one requires any special attention. From the bending initial load term of Eq. (A-36), an additional cos θ appears. The resulting θ-integral is not included in the usual integral tables. Thus, it is evaluated here.

A closed-form solution could not be found for several of the x-integrals; however, it was possible to transform them into the forms $\int [(\sin x)/x]dx$ and $\int [(\cos x)/x]dx$ for which IBM has a standard algorithm, SICI [45], which is based on a series solution. It was found that for some half-wave number combinations, the symmetric integral forms sin( )-sin( ) and cos( )-cos( ) gave considerably different results when the $m_i$ and $m_j$ were interchanged. This algorithm was rewritten in double precision, thus correcting the problem.

The actual algorithm used for each integral is contained in the subroutine INTEG listed in Appendix H.

The integrals I1, I2 and I3 are listed in most common integral tables.

The evaluation and transformation of the integrals follows;

$$\int_0^{2\pi} \cos\theta \cos n_i\theta \cos n_j\theta \, d\theta = \tag{C-1}$$

$$(1/2)\int_0^{2\pi} \cos\theta \, [\cos(n_i + n_j)\theta + \cos(n_i - n_j)\theta]d\theta$$

$$= \pi/2 \qquad \text{for } n_i - n_j = \pm 1$$

$$= \pi \qquad \text{for } n_i = 1,\ n_j = 0$$

$$= 0 \qquad \text{otherwise}$$

$$I1(j,k) = \int_0^L \sin(m_j \pi x/L)\ \sin(m_k \pi x/L)\ dx \tag{C-2}$$

$$= L/2 \quad \text{if } m_j = m_k$$

$$= 0 \quad \text{if } m_j \neq m_k \text{ or if } m_j = 0$$

$$I2(j,k) = \int_0^L \cos(m_j \pi x/L)\ \cos(m_k \pi x/L)\ dx \tag{C-3}$$

$$= L/2 \quad \text{if } m_j = m_k,\ m_j \neq 0$$

$$= L \quad \text{if } m_j = m_k = 0$$

$$= 0 \quad \text{if } m_j \neq m_k$$

$$I3(j,k) = \int_0^L \sin(m_j \pi x/L)\ \cos(m_k \pi x/L)\ dx \tag{C-4}$$

$$= 2Lm_j/[\pi(m_j^2 - m_k^2)] \text{ for } m_j \neq m_k \quad \text{and}$$

$$m_j \pm m_k = \text{odd integer}$$

$$= 0 \quad \text{if } m_j = 0$$

$$= 0 \quad \text{otherwise}$$

$$IR1(j,k) = \int_0^L R\ \sin(m_j \pi x/L)\ \sin(m_k \pi x/L)\ dx \tag{C-5}$$

$$= R_o \int_0^L \sin(m_j \pi x/L)\ \sin(m_k \pi x/L)\ dx$$

$$+ \sin\alpha \int_0^L x\ \sin(m_j \pi x/L)\ \sin(m_k \pi x/L)\ dx$$

$$= R_o\ I1(j,k) \qquad \text{for } \sin\alpha = 0$$

$$= 0 \qquad \text{if } m_j \text{ or } m_k = 0$$

It is convenient to define the expressions

$$\Delta \equiv (m_j - m_k)\pi/L \quad \text{and} \quad \Delta' \equiv (m_j + m_k)\pi/L \tag{C-6}$$

For sin α ≠ 0, it is necessary to integrate by parts. If $m_j = m_k \neq 0$

$$IR1(j,k) = R_o L/2 + (\sin\alpha)L^2/4$$

For $m_j \neq m_k$,

$$IR1(j,k) = (R_o/2 + L\sin\alpha)[(\sin\Delta x)/\Delta - (\sin\Delta' x)/\Delta']\Big|_0^L$$

$$- [(\sin\alpha)/2]\int_0^L [(\sin\Delta x)/\Delta - (\sin\Delta' x)/\Delta']dx$$

$$= -4m_j m_k L^2(\sin\alpha)[\pi^2(m_j^2 - m_k^2)^2]^{-1} \qquad \text{for } m_j \pm m_k = \text{odd}$$

$$= 0 \qquad \text{otherwise}$$

As in the previous case,

$$IR2(j,k) = \int_0^L R\cos(m_j\pi x/L)\cos(m_k\pi x/L)\,dx \tag{C-7}$$

$$= R_o\; I2(j,k) \qquad \text{for } \sin\alpha = 0$$

$$= R_o L/2 + (\sin\alpha)L^2/4 \qquad \text{for } m_j = m_k \neq 0$$

$$= R_o L - (\sin\alpha)L^2/2 \qquad \text{for } m_j = m_k = 0$$

$$= -2L^2(m_j^2 + m_k^2)(\sin\alpha)[\pi^2(m_j^2 - m_k^2)^2]^{-1} \quad \text{for } m_j \neq m_k, \text{ and}$$

$$m_j \pm m_k = \text{odd}$$

Similarly,

$$IR3(j,k) = \int_0^L R\sin(m_j\pi x/L)\cos(m_k\pi x/L)\,dx \tag{C-8}$$

$$= R_o\; I3(j,k) \quad \text{for } \sin\alpha = 0$$

$$= 0 \qquad \text{for } m_j = m_k \text{ or } m_j = 0$$

$$= (Lm_j/\pi)(2R_o + m_j L\sin\alpha)[m_j^2 - m_k^2]^{-1}$$

$$\text{for } m_j \pm m_k = \text{odd}$$

$$= -m_j L^2 \sin\alpha\;[\pi(m_j^2 - m_k^2)]^{-1}$$

$$\text{for } m_j \pm m_k = \text{even}$$

$$IIR1(j,k) = \int_0^L \zeta \ \sin(m_j \pi x/L) \ \sin(m_k \pi x/L) \ dx \qquad \text{(C-9a)}$$

$$= I1(j,k)/R_o \quad \text{for } \sin\alpha = 0$$

$$= 0 \qquad\qquad \text{if } m_j \text{ or } m_k = 0$$

If $\sin\alpha \neq 0$,

$$IIR1(j,k) = (1/2)\int_0^L \zeta[\cos\ (m_j - m_k)\pi x/L - \cos\ (m_j + m_k)\pi x/L]\ dx$$

Making a change of variable such that $u = R_o + \sin\alpha$ , $du = \sin\alpha\ dx$ and $x = (u - R_o)/\sin\alpha$ ,

$$IIR1 = (1/2\ \sin\alpha)\int\{[\cos\ (u - R_o)\Delta/\sin\alpha\ - \cos\ (u - R_o)\Delta'/\sin\alpha]/u\}\ du$$

$$= (1/2\ \sin\alpha)\int\{[\cos Au \cos AR_o + \sin Au \sin AR_o - \cos Bu \cos BR_o -$$

$$\sin Bu \sin BR_o]/u\}\ du$$

where $A = \Delta/L$ and $B = \Delta'/L$.

Substituting $C_A = \cos AR_o$, $S_A = \sin AR_o$, $C_B = \cos BR_o$, and $S_B = \sin BR_o$ and making the variable changes $W_1 = Au$ and $W_2 = Bu$,

$$IIR1 = (1/2\ \sin\alpha)\{C_A\int[(\cos W_1)/W_1]dW_1 + S_A\int[(\sin W_1)/W_1]dW_1$$

$$- C_B\int[(\cos W_2)/W_2]dW_2 - S_B\int[(\sin W_2)/W_2]dW_2\} \qquad \text{(C-9b)}$$

For $m_j = m_k$, the $C_A$ and $S_A$ terms are replaced by $\ln[(R_o + L\sin\alpha)/R_o]$. The integral limits are $W_1 = A(R_o + L\sin\alpha)$ ; $W_2 = B(R_o + L\sin\alpha)$ at $x = L$, and $W_1 = AR_o$ ; $W_2 = BR_o$ at $x = 0$. Eq. (C-9b) can be solved using the IBM algorithm, SICI [45].

Following the same procedure as used for IIR1(j,k),

$$IIR2(j,k) = \int_0^L \zeta \cos(m_j\pi x/L) \cos(m_k\pi x/L)\, dx \tag{C-10}$$

$$= I2(j,k)/R_o \qquad \text{for } \sin\alpha = 0$$

$$= \{\ln[(R_o + L \sin\alpha)/R_o]\}/\sin\alpha \qquad \text{for } m_j = m_k = 0$$

If $\sin\alpha \neq 0$,

$$IIR2(j,k) = (1/2 \sin\alpha)\{C_A\int[(\cos W_1)/W_1]dW_1 + S_A\int[(\sin W_1)/W_1]dW_1$$

$$+ C_B\int[(\cos W_2)/W_2]dW_2 + S_B\int[(\sin W_2)/W_2]dW_2\}$$

where the integral limits are the same as for IIR1 and the $C_A$ and $S_A$ terms are again replaced by $\ln[(R_o + L \sin\alpha)/R_o]$ when $m_j = m_k$.

$$IIR3(j,k) = \int_0^L \zeta \sin(m_j\pi x/L) \cos(m_k\pi x/L)\, dx \tag{C-11}$$

$$= I3(j,k)/R_o \qquad \text{for } \sin\alpha = 0$$

$$= 0 \qquad \text{for } m_j = 0$$

If $\sin\alpha \neq 0$,

$$IIR3(j,k) = (1/2 \sin\alpha)\{C_A\int[(\sin W_1)/W_1]dW_1 - S_A\int[(\cos W_1)/W_1]dW_1$$

$$+ C_B\int[(\sin W_2)/W_2]dW_2 - S_B\int[(\cos W_2)/W_2]dW_2\}$$

where the $C_A$ and $S_A$ terms are replaced by zero when $m_j = m_k$. The limits for $W_1$ and $W_2$ are again the same as used in IIR1.

$$IIRR1(j,k) = \int_0^L \zeta^2 \sin(m_j\pi x/L) \sin(m_k\pi x/L)\, dx \tag{C-12}$$

$$= I1(j,k)/R_o^2 \quad \text{for } \sin\alpha = 0$$

For $\sin\alpha \neq 0$, IIRR1 can be integrated by parts. Then

$$\text{IIRR1}(j,k) = (1/\sin\alpha)\{ -\zeta \sin(m_j\pi x/L)\sin(m_k\pi x/L)\Big|_0^L$$

$$+ \int_0^L \zeta[\ (m_j\pi/L)\cos(m_j\pi x/L)\sin(m_k\pi x/L)$$

$$+ (m_k\pi/L)\sin(m_j\pi x/L)\cos(m_k\pi x/L)]\ dx\}$$

$$= (m_j\pi/L\sin\alpha)\int_0^L \zeta\cos(m_j\pi x/L)\sin(m_k\pi x/L)\ dx$$

$$+ (m_k\pi/L\sin\alpha)\int_0^L \zeta\sin(m_j\pi x/L)\cos(m_k\pi x/L)\ dx$$

$$= (\pi/L\sin\alpha)[m_j\text{IIR3}(k,j) + m_k\text{IIR3}(j,k)]$$

Using the same procedure as above,

$$\text{IIRR2}(j,k) = \int_0^L \zeta^2\cos(m_j\pi x/L)\cos(m_k\pi x/L)\ dx \qquad \text{(C-13)}$$

$$= \text{I2}(j,k)/R_o^2 \quad \text{for } \sin\alpha = 0$$

$$= (1/\sin\alpha)[1/R_o - (-1)^{(m_j + m_k)}/(R_o + L\sin\alpha)]$$

$$- (\pi/L\sin\alpha)[m_j\text{IIR3}(j,k) + m_k\text{IIR3}(k,j)] \quad \text{for } \sin\alpha = 0$$

$$\text{IIRR3}(j,k) = \int_0^L \zeta^2\sin(m_j\pi x/L)\cos(m_k\pi x/L)\ dx \qquad \text{(C-14)}$$

$$= \text{I3}(j,k)/R_o^2 \quad \text{for } \sin\alpha = 0$$

$$= (\pi/L\sin\alpha)[m_j\text{IIR2}(j,k) - m_k\text{IIR1}(j,k)] \quad \text{for } \sin\alpha \neq 0$$

$$\text{IIRRR1}(j,k) = \int_0^L \zeta^3\sin(m_j\pi x/L)\sin(m_k\pi x/L)\ dx \qquad \text{(C-15)}$$

$$= \text{I1}(j,k)/R_o^3 \quad \text{for } \sin\alpha = 0$$

As in IIRR1 when $\sin\alpha \neq 0$, IIRRR1 can be integrated by parts. Then,

$$\text{IIRRR1}(j,k) = (1/\sin\alpha)\{ -(\zeta^2/2)\sin(m_j\pi x/L)\sin(m_k\pi x/L)\Big|_0^L$$

$$+ (m_j\pi/2L)\int_0^L \zeta^2\cos(m_j\pi x/L)\sin(m_k\pi x/L)\ dx$$

$$+ (m_k\pi/2L)\int_0^L \zeta^2 \sin(m_j\pi x/L)\cos(m_k\pi x/L)\,dx$$

$$= (\pi/2L \sin\alpha)[m_j IIRR3(k,j) + m_k IIRR3(j,k)]$$

Using the same procedure as in IIRRR1(j,k),

$$IIRRR2(j,k) = \int_0^L \zeta^3 \cos(m_j\pi x/L)\cos(m_k\pi x/L)\,dx \tag{C-16}$$

$$= I2(j,k)/R_o^3 \quad \text{for } \sin\alpha = 0$$

$$= (1/2\sin\alpha)[1/R_o^2 - (-1)^{(m_j + m_k)}/(R_o + L\sin\alpha)^2]$$

$$- (\pi/2L\sin\alpha)[m_j IIRR3(j,k) + m_k IIRR3(k,j)] \quad \text{for } \sin\alpha \neq 0$$

$$IIRRR3(j,k) = \int_0^L \zeta^3 \sin(m_j\pi x/L)\cos(m_k\pi x/L)\,dx \tag{C-17}$$

$$= I3(j,k)/R_o^3 \quad \text{for } \sin\alpha = 0$$

$$= (\pi/2L\sin\alpha)[m_j IIRR2(j,k) - m_k IIRR1(j,k)] \quad \text{for } \sin\alpha \neq 0$$

## APPENDIX D

## SIMPLE-SUPPORT MATRIX ELEMENTS

### D.1 w - w Stiffness Partition

The coefficient of $D^w_{\ell j}$ in the $(\partial\ /\partial D^w_{\ell i})$ row is as follows:

$$2(B_1 + n^2B_4)IIR1(i,j) + 2(n^2B_5 + n^4B_7)IIRRR1(i,j) + 2(\pi^2/L^2)B_2m_im_j \cdot IR2(i,j) + 2(\pi^2/L^2)(B_3 + n^2B_8)m_im_jIIR2(i,j) + 2(\pi^4/L^4)B_6m_i^2m_j^2IR1(i,j) + (\pi/L)n^2(B_{40} - B_{34})m_jIIRR3(i,j) + (\pi/L)n^2(B_{40} - B_{34})m_iIIRR3(j,i) + (\pi^2/L^2)n^2B_{48}(m_j^2 + m_i^2)IIR1(i,j) - (\pi^3/L^3)B_{33}m_im_j^2I3(j,i) - (\pi^3/L^3)B_{33}m_i^2m_jI3(i,j)$$

### D.2 w - v Stiffness Partition

The coefficient of $D^v_{\ell j}$ in the $(\partial\ /\partial D^w_{\ell i})$ row is as follows:

$$n(B_{31} - B_{41})IIR1(i,j) - n(B_{42} + n^2B_{53})IIRRR1(i,j) - nB_{43}(\pi/L)m_jIIRR3(i,j) + n(B_{35} - B_{57})(\pi/L)m_iIIRR3(j,i) - nB_{58}(\pi^2/L^2)m_im_jIIR2(i,j) - nB_{49}(\pi^2/L^2)m_i^2IIR1(i,j)$$

### D.3 w - u Stiffness Partition

The coefficient of $D^u_{\ell j}$ in the $(\partial\ /\partial D^w_{\ell i})$ row is as follows:

$$B_{29}IIR3(i,j) - B_{30}(\pi/L)m_jI1(i,j)$$

### D.4 w - $\Psi_x$ Stiffness Partition

The coefficient of $D^x_{\ell j}$ in the $(\partial\ /\partial D^w_{\ell i})$ row is as follows:

$$n^2(B_{44} - B_{54})IIRR3(i,j) + n^2B_{55}(\pi/L)m_jIIR1(i,j) + B_{36}(\pi/L)m_iIR2(i,j) +$$

$(B_{37} + n^2B_{59})(\pi/L)m_i IIR2(i,j) - B_{38}(\pi^2/L^2)m_i m_j I3(j,i) - B_{50}(\pi^2/L^2)m_i^2 \cdot$
$I3(i,j) + B_{51}(\pi^3/L^3)m_i^2 m_j IR1(i,j)$

## D.5 w - $\Psi_\theta$ Stiffness Partition

The coefficient of $D^\theta_{\ell j}$ in the $(\partial\ /\partial D^w_{\ell i})$ row is as follows:

$- nB_{45}I1(i,j) + n(B_{32} - B_{46} - n^2B_{56})IIRR1(i,j) - nB_{47}(\pi/L)m_j IIR3(i,j)$
$+ n(B_{39} - B_{60})(\pi/L)m_i IIR3(j,i) - nB_{61}(\pi^2/L^2)m_i m_j I2(i,j) -$
$nB_{52}(\pi^2/L^2)m_i^2 I1(i,j)$

## D.6 v - v Stiffness Partition

The coefficient of $D^v_{\ell j}$ in the $(\partial\ /\partial D^v_{\ell i})$ row is as follows:

$2(B_{12} + n^2B_{16})IIR1(i,j) + 2(B_{13} + n^2B_{17})IIRRR1(i,j) + 2B_{14}(\pi^2/L^2)m_i m_j \cdot$
$IR2(i,j) + 2B_{15}(\pi^2/L^2)m_i m_j IIR2(i,j) + B_{71}(\pi/L)m_j I3(i,j) + B_{71}(\pi/L) \cdot$
$m_i I3(j,i) + B_{72}(\pi/L)m_j IIRR3(i,j) + B_{72}(\pi/L)m_i IIRR3(j,i)$

## D.7 v - u Stiffness Partition

The coefficient of $D^u_{\ell j}$ in the $(\partial\ /\partial D^v_{\ell i})$ row is as follows:

$n(B_{63} - B_{67})IIR3(i,j) - nB_{65}(\pi/L)m_j I1(i,j) - nB_{68}(\pi/L)m_i I2(i,j)$

## D.8 v - $\Psi_x$ Stiffness Partition

The coefficient of $D^x_{\ell j}$ in the $(\partial\ /\partial D^v_{\ell i})$ row is as follows:

$n(B_{80} - B_{73})IIRR3(i,j) - nB_{81}(\pi/L)m_j IIR1(i,j) - nB_{77}(\pi/L)m_i IIR2(i,j)$

## D.9 v - $\Psi_\theta$ Stiffness Partition

The coefficient of $D^\theta_{\ell j}$ in the $(\partial\ /\partial D^v_{\ell i})$ row is as follows:

$B_{74}I1(i,j) + (B_{75} + n^2B_{82})IIRR1(i,j) + B_{76}(\pi/L)m_j IIR3(i,j) + B_{78}(\pi/L) \cdot$
$m_i IIR3(j,i) + B_{79}(\pi^2/L^2)m_i m_j I2(i,j)$

## D.10 u - u Stiffness Partition

The coefficient of $D^{u}_{\ell j}$ in the $(\partial\ /\partial D^{u}_{\ell i})$ row is as follows:

$$2(B_9 + n^2B_{11})IIR2(i,j) + 2B_{10}(\pi^2/L^2)m_i m_j IR1(i,j) - B_{62}(\pi/L)m_j I3(j,i)$$
$$- B_{62}(\pi/L)m_i I3(i,j)$$

## D.11 u - $\Psi_\theta$ Stiffness Partition

The coefficient of $D^{\theta}_{\ell j}$ in the $(\partial\ /\partial D^{u}_{\ell i})$ row is as follows:

$$n(B_{64} - B_{69})IIRR3(j,i) - nB_{70}(\pi/L)m_j IIR2(i,j) - nB_{66}(\pi/L)m_i IIR1(i,j)$$

## D.12 $\Psi_x$ - $\Psi_x$ Stiffness Partition

The coefficient of $D^{x}_{\ell j}$ in the $(\partial\ /\partial D^{x}_{\ell i})$ row is as follows:

$$2B_{18}IR2(i,j) + 2(B_{19} + n^2B_{21})IIR2(i,j) + 2B_{20}(\pi^2/L^2)m_i m_j IR1(i,j) -$$
$$B_{83}(\pi/L)m_j I3(j,i) - B_{83}(\pi/L)m_i I3(i,j)$$

## D.13 $\Psi_x$ - $\Psi_\theta$ Stiffness Partition

The coefficient of $D^{\theta}_{\ell j}$ in the $(\partial\ /\partial D^{x}_{\ell i})$ row is as follows:

$$n(B_{84} - B_{86})IIR3(j,i) - nB_{87}(\pi/L)m_j I2(i,j) - nB_{85}(\pi/L)m_i I1(i,j)$$

## D.14 $\Psi_\theta$ - $\Psi_\theta$ Stiffness Partition

The coefficient of $D^{\theta}_{\ell j}$ in the $(\partial\ /\partial D^{\theta}_{\ell i})$ row is as follows:

$$2B_{22}IR1(i,j) + 2(B_{23} + n^2B_{27})IIR1(i,j) + 2(B_{24} + n^2B_{28})IIRRR1(i,j)$$
$$+ 2B_{25}(\pi^2/L^2)m_i m_j IR2(i,j) + 2B_{26}(\pi^2/L^2)m_i m_j IIR2(i,j) + B_{88}(\pi/L)m_j I3(i,j)$$
$$+ B_{88}(\pi/L)m_i I3(j,i) + B_{89}(\pi/L)m_j IIRR3(i,j) + B_{89}(\pi/L)m_i IIRR3(j,i)$$

## D.15 w - w Load Partition

The coefficient of $D^{w}_{\ell j}$ in the $(\partial\ /\partial D^{w}_{\ell i})$ row is as follows:

$$2\bar{V}_1(\pi^2/L^2)m_i m_j I2(i,j)$$

## APPENDIX E

## CLAMPED-SUPPORT MATRIX ELEMENTS

### E.1 Axial Compression and Bending Loading

#### E.1.1 General Considerations

There are three populated subpartitions in each of the displacement partitions. All three of these subpartitions are identical with the exception of circumferential dependence. Thus, a typical generating term for the middle subpartition, n partition, will be given. The $A_{6k} - A_{3k}$ subpartition can be found by replacing n by n+1, and the $A_{4k} - A_{1k}$ subpartition can be found by replacing n by n-1. Both the load and stiffness matrices are symmetric about their main diagonals.

#### E.1.2 w - w Stiffness Partition

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A^w_{5i})$ row is as follows:

$2(B_4n^2 + B_1)[IIR2(i,j) - IIR2(i,j2)^* - IIR2(i2,j) + IIR2(i2,j2)] +$
$2n^2(B_5 + n^2B_7)[IIRRR2(i,j) - IIRRR2(i,j2) - IIRRR2(i2,j) +$
$IIRRR2(i2,j2)] + 2B_2(\pi^2/L^2)[m_im_jIR1(i,j) - m_i(m_j + 2)IR1(i,j2) -$
$(m_i + 2)m_jIR1(i2,j) + (m_i + 2)(m_j + 2)IR1(i2,j2)] + 2(\pi^2/L^2)\cdot$

*The j2 and i2 positions are those associated with the $m_j + 2$ and $m_i + 2$ integers respectively.

$(B_3 + n^2B_8)[m_i m_j \mathrm{IIR1}(i,j) - m_i(m_j + 2)\mathrm{IIR1}(i,j2) - (m_i + 2)m_j\mathrm{IIR1}(i2,j)$
$+ (m_j + 2)(m_i + 2)\mathrm{IIR1}(i2,j2)] + 2(\pi^4/L^4)B_6[m_i^2 m_j^2\mathrm{IR2}(i,j) -$
$m_i^2(m_j + 2)^2\mathrm{IR2}(i,j2) - (m_i + 2)^2 m_j^2\mathrm{IR2}(i2,j) + (m_i + 2)^2(m_j + 2)^2\mathrm{IR2}$
$(i2,j2)] + n^2(\pi/L)(B_{40} - B_{34})[ - m_j\mathrm{IIRR3}(j,i) + m_j\mathrm{IIRR3}(j,i2) +$
$(m_j + 2)\mathrm{IIRR3}(j2,i) - (m_j + 2)\mathrm{IIRR3}(j2,i2)] + n^2(\pi/L)(B_{40} - B_{34})\cdot$
$[ - m_i\mathrm{IIRR3}(i,j) + m_i\mathrm{IIRR3}(i,j2) + (m_i + 2)\mathrm{IIRR3}(i2,j) -$
$(m_i + 2)\mathrm{IIRR3}(i2,j2)] + B_{48}n^2(\pi^2/L^2)[m_j^2\mathrm{IIR2}(i,j) - (m_j + 2)^2\mathrm{IIR2}(i,j2) -$
$m_j^2\mathrm{IIR2}(i2,j) + (m_j + 2)^2\mathrm{IIR2}(i2,j2) + m_i^2\mathrm{IIR2}(i,j) - m_i^2\mathrm{IIR2}(i,j2) -$
$(m_i + 2)^2\mathrm{IIR2}(i2,j) + (m_i + 2)^2\mathrm{IIR2}(i2,j2)] + B_{33}(\pi^3/L^3)[m_i m_j^2\mathrm{I3}(i,j) -$
$m_i(m_j + 2)^2\mathrm{I3}(i,j2) - (m_i + 2)m_j^2\mathrm{I3}(i2,j) + (m_i + 2)(m_j + 2)^2\mathrm{I3}(i2,j2) +$
$m_i^2 m_j\mathrm{I3}(j,i) - m_i^2(m_j + 2)\mathrm{I3}(j2,i) - (m_i + 2)^2 m_j\mathrm{I3}(j,i2) + (m_i + 2)^2\cdot$
$(m_j + 2)\mathrm{I3}(j2,i2)]$

### E.1.3 v - v Stiffness Partition

The coefficient of $A^v_{2j}$ in the $(\partial\ /\partial A^v_{2i})$ row is as follows:

$2(B_{12} + n^2B_{16})\mathrm{IIR1}(i,j) + 2(B_{13} + n^2B_{17})\mathrm{IIRRR1}(i,j) + 2B_{14}(\pi^2/L^2)\cdot$
$m_i m_j\mathrm{IR2}(i,j) + 2B_{15}(\pi^2/L^2)m_i m_j\mathrm{IIR2}(i,j) + B_{71}(\pi/L)m_j\mathrm{I3}(i,j) +$
$B_{71}(\pi/L)m_i\mathrm{I3}(j,i) + B_{72}(\pi/L)m_j\mathrm{IIRR3}(i,j) + B_{72}(\pi/L)m_i\mathrm{IIRR3}(j,i)$

### E.1.4 v - w Stiffness Partition

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A^v_{2i})$ row is as follows:

$n(B_{31} - B_{41})[\mathrm{IIR3}(i,j) - \mathrm{IIR3}(i,j2)] - n(B_{42} + n^2B_{53})[\mathrm{IIRRR3}(i,j) -$
$\mathrm{IIRRR3}(i,j2)] - B_{43}n(\pi/L)m_i[\mathrm{IIRR2}(i,j) - \mathrm{IIRR2}(i,j2)] + n(B_{35} - B_{57})\cdot$
$(\pi/L)[(m_j + 2)\mathrm{IIRR1}(i,j2) - m_j\mathrm{IIRR1}(i,j)] - nB_{58}(\pi^2/L^2)m_i[(m_j + 2)\cdot$
$\mathrm{IIR3}(j2,i) - m_j\mathrm{IIR3}(j,i)] + nB_{49}(\pi^2/L^2)[(m_j + 2)^2\mathrm{IIR3}(i,j2) -$
$m_j^2\mathrm{IIR3}(i,j)]$

### E.1.5 u - u Stiffness Partition

The coefficient of $A^u_{5j}$ in the $(\partial\ /\partial A^u_{5i})$ row is as follows:

$2(B_9 + n^2B_{11})IIR2(i,j) + 2B_{10}(\pi^2/L^2)m_im_jIR1(i,j) -$
$B_{62}(\pi/L)m_jI3(j,i) - B_{62}(\pi/L)m_iI3(i,j)$

### E.1.6 u - v Stiffness Partition

The coefficient of $A^v_{2j}$ in the $(\partial\ /\partial A^u_{5i})$ row is as follows:

$n(B_{63} - B_{67})IIR3(j,i) - nB_{68}(\pi/L)m_jI2(i,j) - nB_{65}(\pi/L)m_iI1(i,j)$

### E.1.7 u - w Stiffness Partition

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A^u_{5i})$ row is as follows:

$B_{29}[IIR2(i,j) - IIR2(i,j2)] + B_{30}(\pi/L)m_i[I3(i,j2) - I3(i,j)]$

### E.1.8 $\Psi_x$ - $\Psi_x$ Stiffness Partition

The coefficient of $A^x_{5j}$ in the $(\partial\ /\partial A^x_{5i})$ row is as follows:

$2B_{18}IR2(i,j) + 2(B_{19} + n^2B_{21})IIR2(i,j) + 2B_{20}(\pi^2/L^2)m_im_jIR1(i,j) -$
$B_{83}(\pi/L)m_jI3(j,i) - B_{83}(\pi/L)m_iI3(i,j)$

### E.1.9 $\Psi_x$ - v Stiffness Partition

The coefficient of $A^v_{2j}$ in the $(\partial\ /\partial A^x_{5i})$ row is as follows:

$n(B_{80} - B_{73})IIRR3(j,i) - nB_{77}(\pi/L)m_jIIR2(i,j) - nB_{81}(\pi/L)m_iIIR1(i,j)$

### E.1.10 $\Psi_x$ - w Stiffness Partition

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A^x_{5i})$ row is as follows:

$n^2(B_{44} - B_{54})[IIRR2(i,j) - IIR2(i,j2)] + n^2B_{55}(\pi/L)m_i[IIR3(i,j) -$
$IIR3(i,j2)] + B_{36}(\pi/L)[(m_j + 2)IR3(j2,i) - m_jIR3(j,i)] +$
$(B_{37} + n^2B_{59})(\pi/L)[(m_j + 2)IIR3(j2,i) - m_jIIR3(j,i)] + B_{38}(\pi^2/L^2)m_i\cdot$
$[m_jI1(i,j) - (m_j + 2)I1(i,j2)] + B_{50}(\pi^2/L^2)[(m_j + 2)^2I2(i,j2) -$

$m_j^2 I2(i,j)] + B_{51}(\pi^3/L^3)m_i[m_j^2 IR3(i,j) - (m_j + 2)^2 IR3(i,j2)]$

E.1.11 $\Psi_\theta - \Psi_\theta$ Stiffness Partition

The coefficient of $A^\theta_{2j}$ in the $(\partial\ /\partial A^\theta_{2i})$ row is as follows:

$2B_{22}IR1(i,j) + 2(B_{23} + n^2B_{27})IIR1(i,j) + 2(B_{24} + n^2B_{28})IIRRR1(i,j) +$
$2B_{25}(\pi^2/L^2)m_im_jIR2(i,j) + 2B_{26}(\pi^2/L^2)m_im_jIIR2(i,j) + B_{88}(\pi/L)m_jI3(i,j) +$
$B_{88}(\pi/L)m_iI3(j,i) + B_{89}(\pi/L)m_jIIRR3(i,j) + B_{89}(\pi/L)m_iIIRR3(j,i)$

E.1.12 $\Psi_\theta - \Psi_x$ Stiffness Partition

The coefficient of $A^x_{5j}$ in the $(\partial\ /\partial A^\theta_{2i})$ row is as follows:

$n(B_{84} - B_{86})IIR3(i,j) - nB_{85}(\pi/L)m_jI1(i,j) - nB_{87}(\pi/L)m_iI2(i,j)$

E.1.13 $\Psi_\theta$ - u Stiffness Partition

The coefficient of $A^u_{5j}$ in the $(\partial\ /\partial A^\theta_{2i})$ row is as follows:

$n(B_{64} - B_{69})IIRR3(i,j) - nB_{66}(\pi/L)m_jIIR1(i,j) - nB_{70}(\pi/L)m_iIIR2(i,j)$

E.1.14 $\Psi_\theta$ - v Stiffness Partition

The coefficient of $A^v_{2j}$ in the $(\partial\ /\partial A^\theta_{2i})$ row is as follows:

$B_{74}I1(i,j) + (B_{75} + n^2B_{82})IIRR1(i,j) + B_{76}(\pi/L)m_iIIR3(j,i) +$
$B_{78}(\pi/L)m_jIIR3(i,j) + B_{79}(\pi^2/L^2)m_im_jI2(i,j)$

E.1.15 $\Psi_\theta$ - w Stiffness Partition

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A^\theta_{2i})$ row is as follows:

$- nB_{45}[I3(i,j) - I3(i,j2)] + n(B_{32} - B_{46} - n^2B_{56})[IIRR3(i,j) -$
$IIRR3(i,j2)] - nB_{47}(\pi/L)m_i[IIR2(i,j) - IIR2(i,j2)] + n(B_{39} - B_{60})\cdot$
$(\pi/L)[(m_j + 2)IIR1(i,j2) - m_jIIR1(i,j)] - nB_{61}(\pi^2/L^2)m_i[(m_j + 2)I3(j2,i) -$
$m_jI3(j,i)] + nB_{52}(\pi^2/L^2)[(m_j + 2)^2I3(i,j2) - m_j^2I3(i,j)]$

E.1.16 w - w Load Partition

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A^w_{5i})$ row is as follows:

$2\bar{V}_1(\pi^2/L^2)[m_i m_j I1(i,j) - m_i(m_j + 2)I1(i,j2) - (m_i + 2)m_j I1(i2,j) + (m_i + 2)(m_j + 2)I1(i2,j2)]$

The coefficient of $A^w_{6j}$ in the $(\partial\ /\partial A^w_{5i})$ row is as follows:

$\delta\bar{V}_2(\pi^2/L^2)[m_i m_j IIR1(i,j) - m_i(m_j + 2)IIR1(i,j2) - (m_i + 2)m_j IIR1(i2,j) + (m_i + 2)(m_j + 2)IIR1(i2,j)]$

The factor $\delta$ is equal to one unless $n-1 = 0$. If $n-1 = 0$, $\delta = 2$. The other off diagonal load subpartitions are equal to the $A^w_{6j}$ coefficient above with $\delta$ set equal to one.

## E.2 Axial Compression (Axisymmetric Case)

The generating terms for this particular case can be determined as a subset of the axial compression and bending case. The following terms are retained in the respective partitions:

1. w - w Stiffness Partition - $B_1$, $B_2$, $B_3$, $B_6$, and $B_{33}$
2. u - u Stiffness Partition - $B_9$, $B_{10}$, and $B_{62}$
3. $\Psi_x$ - $\Psi_x$ Stiffness Partition - $B_{18}$, $B_{19}$, $B_{20}$, and $B_{83}$
4. u - w Stiffness Partition - $B_{29}$, and $B_{30}$
5. $\Psi_x$ - w Stiffness Partition - $B_{36}$, $B_{37}$, $B_{38}$, $B_{50}$, and $B_{51}$
6. w - w Load Partition - $\bar{V}_1$

## E.3 Axial Compression and Torsion Loading

### E.3.1 General Considerations

Due to symmetry about the main diagonal, only the partitions on and above the diagonal need to be computed.

E.3.2 w - w Stiffness Partition

The coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A^w_{2i})$ row is the same as the generating term of Section E.1.2.

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A^w_{5i})$ row is the same as the generating term of Section E.1.2.

E.3.3 v - v Stiffness Partition

The coefficient of $A^v_{2j}$ in the $(\partial\ /\partial A^v_{2i})$ row and $A^v_{5j}$ in the $(\partial\ /\partial A^v_{5i})$ row is as follows:

Same as generating function of Section E.1.3.

The coefficient of $A'^v_{2j}$ in the $(\partial\ /\partial A'^v_{2i})$ row and $A'^v_{5j}$ in the $(\partial\ /\partial A'^v_{5i})$ row is as follows:

$2(B_{12} + n^2B_{16})IIR2(i,j) + 2(B_{13} + n^2B_{17})IIRRR2(i,j) + 2B_{14}(\pi^2/L^2)\cdot$
$m_i m_j IR1(i,j) + 2B_{15}(\pi^2/L^2)m_i m_j IIR1(i,j) - B_{71}(\pi/L)m_j I3(j,i) -$
$B_{71}(\pi/L)m_i I3(i,j) - B_{72}(\pi/L)m_j IIRR3(j,i) - B_{72}(\pi/L)m_i IIRR3(i,j)$

The coefficient of $A^v_{2j}$ in the $(\partial\ /\partial A'^v_{2i})$ row and $A^v_{5j}$ in the $(\partial\ /\partial A'^v_{5i})$ row is as follows:

$2(B_{12} + n^2B_{16})IIR3(j,i) + 2(B_{13} + n^2B_{17})IIRRR3(j,i) - 2B_{14}(\pi^2/L^2)\cdot$
$m_i m_j IR3(i,j) - 2B_{15}(\pi^2/L^2)m_i m_j IIR3(i,j) + B_{71}(\pi/L)m_j I2(i,j) -$
$B_{71}(\pi/L)m_i I1(i,j) + B_{72}(\pi/L)m_j IIRR2(i,j) - B_{72}(\pi/L)m_i IIRR1(i,j)$

E.3.4 v - w Stiffness Partition

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A^v_{2i})$ row is as follows:

Same as generating function of Section E.1.4.

The coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A^v_{5i})$ row is as follows:

Equal to negative of generating function of Section E.1.4.

The coefficient of $A^{w}_{5j}$ in the $(\partial\ /\partial A'^{v}_{2i})$ row is as follows:

$n(B_{31} - B_{41})[IIR2(i,j) - IIR2(i,j2)] - n(B_{42} + n^2B_{53})[IIRRR2(i,j) - IIRRR2(i,j2)] + nB_{43}(\pi/L)m_i[IIRR3(i,j) - IIRR3(i,j2)] + n(B_{35} - B_{57})\cdot(\pi/L)[(m_j + 2)IIRR3(j2,i) - m_jIIRR3(j,i)] + nB_{58}(\pi^2/L^2)m_i[(m_j + 2)\cdot IIR1(i,j2) - m_jIIR1(i,j)] + nB_{49}(\pi^2/L^2)[(m_j + 2)^2IIR2(i,j2) - m_j^2IIR2(i,j)]$

The coefficient of $A^{w}_{2j}$ in the $(\partial\ /\partial A'^{v}_{5i})$ row is equal to the negative of the preceding coefficient.

### E.3.4 u - u Stiffness Partition

The coefficient of $A^{u}_{2j}$ in the $(\partial\ /\partial A^{u}_{2i})$ row and $A^{u}_{5j}$ in the $(\partial\ /\partial A^{u}_{5i})$ row is as follows:

Same as generating function of Section E.1.5.

The coefficient of $A'^{u}_{2j}$ in the $(\partial\ /\partial A'^{u}_{2i})$ row and $A'^{u}_{5j}$ in the $(\partial\ /\partial A'^{u}_{5i})$ row is as follows:

$2(B_9 + n^2B_{11})IIR1(i,j) + 2B_{10}(\pi^2/L^2)m_im_jIR2(i,j) + B_{62}(\pi/L)m_jI3(i,j) + B_{62}(\pi/L)m_iI3(j,i)$

The coefficient of $A^{u}_{2j}$ in the $(\partial\ /\partial A'^{u}_{2i})$ row and $A^{u}_{5j}$ in the $(\partial\ /\partial A'^{u}_{5i})$ row is as follows:

$2(B_9 + n^2B_{11})IIR3(i,j) - 2B_{10}(\pi^2/L^2)m_im_jIR3(j,i) - B_{62}(\pi/L)m_jI1(i,j) + B_{62}(\pi/L)m_iI2(i,j)$

### E.3.5 u - v Stiffness Partition

The coefficient of $A^{v}_{2j}$ in the $(\partial\ /\partial A^{u}_{5i})$ row is equal to the generating function of Section E.1.6.

The coefficient of $A^{v}_{5j}$ in the $(\partial\ /\partial A^{u}_{2i})$ row is equal to the negative of the generating function of Section E.1.6.

The coefficient of $A'^{v}_{5j}$ in the $(\partial\ /\partial A^{u}_{2i})$ row and the negative of the coefficient of $A'^{v}_{2j}$ in the $(\partial\ /\partial A^{u}_{5i})$ row is as follows:

$n(B_{67} - B_{63})IIR2(i,j) - nB_{68}(\pi/L)m_j I3(j,i) + nB_{65}(\pi/L)m_i I3(i,j)$

The coefficient of $A^{v}_{5j}$ in the $(\partial\ /\partial A'^{u}_{2i})$ row and the negative of the coefficient of $A^{v}_{2j}$ in the $(\partial\ /\partial A'^{u}_{5i})$ row is as follows:

$n(B_{67} - B_{63})IIR1(i,j) + nB_{68}(\pi/L)m_j I3(i,j) - nB_{65}(\pi/L)m_i I3(j,i)$

The coefficient of $A'^{v}_{5j}$ in the $(\partial\ /\partial A'^{u}_{2i})$ row and the negative of the coefficient of $A'^{v}_{2i}$ in the $(\partial\ /\partial A'^{u}_{5i})$ row is as follows:

$n(B_{67} - B_{63})IIR3(i,j) - nB_{68}(\pi/L)m_j I1(i,j) - nB_{65}(\pi/L)m_i I2(i,j)$

### E.3.6 u - w Stiffness Partition

The coefficient of $A^{w}_{5j}$ in the $(\partial\ /\partial A^{u}_{5i})$ row and $A^{w}_{2j}$ in the $(\partial\ /\partial A^{u}_{2i})$ row is equal to the coefficient defined in Section E.1.7.

The coefficient of $A^{w}_{2j}$ in the $(\partial\ /\partial A'^{u}_{2i})$ row and $A^{w}_{5j}$ in the $(\partial\ /\partial A'^{u}_{5i})$ row is as follows:

$B_{29}[IIR3(i,j) - IIR3(i,j2)] + B_{30}(\pi/L)m_i[I2(i,j) - I2(i,j2)]$

### E.3.7 $\Psi_x - \Psi_x$ Stiffness Partition

The coefficient of $A^{x}_{2j}$ in the $(\partial\ /\partial A^{x}_{2i})$ row and $A^{x}_{5j}$ in the $(\partial\ /\partial A^{x}_{5i})$ row is equal to the coefficient defined in Section E.1.8.

The coefficient of $A'^{x}_{2j}$ in the $(\partial\ /\partial A'^{x}_{2i})$ row and $A'^{x}_{5j}$ in the $(\partial\ /\partial A'^{x}_{5i})$ row is as follows:

$2B_{18}IR1(i,j) + 2(B_{19} + n^2B_{21})IIR1(i,j) + 2B_{20}(\pi^2/L^2)m_i m_j IR2(i,j) +$
$B_{83}(\pi/L)m_j I3(i,j) + B_{83}(\pi/L)m_i I3(j,i)$

The coefficient of $A^{x}_{2j}$ in the $(\partial\ /\partial A'^{x}_{2i})$ row and $A^{x}_{5j}$ in the $(\partial\ /\partial A'^{x}_{5i})$ row is as follows:

$2B_{18}IR1(i,j) + 2(B_{19} + n^2B_{21})IIR1(i,j) + 2B_{20}(\pi^2/L^2)m_im_jIR2(i,j) +$
$B_{83}(\pi/L)m_jI3(i,j) + B_{83}(\pi/L)m_iI3(j,i)$

The coefficient of $A^x_{2j}$ in the $(\partial\ /\partial A'^x_{2i})$ row and $A^x_{5j}$ in the $(\partial\ /\partial A'^x_{5i})$ row is as follows:

$2B_{18}IR3(i,j) + 2(B_{19} + n^2B_{21})IIR3(i,j) - 2B_{20}(\pi^2/L^2)m_im_jIR3(j,i) -$
$B_{83}(\pi/L)m_jI1(i,j) + B_{83}(\pi/L)m_iI2(i,j)$

### E.3.8 $\Psi_x$ - v Stiffness Partition

The coefficient of $A^v_{2j}$ in the $(\partial\ /\partial A^x_{5i})$ row and the negative of the coefficient of $A^v_{5j}$ in the $(\partial\ /\partial A^x_{2i})$ row is equal to the coefficient defined in Section E.1.9.

The coefficient of $A'^v_{5j}$ in the $(\partial\ /\partial A^x_{2i})$ row and the negative of the coefficient of $A'^v_{2j}$ in the $(\partial\ /\partial A^x_{5i})$ row is as follows:

$- n(B_{80} - B_{73})IIRR2(i,j) + nB_{81}(\pi/L)m_iIIR3(i,j) - nB_{77}(\pi/L)m_jIIR3(j,i)$

The coefficient of $A^v_{5j}$ in the $(\partial\ /\partial A'^x_{2i})$ row and the negative of the coefficient of $A^v_{2j}$ in the $(\partial\ /\partial A'^x_{5i})$ row is as follows:

$- n(B_{80} - B_{73})IIRR1(i,j) - nB_{81}(\pi/L)m_iIIR3(j,i) + nB_{77}(\pi/L)m_jIIR3(i,j)$

The coefficient of $A'^v_{5j}$ in the $(\partial\ /\partial A'^x_{2i})$ row and the negative of the coefficient of $A'^v_{2j}$ in the $(\partial\ /\partial A'^x_{5i})$ row is as follows:

$- n(B_{80} - B_{73})IIRR3(i,j) - nB_{81}(\pi/L)m_iIIR2(i,j) - nB_{77}(\pi/L)m_jIIR1(i,j)$

### E.3.9 $\Psi_x$ - w Stiffness Partition

The coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A^x_{2i})$ row and $A^w_{5j}$ in the $(\partial\ /\partial A^x_{5i})$ row is equal to the coefficient defined in Section E.1.10.

The coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A'^x_{2i})$ row and $A^x_{5j}$ in the $(\partial\ /\partial A'^x_{5i})$ row is as follows:

$n^2(B_{44} - B_{54})[IIRR3(i,j) - IIRR3(i,j2)] - n^2B_{55}(\pi/L)m_i[IIR2(i,j) -$

$IIR2(i,j2)] + B_{36}(\pi/L)[(m_j + 2)IR1(i,j2) - m_jIR1(i,j)] + (B_{37} + n^2B_{59})\cdot$
$(\pi/L)[(m_j + 2)IIR1(i,j2) - m_jIIR1(i,j)] + B_{38}(\pi^2/L^2)m_i[(m_j + 2)I3(j2,i) -$
$m_jI3(j,i)] + B_{50}(\pi^2/L^2)[(m_j + 2)^2I3(i,j2) - m_j^2I3(i,j)] +$
$B_{51}(\pi^3/L^3)m_i[(m_j + 2)^2IR2(i,j2) - m_j^2IR2(i,j)]$

### E.3.10 $\Psi_\theta - \Psi_\theta$ Stiffness Partition

The coefficient of $A^\theta_{2j}$ in the $(\partial\ /\partial A^\theta_{2i})$ row and $A^\theta_{5j}$ in the $(\partial\ /\partial A^\theta_{5i})$ row is equal to the coefficient defined in Section E.1.11.

The coefficient of $A'^\theta_{2j}$ in the $(\partial\ /\partial A'^\theta_{2i})$ row and $A'^\theta_{5j}$ in the $(\partial\ /\partial A'^\theta_{5i})$ row is as follows:

$2B_{22}IR2(i,j) + 2(B_{23} + n^2B_{27})IIR2(i,j) + 2(B_{24} + n^2B_{28})IIRRR2(i,j) +$
$2B_{25}(\pi^2/L^2)m_im_jIR1(i,j) + 2B_{26}(\pi^2/L^2)m_im_jIIR1(i,j) - B_{88}(\pi/L)m_jI3(j,i) -$
$B_{88}(\pi/L)m_iI3(i,j) - B_{89}(\pi/L)m_jIIRR3(j,i) - B_{89}(\pi/L)m_iIIRR3(i,j)$

The coefficient of $A^\theta_{2j}$ in the $(\partial\ /\partial A'^\theta_{2i})$ row and $A^\theta_{5j}$ in the $(\partial\ /\partial A'^\theta_{5i})$ row is as follows:

$2B_{22}IR3(j,i) + 2(B_{23} + n^2B_{27})IIR3(j,i) + 2(B_{24} + n^2B_{28})IIRRR3(j,i) -$
$2B_{25}(\pi^2/L^2)m_im_jIR3(i,j) - 2B_{26}(\pi^2/L^2)m_im_jIIR3(i,j) + B_{88}(\pi/L)m_jI2(i,j) -$
$B_{88}(\pi/L)m_iI1(i,j) + B_{89}(\pi/L)m_jIIRR2(i,j) - B_{89}(\pi/L)m_iIIRR1(i,j)$

### E.3.11 $\Psi_\theta - \Psi_x$ Stiffness Partition

The coefficient of $A^x_{5j}$ in the $(\partial\ /\partial A^\theta_{2i})$ row is equal to the coefficient defined in Section E.1.12.

The coefficient of $A^x_{2j}$ in the $(\partial\ /\partial A^\theta_{5i})$ row is equal to the negative of the coefficient defined in Section E.1.12.

The coefficient of $A'^x_{5j}$ in the $(\partial\ /\partial A^\theta_{2i})$ row and the negative of the coefficient of $A'^x_{2j}$ in the $(\partial\ /\partial A^\theta_{5i})$ row is as follows:

$n(B_{84} - B_{86})IIR1(i,j) - nB_{87}(\pi/L)m_iI3(j,i) + nB_{85}(\pi/L)m_jI3(i,j)$

The coefficient of $A^{x}_{5j}$ in the $(\partial\ /\partial A'^{\theta}_{2i})$ row and the negative of the coefficient of $A^{x}_{2j}$ in the $(\partial\ /\partial A'^{\theta}_{5i})$ row is as follows:

$n(B_{84} - B_{86})IIR2(i,j) + nB_{87}(\pi/L)m_i I3(i,j) - nB_{85}(\pi/L)m_j I3(j,i)$

The coefficient of $A'^{x}_{5j}$ in the $(\partial\ /\partial A'^{\theta}_{2i})$ row and the negative of the coefficient of $A'^{x}_{2j}$ in the $(\partial\ /\partial A'^{\theta}_{5i})$ row is as follows:

$n(B_{84} - B_{86})IIR3(j,i) + nB_{87}(\pi/L)m_i I1(i,j) + nB_{85}(\pi/L)m_j I2(i,j)$

### E.3.12 $\Psi_\theta$ - u Stiffness Partition

The coefficient of $A^{u}_{5j}$ in the $(\partial\ /\partial A^{\theta}_{2i})$ row is equal to the coefficient defined in Section E.1.13.

The coefficient of $A^{u}_{2j}$ in the $(\partial\ /\partial A^{\theta}_{5i})$ row is equal to the negative of the coefficient defined in Section E.1.13.

The coefficient of $A'^{u}_{5j}$ in the $(\partial\ /\partial A^{\theta}_{2i})$ row and the negative of the coefficient of $A'^{u}_{2j}$ in the $(\partial\ /\partial A^{\theta}_{5i})$ row is as follows:

$n(B_{64} - B_{69})IIRR1(i,j) - nB_{70}(\pi/L)m_i IIR3(j,i) + nB_{66}(\pi/L)m_j IIR3(i,j)$

The coefficient of $A^{u}_{5j}$ in the $(\partial\ /\partial A'^{\theta}_{2i})$ row and the negative of the coefficient of $A^{u}_{2j}$ in the $(\partial\ /\partial A'^{\theta}_{5i})$ row is as follows:

$n(B_{64} - B_{69})IIRR2(i,j) + nB_{70}(\pi/L)m_i IIR3(i,j) - nB_{66}(\pi/L)m_j IIR3(j,i)$

The coefficient of $A'^{u}_{5j}$ in the $(\partial\ /\partial A'^{\theta}_{2i})$ row and the negative of the coefficient of $A'^{u}_{2j}$ in the $(\partial\ /\partial A'^{\theta}_{5i})$ row is as follows:

$n(B_{64} - B_{69})IIRR3(j,i) + nB_{70}(\pi/L)m_i IIR1(i,j) + nB_{66}(\pi/L)m_j IIR2(i,j)$

### E.3.13 $\Psi_\theta$ - v Stiffness Partition

The coefficient of $A^{v}_{2j}$ in the $(\partial\ /\partial A^{\theta}_{2i})$ row and $A^{v}_{5j}$ in the $(\partial\ /\partial A^{\theta}_{5i})$ row is equal to the coefficient defined in Section E.1.14.

The coefficient of $A'^{v}_{2j}$ in the $(\partial\ /\partial A^{\theta}_{2i})$ row and $A'^{v}_{5j}$ in the $(\partial\ /\partial A^{\theta}_{5i})$ row is as follows:

$B_{74}I3(i,j) + (B_{75} - n^2B_{82})IIRR3(i,j) + B_{76}(\pi/L)m_iIIR2(j,i) -$
$B_{78}(\pi/L)m_jIIR1(i,j) - B_{79}(\pi^2/L^2)m_im_jI3(j,i)$

The coefficient of $A^v_{2j}$ in the $(\partial\ /\partial A'^{\theta}_{2i})$ row and $A^v_{5j}$ in the $(\partial\ /\partial A'^{\theta}_{5i})$ row is as follows:

$B_{74}I3(j,i) + (B_{75} + n^2B_{82})IIRR3(j,i) - B_{76}(\pi/L)m_iIIR1(i,j) +$
$B_{78}(\pi/L)m_jIIR2(i,j) - B_{79}(\pi^2/L^2)m_im_jI3(i,j)$

The coefficient of $A'^v_{2j}$ in the $(\partial\ /\partial A'^{\theta}_{2i})$ row and $A'^v_{5j}$ in the $(\partial\ /\partial A'^{\theta}_{5i})$ row is as follows:

$B_{74}I2(i,j) + (B_{75} + n^2B_{82})IIRR2(i,j) - B_{76}(\pi/L)m_iIIR3(i,j) -$
$B_{78}(\pi/L)m_jIIR3(j,i) + B_{79}(\pi^2/L^2)m_im_jI1(i,j)$

### E.3.14 $\Psi_\theta$ - w Stiffness Partition

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A^{\theta}_{2i})$ row is equal to the coefficient defined in Section E.1.15.

The coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A^{\theta}_{5i})$ row is equal to the negative of the coefficient defined in Section E.1.15.

The coefficient of $A^w_{5j}$ in the $(\partial\ /\partial A'^{\theta}_{2i})$ row and the negative of the coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A'^{\theta}_{5i})$ row is as follows:

$- nB_{45}[I2(i,j) - I2(i,j2)] + n(B_{32} - B_{46} - n^2B_{56})[IIRR2(i,j) -$
$IIRR2(i,j2)] - nB_{47}(\pi/L)m_i[IIR3(i,j2) - IIR3(i,j)] + n(B_{39} - B_{60})\cdot$
$(\pi/L)[(m_j + 2)IIR3(j2,i) - m_jIIR3(j,i)] - nB_{61}(\pi^2/L^2)m_i[m_jI1(i,j) -$
$(m_j + 2)I1(i,j2)] + nB_{52}(\pi^2/L^2)[(m_j + 2)^2I2(i,j2) - m_j^2I2(i,j)]$

### E.3.15 w - w Load Partition

The coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A^w_{2i})$ row and $A^w_{5j}$ in the $(\partial\ /\partial A^w_{5i})$ row is as follows:

$2\bar{V}_1(\pi^2/L^2)[m_im_jI1(i,j) - m_i(m_j + 2)I1(i,j2) - (m_i + 2)m_jI1(i2,j) +$

$(m_i + 2)(m_j + 2)I1(i2,j2)]$

The coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A^w_{5i})$ row is as follows:

$- n\bar{V}_3(\pi/L)[ - m_j IIRR3(j,i) + (m_j + 2)IIRR3(j2,i) + m_j IIRR3(j,i2) - (m_j + 2)IIRR3(j2,i2)] + n\bar{V}_3(\pi/L)[ - m_i IIRR3(i,j) + m_i IIRR3(i,j2) + (m_i + 2)IIRR3(i2,j) - (m_i + 2)IIRR3(i2,j2)]$

E.3.16 v - w Load Partition

The coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A^v_{2i})$ row and $A^w_{5j}$ in the $(\partial\ /\partial A^v_{5i})$ row is as follows:

$\bar{V}_4(\pi/L)[(m_j+2)IIRR1(i,j2) - m_j IIRR1(i,j)]$

The coefficient of $A^w_{2j}$ in the $(\partial\ /\partial A'^v_{2i})$ row and $A^w_{5j}$ in the $(\partial\ /\partial A'^v_{5i})$ row is as follows:

$\bar{V}_4(\pi/L)[(m_j + 2)IIRR3(j2,i) - m_j IIRR3(j,i)]$

## APPENDIX F

## BUCKLING OF AXIALLY COMPRESSED SANDWICH CYLINDERS WITH ORTHOTROPIC FACINGS AND CORE

### F.1 Linear Buckling Analysis

The critical buckling stress under uniform axial compression for a sandwich circular cylindrical shell having simply supported edges and orthotropic facings and core can be expressed as follows [22,25]:

$$\sigma_{cr} = N_{cr}/4t = (a_o + a_1)/4t \tag{F-1}$$

where

$$\begin{aligned}
a_o &= (a_2^2\, a_6 + a_3^2\, a_5 - 2a_2a_3a_4)\;(a_4^2 - a_5a_6)^{-1}\tilde{m}^{-2} \\
a_1 &= \tilde{m}^{-2}[D_x\tilde{m}^4 + 2(\nu_{yx}D_x + D_{xy})\tilde{m}^2\tilde{n}^2 + D_y\tilde{n}^4 \\
&\quad + (S_xS_y\tilde{m}^4/R^2)\;\; [S_x\tilde{m}^4 + (S_xS_yS_{xy}^{-1} - 2\nu_{xy}S_y)\tilde{m}^2\tilde{n}^2 \\
&\quad + S_y\tilde{n}^4]^{-1}] \\
a_2 &= D_x\tilde{m}^3 + (\nu_{yx}D_x + D_{xy})\;\tilde{m}\tilde{n}^2 \\
a_3 &= D_y\tilde{n}^3 + (\nu_{xy}D_y + D_{xy})\;\tilde{m}^2\tilde{n} \\
a_4 &= [(\nu_{xy}D_y) + (D_{xy}/2)]\;\tilde{m}\tilde{n} \\
a_5 &= D_{qx} + D_x\tilde{m}^2 + (D_{xy}/2)\;\tilde{n}^2 \\
a_6 &= D_{qy} + D_y\tilde{n}^2 + (D_{xy}/2)\;\tilde{m}^2
\end{aligned} \tag{F-2}$$

The composite shell stiffnesses are calculated as follows:

$$S_x = 4t\;E_x\;\;,\quad S_y = 4t\;E_y\;\;,\quad S_{xy} = 4t\;G_{xy}$$

$$D_x = a^2 S_x/(1-\nu_{xy}\nu_{yx}) \quad , \quad D_y = a^2 S_y/(1-\nu_{xy}\nu_{yx}) \qquad \text{(F-3)}$$
$$D_{xy} = 2a^2 S_{xy}$$

and the core stiffnesses by the following expressions:

$$D_{qx} = G_{xz}\ (h+t)^2/h \quad , \quad D_{qy} = G_{yz}\ (h+t)^2/h \qquad \text{(F-4)}$$

Equation (F-1) can be rewritten in terms of K, the theoretical buckling coefficient, as

$$\sigma_{cr} = (2K\ E_x a/R)[1-\nu_{xy}\ \nu_{yx}]^{-1/2} \qquad \text{(F-5)}$$

Combining Eqs. (F-1) and (F-5):

$$K = [(a_o + a_1)(R/\ 8tE_x a)][1-\nu_{xy}\ \nu_{yx}]^{1/2} \qquad \text{(F-6)}$$

Even though Eq. (F-6) could be minimized with respect to the wave parameters, m and n, a solution to the resulting equations is not apparent except in the axisymmetric case where n = 0. One alternative method of solution is to calculate the critical buckling coefficient for all practical integer combinations of m and n. The lowest such value would then correspond to the actual critical stress. Such a solution is feasible only with the aid of a high-speed computer.

## F.2 Discussion of Numerical Results

In order to determine the interaction between core and facing shear flexibilities, a numerical analysis was undertaken in which all of the parameters, except $G_{xy}$, $G_{xz}$, and $G_{yz}$, were held constant. In this analysis the core shear moduli, $G_{xz}$ and $G_{yz}$, were kept at a constant ratio typical of honeycomb core material. The fixed parameters corresponded to the composite shells reported in Ref. [22] with the exception of shear moduli. The fixed parameters used were as follows:

$E_x$ = 3,280,000 psi

$E_y$ = 3,140,000 psi

$\nu_{xy}$ = 0.13

t = 0.01 inches

R = 21.94 inches

h = 0.15 inches

L = 72 inches

$G_{yz}/G_{xz}$ = 0.572

The resulting values of the buckling coefficient K can be depicted as a function of the dimensionless moduli ratios, $E_x/G_{xz}$ and $E_x/G_{xy}$, by the three-dimensional surface shown in Fig. F.1. Figures F.2 and F.3 show various cutting planes through the surface.

The surface shown in Fig. F.1 consists of three distinct regions or subsurfaces. These regions are as follows: Surface 1, axisymmetric facing mode of buckling; Surface 2, axisymmetric core shear buckling mode; and Surface 3, nonsymmetric facing buckling mode.

The axisymmetric facing buckling mode, shown as Surface 1, can be found by setting n = 0 in Eq. (F-1). The resulting equation can then be minimized with respect to the axial wave parameter such that

$$\sigma_{cr} = (2a/R)\left[\frac{E_x\ E_y}{1-\nu_{xy}\ \nu_{yx}}\right]^{1/2}\left[1 - \frac{ht}{aR}\left(\frac{E_x\ E_y/G_{xz}^2}{1-\nu_{xy}\ \nu_{yx}}\right)^{1/2}\right] \qquad \text{(F-7)}$$

This agrees with the findings of Ref. [35] for the axisymmetric case. Then Eq. (F-5) becomes

$$K = [E_y/E_x]^{1/2}\left[1 - \frac{ht}{a\ R\ G_{xz}}\left(\frac{E_x\ E_y}{1-\nu_{xy}\ \nu_{yx}}\right)^{1/2}\right] \qquad \text{(F-8)}$$

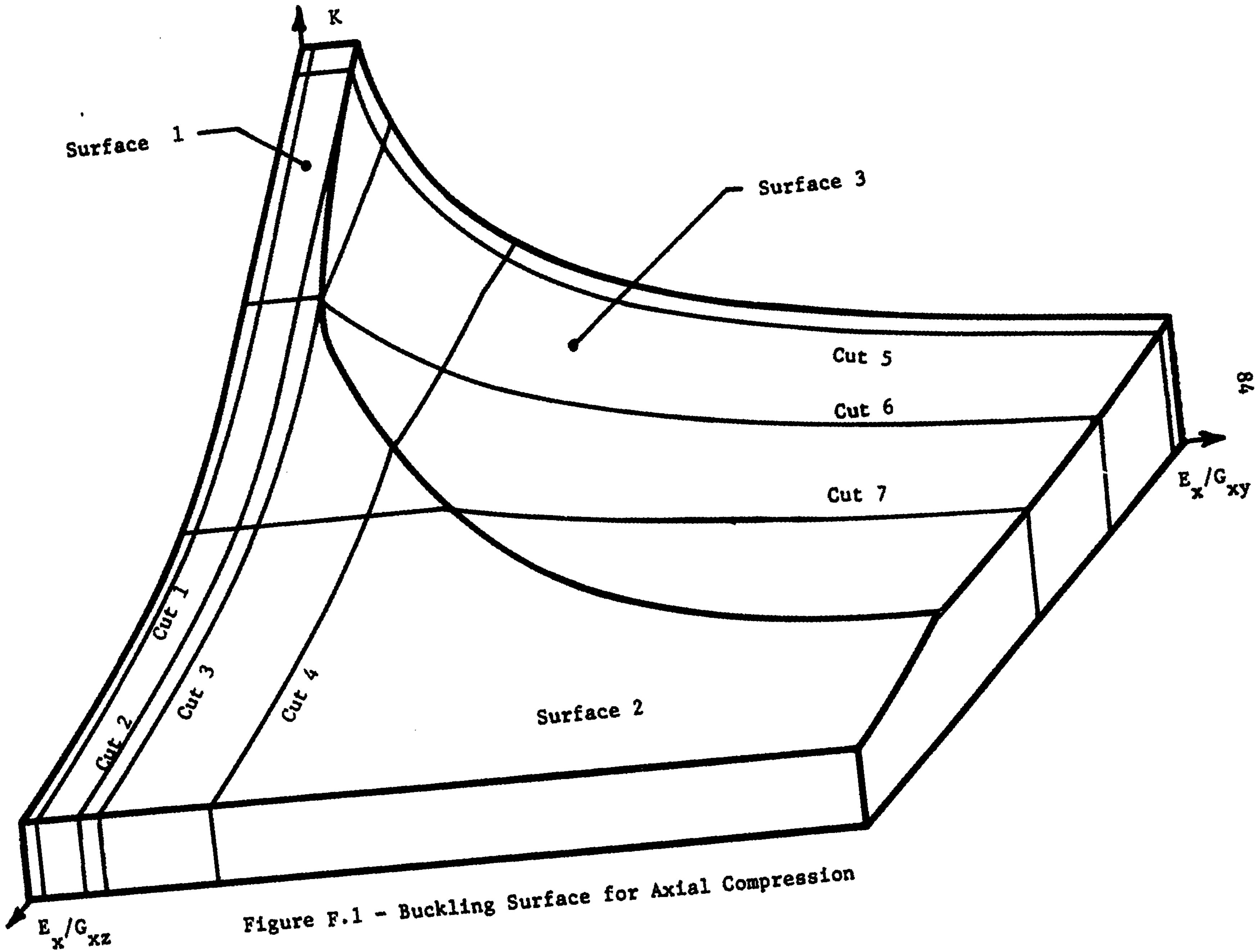


Figure F.1 - Buckling Surface for Axial Compression

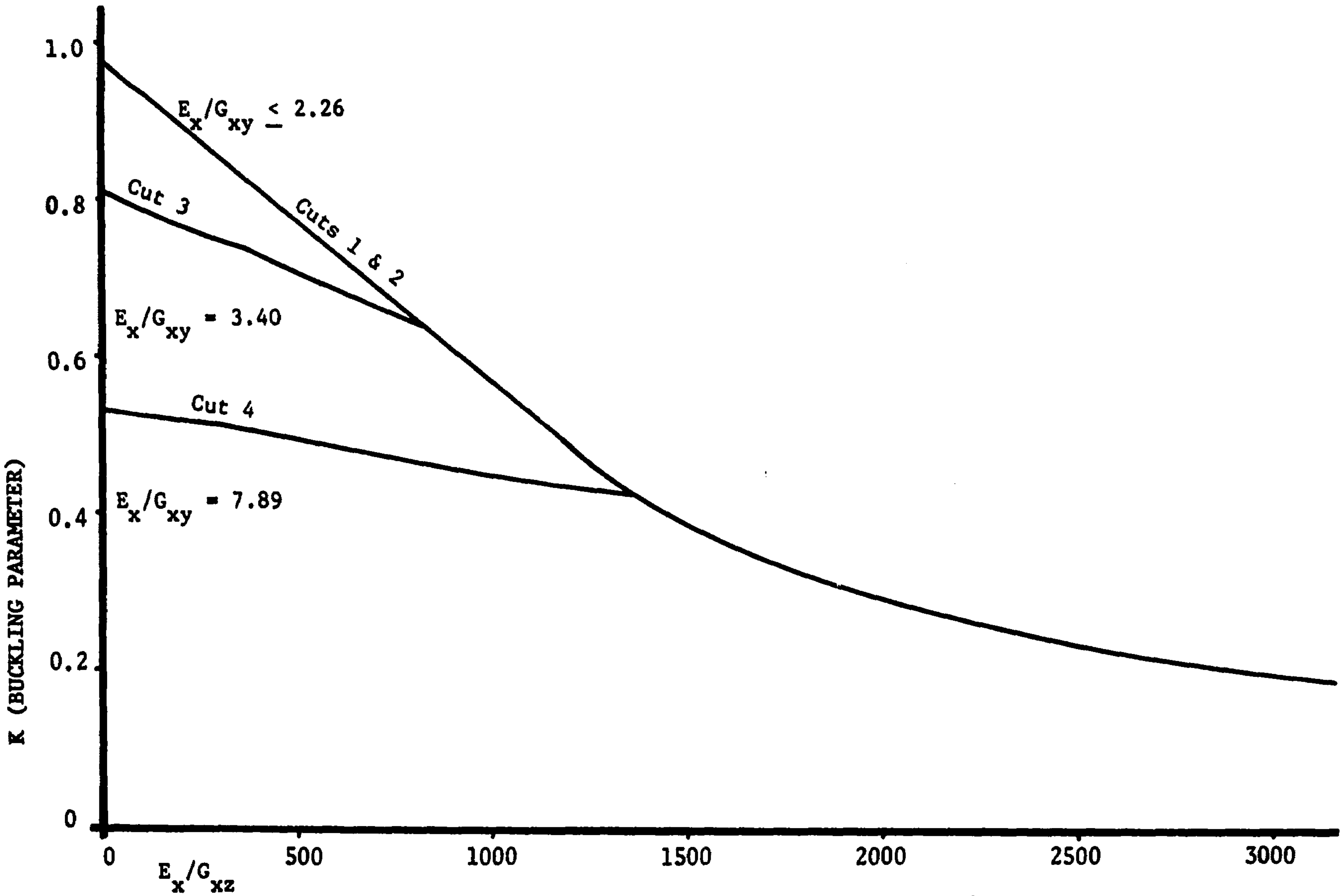


Figure F.2 - Buckling Parameter, K, vs. Ratio of $E_x/G_{xz}$

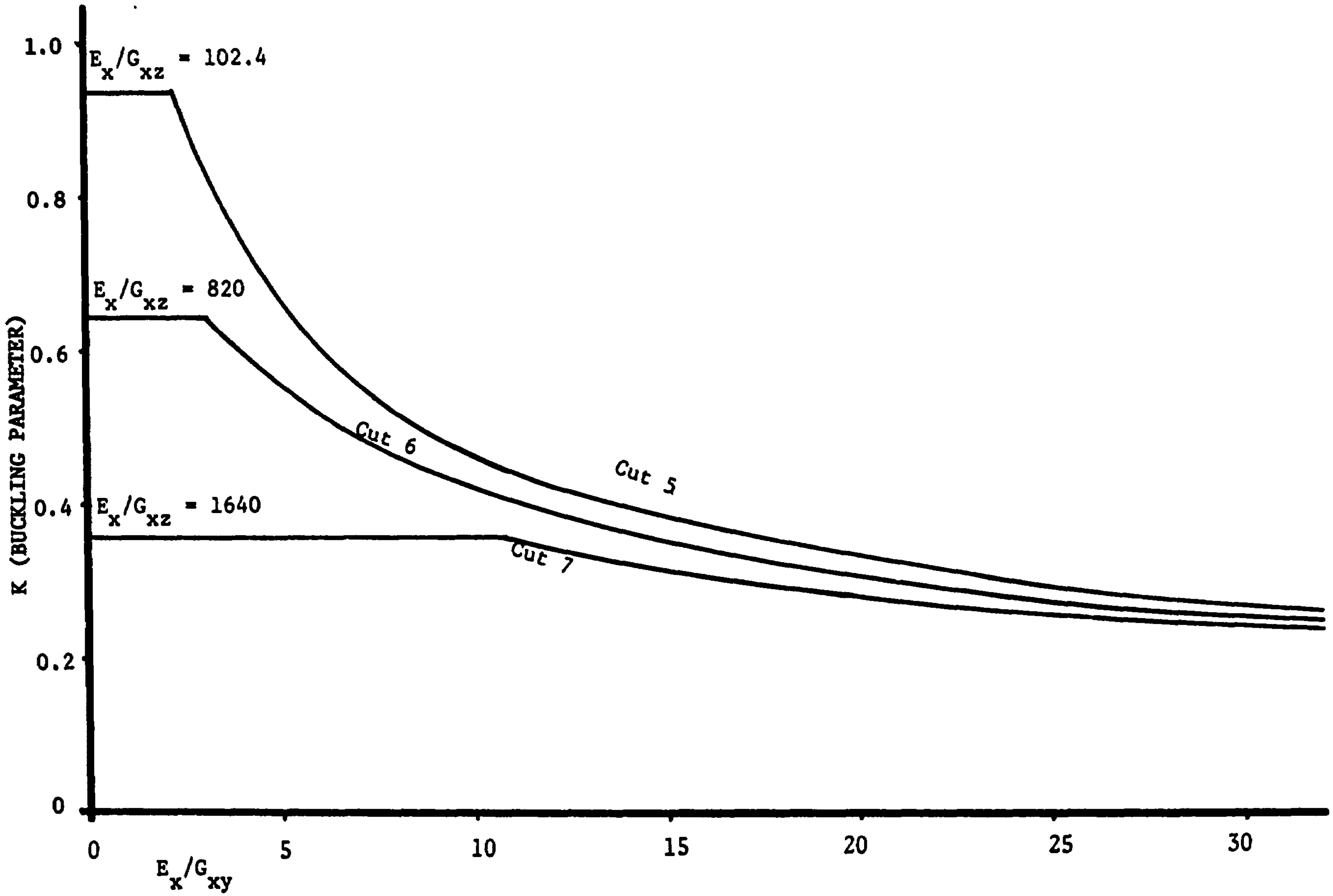


Figure F.3 - Buckling Parameter, K, vs. Ratio of $E_x/G_{xy}$

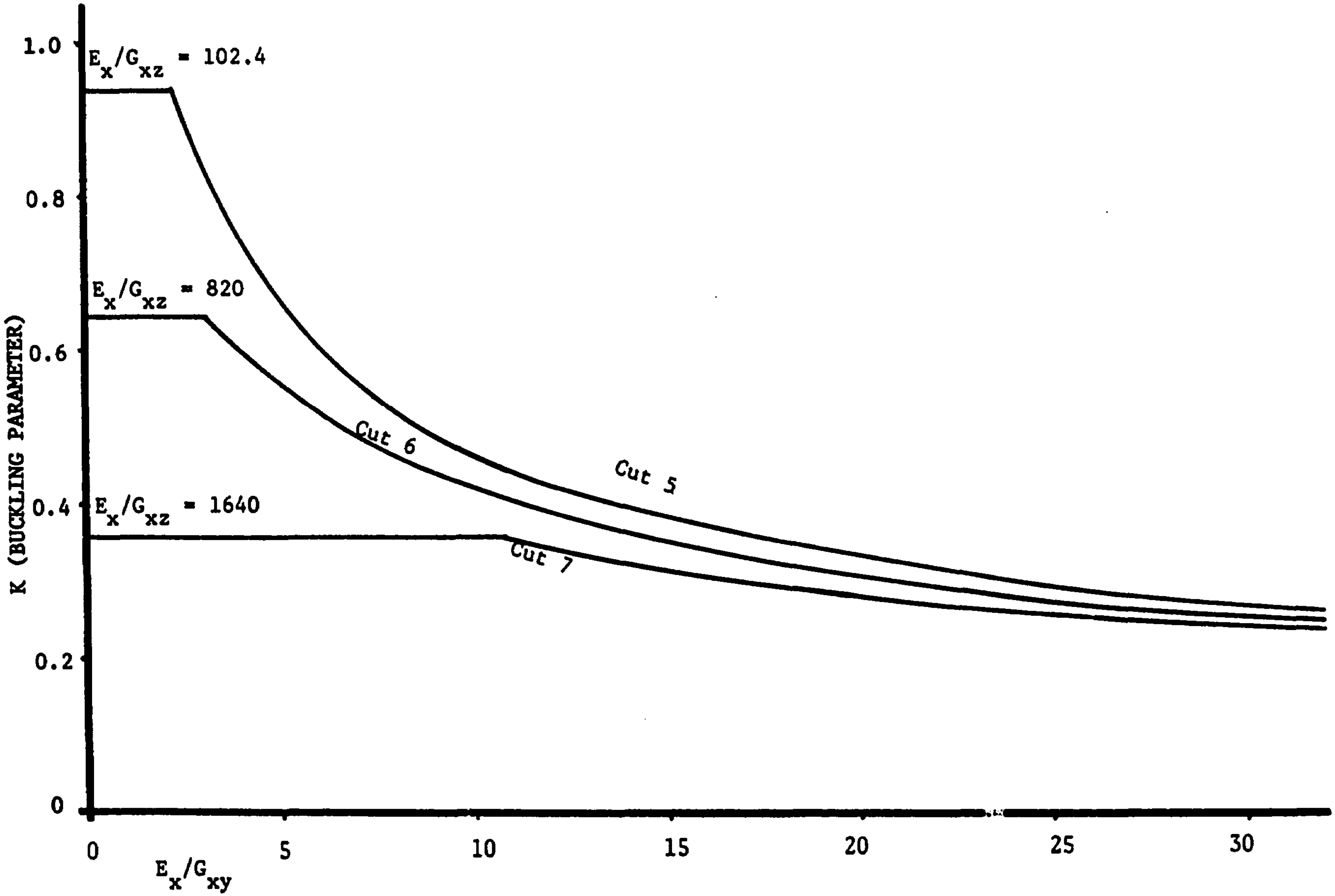


**Figure F.3 - Buckling Parameter, K, vs. Ratio of $E_x/G_{xy}$**

This mode of buckling is primarily a function of $E_x$ and $E_y$, and secondarily a function of $G_{xz}$.

The axisymmetric core shear buckling mode, Surface 2, occurs when $n = 0$ and $m$ becomes very large. Then Eq. (F-1) reduces to the following:

$$\sigma_{cr} = D_{qx}/4t = G_{xz}a^2/2ht \tag{F-9}$$

Eq. (F-9) can be further simplified for a sandwich cylinder with very thin facings by noting that "h" is approximately equal to "a" in that case. Then Eq. (F-9) becomes

$$\sigma_{cr} = G_{xz}a/2t \tag{F-10}$$

This is the same result as was reported in Ref. [46] for the case in which buckling occurs by the faces sliding relative to one another (crimping). As seen by Eqs. (F-9) and (F-10), the core shear mode (crimping mode) is a function of only one material property, $G_{xz}$.

The nonsymmetric facing buckling mode is primarily a function of $G_{xy}$ and secondarily a function of $G_{xz}$. This is the most complicated type of buckling and no simplifications of the general solution are apparent. It is important to note that this buckling mode cannot take place in shells with isotropic facings, yet it is the most predominant buckling mode in the case of facings of composite materials, since they usually have high values of $E_x/G_{xy}$.

In the example used here, the line separating Surfaces 1 and 2 is not obvious due to the smooth transition between modes. For this case, it occurs at an $E_x/G_{xz}$ ratio of approximately 1100. For other choices of the material and geometrical parameters, a more distinct transition might occur.

Figure F.2 shows various cuts through the buckling surface parallel to the $E_x/G_{xz}$ axis. It is apparent from this figure that any plane parallel to the $E_x/G_{xz}$ axis will reveal identical curves for the axisymmetric modes (Surfaces 1 and 2). This corresponds to cuts 1 and 2 shown in Fig. F.2. Although this axisymmetric buckling line provides a bound on buckling, it is an upper bound to the non-symmetric mode and is of little interest in this regard.

Cut 4 shown in Fig. F.2 is based upon an actual set of material and geometrical parameters and thus helps to give some insight into the design aspects of the buckling surface. This curve was calculated for the facing material of Ref. [22]. It shows the dependency of the buckling stress upon the core shear modulus.

It can also be noted in Fig. F.2 that the curves have a double curvature in the axisymmetric facing (Surface 1) and the nonsymmetric facing (Surface 3) buckling regions. This characteristic is probably due to different terms in Eqs. (F-1) and (F-2) predominating for different ranges of $G_{xz}$.

Fig. F.3 shows the cuts through the surface parallel to $E_x/G_{xy}$ axis. The axisymmetric regions (Surface 1 and 2) show up here as straight horizontal lines. Again this emphasizes that the solution is independent of $G_{xy}$ in the axisymmetric regions.

## F.3 Simplified Design Equations

Since the designer must normally make numerous calculations before arriving at the optimal design in terms of material and geometrical parameters, a computer solution such as used here is of little value except possibly to check the final design. With this in mind, an attempt has been made to approximate this more complicated analysis with a set of design equations which can be solved without the aid of a high-speed computer.

It must be pointed out that the solution discussed here considers general instability only and the final design must also be checked for column buckling and face dimpling or wrinkling. It is recommended that the designer follow the procedure outlined in Ref. [47] except that the equations presented here should be substituted for the general instability portion of the analysis.

Allowing $G_{xz}$ to become large in Eq. (F-7) results in the rigid-core axisymmetric solution

$$(\sigma_{cr})_{rc} = (2a/R) \left[\frac{E_x\ E_y}{1-\nu_{xy}\ \nu_{yx}}\right]^{1/2} \qquad \text{(F-11)}$$

Then Eq. (F-7) can be rewritten in the following form:

$$\sigma_{cr} = (\sigma_{cr})_{rc} \left[1 - (1/2)\frac{(\sigma_{cr})_{rc}\ th}{G_{xz}a^2}\right] \qquad \text{(F-12)}$$

Since Eq. (F-11) was found from an axisymmetric solution, the effect of facing shear is not present. Ref. [32] presented a rigid-core analysis for the nonsymmetric mode of buckling in which the equation analogous to Eq. (F-11) was found to be

$$(\sigma_{cr})^{*}_{rc} = (4t/R)\ \ [(1+h/2t)^3 - (h/2t)^3]^{1/2} \left[\frac{E_x\ E_y}{3(1-\nu_{xy}\ \nu_{yx})}\right]^{1/2} \Phi \qquad \text{(F-13)}$$

where $\Phi$ is given as $\Phi = 1$

$$\text{or} \quad \Phi = \left[ \frac{2\, G_{xy}\, (1 + \nu_{xy}\, \nu_{yx})}{E_x\, E_y} \right]^{1/2} \qquad \text{whichever is smaller.}$$

For a sandwich cylinder which has thin facings in comparison with the core thickness it can be shown by expanding the radical that

$$a \doteq \frac{2t}{\sqrt{3}} \left[ \left( 1 + \frac{h}{2t} \right)^3 - \left( \frac{h}{2t} \right)^3 \right]^{1/2} \qquad \text{(F-13)}$$

Thus, Eqs. (F-13) and (F-11) are essentially the same for the axisymmetric case ($\Phi = 1$). This suggests that if Eq. (F-13) is combined with Eq. (F-12), the resulting solution would provide a simple approximation which could be utilized in the nonsymmetric buckling region. Then

$$\sigma_{cr} = \Phi(\sigma_{cr})_{rc} \left[ 1 - (1/2) \frac{\Phi(\sigma_{cr})_{rc}\, th}{G_{xz}\, a^2} \right] \qquad \text{(F-14)}$$

Since a simple solution is already available for the core shear mode of buckling, Eq. (F-9), it is necessary only to determine which equation should be used for any particular set of design parameters. By comparing Eqs. (F-9) and (F-12), a simple criterion was found for determining which equation to use:

1. If the estimated stress given by Eq. (F-14) is less than one half of the rigid-core buckling stress, Eq. (F-13) then Eq. (F-9) should be used for estimating the critical stress.

2. Otherwise, Eq. (F-14) gives the critical estimate.

To determine how accurate an estimate the simplified method suggested above provides, the critical stresses as calculated by the simplified equation for several facing materials were compared to the results of the improved theory, Eqs. (F-1) and (F-2), using a computer. The best available analysis prior to this time was provided by the rigid-core analysis of Ref. [32]. The results obtained by the rigid-core analysis were also compared with the improved theory results. This information is tabulated in Tables 1 through 7 in Ref. [25] for facings of epoxy reinforced with 181-style E-glass cloth (cut 4 in Figures F.1 and F.2), unidirectional S-glass oriented longitudinally, unidirectional S-glass oriented circumferentially, unidirectional boron oriented longitudinally, unidirectional boron oriented circumferentially, unidirectional Thornel oriented longitudinally, and unidirectional Thornel oriented circumferentially. The geometric parameters used were the same as used before. Table 1 of Ref. [25] for the 181-style E-glass is considered representative and is repeated here as Table F.1.

The negative values shown in the table indicate an unconservative error, while positive values represent conservative variations. As expected, the rigid-core analysis is normally unconservative. In those few cases where it is unconservative, it is considerably closer than the rigid-core analysis. All of the points in the table are for values in the nonsymmetric mode and the axisymmetric core shear mode regions. There was no necessity to calculate data for the axisymmetric facing buckling mode, since the equations compared here must agree in this area due to the bases of the formulation.

Table F.1 Comparison of Error in Estimates of Buckling Coefficient for Honeycomb-Core Cylinders with Facings of 181-Style Cloth E-Glass/Epoxy

Facing Properties (Ref. [22])

$E_x = 3.28 \times 10^6$ psi $\quad G_{xy} = 0.416 \times 10^6$ psi

$E_y = 3.14 \times 10^6$ psi $\quad \nu_{xy} = 0.13$

| $E_x/G_{xz}$ | Critical Stress: Linear Analysis (Ref. [22]) psi | Present Estimate Eqs. (F-9) and (F-14) psi | Present Estimate % Error | Rigid-Core Analysis Eq. (F-13) psi | Rigid-Core Analysis % Error |
|---|---|---|---|---|---|
| 3280. | 8540 | 8540 | 0.0 | 25510 | -198.7 |
| 1640. | 17100 | 15977 | 6.6 | 25510 | -49.2 |
| 1093. | 21269 | 19155 | 9.9 | 25510 | -19.9 |
| 820. | 22325 | 20744 | 7.1 | 25510 | -14.3 |
| 547. | 23591 | 22333 | 5.3 | 25510 | -8.1 |
| 273.5 | 24756 | 23922 | 3.4 | 25510 | -3.0 |
| 164.2 | 25089 | 24557 | 2.1 | 25510 | -1.7 |
| 102.4 | 25285 | 24915 | 1.5 | 25510 | -0.9 |
| 65.6 | 25406 | 25129 | 1.1 | 25510 | -0.4 |
| 6.56 | 25606 | 25472 | 0.5 | 25510 | 0.4 |

## APPENDIX G

## COMPUTER PROGRAM DOCUMENTATION

The programs are written in G-level Fortran IV, and were run using an IBM System 360, Model 40, computer operating under the OS System.

Three different programs were written for the different cases; however, the difference between cases occurs in the main program only. All three programs are identified by the same name, BOSS, which denotes Buckling of Orthotropic Sandwich Shells. The three programs are identified as follows:

| | |
|---|---|
| BOSS-SS | Simple Supports, Axial Compression |
| BOSS-AA | Clamped Supports, Axial Compression Axisymmetric Buckling |
| BOSS-AB | Clamped Supports, Axial Compression and Bending Loading |

In general, the flow of the program can be summarized as follows: The main program reads the input data which defines the various material and geometric parameters, defines the number of assumed modal functions and their values including the circumferential wave number, and defines the type of loading considered. Then the main program computes the various coefficients and calls the subroutine INTEG which computes the various integrals and stores them in array form for later usage. Since the integrals are dependent only on the assumed longitudinal modal functions, their storage arrays can be maintained for multiple circumferential runs. Next, the main program computes the stiffness and load matrices. Finally the main program calls the subroutine EIG2 which solves the eigenvalue problem and returns both the eigenvalues and eigenvectors to the main program. The program prints out the reciprocal of $\lambda$. The critical stress can be found from Eqs. (A-40) and (A-42).

A flow diagram for the main program is shown in Fig. G.1. The subroutine SICI called by the subroutine INTEG is the IBM algorithm [45] for evaluating the sine-cosine integrals. The subroutine EIG2 is a modified NASA, Langley Research Center, subroutine that solves the eigenvalue problem:

$$\frac{1}{\lambda}\underbrace{\left[\begin{array}{c|c} A1 & A2 \\ \hline A2^T & A3 \end{array}\right]}_{\text{Stiffness Matrix}} = \underbrace{\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & B \end{array}\right]}_{\text{Load Matrix}} \qquad \text{(G-1)}$$

The subroutines DMATIN, EIGN, and JACOBI are called by EIG2 in solving Eq. (G-1). A description of each of these subroutines is contained in comment cards in the respective subroutine in Appendix H.

The input data decks, including format, for the respective programs are as follows:

BOSS-SS

1. FORMAT (4F20.8) ANG, RO, XL, T, H, MUX, MUT, EX, ET, GXT, GZX, GTZ, KX, KT

   ANG = Shell semi-vertex angle, $\alpha$. (degrees)

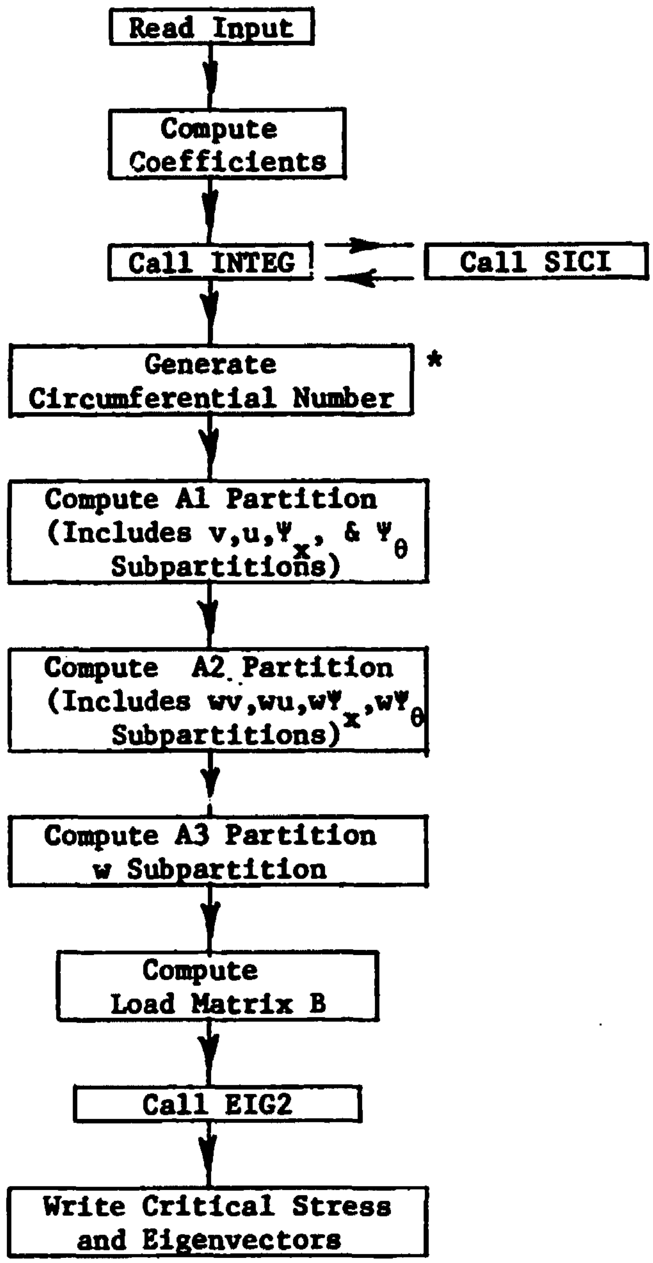


Figure G.1 - Typical Main Program Flow Diagram

---

*Optional. Since the integrals are independent of the circumferential mode number, NNN, it is generally desirable to insert a generation matrix at this location for NNN. This portion of the program is identified by comment cards in Appendix H. The input information uses only one value of NNN and can be overriden by this optional generation block.

$\alpha = 0.0$ for a cylinder

RO = Shell small-end radius, R. (inches)

XL = Shell slant length, L. (inches)

T = Facing half-thickness, t. (inches)

H = Core half-thickness, h. (inches)

MUX = Facing Poisson's ratio, $\nu_{x\theta}$. (dimensionless)

MUT = Facing Poisson's ratio, $\nu_{\theta x}$. (dimensionless)

EX = Facing elastic modulus in x-direction, $E_x$. (psi)

ET = Facing elastic modulus in θ-direction, $E_\theta$. (psi)

GXT = Facing shear modulus in x-θ plane, $G_{x\theta}$. (psi)

GZX = Core shear modulus in the z-x plane, $G_{zx}$. (psi)

GTZ = Core shear modulus in the θ-z plane, $G_{\theta z}$. (psi)

KX = Core shear coefficient in the z-x plane, $K_x$. (dimensionless)

KT = Core shear coefficient in the θ-z plane, $K_\theta$. (dimensionless)

2. FORMAT (I5/(8I10)) LLL, (N(I), I = 1,LLL)

   LLL = Number of longitudinal terms

   N(I) = Longitudinal half-wave numbers, $m_j$

3. FORMAT (I10) NNN

   NNN = Circumferential full-wave number (can be overriden by optional block of Fig. (G.1)

BOSS-AA

1. Same as Card 1. of BOSS-SS
2. Same as Card 2. of BOSS-SS

BOSS-AB

1. Same as Card 1. of BOSS-SS
2. Same as Card 2. of BOSS-SS
3. Same as Card 3. of BOSS-SS
4. FORMAT (4F20.8) PLOAD, BMOM

   PLOAD = Axial load factor defined in Eq. (A-42),
   P'. (dimensionless)

   BMOM = Bending load factor defined in Eq. (A-42),
   M'. (dimensionless)

5. FORMAT (I5) NCIR

   NCIR = Number of circumferential terms:

   3 - Bending

   1 - Axial Compression

   3 - Combined Loading

## APPENDIX H

## COMPUTER PROGRAM LISTING

```
C     MAIN PROGRAM FOR BOSS-SS
C     CONTROL CARDS ARE NOT INCLUDED IN THIS LISTING
      REAL*8 I1(16,16),I2(16,16),I3(16,16),IR1(16,16),IR2(16,16),
     1IR3(16,16),IIR1(16,16),IIR2(16,16),IIR3(16,16),IIRR1(16,16),
     2IIRR2(16,16),IIRR3(16,16),IIRRR1(16,16),IIRRR2(16,16),
     3IIRRR3(16,16)
      DOUBLE PRECISION PI,RAD,SA,CA,AA,    ANG,RO,XL,T,H,MUX,MUT,EX,ET,
     1GXT,GZX,GTZ,KX,KT,EXB,ETB,ROLSA,BBB,CBBB,SBBB,BRL,BRU,CW2,SW2,
     2AAA,CAAA,SAAA,ARL,ARU,CW1,SW1,TH2,T3,HT2,T3H2,CV1,CCV1,V1B,PIL,
     3PIL2,PIL4,NN,NN2,NN4
      DOUBLE PRECISION B(12,12),EVAL(12),EVEC(60,12),A1(48,48),A(60,60),
     1A2(48,12),A3(12,12),W4(48,12),W5(12,12),W6(12,12)          ,XN(16)
      DIMENSION IW8(48),IW9(48,2),IW10(12),N(16)
      COMMON /INT/I1,I2,I3,IR1,IR2,IR3,IIR1,IIR2,IIR3,IIRR1,IIRR2,
     1IIRR3,IIRRR1,IIRRR2,IIRRR3,XL,XN,PI,ANG,SA,RO,N,LLL
      READ(5,3) ANG,RO,XL,T,H,MUX,MUT,EX,ET,GXT,GZX,GTZ,KX,KT
    3 FORMAT(4F20.8)
      READ(5,4) LLL,(N(I),I=1,LLL)
    4 FORMAT(I5/(8I10))
      READ(5,999) NNN
  999 FORMAT(I10)
      WRITE(6,5) ANG,RO,XL,T,H,MUX,MUT,EX,ET,GXT,GZX,GTZ,KX,KT
    5 FORMAT(/' ANG =',D14.7,5X,'RO  =',D14.7,5X,'XL  =',D14.7,5X,'T  =
     1',D14.7,5X,'H   =',D14.7/' MUX =',D14.7,5X,'MUT =',D14.7,5X,'EX  =
     2',D14.7,5X,'ET  =',D14.7,5X,'GXT =',D14.7/' GZX =',D14.7,5X,'GTZ =
     3',D14.7,5X,'KX  =',D14.7,5X,'KT  =',D14.7)
      WRITE(6,6) LLL,(N(I),I=1,LLL)
    6 FORMAT(I5/' THE FUNCTION NUMBERS USED FOR W,V,U, AND CORE ROT. ARE
     1AS FOLLOWS'/20I6)
      PI=3.1415926536D0
      RAD=57.29578D0
      SA=DSIN(ANG/RAD)
      CA=DCOS(ANG/RAD)
      AA=H+T
      EXB=EX/(1.D0-MUX*MUT)
      ETB=ET/(1.D0-MUX*MUT)
```

```
      DO 1 I=1,LLL
      AN(I)=DFLOAT(N(I))
    1 CONTINUE
      TH2=2.D0*T*H*H
      T3=8.D0*T**3/3.D0
      HT2=2.D0*H*T*T
      T32=8.D0*H*H*T*T*T/3.D0
      LLLS=LLL*LLL
      KKK=4*LLL
      PIL=PI/XL
      PIL2=PIL*PIL
      PIL4=PIL2*PIL2
C
C     CALCULATE COEFFICIENTS
C
      B1 =2.D0*T*ETB*CA*CA
      B2 =H*KX*GZX
      B3 =T3*ETB*SA*SA
      B4 = KKT*GTZ
      B5 =4.D0*T3*GXT*SA*CA
      B6 =T3*EXB
      B7 =T3*ETB
      B8 =4.D0*T3*GXT
      B9 =2.D0*T*ETB*SA*SA
      B10=2.D0*T*EXB
      B11=2.D0*T*GXT
      B12=B4*CA*CA+B11*SA
      B13=B5*CA*CA
      B14=B11
      B15=T3*GXT*CA*CA
      B16=2.D0*T*ETB
      B17=B7*CA*CA
      B18=B2
      B19=TH2*ETB*SA*SA
      B20=TH2*EXB
      B21=TH2*GXT
```

```
B22=B4
B23=B21*SA*SA
B24=4.D0*T3H2*GXT*SA*SA*CA*CA
B25=B21
B26=T3H2*GXT*CA*CA
B27=TH2*ETB
B28=T3H2*ETB*CA*CA
B29=2.D0*B16*SA*CA
B30=2.D0*B10*MUT*CA
B31=2.D0*B16*CA
B32=2.D0*HT2*CA*CA*ETB
B33=2.D0*B5*MUT*SA
B34=2.D0*B7*SA
B35=-B34*CA
B36=2.D0*B2
B37=-2.D0*HT2*ETB*SA*SA
B38=-2.D0*HT2*EXB*MUT*SA
B39=-2.D0*HT2*ETB*SA
B40=-2.D0*B5*SA
B41=-2.D0*B4*CA
B42=-2.D0*B5*CA
B43=B8*SA*CA
B44=4.D0*HT2*GXT*SA
B45=2.D0*B4
B46=-B44*SA
B47=B44
B48=2.D0*B4*MUT
B49=-B48*CA
B50=-2.D0*HT2*EXB*MUT*SA
B51=-2.D0*HT2*EXB
B52=B51*MUT
B53=-2.D0*B7*CA
B54=B
B55=B2
B56=-2.D0*HT2*ETB
B57=2.D0*B3*SA*CA
```

```
      B58=-B8*CA
      B59=-4.D0*HT2*GXT
      B60=-B59*SA
      B61=B59
      B62=2.D0*B10*MUT*SA
      B63=2.D0*B16*SA
      B64=-B56*SA*CA
      B65=2.D0*B10*MUT
      B66=-B55*CA
      B67=-2.D0*B11*SA
      B68=2.D0*B11
      B69=-B60*CA
      B70=-B59*CA/2.D0
      B71=B67
      B72=-B43*CA
      B73=-B60*CA
      B74=-B45*CA
      B75=-2.D0*B73*SA
      B76=1.5D0*B73
      B77=B70
      B78=B76
      B79=2.D0*B70
      B80=B64
      B81=B66
      B82=-2.D0*B56*CA
      B83=2.D0*B20*MUT*SA
      B84=2.D0*B27*SA
      B85=2.D0*B20*MUT
      B86=-2.D0*B25*SA
      B87=2.D0*B25
      B88=B86
      B89=-4.D0*T3H2*GXT*SA*CA*CA
      V1B= 1.D0/(2.D0*PI*CA)
C
C     SUBROUTINE INTEG USES SUBROUTINE  SICI
C
```

```
      MNUMB=1
      CALL INTEG(MNUMB)
C
C     INSERT DO LOOP FOR GENERATING NNN, RUN TO 600,
C
      WRITE(6,25) NNN
   25 FORMAT(//' THE CIRCUMFERENTIAL WAVE NUMBER IS  ',I5)
      NN=DFLOAT(NNN)
      NN2=NN*NN
      NN4=NN2*NN2
      C1=2.D0*(B1+NN2*B4)
      C2=2.D0*NN2*(B5+NN2*B7)
      C3=2.D0*PIL2*B2
      C4=2.D0*PIL2*(B3+NN2*B8)
      C5=2.D0*PIL4*B6
      C6=NN2*PIL*(B40-B34)
      C7=PIL2*NN2*B48
      C8=-PIL2*PIL*B33
      C9=NN*(B31-B41)
      C10=-NN*(B42+NN2*B53)
      C11=-NN*B43*PIL
      C12=NN*(B35-B57)*PIL
      C13=-NN*B58*PIL2
      C14=-NN*B49*PIL2
      C15=B29
      C16=-B30*PIL
      C17=NN2*(B44-B54)
      C18=NN2*B55*PIL
      C19=B36*PIL
      C20=(B37+NN2*B59)*PIL
      C21=-B38*PIL2
      C22=-B50*PIL2
      C23=B51*PIL2*PIL
      C24=-NN*B45
      C25=NN*(B32-B46-NN2*B56)
      C26=-NN*B47*PIL
```

```
C27=NN*(B39-B60)*PIL
C28=-NN*B61*PIL2
C29=-NN*B52*PIL2
C30=2.D0*(B12+NN2*B16)
C31=2.D0*(B13+NN2*B17)
C32=2.D0*B14*PIL2
C33=2.D0*B15*PIL2
C34=B71*PIL
C35=B72*PIL
C36=NN*(B63-B67)
C37=-NN*B65*PIL
C38=-NN*B68*PIL
C39=NN*(B80-B73)
C40=-NN*B81*PIL
C41=-NN*B77*PIL
C42=B74
C43=B75+NN2*B82
C44=B76*PIL
C45=B78*PIL
C46=B79*PIL2
C47=2.D0*(B9+NN2*B11)
C48=2.D0*B10*PIL2
C49=-B62*PIL
C50=NN*(B64-B69)
C51=-NN*B70*PIL
C52=-NN*B66*PIL
C53=2.D0*B18
C54=2.D0*(B19+NN2*B21)
C55=2.D0*B20*PIL2
C56=-B83*PIL
C57=NN*(B84-B86)
C58=-NN*B87*PIL
C59=-NN*B85*PIL
C60=2.D0*B22
C61=2.D0*(B23+NN2*B27)
C62=2.D0*(B24+NN2*B28)
```

```
      C63=2.D0*PIL2*B25
      C64=2.D0*PIL2*B26
      C65=B88*PIL
      C66=B89*PIL
      CV1=V18*PIL2
C
C     COMPUTE STIFFNESS MATRIX  A(J,K) WHERE J AND K =5*LLL
C
C     W - W STIFFNESS MATRIX
C
      DO 300 I=1,LLL
      J=5*LLL+1-I
      CC3=C3*XN(I)
      CC4=C4*XN(I)
      CC5=C5*XN(I)*XN(I)
      CC6=C6*XN(I)
      CC7=C7*XN(I)*XN(I)
      CC8=C8*XN(I)
      CCC8=C8*XN(I)*XN(I)
      DO 300 M=1,LLL
      K=5*LLL+1-M
      A(J,K)=C1*IIR1(I,M)+C2*IIRRR1(I,M)+CC3*XN(M)*IR2(I,M)+CC4*XN(M)
     1*IIR2(I,M)+CC5*XN(M)*XN(M)*IR1(I,M)+C6*XN(M)*IIRR3(I,M)+CC6*IIRR
     23(M,I)+C7*XN(M)*XN(M)*IIR1(I,M)+CC7*IIR1(I,M)+CC8*XN(M)*XN(M)*
     3I3(M,I)+CCC8*XN(M)*I3(I,M)
  300 CONTINUE
C
C     W - V STIFFNESS MATRIX
C
      DO 301 I=1,LLL
      J=5*LLL+1-I
      CC12=C12*XN(I)
      CC13=C13*XN(I)
      CC14=C14*XN(I)*XN(I)
      DO 301 M=1,LLL
      K=4*LLL+1-M
```

```
      A(J,K)=C9*IIR1(I,M)+C10*IIRR1(I,M)+C11*XN(M)*IIRR3(I,M)
     1+CC12*IIRR3(M,I)+CC13*XN(M)*IIR2(I,M)+CC14*IIR1(I,M)
  301 CONTINUE
C
C     W - U STIFFNESS MATRIX
C
      DO 302 I=1,LLL
      J=5*LLL+1-I
      DO 302 M=1,LLL
      K=3*LLL+1-M
      A(J,K)=C15*IIR3(I,M)+C16*XN(M)*I1(I,M)
  302 CONTINUE
C
C     W - PSI X STIFFNESS MATRIX
C
      DO 303 I=1,LLL
      J=5*LLL+1-I
      CC19=C19*XN(I)
      CC20=C20*XN(I)
      CC21=C21*XN(I)
      CC22=C22*XN(I)*XN(I)
      CC23=C23*XN(I)*XN(I)
      DO 303 M=1,LLL
      K=2*LLL+1-M
      A(J,K)=C17*IIRR3(I,M)+C18*XN(M)*IIR1(I,M)+CC19*IR2(I,M)+CC20*IIR2
     1(I,M)+CC21*XN(M)*I3(M,I)+CC22*I3(I,M)+CC23*XN(M)*IR1(I,M)
  303 CONTINUE
C
C     W - PSI THETA STIFFNESS MATRIX
C
      DO 304 I=1,LLL
      J=5*LLL+1-I
      CC27=C27*XN(I)
      CC28=C28*XN(I)
      CC29=C29*XN(I)*XN(I)
      DO 304 M=1,LLL
```

```
      K=LLL+1-M
      A(J,K)=C24*I1(I,M)+C25*IIRR1(I,M)+C26*XN(M)*IIR3(I,M)+CC27*IIR3(M,
     1I)+CC28*XN(M)*I2(I,M)+CC29*I1(I,M)
  304 CONTINUE
C
C     V - W STIFFNESS MATRIX
C
      DO 305 I=1,LLL
      J=4*LLL+1-I
      CC11=C11*XN(I)
      CC13=C13*XN(I)
      DO 305 M=1,LLL
      K=5*LLL+1-M
      A(J,K)=C9*IIR1(I,M)+C10*IIRRR1(I,M)+C12*XN(M)*IIRR3(I,M)+C14*XN(M)
     1*XN(M)*IIR1(I,M)+CC11*IIRR3(M,I)+CC13*XN(M)*IIR2(I,M)
  305 CONTINUE
C
C     V - V STIFFNESS MATRIX
C
      DO 306 I=1,LLL
      J=4*LLL+1-I
      CC32=C32*XN(I)
      CC33=C33*XN(I)
      CC34=C34*XN(I)
      CC35=C35*XN(I)
      DO 306 M=1,LLL
      K=4*LLL+1-M
      A(J,K)=C30*IIR1(I,M)+C31*IIRRR1(I,M)+CC32*XN(M)*IR2(I,M)
     1+CC33*XN(M)*IIR2(I,M)+C34*XN(M)*I3(I,M)+CC34*I3(M,I)
     2+C35*XN(M)*IIRR3(I,M)+CC35*IIRR3(M,I)
  306 CONTINUE
C
C     V - U STIFFNESS MATRIX
C
      DO 307 I=1,LLL
      J=4*LLL+1-I
```

```
      CC38=C38*XN(I)
      DO 307 M=1,LLL
      K=3*LLL+1-M
      A(J,K)=C36*IIR3(I,M)+C37*XN(M)*I1(I,M)+CC38*I2(I,M)
  307 CONTINUE
C
C     V - PSI X STIFFNESS MATRIX
C
      DO 308 I=1,LLL
      J=4*LLL+1-I
      CC41=C41*XN(I)
      DO 308 M=1,LLL
      K=2*LLL+1-M
      A(J,K)=C39*IIRR3(I,M)+C40*XN(M)*IIR1(I,M)+CC41*IIR2(I,M)
  308 CONTINUE
C
C     V - PSI THETA STIFFNESS MATRIX
C
      DO 309 I=1,LLL
      J=4*LLL+1-I
      CC45=C45*XN(I)
      CC46=C46*XN(I)
      DO 309 M=1,LLL
      K=LLL+1-M
      A(J,K)=C42*I1(I,M)+C43*IIRR1(I,M)+C44*XN(M)*IIR3(I,M)+
     1CC45*IIR3(M,I)+CC46*XN(M)*I2(I,M)
  309 CONTINUE
C
C     U - W STIFFNESS MATRIX
C
      DO 310 I=1,LLL
      J=3*LLL+1-I
      CC16=C16*XN(I)
      DO 310 M=1,LLL
      K=5*LLL+1-M
      A(J,K)=C15*IIR3(M,I)+CC16*I1(I,M)
```

```
  310 CONTINUE
C
C     U - V STIFFNESS MATRIX
C
      DO 311 I=1,LLL
      J=3*LLL+1-I
      CC37=C37*XN(I)
      DO 311 M=1,LLL
      K=4*LLL+1-M
      A(J,K)=C36*IIR3(M,I)+C38*XN(M)*I2(I,M)+CC37*I1(I,M)
  311 CONTINUE
C
C     U - U STIFFNESS MATRIX
C
      DO 312 I=1,LLL
      J=3*LLL+1-I
      CC48=C48*XN(I)
      CC49=C49*XN(I)
      DO 312 M=1,LLL
      K=3*LLL+1-M
      A(J,K)=C47*IIR2(I,M)+CC48*XN(M)*IR1(I,M)+C49*XN(M)*I3(M,I)+
     1CC49*I3(I,M)
  312 CONTINUE
C
C     U - PSI X STIFFNESS MATRIX
C
      DO 313 I=1,LLL
      J=3*LLL+1-I
      DO 313 M=1,LLL
      K=2*LLL+1-M
      A(J,K)=0.D0
  313 CONTINUE
C
C     U - PSI THETA STIFFNESS MATRIX
C
      DO 314 I=1,LLL
```

```
      J=3*LLL+1-I
      CC52=C52*XN(I)
      DO 314 M=1,LLL
      K=LLL+1-M
      A(J,K)=C50*IIRR3(M,I)+C51*XN(M)*IIR2(I,M)+CC52*IIR1(I,M)
  314 CONTINUE
C
C     PSI X - W STIFFNESS MATRIX
C
      DO 315 I=1,LLL
      J=2*LLL+1-I
      CC18=C18*XN(I)
      CC21=C21*XN(I)
      CC23=C23*XN(I)
      DO 315 M=1,LLL
      K=5*LLL+1-M
      A(J,K)=C17*IIRR3(M,I)+C20*XN(M)*IIR2(I,M)+C19*XN(M)*IR2(I,M)+
     1C22*XN(M)*XN(M)*I3(M,I)+CC18*IIR1(I,M)+CC21*XN(M)*I3(I,M)+
     2CC23*XN(M)*XN(M)*IR1(I,M)
  315 CONTINUE
C
C     PSI X - V STIFFNESS MATRIX
C
      DO 316 I=1,LLL
      J=2*LLL+1-I
      CC40=C40*XN(I)
      DO 316 M=1,LLL
      K=4*LLL+1-M
      A(J,K)=C39*IIRR3(M,I)+CC40*IIR1(I,M)+C41*XN(M)*IIR2(I,M)
  316 CONTINUE
C
C     PSI X - U STIFFNESS MATRIX
C
      DO 317 I=1,LLL
      J=2*LLL+1-I
      DO 317 M=1,LLL
```

```
      K=3*LLL+1-M
      A(J,K)=0.D0
  317 CONTINUE
C
C     PSI X - PSI X STIFFNESS MATRIX
C
      DO 318 I=1,LLL
      J=2*LLL+1-I
      CC55=C55*XN(I)
      CC56=C56*XN(I)
      DO 318 M=1,LLL
      K=2*LLL+1-M
      A(J,K)=C53*IR2(I,M)+C54*IIR2(I,M)+CC55*XN(M)*IR1(I,M)+
     1C56*XN(M)*I3(M,I)+CC56*I3(I,M)
  318 CONTINUE
C
C     PSI X - PSI THETA STIFFNESS MATRIX
C
      DO 319 I=1,LLL
      J=2*LLL+1-I
      CC59=C59*XN(I)
      DO 319 M=1,LLL
      K=LLL+1-M
      A(J,K)=C57*IIR3(M,I)+C58*XN(M)*I2(I,M)+CC59*I1(I,M)
  319 CONTINUE
C
C     PSI THETA - W STIFFNESS MATRIX
C
      DO 320 I=1,LLL
      J=LLL+1-I
      CC26=C26*XN(I)
      CC28=C28*XN(I)
      DO 320 M=1,LLL
      K=5*LLL+1-M
      A(J,K)=C24*I1(I,M)+C25*IIRR1(I,M)+C27*XN(M)*IIR3(I,M)+C29*XN(M)*
     1XN(M)*I1(I,M)+CC26*IIR3(M,I)+CC28*XN(M)*I2(I,M)
```

```
  320 CONTINUE
C
C     PSI THETA - V STIFFNESS MATRIX
C
      DO 321 I=1,LLL
      J=LLL+1-I
      CC44=C44*XN(I)
      CC46=C46*XN(I)
      DO 321 M=1,LLL
      K=4*LLL+1-M
      A(J,K)=C42*I1(I,M)+C43*IIRR1(I,M)+C45*XN(M)*IIR3(I,M)+
     1CC44*IIR3(M,I)+CC46*XN(M)*I2(I,M)
  321 CONTINUE
C
C     PSI THETA - U STIFFNESS MATRIX
C
      DO 322 I=1,LLL
      J=LLL+1-I
      CC51=C51*XN(I)
      DO 322 M=1,LLL
      K=3*LLL+1-M
      A(J,K)=C50*IIRR3(I,M)+C52*XN(M)*IIR1(I,M)+CC51*IIR2(I,M)
  322 CONTINUE
C
C     PSI THETA - PSI X STIFFNESS MATRIX
C
      DO 323 I=1,LLL
      J=LLL+1-I
      CC58=C58*XN(I)
      DO 323 M=1,LLL
      K=2*LLL+1-M
      A(J,K)=C57*IIR3(I,M)+C59*XN(M)*I1(I,M)+CC58*I2(I,M)
  323 CONTINUE
C
C     PSI THETA - PSI THETA STIFFNESS MATRIX
C
```

```
      DO 324 I=1,LLL
      J=LLL+1-I
      CC63=C63*XN(I)
      CC64=C64*XN(I)
      CC65=C65*XN(I)
      CC66=C66*XN(I)
      DO 324 M=1,LLL
      K=LLL+1-M
      A(J,K)=C60*IR1(I,M)+C61*IIR1(I,M)+C62*IIRRR1(I,M)+CC63*XN(M)*
     1IR2(I,M)+CC64*XN(M)*IIR2(I,M)+C65*XN(M)*I3(I,M)+CC65*I3(M,I)+
     2C66*XN(M)*IIRR3(I,M)+CC66*IIRR3(M,I)
  324 CONTINUE
      WRITE(6,26) ((A(J,K),K=1,LLL5),J=1,LLL5)
   26 FORMAT(//' STIFFNESS MATRIX '//(6D20.8))
C
C     COMPUTE LOAD MATRIX B(J,K) WHERE  J AND K = LLL
C
      DO 325 I=1,LLL
      J=LLL+1-I
      CCV1=CV1*XN(I)
      DO 325 M=1,LLL
      K=LLL+1-M
      B(J,K)=CCV1*XN(M)*I2(I,M)
  325 CONTINUE
      WRITE(6,27) ((B(J,K),K=1,LLL),J=1,LLL)
   27 FORMAT(//' LOAD MATRIX '//(6D20.8))
C
C     PARTITION A MATRIX INTO A1, A2, AND A3 MATRICES
C
      LIMA1=4*LLL
      LIMA3=LLL
      DO 721 I=1,LIMA1
      DO 721 J=1,LIMA1
  721 A1(I,J)=A(I,J)
      DO 722 I=1,LIMA1
      DO 722 J=1,LIMA3
```

```
      J1=J+LIMA1
  722 A2(I,J)=A(I,J1)
      DO 723 I=1,LIMA3
      I1=I+LIMA1
      DO 723 J=1,LIMA3
      J1=J+LIMA1
  723 A3(I,J)=A(I1,J1)
      CALL EIG2(LIMA1,LIMA3,60,20,40,EVAL,EVEC,A1,A2,A3,W4,   IW8,IW9,
     1IW10,NERR,3,W5,W6)
      IF(NERR)200,201,200
  200 WRITE(6,400)
  400 FORMAT(//34H MASS MATRIX NOT POSITIVE DEFINITE)
      GO TO 600
  201 CONTINUE
C
C     OUTPUT OF EIGENVALUES AND EIGENVECTORS
C
      DO 600 I=1,LLL
      WRITE(6,601) I,EVAL(I),(EVEC(J,I),J=1,LLL5)
  601 FORMAT(//' EIGENVALUE  ',I5,'   EQUALS  ',D16.8/(6D20.8))
  600 CONTINUE
      STOP
      END
```

```
C      MAIN PROGRAM FOR BOSS-AA
C      CONTROL CARDS ARE NOT INCLUDED IN THIS LISTING
       REAL*8 I1(16,16),I2(16,16),I3(16,16),IR1(16,16),IR2(16,16),
      1IR3(16,16),IIR1(16,16),IIR2(16,16),IIR3(16,16),IIRR1(16,16),
      2IIRR2(16,16),IIRR3(16,16),IIRRR1(16,16),IIRRR2(16,16),IIRRR3(16,
      316)
       DOUBLE PRECISION PI,RAD,SA,CA,NNN,   ANG,RO,XL,T,H,MUX,MUT,EX,ET,
      1GXT,GZX,GTZ,KX,KT,EXB,ETB,ROLSA,BBB,CBBB,SBBB,BRL,BRU,CW2,SW2,
      2AAA,CAAA,SAAA,ARL,ARU,CW1,SW1,TH2,T3,HT2,T3H2
       DOUBLE PRECISION B(12,12),EVAL(12),EVEC(60,12),A1(48,48),
      1A2(48,12),A3(12,12),W4(48,12),W5(12,12),W6(12,12)          ,XN(16)
       DIMENSION IW8(48),IW9(48,2),IW10(12),N(16)
       COMMON /INT/I1,I2,I3,IR1,IR2,IR3,IIR1,IIR2,IIR3,IIRR1,IIRR2,
      1IIRR3,IIRRR1,IIRRR2,IIRRR3,XL,XN,PI,ANG,SA,RO,N,LLL
       READ(5,3) ANG,RO,XL,T,H,MUX,MUT,EX,ET,GXT,GZX,GTZ,KX,KT
     3 FORMAT(4F20.8)
       READ(5,4) LLL,(N(I),I=1,LLL)
     4 FORMAT(I5/(8I10))
       PLOAD=1.D0
       WRITE(6,5) ANG,RO,XL,T,H,MUX,MUT,EX,ET,GXT,GZX,GTZ,KX,KT
     5 FORMAT(/' ANG =',D14.7,5X,'RO  =',D14.7,5X,'XL  =',D14.7,5X,'T  =
      1',D14.7,5X,'H   =',D14.7/' MUX =',D14.7,5X,'MUT =',D14.7,5X,'EX  =
      2',D14.7,5X,'ET  =',D14.7,5X,'GXT =',D14.7/' GZX =',D14.7,5X,'GTZ =
      3',D14.7,5X,'KX  =',D14.7,5X,'KT  =',D14.7)
       WRITE(6,6) LLL,(N(I),I=1,LLL)
     6 FORMAT(I5/' THE FUNCTION NUMBERS USED FOR W,V,U, AND CORE ROT. ARE
      1AS FOLLOWS'/20I6)
       PI=3.141592653589793
       RAD=57.29577951308232
       SA=DSIN(ANG/RAD)
       CA=DCOS(ANG/RAD)
       EXB=EX/(1.D0-MUX*MUT)
       ETB=ET/(1.D0-MUX*MUT)
       DO 1 I=1,LLL
     1 XN(I)=DFLOAT(N(I))
       DO 2 I=1,LLL
```

```
      J=LLL+I
      N(J)=N(I)+2
    2 XN(J)=DFLOAT(N(J))
      DO 9 I=1,LLL
      J=2*LLL+I
      K=3*LLL+I
      N(J)=N(I)-1
      N(K)=N(I)+1
      XN(J)=DFLOAT(N(J))
    9 XN(K)=DFLOAT(N(K))
      TH2=2.D0*T*H*H
      T3=8.D0*T**3/3.D0
      HT2=2.D0*H*T*T
      T3H2=8.D0*H*H*T*T*T/3.D0
      LIMA1=2*LLL
      LIMA3=LLL
      LLLALL=LIMA1+LIMA3
      PIL=PI/XL
      PIL2=PIL*PIL
      PIL4=PIL2*PIL2
C     CALCULATE COEFFICIENTS
C
      B1 =2.D0*T*ETH*CA*CA
      B2 =H*KX*GZX
      B3 =T3*ETB*SA*SA
      B6 =T3*EXB
      B9 =2.D0*T*ETB*SA*SA
      B10=2.D0*T*EXB
      B16=2.D0*T*ETB
      B18=B2
      B19=TH2*ETB*SA*SA
      B20=TH2*EXB
      B29=2.D0*B16*SA*CA
      B30=2.D0*B10*MUT*CA
      B33=2.D0*B6*MUT*SA
      B36=2.D0*B2
```

```
      B37=-2.DO*HT2*ETB*SA*SA
      B38=-2.DO*HT2*EXB*MUT*SA
      B50=-2.DO*HT2*EXB*MUT*SA
      B51=-2.DO*HT2*EXB
      B62=2.DO*B10*MUT*SA
      B83=2.DO*B20*MUT*SA
      V1=PLOAD/(4.DO*PI*CA)
C
C     SUBROUTINE INTEG IS CALLED, IT IS INDEPENDENT OF NNN
C
      MNUMB=4
      CALL INTEG(MNUMB)
C
C     INIALIZATION OF A1,A2,A3, AND B MATRICES TO ZERO
C
      DO 370 J=1,LIMA1
      DO 370 K=1,LIMA1
  370 A1(J,K)=0.DO
      DO 371 J=1,LIMA1
      DO 371 K=1,LIMA3
  371 A2(J,K)=0.DO
      DO 372 J=1,LIMA3
      DO 372 K=1,LIMA3
      A3(J,K)=0.DO
  372 B(J,K)=0.DO
C
C     U - U STIFFNESS MATRIX
C
      C1=2.DO*B9
      DO 301 I=1,LLL
      J=2*LLL+1-I
      C2=2.DO*PIL2*XN(I)*B10
      C3=-B62*PIL
      C4=C3*XN(I)
      DO 301 M=1,LLL
      K=2*LLL+1-M
```

```
  301 A1(J,K)=C1*IIR2(I,M)+C2*XN(M)*IR1(I,M)+C3*XN(M)*I3(M,I)+C4*I3(I,M)
C
C     PSI X - PSI X  STIFFNESS MATRIX
C
      C1=2.D0*B19
      DO 302 I=1,LLL
      J=LLL+1-I
      C2=2.D0*B20*XN(I)*PIL2
      C3=-B83*XN(I)*PIL
      DO 302 M=1,LLL
      K=LLL+1-M
  302 A1(J,K)=2.D0*B18*IR2(I,M)+C1*IIR2(I,M)+C2*XN(M)*IR1(I,M)-PIL*B83*
     1XN(M)*I3(M,I)+C3*I3(I,M)
C
C     W - U  STIFFNESS MATRIX
C
      DO 310 I=1,LLL
      J=2*LLL+1-I
      C1=B30*XN(I)*PIL
      DO 310 M=1,LLL
      Z=2*LLL+M
      K=LLL+1-M
      LM=3*LLL+M
  310 A2(J,K)=B29*(IIR2(I,Z)-IIR2(I,LM))+C1*(I3(I,LM)-I3(I,Z))
C
C     W - PSI X  STIFFNESS MATRIX
C
      C3=PIL*B37
      DO 311 I=1,LLL
      J=LLL+1-I
      C4=B38*XN(I)*PIL2
      C5=B51*XN(I)*PIL2*PIL
      DO 311 M=1,LLL
      Z=2*LLL+M
      K=LLL+1-M
      BM2=XN(Z)+2.D0
```

```
      LM=3*LLL+M
  311 A2(J,K)=PIL*B36*(BM2*IR3(LM,I)-XN(Z)*IR3(Z,I))+C3*(BM2*IIR3(LM,I)-
     1XN(Z)*IIR3(Z,I))+C4*(XN(Z)*I1(I,Z)-BM2*I1(I,LM))+B50*PIL2*
     2(BM2*BM2*I2(I,LM)-XN(Z)*XN(Z)*I2(I,Z))+C5*(XN(Z)*XN(Z)*IR3(I,Z)-
     3BM2*BM2*IR3(I,LM))
C
C     W - W STIFFNESS MATRIX
C
      C3=2.DO*B2*PIL2
      C5=2.DO*B6*PIL4
      C8=B33*PIL2*PIL
      C1=2.DO*B1
      C4=2.DO*B3*PIL2
      DO 313 I=1,LLL
      Y=2*LLL+I
      J=LLL+1-I
      BI2=XN(Y)+2.DO
      LI=3*LLL+I
      DO 313 M=1,LLL
      Z=2*LLL+M
      K=LLL+1-M
      BM2=XN(Z)+2.DO
      LM=3*LLL+M
  313 A3(J,K)=C1*(IIR2(Y,Z)-IIR2(Y,LM)-IIR2(LI,Z)+IIR2(LI,LM))
     1+C3*(XN(Y)*XN(Z)*IR1(Y,Z)-XN(Y)*BM2*IR1(Y,LM)-BI2*XN(Z)*IR1(LI,Z)+
     2BI2*BM2*IR1(LI,LM))+C4*(XN(Y)*XN(Z)*IIR1(Y,Z)-XN(Y)*BM2*IIR1(Y,LM)
     3-BI2*XN(Z)*IIR1(LI,Z)+BI2*BM2*IIR1(LI,LM))+C5*(XN(Y)*XN(Z)*XN(Y)
     4*XN(Z)*IR2(Y,Z)-XN(Y)*XN(Y)*BM2*BM2*IR2(Y,LM)-BI2*BI2*XN(Z)*XN(Z)
     5*IR2(LI,Z)+BI2*BI2*BM2*BM2*IR2(LI,LM))
     6+C8*(XN(Z)*XN(Z)*(XN(Y)*I3(Y,Z)-BI2*I3(LI,Z)) +
     7BM2*BM2*(BI2*I3(LI,LM)-XN(Y)*I3(Y,LM))+XN(Y)*XN(Y)*(XN(Z)*I3(Z,Y)
     8-BM2*I3(LM,Y))+BI2*BI2*(BM2*I3(LM,LI)-XN(Z)*I3(Z,LI)))
C
C     W - W LOAD MATRIX
      C2=2.DO*V1*PIL2
      DO 315 I=1,LLL
```

```
      Y=2*LLL+I
      J=LLL+1-I
      BI2=XN(Y)+2.DO
      LI=3*LLL+I
      DO 315 M=1,LLL
      Z=2*LLL+M
      K=LLL+1-M
      BM2=XN(Z)+2.DO
      LM=3*LLL+M
  315 B(J,K)=C1*(XN(Y)*(XN(Z)*I1(Y,Z)-BM2*I1(Y,LM))+BI2*(BM2*I1(LI,LM)
     1-XN(Z)*I1(LI,Z)))
C
C     SETTING THE TRANSPOSE OF A1, A3, AND B MATRICES  EQUAL
C
      DO 350 J=2,LIMA1
      J1=J-1
      DO 350 K=1,J1
  350 A1(J,K)=A1(K,J)
      DO 351 J=2,LIMA3
      J1=J-1
      DO 351 K=1,J1
      B(J,K)=B(K,J)
  351 A3(J,K)=A3(K,J)
      WRITE(6,26) ((A1(J,K),K=1,LIMA1),J=1,LIMA1)
   26 FORMAT(//'  A1(J,K) STIFFNESS MATRIX = '//(6D20.8))
      WRITE(6,27) ((A2(J,K),K=1,LIMA3),J=1,LIMA1)
   27 FORMAT(//'  A2(J,K) STIFFNESS MATRIX = '//(6D20.8))
      WRITE(6,28) ((A3(J,K),K=1,LIMA3),J=1,LIMA3)
   28 FORMAT(//'  A3(J,K) STIFFNESS MATRIX =  '//(6D20.8))
      WRITE(6,29) ((B(J,K),K=1,LIMA3),J=1,LIMA3)
   29 FORMAT(//'  LOAD MATRIX =  '//(6D20.8))
      CALL EIG2(LIMA1,LIMA3,60,20,40,EVAL,EVEC,A1,A2,A3,W4,IW8,IW9,
     1IW10,NERR,B,W5,W6)
      IF(NERR)200,201,200
  200 WRITE(6,400)
  400 FORMAT(//'   REDUCED STIFFNESS MATRIX IS NOT POSITIVE DEFINITE')
```

```
      GO TO 600
  201 CONTINUE
C
C     OUTPUT OF EIGENVALUES AND EIGENVECTORS
C
      DO 600 I=1,LIMA3
      WRITE(6,601) I,EVAL(L),(EVEC(J,I),J=1,LLLALL)
  601 FORMAT(//' EIGENVALUE  ',I5,'   EQUALS  ',D16.8/(6D20.8))
  600 CONTINUE
      STOP
      END
```

```
C     MAIN PROGRAM FOR BOSS-AB
C     CONTROL CARDS ARE NOT INCLUDED IN THIS LISTING
      REAL*8 I1(16,16),I2(16,16),I3(16,16),IR1(16,16),IR2(16,16),
     1IR3(16,16),IIR1(16,16),IIR2(16,16),IIR3(16,16),IIRR1(16,16),
     2IIRR2(16,16),IIRR3(16,16),IIRRR1(16,16),IIRRR2(16,16),
     3IIRRR3(16,16)
      DOUBLE PRECISION PI,RAD,SA,CA,AA,    ANG,RO,XL,T,H,MUX,MUT,EX,ET,
     1GXT,GZX,GTZ,KX,KT,EXB,ETB,ROLSA,BBB,CBBB,SBBB,BRL,BRU,CW2,SW2,
     2AAA,CAAA,SAAA,ARL,ARU,CW1,SW1,TH2,T3,HT2,T3H2,CV1,CCV1,V1B,PIL,
     3PIL2,PIL4,NN,NN2,NN4
      DOUBLE PRECISION B(12,12),EVAL(12),EVEC(60,12),A1(48,48),
     1A2(48,12),A3(12,12),W4(48,12),W5(12,12),W6(12,12)          ,XN(16)
      DIMENSION IW8(48),IW9(48,2),IW10(12),N(16)
      COMMON /INT/I1,I2,I3,IR1,IR2,IR3,IIR1,IIR2,IIR3,IIRR1,IIRR2,
     1IIRR3,IIRRR1,IIRRR2,IIRRR3,XL,XN,PI,ANG,SA,RO,N,LLL
      INTEGER Y,Z
      READ(5,3) ANG,RO,XL,T,H,MUX,MUT,EX,ET,GXT,GZX,GTZ,KX,KT
    3 FORMAT(4F20.8)
      READ(5,4) LLL,(N(I),I=1,LLL)
    4 FORMAT(I5/(8I10))
      READ(5,999) NNN
  999 FORMAT(I10)
      READ(5,3) PLOAD, BMOM
      READ(5,8) NCIR
    8 FORMAT(I5)
      WRITE(6,7) PLOAD,BMOM,NCIR
    7 FORMAT(/' AXIAL LOAD FACTOR =   ',D14.7,'         MOMENT FACTOR =
     1 ',D14.7,'   NO. CIRCUM. TERMS = ',I5)
      WRITE(6,5) ANG,RO,XL,T,H,MUX,MUT,EX,ET,GXT,GZX,GTZ,KX,KT
    5 FORMAT(/' ANG =',D14.7,5X,'RO  =',D14.7,5X,'XL  =',D14.7,5X,'T   =
     1',D14.7,5X,'H   =',D14.7/' MUX =',D14.7,5X,'MUT =',D14.7,5X,'EX  =
     2',D14.7,5X,'ET  =',D14.7,5X,'GXT =',D14.7/' GZX =',D14.7,5X,'GTZ =
     3',D14.7,5X,'KX  =',D14.7,5X,'KT  =',D14.7)
      WRITE(6,6) LLL,(N(I),I=1,LLL)
    6 FORMAT(I5/' THE FUNCTION NUMBERS USED FOR W,V,U, AND CORE ROT. ARE
     1AS FOLLOWS'/20I6)
```

```
      PI=3.1415926536D0
      RAD=57.29578D0
      SA=DSIN(ANG/RAD)
      CA=DCOS(ANG/RAD)
      AA=H+T
      EXB=EX/(1.D0-MUX*MUT)
      ETB=ET/(1.D0-MUX*MUT)
      DO 1 I=1,LLL
    1 XN(I)=DFLOAT(N(I))
      DO 2 I=1,LLL
      J=LLL+I
      N(J)=N(I)+2
    2 XN(J)=DFLOAT(N(J))
      DO 9 I=1,LLL
      J=2*LLL+I
      K=3*LLL+I
      N(J)=N(I)-1
      N(K)=N(I)+1
      XN(J)=DFLOAT(N(J))
    9 XN(K)=DFLOAT(N(K))
      TH2=2.D0*T*H*H
      T3=8.D0*T**3/3.D0
      HT2=2.D0*H*T*T
      T3H2=8.D0*H*H*T*T*T/3.D0
      LLL5=5*LLL*NCIR
      KKK=4*LLL*NCIR
      PIL=PI/XL
      PIL2=PIL*PIL
      PIL4=PIL2*PIL2
C
C     CALCULATE COEFFICIENTS
C
      B1 =2.D0*T*ETB*CA*CA
      B2 =H*KX*GZX
      B3 =T3*ETB*SA*SA
      B4 =H*KT*GTZ
```

```
B5 =4.D0*T3*GXT*SA*SA
B6 =T3*EXB
B7 =T3*ETB
B8 =4.D0*T3*GXT
B9 =2.D0*T*ETB*SA*SA
B10=2.D0*T*EXB
B11=2.D0*T*GXT
B12=B4*CA*CA+B11*SA*SA
B13=B5*CA*CA
B14=B11
B15=T3*GXT*CA*CA
B16=2.D0*T*ETB
B17=B7*CA*CA
B18=B2
B19=TH2*ETB*SA*SA
B20=TH2*EXB
B21=TH2*GXT
B22=B4
B23=B21*SA*SA
B24=4.D0*T3H2*GXT*SA*SA*CA*CA
B25=B21
B26=T3H2*GXT*CA*CA
B27=TH2*ETB
B28=T3H2*ETB*CA*CA
B29=2.D0*B16*SA*CA
B30=2.D0*B10*MUT*CA
B31=2.D0*B16*CA
B32=2.D0*HT2*CA*CA*ETB
B33=2.D0*B6*MUT*SA
B34=2.D0*B7*SA
B35=-B34*CA
B36=2.D0*B2
B37=-2.D0*HT2*ETB*SA*SA
B38=-2.D0*HT2*EXB*MUT*SA
B39=-2.D0*HT2*ETB*SA
B40=-2.D0*B8*SA
```

```
B41=-2.D0*B4*CA
B42=-2.D0*B5*CA
B43=B8*SA*CA
B44=4.D0*HT2*GXT*SA
B45=2.D0*B4
B46=-B44*SA
B47=B44
B48=2.D0*B6*MUT
B49=-B48*CA
B50=-2.D0*HT2*EXB*MUT*SA
B51=-2.D0*HT2*EXB
B52=B51*MUT
B53=-2.D0*B7*CA
B54=B39
B55=B52
B56=-2.D0*HT2*ETB
B57=2.D0*B8*SA*CA
B58=-B8*CA
B59=-4.D0*HT2*GXT
B60=-B59*SA
B61=B59
B62=2.D0*B10*MUT*SA
B63=2.D0*B16*SA
B64=-B56*SA*CA
B65=2.D0*B10*MUT
B66=-B55*CA
B67=-2.D0*B11*SA
B68=2.D0*B11
B69=-B60*CA
B70=-B59*CA/2.D0
B71=B67
B72=-B43*CA
B73=-B60*CA
B74=-B45*CA
B75=-2.D0*B73*SA
B76=1.5D0*B73
```

```
      B77=B70
      B78=B76
      B79=2.D0*B70
      B80=B64
      B81=B66
      B82=-2.D0*B56*CA
      B83=2.D0*B20*MUT*SA
      B84=2.D0*B27*SA
      B85=2.D0*B20*MUT
      B86=-2.D0*B25*SA
      B87=2.D0*B25
      B88=B86
      B89=-4.D0*T3H2*GXT*SA*CA*CA
      V1=PLOAD/(4.D0*PI*CA)
      V2=-BMOM/(2.D0*PI*CA)
C
C     SUBROUTINE INTEG USES SUBROUTINE  SICI
C
      MNUMB=4
      CALL INTEG(MNUMB)
C
C     INSERT DO LOOP FOR GENERATING NNN, RUN TO 600,
C
      WRITE(6,25) NNN
   25 FORMAT(//' THE CIRCUMFERENTIAL WAVE NUMBER IS  ',I5)
      LIMA1=4*LLL*NCIR
      DO 299 J=1,LIMA1
      DO 299 K=1,LIMA1
  299 A1(J,K)=0.D0
      GO TO (20,21,21),NCIR
   20 N1=NNN
      N2=NNN
      N3=NNN
      GO TO 22
   21 N1=NNN-1
      N2=NNN
```

```
      N3=NNN+1
   22 LIMA3=LLL*NCIR
      DO 298 J=1,LIMA3
      DO 298 K=1,LIMA3
  298 A3(J,K)=0.D0
      DO 297 J=1,LIMA1
      DO 297 K=1,LIMA3
  297 A2(J,K)=0.D0
C
C     COMPUTE STIFFNESS MATRIX  A(J,K) WHERE J AND K =5*LLL
C
C
C     V - V STIFFNESS MATRIX
C
      NC=-1
      DO 300 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=2.D0*(B16*XN4*XN4+B12)
      C2=2.D0*(B17*XN4*XN4+B13)
      DO 300 I=1,LLL
      J=LLL*(4*NCIR-NC)+1-I
      C3=2.D0*B14*XN(I)*PIL2
      C4=2.D0*B15*XN(I)*PIL2
      C5=B71*PIL
      C6=C5*XN(I)
      C7=B72*PIL
      C8=C7*XN(I)
      DO 300 M=1,LLL
      K=LLL*(4*NCIR-NC)+1-M
  300 A1(J,K)=C1*IIR1(I,M)+C2*IIRRR1(I,M)+C3*XN(M)*IR2(I,M)+C4*XN(M)
     1*IIR2(I,M)+C5*XN(M)*I3(I,M)+C6*I3(M,I)+C7*XN(M)*IIRR3(I,M)+
     2C8*IIRR3(M,I)
C
C     U - U STIFFNESS MATRIX
C
```

```
      NC=-1
      DO 301 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=2.D0*(B11*XN4*XN4+B9)
      DO 301 I=1,LLL
      J=LLL*(3*NCIR-NC)+1-I
      C2=2.D0*PIL2*XN(I)*B10
      C3=-B62*PIL
      C4=C3*XN(I)
      DO 301 M=1,LLL
      K=LLL*(3*NCIR-NC)+1-M
  301 A1(J,K)=C1*(IR2(I,M)+C2*XN(M)*IR1(I,M)+C3*XN(M)*I3(M,I)+C4*I3(I,M)
C
C     PSI X - PSI X STIFFNESS MATRIX
C
      NC=-1
      DO 302 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=2.D0*(B21*XN4*XN4+B19)
      DO 302 I=1,LLL
      J=LLL*(2*NCIR-NC)+1-I
      C2=2.D0*B20*XN(I)*PIL2
      C3=-B83*XN(I)*PIL
      DO 302 M=1,LLL
      K=LLL*(2*NCIR-NC)+1-M
  302 A1(J,K)=2.D0*B18*IR2(I,M)+C1*IIR2(I,M)+C2*XN(M)*IR1(I,M)-PIL*B83*
     1XN(M)*I3(M,I)+C3*I3(I,M)
C
C     PSI THETA - PSI THETA STIFFNESS MATRIX
C
      NC=-1
      DO 303 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
```

```
      C1=2.D0*(B27*XN4*XN4+B23)
      C2=2.D0*(B28*XN4*XN4+B24)
      DO 303 I=1,LLL
      J=LLL*(   NCIR-NC)+1-I
      C3=2.D0*B25*PIL2*XN(I)
      C4=2.D0*B26*PIL2*XN(I)
      C5=B88*PIL*XN(I)
      C6=B89*PIL*XN(I)
      DO 303 M=1,LLL
      K=LLL*(   NCIR-NC)+1-M
  303 A1(J,K)=2.D0*B22*IR1(I,M)+C1*IIR1(I,M)+C2*IIRRR1(I,M)+C3*XN(M)*
     1IR2(I,M)+C4*XN(M)*IIR2(I,M)+B88*PIL*XN(M)*I3(I,M)+C5*I3(M,I)+
     2B89*PIL*XN(M)*IIRR3(I,M)+C6*IIRR3(M,I)
C
C     U - V STIFFNESS MATRIX
C
      NC=-1
      DO 304 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=XN4*(B63-B67)
      C2=-B68*XN4*PIL
      DO 304 I=1,LLL
      J=LLL*(3*NCIR-NC)+1-I
      C3=-B65*XN4*XN(I)*PIL
      DO 304 M=1,LLL
      K=LLL*(4*NCIR-NC)+1-M
      A1(J,K)=C1*IIR3(M,I)+C2*XN(M)*I2(I,M)+C3*I1(I,M)
  304 A1(K,J)=A1(J,K)
C
C     PSI THETA - PSI X STIFFNESS MATRIX
C
      NC=-1
      DO 305 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
```

```
      C1=XN4*(B84-B86)
      C2=-B85*XN4*PIL
      DO 305 I=1,LLL
      J=LLL*(   NCIR-NC)+1-I
      C3=-B87*XN4*PIL*XN(I)
      DO 305 M=1,LLL
      K=LLL*(2*NCIR-NC)+1-M
      A1(J,K)=C1*IIR3(I,M)+C2*I1(I,M)*XN(M)+C3*I2(I,M)
  305 A1(K,J)=A1(J,K)
C
C     PSI X - V STIFFNESS MATRIX
C
      NC=-1
      DO 306 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=XN4*(B80-B73)
      C2=-B77*XN4*PIL
      DO 306 I=1,LLL
      J=LLL*(2*NCIR-NC)+1-I
      C3=-B81*XN4*PIL*XN(I)
      DO 306 M=1,LLL
      K=LLL*(4*NCIR-NC)+1-M
      A1(J,K)=C1*IIRR3(M,I)+C2*XN(M)*IIR2(I,M)+C3*IIR1(I,M)
  306 A1(K,J)=A1(J,K)
C
C     PSI THETA - U STIFFNESS MATRIX
C
      NC=-1
      DO 307 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=XN4*(B64-B69)
      C2=-B66*XN4*PIL
      DO 307 I=1,LLL
      J=LLL*(   NCIR-NC)+1-I
```

```
      C3=-B70*XN4*XN(I)*PIL
      DO 307 M=1,LLL
      K=LLL*(3*NCIR-NC)+1-M
      A1(J,K)=C1*IIRR3(I,M)+C2*XN(M)*IIR1(I,M)+C3*IIR2(I,M)
  307 A1(K,J)=A1(J,K)
C
C     PSI THETA - V STIFFNESS MATRIX
C
      NC=-1
      DO 308 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=B82*XN4*XN4+B75
      DO 308 I=1,LLL
      J=LLL*(   NCIR-NC)+1-I
      C2=B76*XN(I)*PIL
      C3=B79*XN(I)*PIL2
      DO 308 M=1,LLL
      K=LLL*(4*NCIR-NC)+1-M
      A1(J,K)=B74*I1(I,M)+C1*IIRR1(I,M)+C2*IIR3(M,I)+
     1B78*PIL*XN(M)*IIR3(I,M)+C3*XN(M)*I2(I,M)
  308 A1(K,J)=A1(J,K)
      WRITE(6,26) ((A1(J,K),K=1,LIMA1),J=1,LIMA1)
   26 FORMAT(//' STIFFNESS MATRIX '//(6D20.8))
C
C     V - W STIFFNESS MATRIX
C
      NC=-1
      DO 309 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=XN4*(B31-B41)
      C2=-XN4*(B42+XN4*XN4*B53)
      C4=XN4*PIL*(B35-B57)
      C6=XN4*B49*PIL2
      DO 309 I=1,LLL
```

```
      J=LLL*(4*NCIR-NC)+1-I
      C3=-B43*XN4*XN(I)*PIL
      C5=-B58*XN4*XN(I)*PIL2
      DO 309 M=1,LLL
      Z=2*LLL+M
      K=LLL*(   NCIR-NC)+1-M
      BM2=XN(Z)+2.D0
      LM=3*LLL+M
  309 A2(J,K)=C1*(IIR3(I,Z)-IIR3(I,LM))+C2*(IIRRR3(I,Z)-IIRRR3(I,LM))+
     1C3*(IIRR2(I,Z)-IIRR2(I,LM))+C4*(BM2*IIRR1(I,LM)-XN(Z)*IIRR1(I,Z))
     2+C5*(BM2*IIR3(LM,I)-XN(Z)*IIR3(Z,I))+C6*(BM2*BM2*IIR3(I,LM)-
     3XN(Z)*XN(Z)*IIR3(I,Z))
C
C     U - W STIFFNESS MATRIX
C
      NC=-1
      DO 310 N4=N1,N3
      NC=NC+1
      DO 310 I=1,LLL
      J=LLL*(3*NCIR-NC)+1-I
      C1=B30*XN(I)*PIL
      DO 310 M=1,LLL
      Z=2*LLL+M
      K=LLL*(   NCIR-NC)+1-M
      LM=3*LLL+M
  310 A2(J,K)=B29*(IIR2(I,Z)-IIR2(I,LM))+C1*(I3(I,LM)-I3(I,Z))
C
C     PSI X - W STIFFNESS MATRIX
C
      NC=-1
      DO 311 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=XN4*XN4*(B44-B54)
      C3=PIL*(B37+B59*XN4*XN4)
      DO 311 I=1,LLL
```

```
      J=LLL*(2*NCIR-NC)+1-I
      C2=B55*XN4*XN4*PIL*XN(I)
      C4=B38*XN(I)*PIL2
      C5=B51*XN(I)*PIL2*PIL
      DO 311 M=1,LLL
      Z=2*LLL+M
      K=LLL*(   NCIR-NC)+1-M
      BM2=XN(Z)+2.D0
      LM=3*LLL+M
  311 A2(J,K)=C1*(IIRR2(I,Z)-IIRR2(I,LM))+C2*(IIR3(I,Z)-IIR3(I,LM))+
     1PIL*B36*(BM2*IR3(LM,I)-XN(Z)*IR3(Z,I))+C3*(BM2*IIR3(LM,I)-
     2XN(Z)*IIR3(Z,I))+C4*(XN(Z)*I1(I,Z)-BM2*I1(I,LM))+B50*PIL2*
     3(BM2*BM2*I2(I,LM)-XN(Z)*XN(Z)*I2(I,Z))+C5*(XN(Z)*XN(Z)*IR3(I,Z)-
     4BM2*BM2*IR3(I,LM))
C
C     PSI THETA - W STIFFNESS MATRIX
C
      NC=-1
      DO 312 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=-B45*XN4
      C2=XN4*(B32-B46-B56*XN4*XN4)
      C4=XN4*PIL*(B39-B60)
      C6=B52*XN4*PIL2
      DO 312 I=1,LLL
      J=LLL*(   NCIR-NC)+1-I
      C3=-B47*XN4*XN(I)*PIL
      C5=-B61*XN4*XN(I)*PIL2
      DO 312 M=1,LLL
      Z=2*LLL+M
      K=LLL*(   NCIR-NC)+1-M
      BM2=XN(Z)+2.D0
      LM=3*LLL+M
  312 A2(J,K)=C1*(I3(I,Z)-I3(I,LM))+C2*(IIRR3(I,Z)-IIRR3(I,LM))+
     1C3*(IIR2(I,Z)-IIR2(I,LM))+C4*(BM2*IIR1(I,LM)-XN(Z)*IIR1(I,Z))+
```

```
     2C5*(BM2*I3(LM,I)-XN(Z)*I3(Z,I))+C6*(BM2*BM2*I3(I,LM)-
     3XN(Z)*XN(Z)*I3(I,Z))
      WRITE(6,26) ((A2(J,K),K=1,LIMA3),J=1,LIMA1)
C
C     W - W STIFFNESS MATRIX
C
      C3=2.DO*B2*PIL2
      C5=2.DO*B6*PIL4
      C8=B33*PIL2*PIL
      NC=-1
      DO 313 N4=N1,N3
      NC=NC+1
      XN4=DFLOAT(N4)
      C1=2.DO*(B4*XN4*XN4+B1)
      C2=2.DO*XN4*XN4*(B7*XN4*XN4+B5)
      C4=2.DO*(B8*XN4*XN4+B3)*PIL2
      C6=PIL2*XN4*XN4*(B40-B34)
      C7=-B48*XN4*XN4*PIL2
      DO 313 I=1,LLL
      Y=2*LLL+I
      J=LLL*(   NCIR-NC)+1-I
      BI2=XN(Y)+2.DO
      LI=3*LLL+I
      DO 313 M=1,LLL
      Z=2*LLL+M
      K=LLL*(   NCIR-NC)+1-M
      BM2=XN(Z)+2.DO
      LM=3*LLL+M
  313 A3(J,K)=C1*(IIR2(Y,Z)-IIR2(Y,LM)-IIR2(LI,Z)+IIR2(LI,LM))
     1+C2*(IIRRR2(Y,Z)-IIRRR2(Y,LM)-IIRRR2(LI,Z)+IIRRR2(LI,LM))+C3*
     2(XN(Y)*XN(Z)*IR1(Y,Z)-XN(Y)*BM2*IR1(Y,LM)-BI2*XN(Z)*IR1(LI,Z)+
     3BI2*BM2*IR1(LI,LM))+C4*(XN(Y)*XN(Z)*IIR1(Y,Z)-XN(Y)*BM2*IIR1(Y,LM)
     4-BI2*XN(Z)*IIR1(LI,Z)+BI2*BM2*IIR1(LI,LM))+C5*(XN(Y)*XN(Z)*XN(Y)
     5*XN(Z)*IR2(Y,Z)-XN(Y)*XN(Y)*BM2*BM2*IR2(Y,LM)-BI2*BI2*XN(Z)*XN(Z)
     6*IR2(LI,Z)+BI2*BI2*BM2*BM2*IR2(LI,LM))+C6*(-XN(Z)*IIRR3(Z,Y)+
     7XN(Z)*IIRR3(Z,LI)+BM2*IIRR3(LM,Y)-BM2*IIRR3(LM,LI)-XN(Y)*
```

```
     8IIRR3(Y,Z)+XN(Y)*IIRR3(Y,LM)+BI2*IIRR3(LI,Z)-BI2*IIRR3(LI,LM))+
     9C7*(XN(Z)*XN(Z)*(IIR2(LI,Z)-IIR2(Y,Z))+BM2*BM2*(IIR2(Y,LM)-
     AIIR2(LI,LM))+XN(Y)*XN(Y)*(IIR2(Y,LM)-IIR2(Y,Z))+BI2*BI2*(IIR2(LI,
     BM)-IIR2(LI,LM)))+CB*(XN(Z)*XN(Z)*(XN(Y)*I3(Y,Z)-BI2*I3(LI,Z))+
     CBM2*BM2*(BI2*I3(LI,LM)-XN(Y)*I3(Y,LM))+XN(Y)*XN(Y)*(XN(Z)*I3(Z,Y)
     D-BM2*I3(LM,Y))+BI2*BI2*(BM2*I3(LM,LI)-XN(Z)*I3(Z,LI)))
      WRITE(6,26) ((A3(J,K),K=1,LIMA3),J=1,LIMA3)
C
C     COMPUTE LOAD MATRIX B(J,K) WHERE  J AND K = LLL*3
C
      DO 314 J=1,LIMA3
      DO 314 K=1,LIMA3
  314 B(J,K)=0.0D0
      NC=-1
      C1=2.D0*V1*PIL2
      DO 315 N4=N1,N3
      NC=NC+1
      DO 315 I=1,LLL
      Y=2*LLL+I
      J=LLL*(   NCIR-NC)+1-I
      BI2=XN(Y)+2.D0
      LI=3*LLL+I
      DO 315 M=1,LLL
      Z=2*LLL+M
      K=LLL*(   NCIR-NC)+1-M
      BM2=XN(Z)+2.D0
      LM=3*LLL+M
  315 B(J,K)=C1*(XN(Y)*(XN(Z)*I1(Y,Z)-BM2*I1(Y,LM))+BI2*(BM2*I1(LI,LM)
     1-XN(Z)*I1(LI,Z)))
      IF(NCIR-1)319,318,319
  319 CONTINUE
      NC=-1
      C1=V2*PIL2
      DO 316 N4=1,2
      NC=NC+1
      DO 316 I=1,LLL
```

```
      Y=2*LLL+I
      J=LLL*(2-NC)+1-I
      BI2=XN(Y)+2.D0
      LI=3*LLL+I
      DO 316 M=1,LLL
      Z=2*LLL+M
      K=LLL*(3-NC)+1-M
      BM2=XN(Z)+2.D0
      LM=3*LLL+M
      B(J,K)=C1*(XN(Y)*(XN(Z)*IIR1(Y,Z)-BM2*IIR1(Y,LM))+BI2*(BM2*
     1IIR1(LI,LM)-XN(Z)*IIR1(LI,Z)))
  316 B(K,J)=B(J,K)
  318 WRITE(6,27) ((B(J,K),K=1,LIMA3),J=1,LIMA3)
   27 FORMAT(//' LOAD MATRIX '//(6D20.8))
      WRITE(6,800)
  800 FORMAT('  CHECK 0')
      CALL EIG2(KKK,LIMA3,60,20,40,EVAL,EVEC,A1,A2,A3,W4,   IW8,IW9,
     1IW10,NERR,B,W5,W6)
      WRITE(6,801)
  801 FORMAT('  CHECK 1')
      IF(NERR)200,201,200
  200 WRITE(6,400)
  400 FORMAT(//34H LOAD MATRIX NOT POSITIVE DEFINITE)
      GO TO 600
  201 CONTINUE
C
C     OUTPUT OF EIGENVALUES AND EIGENVECTORS
C
      DO 600 I=1,LIMA3
      WRITE(6,601) I,EVAL(I),(EVEC(J,I),J=1,LLL5)
  601 FORMAT(//' EIGENVALUE  ',I5,'   EQUALS  ',D16.8/(6D20.8))
  600 CONTINUE
      STOP
      END
```

```
      SUBROUTINE INTEG (MNUMB)
      REAL*8 I1(16,16),I2(16,16),I3(16,16),IR1(16,16),IR2(16,16),
     1IR3(16,16),IIR1(16,16),IIR2(16,16),IIR3(16,16),IIRR1(16,16),
     2IIRR2(16,16),IIRR3(16,16),IIRRR1(16,16),IIRRR2(16,16),
     3IIRRR3(16,16)
      COMMON /INT/I1,I2,I3,IR1,IR2,IR3,IIR1,IIR2,IIR3,IIRR1,IIRR2,
     1IIRR3,IIRRR1,IIRRR2,IIRRR3,XL,XN,PI,ANG,SA,RO,N,LLL
      DIMENSION N(16)
      DOUBLE PRECISION XN(16),XL,PI,ANG,SA,RO,ROLSA,BBB,CBBB,SBBB,
     1BRL,BRU,CM2,SW2,AAA,CAAA,SAAA,ARL,ARU,CM1,SW1
      DOUBLE PRECISION SIBL,CIBL,SIBU,CIBU,SIAL,CIAL,SIAU,CIAU
C
C     INTEGRAL INITIALIZATION TO ZERO
C
      M=LLL*MNUMB
      DO 99 J=1,M
      DO 99 K=1,M
      I1(J,K)=      0.0D0
      I2(J,K)=      0.0D0
      I3(J,K)=      0.0D0
      IR1(J,K)=     0.0D0
      IR2(J,K)=     0.0D0
      IR3(J,K)=     0.0D0
      IIR1(J,K)=    0.0D0
      IIR2(J,K)=    0.0D0
      IIR3(J,K)=    0.0D0
      IIRR1(J,K)=   0.0D0
      IIRR2(J,K)=   0.0D0
      IIRR3(J,K)=   0.0D0
      IIRRR1(J,K)=  0.0D0
      IIRRR2(J,K)=  0.0D0
      IIRRR3(J,K)=  0.0D0
   99 CONTINUE
C
C     INTEGRAL  I1(J,K)
C
```

```
      DO 100 J=1,M
      DO 100 K=1,M
      IF(N(J).NE.N(K)) GO TO 100
      IF(N(J).EQ.0) GO TO 100
      I1(J,K)=XL/2.D0
  100 CONTINUE
      WRITE(6,7)
    7 FORMAT(//'                    INTEGRAL I1(J,K)      ')
      WRITE(6,8) ((I1(J,K),K=1,M  ),J=1,M  )
    8 FORMAT(6D20.8)
C
C     INTEGRAL  I2(J,K)
C
      DO 101 J=1,M
      DO 101 K=1,M
      IF(N(J).NE.N(K)) GO TO 101
      IF(N(J)) 1101,1102,1101
 1102 I2(J,K)=XL
      GO TO 101
 1101 I2(J,K)=XL/2.D0
  101 CONTINUE
      WRITE(6,9)
    9 FORMAT(//'                    INTEGRAL I2(J,K)      ')
      WRITE(6,8) ((I2(J,K),K=1,M  ),J=1,M  )
C
C     INTEGRAL  I3(J,K)
C
      DO 102 J=1,M
      DO 102 K=1,M
      IF(N(J).EQ.0) GO TO 102
      IF(N(J).EQ.N(K)) GO TO 102
      IA=N(J)+N(K)
      IF(IA/2*2-IA)103,102,102
  103 I3(J,K)=2.D0*XL*XN(J)/(PI*(XN(J)*XN(J)-XN(K)*XN(K)))
  102 CONTINUE
      WRITE(6,10)
```

```
   10 FORMAT(//'                    INTEGRAL I3(J,K)     ')
      WRITE(6,8) ((I3(J,K),K=1,M  ),J=1,M  )
C
C     INTEGRAL  IR1(J,K)
C
      IF(ANG.EQ.0.0) GO TO 105
      DO 104 J=1,M
      DO 104 K=1,M
      IF((N(J)*N(K)).EQ.0) GO TO 104
      IF(N(J).EQ.N(K)) GO TO 106
      IA=N(J)+N(K)
      IF(IA/2*2-IA)107,104,104
  107 IR1(J,K)=-4.D0*XN(J)*XN(K)*XL*XL*SA/(PI*PI*(XN(J)*XN(J)-XN(K)*XN(K
     1))**2)
      GO TO 104
  106 IR1(J,K)=(RO+XL*SA/2.D0)*XL/2.D0
  104 CONTINUE
      GO TO 108
  105 DO 108 J=1,M
      DO 108 K=1,M
      IF(N(J).NE.N(K)) GO TO 108
      IF(N(J).EQ.0) GO TO 108
      IR1(J,K)=RO*XL/2.D0
  108 CONTINUE
      WRITE(6,11)
   11 FORMAT(//'                    INTEGRAL IR1(J,K)    ')
      WRITE(6,8)  ((IR1(J,K),K=1,M  ),J=1,M  )
C
C     INTEGRAL  IR2(J,K)
C
      IF(ANG.EQ.0.0) GO TO 109
      DO 110 J=1,M
      DO 110 K=1,M
      IF(N(J)+N(K))1112,1110,1112
 1112 IF(N(J).EQ.N(K)) GO TO 111
      IA=N(J)+N(K)
```

```
      IF(IA/2*2-IA)112,110,110
  112 IR2(J,K)=-2.DO*XL*XL*SA*(XN(J)*XN(J)+XN(K)*XN(K))/(PI*PI*(XN(J)*
     1XN(J)-XN(K)*XN(K))**2)
      GO TO 110
  111 IR2(J,K)=(RO+XL*SA/2.DO)*XL/2.DO
      GO TO 110
 1110 IR2(J,K)=(RO+XL*SA/2.DO)*XL
  110 CONTINUE
      GO TO 113
  109 DO 113 J=1,M
      DO 113 K=1,M
      IF(N(J).NE.N(K)) GO TO 113
      IF(N(J))1113,1114,1113
 1114 IR2(J,K)=RO*XL
      GO TO 113
 1113 IR2(J,K)=RO*XL/2.DO
  113 CONTINUE
      WRITE(6,12)
   12 FORMAT(//'                    INTEGRAL IR2(J,K)    ')
      WRITE(6,8)  ((IR2(J,K),K=1,M  ),J=1,M  )
C
C     INTEGRAL  IR3(J,K)
C
      DO 114 J=1,M
      DO 114 K=1,M
      IF(N(J).EQ.0) GO TO 114
      IF(N(J).EQ.N(K)) GO TO 114
      IA=N(J)+N(K)
      IF(IA/2*2-IA)115,116,116
  116 IR3(J,K)=-XN(J)*XL*XL*SA/(PI*(XN(J)*XN(J)-XN(K)*XN(K)))
      GO TO 114
  115 IR3(J,K)=(2.DO*XL*XN(J)*RO+XN(J)*XL*XL*SA)/(PI*(XN(J)*XN(J)-XN(K)
     1*XN(K)))
  114 CONTINUE
      WRITE(6,13)
   13 FORMAT(//'                    INTEGRAL IR3(J,K)    ')
```

```
      WRITE(6,8)  ((IR3(J,K),K=1,M  ),J=1,M  )
C
C     INTEGRALS  IIR1(J,K), IIR2(J,K),  AND IIR3(J,K)
C
      IF(ANG.EQ.0.0) GO TO 117
      ROLSA=DLOG(1.D0+XL*SA/RO)
      SAA=PI/(XL*SA)
      DO 118 J=1,M
      DO 118 K=1,M
      IF(N(J)+N(K)) 1116,1119,1116
 1119 IIR2(J,K)=ROLSA/SA
      GO TO 118
 1116 BBB=(XN(J)+XN(K))*SAA
      CBBB=DCOS(BBB*RO)
      SBBB=DSIN(BBB*RO)
      BRL=BBB*RO
      BRU=BBB*(RO+XL*SA)
      CALL SICI(SIBL,CIBL,BRL)
      CALL SICI(SIBU,CIBU,BRU)
      CW2=CIBU-CIBL
      SW2=SIBU-SIBL
      IF(N(J).EQ.N(K)) GO TO 119
      AAA=(XN(J)-XN(K))*SAA
      CAAA=DCOS(AAA*RO)
      SAAA=DSIN(AAA*RO)
      ARL=AAA*RO
      ARU=AAA*(RO+XL*SA)
      CALL SICI(SIAL,CIAL,ARL)
      CALL SICI(SIAU,CIAU,ARU)
      CW1=CIAU-CIAL
      SW1=SIAU-SIAL
      IF(N(J).EQ.0) GO TO 1121
      IF(N(K).EQ.0) GO TO 1120
      IIR1(J,K)=(CAAA*CW1+SAAA*SW1-CBBB*CW2-SBBB*SW2)/(2.D0*SA)
 1120 IIR3(J,K)=(CAAA*SW1-SAAA*CW1+CBBB*SW2-SBBB*CW2)/(2.D0*SA)
 1121 IIR2(J,K)=(CAAA*CW1+SAAA*SW1+CBBB*CW2+SBBB*SW2)/(2.D0*SA)
```

```
      GO TO 118
  119 IIR1(J,K)=(ROLSA-CBBB*CW2-SBBB*SW2)/(2.D0*SA)
      IIR2(J,K)=(ROLSA+CBBB*CW2+SBBB*SW2)/(2.D0*SA)
      IIR3(J,K)=(CBBB*SW2-SBBB*CW2)/(2.D0*SA)
  118 CONTINUE
      GO TO 120
  117 DO 120 J=1,M
      DO 120 K=1,M
      IIR1(J,K)=I1(J,K)/RO
      IIR2(J,K)=I2(J,K)/RO
      IIR3(J,K)=I3(J,K)/RO
  120 CONTINUE
      WRITE(6,14)
   14 FORMAT(//'                 INTEGRAL IIR1(J,K)   ')
      WRITE(6,8)  ((IIR1(J,K),K=1,M  ),J=1,M  )
      WRITE(6,15)
   15 FORMAT(//'                 INTEGRAL IIR2(J,K)   ')
      WRITE(6,8)  ((IIR2(J,K),K=1,M  ),J=1,M  )
      WRITE(6,16)
   16 FORMAT(//'                 INTEGRAL IIR3(J,K)   ')
      WRITE(6,8)  ((IIR3(J,K),K=1,M  ),J=1,M  )
C
C     INTEGRALS  IIRR1(J,K), IIRR2(J,K), AND IIRR3(J,K)
C
      IF(ANG.EQ.0.0) GO TO 121
      DO 122 J=1,M
      DO 122 K=1,M
      IIRR1(J,K)=(XN(J)*IIR3(K,J)+XN(K)*IIR3(J,K))*PI/(XL*SA)
      IIRR2(J,K)=-(XN(J)*IIR3(J,K)+XN(K)*IIR3(K,J))*PI/(XL*SA)
     1+(1.D0/RO-(-1.D0**(N(J)+N(K)))/(RO+XL*SA))/SA
      IIRR3(J,K)=(XN(J)*IIR2(J,K)-XN(K)*IIR1(J,K))*PI/(XL*SA)
  122 CONTINUE
      GO TO 123
  121 DO 123 J=1,M
      DO 123 K=1,M
      IIRR1(J,K)=I1(J,K)/(RO*RO)
```

```
      IIRR2(J,K)=I2(J,K)/(RO*RO)
      IIRR3(J,K)=I3(J,K)/(RO*RO)
  123 CONTINUE
      WRITE(6,17)
   17 FORMAT(//'                   INTEGRAL IIRR1(J,K) ')
      WRITE(6,8)  ((IIRR1(J,K),K=1,M  ),J=1,M  )
      WRITE(6,18)
   18 FORMAT(//'                   INTEGRAL IIRR2(J,K) ')
      WRITE(6,8)  ((IIRR2(J,K),K=1,M  ),J=1,M  )
      WRITE(6,19)
   19 FORMAT(//'                   INTEGRAL IIRR3(J,K) ')
      WRITE(6,8)  ((IIRR3(J,K),K=1,M  ),J=1,M  )
C
C     INTEGRALS  IIRRR1(J,K), IIRRR2(J,K), AND IIRRR3(J,K)
C
      IF(ANG.EQ.0.0) GO TO 124
      DO 125 J=1,M
      DO 125 K=1,M
      IIRRR1(J,K)=(XN(J)*IIRR3(K,J)+XN(K)*IIRR3(J,K))*PI/(2.DO*XL*SA)
      IIRRR2(J,K)=-(XN(J)*IIRR3(J,K)+XN(K)*IIRR3(K,J))*PI/(2.DO*XL*SA)
     1+(1.DO/(RO*RO)-(-1.DO**(N(J)+N(K)))/((RO+XL*SA)**2))/(2.DO*SA)
      IIRRR3(J,K)=(XN(J)*IIRR2(J,K)-XN(K)*IIRR1(J,K))*PI/(2.DO*XL*SA)
  125 CONTINUE
      GO TO 126
  124 DO 126 J=1,M
      DO 126 K=1,M
      IIRRR1(J,K)=I1(J,K)/(RO**3)
      IIRRR2(J,K)=I2(J,K)/(RO**3)
      IIRRR3(J,K)=I3(J,K)/(RO**3)
  126 CONTINUE
      WRITE(6,20)
   20 FORMAT(//'                   INTEGRAL IIRRR1(J,K)')
      WRITE(6,8)  ((IIRRR1(J,K),K=1,M  ),J=1,M  )
      WRITE(6,21)
   21 FORMAT(//'                   INTEGRAL IIRRR2(J,K)')
      WRITE(6,8)  ((IIRRR2(J,K),K=1,M  ),J=1,M  )
```

```
      WRITE(6,22)
   22 FORMAT(//'                    INTEGRAL IIRRR3(J,K)')
      WRITE(6,8)  ((IIRRR3(J,K),K=1,M  ),J=1,M  )
      RETURN
      END
C
C     ...............................................................SICI
C
C     SUBROUTINE DSICI
C
C     PURPOSE
C        COMPUTES THE SINE AND COSINE INTEGRAL IN DOUBLE PRECISION
C
C     USAGE
C        CALL SICI(SI,CI,X)
C
C     DESCRIPTION OF PARAMETERS
C        SI    - THE RESULTANT VALUE SI(X)
C        CI    - THE RESULTANT VALUE CI(X)
C        X     - THE ARGUMENT OF SI(X) AND CI(X)
C
C     REMARKS
C        THE CALL COMMON VALUES MUST BE SPECIFIED AS DOUBLE
C          PRECISION IN THE CALLING PROGRAM
C        THE ARGUMENT VALUE REMAINS UNCHANGED
C
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
C        NONE
C
C     METHOD
C        DEFINITION
C        SI(X)=INTEGRAL(SIN(T)/T, SUMMED OVER T FROM INFINITY TO X)
C        CI(X)=INTEGRAL(COS(T)/T, SUMMED OVER T FROM INFINITY TO X)
C        EVALUATION
C        REDUCTION OF RANGE USING SYMMETRY.
C        DIFFERENT APPROXIMATIONS ARE USED FOR DABS(X) GREATER
```

```
C          THAN 4 AND FOR DABS(X) LESS THAN 4.
C          REFERENCE
C          LUKE AND WIMP, 'POLYNOMIAL APPROXIMATIONS TO INTEGRAL
C          TRANSFORMS', MATHEMATICAL TABLES AND OTHER AIDS TO
C          COMPUTATION, VOL. 15, 1961, ISSUE 74,PP. 174-178.
C
C       ..................................................................
C
      SUBROUTINE SICI(SI,CI,X)
      DOUBLE PRECISION Z,X,SI,CI,Y,U,V
C
C     TEST ARGUMENT RANGE
C
      Z=DABS(X)
      IF(Z-4.D0) 10,10,50
C
C     Z IS NOT GREATER THAN 4
C
   10 Y=Z*Z
      SI=-1.5707963+X*((((((.97942154D-11*Y-.22232633D-8)*Y+.30561233D-6
     1)*Y-.28341460D-4)*Y+.16666582D-2)*Y-.55555547D-1)*Y+1.D0)
C
C     TEST FOR LOGARITHMIC SINGULARITY
C
      IF(Z) 30,20,30
   20 CI=-1.D75
      RETURN
   30 CI=0.57721566+DLOG(Z)-Y*(((((-.13869851D-9*Y+.26945842D-7)*Y-
     1.30952207D-5)*Y+.23146303D-3)*Y-.10416642D-1)*Y+.24999999)
   40 RETURN
C
C     Z IS GREATER THAN 4.D0
C
   50 SI=DSIN(Z)
      Y=DCOS(Z)
      Z=4.D0/Z
```

```
      U=((((((((.40480690D-2*Z-.022791426)*Z+.055150700)*Z-.072616418)*Z
     1+.049877159)*Z-.33325186D-2)*Z-.023146168)*Z-.11349579D-4)*Z
     2+.062500111)*Z+.25839886D-9
      V=((((((((((-.0051086993*Z+.028191786)*Z-.065372834)*Z+.079020335)*
     1Z-.044004155)*Z-.0079455563)*Z+.026012930)*Z-.37640003D-3)*Z
     2-.031224178)*Z-.66464406D-6)*Z+.25000000
      CI=Z*(SI*V-Y*U)
      SI=-Z*(SI*U+Y*V)
C
C     TEST FOR NEGATIVE ARGUMENT
C
      IF(X) 60,40,40
C
C     X IS LESS THAN -4.D0
C
   60 SI=-3.1415927-SI
      RETURN
      END
      SUBROUTINE EIG2(      K,N,NMAX1,NMAX2,NMAX3,EVAL,EVEC,A,B,C,AINVB,
     1       IPIVOT,INDEX,L,NERR,EM,VECT2,BTAB)
C
C     A1,A2, AND A3 MATRICES FORM THE STIFFNESS MATRIX
C     EIG2 REDUCES THIS COMBINED STIFFNESS MATRIX TO SAME ORDER AS THE
C     LOAD MATRIX AND SOLVES THE RESULTING EIGENVALUE PROBLEM
C       REDUCED MATRIX = A3 - (A2 TRANSPOSE)(A1 INVERSE)(A2)
C         K = DIMENSION OF A1
C         N = DIMENSION OF A3
C         NMAX1,NMAX2, AND NMAX3 DUMMY VARIABLES FOR LATER ADDING
C           VARIABLE DIMENSION -  MUST  BE DEFINED IN CALL STATEMENT
C           EVEN THOUGH ARE NOT USED
C         EVEC = MATRIX FOR CALCULATED EIGENVECTORS,  IN COLUMNS
C         EVAL = COLUMN VECTOR OF LENGTH N FOR EIGENVALUES
C         A1,A2, AND A3 STORAGE MATRICES FOR STIFFNESS MATRIX
C         AINVB,IPIVOT, INDEX, L = TEMPORARY STORAGE MATRICES
C         NERR = ERROR CODE ON RETURN 0 = NORMAL RETURN, 1 = REDUCED
C           MATRIX IS SINGULAR
```

```
C         EM = LOAD MATRIX OF DIMENSION N-N
C         VECT2 AND BTAB ARE TEMPORARY STORAGE MATRICES
C
C     SUBROUTINES CALLED BY EIG2 - EIGEN, DMATIN(LRC LIBRARY SUBROUTINE)
C
      DOUBLE PRECISION EM(12,12),EVAL(12),EVEC(60,12),A(48,48),
     1B(48,12),C(12,12),AINVB(48,12),BTAB(12,12),VECT2(12,12)
      DIMENSION IPIVOT(48),INDEX(48,2),L(12)
C
C     INVERT A
C
      WRITE(6,802)
  802 FORMAT('  CHECK 2')
      CALL DMATIN(A,K,1.D0,0,DETERM,IPIVOT,INDEX,NMAX3,ISCALE)
      WRITE(6,803)
  803 FORMAT('  CHECK 3')
C
C     MATRIX PRODUCT A-INVERSE * B
C
      DO 4 I=1,K
      DO 4 J=1,N
      AINVB(I,J)=0.D0
      DO 4 JJ=1,K
    4 AINVB(I,J)=AINVB(I,J)+A(I,JJ)*B(JJ,J)
C
C     MATRIX PRODUCT B-TRANSPOSE * A-INVERSE * B
C
      DO 5 I=1,N
      DO 5 J=1,N
      BTAB(I,J)=0.D0
      DO 5 JJ=1,K
    5 BTAB(I,J)=BTAB(I,J)+B(JJ,I)*AINVB(JJ,J)
C
C     REDUCED MATRIX   C - BT * AINV * B
C
      DO 6 I=1,N
```

```
      DO 6 J=1,N
    6 BTAB(I,J)=C(I,J)-BTAB(I,J)
C
C     SOLVE REDUCED SYSTEM FOR EIGENVALUES AND VECTORS
C
      WRITE(6,804)
  804 FORMAT('  CHECK 4')
      CALL EIGN(N,EVAL,NERR,BTAB,VECT2,EM)
      WRITE(6,805)
  805 FORMAT('  CHECK 5')
      IF(NERR-1)9,12,9
C
C     COMPUTE EIGENVECTORS OF ORIGINAL SYSTEM
C
    9 DO 10 I=1,K
      DO 10 J=1,N
      EVEC(I,J)=0.D0
      DO 10 JJ=1,N
   10 EVEC(I,J)=EVEC(I,J)-AINVB(I,JJ)*VECT2(JJ,J)
      DO 11 I=1,N
      I1=I+K
      DO 11 J=1,N
   11 EVEC(I1,J)=VECT2(I,J)
   12 RETURN
      END
C     DOUBLE PRECISION MATRIX INVERSION WITH ACCOMPANYING SOLUTION OF
C     LINEAR EQUATIONS
C                                                                       F4020002
      SUBROUTINE DMATIN(A,N,B,M,DETERM,IPIVOT,INDEX,NMAX,ISCALE)
C                                                                       F4020004
C     DMATIN SOLVES IN DOUBLE PRECISION THE MATRIX EQUATION AX = B
C        WHERE A IS A SQUARE COEFFICIENT MATRIX AND B IS A MATRIX OF
C        CONSTANT VECTORS.  A INVERSE IS ALSO OBTAINED IN DOUBLE
C        PRECISION AND THE DETERMINANT OF A IS AVAILABLE IN
C        SINGLE PRECISION.
C        A = LOCATION OF MATRIX A
```

```
C         N = ORDER OF A
C         B = LOCATION OF B
C         M = NUMBER OF COLUMN VECTORS IN B. (M=0 SIGNALS THAT THE
C           SUBROUTINE IS TO BE USED ONLY FOR INVERSION, HOWEVER, IN THE
C           CALL STATEMENT AN ENTRY CORRESPONDING TO B MUST STILL BE
C           INCLUDED.
C         DETERM = VALUE OF DEGERMINANT
C         IPIVOT AND INDEX ARE TEMPORARY STORAGE BLOCKS
      DOUBLE PRECISION A,B,AMAX,SWAP,PIVOT,T
      DIMENSION IPIVOT(48),A(48,48),B(48,1),INDEX(48,2)
      EQUIVALENCE (IROW,JROW), (ICOLUM,JCOLUM), (AMAX, T, SWAP)         F4020007
C                                                                       F4020008
C     INITIALIZATION                                                    F4020009
C                                                                       F4020010
    5 ISCALE=0
    6 R1=10.0**18
    7 R2=1.0/R1
   10 DETERM=1.0                                                        F4020011
   15 DO 20 J=1,N                                                       F4020012
   20 IPIVOT(J)=0                                                       F4020013
   30 DO 550 I=1,N                                                      F4020014
C                                                                       F4020015
C     SEARCH FOR PIVOT ELEMENT                                          F4020016
C                                                                       F4020017
   40 AMAX=0.0D0
   45 DO 105 J=1,N                                                      F4020019
   50 IF (IPIVOT(J)-1) 60, 105, 60                                      F4020020
   60 DO 100 K=1,N                                                      F4020021
   70 IF (IPIVOT(K)-1) 80, 100, 740                                     F4020022
   80 IF(DABS(AMAX)-DABS(A(J,K)))85,100,100
   85 IROW=J                                                            F4020024
   90 ICOLUM=K                                                          F4020025
   95 AMAX=A(J,K)                                                       F4020026
  100 CONTINUE                                                          F4020027
  105 CONTINUE                                                          F4020028
      IF (AMAX) 110,106,110
```

```
  106 DETERM=0.0
      ISCALE=0
      GO TO 740
  110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1                                   F4020029
C                                                                       F4020030
C     INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL                 F4020031
C                                                                       F4020032
  130 IF (IROW-ICOLUM) 140, 260, 140                                    F4020033
  140 DETERM=-DETERM                                                    F4020034
  150 DO 200 L=1,N                                                      F4020035
  160 SWAP=A(IROW,L)                                                    F4020036
  170 A(IROW,L)=A(ICOLUM,L)                                             F4020037
  200 A(ICOLUM,L)=SWAP                                                  F4020038
  205 IF(M) 260, 260, 210                                               F4020039
  210 DO 250 L=1, M                                                     F4020040
  220 SWAP=B(IROW,L)                                                    F4020041
  230 B(IROW,L)=B(ICOLUM,L)                                             F4020042
  250 B(ICOLUM,L)=SWAP                                                  F4020043
  260 INDEX(I,1)=IROW                                                   F4020044
  270 INDEX(I,2)=ICOLUM                                                 F4020045
  310 PIVOT=A(ICOLUM,ICOLUM)
C
C     SCALE THE DETERMINANT
C
 1000 PIVOTI=PIVOT
 1005 IF(ABS(DETERM)-R1)1030,1010,1010
 1010 DETERM=DETERM/R1
      ISCALE=ISCALE+1
      IF(ABS(DETERM)-R1)1060,1020,1020
 1020 DETERM=DETERM/R1
      ISCALE=ISCALE+1
      GO TO 1060
 1030 IF(ABS(DETERM)-R2)1040,1040,1060
 1040 DETERM=DETERM*R1
      ISCALE=ISCALE-1
      IF(ABS(DETERM)-R2)1050,1050,1060
```

```
 1050 DETERM=DETERM*R1
      ISCALE=ISCALE-1
 1060 IF(ABS(PIVOTI)-R1)1090,1070,1070
 1070 PIVOTI=PIVOTI/R1
      ISCALE=ISCALE+1
      IF(ABS(PIVOTI)-R1)320,1080,1080
 1080 PIVOTI=PIVOTI/R1
      ISCALE=ISCALE+1
      GO TO 320
 1090 IF(ABS(PIVOTI)-R2)2000,2000,320
 2000 PIVOTI=PIVOTI*R1
      ISCALE=ISCALE-1
      IF(ABS(PIVOTI)-R2)2010,2010,320
 2010 PIVOTI=PIVOTI*R1
      ISCALE=ISCALE-1
  320 DETERM=DETERM*PIVOTI
C                                                                F4020048
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT                          F4020049
C                                                                F4020050
  330 A(ICOLUM,ICOLUM)=1.0D0
  340 DO 350 L=1,N                                               F4020052
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT                              F4020053
  355 IF(M) 380, 380, 360                                        F4020054
  360 DO 370 L=1,M                                               F4020055
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT                              F4020056
C                                                                F4020057
C     REDUCE NON-PIVOT ROWS                                      F4020058
C                                                                F4020059
  380 DO 550 L1=1,N                                              F4020060
  390 IF(L1-ICOLUM) 400, 550, 400                                F4020061
  400 T=A(L1,ICOLUM)                                             F4020062
  420 A(L1,ICOLUM)=0.0D0
  430 DO 450 L=1,N                                               F4020064
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T                              F4020065
  455 IF(M) 550, 550, 460                                        F4020066
  460 DO 500 L=1,M                                               F4020067
```

```
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T                                     F4020068
  550 CONTINUE                                                          F4020069
C                                                                       F4020070
C       INTERCHANGE COLUMNS                                             F4020071
C                                                                       F4020072
  600 DO 710 I=1,N                                                      F4020073
  610 L=N+1-I                                                           F4020074
  620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630                          F4020075
  630 JROW=INDEX(L,1)                                                   F4020076
  640 JCOLUM=INDEX(L,2)                                                 F4020077
  650 DO 705 K=1,N                                                      F4020078
  660 SWAP=A(K,JROW)                                                    F4020079
  670 A(K,JROW)=A(K,JCOLUM)                                             F4020080
  700 A(K,JCOLUM)=SWAP                                                  F4020081
  705 CONTINUE                                                          F4020082
  710 CONTINUE                                                          F4020083
  740 RETURN                                                            F4020084
      END                                                               F4020085
      SUBROUTINE EIGN(N,EVAL,NERR,B,EVEC,A)
C     EIGN SOLVES STANDARD EIGENVALUE PROBLEM
      DOUBLE PRECISION A(12,12),B(12,12),EVAL(12),EVEC(12,12),F
      DOUBLE PRECISION DUM
      DIMENSION L(12)
C
C     SUBROUTINES CALLED BY EIGEN  -  JACOBI
C
      NERR = 0
C
C     SOLVE B*V=LAMBDA*V
C
      NMAX=100
      CALL JACOBI(N,0,100,NMAX,EVAL,L,B,EVEC)
C
C     WRITE EIGENVALUES OF B FOR CHECK
C
      WRITE(6,900)(B(I,I),I=1,N)
```

```
  900 FORMAT(//27H EIGENVALUES OF MASS MATRIX//(E18.8))
C
C     TEST EIGENVALUES OF B
C
      DO 10 I=1,N
      IF(B(I,I))1,1,10
C
C     RETURN IF B IS NOT POSITIVE DEFINITE
C
    1 NERR = 1
      RETURN
   10 CONTINUE
C
C     REDUCE SYSTEM (SEE SUBROUTINE WRITE-UP)
C
C     FORM (U)(D**-1/2)
C
      DO 2 J=1,N
      F=DSQRT(B(J,J))
      DO 2 I=1,N
    2 B(I,J)=EVEC(I,J)/F
C
C     FORM (A)(B) = (A)(U)(D**-1/2)
C
      DO 3 I=1,N
      DO 3 J=1,N
      EVEC(I,J)=0.D0
      DO 3 K=1,N
    3 EVEC(I,J)=EVEC(I,J)+A(I,K)*B(K,J)
C
C     FORM (D**-1/2)(U**T)(A)(U)(D**-1/2) = (B**T)(EVEC)
C
      DO 4 I=1,N
      DO 4 J=1,N
      A(I,J)=0.D0
      DO 4 K=1,N
```

```
    4 A(I,J)=A(I,J)+B(K,I)*EVEC(K,J)
      DO 5 I=1,N
      DO 5 J=1,N
      DUM=A(I,J)
      A(I,J)=B(I,J)
    5 B(I,J)=DUM
C
C   SOLVE REDUCED SYSTEM FOR EIGENVALUES AND VECTORS
C
      CALL JACOBI(N,0,100,NMAX,EVAL,L,B,EVEC)
C
C   COMPUTE EIGENVECTORS OF ORIGINAL PROBLEM
C
      DO 7 I=1,N
      L(I)=I
    7 EVAL(I)=B(I,I)
      DO 8 I=1,N
      DO 8 J=1,N
      B(I,J)=0.DO
      DO 8 K=1,N
    8 B(I,J)=B(I,J)+A(I,K)*EVEC(K,J)
C
C     REARRANGE EIGENVALUES AND CORRESPONDING EIGENVECTORS IN ASCENDING
C     ORDER
C
      NM1=N-1
      DO 11 J=1,NM1
      K = 1
      DO 11 I=1,NM1
      IF(DABS(EVAL(K))-DABS(EVAL(K+1)))13,13,12
   12 A1 = EVAL(K+1)
      L1=L(K+1)
      A2 = EVAL(K)
      L2=L(K)
      EVAL(K) = A1
      L(K)=L1
```

```
      K = K+1
      EVAL(K) = A2
      L(K)=L2
      GO TO 11
   13 K = K+1
   11 CONTINUE
      DO 14 I=1,N
      DO 14 J=1,N
      JL=L(J)
   14 EVEC(I,J)=B(I,JL)
      RETURN
      END
      SUBROUTINE JACOBI(N,IEGEN,NR,NMAX,X,IQ,A,U)
C
C     DIAGONALIZATION OF A REAL SYMMETRIC MATRIX BY THE JACOBI METHOD
C     CALLING SEQUENCE FOR DIAGONALIZATION
C       CALL JACOBI(A,N,IEGEN,U,NR,NMAX)
C        WHERE A IS THE ARRAY TO BE DIAGONALIZED
C              N IS THE ORDER OF THE MATRIX A
C              IEGEN MUST BE SET UNEQUAL TO ZERO IF ONLY EIGENVALUES ARE
C                  TO BE COMPUTED
C              IEGEN MUST BE SET EQUAL TO ZERO IF EIGENVALUES AND
C                  EIGENVECTORS ARE TO BE COMPUTED
C              U IS THE UNITARY MATRIX USED FOR FORMATION OF THE
C                  EIGENVECTORS
C              NR IS THE NUMBER OF ROTATIONS
C              NMAX IS THE MAXIMUM ORDER OF A
C              X,IQ ARE TEMPORARY STORAGE MATRICES
C
C     THE SUBROUTINE OPERATES ONLY ON THE ELEMENTS OF A THAT ARE TO THE
C     RIGHT OF THE MAIN DIAGONAL.  THUS, ONLY A TRIANGULAR SECTION NEED
C     BE STORED IN THE ARRAY A
C
      DOUBLE PRECISION A,U,X,XMAX,HDMIN,HDTEST,TANG,COSINE,SINE,AII,ATEM
     1P,RAP
      DIMENSION A(12,12),U(12,12),X(12),IQ(12)
```

```
      IF(IEGEN)15,10,15
   10 DO 14 I=1,N
      DO 14 J=1,N
      IF(I-J)12,11,12
   11 U(I,J) = 1.D0
      GO TO 14
   12 U(I,J) = 0.D0
   14 CONTINUE
   15 NR = 0
      IF(N-1)1000,1000,17
C
C     SCAN FOR LARGEST OFF DIAGONAL ELEMENT IN EACH ROW
C     X(I) CONTAINS LARGEST ELEMENT IN I-TH ROW
C     IQ(I) CONTAINS SECOND SUBSCRIPT DEFINING POSITION OF ELEMENT
C
   17 NMI1 = N-1
      DO 30 I=1,NMI1
      X(I) = 0.D0
      IPL1 = I+1
      DO 30 J=IPL1,N
      IF(X(I)-DABS(A(I,J)))20,20,30
   20 X(I) = DABS(A(I,J))
      IQ(I) = J
   30 CONTINUE
C
C     SET INDICATOR FOR RETURN
C     RAP = 2**-27
C
      RAP = .745058059692382BD-08
      HDTEST = 1.0D38
C
C     FIND MAXIMUM X(I) FOR PIVOT ELEMENT AND TEST FOR END OF PROBLEM
C
   40 DO 70 I=1,NMI1
      IF(I-1)60,60,45
   45 IF(XMAX-X(I))60,70,70
```

```
   60 XMAX = X(I)
      IPIV = I
      JPIV = IQ(I)
   70 CONTINUE
C
C     IF MAXIMUM X(I) IS EQUAL TO OR LESS THAN HDTEST, REVISE HDTEST
C
      IF(XMAX)1000,1000,80
   80 IF(HDTEST)90,90,85
   85 IF(XMAX-HDTEST)90,90,148
   90 HDMIN = DABS(A(1,1))
      DO 110 I=2,N
      IF(HDMIN-DABS(A(I,I)))110,110,100
  100 HDMIN = DABS(A(I,I))
  110 CONTINUE
      HDTEST = HDMIN*RAP
C
C     RETURN IF MAXIMUM A(I,J) IS LESS THAN (2**-27)*ABS(A(K,K)-MIN)
C
      IF(HDTEST-XMAX)148,1000,1000
  148 NR = NR+1
C
C     COMPUTE TANGENT, SINE, AND COSINE, A(I,I), A(J,J)
C
  150 TANG = DSIGN(2.DO,(A(IPIV,IPIV)-A(JPIV,JPIV)))*A(IPIV,JPIV)/(DABS(
     1A(IPIV,IPIV)-A(JPIV,JPIV))+DSQRT((A(IPIV,IPIV)-A(JPIV,JPIV))**2+4.
     2DO*A(IPIV,JPIV)**2))
      COSINE = 1.DO/DSQRT(1.DO+TANG**2)
      SINE = TANG*COSINE
      AII = A(IPIV,IPIV)
      A(IPIV,IPIV) = COSINE**2*(AII+TANG*(2.DO*A(IPIV,JPIV)+TANG*A(JPIV,
     1JPIV)))
      A(JPIV,JPIV) = COSINE**2*(A(JPIV,JPIV)-TANG*(2.DO*A(IPIV,JPIV)-TAN
     1G*AII))
      A(IPIV,JPIV) = 0.DO
C
```

```
C     PSEUDO RANK THE EIGENVALUES
C     ADJUST SINE AND COSINE FOR COMPUTATION OF A(IK) AND U(IK)
C
      IF(A(IPIV,IPIV)-A(JPIV,JPIV))152,153,153
  152 ATEMP = A(IPIV,IPIV)
      A(IPIV,IPIV) = A(JPIV,JPIV)
      A(JPIV,JPIV) = ATEMP
C
C     RECOMPUTE SINE AND COSINE
C
      ATEMP = DSIGN(1.D0,-SINE)*COSINE
      COSINE = DABS(SINE)
      SINE = ATEMP
  153 CONTINUE
C
C     INSPECT THE IQS BETWEEN I+1 AND N-1 TO DETERMINE WHETHER A NEW
C     MAXIMUM VALUE SHOULD BE COMPUTED SINCE THE PRESENT MAXIMUM IS IN
C     THE I OR J ROW
C
      DO 350 I=1,NMI1
      IF(I-IPIV)210,350,200
  200 IF(I-JPIV)210,350,210
  210 IF(IQ(I)-IPIV)230,240,230
  230 IF(IQ(I)-JPIV)350,240,350
  240 K = IQ(I)
  250 ATEMP = A(I,K)
      A(I,K) = 0.D0
      IPL1 = I+1
      X(I) = 0.D0
C
C     SEARCH IN DEPLETED ROW FOR NEW MAXIMUM
C
      DO 320 J=IPL1,N
      IF(X(I)-DABS(A(I,J)))300,300,320
  300 X(I) = DABS(A(I,J))
      IQ(I) = J
```

```
  320 CONTINUE
      A(I,K) = ATEMP
  350 CONTINUE
      X(IPIV) = 0.D0
      X(JPIV) = 0.D0
C
C     CHANGE THE OTHER ELEMENTS OF A
C
      DO 530 I=1,N
      IF(I-IPIV)370,530,420
  370 ATEMP = A(I,IPIV)
      A(I,IPIV) = COSINE*ATEMP+SINE*A(I,JPIV)
      IF(X(I)-DABS(A(I,IPIV)))380,390,390
  380 X(I) = DABS(A(I,IPIV))
      IQ(I) = IPIV
  390 A(I,JPIV) = -SINE*ATEMP+COSINE*A(I,JPIV)
      IF(X(I)-DABS(A(I,JPIV)))400,530,530
  400 X(I) = DABS(A(I,JPIV))
      IQ(I) = JPIV
      GO TO 530
  420 IF(I-JPIV)430,530,480
  430 ATEMP = A(IPIV,I)
      A(IPIV,I) = COSINE*ATEMP+SINE*A(I,JPIV)
      IF(X(IPIV)-DABS(A(IPIV,I)))440,450,450
  440 X(IPIV) = DABS(A(IPIV,I))
      IQ(IPIV) = I
  450 A(I,JPIV) = -SINE*ATEMP+COSINE*A(I,JPIV)
      IF(X(I)-DABS(A(I,JPIV)))400,530,530
  480 ATEMP = A(IPIV,I)
      A(IPIV,I) = COSINE*ATEMP+SINE*A(JPIV,I)
      IF(X(IPIV)-DABS(A(IPIV,I)))490,500,500
  490 X(IPIV) = DABS(A(IPIV,I))
      IQ(IPIV) = I
  500 A(JPIV,I) = -SINE*ATEMP+COSINE*A(JPIV,I)
      IF(X(JPIV)-DABS(A(JPIV,I)))510,530,530
  510 X(JPIV) = DABS(A(JPIV,I))
```

```
      IQ(JPIV) = I
  530 CONTINUE
C
C     TEST FOR COMPUTATION OF EIGENVECTORS
C
      IF(IEGEN)40,540,40
  540 DO 550 I=1,N
      ATEMP = U(I,IPIV)
      U(I,IPIV) = COSINE*ATEMP+SINE*U(I,JPIV)
  550 U(I,JPIV) = -SINE*ATEMP+COSINE*U(I,JPIV)
      GO TO 40
 1000 RETURN
      END
```