STRICTLY ISOSCELES METRICS ON ARCS

By

ROBERT ALLEN DOOLEY

Bachelor of Arts

University of Texas at Arlington

Arlington, Texas

1966

Submitted to the Faculty of the Graduate College
of the Oklahoma State University
in partial fulfillment of the requirements
for the degree of
MASTER OF SCIENCE
May, 1968

STRICTLY ISOSCELES METRICS ON ARCS

Thesis Approved:

John Jobe
Thesis Adviser

E. K. McLachlan

N Durham
Dean of the Graduate College

PREFACE

This paper was conceived with the purpose of finding necessary and sufficient conditions for a metric on an arc to be a strictly isosceles metric on the arc. Four such conditions have been found and listed. Most of this paper is a study of the properties of two of these conditions, namely, whether an arc has a D-kink and the value of the index k for which an arc is k-flat. Definitions of common topological concepts used in this paper are those of Elementary Topology by Hall and Spencer [3].

I would like to take this opportunity to express my appreciation to several persons to whom I am deeply indebted for the preparation of this paper: to Dr. John Jobe, my adviser, for his generous and invaluable assistance; to Dr. L. Wayne Johnson and Dr. John Jewett, through whom a graduate assistantship and NSF Traineeship have been generously provided; to my wife Kathie for her understanding and encouragement; and to the Lord, without whose help I could accomplish nothing.

TABLE OF CONTENTS

Chapter		Page
I.	SYNOPSIS	1
II.	ISOSCELES METRICS AND STRICTLY ISOSCELES METRICS	4
III.	ARCS AND THEIR D-KINKS	11
IV.	K-FLAT ARCS	23
V.	CONVERGING ARCS	32
A SELECTED BIBLIOGRAPHY		41

CHAPTER I

SYNOPSIS

This paper is an investigation of certain metric properties of arcs. The basic question to be answered is when a given metric on an arc is a strictly isosceles metric. In Chapter II a metric D on a set is defined to be an isosceles metric (i-metric) if for every two distinct points x, z of the set, there is a third point y of the set such that D(x, y) = D(y, z). An isosceles metric is a strictly isosceles metric (si-metric) if the third point in each case is unique.

The properties of i-metrizability and si-metrizability are topological properties. One result of this fact is that every arc is si-metrizable. Also, the metric of a metric space is an i-metric on each connected subset. However, not every subset on which the metric is an i-metric is connected. Any set which contains a simple closed curve is not si-metrizable, and for this reason no locally connected compact continuum which separates the plane is si-metrizable.

While it is true that every arc is si-metrizable, not every metric on an arc is a si-metric. In Chapter III it is shown that a metric D is a si-metric on arc A if and only if A has no D-kink. Arc A is said to have a D-kink at x, z whenever x and z are distinct points of A and D(x, y) = D(y, z) for some point y in A - [x, z]. If an arc has a D-kink at some pair of points, it must have a D-kink at an uncountable number of pairs of points.

Chapter III gives two other ways besides the existence of a D-kink to tell whether D is a si-metric on arc A. Metric D is a si-metric on A if and only if $D(x, z) > D(y, z)$ and $D(x, z) > D(y, x)$ for every three points x, y, z of A with y between x and z, and if and only if for every three points x, y, z of A with y between x and z there is a number $p \geq 0$ such that $[D(x, y)]^p + [D(y, z)]^p \leq [D(x, z)]^p$.

Chapter IV introduces a further classification of arcs. Arc A is said to be k-flat with respect to metric D if and only if k is the infimum of all quotients of the types

$$\frac{D(x, z) - D(y, z)}{D(y, x)} \quad \text{and} \quad \frac{D(x, z) - D(y, x)}{D(y, z)}$$

for every three points x, y, z of A with y between x and z. Arc A is said to be (+) - flat if all such quotients are positive and (-) - flat if there are non-positive quotients of these types. Arc A is said to be at least k-flat if A is m-flat for some $m \geq k$. Each arc is k-flat with respect to a given metric for some unique value of k in the range $-1 \leq k \leq 1$. Metric D is a si-metric on arc A if and only if A is (+) - flat with respect to D.

Values of the index k for several types of arcs in the plane are computed in Chapter IV. For example, it is found that a circular arc of angular measure α, where $0 < \alpha < 2\pi$, is $\cos \alpha/2$ - flat. An arc consisting of the union of two line segments is $(-\cos \beta)$ - flat, where β is the least positive angle between the two segments. If a polygonal arc of $n \geq 2$ line segments is inscribed in a circular arc of angular measure α, then the polygonal arc is $\cos \alpha/2$ - flat.

Let $\{A_n\}$ be a sequence of arcs converging to an arc A in a metric space with metric D. Chapter V investigates when the limit arc A inherits from the sequence $\{A_n\}$ the properties of having a D-kink, having no D-kink, and being k-flat for a given value of k. Examples are given to show that the sequence $\{A_n\}$ may either lose a D-kink or create a new one on the limit arc A. Another example shows that arc A need not be k-flat just because each arc A_n is k-flat, at least for $k < 1$. However, it is true that if $\overline{\cup A_n}$ is compact and if each arc A_n is at least k-flat, then arc A must be at least k-flat also. The following three statements are corollaries to this result. The limit arc of a sequence of 1-flat arcs is 1-flat. If each arc A_n is at least k-flat and $k > 0$, then D is a si-metric on arc A. If D is a si-metric on each arc A_n, then arc A has the following two properties: if y is between x and z on A, then $D(x, z) \geq D(y, z)$ and $D(x, z) \geq D(y, x)$; if y is in $A-[x, z]$ and $D(x, y) = D(y, z)$, then $D(t, y) = D(y, z)$ for every point t in the subarc $[x, z]$ of A.

CHAPTER II

ISOSCELES METRICS AND STRICTLY ISOSCELES METRICS

A topological space S is a metric space with metric D if D is a real-valued function with domain $S \times S$ such that if x, y, and z are points of S,

(i) $D(x, y) \geq 0$

(ii) $D(x, y) = 0$ if and only if $x = y$.

(iii) $D(x, y) = D(y, x)$

(iv) $D(x, z) \leq D(x, y) + D(y, z)$

and the topology of S is precisely the collection of subsets of S which is generated by all spherical neighborhoods determined by D. In this paper the spherical neighborhood about a point x consisting of all points y such that $D(x, y) < \epsilon$ for a given $\epsilon > 0$ is denoted by $N_\epsilon(x)$.

Metrics with various properties may be obtained by adding further requirements to their definition. Menger in [5] defines a metric to be a convex metric if it satisfies the additional requirement

(v) for each pair of points x, y of S there exists a point u of S such that $D(x, u) = D(u, y) = D(x, y)/2$.

Glynn in [2] defines a metric to be a strictly convex metric by the further requirement

(v') for each pair of points x, y of S there exists a unique point u of S such that $D(x, u) = D(u, y) = D(x, y)/2$.

The following definition is suggested by those of Menger and Glynn.

Definition 2.1. Let S be a metric space with metric D. The metric D is an isosceles metric, or simply an i-metric, on S if it satisfies the requirement

(vi) for each pair of points x, y of S there exists a point u of S such that $D(x, u) = D(u, y)$.

The metric D is a strictly isosceles metric, or a si-metric, on S if it satisfies the requirement

(vi') for each pair of distinct points x, y of S there exists a unique point u of S such that $D(x, u) = D(u, y)$.

If for a given subset M of a topological space T there is a metric D which is an i- (or si-) metric on M and for which M is a metric subspace of T, then M is said to be i- (or si-) metrizable.

If follows from these definitions that a (strictly) convex metric is a (strictly) isosceles metric. Although these metrices are defined in a similar manner, it is not the purpose of this paper to compare their properties. However, they do enjoy some properties in common. For instance, it is shown by Glynn in [2] that convex and strictly convex metrizability are topological properties. The same result is obtained for isosceles and strictly isosceles metrics.

Theorem 2.1. Both i-metrizability and si-metrizability are topological properties.

Proof. Let S and T be two homeomorphic topological spaces such that S is i- (or si-) metrizable under metric D. Let f be a homeomorphism from S onto T. Define $D'(x, y) = D(f^{-1}(x), f^{-1}(y))$ for every two points x, y of T. Then f is an isometry from(S, D) onto (T, D') as well as a homeomorphism, and thus D' is an i- (or si-) metric for T.

Before stating a corollary to Theorem 2. 1, some necessary terminology on arcs is given. An arc is defined to be a homeomorphic image of the subspace of real numbers consisting of the closed interval [0, 1] and is characterized by being a compact, connected, separable metric space having exactly two non-cut points [3-p. 168]. If the two non-cut points of an arc A are a and b, A is often called an arc from a to b. If x, y, and z are three distinct points of an arc A, then y is said to be between x and z, written xyz, if A-{y} is the union of two separated sets, one containing x and the other containing z. This property of "betweenness" may also be considered a consequence of a natural linear ordering of the points of the arc. If x and y are distinct points of an arc A, then the set of all points between x and z, together with x and z, is called the subarc of A from x to z and is denoted [x, z] or [z, x]. A subarc of an arc is itself an arc.

Corollary 2. 1. 1. Every arc is si-metrizable.

Proof. The subspace [0, 1] of real numbers has a si-metric which induces its topology, namely, the usual distance function. Since every arc is topologically equivalent to [0, 1], the previous theorem shows that every arc inherits si-metrizability.

A large class of examples of i-metrizable spaces is provided by the following theorem, which is fundamental to the remaining results of this paper.

Theorem 2. 2. Let S be a metric space with metric D. If M is a connected subset of S, then D is an i-metric on M.

Proof. If M is degenerate, then D is vacuously an i-metric on M. If M is non-degenerate, let u and v be distinct points of M. Define a real-valued function f on M in the following way:

$$f(x) = \frac{D(u, x)}{D(u, x) + D(v, x)}$$

for every x in M. Since $D(u, x) + D(v, x) \geq D(u, v) > 0$, f is well defined on M. Now f is also continuous on M, for let x in M and $\epsilon > 0$ be given. Let $\delta = \epsilon D(u, v) > 0$. Whenever $D(x, y) < \delta$ for some y in M, then $|D(u, x) - D(u, y)| \leq D(x, y) < \delta$ and $|D(v, y) - D(v, x)| \leq D(x, y) < \delta$ by the triangle inequality. Therefore,

$$\begin{aligned}
&|D(u, x)\,[D(u, y) + D(v, y)] - D(u, y)\,[D(u, x) + D(v, x)]| \\
&\quad = |D(u, x)\,D(u, y) + D(u, x)\,D(v, y) - D(u, y)\,D(u, x) - D(u, y)\,D(v, x)| \\
&\quad = |D(u, x)\,D(v, y) - D(u, y)\,D(v, x)| \\
&\quad = |D(u, x)\,D(v, y) - D(u, x)\,D(v, x) + D(u, x)\,D(v, x) - D(u, y)\,D(v, x)| \\
&\quad \leq |D(u, x)\,D(v, y) - D(u, x)\,D(v, x)| + |D(u, x)\,D(v, x) - D(u, y)\,D(v, x)| \\
&\quad = D(u, x)\,|D(v, y) - D(v, x)| + D(v, x)\,|D(u, x) - D(u, y)| \\
&\quad < D(u, x)\,\delta + D(v, x)\,\delta \\
&\quad = [D(u, x) + D(v, x)]\,\delta \\
&\quad = [D(u, x) + D(v, x)]\,\epsilon\, D(u, v) \\
&\quad \leq [D(u, x) + D(v, x)]\,[D(u, y) + D(v, y)]\,\epsilon.
\end{aligned}$$

It follows by division that

$$|f(x) - f(y)| = \left| \frac{D(u, x)}{D(u, x) + D(v, x)} - \frac{D(u, y)}{D(u, y) + D(v, y)} \right| < \epsilon.$$

Hence f is continuous on M.

Every real-valued continuous function defined on a connected set has the intermediate value property [4-p. 200]. In particular, f has this property, and since $f(u) = 0$ and $f(v) = 1$, it follows that there exists a point w in M such that $f(w) = 1/2$. That is,

$$\frac{D(u, w)}{D(u, w) + D(v, w)} = \frac{1}{2},$$

and therefore D(u, w) = D(v, w) for this w in M. Thus D is shown to be an i-metric on M, completing the proof.

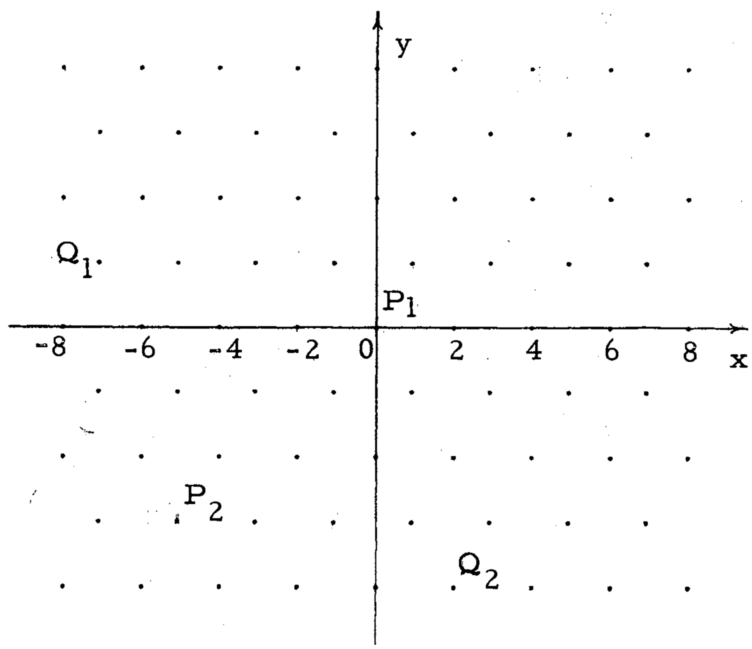

Figure 1.

Example 2.1. Let $M = \{(m, \sqrt{3}n)$: m and n are integers, m+n is even$\}$ be a subset of E^2 with the usual topology. M consists of the vertices of a system of congruent equilateral triangles which, with their interiors, cover the plane, as illustrated in Figure 1. Let P_1 and P_2 be any two points of M. Since the axes may be translated without affecting distances, let $P_1 = (0, 0)$ and $P_2 = (m, \sqrt{3}n)$. P_2 is represented in Figure 1 with m = -5 and n = -3. Consider the following two points, whose coordinates are determined by analytic methods:
$Q_1 = ((m+3n)/2, \sqrt{3}(n-m)/2)$ and $Q_2 = ((m-3n)/2, \sqrt{3}(n+m)/2)$.

Now $(m+3n)/2 = (m\pm n)/2 \pm n$ is an integer since $m\pm n$ is even, and $(n\mp m)/2$ is likewise an integer. Also, $(m+3n)/2 + (n-m)/2 = 2n$ and $(m-3n)/2 + (n+m)/2 = m-n$ are both even integers. Therefore, Q_1 and Q_2 are points of M. Also, direct calculation shows that $D(P_1, Q_1) = D(P_2, Q_1) = D(P_1, Q_2) = D(P_2, Q_2)$ where D is the usual distance function. Hence D is an i-metric on M, but not a si-metric.

It may also be noted that a si-metrizable space need not be connected, for the three vertices of a single equilateral triangle in the plane is si-metrizable but not connected. Two corollaries to Theorem 2.2 follow.

Corollary 2.2.1. If a set contains a simple closed curve, it is not si-metrizable.

Proof. Suppose a subset M of a topological space contains a simple closed curve C, and suppose D is a metric on M which generates the subspace topology. Let a and b be two distinct points of C. Then $C = P \cup Q$, where P and Q are independent arcs from a to b [3-p. 171]. Since P is a connected set, D is an i-metric on P by Theorem 2.2. Therefore there exists a point u of P such that $D(a, u) = D(u, b)$, and u is distinct from a and b since a and b are distinct. Similarly, there exists a point v of Q distinct from a and b such that $D(a, v) = D(v, b)$. Since P and Q are independent, $P \cap Q = \{a, b\}$, and therefore $u \neq v$. Hence D is not a si-metric on C and certainly not on M, which contains C. This shows that M is not si-metrizable.

Corollary 2.2.2. No locally connected compact continuum which separates the plane is si-metrizable.

Proof. If M is a locally connected compact continuum which separates the plane, then M contains a simple closed curve [6-p. 34]. By Corollary 2.2.1, M cannot be si-metrizable.

CHAPTER III

ARCS AND THEIR D-KINKS

In Corollary 2. 1. 1 it was noted that every arc is si-metrizable and therefore i-metrizable. Also, in the light of Theorem 2. 2, every metric on an arc which induces its topology is an i-metric on the arc. However, not every metric on an arc which induces its topology is a si-metric on the arc. This fact is illustrated in Figure 2, where A is an arc in the plane and D is the usual distance between points. For every pair of distinct points x, z of A, since the subarc [x, z] is connected, according to Theorem 2. 2 there will always be a point y in [x, z] such that D(x, y) = D(y, z). However, the arc may bend so much that there is some other point y', which may or may not be in [x, z], such that D(x, y') = D(y', z). (Whenever A is an arc in the plane and D is the usual metric, the points y and y' are found as

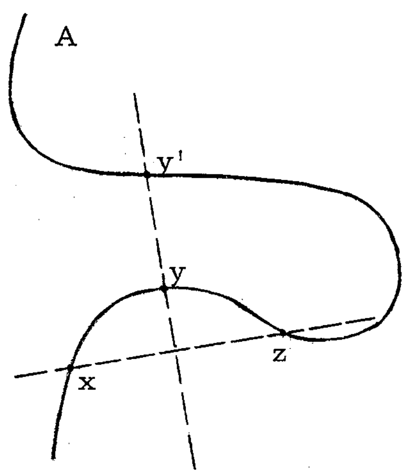

Figure 2.

the intersection of A and the perpendicular bisector of the line segment between x and z, as in Figure 2.) For this arc, D is not a si-metric. Such possibilities raise the fundamental question of this paper: What are necessary and sufficient conditions for a metric on an arc to be a si-metric?

One answer to this question is provided by the notion of a D-kink. Unless otherwise stated, the setting for all definitions and theorems in the remainder of this paper is a metric space S with metric D.

Definition 3.1. An arc A is said to have a D-kink at x, z if and only if x and z are distinct points of A with the property that $D(x, y) = D(y, z)$ for some point y in A - [x, z]. Arc A is said to have a D-kink if and only if A has a D-kink at x, z for some pair of points x, z of A.

Theorem 3.1. If D is a si-metric on arc A, then A has no D-kink.

Proof. The proof is given by contraposition. Suppose that A has a D-kink at x, z for some pair of points x, z. Then there is a point y in A - [x, z] such that $D(x, y) = D(y, z)$. But by Theorem 1.2 there is a point y' in [x, z] such that $D(x, y') = D(y', z)$. Since $y \neq y'$, D cannot be a si-metric on A.

It should be noted that whether a given arc has a D-kink depends only on the metric D rather than on the arc. Since each arc is si-metrizable, the previous theorem implies that there is always some metric D for which it will have no D-kink. The property of having no D-kink actually characterizes when D is a si-metric on an arc.

Theorem 3.2. If an arc A has no D-kink, then D is a si-metric on A.

Proof. The proof is again given by contraposition. If D is not a si-metric on A, then there are four pairwise distinct points x, z, p, q of A such that D(x, p) = D(p, z) and D(x, q) = D(q, z). If one of p or q is in A - [x, z], then A has a D-kink at x, z, and the theorem is proved. If p and q are both between x and z, then without loss of generality let xpq and pqz, as in Figure 3.

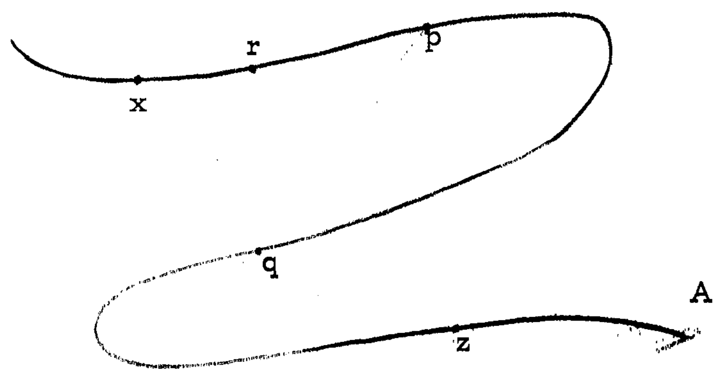

Figure 3.

It cannot be true both that D(x, p) < D(x, q) and D(z, q) < D(z, p), for then D(x, p) < D(x, q) = D(z, q) < D(z, p) implies D(x, p) ≠ D(z, p). Therefore let D(x, p) ≥ D(x, q), again without loss of generality. If D(x, p) = D(x, q), then since x is in A - [p, q], A has a D-kink at p, q and the theorem is proved.

Hence suppose D(x, p) > D(x, q), and let G = {t: t is in [x, p], D(t, p) > D(t, q)} and H = {t: t is in [x, p], D(t, p) < D(t, q)} . Sets G and

H are nonempty since x is in G and p is in H. Also, G and H are open in [x, p], for let s be in G, for instance. Then for $2\epsilon = D(s,p) - D(s,q) > 0$, let $U = N_\epsilon(s) \cap [x, p]$. Set U is open in [x, p] and contains s. If t is in U, then

$$D(t, q) \leq D(t, s) + D(s, q) < \epsilon + D(s, q) = D(s,p) - \epsilon \leq D(s,t) - \epsilon + D(t,p) < D(t,p),$$

which shows that t is in G; that is, $U \subset G$. Hence G is open in [x, p], and similarly H is open in [x, p]. Since $G \cap H = \emptyset$, the equation $[x, p] = G \cup H$ is a separation of the connected set [x, p], which is a contradiction. Therefore there must be a point r in [x, p] which is neither in G nor H; that is, D(p, r) = D(r, q). Since $r \neq x$ and $r \neq p$, then xrp. Therefore rpq, r is in A-[p, q], and A has a D-kink at p, q. This completes the proof.

Theorem 3.3. Metric D is a si-metric on arc A if and only if A has no D-kink.

Proof. Theorem 3.1 gives the necessity, and Theorem 3.2 the sufficiency of the D-kink condition.

There follow two corollaries which state other equivalent conditions for D to be a si-metric on A.

Corollary 3.3.1. Metric D is a si-metric on arc A if and only if D(x, z) > D(y, z) and D(x, z) > D(y, x) for every three points x, y, z of A such that xyz.

Proof. Both sufficiency and necessity are proved by contraposition. If D is not a si-metric on A, then A has a D-kink at some pair of distinct points x, z; that is, there is a point y such that yxz or xzy such that D(x, y) = D(y, z). This contradicts the inequality condition.

Conversely, if there are points x, y, z of A with xyz and $D(x, z) \leq D(y, z)$, for instance, there are two cases to consider. First, if $D(x,z) = D(y, z)$, then A has a D-kink at x, y. Second, if $D(x, z) < D(y, z)$, let $G = \{t: t \text{ is in } [y, z], D(y, t) < D(t, x)\}$ and $H = \{t: t \text{ is in } [y, z], D(y, t) > D(t, x)\}$. Since y is in G and z is in H, G and H are nonempty disjoint sets which are open in [y, z]. Therefore $[y, z] = G \cup H$ would be a separation of the connected set [y, z] unless there is a point r of [y, z] such that $D(y, r) = D(r, x)$. The existence of such a point r, together with xyr, implies that A has a D-kink at x, y. In either case A has a D-kink, and therefore D is not a si-metric on A. This completes the proof.

Corollary 3.3.2. Metric D is a si-metric on arc A if and only if for every three points x, y, z of A with xyz there exists a number $p \geq 0$ such that $[D(x, y)]^p + [D(y, z)]^p \leq [D(x, z)]^p$.

Proof. The necessity is proved directly. If D is a si-metric on A, let x, y, z be points of A with xyz. By Corollary 3.3.1, $D(x, z) > D(y, z)$ and $D(x, z) > D(x, y)$. If $p \geq 0$ is chosen so that $[D(x, z)]^p \geq 2[D(y, z)]^p$ and $[D(x, z)]^p \geq 2[D(x, y)]^p$, then $[D(x, y)]^p + [D(y, z)]^p \leq [D(x, z)]^p$.

The sufficiency is proved by contraposition. If D is not a si-metric on A, then A has a D-kink at x, y for some pair of points x, y. Then $D(x, z) = D(z, y)$ for some z in A-[x, y], and without loss of generality xyz. Therefore $D^p(x, z) = D^p(y, z)$ and hence $D^p(x, y) + D^p(y, z) > D^p(x, z)$ for every number p, which contradicts the exponent condition of the theorem.

Once these equivalences are proved, the question then arises:

May an arc have a D-kink at one pair of points only? The following definition will aid in answering that question.

Definition 3.2. For a point x of an arc A, the set $A'(D;x)$ of all points y of A such that A has a D-kink at x, y is called the deleted D-kink set of x. The set $A(D;x) = A'(D;x) \cup \{x\}$ is called the D-kink set of x.

The following example illustrates how large the D-kink set of a point may be and at how many pairs of points an arc may have a D-kink.

Example 3.1. Let A be an arc in the plane composed of two sides of an equilateral triangle which meet at vertex c, and let D be the usual distance metric. Then $A(D;c) = A$ and $A'(D;x) \neq \emptyset$ for every point x in A.

The next example illustrates that the subset $\{x: A'(D;x) \neq \emptyset\}$ of an arc A need not be closed.

Example 3.2. Let arc A in the plane be constructed in the following way: let $b_i = (1/2^{i-1}, 0)$ and $a_i = (3/2^i, -\sqrt{3}/2^i)$ for $i = 1, 2, \ldots.$ Let $A = \{0\} \cup \bigcup_{i=1}^{\infty} (\overline{a_i b_i} \cup \overline{b_i a_{i+1}})$, where $0 = (0, 0)$ and $\overline{pq}$ denotes the line segment from p to q. A is shown in Figure 4. Triples of the form a_i, b_i, a_{i+1} are vertices of right $30^\circ - 60^\circ$ triangles. If D is the usual distance metric, then $A'(D;b_i) \neq 0$ for every i since $D(a_{i+1}, b_{i+1}) = D(b_i, b_{i+1})$ shows a_{i+1} is in $A'(D;b_i)$. Also, the sequence $\{b_i\}$ converges to 0. However, $A'(D;0) = \emptyset$, for let p be any point in $A-\{0\}$. There exists an n such that p is in $\overline{a_n b_n} \cup \overline{b_n a_{n+1}}$. Let L be the perpendicular bisector of $\overline{0p}$. If $p = a_n$, L is the line passing through b_n and a_{n+1}

and hence $L \cap A = \overline{b_n a_{n+1}}$. That is, $L \cap A \subset [0, p]$. If $a_n p b_n$, then $L \cap A$ is a point between a_{n+1} and b_{n+1}, and again $L \cap A \subset [0, p]$. If $p = b_n$, $L \cap A = \{b_{n+1}\}$, which is in $[0,p]$. If $b_n p a_{n+1}$, then $L \cap A$ is some point between b_{n+1} and a_{n+2}, and again $L \cap A \subset [0, p]$. Hence p is not in $A'(D;0)$ and $A'(D;0) = \emptyset$. Thus the set $\{x: A'(D;x) \neq 0\}$ is not closed since the accumulation point 0 is not included in the set. However, the following theorem shows that the D-kink set of a point is always closed.

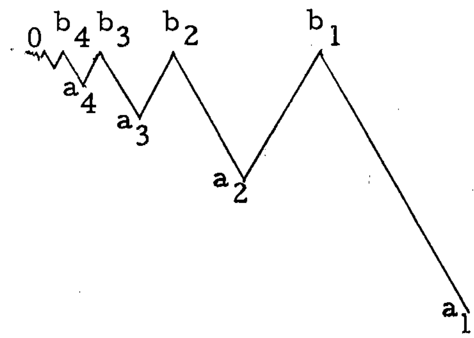

Figure 4.

Theorem 3.4. If x is a point of arc A, then the set A(D;x) is closed.

Proof. Let x_o be an accumulation point of A(D;x). There exists a sequence $\{x_n\}$ of distinct points of A(D;x) which converges to x_o [3-p. 70]. Since the x_n's are distinct, we may assume that $x_n \neq x$ for every n. Since for each n A has a D-kink at x, x_n, then for each n there exists a point z_n in $A-[x, x_n]$ such that $D(x, z_n) = D(z_n, x_n)$. The sequence $\{z_n\}$ in A so defined has a subsequence $\{z_k\}$ which converges to a point z_o of A since A is compact. The associated subsequence $\{x_k\}$ of $\{x_n\}$ converges to x_o.

If $x_o \neq x$, then it can be shown that A has a D-kink at x, x_o, for let $\epsilon > 0$ be given. There exists a number k such that $D(x_o, x_k) < \epsilon /3$

and $D(z_o, z_k) < \epsilon/3$. Since $D(x, z_k) = D(z_k, x_k)$, $D(z_o, x_o) \leq D(z_o, z_k) + D(z_k, x_k) + D(x_k, x_o)$, and $D(x, z_o) \geq D(x, z_k) - D(z_k, z_o)$, it follows that

$$\begin{aligned} D(z_o, x_o) - D(x, z_o) &\leq D(z_o, z_k) + D(z_k, x_k) + D(x_k, x_o) - D(x, z_k) + D(z_k, z_o) \\ &= D(z_o, z_k) + D(x_k, x_o) + D(z_k, z_o) \\ &< \epsilon . \end{aligned}$$

Similarly

$$\begin{aligned} D(x, z_o) - D(z_o, x_o) &\leq D(x, z_k) + D(z_k, z_o) - D(z_k, x_k) + D(z_k, z_o) + D(x_o, x_k) \\ &= D(z_k, z_o) + D(z_k, z_o) + D(x_o, x_k) \\ &< \epsilon . \end{aligned}$$

Therefore $D(x, z_o) = D(z_o, x_o)$, and A has a D-kink at x, x_o once it is shown that z_o is in $A-[x, x_o]$.

Since $D(x, z_o) = D(z_o, x_o)$ and $x \neq x_o$, then $x \neq z_o \neq x_o$ implies that either xz_ox_o or z_o is in $A-[x, x_o]$. If xz_ox_o, let y_o be a point in A such that $z_oy_ox_o$. Let a and b be endpoints of A, and without loss of generality let az_ox_o and z_oy_ob. Since $A-([a, x] \cup [y_o, b])$ and $A-[a, y_o]$ are open sets in A containing z_o and x_o respectively, there exists a number $\epsilon > 0$ such that if s is in $N_\epsilon(z_o) \cap A$ and t is in $N_\epsilon(x_o) \cap A$, then xsy_o and y_otb. Since $\{x_k\}$ converges to x_o and $\{z_k\}$ converges to z_o, there exists a number k such that $D(z_o, z_k) < \epsilon$ and $D(x_o, x_k) < \epsilon$. Therefore xz_ky_o and y_ox_kb, which imply xz_kx_k. But by the definition of D-kink, z_k is in $A-[x, x_k]$. This contradiction shows that xz_ox_o is false. Hence z_o is in $A-[x, x_o]$, A has a D-kink at x, x_o, and the accumulation point x_o is in $A(D;x)$. Therefore $A(D;x)$ is closed.

Now the question which motivated the definition of the D-kink set of a point is to be answered. The answer is that if an arc has a D-kink at some pair of points, it has a D-kink at an uncountable number of such pairs. The following two theorems sharpen this statement in different ways.

Theorem 3.5. If there exist distinct points x, y, z of an arc A such that y is in A-[x, z] and D(x, y) = D(y, z), then there are two subarcs in [x, z] which intersect in only one point and have the property that for every point r in one subarc there is a point s in the other such that D(r, y) = D(y, s).

Proof. Let x, y, z be points of A such that y is in A-[x, z] and D(x, y) = D(y, z). There are three cases to consider.

(1) If there exists a point p in [x, z] such that D(p, y) < D(x, y), then let I = {t: t is in [x, z], D(t, y) $\leq$ D(x, y)}. Let C be the component of I which contains p. The set C is closed in I [3-p. 171]. Since I is closed in A, C is closed in A. Therefore C must be either a subarc or the singleton {p}. But C $\neq$ {p}, for since 0 = {t: t is in [x, z], D(t, y) < D(x, y)} is a set open in A and containing p, there are distinct points c and d such that the set M = {t: ctd} contains p and is contained in 0 and therefore in I. Since M is connected, M $\subset$ C. Therefore C $\neq$ {p}. Hence C is some subarc [u, v] in [x, z], where D(u, y) = D(v, y) = D(x, y) and without loss of generality pvy, as in Figure 5. Since {y} and [u, v] are disjoint compact sets, there is a point q in [u, v] such that D(q, y) $\leq$ D(t, y) for any t in [u, v] [3-p. 91]. Also, since D(q, y) $\leq$ D(p, y) < D(u, y) = D(v, y), then u $\neq$ q $\neq$ v. The subarcs

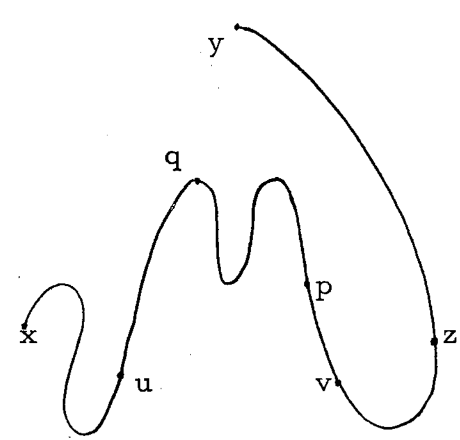

Figure 5.

[u, q] and [q, v] are the ones required in the theorem, for let r be a point in [u, q]. If D(r, y) = D(q, y), then q is the required point in [q, v]. If D(r, y)= D(u, y), then v is the required point in [q, v] since D(u, y) = D(v, y). If D(q, y) < D(r, y) < D(u, y), then let G = {t: t is in [q, v], D(t, y) < D(r, y) } and H = { t: t is in [q, v], D(t, y) > D(r, y)}. If [q, v] = G ∪ H, then this representation is a separation of the subarc [q, v] since G and H are both open in [q, v] and contain q and v, respectively. The impossibility of such a separation implies the existence of a point s in [q, v] such that D(r, y) = D(y, s). Similarly, for any given point r in [q, v] there is a point s in [u, q] such that D(r, y) = D(y, s).

(2) If there exists a point p in [x, z] such that D(p, y) > D(x, y), then the proof is entirely similar to the preceding argument.

(3) If every point p of [x, z] has the property that D(p, y) = D(x,y), then for any point q between x and z the subarcs [x, q] and [q, z] possess the property required by the theorem.

Theorem 3.6. If arc A has a D-kink at x, z, then either A'(D;x) or A'(D;z) contains a subarc of [x, z], and [x, z] ⊂ A'(D;x) ∪ A'(D;z).

Proof. If A has a D-kink at x, z, then there is a point y in A-[x, z] such that D(x, y) = D(y, z). Without loss of generality let xzy. As in the proof of the previous theorem, there are three cases to consider.

(1) If p is a point of [x, z] such that D(p, y) > D(y, x) then since p is in the set M = { t: t is in [x, z], D(t, y) > D(y, x)} , which is open in [x, z], and x ≠ p ≠ z, then there are distinct points c and d of M such that { t: ctd} contains p and is contained in M. There are points u and v such that cup and pvd, so that the subarc [u, v] is in M, as illustrated

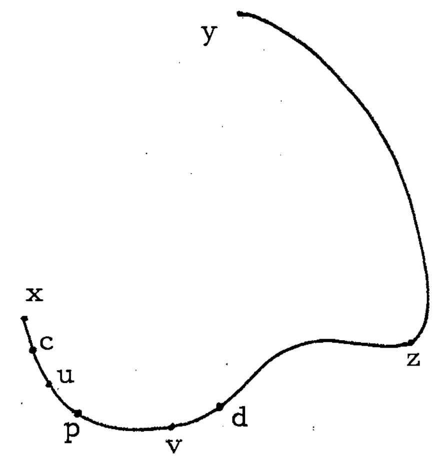

Figure 6

in Figure 6. Subarc [u, v] is contained in A'(D;x), for let r be a point of [u, v]. Let G = {t: t is in [r, y], D(t, x) < D(t, r)} and H = {t: t is in [r, y], D(t, x) > D(t, r)}. Since G and H are both open sets in [r, y] and contain y and r, respectively, then if [r, y] = G ∪ H, this representation is a separation of the subarc [r, y]. Hence there is a point s in [r,y] such that D(s, x) = D(s, r), and since r ≠ x, then rsy and s is in A-[x, r]. Hence r is in A-[x, r]. Hence r is in A'(D;x), and [u, v] ⊂ A'(D;x).

(2) If p is a point of [x, z] such that D(p, y) < D(y, x), then it can be shown similarly that there is a subarc [u, v] of [x, z] which contains p and is contained in A'(D;z).

(3) If D(p, y) = D(y, x) = S(y, z) for every point p in [x, z], then the subarc [x, z] is contained in both A'(D;x) and A'(D; z).

In fact, if p is a point of [x, z] such that D(p, y) = D(y, x), then p is in A'(D;x) and A'(D;z). This completes the proof.

<u>Corollary 3.6.1.</u> It is not possible for the following two conditions to hold simultaneously on an arc A:

(i) A'(D;x) ∩ A'(D;y) = ∅ for every two distinct points x, y of A.

(ii) A'(D;x) ≠ ∅ for every x in A.

Proof. Suppose (i) and (ii) both hold. For each x in A, $A'(D;x) \neq \emptyset$. If y and z are points of $A'(D;x)$, then A has a D-kink at x, y, and A has a D-kink at x, z; that is, x is in $A'(D;y) \cap A'(D;z)$. But since this intersection is empty if y and z are distinct, then y = z. Hence $A'(D;x)$ is a singleton for each x in A. But by Theorem 3.6, if $A'(D;x) = \{y\}$, then $A'(D;y)$ must contain a subarc, which a singleton cannot do. Hence (i) and (ii) cannot be satisfied simultaneously.

CHAPTER IV

K-FLAT ARCS

The present chapter introduces a further classification of arcs according to metric properties.

Definition 4. 1. Arc A is said to be k-flat with respect to metric D if and only if k is the infimum of all quotients of the types

$$\frac{D(x, z) - D(y, z)}{D(y, x)} \text{ and } \frac{D(x, z) - D(y, x)}{D(y, z)}$$

for every three points x, y, z of A such that xyz. Arc A is said to be (+) - flat if all such quotients are positive and (-) - flat if there are non-positive quotients of these types for points of A. Arc A is said to be at least k-flat if and only if A is m-flat for some $m \geq k$.

Since $D(x, z) - D(y, z) \geq -1 \cdot D(y, x)$ for any three points of A by the triangle inequality, all quotients of the types given in Definition 4. 1 will be bounded below by -1. Hence an infimum of such quotients will exist, and every arc will be k-flat for some unique value of k. Also, since $D(y, x) \geq D(x, z) - D(y, z)$ by the triangle inequality, such quotients are bounded above by 1. Therefore, the range of values of k for which an arc may be k-flat is $-1 \leq k \leq 1$.

Theorem 4. 1. Metric D is a si-metric on an arc A if and only if A is (+) - flat with respect to D.

Proof. This theorem is simply a restatement of Corollary 3.3.1, for quotients such as

$$\frac{D(x,z) - D(y,z)}{D(y,x)} \text{ and } \frac{D(x,z) - D(y,x)}{D(y,z)}$$

are positive if and only if the numerators are positive.

It should be noted that if an arc A is k-flat with respect to metric D, then if $k > 0$, D is a si-metric on A; if $k < 0$, D is not a si-metric on A; if $k = 0$, D is a si-metric on A if and only if D is a proper infimum for the quotients in question on A. Thus for an arc A which is 0-flat with respect to metric D, the metric may or may not be a si-metric on A. Both types of 0-flat arcs will be exhibited later, in the first two examples of Chapter V.

As in the case of arcs with D-kinks, whether a given arc is k-flat for a given value of k depends upon the metric under consideration. In fact, for any given arc A and for any value of k in the range $-1 \leq k \leq 1$, A can be made into a k-flat arc with respect to some metric D which is defined in an appropriate way. One such way to define metric D is as follows. Let $\overline{A}$ be an arc which is k-flat with respect to a metric $\overline{D}$ and let f be a homeomorphism from A onto $\overline{A}$. For any two points x, y of A define $D(x, y) = \overline{D}(f(x), f(y))$. Then A is a metric space with metric D, and A is k-flat with respect to D. The metric D was constructed under the assumption that for the given value of k there is another arc $\overline{A}$ which is k-flat already with respect to some metric $\overline{D}$. The remainder of this chapter is devoted to exhibiting families of arcs in the plane which have k-flat arcs for a wide range of values of k.

Example 4.1. Let A_α be a circular arc in the plane of radius ρ_0 and angular measure α, where $0 < \alpha < 2\pi$. By means of polar coordinates it can be shown that A_α is cos $\alpha/2$ - flat. Let the arc lie on the circle $\rho = \rho_0$ and let the endpoints be given by $\bar{a} = (\rho_0, 0)$ and $\bar{b} = (\rho_0, \alpha)$, as in Figure 7. Let $\bar{y} = (\rho_0, \beta)$ be any point of A_α between $\bar{a}$ and $\bar{b}$, so that $0 < \beta < \alpha < 2\pi$. It will be

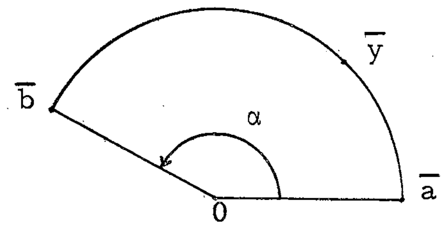

Figure 7.

shown first that the limiting value of the quotient $\dfrac{D(\bar{a},\bar{b}) - D(\bar{y},\bar{b})}{D(\bar{y},\bar{a})}$ as $\bar{y}$ approaches $\bar{a}$ is cos $\alpha/2$, and then that cos $\alpha/2$ is a lower bound of such quotients.

$$\text{Let } f(\beta) = \frac{D(\bar{a},\bar{b}) - D(\bar{y},\bar{b})}{D(\bar{y},\bar{a})}$$

$$= \frac{\sqrt{2\rho_0^2 - 2\rho_0^2 \cos\alpha} - \sqrt{2\rho_0^2 - 2\rho_0^2 \cos(\alpha-\beta)}}{\sqrt{2\rho_0^2 - 2\rho_0^2 \cos\beta}}$$

$$= \frac{\sqrt{1-\cos\alpha} - \sqrt{1-\cos(\alpha-\beta)}}{\sqrt{1-\cos\beta}}$$

$$= \frac{\sqrt{2}\sin\alpha/2 - \sqrt{2}\sin(\alpha-\beta)/2}{2\sin\beta/2}$$

$$= \frac{\sin\alpha/2 - \sin(\alpha-\beta)/2}{\sin\beta/2}$$

Thus $f(\beta)$ is independent of the value of ρ_0.

$$\lim_{\beta\to 0^+} f(\beta) = \lim_{\beta\to 0^+} \frac{-(-1/2)\cos(\alpha-\beta)/2}{(1/2)\cos\beta/2} = \cos\alpha/2$$ by the use of L'Hospital's rule. Now if it can be shown that $f(\beta) \geq \cos\alpha/2$ for $0 < \beta < \alpha < 2\pi$, then $\cos\alpha/2$ is actually the infimum of $f(\beta)$.

$$f'(\beta) = \frac{\sin \beta/2 \,[-\cos(\alpha-\beta)/2](-1/2) - [\sin \alpha/2 - \sin(\alpha-\beta)/2]\cos \beta/2 \,(1/2)}{\sin^2 \beta/2}$$

$$= \frac{\sin \beta/2 \cos(\alpha-\beta)/2 - \sin \alpha/2 \cos \beta/2 + \sin(\alpha-\beta)/2 \cos \beta/2}{2 \sin^2 \beta/2}$$

$$= \frac{\sin \alpha/2 \,(1 - \cos \beta/2)}{2 \sin^2 \beta/2} > 0 .$$

Hence $f(\beta)$ increases as β increases in $0 < \beta < \alpha$, and therefore $f(\beta) \geq \cos \alpha/2$.

In fact, A_α is $\cos \alpha/2$-flat, for let $\overline{y}$ be between $\overline{x}$ and $\overline{z}$ on A_α. Without loss of generality $\overline{x}$ has a smaller angular coordinate than $\overline{z}$, and without loss of generality $\overline{x} = (\rho_0, 0)$ for computational purposes since the value of the quotient

$$\frac{D(\overline{x}, \overline{z}) - D(\overline{y}, \overline{z})}{D(\overline{y}, \overline{x})}$$

will be unchanged whenever these points are equally rotated. Then $\overline{z}$ is given by $\overline{z} = (\rho_0, \gamma)$, where $\gamma \leq \alpha$. But since $\overline{x}$ and $\overline{z}$ may now be considered as endpoints of the arc A_γ, then

$$\frac{D(\overline{x}, \overline{z}) - D(\overline{y}, \overline{z})}{D(\overline{y}, \overline{x})} \geq \cos \gamma/2 \geq \cos \alpha/2$$

by the previous computation. Hence A_α is $\cos \alpha/2$-flat.

Example 4.2. Let arc B consist of the union of two line segments in the plane with a common endpoint and β the least positive angle between them, so that $0 < \beta \leq \pi$. The following computation shows that B is $(-\cos \beta)$-flat.

By means of polar coordinates B may be represented as an arc from $\overline{a} = (a, 0)$ to $\overline{b} = (b, \beta)$ with the origin $\overline{0} = (0, 0)$ as the common end-

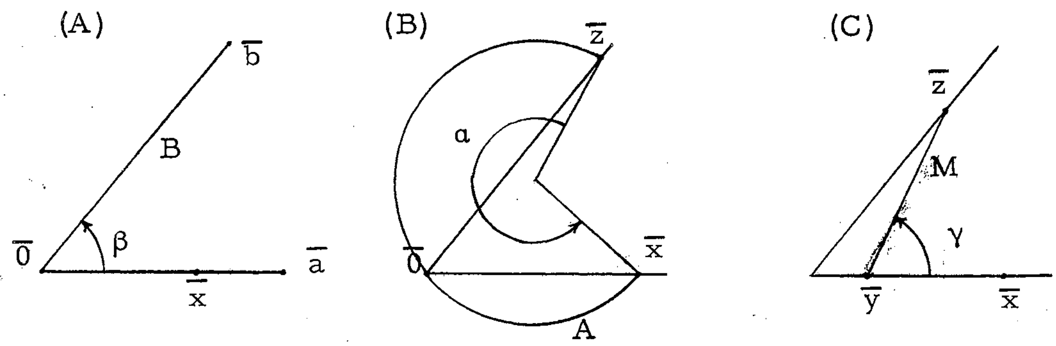

Figure 8.

point of the segments, as in Figure 8 (A). The proof that B is $(-\cos\beta)$-flat consists of two steps: one, showing that quotients of the types

$$\frac{D(\bar{x}, \bar{z}) - D(\bar{y}, \bar{z})}{D(\bar{y}, \bar{x})} \text{ and } \frac{D(\bar{x}, \bar{z}) - D(\bar{y}, \bar{x})}{D(\bar{y}, \bar{z})}$$

come arbitrarily close to the value $-\cos\beta$ for $\overline{x\,y\,z}$; two, showing that $-\cos\beta$ is a lower bound for all quotients of these types.

First, let $\bar{0}$ be the point between $\bar{x} = (x, 0)$ and $\bar{b} = (b, \beta)$. The quotient

$$\frac{D(\bar{x}, \bar{b}) - D(\bar{0}, \bar{b})}{D(\bar{0}, \bar{x})}$$

is given by

$$F(x) = \frac{\sqrt{x^2 + b^2 - 2xb\cos\beta} - b}{x},$$

where $\bar{x}$ approaches $\bar{0}$ from the right. Then

$$\underset{x \to 0^+}{\text{limit}}\ F(x) = \underset{x \to 0^+}{\text{limit}} \frac{x - b\cos\beta}{\sqrt{x^2 + b^2 - 2xb\cos\beta}} = -\cos\beta$$

by the use of L'Hospital's Rule. Thus quotients of the type

$$\frac{D(\overline{x}, \overline{z}) - D(\overline{y}, \overline{z})}{D(\overline{y}, \overline{x})}$$

come arbitrarily close to the value $-\cos\beta$.

Next, it will be shown that $-\cos\beta$ is a lower bound for all quotients of the types

$$\frac{D(x, z) - D(y, z)}{D(\overline{y}, \overline{x})} \text{ and } \frac{D(x, z) - D(y, x)}{D(\overline{y}, \overline{z})}$$

for $\overline{x\,y\,z}$. If $\overline{x}, \overline{y}, \overline{z}$ are on the same line segment of B, then distances between the points are additive, and these quotients have the value $1 \geq -\cos\beta$. If $\overline{y} = \overline{0}$, then a circular arc A from $\overline{x}$ to $\overline{z}$ may be constructed passing through $\overline{y}$, as in Figure 8(B). If A has angular measure α, then the geometrical relationship $\alpha/2 = \pi - \beta$ holds. Since A is $\cos \alpha/2$-flat, then

$$\frac{D(\overline{x}, \overline{z}) - D(\overline{y}, \overline{z})}{D(\overline{y}, \overline{x})} \geq \cos \alpha/2 = -\cos\beta.$$

If $\overline{y}$ is not at the vertex of B, then let $\overline{x}$ and $\overline{y}$ be together on one segment with $\overline{z}$ on the other, as in Figure 8(C). By connecting $\overline{y}$ to $\overline{z}$ with the line segment M, a new arc $M \cup [\overline{x}, \overline{y}]$ with vertex $\overline{y}$ and angular measure $\gamma \geq \beta$ is formed. With $\overline{y}$ at the vertex, it follows that

$$\frac{D(\overline{x}, \overline{z}) - D(\overline{y}, \overline{z})}{D(\overline{y}, \overline{x})} \geq -\cos\gamma \geq -\cos\beta.$$

Thus $-\cos\beta$ is a lower bound for all quotients in question on B, and B is $(-\cos\beta)$-flat.

In the proof of the previous example, a circular arc was circumscribed about a polygonal arc with two segments, as in Figure 8(B). The following example shows that a simple relationship in terms of the index k exists in a more general situation.

Example 4.3. Let B be a polygonal arc in the plane composed of $n \geq 2$ line segments inscribed in a circular arc A of angular measure α, where $0 < \alpha < 2\pi$. The following geometrical argument shows that B is $\cos \alpha/2$-flat.

For n=2, let β denote the angle between the two segments of B. The geometrical relationship $\alpha/2 = \pi - \beta$ and the results obtained in the previous example show that B is $\cos \alpha/2$-flat, since $\cos \alpha/2 = -\cos \beta$.

For $n > 2$, let the line segments of B be denoted in order by $\overline{a_1a_2}, \overline{a_2a_3}, \ldots, \overline{a_na_{n+1}}$. The ith segment $\overline{a_ia_{i+1}}$ is a chord of A which subtends the ith circular subarc of A, denoted by $\widehat{a_ia_{i+1}}$. Figure 9 illustrates this situation when n = 5.

First, for any three points x, y, z of B with xyz it will be shown that the quotient

$$\frac{D(x, z) - D(y, z)}{D(y, x)}$$

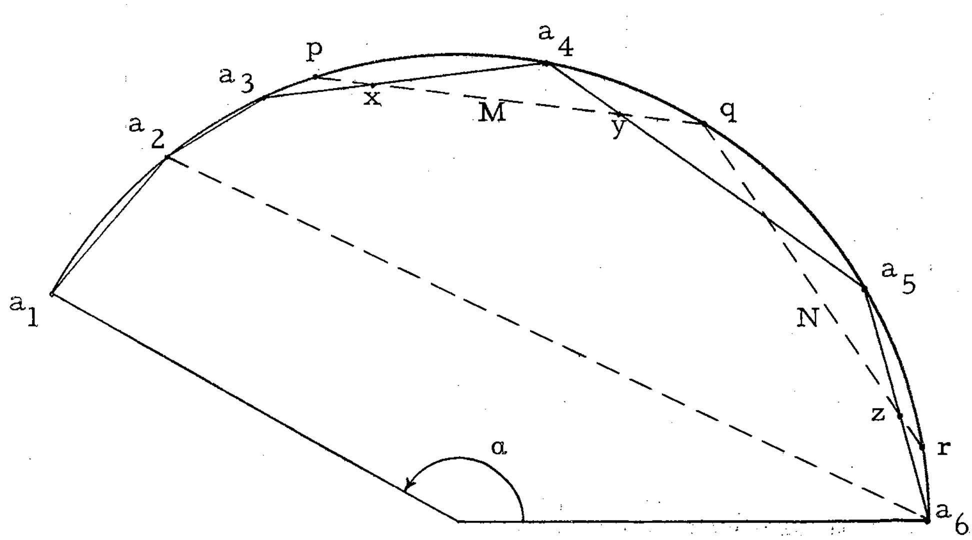

Figure 9.

exceeds cos $\alpha/2$. If x, y, z lie on the same segment of B, then the quotient is $1 > \cos \alpha/2$. If these points are not on the same segment, without loss of generality let x be on the ith segment, y be on the jth segment, and z be on the kth segment, where $i \leq j < k$. If M is the chord of A passing through x and y, the endpoints p and q of M are in $\widehat{a_i a_{i+1}}$ and $\widehat{a_j a_{j+1}}$, respectively. If N is the chord of A starting at q and passing through z, the other endpoint r of N is in $\widehat{a_k a_{k+1}}$. The circular subarc $\widehat{pr}$ of A is of some angular measure $\beta \leq \alpha$, and the arc $M \cup N$ is inscribed in $\widehat{pr}$. Hence $M \cup N$ is cos $\beta/2$-flat, and therefore

$$\frac{D(x, z) - D(y, z)}{D(y, x)} \geq \cos \beta/2 \geq \cos \alpha/2.$$

Therefore B is at least cos $\alpha/2$-flat.

To show that B is exactly cos $\alpha/2$-flat, construct the line segment $\overline{a_2 a_{n+1}}$. The arc $\overline{a_1 a_2} \cup \overline{a_2 a_{n+1}}$ is cos $\alpha/2$-flat, and by the methods of Example 4.2, the value cos $\alpha/2$ is obtained as

$$\lim_{t \to a_2} \frac{D(a_{n+1}, t) - D(a_{n+1}, a_2)}{D(a_2, t)},$$

where t is a point of $\overline{a_1 a_2}$. Hence quotients of B come arbitrarily close to the value cos $\alpha/2$, and therefore B is cos $\alpha/2$-flat.

The previous examples of this chapter exhibit k-flat arcs for k in the range $-1 < k \leq 1$. A simple example of a (-1)-flat arc is now given.

<u>Example 4.4</u>. Let A be an arc in the plane composed of the upper half of the unit circle together with the segment $[-1, 0]$ of the x-axis, as shown in Figure 10. The quotient $\frac{D(x, z) - D(y, z)}{D(y, x)}$, when $x = (0, 0)$, $y = (-1, 0)$, and $z = (1, 0)$, has the value -1. Thus arc A is

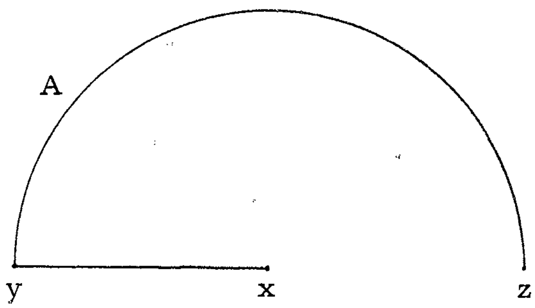

Figure 10.

(-1)-flat.

In this paper, the greatest contribution of the index k is to be found in the following chapter.

CHAPTER V

CONVERGING ARCS

Thus far two basic properties of an arc A with respect to a metric D have been studied: whether or not A has a D-kink, and the value of k such that A is k-flat with respect to D. The present chapter presents various answers to the following question: When a sequence of arcs converges to an arc, which of these properties are preserved? The most general answer is that none of them are. The two following examples show that converging arcs may either lose a D-kink or create a new one on the limit arc.

Example 5.1. For each natural number n let A_n be an arc in the plane composed of the union of two line segments of length 1 which intersect at an angle of $\pi(1-2^{-n})/2$, so that $A = \text{Lim } A_n$, as shown in Figure 11, is an arc whose line segments intersect at an angle of $\pi/2$. Each arc A_n has a D-kink, where D is the usual distance metric. In fact, $A_n(D; c_n) = A_n$, where c_n is the point of intersection of the two line segments that compose A_n. However, A has no D-kink. Therefore

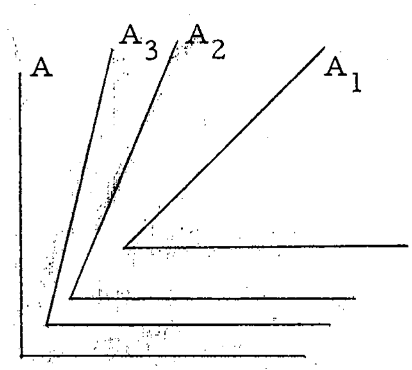

Figure 11.

D is a si-metric on A, and by Theorem 4. 1 arc A is (+)-flat with respect to D. Since Example 4. 2 shows that A is 0-flat, A is an example of a 0-flat arc which is (+)-flat.

Example 5. 2. Let each arc A_n in the complex plane be the union of two circular arcs, each of angular measure $\pi/3$ and radius 1, which meet at the origin, and let the endpoints of each arc A_n be 1 and $e^{i(1+2^{-n+1})\pi/3}$. Then $A = \text{Lim } A_n$ will be the arc from 1 to $e^{i\pi/3}$ composed of two circular arcs of angular measure $\pi/3$ and radius 1 with centers at 1 and $e^{i\pi/3}$, as in Figure 12. No arc A_n has a D-kink. However, A has a D-kink; in fact, $A(D;0) = A$. Therefore D is not a si-metric on A, and by Theorem 4. 1 arc A is (-)-flat with respect to D. Since Theorem 5. 1 will show that A must be at least 0-flat with respect to D, then A is exactly 0-flat and is therefore an example of a 0-flat arc which is (-)-flat.

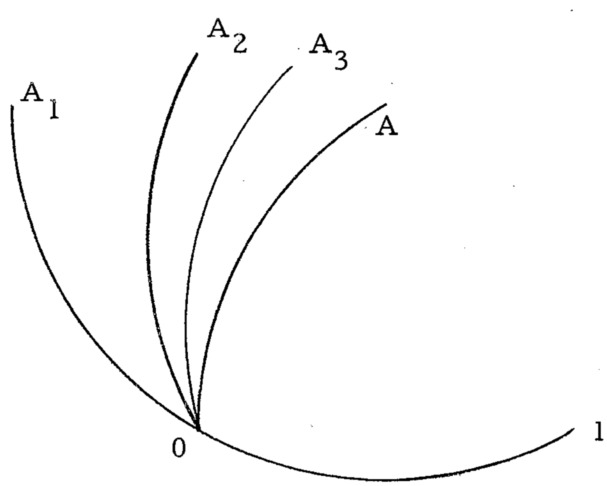

Figure 12.

The next example shows that, at least for $k < 1$, a sequence of k-flat arcs converging to an arc does not force the limit arc to be k-flat.

Example 5.3. For each n let A_n be an arc in the plane from $(0, 1/n)$ to $(2, 1/n)$ consisting of a line segment broken by a semicircle with center $(1, 1/n)$ and radius $1/n$, as shown in Figure 13. The limit arc A is the line segment from (0, 0) to (2, 0). With respect to the usual distance metric each arc A_n is 0-flat, but A is 1-flat.

In this example each arc A_n could have been constructed k-flat for any $0 < k < 1$ by reducing the semicircle to a circular arc of lesser angular measure. Further deformations in the middle section of each A_n could have given each arc A_n any chosen permissible negative value of the index k. However, if each A_n is k-flat, the limit arc in any case is at least k-flat. The next theorem, which is the fundamental result of this chapter, shows that this is inevitable.

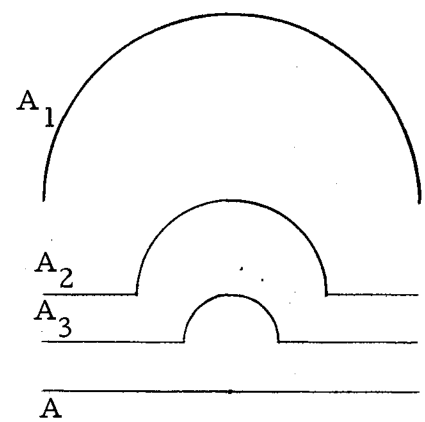

Figure 13.

Theorem 5.1. Let $\{A_n\}$ be a sequence of arcs converging to an arc A such that $\overline{\bigcup A_n}$ is compact. If each arc A_n is at least k-flat with respect to metric D for some fixed value k, then A is at least k-flat with respect to D.

Proof. If $k = -1$, the theorem is obvious since every arc is at least (-1)-flat. For $k > -1$ the proof is given by contraposition. If A is not at least k-flat, then there are points x, y, z of A with xyz and $D(x, z) - D(y, z) < k D(y, x)$. Since $A = \text{Lim } A_n$, there are sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converging respectively to x, y, and z, where x_n, y_n, z_n are points of arc A_n for each n.

Let $6\epsilon = \min\{D(x, y),\ D(x, z),\ D(y, z),\ kD(y, x) - D(x, z) + D(y, z),$

$$2(1+k)\ D(y, z)/3,\ 2(1+k)\ D(y, x)/3\}.$$

There is a number N such that

$D(x, x_n) < \epsilon$, $D(y, y_n) < \epsilon$, $D(z, z_n) < \epsilon$ for all $n > N$. If follows that $D(x_n, z_n) \leq D(x, x_n) + D(x, z) + D(z, z_n) < D(x, z) + 2\epsilon$ and $D(y, z) \leq D(y, y_n) + D(y_n, z_n) + D(z, z_n) < D(y_n, z_n) + 2\epsilon$ for such n. Hence $D(x_n, z_n) - D(y_n, z_n) < D(x, z) - D(y, z) + 4\epsilon \leq k D(x, y) - 2\epsilon$. If $k \geq 0$, then since $D(x, y) \leq D(x, x_n) + D(x_n, y) + D(y, y_n) < D(x_n, y_n) + 2\epsilon$, it follows that $k\ D(x, y) \leq k D(x_n, y_n) + 2k\epsilon$, and therefore $k D(x, y) - 2\epsilon \leq k D(x, y) - 2k\epsilon \leq k D(x_n, y_n)$ since $k \leq 1$. If $k < 0$, then since $D(x_n, y_n) \leq D(x, x_n) + D(x, y) + D(y, y_n) < D(x,y) + 2\epsilon$, it follows that $k D(x, y) - 2\epsilon \leq k D(x, y) + 2k\epsilon \leq k D(x_n, y_n)$ since $k \geq -1$. In either case $k D(x, y) - 2\epsilon \leq k D(x_n, y_n)$, and therefore $D(x_n, z_n) - D(y_n, z_n) < k D(x_n, y_n)$ when combined with the previous inequality. This last result contradicts the fact that A_n is at least k-flat if it can be shown that $x_n y_n z_n$ on A_n, which is in fact the case for large enough n.

There exist points p and q of $A \cap N_\epsilon(y)$ such that xpy and yqz, as in Figure 14. If A is represented as [a, b], where x is in [a, y] and z is in [y, b], then the subarcs [a, p] and [q, b] are disjoint, non-empty, compact sets, and therefore $D([a, p], [q, b])$ is a positive number. Let $\delta = \min\{\epsilon,\ D([a, p], [q, b])/2\}$, and let $U = \bigcup\{N_\delta(t) : t \text{ is in } [a, p]\}$ and $V = \bigcup\{N_\delta(t) : t \text{ is in } [q, b]\}$.

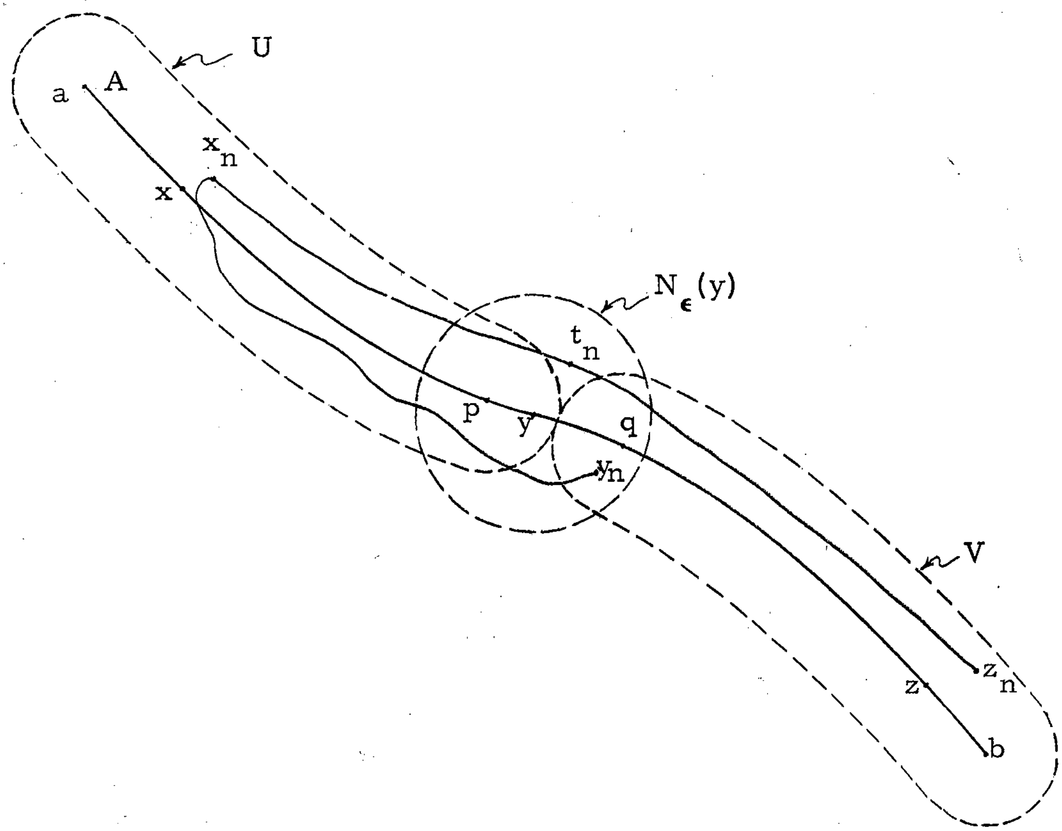

Figure 14.

The sets U and V are disjoint, for suppose s is in $U \cap V$. Then there are points t' in [a, p] and t'' in [q, b] such that $D(t', s) < \delta$ and $D(t'', s) < \delta$. Hence $D(t', t'') \leq D(t', s) + D(t'', s) < 2\delta \leq D([a, p], [q, b])$, and the strict inequality contradicts the definition of distance between two sets. Hence U and V are disjoint. Let $W = U \cup N_\epsilon(y) \cup V$. Since W is an open set containing A and since $\overline{\cup A_n}$ is compact, there is a number M such that if $n > M$, then $A_n \subset W$ [3-p. 105]. Let $n > N+M$ be an integer such that $D(x, x_n) < \delta$, $D(y, y_n) < \delta$, and $D(z, z_n) < \delta$.

Suppose that y_n is not between x_n and z_n. The choice of the constant ϵ will lead to a contradiction in this event. First of all, since

$D(y_n, x_n) \geq D(y, x) - D(y, y_n) - D(x, x_n) > D(y, x) - 2\delta \geq D(y, x) - 2\epsilon \geq 4\epsilon > 0$, y_n and x_n are distinct. Similarly y_n and z_n are distinct, and x_n and z_n are distinct. Therefore if y_n is not between x_n and z_n, then either $y_n x_n z_n$ or $x_n z_n y_n$. For the sake of definiteness, let $y_n x_n z_n$. The proof for $x_n z_n y_n$ is completely analogous.

Since $A_n \subset W$, the subarc $[x_n, z_n]$ of A_n is contained in W. If $[x_n, z_n] \subset U \cup V$, then, $[x_n, z_n] = U_n \cup V_n$, where $U_n = U \cap [x_n, z_n]$ and $V_n = V \cap [x_n, z_n]$. Since $D(x, x_n) < \delta$ and $D(z, z_n) < \delta$, x_n is in U_n and z_n is in V_n; and therefore U_n and V_n are nonempty. U_n and V_n are disjoint since $U_n \cap V_n \subset U \cap V = \emptyset$. Also, U_n and V_n are both open in $[x_n, z_n]$ since U and V are open sets. Therefore $[x_n, z_n] = U_n \cup V_n$ is a separation of $[x_n, z_n]$, which is impossible. Hence there is a point t_n of $[x_n, z_n]$ in $N_\epsilon(y)$. Now $x_n \neq t_n$, for $D(x_n, t_n) \geq D(x_n, y_n) - D(y_n, t_n) > 4\epsilon - \epsilon = 3\epsilon > 0$. Therefore it is true that $y_n x_n t_n$.

Now since $6\epsilon \leq D(y, x)$ and $6\epsilon \leq 2(1+k)\, D(y, x)/3$, it follows that $2D(y, x)/3 \leq D(y, x) - 2\epsilon$, and therefore that $6\epsilon \leq (1+k)\,[D(y, x) - 2\epsilon]$. But since $D(y, x) \leq D(y, y_n) + D(y_n, x_n) + D(x, x_n) < D(y_n, x_n) + 2\delta \leq D(y_n, x_n) + 2\epsilon$, then $D(y, x) - 2\epsilon < D(y_n, x_n)$. Hence $6\epsilon \leq (1+k)\, D(y_n, x_n)$, or $6\epsilon - D(y_n x_n) \leq k\, D(y_n, x_n)$. The following inequalities also hold:

$$D(y_n, t_n) \leq D(y, y_n) + D(y, t_n) < \delta + \epsilon \leq 2\epsilon$$

$$D(y, x) \leq D(y, t_n) + D(x_n, t_n) + D(x, x_n) < D(x_n, t_n) + \epsilon + \delta \leq D(x_n, t_n) + 2\epsilon$$

$$D(y_n, x_n) \leq D(y, y_n) + D(y, x) + D(x, x_n) < D(y, x) + 2\delta \leq D(y, x) + 2\epsilon.$$

It follows that

$$D(y_n, t_n) - D(x_n, t_n) < 4\epsilon - D(y, x) < 6\epsilon - D(y_n, x_n) \leq k\, D(y_n, x_n).$$

That is,

$$\frac{D(y_n, t_n) - D(x_n, t_n)}{D(y_n, x_n)} < k$$

for $y_n x_n t_n$ on arc A_n. But this inequality contradicts the fact that A_n is at least k-flat. This contradiction to the hypothesis completes the proof.

Corollary 5.1.1. Let $\{A_n\}$ be a sequence of arcs converging to an arc A such that $\overline{\bigcup A_n}$ is compact. If each arc A_n is 1-flat with respect to metric D, then A is 1-flat with respect to D.

Proof. By Theorem 5.1 arc A is at least 1-flat, that is, A is 1-flat.

Thus the value 1 is the only value of k which is always preserved under convergence.

Corollary 5.1.2. Let $\{A_n\}$ be a sequence of arcs converging to an arc A such that $\overline{\bigcup A_n}$ is compact. If there is a value $k > 0$ such that each arc A_n is at least k-flat, then D is a si-metric on A.

Proof. Arc A is at least k-flat and is therefore (+)-flat. By Theorem 4.1, D is a si-metric on A.

Corollary 5.1.3. Let $\{A_n\}$ be a sequence of arcs converging to an arc A such that $\overline{\bigcup A_n}$ is compact. If D is a si-metric on each arc A_n, then $D(x, z) \geq D(y, z)$ and $D(x, z) \geq D(y, x)$ for every three points x, y, z of A with xyz.

Proof. By Theorem 4.1 each arc A_n is at least 0-flat, and by Theorem 5.1 arc A is at least 0-flat. Therefore all quotients

$$\frac{D(x, z) - D(y, z)}{D(y, x)} \quad \text{and} \quad \frac{D(x, z) - D(y, x)}{D(y, z)}$$

for xyz on A are non-negative, and the corollary follows.

The previous corollary shows that the limit arc of a sequence of arcs which have no D-kink must come very close, in a sense, to being an arc with no D-kink. That is, Corollary 3. 3. 1 shows that D will be a si-metric on the limit arc A if $D(x, z) > D(y, z)$ and $D(x, z) > D(y, x)$ for every three points x, y, z of A with xyz, and Corollary 5. 1. 3 almost insures this condition on A. In effect, Corollary 5. 1. 3 says that if such a limit arc A does have a D-kink, it must be a rather special kind of arc. Corollary 5. 1. 4 will describe such an arc explicitly. An illustration of both the previous corollary and the following one is found in Example 5. 2.

<u>Corollary 5. 1. 4</u>. Let $\{A_n\}$ be a sequence of arcs converging to arc A such that $\overline{\cup A_n}$ is compact. If D is a si-metric on each arc A_n and if there are points x, y, z of A with y in $A-[x, z]$ and $D(x, y) = D(y, z)$, then $D(t, y) = D(y, z)$ for every point t in the subarc $[x, z]$ of A.

<u>Proof</u>. Without loss of generality let xzy. According to Corollary 5. 1. 3, $D(y, t) \leq D(x, y)$ since xty and $D(z, y) \leq D(y, t)$ since tzy. Therefore, $D(z, y) = D(y, t) = D(x, y)$.

In conclusion, the author would like to mention some questions for further study which remain, to his knowledge, unanswered. With reference to the characterization given in Corollary 3. 3. 2 of when a metric is a si-metric on an arc, do the exponents p yield some index which describes the geometrical configuration of the arc? If so, how does this index relate to the index k developed in Chapter IV? May the index k for sections of the graphs of algebraic and transcendental functions be determined analytically? In particular, what relationship, if any, is there between the index k and the derivatives of such functions? Finally, for a sequence $\{A_n\}$ of arcs converging to an arc A, what is a necessary and sufficient condition on the arcs A_n so that the metric for the space will be si-metric on A?

A SELECTED BIBLIOGRAPHY

1. Blumenthal, Leonard M. Theory and Applications of Distance Geometry. Oxford: Clarendon Press, 1953.

2. Glynn, William A. "Strictly Convex Metrics and Natural Delta Functions." (Ph.D. Thesis, Oklahoma State University, 1965).

3. Hall, Dick Wick and Guilford L. Spencer II. Elementary Topology. New York: John Wiley and Sons, 1955.

4. Kuratowski, Kazimierz. Introduction to Set Theory and Topology. New York: Pergamon Press, 1961, International Series of Monographs of Pure and Applied Mathematics, v. 13.

5. Menger, Karl. "Untersuchungen über allgemeine Metrik." Math. Ann., Vol. 100 (1928), 75-163.

6. Whyburn, Gordon Thomas. Topological Analysis. Princeton: Princeton University Press, 1964, Princeton Mathematical Series, No. 23.

VITA

Robert Allen Dooley

Candidate for the Degree of

Master of Science

Thesis: STRICTLY ISOSCELES METRICS ON ARCS

Major Field: Mathematics

Biographical:

Personal Data: Born in Compton, California, January 9, 1944, the son of Mr. and Mrs. William S. Dooley.

Education: Graduated from Castleberry High School, Fort Worth, Texas, in May, 1962; received the Bachelor of Arts degree from The University of Texas at Arlington in 1966, with a major in mathematics; completed requirements for the Master of Science degree at Oklahoma State University in May, 1968, as a National Science Foundation Trainee.

Professional Experience: Graduate Teaching Assistant, Department of Mathematics and Statistics, Oklahoma State University, 1966-68.